Limits and Continuity

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Introduction

- The development of calculus in the seventeenth century by Newton and Leibniz provided scientists with their first real understanding of what is meant by an "instantaneous rate of change" such as velocity and acceleration.
- The fundamental building block on which rates of change rest is the concept of a "**limit**," an idea that is so important that all other calculus concepts are now based on it.
- In this chapter we will develop the concept of a limit in stages, proceeding from an informal,

intuitive notion to a precise mathematical definition.

- We will also develop theorems and procedures for calculating limits, and we will conclude the chapter by using the limits to study "continuous" curves.
- We use limits to describe the way a function varies. Some functions vary continuously; small changes in x produce only small changes in $f(x)$. Other functions can have values that jump, vary erratically, or tend to increase or decrease without bound. The notion of limit gives a precise

way to distinguish among these behaviors.

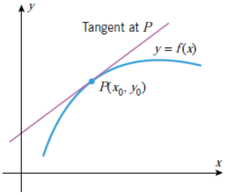
- The concepts of limits and continuity are fundamental to the main subjects of calculus: the derivative and the integral. This chapter opens with a discussion of two seemingly unrelated topics–the line **tangent** to the graph of a function and the **velocity** of a moving object.
- It turns out that these two problems have similar mathematical formulations that lead to the definition of limit. After defining limits and giving rules for evaluating them, we turn to properties of continuous functions, and end the chapter

with a discussion of approximating zeros of functions.

Tangent Lines and Velocity

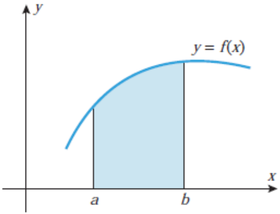
- Many of the ideas of calculus originated with the following two geometric problems:

  The tangent line problem: Given a function f and a point $P(x_0, y_0)$ on its graph, find an equation of the line that is tangent to the graph at P.



The area problem: Given a function f, find the area between the graph of f and an interval $[a, b]$ on the $x-$axis.

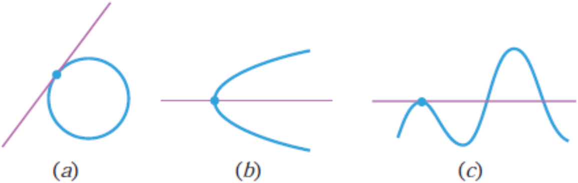
- Traditionally, that portion of calculus arising from the tangent line problem is called **differential calculus** and that arising from the area problem is called **integral calculus**.



Tangent lines and limits

- Since the time of ancient Greece, mathematicians have been interested in tangent lines. Early mathematicians studied tangents to simple curves such as circles and spirals. Euclid, who was the most prominent of them, conceived of a line tangent to a circle as a line that touches the circle at exactly one point.
- In plane geometry, a line is called **tangent** to a circle if it meets the circle at precisely one point (figure (a)). Although this definition is adequate for circles, it is not appropriate for more general

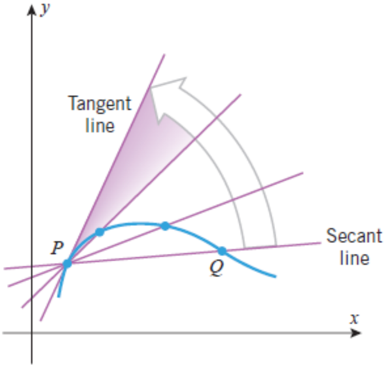
curves. For example, in figure (b), the line meets the curve exactly once but is obviously not what we would regard to be a tangent line; and in figure (c), the line appears to be tangent to the curve, yet it intersects the curve more than once.



- To obtain a definition of a tangent line that applies to curves other than circles, we must view tangent lines another way. For this purpose, suppose that we are interested in the tangent line at a point P on a curve in the $xy-$plane and that Q is any point that lies on the curve and is different from P. The line through P and Q is called a **secant line** for the curve at P.
- Intuition suggests that if we move the point Q along the curve toward P, then the secant line will rotate toward a limiting position. The line in this limiting position is what we will consider

to be the tangent line at P (figure below).

- Note that this new concept of a tangent line coincides with the traditional concept when applied to circles.



- It turns out that the line tangent to the curve at P is the **limit** of secant lines through P, in the sense that the slope of the secant line approaches the slope of the tangent line as Q approaches P.
- This property of the tangent line provides a method for finding an equation of the tangent line, even when the curve to which it is tangent is a circle or not.

Example. Find an equation for the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

Solution: If we can find the slope m_{tan} of the tangent line at P, then we can use the point P

and the point-slope formula for a line to write the equation of the tangent line as

$$y - 1 = m_{tan}(x - 1)$$

To find the slope m_{tan}, consider the secant line through P and a point $Q(x, f(x)) = Q(x, x^2)$ on the parabola that is distinct from P. The slope m_{sec} of this secant line is

$$m_{sec} = \frac{x^2 - 1}{x - 1} \qquad (*)$$

If we now let Q move along the parabola, getting closer and closer to P, then the limiting position of the secant line through P and Q will coincide with that of the tangent line at P.

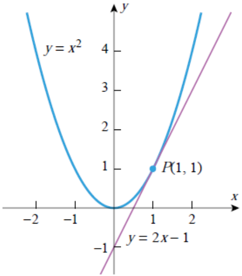
This in turn suggests that the value of m_{sec} will get closer and closer to the value of m_{tan} as P moves toward Q along the curve. However, to say that $Q(x, x^2)$ gets closer and closer to $P(1, 1)$ is algebraically equivalent to saying that x gets closer and closer to 1.

Thus, the problem of finding m_{tan} reduces to finding the "limiting value" of m_{sec} in formula $(*)$ as x gets closer and closer to 1 (but with $x \neq 1$ to ensure that P and Q remain distinct).

We can rewrite $(*)$ as

$$m_{sec} = \frac{x^2 - 1}{x - 1} = \frac{(x-1)(x+1)}{x-1} = x + 1$$

It is now evident that m_{sec} gets closer and closer to 2 as x gets closer and closer to 1. Thus, $\boldsymbol{m_{tan} = 2}$ and the equation of the tangent line is $y - 1 = 2(x - 1)$ or $\boldsymbol{y = 2x - 1}$. ❑



- More generally, if f is any function, then the **slope of the secant line** through $(a, f(a))$ and any other point $(x, f(x))$ on the graph of f is

$$m_{sec} = \frac{f(x) - f(a)}{x - a}$$

- In case the numbers $\frac{f(x)-f(a)}{x-a}$ approach a limiting value as x approaches a, we will define the line **tangent** to the graph of f at $(a, f(a))$ to be the line through $(a, f(a))$ whose **slope** is

$$m_{tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \qquad (**)$$

Velocity and limits

- The limit in (∗∗) also appears in the study of velocity. Suppose a spacecraft is launched from earth and travels vertically upward. If the rocket travels 32 miles during a 2−minute time interval, then the average velocity during that interval is 16 miles per minute.
- In general, suppose that a moving object has traveled distance $f(t)$ at time t. The **average velocity** of the object during an interval of time $[t_1, t_2]$ is defined by

$$\vec{v}_{av} = \frac{\text{distance traveled}}{\text{elapsed time}} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

- Instead of finding average velocity we would like to be able to calculate the velocity at a particular moment (sometimes called **instantaneous velocity**). This is the number that can be read from a speedometer.
- One way to calculate this velocity is to consider it as the limit of average velocities. In particular, let $f(t)$ be the height (in miles) of the rocket t minutes after launch. If t is a little larger than 2, then the distance traveled in the interval from

2 to t is $f(t) - f(2)$, and the elapsed time is $t - 2$. Consequently the average velocity during the time interval from 2 to t is given by

$$\frac{f(t) - f(2)}{t - 2}$$

The closer t is to 2, the closer we would expect the average velocity to be to the velocity at 2.

- It is therefore natural to define the velocity of the rocket at time 2 to be the limit of $\frac{f(t)-f(2)}{t-2}$ as t approaches 2, which in limit notation we would write as

$$\lim_{t \to 2} \frac{f(t) - f(2)}{t - 2}$$

- More generally, the **velocity $\boldsymbol{v(t_0)}$** at time t_0 of an object traveling in a straight line with position $f(t)$ at time t is given by

$$\boxed{v(t_0) = \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0}}$$

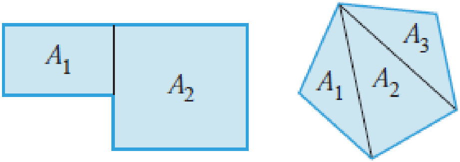
- For **example**, suppose a spacecraft is headed vertically upward, and that $f(t)$ is its height (in miles) t minutes after launch, with $f(t) = t^2$ for $0 \leq t \leq 2$. Then the velocity $v(1)$ of the spacecraft at time $t = 1$ is given by

$$v(1) = \lim_{t\to 1} \frac{f(t) - f(1)}{t - 1} = \lim_{t\to 1} \frac{t^2 - 1}{t - 1}$$
$$= \lim_{t\to 1}(t + 1) = 2$$

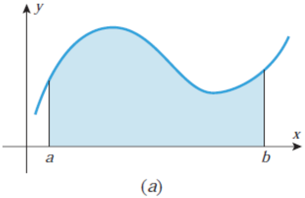
Therefore $\boldsymbol{v(1) = 2}$ mil/min. ❑

Areas and limits

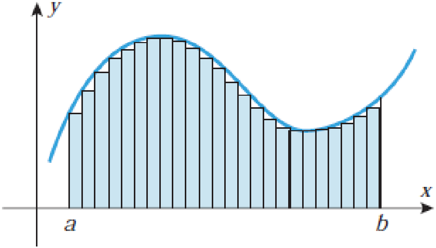
- Just as the general notion of a tangent line leads to the concept of limit, so does the general notion of area. For plane regions with straight-line boundaries, areas can often be calculated by subdividing the region into rectangles or triangles and adding the areas of the constituent parts.



- However, for regions with curved boundaries, such as that in figure (a) below, a more general approach is needed. One such approach is to begin by approximating the area of the region by inscribing a number of rectangles of equal width under the curve and adding the areas of these rectangles (figure (b)).



- Intuition suggests that if we repeat that approximation process using more and more rectangles, then the rectangles will tend to fill in the gaps under the curve, and the approximations will get closer and closer to the exact area under the curve (figure (c) below).



(c)

- This suggests that we can define the area under the curve to be the **limiting value** of these approximations. This idea will be considered in detail later, but the point to note here is that once again the concept of a limit comes into play.

Decimals and limits

- Limits also arise in the familiar context of decimals. For example, the decimal expansion of the fraction $\frac{1}{3}$ is $0.33333\cdots$
- Although we may not have thought about decimals in this way, we can write the expansion as

$$\frac{1}{3} = 0.3333\cdots = 0.3 + 0.03 + 0.003 + 0.0003 + \cdots$$

which is a sum with "infinitely many" terms.

- As we will discuss in more detail later, we interpret this to mean that the succession of finite sums 0.3, $0.3 + 0.03$, $0.3 + 0.03 + 0.003$, $0.3 + 0.03 + 0.003 + 0.0003, \cdots$ gets **closer and closer** to a **limiting value** of $\frac{1}{3}$ as more and more terms are included.
- Thus, limits even occur in the familiar context of decimal representations of real numbers.