Econometric Methods III - Problem Set 1

Álvaro Corrales Cano and Emilio Esguerra

Question 1. Stationarity of AR(2) process

$$y_t = 1 + 0.8y_{t-1} - 1.2y_{t-2} + \varepsilon_t \tag{1}$$

a. Companion form of the model

The companion form of model 1 is defined as follows:

$$\begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0.8 & -1.2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \end{bmatrix} + \begin{bmatrix} \varepsilon_{t-1} \end{bmatrix} \tag{2}$$

b. Eigenvalues of companion matrix

The companion matrix of model 2 is defined as A = $\begin{bmatrix} 0.8 & -1.2 \\ 1 & 0 \end{bmatrix}$ The process will be stationary if the eigenvalues of A, λ_i ($i \leq 2$), are all less than one in absolute value. To find its eigenvalues, we proceed as follows:

$$det[A - \lambda I] = \det \begin{bmatrix} 0.8 - \lambda & -1.2 \\ 1 & -\lambda \end{bmatrix} = (0.8 - \lambda)(-\lambda) + 1.2 = 1.2 - 0.8\lambda + \lambda^2 = 0$$

$$\lambda_{i,j} = \frac{0.8 \pm \sqrt{0.8^2 - 4(1.2)}}{2}$$

This yields two complex numbers as eigenvalues:

$$\lambda_1 = 0.4 + 1.02i$$
$$\lambda_2 = 0.4 - 1.02i$$

Note that the modulus of both complex eigenvalues is more than one in absolute value, which means that the process is not stationary:

$$modulus(\lambda_i) = \sqrt{0.4^2 + 1.0198^2} = |1.095| > 1$$

c. Roots of the lag polynomial

The lag polynomial of the process is defined as:

$$(1 - 0.8L + 1.2L^2)y_t = 1 + \varepsilon_t$$

Solving for the roots of the lag polynomial yields:

$$1 - 0.8z + 1.2z^2 = 0$$
$$z_{i,j} = \frac{0.8 \pm \sqrt{0.8^2 - 4(1.2)}}{2}$$

Which is the same expression that we got when solving for the eigenvalues of the companion matrix. We already now that this expression yields two solutions that imply a non-stationary process.

Question 2. Sum of covariance-stationary processes

$$y_{1,t} = 1 + \varepsilon_t + 0.2\varepsilon_{t-1} + 0.4\varepsilon_{t-2} \tag{3}$$
$$y_{2,t} = 0.5 + \xi_t + 0.8\xi_{t-1} \tag{4}$$
$$x_t = y_{1,t} + y_{2,t} \tag{5}$$

Where ε is $i.i.d.$ $N(0, 0.1)$ and ξ is $i.i.d.$ $N(0, 1)$.

A process x_t is covariance-stationary if three conditions are met:

- $\mathrm{E}[x_t] = \mu$
- $\mathrm{E}[x_t^2] < \infty$
- $cov(\mathrm{x}_t, x_{t-j}) = \mathrm{cov}(x_s, x_{s-j})\ \forall s, t$

In our case, as both ε and ξ are white noise with zero mean, we have that

$$E[x_t] = E[1 + \varepsilon_t + 0.2\varepsilon_{t-1} + 0.4\varepsilon_{t-2} + 0.5 + \xi_t + 0.8\xi_{t-1}] = E[1 + 0.5] = 1.5$$

For the autocovariance of order j of the process, we have that

$$\gamma_j = E[(x_t - E[x_t])(x_{t-j} - E[x_{t-j}])]$$

$$= E[(\varepsilon_t + 0.2\varepsilon_{t-1} + 0.4\varepsilon_{t-2} + \xi_t + 0.8\xi_{t-1})(\varepsilon_{t-j} + 0.2\varepsilon_{t-1-j} + 0.4\varepsilon_{t-2-j} + \xi_{t-j} + 0.8\xi_{t-1-j})]$$

$$= E[\varepsilon_t\varepsilon_{t-j} + 0.2\varepsilon_{t-1}\varepsilon_{t-j} + 0.4\varepsilon_{t-2}\varepsilon_{t-j} + 0.2^2\varepsilon_{t-1-j}\varepsilon_{t-1} + 0.2\varepsilon_{t-1-j}0.4\varepsilon_{t-2} + 0.4^2\varepsilon_{t-2-j}\varepsilon_{t-2}$$
$$+\xi_t\xi_{t-j} + 0.8\xi_{t-1}\xi_{t-j} + 0.8^2\xi_{t-1}\xi_{t-1-j}]$$

As ε_t and ξ_t are i.i.d., the above expression implies that the value of γ_j will depend on the lag, j. In particular, if we define $var(\varepsilon) = \sigma_\varepsilon^2$ and $var(\xi) = \sigma_\xi^2$, then:

For $j = 0$:

$$\gamma_0 = \sigma_\varepsilon^2 + 0.2^2\sigma_\varepsilon^2 + 0.4^2\sigma_\varepsilon^2 + \sigma_\xi^2 + 0.8^2\sigma_\xi^2 = 1.2\sigma_\varepsilon^2 + 1.64\sigma_\xi^2 = 1.76$$

For $j = 1$:

$$\gamma_1 = 0.2\sigma_\varepsilon^2 + 0.2 * 0.4\sigma_\varepsilon^2 + 0.8\sigma_\xi^2 = 0.28\sigma_\varepsilon^2 + 0.8\sigma_\xi^2 = 0.828$$

For $j = 2$:

$$\gamma_2 = 0.4\sigma_\varepsilon^2 = 0.04$$

For $j > 2$:

$$\gamma_j = 0$$

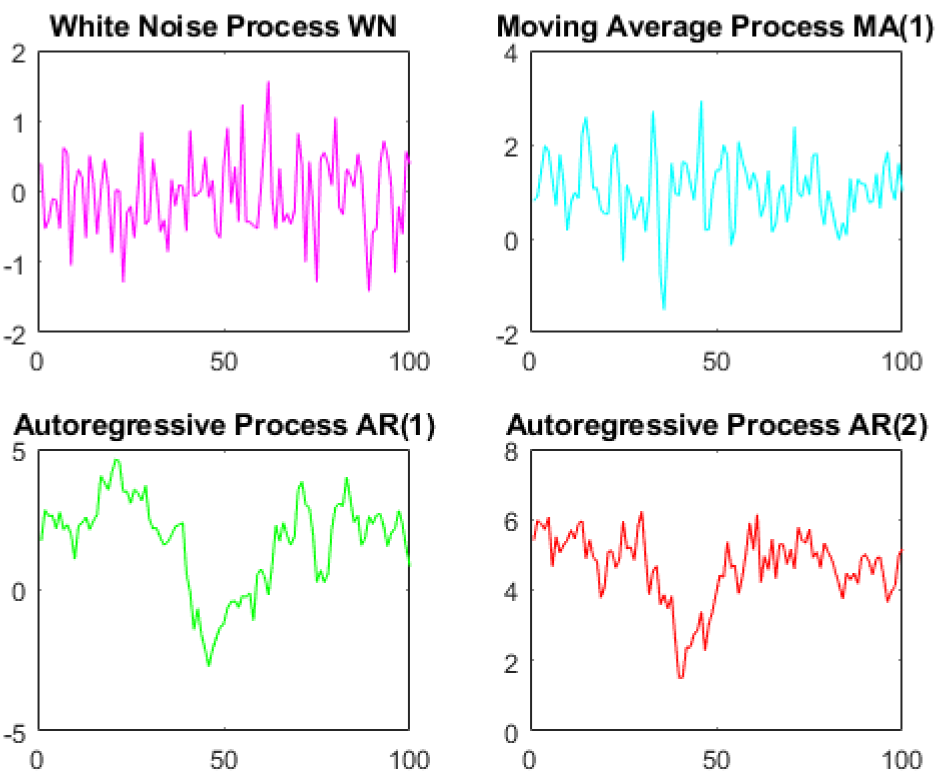
The autocovariance of the process decays as the observations are further apart in time.

Note that checking the condition that $\mathrm{E}[x_t^2] < \infty$ requires to solve the same operation than the for the case of the autocovariance, with the sole difference that we would have to add 1.5^2 to the final result. Nonetheless, the result would still be less than infinity. Hence, the three necessary conditions for covariance stationarity are met and we can conclude that the process is covariance stationary.

Question 3. Matlab exercise (selected exercises)

b. (i) Figure 1 shows the plots of the four time series processes. Visually, one can easily check that they all float around the unconditional mean of the process (shown in Figure 2)

Figure 1: Plots of models



b. (ii) see results shown in Figure 2

b. (iii) In the table below, we can find the population (theoretical) moments against the sample moments of our generated time series.

b. (iv) Figure 3 shows the comparison between the sample autocorrelation (shown by the red dots) and the population autocorrelation (blue dots) for the four times series

For questions (c) and (d) see Matlab Code and Output.

Figure 2: Population and Sample moments

	White noise		MA(1)		AR(1)		AR(2)	
	Population	Sample	Population	Sample	Population	Sample	Population	Sample
Mean	0	-0.05	1.2	1.03	2.00	1.72	5.00	4.63
Variance	0.50	0.32	0.7	0.52	2.63	2.64	1.19	0.97
Autocovariance (lags)								
0	1	0.32	0.7	0.52	2.63	2.64	1.19	0.97
1	0	0.01	0.3	0.16	2.37	2.40	0.89	0.75
2	0	-0.06	0.0	-0.12	2.13	2.17	0.77	0.67
3	0	0.00	0.0	-0.05	1.92	1.94	0.64	0.56
4	0	-0.04	0.0	0.03	1.73	1.74	0.54	0.55
5	0	-0.09	0.0	-0.04	1.55	1.54	0.45	0.45
6	0	0.01	0.0	-0.01	1.40	1.40	0.38	0.38
7	0	0.08	0	0.06	1.26	1.27	0.32	0.35
8	0	-0.01	0	0.01	1.13	1.14	0.27	0.31
9	0	0.01	0	-0.07	1.02	1.08	0.22	0.22
10	0	0.04	0	0.00	0.92	0.98	0.19	0.07

Figure 3: Sample (red) and Population (blue) Autocorrelation