Para el hamiltoniano del fluxonio :

$$\hat{H} = 4E_c \hat{N}^2 + \frac{1}{2} E_L \hat{\varphi}^2 - E_J \cos(\hat{\varphi} - \varphi_{ext})$$

requiero usar una base adecuada para encontrar la forma matricial del operador.

Tomando como base en $H = 4E_c \hat{N}^2 + \frac{1}{2} E_L \hat{\varphi}^2$, debería adimensionalizar los operadores $\hat{N}$ y $\hat{\varphi}$, si luego

$$\hat{n} = \frac{\hat{N}}{a} \quad , \quad \hat{f} = \frac{\hat{\varphi}}{b}$$

tal que

$$\begin{aligned} H &= 4E_C (a\hat{n})^2 + \frac{1}{2} E_L (b\hat{f})^2 \\ &= 4E_C a^2 (\hat{n}^2 + \frac{b^2 E_L}{8a^2 E_C} \hat{f}^2) \quad , \quad b^2 = 8a^2 E_C / E_L \\ &= 4E_C a^2 (\hat{n}^2 + \hat{f}^2) \quad , \text{ con } \hat{n} \text{ y } \hat{f} \text{ adimensionales.} \end{aligned}$$

luego, como en el oscilador armónico, busco $\hat{b}$ tal que

$$\hat{b} = \frac{1}{\sqrt{2}} (\hat{f} + i\hat{n}) \quad , \quad \hat{b}^+ = \frac{1}{\sqrt{2}} (\hat{f} - i\hat{n})$$

$$\hookrightarrow \hat{b}^+ \hat{b} = \frac{1}{2} (\hat{f}^2 + \hat{n}^2 + i(\hat{f}\hat{n} - \hat{n}\hat{f})) = \frac{1}{2} (\hat{f}^2 + \hat{n}^2 + i[\hat{f}, \hat{n}])$$

$$* \; [\hat{f}, \hat{n}] = \left[\frac{\hat{\varphi}}{b}, \frac{\hat{N}}{a} \right] = \frac{1}{ab} [\hat{\varphi}, \hat{N}] = \frac{i}{ab}$$

$$\hookrightarrow \hat{b}^+ \hat{b} = \frac{1}{2} (\hat{f}^2 + \hat{n}^2 - \frac{1}{ab})$$

$$\therefore \hat{H} = 4E_c a^2 (2\hat{b}^+ \hat{b} + \frac{1}{ab}) = 8E_c a^2 (\hat{b}^+ \hat{b} + \frac{1}{2ab}) \quad , \quad \omega = 8E_c a^2$$

$$\text{si } \hat{B} = \hat{b}^+ \hat{b} \quad , \quad [\hat{B}, \hat{H}] = \left[\frac{\hat{H}}{8E_c a^2} - \frac{1}{2ab}, \hat{H} \right] = \frac{1}{8E_c a^2} [\hat{H}, \hat{H}] = 0$$

también $\hat{H}|n\rangle = E_n|n\rangle$.

Otro conjunto de conmutadores útiles son:

$(a+b, c)$

$ac + bc - ca - cb$

$a,c + b,c$

$$[\hat{b}, \hat{b}^+] = [\tfrac{1}{\sqrt{2}}(\hat{f} + i\hat{n}), \tfrac{1}{\sqrt{2}}(\hat{f} - i\hat{n})]$$
$$= \tfrac{1}{2}([\hat{f}, \hat{f} - i\hat{n}] + [i\hat{n}, \hat{f} - i\hat{n}]) \quad , \quad [\hat{f}, \hat{n}] = \tfrac{i}{ab}$$
$$= \tfrac{1}{2}([\hat{f}, -i\hat{n}] + [i\hat{n}, \hat{f}])$$
$$= \tfrac{1}{2}(-i(i) + i(-i)) = 1/ab$$
$$= 1/ab$$

$$[\hat{b}, \hat{H}] = [\hat{b}, \omega(\hat{B} + \tfrac{1}{2ab})] = \omega[\hat{b}, \hat{B}] = \omega[\hat{b}, \hat{b}^+\hat{b}]$$
$$= \omega(\hat{b}\hat{b}^+\hat{b} - \hat{b}^+\hat{b}\hat{b}) = \omega[\hat{b}, \hat{b}^+]\hat{b} = \omega\hat{b}/ab$$
$$[\hat{b}^+, \hat{H}] = [\hat{b}^+, \omega(\hat{B} + \tfrac{1}{2ab})] = \omega[\hat{b}^+, \hat{B}] = \omega[\hat{b}^+, \hat{b}^+\hat{b}]$$
$$= \omega(\hat{b}^+\hat{b}^+\hat{b} - \hat{b}^+\hat{b}\hat{b}^+) = \omega\hat{b}^+[\hat{b}^+, \hat{b}] = -\omega\hat{b}^+/ab$$

luego,

$$\hat{H}(\hat{b}|n\rangle) = (-[\hat{b}, \hat{H}] + \hat{b}\hat{H})|n\rangle = (-\tfrac{\omega\hat{b}}{ab} + \hat{b}E_n)|n\rangle = (E_n - \tfrac{\omega}{ab})(\hat{b}|n\rangle)$$
$$\hat{H}(\hat{b}^+|n\rangle) = (-[\hat{b}^+, \hat{H}] + \hat{b}^+\hat{H})|n\rangle = (\tfrac{\omega\hat{b}^+}{ab} + \hat{b}^+E_n)|n\rangle = (E_n + \tfrac{\omega}{ab})(\hat{b}^+|n\rangle)$$

como

$$[\hat{B}, \hat{b}] = [\hat{b}^+\hat{b}, \hat{b}] = \hat{b}^+(0) + [\hat{b}^+, \hat{b}]\hat{b} = -\hat{b}$$
$$[\hat{B}, \hat{b}^+] = \hat{b}^+$$

$\hookrightarrow$ * $\hat{B}\hat{b} - \hat{b}\hat{B} = -\hat{b} \rightarrow \hat{B}\hat{b} = \hat{b}(\hat{B} - 1)$

* $\hat{B}\hat{b}^+ - \hat{b}^+\hat{B} = \hat{b}^+ \rightarrow \hat{B}\hat{b}^+ = \hat{b}^+(\hat{B} + 1)$

entonces, aplicar $\hat{b}$ y $\hat{b}^+$ sobre un estado $|n\rangle$ lo llevan a un estado distinto $|m\rangle$ (suponiendo que no hay degeneración) el cual posee una energía distinta.

Esto indica que existe el conjunto contable de estados tal que $i \in \mathbb{Z}$, $H|i\rangle = E_i|i\rangle$, estas energías pueden ordenar los estados, de modo que $E_i < E_{i+1}$, $\forall\ i \in \mathbb{Z}$

Pero el $\hat{B} = \hat{b}^+\hat{b}$ es definido positivo para todo $|i\rangle$, como $H = \omega(\hat{B} + \frac{1}{2ab}) \rightarrow \hat{B} = \frac{H}{\omega} - \frac{1}{2ab}$

$$\rightarrow \hat{B}|i\rangle = \left(\frac{\hat{H}}{\omega} - \frac{1}{2ab}\right)|i\rangle = \left(\frac{E_i}{\omega} - \frac{1}{2ab}\right)|i\rangle \rightarrow \frac{E_i}{\omega} - \frac{1}{2ab} \geq 0$$

$E_i \geq \frac{\omega}{2ab}$, entonces existe un $E_{i\,min}$ tal que cumple la condición.
Por ello, en algun momento, aplicar $\hat{b}$ en un estado $|i_{min}\rangle$ no lleva a ningun otro estado (ya que $\hat{b}$ no conmuta con $\hat{H}$), así que $\hat{b}|i_{min}\rangle = 0$.
Asignando $i_{min} = 0$, llamo a $|i_{min}\rangle = |0\rangle$ estado base.
La cadena $\{|0\rangle, |1\rangle, |2\rangle, \dots\}$ tiene energías $\{E_0, E_1, E_2, \dots\}$ y se accede a ellos mediante los operadores $\hat{b}$ y $\hat{b}^+$.

Luego, $\hat{b}|n\rangle = \alpha_n|n-1\rangle$ y $\hat{b}^+|n\rangle = \beta_n|n+1\rangle$

$\hookrightarrow \langle n|\hat{b}^+ = \langle n-1|\alpha_n^*$, $\langle n|\hat{b} = \langle n+1|\beta_n^*$

$$* \ \langle n|\hat{b}^+\hat{b}|n\rangle = \langle n|\frac{H}{\omega} - \frac{1}{2ab}|n\rangle = \left(\frac{E_n}{\omega} - \frac{1}{2ab}\right)\langle n|n\rangle = \langle n-1|\alpha_n^*\alpha_n|n-1\rangle$$

$$\frac{E_n}{\omega} - \frac{1}{2ab} = |\alpha_n^*|$$

Ponemos E_n en función de n: $E_n = E_0 + \omega n = \frac{\omega}{2ab} + \frac{\omega n}{ab}$ (aplicando $\hat{b}^+$ sucesivo)

$$\hookrightarrow \frac{E_n}{\omega} - \frac{1}{2ab} = \left(\frac{1}{2ab} + \frac{n}{ab}\right) - \frac{1}{2ab} = n/ab \rightarrow \alpha_n = \sqrt{\frac{n}{ab}}$$

$$\therefore \ \hat{b}|n\rangle = \sqrt{\frac{n}{ab}}|n-1\rangle$$

$$* \langle n | \hat{b}\hat{b}^{+} | n \rangle = \langle n | -[\hat{b}^{+}, \hat{b}] + \hat{b}^{+}\hat{b} | n \rangle$$

$$= \langle n | \tfrac{1}{ab} + \tfrac{n}{ab} | n \rangle \quad = \langle n+1 | \beta_n^{*} \beta_n | n+1 \rangle$$

$$\hookrightarrow \beta_n = \sqrt{\tfrac{n+1}{ab}}$$

$$\therefore \hat{b}^{+} | n \rangle = \sqrt{\tfrac{n+1}{ab}} \, | n+1 \rangle$$

Recordamos que

$$\hat{b} = \frac{1}{\sqrt{2}} (\hat{f} + i\hat{n}) \quad , \quad \hat{b}^{+} = \frac{1}{\sqrt{2}} (\hat{f} - i\hat{n})$$

$$\hookrightarrow \hat{f} = \sqrt{2} (\hat{b} + \hat{b}^{+}) \quad , \quad \hat{n} = i\sqrt{2} (\hat{b} - \hat{b}^{+})$$

$$\hat{\varphi} = \sqrt{2}\, b (\hat{b} + \hat{b}^{+}) \qquad \hat{N} = i\sqrt{2}\, a (\hat{b} - \hat{b}^{+})$$

para preservar la estructura canónica, debería forzar a que $[\hat{n}, \hat{f}] = i$, como yo tengo $[\hat{n}, \hat{f}] = i/ab$

$\rightarrow ab = 1$. luego, usando la relación:

$$b^2 = 8 a^2 E_C / E_L$$

$$a^2 b^2 = 8 a^4 E_C / E_L = 1$$

$$\rightarrow a = (E_L / 8E_C)^{1/4}$$

$$\wedge \quad b = (8E_C / E_L)^{1/4}$$

$$\rightarrow \hat{\varphi} = (32 E_C / E_L)^{1/4} (\hat{b} + \hat{b}^{+}) = \varphi_c (\hat{b} + \hat{b}^{+})$$

$$\hat{N} = i (E_L / 2E_C)^{1/4} (\hat{b} - \hat{b}^{+}) = i N_c (\hat{b} - \hat{b}^{+})$$

Ojalá que sea eso, jaja.

$$\hat{\varphi} = \varphi_c (\hat{b} + \hat{b}^+) \qquad \hat{H} = 4E_c(\hat{N} - N_{ext})^2 + \frac{1}{2} E_L \hat{\varphi}^2 - E_J \cos(\hat{\varphi} - \varphi_{ext})$$
$$\hat{N} = i N_c (\hat{b} - \hat{b}^+)$$

$$* \ (\hat{N} - N_{ext})^2 = \hat{N}^2 - 2N_{ext}\hat{N} + N_{ext}^2 \hat{I}$$
$$\hat{N}|n\rangle = iN_c(b - b^+)|n\rangle = iN_c\left(\sqrt{n}\,|n-1\rangle - \sqrt{n+1}\,|n+1\rangle\right)$$
$$\hat{N}^2|n\rangle = iN_c\sqrt{n}\left(\hat{N}|n-1\rangle\right) - iN_c\sqrt{n+1}\left(\hat{N}|n+1\rangle\right)$$
$$= iN_c\left(\sqrt{n}(iN_c)(\sqrt{n-1}\,|n-2\rangle - \sqrt{n}\,|n\rangle)\right.$$
$$\left. - \sqrt{n+1}(iN_c)(\sqrt{n+1}\,|n\rangle - \sqrt{n+2}\,|n+2\rangle)\right)$$
$$= -N_c^2\left(\sqrt{n(n-1)}\,|n-2\rangle + \sqrt{(n+1)(n+2)}\,|n+2\rangle - (2n+1)|n\rangle\right)$$

$$* \ \hat{\varphi}|n\rangle = \varphi_c(b + b^+)|n\rangle = \varphi_c\left(\sqrt{n}\,|n-1\rangle + \sqrt{n+1}\,|n+1\rangle\right)$$
$$\hat{\varphi}^2|n\rangle = \varphi_c^2\left(\sqrt{n}\,(b+b^+)|n-1\rangle + \sqrt{n+1}\,(b+b^+)|n+1\rangle\right)$$
$$= \varphi_c^2\left(\sqrt{n}(\sqrt{n-1}\,|n-2\rangle + \sqrt{n}\,|n\rangle) + \sqrt{n+1}(\sqrt{n+1}\,|n\rangle + \sqrt{n+2}\,|n+2\rangle)\right)$$
$$= \varphi_c^2\left(\sqrt{n(n-1)}\,|n-2\rangle + \sqrt{(n+1)(n+2)}\,|n+2\rangle + (2n+1)|n\rangle\right)$$

$$* \ \cos(\hat{\varphi} - \varphi_{ext}) = \frac{1}{2}\left(e^{i(\hat{\varphi} - \varphi_{ext})} + e^{-i(\hat{\varphi} - \varphi_{ext})}\right) = \frac{1}{2}\left(e^{-i\varphi_{ext}} e^{i\hat{\varphi}} + e^{i\varphi_{ext}} e^{-i\hat{\varphi}}\right), \quad e^{i\varphi_{ext}\hat{I}} = \hat{I} e^{i\varphi_{ext}}$$

$$\circ \ e^{\pm i\hat{\varphi}} = e^{\pm i\varphi_c(b+b^+)}, \text{ si } \hat{P} = \pm i\varphi_c \hat{b} \ \text{ y } \ \hat{Q} = \pm i\varphi_c \hat{b}^+$$
$$\rightarrow [\hat{P}, \hat{Q}] = -\varphi_c^2 [\hat{b}, \hat{b}^+] = -\varphi_c^2$$

En la formula de Dynkin

$$\ln(\exp(\hat{P})\exp(\hat{Q})) = \hat{P} + \hat{Q} + \frac{1}{2}[\hat{P}, \hat{Q}]$$, ya que $\hat{P}, \hat{Q}$ conmutan, los demas

$$= \hat{P} + \hat{Q} - \varphi_c^2/2 \cdot \hat{I}$$ términos se anulan.

$$\exp(\hat{P})\exp(\hat{Q}) = \exp(\hat{P} + \hat{Q})\exp(-\varphi_c^2/2)$$
$$\exp(\hat{P} + \hat{Q}) = \exp(\varphi_c^2/2)\exp(\hat{P})\exp(\hat{Q})$$

$$\rightarrow \exp(\pm i\hat{\varphi}) = \exp(\varphi_c^2/2)\exp(\pm i\varphi_c\hat{b})\exp(\pm i\varphi_c\hat{b}^+)$$

$$\exp(\hat{\alpha}) = \sum_k \frac{(\hat{\alpha})^k}{k!}$$

$\hookrightarrow \exp(\pm i\varphi_c \hat{b}) = \sum_k \frac{(\pm i\varphi_c)^k}{k!} \hat{b}^k \quad \wedge \quad \exp(\pm i\varphi_c \hat{b}^\dagger) = \sum_k \frac{(\pm i\varphi_c)^k}{k!} \hat{b}^{\dagger k}$

$$\exp(\pm i\varphi_c \hat{b}^\dagger)|n\rangle = \sum_k \frac{(\pm i\varphi_c)^k}{k!} \hat{b}^{\dagger k}|n\rangle = \sum_k \frac{(\pm i\varphi_c)^k}{k!}\sqrt{\frac{(n+k)!}{n!}}|n+k\rangle$$

$\leftarrow l \leq n+k$

$$\exp(\pm i\varphi_c \hat{b})\exp(\pm i\varphi_c \hat{b}^\dagger)|n\rangle = \sum_l \frac{(\pm i\varphi_c)^l}{l!}\sum_k \frac{(\pm i\varphi_c)^k}{k!}\sqrt{\frac{(n+k)!}{n!}}\,\hat{b}^l|n+k\rangle$$

$$= \sum_{k=0}^{\infty}\sum_{l=0}^{n+k} \frac{(\pm i\varphi_c)^{k+l}}{k!\,l!}\sqrt{\frac{(n+k)!}{n!}}\sqrt{\frac{(n+k)!}{(n+k-l)!}}|n+k-l\rangle \quad , \quad m = n+k-l$$

$$l = n+k-m$$

$$0 \leq l \leq n+k \qquad \hookrightarrow l = n+k-m \geq 0$$

$$k \geq m-n$$

$\hookrightarrow \max(0, m-n) \leq k$

$$e^{\pm i\hat{\varphi}}|n\rangle = \sum_{k=\max(0,m-n)}^{\infty} e^{\varphi_c^2/2}\frac{(\pm i\varphi_c)^{n+2k-m}}{k!\,(n-m+k)!}\sqrt{\frac{(n+k)!^2}{n!\,m!}}\,|m\rangle$$

$$\hookrightarrow P_{mn\pm} = \langle m|e^{\pm i\hat{\varphi}}|n\rangle = e^{\varphi_c^2/2}\frac{(\pm i\varphi_c)^{n-m}}{\sqrt{n!\,m!}}\sum_{k=\max(0,m-n)}^{\infty}\frac{(-\varphi_c^2)^k}{k!\,(n-m+k)!}(n+k)!$$

Por lo tanto, la matriz del hamiltoniano $\hat{H}$ es

$$[\hat{H}]_{mn} = 4E_c\Big(-N_c^2\big(\sqrt{n(n-1)}\,\delta_{m,n-2} + \sqrt{(n+1)(n+2)}\,\delta_{m,n+2} - (2n+1)\delta_{m,n}$$
$$-2N_{ext}(iN_c)(\sqrt{n}\,\delta_{m,n-1} - \sqrt{n+1}\,\delta_{m,n+1}) + N_{ext}^2\,\delta_{m,n}\Big)$$
$$+\frac{1}{2}E_L(\varphi_c^2)\big(\sqrt{n(n-1)}\,\delta_{m,n-2} + \sqrt{(n+1)(n+2)}\,\delta_{m,n+2} + (2n+1)\delta_{m,n}\big)$$
$$-\frac{E_J}{2}\big(e^{-i\varphi_{ext}}P_{mn+} + e^{i\varphi_{ext}}P_{mn-}\big)$$

Tomo $P_{mn\pm}$, cuando $m \geq n$, $d = m-n \geq 0 \rightarrow \max(0, m-n) = d$ y $s = k-d$

$$\hookrightarrow P_{mn\pm} = e^{\varphi_c^2/2}\frac{(\pm i\varphi_c)^{-d}}{\sqrt{n!\,m!}}\sum_{s=0}^{\infty}\frac{(-\varphi_c^2)^{s+d}(n+s+d)!}{(s+d)!\,s!}$$

$$= e^{\varphi_c^2/2}\frac{\left(\frac{-\varphi_c^2}{\pm i\varphi_c}\right)^d}{\sqrt{n!\,m!}}\sum_{s=0}^{\infty}\frac{(n+s+d)!}{(s+d)!\,s!}(-\varphi_c^2)^s = e^{\varphi_c^2/2}\frac{(\pm i\varphi_c)^d}{\sqrt{n!\,m!}}\sum_{s=0}^{\infty}\frac{(n+s+d)!}{(s+d)!\,s!}(-\varphi_c^2)^s$$

$$* \quad \frac{(n+s+d)!}{(s+d)!} = \frac{\Gamma(n+d+1+s)}{\Gamma(s+d+1)} = \qquad , \quad (x+1)_s = \frac{\Gamma(x+s+1)}{\Gamma(x+1)}$$

$$= \frac{\Gamma(s+d+n+1)}{\Gamma(d+n+1)} \cdot \Gamma(d+n+1) \cdot \left(\frac{\Gamma(s+d+1)}{\Gamma(d+1)} \cdot \Gamma(d+1) \right)^{-1}$$

$$= (d+n+1)_s \, \Gamma(d+n+1) \left((d+1)_s \cdot \Gamma(d+1) \right)^{-1} = \frac{\Gamma(d+n+1)}{\Gamma(d+1)} \frac{(d+n+1)_s}{(d+1)_s}$$

$$= \frac{(d+n)!}{d!} \frac{(d+n+1)_s}{(d+1)_s}$$

$$\hookrightarrow \sum_{s=0}^{\infty} \frac{(n+s+d)!}{(s+d)!\,s!} (-\varphi_c^2)^s = \frac{(d+n)!}{d!} \sum_{s=0}^{\infty} \frac{(d+n+1)_s}{(d+1)_s\, s!} (-\varphi_c^2)^s$$

$$= \frac{(d+n)!}{d!} {}_1F_1(d+n+1, d+1, -\varphi_c^2)$$

$$L_n^{\alpha} = \frac{(n+\alpha)!}{n!\,\alpha!} {}_1F_1(-n; \alpha+1; x)$$

$${}_1F_1(a,b,c) = e^{c} {}_1F_1(b-a, b, -c)$$

$$= \frac{(d+n)!}{d!} e^{(-\varphi_c^2)} {}_1F_1(-n, d+1, +\varphi_c^2) \cdot \frac{n!}{n!}$$

$$= n!\, e^{-\varphi_c^2} \left(\frac{(d+n)!}{d!\,n!} {}_1F_1(-n, d+1, \varphi_c^2) \right)$$

$$= n!\, e^{-\varphi_c^2} L_n^{d}(\varphi_c^2)$$

$$\widehat{\text{so}} \quad P_{mn\pm} = e^{\varphi_c^2/2} \frac{(\pm i\varphi_c)^d}{\sqrt{m!\,n!}} n!\, e^{-\varphi_c^2} L_n^d(\varphi_c^2)$$

$$= e^{-\varphi_c^2/2} (\pm i\varphi_c)^d \sqrt{\frac{n!}{m!}} L_n^d(\varphi_c^2)$$

$$\longrightarrow e^{-i\varphi_{ext}} P_{mn+} + e^{i\varphi_{ext}} P_{mn-} = e^{-i\varphi_{ext}} e^{-i\varphi_c^2/2} (i\varphi_c)^d \sqrt{\frac{n!}{m!}} L_n^d(\varphi_c^2)$$

$$+ e^{i\varphi_{ext}} e^{-i\varphi_c^2/2} (-i\varphi_c)^d \sqrt{\frac{n!}{m!}} L_n^d(\varphi_c^2)$$

$$= e^{-i\varphi_c^2/2} (i\varphi_c)^d \sqrt{\frac{n!}{m!}} L_n^d(\varphi_c^2) \left(e^{-i\varphi_{ext}} + (-1)^d e^{i\varphi_{ext}} \right)$$

5. $m \geq n$

$$[\hat{H}]_{mn} = 4E_C\Big(-N_c^2\big(\sqrt{n(n-1)}\,\delta_{m,n-2} + \sqrt{(n+1)(n+2)}\,\delta_{m,n+2} - (2n+1)\delta_{m,n}\big)$$
$$-2N_{ext}(iN_c)(\sqrt{n}\,\delta_{m,n-1} - \sqrt{n+1}\,\delta_{m,n+1}) + N_{ext}^2\,\delta_{m,n}\Big)$$
$$+\frac{1}{2}E_L(\varphi_c^2)\big(\sqrt{n(n-1)}\,\delta_{m,n-2} + \sqrt{(n+1)(n+2)}\,\delta_{m,n+2} + (2n+1)\delta_{m,n}\big)$$
$$-\frac{E_J}{2}\Big(e^{-i\varphi_c^2/2}(i\varphi_c)^d\sqrt{\frac{n!}{m!}}\,L_n^d(\varphi_c^2)\big(e^{-i\varphi_{ext}} + (-1)^d e^{i\varphi_{ext}}\big)\Big)$$

En esta base, truncaré la matriz, limitando n de 0 a un N_{max}, también usaré $\varphi_{ext} = 2\pi\frac{\Phi}{\Phi_0}$, con $\frac{\Phi}{\Phi_0} \in [-1, 1]$, los otros parámetros son $E_J/E_L = r_1$ ^ $E_J/E_C = r_2$, ya que graficaré E/E_J, elijo $E_J = 1$ $\rightarrow r_1 = 1/E_L$, $r_2 = 1/E_C$. Por si acaso, usaré $N_{ext} = 0$.

Recordando que:

$$N_c = (E_L/2E_C)^{1/4} = ((1/r_1)/2(1/r_2))^{1/4} = (r_2/2r_1)^{1/4}$$
$$\varphi_c = (32E_C/E_L)^{1/4} = (32(1/r_2)/(1/r_1))^{1/4} = (32r_1/r_2)^{1/4}$$

Como solo calcularé para $m \geq n$, ignoro $\delta_{m,n-2}$ y $\delta_{m,n-1}$

$$\rightarrow [\hat{H}]_{mn} = \sqrt{(n+1)(n+2)}\,\delta_{m,n+2}\left(-4E_C N_c^2 + \frac{1}{2}E_L\varphi_c^2\right) +$$
$$\sqrt{n+1}\,\delta_{m,n+1}(i8E_C N_c N_{ext}) +$$
$$\delta_{m,n}\left(4(2n+1)E_C N_c^2 + 4E_C N_{ext}^2 + \frac{1}{2}(2n+1)E_L(\varphi_c^2)\right) +$$
$$-\frac{E_J}{2}\left(e^{-i\varphi_c^2/2}(i\varphi_c)^d\sqrt{\frac{n!}{m!}}\,L_n^d(\varphi_c^2)\left(e^{-i\varphi_{ext}} + (-1)^d e^{i\varphi_{ext}}\right)\right), \quad d = m-n$$