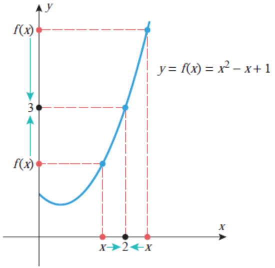
Limits

- Now that we have seen how limits arise in various ways, let us focus on the limit concept itself. The most basic use of limits is to describe how a function behaves as the independent variable approaches a given value.
- For example, let us examine the behavior of the function $f(x) = x^2 - x + 1$ for $x-$values closer and closer to 2.
- It is evident from the graph and table below that the values of $f(x)$ get closer and closer to 3 as values of x are selected closer and closer to 2 on either the left or the right side of 2.

- We describe this by saying that the " limit of $x^2 - x + 1$ is 3 as x approaches 2 from either side," and we write $\lim_{x \to 2}(x^2 - x + 1) = 3$.



x	1.0	1.5	1.9	1.95	1.99	1.995	1.999	2	2.001	2.005	2.01	2.05	2.1	2.5	3.0
$f(x)$	1.000000	1.750000	2.710000	2.852500	2.970100	2.985025	2.997001		3.003001	3.015025	3.030100	3.152500	3.310000	4.750000	7.000000

Left side → ← Right side

Definition (Intuitive/Informal Approach)

Suppose that $f(x)$ is defined on an open interval about a number a, except possibly at a itself. If the values of $f(x)$ can be made as close as we like to the number L by taking values of x sufficiently close to a from both sides (but not equal to a), then we write

$$\lim_{x \to a} f(x) = L$$

which is read "the limit of $f(x)$ as x approaches a is L" or "$f(x)$ approaches L as x approaches a." This expression can also be written as

$$f(x) \to L \text{ as } x \to a$$

Example 1. Use numerical evidence to make a conjecture about the value of $\lim\limits_{x \to 1} \dfrac{x-1}{\sqrt{x}-1}$.

Solution: Although the given function is undefined at $x = 1$, this has no bearing on the limit. Table below shows sample $x-$values approaching 1 from the left side and from the right side. In both cases the corresponding values of $f(x)$, calculated to six decimal places, appear to get closer and closer to 2, and hence we conjecture

x	0.99	0.999	0.9999	0.99999	**1**	1.00001	1.0001	1.001	1.01
$f(x)$	1.994987	1.999500	1.999950	1.999995		2.000005	2.000050	2.000500	2.004988

Left side → ← Right side

that $\lim\limits_{x\to 1} \dfrac{x-1}{\sqrt{x}-1} = \mathbf{2}$.

In the next section we will show how to obtain this result algebraically. ❑

- In our discussion of tangent lines and velocity in previous section we were led to limits of the form $\lim\limits_{x\to a} \frac{f(x)-f(a)}{x-a}$. Some examples, such as $\lim\limits_{x\to 1} \frac{x^2-1}{x-1}$ are easy to evaluate. However, other limits in this form, such as $\lim\limits_{x\to 0} \frac{\sin x - \sin 0}{x-0} = \lim\limits_{x\to 0} \frac{\sin x}{x}$ are not so accessible.
- Moreover, we will encounter many diverse limits that don't involve fractions. As a result, we need

a way of defining limits that allows us to evaluate limits not only arising from tangents and velocity but in many other areas.

Example 2. Use numerical evidence to make a conjecture about the value of

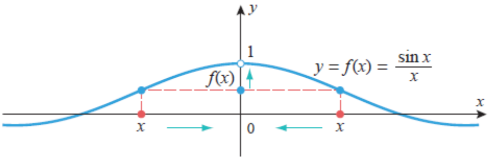
$$\lim_{x \to 0} \frac{\sin x}{x}$$

Solution: With the help of a calculating utility set in radian mode, we obtain the following table. The data in the table suggest that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

The result is consistent with the graph of $f(x) = (\sin x)/x$ shown in figure below. Later we will give a geometric argument to prove that our conjecture is correct. ❑

x (RADIANS)	$y = \frac{\sin x}{x}$
± 1.0	0.84147
± 0.9	0.87036
± 0.8	0.89670
± 0.7	0.92031
± 0.6	0.94107
± 0.5	0.95885
± 0.4	0.97355
± 0.3	0.98507
± 0.2	0.99335
± 0.1	0.99833
± 0.01	0.99998



- Note that numerical evidence can sometimes lead to **incorrect conclusions** about limits because of roundoff error or because the sample values chosen do not reveal the true limiting behavior.

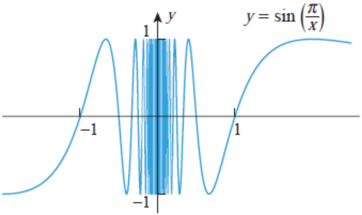
- For example, one might incorrectly conclude from the following table that

$$\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right) = 0$$

- The fact that this is not correct is evidenced by the graph of f in figure below. The graph reveals that the values of f oscillate between -1 and 1 with increasing rapidity as $x \to 0$ and hence do not approach a limit.

- The data in the table deceived us because the $x-$values selected all happened to be $x-$intercepts for $f(x)$. This points out the need for having alternative methods for corroborating limits conjectured from numerical evidence.

x	$\frac{\pi}{x}$	$f(x) = \sin\left(\frac{\pi}{x}\right)$
$x = \pm 1$	$\pm\pi$	$\sin(\pm\pi) = 0$
$x = \pm 0.1$	$\pm 10\pi$	$\sin(\pm 10\pi) = 0$
$x = \pm 0.01$	$\pm 100\pi$	$\sin(\pm 100\pi) = 0$
$x = \pm 0.001$	$\pm 1000\pi$	$\sin(\pm 1000\pi) = 0$
$x = \pm 0.0001$	$\pm 10{,}000\pi$	$\sin(\pm 10{,}000\pi) = 0$
$\vdots$	$\vdots$	$\vdots$



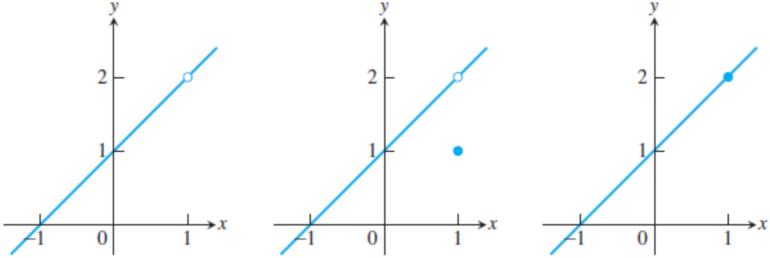
Notice:

- Our definition above is informal, because phrases like arbitrarily close and sufficiently close are imprecise; their meaning depends on the context.
- Nevertheless, the definition is clear enough to enable us to recognize and evaluate limits of many specific functions.
- We will need the precise definition given in the other section, when we set out to prove theorems about limits or study complicated functions.
- The value of the function at 'a' itself is not considered. That is,

- The limit of a function does not depend on how the function is defined at the point being approached.

- Consider the three functions in figure below. The function f has limit 2 as $x \to 1$ even though f is not defined at $x = 1$. The function g has limit 2 as $x \to 1$ even though $2 \neq g(1)$. The function h is the only one of the functions in the figure whose limit as $x \to 1$ equals its value at $x = 1$.
- That is, the limits of $f(x), g(x)$, and $h(x)$ all equal 2 as x approaches 1. However, only $h(x)$ has the same function value as its limit at $x = 1$.

- If $f(x)$ approaches to different numbers as x approaches to a from the right and from the left, then we conclude that $\lim\limits_{x \to a} f(x)$ **does not exist**.



(a) $f(x) = \dfrac{x^2 - 1}{x - 1}$

(b) $g(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$

(c) $h(x) = x + 1$

Two Basic Limits

- The process of finding a limit can be broken up into a series of steps involving limits of basic functions, which are combined using a sequence of simple operations that we will develop.

i. If f is the **constant function** $f(x) = k$, then for any value of a

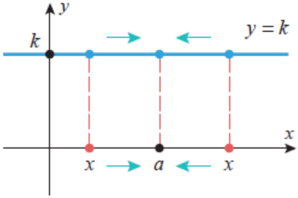
$$\lim_{x\to a} f(x) = \lim_{x\to a} k = k$$

ii. If f is the **identity function** $f(x) = x$, then for any value of a

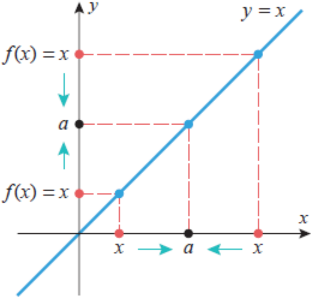
$$\lim_{x\to a} f(x) = \lim_{x\to a} x = a$$

Example. $\lim_{x\to 2} 5 = \lim_{x\to -3} 5 = \lim_{x\to \pi} 5 = 5$, and
$\lim_{x\to 6} x = 6$, $\lim_{x\to 0} x = 0$, $\lim_{x\to -2} x = -2$.

These rules are illustrated in figure below.



$\lim_{x\to a} k = k$



$\lim_{x\to a} x = a$

The Limit Laws/Properties
(Basic Limit Theorems)

- Limits appear, either explicitly or implicitly, throughout calculus. Fortunately it is not necessary to apply the definition of limit every time a limit is to be evaluated.
- The reason is that most of the functions that appear in calculus are combinations (such as sums, products, quotients, and composites) of simpler functions, and there are rules for finding limits of combinations of functions whose limits are already known.

- For **example**, the function $x+3$ is the sum of the functions x and 3, and we know how to find the limits of x and 3 from the two basic limits.
- Similarly, the function x^2 is the product $x \cdot x$, and we know the limit of x at any number a. Once we know the rules for finding limits of combinations, we will be able to calculate the limits of $x+3$ and x^2 with ease.
- Now we will develop a repertoire of theorems that will enable us to use the limits of the basic simple functions as building blocks for finding limits of more complicated functions.

- The following theorem will be our basic tool for finding limits algebraically.

  Theorem (Limit Laws): Let a and k be real numbers, and suppose that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. That is, the limits exist and have values L and M, respectively. Then

  $$\lim_{x\to a}[f(x)+g(x)], \quad \lim_{x\to a} kf(x), \quad \lim_{x\to a}[f(x)-g(x)],$$

  $$\lim_{x\to a}[f(x)g(x)], \text{ and } \lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] \text{ (provided that}$$

  $\lim_{x\to a} g(x) \neq 0$) exist, and

a) $\lim\limits_{x\to a}[f(x)+g(x)] = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x) = L + M$

(sum rule)

b) $\lim\limits_{x\to a} kf(x) = k\lim\limits_{x\to a} f(x) = kL$

(constant multiple rule)

c) $\lim\limits_{x\to a}[f(x)-g(x)] = \lim\limits_{x\to a} f(x) - \lim\limits_{x\to a} g(x) = L - M$

(difference rule)

d) $\lim\limits_{x\to a}[f(x)g(x)] = \lim\limits_{x\to a} f(x) \lim\limits_{x\to a} g(x) = LM$

(product rule)

e) $$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim\limits_{x\to a}f(x)}{\lim\limits_{x\to a}g(x)}=\frac{L}{M}\text{ (provided } M\neq 0)$$

(**quotient rule**)

f) $$\lim_{x\to a}[f(x)]^n=\left[\lim_{x\to a}f(x)\right]^n=L^n,\ n\text{ a +ve integer}$$

(**power rule**)

g) $$\lim_{x\to a}\sqrt[n]{f(x)}=\sqrt[n]{\lim_{x\to a}f(x)}=\sqrt[n]{L},\ n\text{ a +ve integer,}$$

and provided that $L>0$ if n is even or if n is even, we assume that $f(x)\geq 0$ for x near a.

(**root rule**)

- Moreover, these statements are also true for the one-sided limits as $x \to a^-$ or as $x \to a^+$.
- This theorem can be stated informally as: the limit of a sum is the sum of the limits, and so on. Part b) can be rephrased as: a constant factor can be moved through a limit symbol.
- Although parts a), c) and d) of the theorem are stated for two functions, the results hold for any **finite number** of functions. Moreover, the various parts of the theorem can be used in combination to reformulate expressions involving limits.
- Note that $\lim_{x \to a} |x| = |a|$.

Example. Evaluate the following limits.

a. $\lim\limits_{x\to -3} x^2$

b. $\lim\limits_{x\to 1}(x^3 - 4x^2 + 7)$

c. $\lim\limits_{x\to 2} \dfrac{5x^3 + 4}{x - 3}$

d. $\lim\limits_{x\to -2} \sqrt{4x^2 - 3}$

Solution:

a) Using the product rule $\lim\limits_{x\to -3} x^2 = \lim\limits_{x\to -3}(x \cdot x)$

$$= \lim_{x\to -3} x \lim_{x\to -3} x = (-3)(-3) = \mathbf{9}.$$

* In general $\boldsymbol{\lim\limits_{x\to a} x^n = a^n}$ for any +ve integer n.

b) $\lim\limits_{x\to1}(x^3-4x^2+7)=\lim\limits_{x\to1}x^3-\lim\limits_{x\to1}4x^2+\lim\limits_{x\to1}7=$

$\lim\limits_{x\to1}x^3-4\lim\limits_{x\to1}x^2+\lim\limits_{x\to1}7=1^3-4(1^2)+7=\mathbf{4}.$

* In general, for a **polynomial**

$$p(x)=c_0+c_1x+\cdots+c_nx^n$$

and any real number a,

$$\boldsymbol{\lim_{x\to a}p(x)=p(a)}$$

c) $\lim\limits_{x\to2}\dfrac{5x^3+4}{x-3}=\dfrac{\lim\limits_{x\to2}(5x^3+4)}{\lim\limits_{x\to2}(x-3)}=\dfrac{5\cdot2^3+4}{2-3}=\mathbf{-44}.$

* Limit of a **rational** function:

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{f(a)}{g(a)} \text{ provided that } g(a)\neq 0.$$

d) $\displaystyle\lim_{x\to -2}\sqrt{4x^2-3}=\sqrt{\lim_{x\to -2}(4x^2-3)}=$

$$\sqrt{\lim_{x\to -2}4x^2-\lim_{x\to -2}3}=\boldsymbol{\sqrt{13}}.$$

Exercise: Find the following limits.

i. $\displaystyle\lim_{x\to 1}(x^7-2x^5+1)^{35}$ **Ans.** 0

ii. $\displaystyle\lim_{x\to -2}\frac{x^2}{|x|+3}$ **Ans.** $4/5$

iii. $\lim\limits_{x \to -1} \dfrac{x^3 + 4x^2 - 3}{x^2 + 5}$ **Ans.** 0

iv. $\lim\limits_{x \to 1} \sqrt[3]{1 - 10x + x^2}$ **Ans.** -2

Eliminating Common Factors from Zero Denominators

- Although the quotient rule does not guarantee the existence of $\lim_{x\to a} \frac{f(x)}{g(x)}$ when $\lim_{x\to a} g(x) = 0$, it is still possible to evaluate such limits.
- That is, the method used in the limit of a rational function applies only if the denominator of the rational function is not zero at the limit point a.
- If the denominator is zero, canceling common factors in the numerator and denominator may reduce the fraction to one whose denominator is

no longer zero at a.

- If this happens, we can find the limit by substitution in the simplified fraction.

Theorem: Let f and g be functions. Suppose $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist and $f(x) = g(x)$, for all $x \neq a$. Then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$.

Example 1. Find $\lim_{x\to 1} \dfrac{x^2 - 1}{x - 1}$.

Solution: We cannot substitute $x = 1$ because it makes the denominator zero. We test the numerator to see if it, too, is zero at $x = 1$.

It is, so it has a factor of $(x-1)$ in common with the denominator. Canceling this common factor gives a simpler fraction with the same values as the original for $x \neq 1$:

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x\to 1} x+1 = \mathbf{2}.$$

Example 2. Evaluate $\displaystyle\lim_{x\to 3} \frac{x^2-6x+9}{x-3}$.

Solution: Since $\lim\limits_{x\to 3}(x-3) = 0$, we cannot apply the quotient rule to this function in its original form. However, since $x^2-6x+9 = (x-3)^2$,

$$\lim_{x\to 3}\frac{x^2-6x+9}{x-3}=\lim_{x\to 3}\frac{(x-3)(x-3)}{x-3}$$
$$=\lim_{x\to 3}(x-3)=\mathbf{0}.\ \square$$

Example 3. Find $\displaystyle\lim_{x\to -4}\frac{2x+8}{x^2+x-12}$.

Solution: The numerator and the denominator both have a zero at $x=-4$, so there is a common factor of $x-(-4)=x+4$. Then

$$\lim_{x\to -4}\frac{2x+8}{x^2+x-12}=\lim_{x\to -4}\frac{2(x+4)}{(x+4)(x-3)}$$

$$= \lim_{x \to -4} \frac{2}{x-3} = -\frac{2}{7}. \quad \square$$

Remark:

- A quotient $f(x)/g(x)$ in which the numerator and denominator both have a limit of zero as $x \to a$ is called an **indeterminate form of type 0/0**.
- The problem with such limits is that it is difficult to tell by inspection whether the limit exists, and, if so, its value.
- Sometimes, limits of indeterminate forms of type $0/0$ can be found by algebraic simplification, as

in the above examples, but frequently this will not work and other methods must be used. We will study such methods later.

- The following theorem summarizes our observations about limits of rational functions.

Theorem: Let $f(x) = \frac{p(x)}{q(x)}$ be a rational function, and let a be any real number.

i. If $q(a) \neq 0$, then $\lim\limits_{x \to a} f(x) = f(a)$

ii. If $q(a) = 0$ but $p(a) \neq 0$, then $\lim\limits_{x \to a} f(x)$ does not exist.

Exercise: Evaluate the following limits.

a) $\displaystyle\lim_{x\to 1}\frac{x^2+x-2}{x^2-x}$ **Ans.** 3

b) $\displaystyle\lim_{x\to -3}\frac{x^3+27}{x+3}$ **Ans.** 27

c) $\displaystyle\lim_{x\to 5}\frac{x^2-3x-10}{x^2-10x+25}$ **Ans.** does not exist

Limits Involving Radicals

Example. Find $\lim_{x\to 1} \dfrac{x-1}{\sqrt{x}-1}$.

Solution: In the previous section we used numerical evidence to conjecture that this limit is 2. Here we will confirm this algebraically. Since this limit is an indeterminate form of type $0/0$, we will need to devise some strategy for making the limit (if it exists) evident. One such strategy is to **rationalize the denominator** of the function:

$$\frac{x-1}{\sqrt{x}-1} = \frac{(x-1)(\boldsymbol{\sqrt{x}+1})}{(\sqrt{x}-1)(\boldsymbol{\sqrt{x}+1})} =$$

$$\frac{(x-1)(\sqrt{x}+1)}{x-1} = \sqrt{x}+1 \quad (x \neq 1)$$

Therefore, $\displaystyle\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1} = \lim_{x\to 1}(\sqrt{x}+1) = 2.$ ❑

Exercise. Evaluate the following limits.

a) $\displaystyle\lim_{x\to 1} \frac{x-1}{\sqrt{x+3}-2}$ **Ans.** 4

b) $\displaystyle\lim_{t\to 4} \frac{t-\sqrt{3t+4}}{4-t}$ **Ans.** $-\frac{5}{8}$

c) $\lim_{x \to 9} \frac{x(\sqrt{x} - 3)}{x - 9}$ **Ans.** $\frac{3}{2}$

d) $\lim_{x \to 0} \frac{\sqrt{x^2 + 100} - 10}{x^2}$ **Ans.** $\frac{1}{20}$

e) $\lim_{x \to 1} \frac{\sqrt[3]{x} - 1}{x - 1}$ **Ans.** $\frac{1}{3}$

(hint: use the substitution $t = \sqrt[3]{x}$)

Some Prominent Limits

There are several limits that will appear frequently in the remainder of this course.

1. If r is any fixed **rational number**, then

$$\lim_{x\to a} x^r = a^r$$

which is valid for any nonzero number a in the domain of x^r and is even valid for $a = 0$ if $r = m/n$, where m and n are positive integers, with n odd.

- For **example**, $\displaystyle\lim_{x\to -32} x^{1/5} = (-32)^{1/5} = -2$ and

$$\lim_{x\to 9} x^{3/2} = 9^{3/2} = 27.$$

2. For any real number a,

$$\lim_{x\to a} e^x = e^a$$

3. One can also show that

$$\lim_{x\to a} \ln x = \ln a \quad \text{for } a > 0$$

4. For any number a,

$$\lim_{x\to a} \sin x = \sin a \quad \text{and} \quad \lim_{x\to a} \cos x = \cos a$$

Note:

$$\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a}(f(x) - L) = 0$$

and

$$\lim_{x\to a} f(x) = L \text{ if and only if } \lim_{h\to 0} f(a+h) = L$$

- Thus the following three statements are equivalent:

$$\lim_{x\to -1} x^2 = 1, \quad \lim_{x\to -1}(x^2-1) = 0, \text{ and}$$

$$\lim_{h\to 0}(-1+h)^2 = 1$$

- The equivalent forms of the limit statement given above will be important when we evaluate limits of exponential and trigonometric functions.

- From (4.) it follows that the sine and cosine functions have the property that the limit at any number a is the value of the function at a.
- The other trigonometric functions have the same property, as can be verified from (4.) by using the limit rules. For **example**,

$$\lim_{x\to a}\tan x = \lim_{x\to a}\frac{\sin x}{\cos x} = \frac{\lim\limits_{x\to a}\sin x}{\lim\limits_{x\to a}\cos x} = \frac{\sin a}{\cos a}$$
$$= \tan a$$

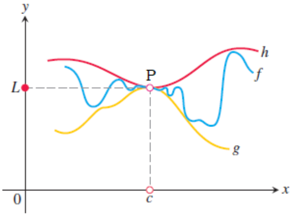
for any number a in the domain of the tangent function.

- In summary, we can now find the limits of many kinds of functions: polynomials, rational functions, basic trigonometric functions, e^x, $\ln x$, x^n (for any integer n).
- More precisely, if f represents any of these functions, then it has a limit at any point a in its domain, which is $f(a)$. The same is true for x^r, where r is any rational number, with some stipulations on a.

The Squeezing/Sandwich Theorem

- The limit rules presented thus far are effective for finding limits of many common functions. However, they don't apply in the evaluation of limits like limit of $f(x) = \frac{\sin x}{x}$ as $x \to 0$.
- Although the limit rules studied so far don't yield definitive information about the limit, and we cannot just evaluate the function $(\sin x)/x$ at 0 because it is not defined at 0, nevertheless we will be able to prove that the limit is 1 with help of the Squeezing Theorem.

- The theorem enables us to calculate a variety of limits. It is called the Sandwich Theorem because it refers to a function f whose values are sandwiched between the values of two other functions g and h that have the same limit L at a point c. Being trapped between the values of two functions that approach L, the values of f must also approach L (figure below).



- The theorem says that if the graphs of g and h converge at a point P in the plane (figure above) and if the graph of f is "squeezed" between the graphs of g and h, then the graph of f converges with the graphs of g and h at P. This result is sometimes called the pinching theorem.

  Squeezing Theorem: Assume that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself.

  Suppose also that $\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$.

  Then $\lim_{x \to c} f(x)$ exists and $\lim_{x \to c} f(x) = L$.

Example. Evaluate $\lim\limits_{x \to 0} x^2 \cos\left(\dfrac{1}{x}\right)$.

Solution: In this example none of the previous examples can help us. There's no factoring or simplifying to do. We can't rationalize and one-sided limits won't work. There's even a question as to whether this limit will exist since we have division by zero inside the cosine at $x = 0$.

The first thing to notice is that we know the following fact about cosine: $-1 \leq \cos x \leq 1$. Our function doesn't have just an x in the cosine, but as long as we avoid $x = 0$ we can say the same thing for our cosine: $-1 \leq \cos\left(\frac{1}{x}\right) \leq 1$.

It's okay for us to ignore $x = 0$ here because we are taking a limit and we know that limits don't care about what's actually going on at the point in question, $x = 0$ in this case.

Now if we have the above inequality for our cosine we can just multiply everything by an x^2 and get $-x^2 \leq x^2 \cos\left(\frac{1}{x}\right) \leq x^2$.

In other words we've managed to squeeze the function that we were interested in between two other functions that are very easy to deal with.

So, the limits of the two outer functions are

$$\lim_{x\to 0} x^2 = 0 \quad \text{and} \quad \lim_{x\to 0}(-x^2) = 0$$

These are the same and so by the squeeze theorem we must also have,

$$\lim_{x\to 0} x^2 \cos\left(\frac{1}{x}\right) = 0. \ \square$$

Fact: If $f(x) \leq g(x)$ for all x on $[a, b]$ (except possibly at $x = c$) and $a \leq c \leq b$ then,

$$\lim_{x\to c} f(x) \leq \lim_{x\to c} g(x)$$

Assignment 1.

1. Using the squeezing theorem show that
$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$
2. Using 1) show that $\displaystyle\lim_{x\to 0} \frac{\cos x - 1}{x} = 0.$

Note: The squeezing theorem helps us establish several important limit rules, for instance, for any function f,

$$\lim_{x\to a} |f(x)| = 0 \text{ implies } \lim_{x\to a} f(x) = 0.$$

Exercise:

1. Find $\lim_{x\to 0} \dfrac{\sin 5x}{x}$. **Ans.** 5

2. Evaluate $\lim_{x\to 0} \dfrac{\sin x}{x^{2/3}}$. **Ans.** 0

Substitution Rule for Limits

- It allows us to evaluate limits of composite functions such as

$$\lim_{x\to 3}\sqrt{25-x^2} \quad \text{and} \quad \lim_{x\to 0}\frac{\sin 2x}{x}$$

- As we would expect, the substitution rule involves a substitution. Let us see informally how the rule works. Consider the limit

$$\lim_{x\to 3}\sqrt{25-x^2}$$

First notice that $\sqrt{25-x^2}$ is the composite $(g\circ f)(x)=g(f(x))$, where $f(x)=25-x^2$ and

$g(x) = \sqrt{x}$. First we substitute y for $f(x)$, that is, we let $y = 25 - x^2$. Next we find that if x approaches 3, then $25 - x^2$ approaches $25 - 9 = 16$, that is, y approaches 16. Therefore

$$\lim_{x \to 3} \sqrt{25 - x^2} = \lim_{y \to 16} \sqrt{y} = \mathbf{4}.$$

- In general, to evaluate $\lim_{x \to a} g(f(x))$, we may substitute y for $f(x)$ and apply the following result.

Substitution Rule: Suppose $\lim_{x\to a} f(x) = c$ and $f(x) \neq c$ for all x in some open interval about a, with the possible exception of a itself. Suppose also that $\lim_{y\to c} g(y)$ exists. Then

$$\lim_{x\to a} g(f(x)) = \lim_{y\to c} g(y)$$

- When we use the substitution rule to find $\lim_{x\to a} g(f(x))$, we first substitute y for $f(x)$, then determine c by the formula $\lim_{x\to a} y = \lim_{x\to a} f(x) = c$, and finally compute $\lim_{y\to c} g(y)$.

Example 1. Evaluate $\lim\limits_{x\to -1} e^{(x^4)}$.

Solution: Let $y = x^4$, and notice that

$$\lim_{x\to -1} y = \lim_{x\to -1} x^4 = (-1)^4 = 1.$$

Then by the substitution rule and limit of exponential function,

$$\lim_{x\to -1} e^{(x^4)} = \lim_{y\to 1} e^y = \boldsymbol{e}. \quad \square$$

- With experience we often will be able to apply the substitution rule mentally, without having to write down all the steps, as we did in example 1.

- That will be especially true if f is **continuous** at a and g is continuous at $f(a)$. In that case, the substitution rule reduces to

$$\lim_{x\to a} g(f(x)) = g(f(a)) \qquad (*)$$

- In particular, $(*)$ applies to the limit in example 1, because x^4 is continuous at -1 and e^x is continuous at 1. Thus $\lim\limits_{x\to -1} e^{(x^4)} = e^1 = e$.
- More generally, since e^x is continuous at each number, we can show that if $\lim\limits_{x\to a} f(x)$ exists,

$$\boxed{\lim_{x\to a} e^{f(x)} = e^{\lim\limits_{x\to a} f(x)}}$$

We will use this formula from time to time.

- In contrast to example 1, we cannot use $(*)$ to evaluate the limit in the next example because we cannot substitute 0 for x in the expression $(\sin 2x)/x$.

Example 2. Find $\lim\limits_{x\to 0} \dfrac{\sin 2x}{x}$.

Solution: Because of the appearance of $2x$ in the numerator, we substitute $y = 2x$ and notice that $\lim\limits_{x\to 0} y = \lim\limits_{x\to 0} 2x = 0$.

From the fact that $x = y/2$, the substitution and constant multiple rules, and previous discussions, we conclude that

$$\lim_{x\to 0}\frac{\sin 2x}{x}=\lim_{y\to 0}\frac{\sin y}{y/2}=2\lim_{y\to 0}\frac{\sin y}{y}=2\cdot 1=\mathbf{2}.$$