Week Three: Discrete Distributions

CS 217

Random Variables

- **Random Variable**: A variable whose possible values are outcomes of a ‘random’ phenomenon
- Throwing a dice or flipping a coin is inherently random but the probable outcomes of each result are not random
- A **probability distribution** is a mathematical distribution that provides the probabilities of occurrence of different outcomes of an experiment

Probability

- In probability there is a clearly defined **experiment**.
  - We will toss exactly four die.
- There is also a clearly defined **sample space**, or range of possible outcomes.
  - If we toss four die, they can add add up to anywhere from 4 (1 * 4) to 24 (6 * 4)
- There may be an **event** that we're looking for.
  - The event that we are looking for here is that our four die add up to exactly 7.
- There is a **probability function**, or a probability of each outcome in our **sample space** occuring.
  - Each of the possible events in our sample space has a predefined probability of occuring.
- **There is a probability distribution, or a** mathematical function that provides the probabilities of occurrence of different possible outcomes in an experiment

Probability

- Let's flip a coin.

Probability

- Let’s flip a coin.
- What is our experiment?

Probability

- Let's flip a coin.
- What is our experiment?
  - We flip a single coin.

Probability

- Let's flip a coin.
- What is our experiment?
  - We flip a single coin.
- What is our sample space?

Probability

- Let's flip a coin.
- What is our experiment?
  - We flip a single coin.
- What is our sample space?
  - Heads and tails.

Probability

- Let's flip a coin.
- What is our experiment?
  - We flip a single coin.
- What is our sample space?
  - Heads and tails.
- What is the event we're looking for?

Probability

- Let's flip a coin.
- What is our experiment?
  - We flip a single coin.
- What is our sample space?
  - Heads and tails.
- What is the event we're looking for?
  - We land on heads.

Probability

- Let's flip a coin.
- What is our experiment?
  - We flip a single coin.
- What is our sample space?
  - Heads and tails.
- What is the event we're looking for?
  - We land on heads.
- What is the probability function of our event occurring?

Probability

- Let's flip a coin.
- What is our experiment?
    - We flip a single coin.
- What is our sample space?
    - Heads and tails.
- What is the event we're looking for?
    - We land on heads.
- What is the probability function of our event occurring?
    - 0.5

Probability

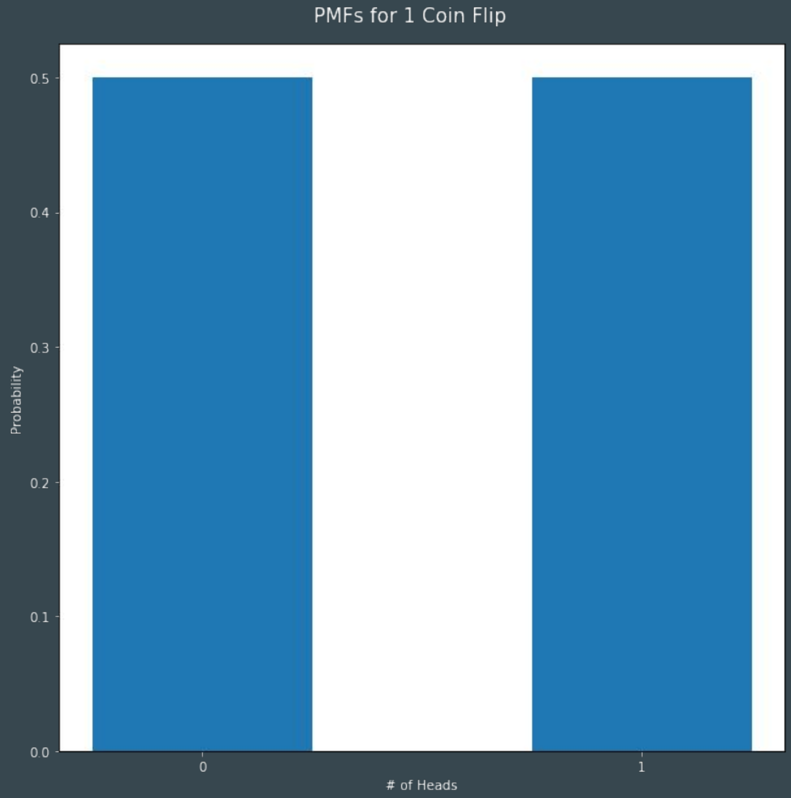
- Let's flip a coin.
- What is our experiment?
  - We flip a single coin.
- What is our sample space?
  - Heads and tails.
- What is the event we're looking for?
  - We land on heads.
- What is the probability function of our event occurring?
  - 0.5
- What is the probability function of our event *not* occurring?

Probability

- Let's flip a coin.
- What is our experiment?
  - We flip a single coin.
- What is our sample space?
  - Heads and tails.
- What is the event we're looking for?
  - We land on heads.
- What is the probability function of our event occurring?
  - 0.5
- What is the probability function of our event *not* occurring?
  - 0.5

Bernoulli Distribution

- Flipping a single coin is an example of a **Bernoulli distribution**, where there is a probability of an event occurring in a single trial
- It has one input - *p*, or the probability of the event occurring.
- The *expected value* for a given Bernoulli experiment is p.



Probability

- Let's flip three coins.

Probability

- Let's flip three coins.
- What is our experiment?

Probability

- Let’s flip three coins.
- What is our experiment?
  - We flip three coins.

Probability

- Let's flip three coins.
- What is our experiment?
  - We flip three coins.
- What is our sample space?

Probability

- Let's flip three coins.
- What is our experiment?
  - We flip three coins.
- What is our sample space?
  - The eight possibilities to our right.

HHH	HHT
HTH	THH
THT	HTT
TTH	TTT

Probability

- Let's flip three coins.
- What is our experiment?
  - We flip three coins.
- What is our sample space?
  - The eight possibilities to our right.
- What is the event we're looking for?

HHH	HHT
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Probability

- Let's flip three coins.
- What is our experiment?
  - We flip three coins.
- What is our sample space?
  - The eight possibilities to our right.
- What is the event we're looking for?
  - We land on heads.

HHH	HHT
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Probability

- Let's flip three coins.
- What is our experiment?
    - We flip three coins.
- What is our sample space?
    - The eight possibilities to our right.
- What is the event we're looking for?
    - We land on heads.
- What is the probability function of our event occurring?

HHH	HHT
HTH	THH
THT	HTT
TTH	TTT

Probability

- 0 heads - ⅛

HHH	HHT
HTH	THH
THT	HTT
TTH	TTT

Probability

- 0 heads - ⅛
- 1 head - ⅜

HHH	HHT
HTH	THH
THT	HTT
TTH	TTT

Probability

- 0 heads - ⅛
- 1 head - ⅜
- 2 heads - ⅜

HHH	HHT
HTH	THH
THT	HTT
TTH	TTT

Probability

- 0 heads - ⅛
- 1 head - ⅜
- 2 heads - ⅜
- 3 heads - ⅛

HHH	HHT
HTH	THH
THT	HTT
TTH	TTT

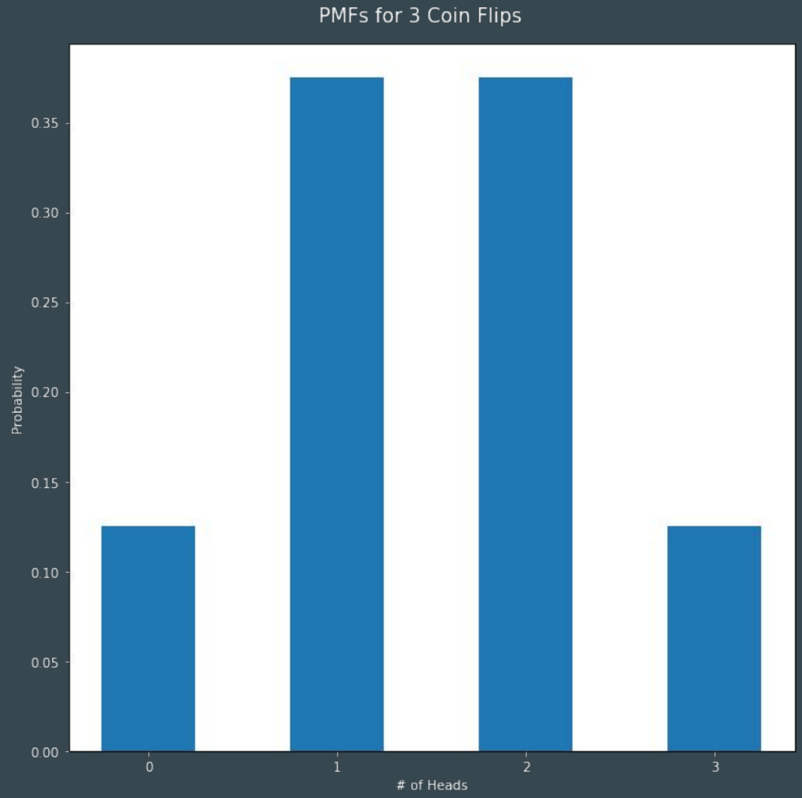
Probability

- 0 heads - ⅛
- 1 head - ⅜
- 2 heads - ⅜
- 3 heads - ⅛
- The probability that a given result occurs in a **discrete distribution** is called the **probability mass function.**

HHH	HHT
HTH	THH
THT	HTT
TTH	TTT

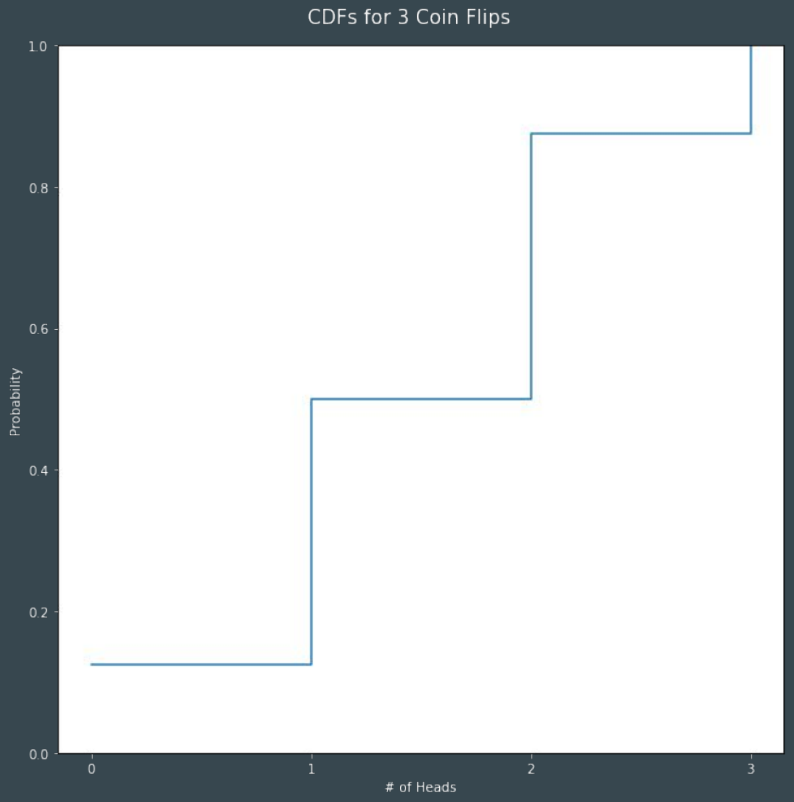
Probability

- 0 heads - ⅛
- 1 head - ⅜
- 2 heads - ⅜
- 3 heads - ⅛
- The probability that a given result occurs in a **discrete distribution** is called the **probability mass function.**



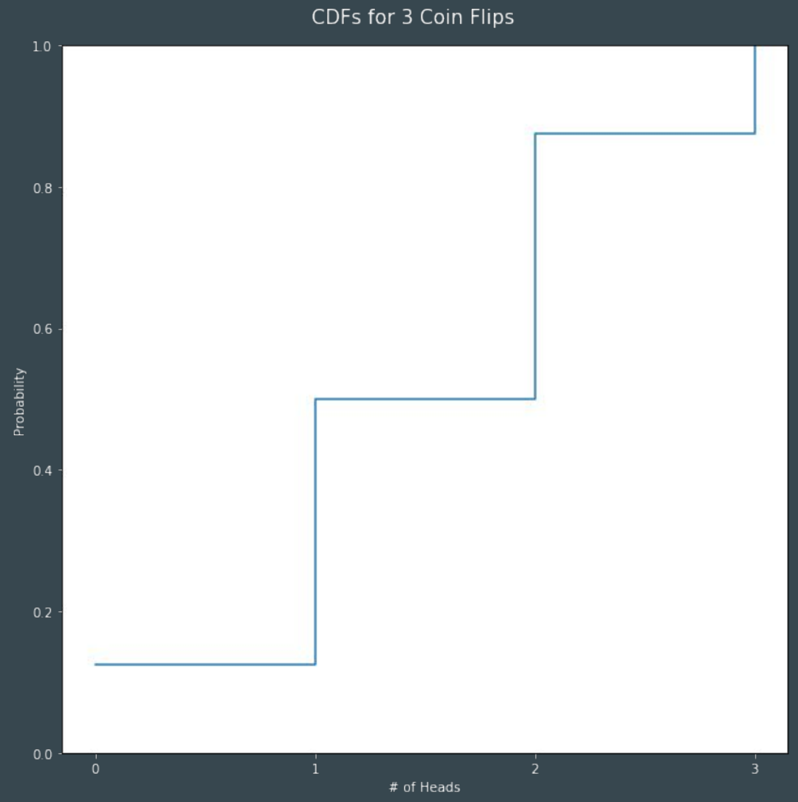
Probability

- The probability that **less than or equal to** a given event occurs is called the **cumulative distribution function**.
- The cumulative distribution function is simply the aggregate of all of the probability mass functions up to and including a given value.
- It can be found by adding up all of the probability mass functions up to and including a given value.



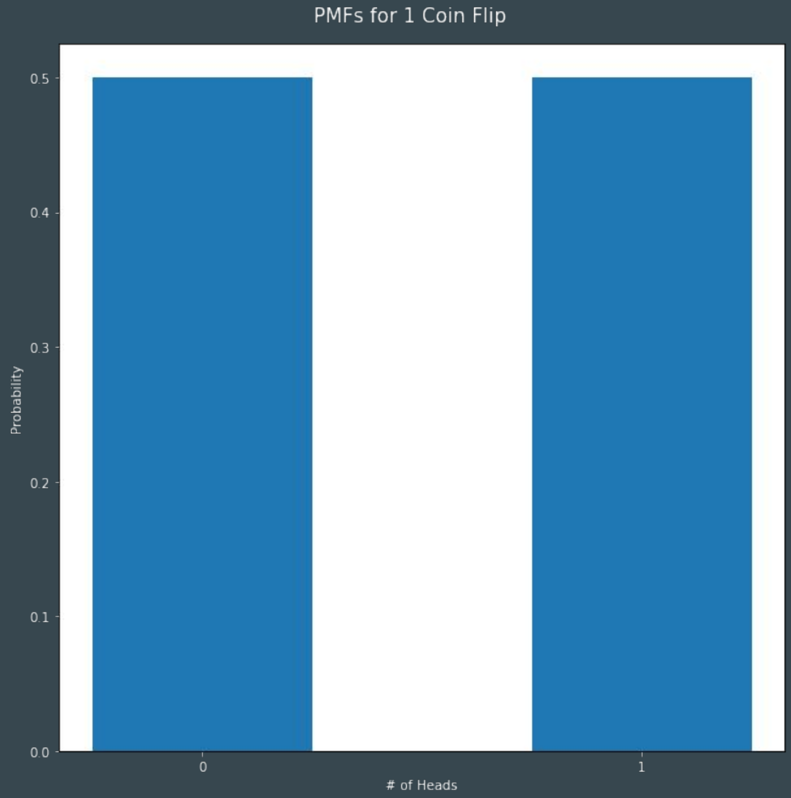
Probability

Heads	PMF	CDF
0	1/8	1/8
1	3/8	4/8
2	3/8	7/8
3	1/8	1



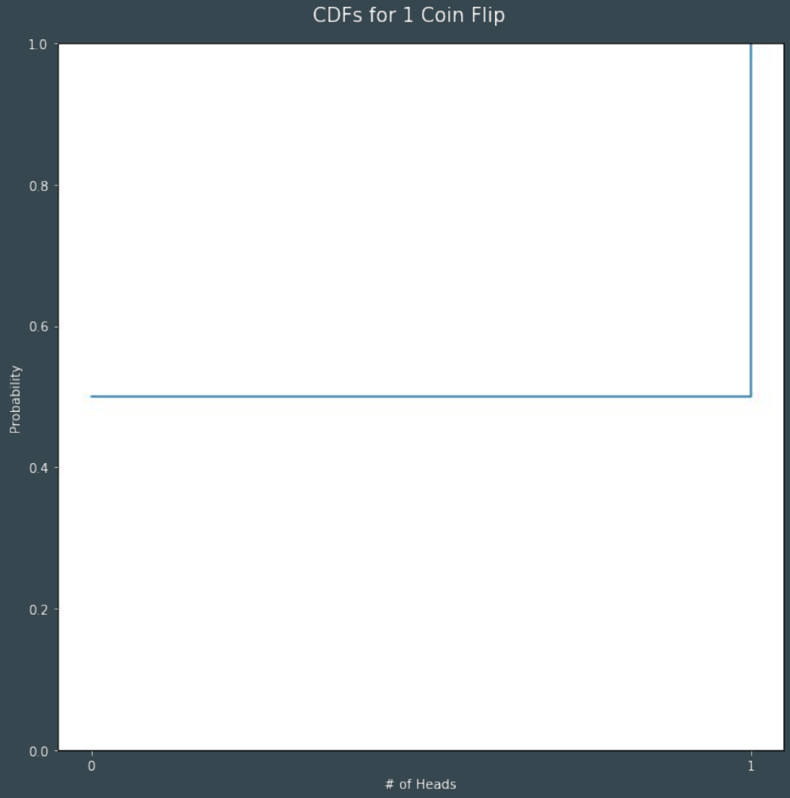
Probability

Heads	PMF	CDF
0	1/2	1/2
1	1/2	2/2



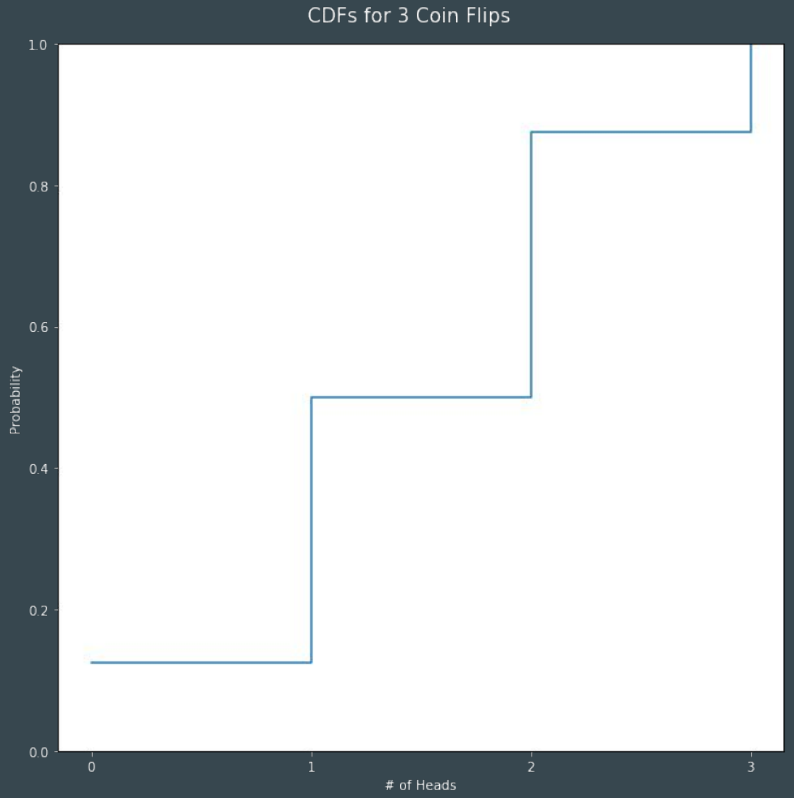
Probability

Heads	PMF	CDF
0	1/2	1/2
1	1/2	2/2



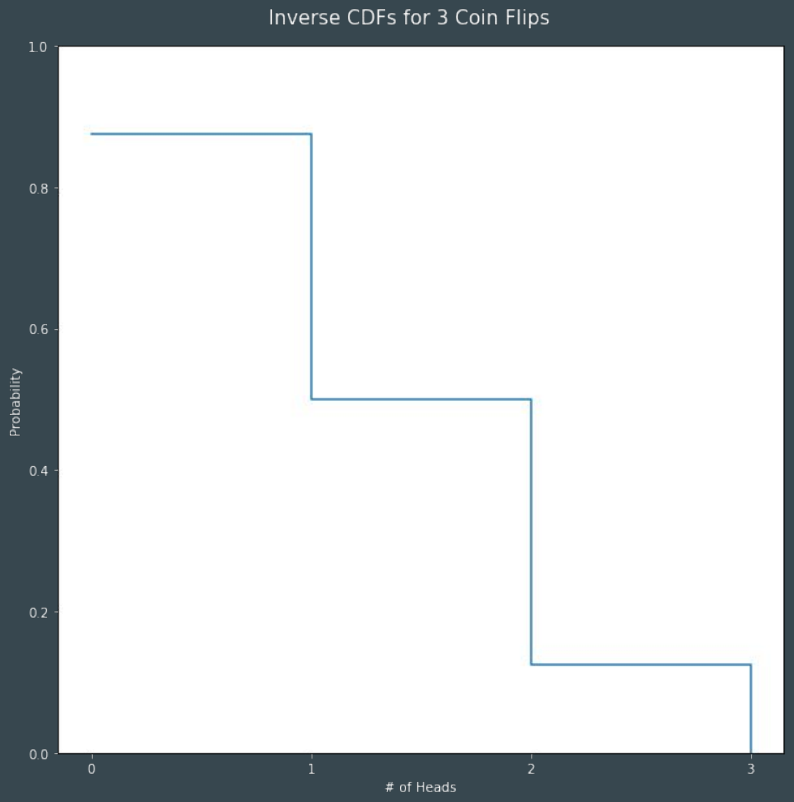
Probability

- The cumulative distribution function is simply the aggregate of all of the probability mass functions up to and including a given value.
- It can be found by adding up all of the probability mass functions up to and including a given value.
- Of course we can use 1 minus this value to find the probability of getting a result *greater* than a given value. This is called the **inverse CDF**.



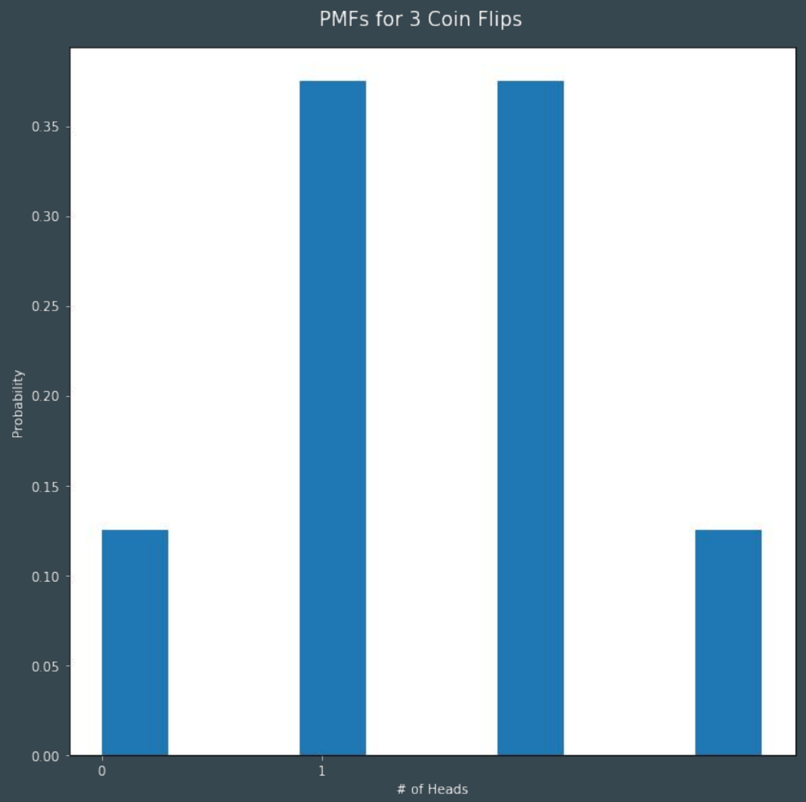
Probability

Heads	PMF	CDF	Inverse CDF
0	1/8	1/8	7/8
1	3/8	4/8	3/8
2	3/8	7/8	1/8
3	1/8	1	0



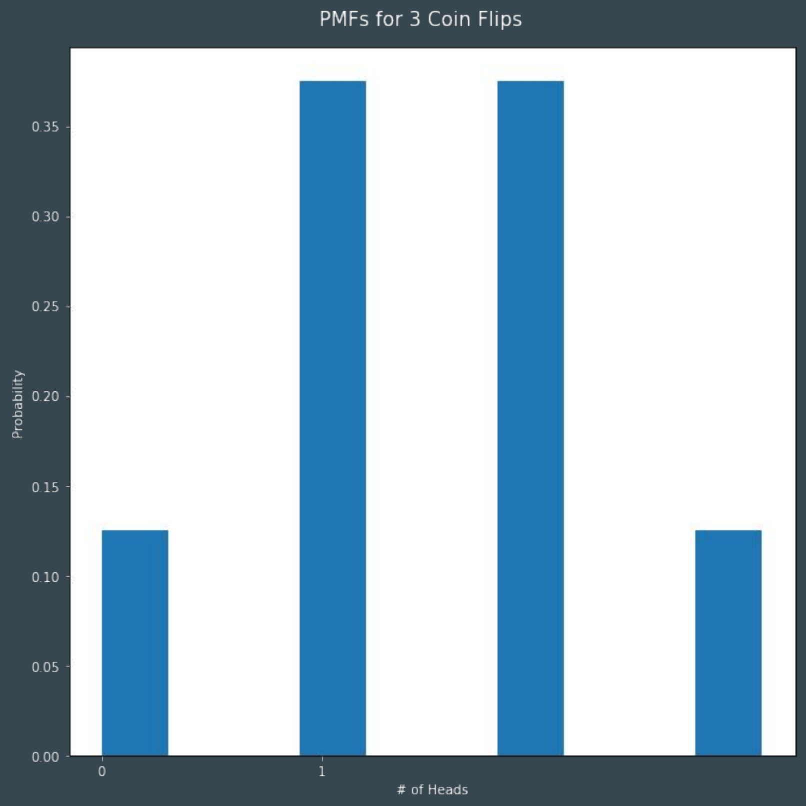
Binomial Distribution

- Flipping multiple coins is an example of a **Binomial distribution**, where **multiple independent Bernoulli trials** occur.
- It has two inputs - *p*, or the probability of the event occurring in a single trial, and *n*, the number of trials held.
- The *expected value* for a given binomial distribution is n * p.
    - What is the expected value for three coin flips?



Binomial Distribution

- Note that the trials must be independent from each other or the Binomial distribution doesn't hold!
- If getting a heads on this coin flip somehow influences whether I get heads on the next coin flip, this is not a binomial distribution.



Bernoulli Distribution

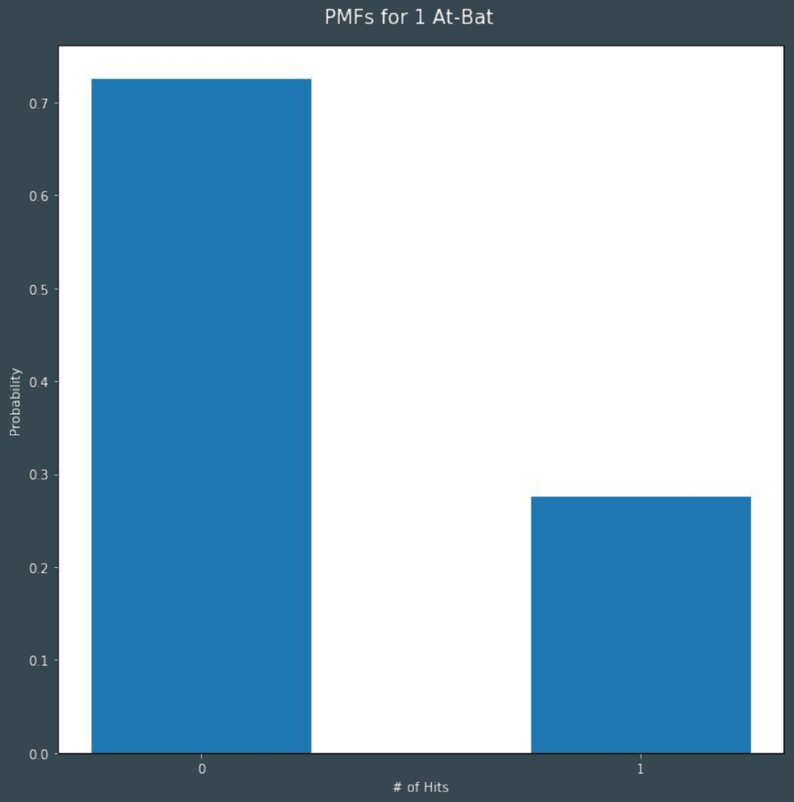
- Now let’s look at Aaron Judge in a single at-bat.
- Say that success is constituted by a hit, and a ‘failure’ is constituted by not a hit (we don’t care about any other outcomes).
- Aaron Judge has a batting average of 0.275

Bernoulli Distribution

- Aaron Judge is up to bat.
- What is our experiment?
  - A single at-bat from Aaron Judge.
- What is our sample space?
  - A hit and not a hit.
- What is the event we're looking for?
  - A hit.
- What is the probability function of our event occurring?
  - 0.275
- What is the probability function of our event *not* occurring?
  - 0.725

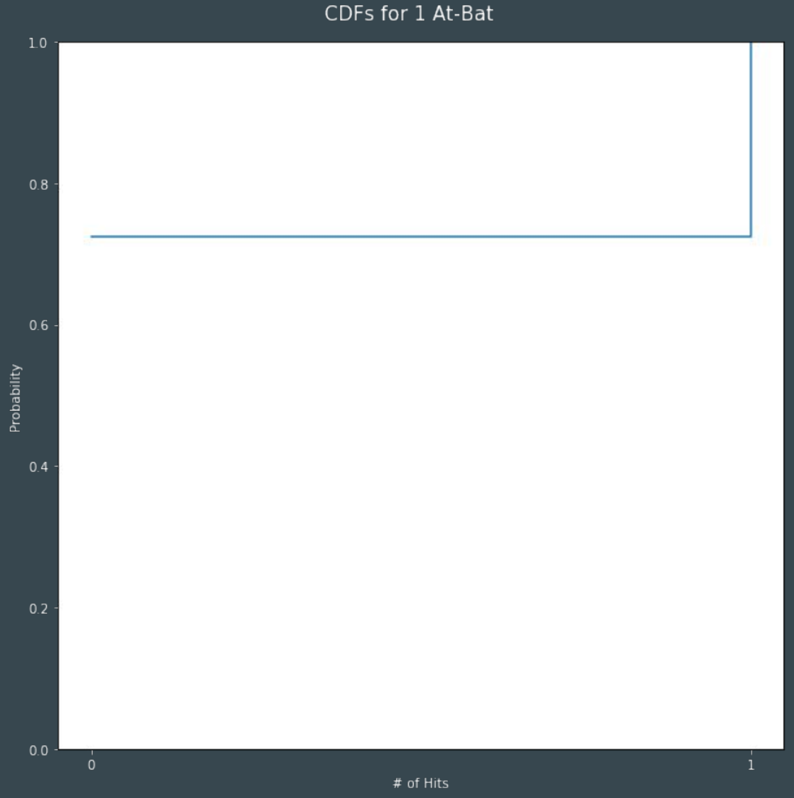
Probability

Hits	PMF	CDF
0	0.725	0.725
1	0.275	1



Probability

Hits	PMF	CDF
0	0.725	0.725
1	0.275	1

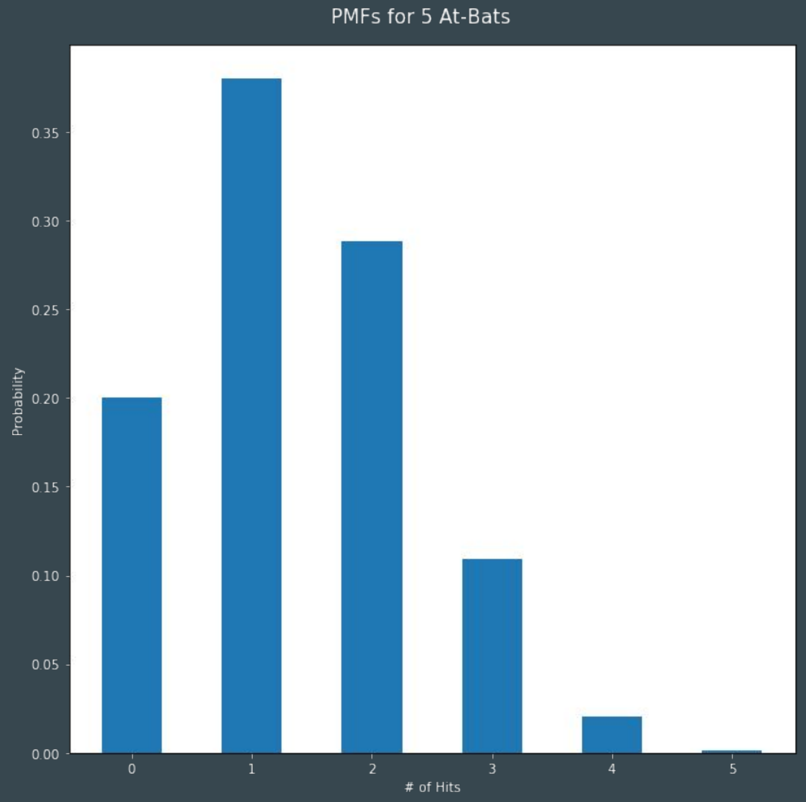


Example

- What if Aaron Judge gets up to bat five times in a game?
- This is an example of a binomial distribution with $p=0.275$ and $N=5$
- The *expected value* is $5 * 0.275$, or 1.375 hits.
- What are the odds that he gets 0, 1, 2, 3, 4, and 5 hits?

Probability

Hits	PMF	CDF
0	0.2	0.2
1	0.38	0.58
2	0.29	0.87
3	0.11	0.98
4	0.02	1
5	0.00	1



Probability

Hits	PMF	CDF
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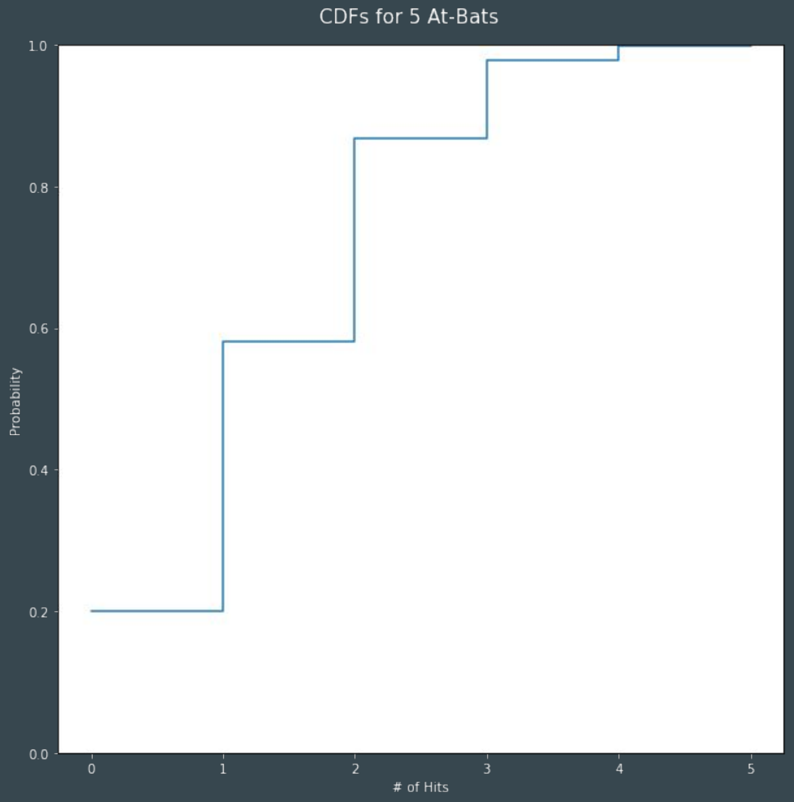


Expected Value

- We saw earlier that the *expected value* is 5 * 0.275, or 1.375 hits.
- This is the formula for the expected value for the binomial distribution.
- We can also find this value *for any distribution* by adding the sums of all values times their respective PMFs
- The EV values to the right are the number of hits times their respective PMFs.
- Add up all of these values and we get 1.37 (slightly lower due to rounding error)

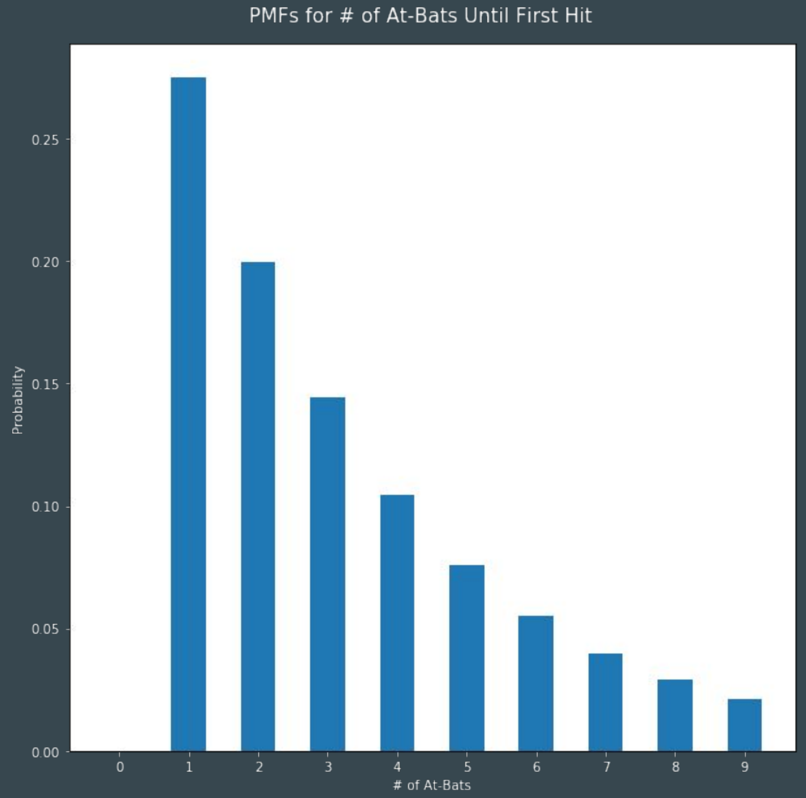
Hits	PMF	EV
0	0.2	0
1	0.38	0.38
2	0.29	0.58
3	0.11	0.33
4	0.02	0.08
5	0.00	0.00

Example

- How many at-bats will it take for Aaron Judge to get his first hit?
- This is an example of the *geometric distribution*.
- The geometric distribution only has one input - p, which again here is 0.275.
- The *expected value* is 1/p, or 3.6

Probability

At-Bats	PMF	CDF
1	0.28	0.28
2	0.20	0.48
3	0.14	0.62
4	0.10	0.72
5	0.08	0.80



Probability

At-Bats	PMF	CDF
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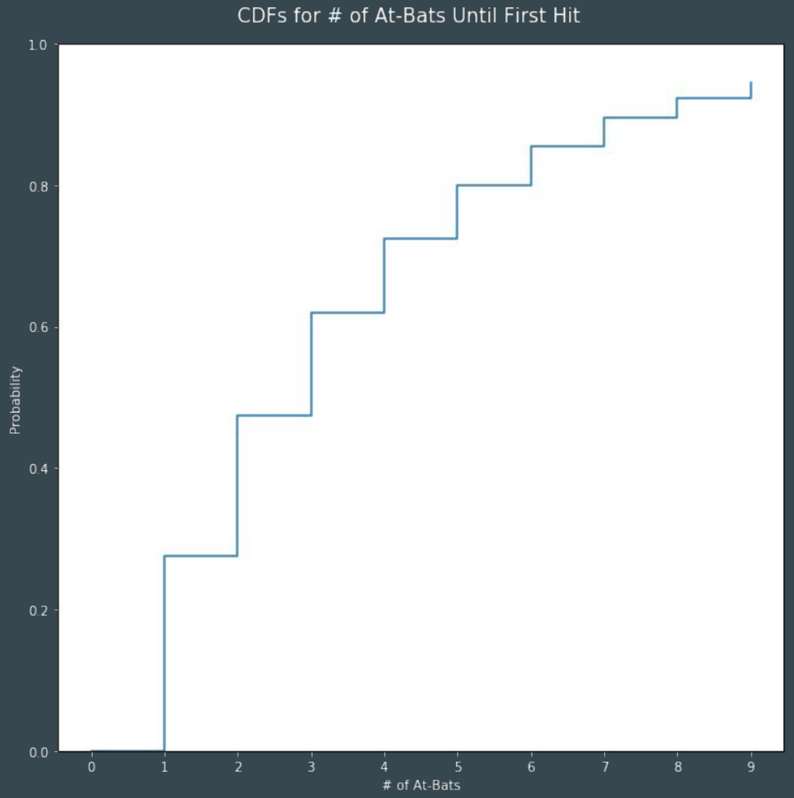

Geometric Distribution

- What are the odds someone has the same birthday as you?
- How many people do you have to meet, on average, to find someone with the same birthday as you?
- What is the probability of the 100th person you meet being the first to share the same birthday as you?
- What is the probability that one of the first 100 people you meet with share the same birthday as you?

Geometric Distribution

- What are the odds someone has the same birthday as you?
  - 1/365
- How many people do you have to meet, on average, to find someone with the same birthday as you?
- What is the probability of the 100th person you meet being the first to share the same birthday as you?
- What is the probability that one of the first 100 people you meet with share the same birthday as you?

Geometric Distribution

- What are the odds someone has the same birthday as you?
    - 1/365
- How many people do you have to meet, on average, to find someone with the same birthday as you?
    - 365
- What is the probability of the 100th person you meet being the first to share the same birthday as you?
- What is the probability that one of the first 100 people you meet with share the same birthday as you?

Geometric Distribution

- What are the odds someone has the same birthday as you?
    - 1/365
- How many people do you have to meet, on average, to find someone with the same birthday as you?
    - 365
- What is the probability of the 100th person you meet being the first to share the same birthday as you?
    - That the first 99 people *don't* share the same birthday as you and the 100th does, or $(364/365)^{99} * (1/365) = 0.0002$
- What is the probability that one of the first 100 people you meet with share the same birthday as you?

Geometric Distribution

- What are the odds someone has the same birthday as you?
    - 1/365
- How many people do you have to meet, on average, to find someone with the same birthday as you?
    - 365
- What is the probability of the 100th person you meet being the first to share the same birthday as you?
    - That the first 99 people *don't* share the same birthday as you and the 100th does, or $(364/365)^{99} * (1/365) = 0.0002$
- What is the probability that one of the first 100 people you meet with share the same birthday as you?
    - This is the CDF value of 100, or $1 - (364/365)^{100} = 0.24$

Discrete Distributions

- Of course a discrete event can occur that doesn’t follow a common distribution.
- In that case we can use the traditional measures for mean, PMF, and CDF, and expected value.

Discrete Distributions

- Say we roll two dice. Below is the sample space of all possible outcomes.

	1	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Discrete Distributions

- We can obtain our metrics via **counting**.

Outcome	PMF	CDF
2	1/36	1/36
3	2/36	3/36
4	3/36	6/36
5	4/36	10/36
6	5/36	15/36
7	6/36	21/36

Outcome	PMF	CDF
8	5/36	26/36
9	4/36	30/36
10	3/36	33/36
11	2/36	35/36
12	1/36	16/36

Discrete Distributions

- Like we did earlier, we can find the **expected value** by adding the value of each outcome multiplied by its respective PDF.

Outcome	PMF	CDF	EV
2	1/36	1/36	2/36
3	2/36	3/36	6/36
4	3/36	6/36	12/36
5	4/36	10/36	20/36
6	5/36	15/36	30/36
7	6/36	21/36	42/36

Outcome	PMF	CDF	EV
8	5/36	26/36	40/36
9	4/36	30/36	36/36
10	3/36	33/36	30/36
11	2/36	35/36	22/36
12	1/36	16/36	12/36

Discrete Distributions

- The expected value here is 7 (try it yourself!)

Outcome	PMF	CDF	EV
2	1/36	1/36	2/36
3	2/36	3/36	6/36
4	3/36	6/36	12/36
5	4/36	10/36	20/36
6	5/36	15/36	30/36
7	6/36	21/36	42/36

Outcome	PMF	CDF	EV
8	5/36	26/36	40/36
9	4/36	30/36	36/36
10	3/36	33/36	30/36
11	2/36	35/36	22/36
12	1/36	16/36	12/36

Expected Value

- Say we win \$500 if we get seven from rolling a pair of dice, but lose \$100 otherwise. Should we take this bet?

Expected Value

- Say we win $600 if we get seven from rolling a pair of dice, but lose $100 otherwise. Should we take this bet?
- Yes!

Event	Odds	Payout	Total Value
Hit 7	1/6	600	$100
Anything Else	5/6	-100	$-83.33
Net Value			**$16.67**