Theorem: Each nested sequence of non-null compact sets has a common point.

Proof: Let M_n be a nested sequence of non-empty compact sets, i.e., $M_1 \supseteq M_2 \supseteq M_3 \supseteq \cdots$. We want to show that $\bigcap_{n=1}^{\infty} M_n \neq \emptyset$.

Note: M_1= $\bigcup_{n=1}^{\infty} M_n$. Then , Since M_n is compact for all n, then it is also bounded according to Theorem 31. Therefore, M_1 is also bounded.

$$\begin{aligned}
\text{Let } x_1 &\in M_1, \\
x_2 &\in M_2, \\
x_3 &\in M_3, \\
&\vdots \\
x_n &\in M_n
\end{aligned}$$

Now, consider the infinite collection $Z = \{x_1, x_2, x_3..\}$. If Z is finite , x would be a common point. Since M_1 is compact, there exists a limit point $x \in M_1$ of the collection.

Claim: x in M_n for each n

> Proof: Suppose not. Then there is some ℓ that $x \notin M_\ell$. Since the M_n 's are nested, this means that $x \notin M_w$ for all $w \geq \ell$. Define $\epsilon = min \{d(x, M_\ell)/2, x_1, x_2, \ldots, m_\ell\}$
>
> Now, consider the neighborhood $N_\epsilon(x)$. By construction, $N_\epsilon(x) \cap M_\ell = \emptyset$. Thus, for all $1 \leq k \leq \ell - 1$, $x_k \notin N_\epsilon(x)$, because the distance between x_k and x is greater than or equal to ϵ.
>
> Since x is a limit point of the collection Z, there exists an $x_n \in Z$ such that $x_n \in N_\epsilon(x)$. By construction, $x_n \notin M_\ell$, so $x_n \notin M_n$ However, we know that $x_n \in M_n$, and since $M_n \subseteq M_\ell$ for all $n \geq \ell$, we arrive at a contradiction.

Thus, $x \in M_n$ for each n, and so $\bigcap_{n=1}^{\infty} M_n \neq \emptyset$. This completes the proof.

$\therefore$ Each nested sequence of non-null compact sets has a common point. $\square$