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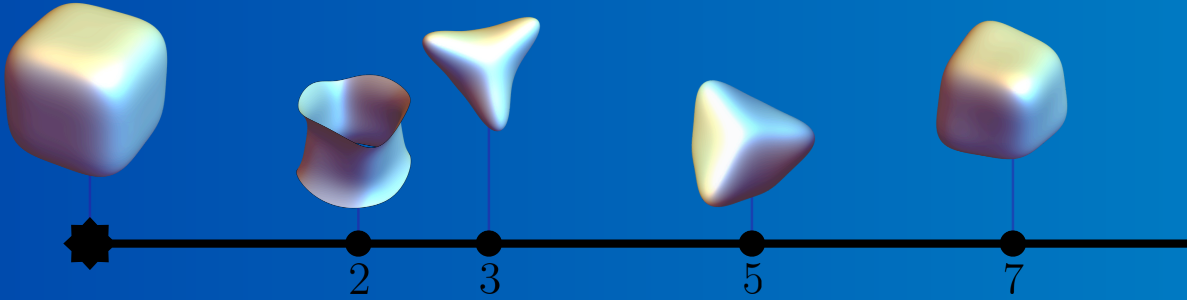
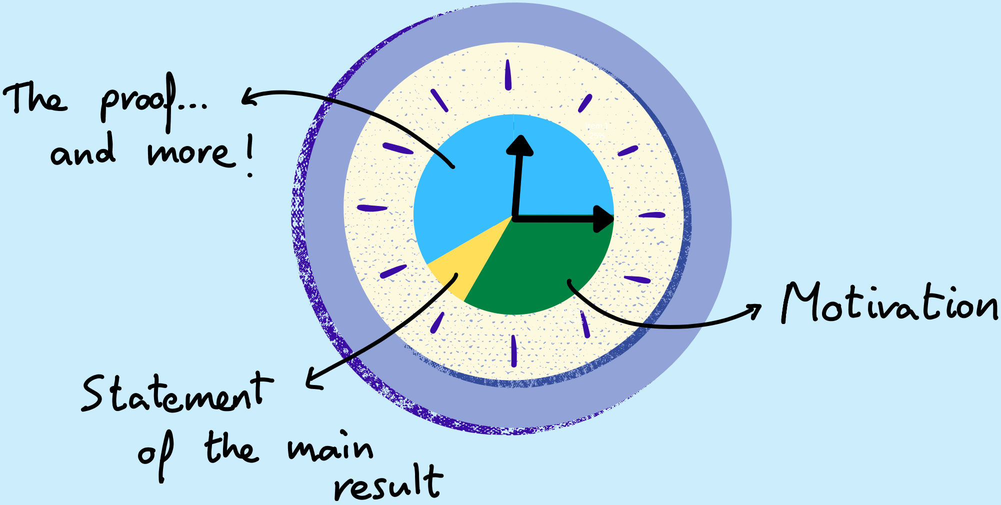
OBERSEMINAR ON ARITHMETISCHE GEOMETRIE, 18TH JULY 2024

Explicit construction of K3 surfaces with everywhere good reduction



Plan for the talk



1. Motivation

Reduction of elliptic curves

$E/\mathbb{Q}:$ $\quad y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$

$\begin{cases} x \mapsto c^2x \\ y \mapsto c^3x \end{cases}$ $\qquad c \in \mathbb{Q}$

$c = \sqrt{q} \quad q$ prime

$E/\mathbb{Z}:$ $\quad y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$

mod p $\qquad p \nmid \Delta_E$

$q \nmid \Delta_E$

$E/\mathbb{F}_p:$ $\quad y^2 + \bar{a}_1xy + \bar{a}_3y = x^3 + \bar{a}_2x^2 + \bar{a}_4x + \bar{a}_6$

p prime of good reduction

q prime of potential good reduction

Why is studying good reduction of elliptic curves "easy"?

1- Existence of a notion of discriminant

2- Well-studied minimal regular models

↳ Tate's algorithm

3- Easy access to Galois representations

For other types of varieties good reduction = hard

Notation

- X projective algebraic variety
- $K/\mathbb{Q}$ number field $\quad \begin{cases} p \text{ prime of } \mathbb{Q} \\ \mathfrak{p} \supseteq (p) \text{ in } K \end{cases}$
- $K_{\mathfrak{p}}/\mathbb{Q}_p$ discretely valued field
- k residue field (perfect and $\mathrm{char}(k) = p$)
- $\mathcal{O}_{K_{\mathfrak{p}}}$ discrete valuation ring

Defining good reduction

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Good reduction

A variety X/K has **good reduction** at a prime $\mathfrak{p}$ if there exists a scheme or algebraic space $\mathcal{X}$ smooth and proper over $\mathcal{O}_{K_\mathfrak{p}}$ with generic fibre $\mathcal{X}_{K_\mathfrak{p}} \cong X \otimes_K K_\mathfrak{p}$.

A variety X/K is said to have **potential good reduction** at $\mathfrak{p}$ if there exists a field extension L/K such that X/L has good reduction.

A variety X/K is said to have **everywhere good reduction** if it has good reduction at all primes.

Serre and Tate provided the fundations for studying the reduction of abelian varieties in their landmark paper titled

Good reduction of abelian varieties

where they proved the Néron-Ogg-Shafarevich criterion.

Is there any abelian variety with everywhere good reduction over $\mathbb{Q}$?

No !

Fontaine and Abrashkin proved independently that there are <u>no abelian varieties with everywhere good reduction defined over the rationals.</u>

What about other number fields?

There are still none over $\mathbb{Q}(i), \mathbb{Q}(\sqrt{-3})$, **or** $\mathbb{Q}(\sqrt{5})$.

But there is over $\mathbb{Q}(\sqrt{29})$:

$E/\mathbb{Q}(\sqrt{29})$: $\quad y^2 + xy + \left(\frac{5+\sqrt{29}}{2}\right)^2 y = x^3 \qquad \Delta_E = \text{unit}$

Varieties with everywhere good reduction over the rationals

- **Curves**
  1. **Genus 0** ✓ e.g. $\mathbb{P}^1$
  2. **Higher genus** ✗ C g.r $\Rightarrow$ Jac(C) g.r
     Contradiction!
- **Surfaces** **(maybe?)**
  1. **Rational surfaces** ✓ e.g $\mathbb{P}^2$
  2. **Abelian surfaces** ✗

What else is there?

What is a K3 surface?

K3 surfaces

A **K3 surface** X is a simply connected surface with trivial canonical bundle, meaning

$$h^1(X, \mathcal{O}_X) = 0,$$
$$\omega_X = \wedge^2 \Omega_X \simeq \mathcal{O}_X.$$

2nd tallest & 1/5 mortality rate

K3 → Kummer, Kähler, Kodaira

also K3 are more difficult than K2 mountain

Examples of K3 surfaces

1- X_4 smooth quartic in $\mathbb{P}^3$

2- $X_{2,3}$ smooth complete intersection in $\mathbb{P}^4$

3- $X_{2,2,2}$ smooth complete intersection in $\mathbb{P}^5$

4- Kummer surfaces

(*) If any of the previous examples has "nice" singularities, (ADE) rational double points, the desingularisation is also a K3 surface.

What is a Kummer surface?

$G \curvearrowright X \Rightarrow Y = X/G$ quotient variety $\mathrm{Sing}(Y) \leftrightarrow \mathrm{Fix}_G(X)$

Kummer surfaces

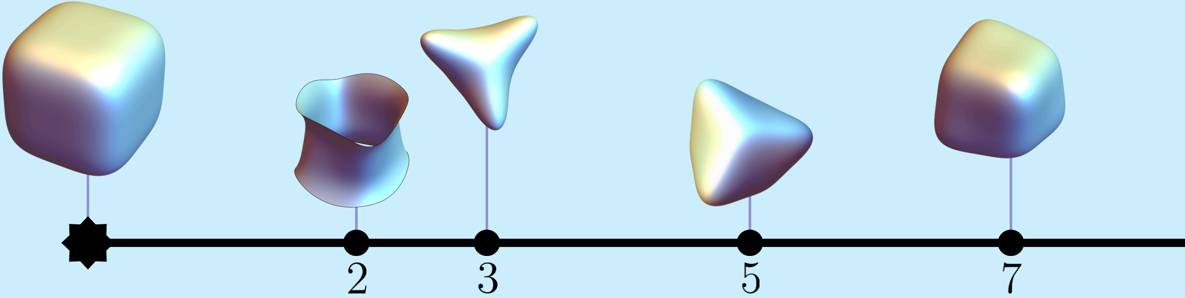
Let A be an abelian surface and let ι be the involution in A that sends an element to its inverse. Then, the **Kummer surface** associated to A, $\mathrm{Kum}(A)$ is the quotient variety $A/\langle\iota\rangle$.

$\mathrm{Sing}(\mathrm{Kum}(A)) \leftrightarrow \mathrm{Fix}_{\langle\iota\rangle}(A) = \{P \in A : P = -P\} = A[2]$

$\mathrm{char}(K) \neq 2$	$\mathrm{char}(K) = 2$		
	p-rank $= 2$	p-rank $= 1$	p-rank $= 0$
$\#A[2](\bar{K}) = 16$	$\#A[2](\bar{K}) = 4$	$\#A[2](\bar{K}) = 2$	$\#A[2](\bar{K}) = 1$
16 A_1 singularities	4 D_4 singularities	2 D_8 singularities	1 elliptic singularity

So, are there examples of K3 surfaces with everywhere good reduction over the rationals?

No!



Abrashkin and **Fontaine** proved that there are no K3 surfaces with everywhere good reduction over the rationals.

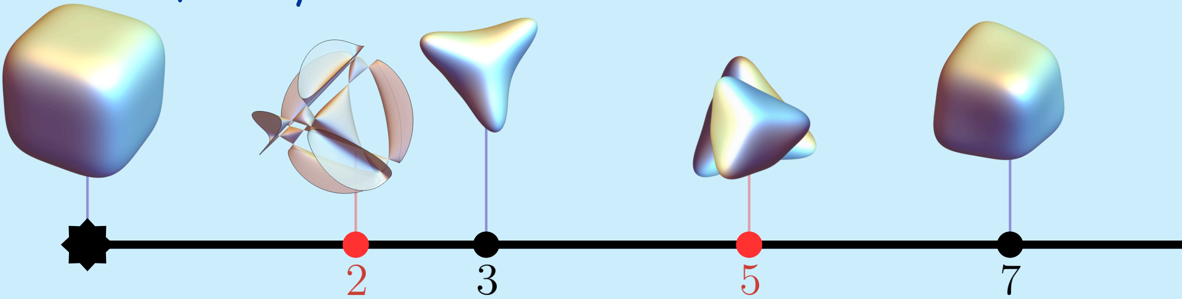
Thm

X good reduction everywhere over $\mathbb{Q}$ or $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-3})$, $\mathbb{Q}(\sqrt{-5})$ $\Rightarrow$ $i+j \leq 3$ $i \neq j$ $h^i(X, \Omega^j_X) = 0$

$\hookrightarrow h^{ij}$

Hodge diamond of a K3 surface =

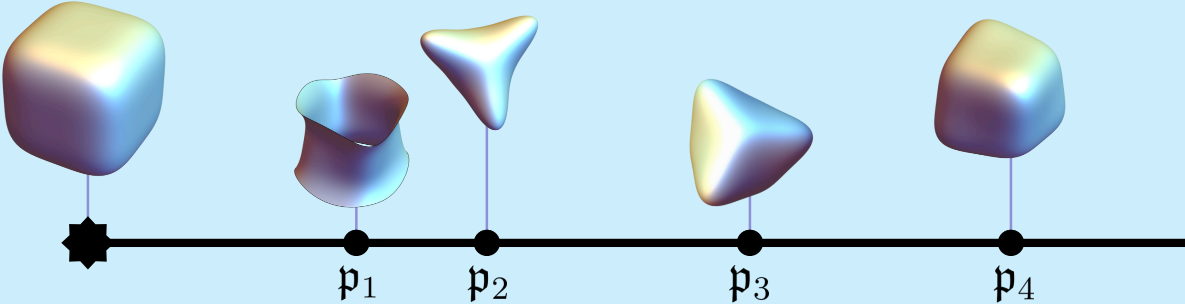
$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$



So, are there examples of K3 surfaces with everywhere good reduction over ~~the rationals?~~

a number field ?
quadratic ?

Yes !



2. The theorem

There are examples of K3 surfaces with everywhere good reduction

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Theorem (G.)

There exists a K3 surface Z defined over $\mathbb{Z}[\frac{1+\sqrt{353}}{2}] = \mathcal{O}_{\mathbb{Q}(\sqrt{353})}$ that has good reduction at all primes.

Furthermore, a model for this surface can be found as the blow-up of 4 lines of a smooth K3 surface $X \subset \mathbb{P}^5$.

$\hookrightarrow$ what is X ?

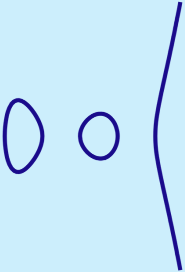
$X \subset \mathbb{P}^5_{\{b_1,b_2,b_3,b_4,b_5,b_6\}}$ **where** $w = \frac{1-\sqrt{353}}{2}$.

$$\begin{gathered}
(1452709374948488612467 5+ 16333318488259047565 49w)b_1^2 + (528456860823969752651 6+ 594162491410199983308w)b_1b_2 + (2100204515307551826 87+ 236133323227258194 33w)b_2^2 \\
+(5418448047821790138 76+ 609215024032573186 68w)b_1b_3 + (1936422246406439639 7+ 217718711145702593 1w)b_2b_3 + (212993331163719105 + 23947583555008119w)b_3^2 \\
+(5742287842316765151 + 645625462306548969w)b_1b_4 + (50397155427910737 + 5666328067412295w)b_2b_4 + (-1787709049454413 - 200998367413419w)b_1b_5 + (16252032444984 + 182727272726280w)b_2b_5 \\
+(581492979 + 65379285w)b_3b_5 + (-1162985958 - 130758570w)b_4b_5 + (21268042436 + 2391240236w)b_1b_6 + (-464255322 - 52197846w)b_2b_6 + (5915097 + 665055w)b_3b_6 = 0,
\end{gathered}$$

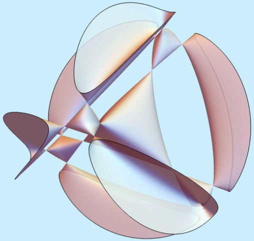
$$\begin{gathered}
(484131085002349914903 7+ 54432547471445788580 3w)b_1^2 + (176113096402343583970 6+ 198010100512701953062w)b_1b_2 + (6998923615886728607 1+ 786913407902589406 5w)b_2^2 \\
+(1805748766556928896 69+ 203026635764645574 11w)b_1b_3 + (6452945795759781270 + 725527216160176794w)b_2b_3 + (70977140699228925 + 7980207634218555w)b_3^2 \\
+(1913671421321646039 + 215160756482583681w)b_1b_4 + (16791472437285390 + 1887923847223266w)b_2b_4 + (193830993 + 21793095w)b_4^2 \\
+(-595770851724353 - 66984596059223w)b_1b_5 + (5417342176629 + 609090687075w)b_2b_5 + (172351014 + 19378026w)b_3b_5 \\
+(-387661986 - 43586190w)b_4b_5 + (7087775171 + 796903293w)b_1b_6 + (-154731717 - 17397027w)b_2b_6 + (1971699 + 221685w)b_3b_6 = 0,
\end{gathered}$$

$$\begin{gathered}
(484236458091975404041 9+ 54444394935657312750 9w)b_1^2 + (176152286855634721172 0+ 198054163707057391280w)b_1b_2 + (7000681713303829320 1+ 787111076930834691 9w)b_2^2 \\
+(1806149348398025904 28+ 203071674579043448 12w)b_1b_3 + (6454740816314215349 + 725729036585611587w)b_2b_3 + (70997776977378135 + 7982527842990081w)b_3^2 \\
+(1914095946510647319 + 215208487331147193w)b_1b_4 + (16799051780893503 + 1888776019276521w)b_2b_4 + (-361974 - 40698w)b_3b_4 + (-595903016195803 - 66999455771981w)b_1b_5 \\
+(5417344152010 + 609090909174w)b_2b_5 + (193830993 + 21793095w)b_3b_5 + (-387661986 - 43586190w)b_4b_5 \\
+(7089347478 + 797080078w)b_1b_6 + (-154751774 - 17399282w)b_2b_6 + (1971699 + 221685w)b_3b_6 = 0
\end{gathered}$$

How did I compute it?



Jacobian of a genus two curve with everywhere good reduction and ordinary reduction at 2



Its associated Kummer surface (which is a K3)

3. The proof

Sketch of the steps of the proof

1. Find abelian surfaces with everywhere good reduction

2. Check the reduction of its associated Kummer surface

3. Construct an explicit scheme model for the example

1. Find abelian surfaces with everywhere good reduction over a number field

Examples over a quadratic field have been computed by Dembélé and Kumar (2016) and Dembélé (2020).

$g(x)$	$f(x)$	w
$wx^3 + wx^2 + w + 1$	$-4x^6 + (w-17)x^5 + (12w-27)x^4 + (5w-122)x^3 + (45w-25)x^2 + (-9w-137)x + 14w + 9$	$\frac{1+\sqrt{53}}{2}$
$x^3 + x + 1$	$(w-5)x^6 + (3w-14)x^5 + (3w-19)x^4 + (4w-3)x^3 - (3w+16)x^2 + (3w+11)x - (w+4)$	$\frac{1+\sqrt{73}}{2}$
$w\left(x^3+1\right)$	$-2(4414w+43089)x^6 + (31147w+303963)x^5 - 10(4522w+44133)x^4 + 2(17290w+168687)x^3 - 18(816w+7967)x^2 + 27(122w+1189)x - (304w+3003)$	$\frac{1+\sqrt{421}}{2}$
$x^3 + x^2 + 1$	$-2x^6 + (-3w+1)x^5 - 219x^4 + (-83w+41)x^3 - 1806x^2 + (-204w+102)x - 977$	$\frac{1+\sqrt{409}}{2}$
$x^3 + x + 1$	$-134x^6 - (146w-73)x^5 - 13427x^4 - (3255w-1627)x^3 - 89746x^2 - (6523w-3261)x - 39941$	$\frac{1+\sqrt{809}}{2}$
$x^3 + x + 1$	$23x^6 + (90w-45)x^5 + 33601x^4 + (28707w-14354)x^3 + 3192149x^2 + (811953w-405977)x + 19904990$	$\frac{1+\sqrt{929}}{2}$

2. Check the reduction of its associated Kummer surface

A good property of Kummer surfaces is that good reduction is preserved at all primes that do not lie above 2. $\# \mathrm{Sing}(\mathrm{Kum}(A)) = 16$ when $\mathrm{char}(k) \neq 2$

At the primes lying above 2, a result by Lazda and Skorobogatov ensures that if the abelian surface has good reduction, then the Kummer surface has <u>potential good reduction.</u>

(Assuming p-rank$(A) \neq 0$, if A is supersingular, then the result is due to Y. Matsumoto (2023))

Good reduction of a Kummer surface at 2

Theorem (Lazda, Skorobogatov, 2022)

Let $A = \mathrm{Jac}(\mathcal{C})$ be an abelian surface with good ordinary reduction at 2, let $K_{\mathfrak{p}}$ be a discretely valued field with perfect residue field k of characteristic 2, and let $\mathcal{A}/\mathcal{O}_{K_{\mathfrak{p}}}$ be the Néron model of $A/K_{\mathfrak{p}}$, which is an abelian scheme with generic fiber $\mathcal{A}_{K_{\mathfrak{p}}} \cong A$. Let us fix an algebraic closure $\bar{K}_{\mathfrak{p}}$ of $K_{\mathfrak{p}}$, with residue field $\bar{k}$, and let $\Gamma_{K_{\mathfrak{p}}}$ denote the Galois group of $\bar{K}_{\mathfrak{p}}/K_{\mathfrak{p}}$. Then, we have the exact sequence of $\Gamma_{K_{\mathfrak{p}}}$-modules:

$$0 \longrightarrow \mathcal{A}[2]^{\circ}(\bar{K}_{\mathfrak{p}}) \longrightarrow \mathcal{A}[2](\bar{K}_{\mathfrak{p}}) \longrightarrow \mathcal{A}[2](\bar{k}) \longrightarrow 0$$

where $\mathcal{A}[2]^{\circ}$ is the connected component of the identity of the 2-torsion subscheme $\mathcal{A}[2] \subseteq \mathcal{A}$.

Then, the Kummer surface associated to A has good reduction at $\mathfrak{p}$ if and only if the previous exact sequence of $\Gamma_{K_{\mathfrak{p}}}$-modules split.

The particular example I analised

$353 \equiv 1 \pmod 8 \Rightarrow 353 = \square$ in $\mathbb{Q}_2$

2 is inert in $\mathbb{Q}(\sqrt{353})$

$\Rightarrow K_2 = \mathbb{Q}_2$

$\sqrt{353} = 1 + O(2^2)$

$\Rightarrow w = O(2)$

Let $K = \mathbb{Q}(\sqrt{353})$, let $w = \frac{1-\sqrt{353}}{2}$ and let

$$\mathcal{C} : y^2 + g(x)y = f(x)$$

where $\bar{\mathcal{C}} / \mathbb{F}_2 : y^2 + (x^3 + x^2 + 1)y = x^6 + x^4 + x^2 + x$

$$\begin{aligned} g(x) &= (w+1)x^3 + x^2 + wx + 1, \\ f(x) &= (-15w + 149)x^6 - (1119w + 9948)x^5 - (36545w + 325409)x^4 \\ &\quad - (363632w + 5659370)x^3 - (622714w + 5538975)x^2 \\ &\quad - (3284000w + 288867915)x - 70532813w - 627353458. \end{aligned}$$

$A = \mathrm{Jac}(\mathcal{C}) = \{ P + Q - \infty_+ - \infty_- \text{ where } P, Q \in \mathcal{C} \} / \cong$

If P, Q are Weierstrass points $\Rightarrow (P, Q) \in A[2] \setminus \{ id \}$

Proving good reduction at 2 in our example

Short exact sequence of $\Gamma_{K_\mathfrak{p}}$-modules, wwtp that it splits

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{A}[2]^\circ(\bar{K}_\mathfrak{p}) \arrow[r] & \mathcal{A}[2](\bar{K}_\mathfrak{p}) \arrow[r] & \mathcal{A}[2](\bar{k}) \arrow[r] \arrow[l, bend left, "\sigma"] & 0 \end{tikzcd}$$

Over $\bar{K}_\mathfrak{p}$, $\mathcal{C}$ has 6 Weierstrass points, corresponding to the roots of $f(x) + \frac{1}{4} g(x)^2$

$\binom{6}{2} = 15 = 16 - 1$

Over $\bar{k}$, $\mathcal{C}$ has 3 Weierstrass points, corresponding to the roots of $g(x)$

$\binom{3}{2} = 3 = 4 - 1.$

The action of $\Gamma_{K_\mathfrak{p}}$ on $A[2]$ is described completely by the action on the Weierstrass points.

Proving good reduction at 2 in our example

$$f(x) + \frac{1}{4}g(x)^2 = \frac{1}{4}q_1(x)q_2(x)$$

where

$$q_1(x) = x^3 + (2088841801 + O(2^{32}))x^2 + (1097586240 + O(2^{32}))x + 553607353 + O(2^{32}),$$
$$q_2(x) = x^3 + (1373013921 + O(2^{32}))x^2 - (1548938988 + O(2^{32}))x - 856394843 + O(2^{32}).$$

$g(x)$ is irreducible in $\mathbb{Q}_2$ + $\bar{g}(x)$ is irreducible in $\mathbb{F}_2$
$\bar{g}(x) = x^3+x^2+1$

Consider $\mathbb{Q}_2(\gamma)$ where $g(\gamma)=0$. Then g completely splits in $\mathbb{Q}_2(\gamma)$ and $\bar{g}$ completely splits over $\mathbb{F}_2(\bar{\gamma}) = \mathbb{F}_8$.

$\mathbb{Q}_2(\gamma)$ is the unique unramified extension of deg 3 of $\mathbb{Q}_2$

Proving good reduction at 2 in our example

Over $\mathbb{Q}_2(\gamma)$

$$f(x) + \frac{1}{4}g(x)^2 = h_1(x)h_2(x)h_3(x)h_4(x)h_5(x)h_6(x)$$

where

mod 2

$$\left\{\begin{aligned}
h_1(x) &= x - 406904280\gamma^2 + 435522127\gamma - 1230442616 + O(2^{32}), &&\longrightarrow x + \bar{\gamma}\\
h_2(x) &= x + 394057577\gamma^2 - 1606502354\gamma + 490223466 + O(2^{32}), &&\longrightarrow x + \bar{\gamma}^2\\
h_3(x) &= x - 1060895121\gamma^2 - 976503421\gamma + 681577303 + O(2^{32}), &&\longrightarrow x + \bar{\gamma}^2 + \bar{\gamma} + 1
\end{aligned}\right.$$

$$\begin{aligned}
h_4(x) &= x + 1307484884\gamma^2 + 1755128143\gamma - 56114964 + O(2^{32}),\\
h_5(x) &= x + 914512901\gamma^2 + 842339586\gamma - 1344868422 + O(2^{32}),\\
h_6(x) &= x - 1148255961\gamma^2 - 449984081\gamma + 626513659 + O(2^{32}).
\end{aligned}$$

$h_i(r_i) = 0$

Proving good reduction at 2 in our example

If $K_\mathfrak{p} = \mathbb{Q}_2(\gamma)$, then this is a trivial exact sequence of $\Gamma_{K_\mathfrak{p}}$-modules

$$0 \longrightarrow \mathcal{A}[2]^\circ(\bar{K}_\mathfrak{p}) \longrightarrow \mathcal{A}[2](\bar{K}_\mathfrak{p}) \underset{\nwarrow s}{\longrightarrow} \mathcal{A}[2](\bar{k}) \longrightarrow 0$$

But even if $K_\mathfrak{p} = \mathbb{Q}_2$, this still splits!

$\Gamma_{\mathbb{Q}_2} = \mathrm{Gal}(\bar{\mathbb{Q}}_2/\mathbb{Q}_2)$ factors through $\mathrm{Gal}(\mathbb{Q}_2(\gamma)/\mathbb{Q}_2) \cong \mathbb{Z}/3\mathbb{Z}$

As $\mathbb{Q}_2(\gamma)$ is unramified, this is "compatible" with $\mathrm{Gal}(\mathbb{F}_2(\bar{\gamma})/\mathbb{Q}_2) \cong \mathbb{Z}/3\mathbb{Z}$

We can construct a section

$$\bar{g}(x) = \underbrace{(x+\bar{\gamma})}_{\bar{r}_1}\underbrace{(x+\bar{\gamma}^2)}_{\bar{r}_2}\underbrace{(x+\bar{\gamma}^2+\bar{\gamma}+1)}_{\bar{r}_3}$$

$$\begin{aligned} A[2](\mathbb{F}_2) &\longrightarrow (\mathbb{Z}/2\mathbb{Z})^2 \leq A[2](\bar{\mathbb{Q}}_2) \\ 0 &\longmapsto 0 \\ (\bar{r}_i, \bar{r}_j) &\longmapsto (r_i, r_j) \end{aligned}$$

Why do we need a field extension for the other examples?

Consider the example ★

$$0 \longrightarrow \mathcal{A}[2]^\circ(\bar{K}_\mathfrak{p}) \longrightarrow \mathcal{A}[2](\bar{K}_\mathfrak{p}) \longrightarrow \mathcal{A}[2](\bar{k}) \longrightarrow 0$$

Over $\mathbb{Q}_2\left(\frac{1+\sqrt{53}}{2}\right)$,

$$g(x) = c_1(x)c_2(x)c_3(x) \quad \text{linear}$$

$$f(x) + \frac{1}{4}g(x)^2 = q_1(x)q_2(x)q_3(x) \quad \text{quadratic}$$

Therefore, $A[2](\bar{k})$ is trivial as a Γ_{u_3}-module, but $A[2](\bar{K}_\mathfrak{p})$ is <u>not</u> as there are actions interchanging the roots of q_1. This implies that the sequence cannot split over $\mathbb{Q}_2\left(\frac{1+\sqrt{53}}{2}\right)$.

Considering the extension $\mathbb{Q}_2\left(\frac{1+\sqrt{53}}{2}, i\right)$, then $A[2](\bar{K}_\mathfrak{p})$ becomes trivial and the sequence splits.

3. Construct an explicit smooth scheme model for the example

Characteristic zero

$\mathrm{Kum}(\mathcal{C}) = X_4 \subset \mathbb{P}^3$

Model with 16 A_1 singularities

Blow-up of singular locus

$X_{2,2,2} \subset \mathbb{P}^5$

Smooth model

Blow-up of 4 lines

Smooth model with four (-1) curves

mod 2

Characteristic 2 ordinary case

$\mathrm{Kum}(\mathcal{C}) = X_4 \subset \mathbb{P}^3$

Model with 4 D_4^1 singularities

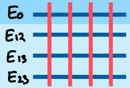
Blow-up of singular locus

$X_{2,2,2} \subset \mathbb{P}^5$

Model with 12 A_1 singularities

Blow-up of 4 lines

Smooth model



Given $\mathcal{C} : y^2 + g(x)y = f(x)$ with $g(x) = \sum_{i=0}^{3} g_i x^i$, $f(x) = \sum_{i=0}^{6} f_i x^i$

we can construct $\mathrm{Sym}^2 \mathcal{C}$.

Let $\Theta_+ = \mathcal{C} \times \{\infty_+\}$, $\Theta_- = \mathcal{C} \times \{\infty_-\}$, then, the theory of theta divisors shows that

1- $\mathcal{L}(\Theta_+ + \Theta_-)$ generates an embedding of $\mathrm{Kum}(\mathcal{C}) \hookrightarrow \mathbb{P}^3$

2- $\mathcal{L}(2(\Theta_+ + \Theta_-))$ generates an embedding of $\mathrm{Jac}(\mathcal{C}) \hookrightarrow \mathbb{P}^{15}$

Studying the desingularisation of Kummer surfaces

$\iota_H : \mathcal{C} \longrightarrow \mathcal{C}$ induces a linear action on
$(x,y) \longmapsto (x, -y-g(x))$ the elements of $\mathcal{L}(2(\Theta_+ + \Theta_-))$

Diagonalising with respect of this action

$$\mathcal{L}(2(\Theta_+ + \Theta_-)) = \{\text{even functions}\} + \{\text{odd functions}\}$$

10 of them, they are quadratic expressions on the functions of $\mathcal{L}(\Theta_+ + \Theta_-)$

6 of them. There are 2 quadratic relations among them $\mathrm{Des}(\mathrm{Kum}(\mathcal{C})) \hookrightarrow \mathbb{P}^5$

S. Müller computed a basis of $\mathcal{L}(\Theta_+ + \Theta_-)$ in characteristic zero that reduces well when reducing modulo 2.

I've extended it to a basis of $\mathcal{L}(2(\Theta_+ + \Theta_-))$, and so, I've computed the equations for the Jacobian!

$$\mathcal{L}(2(\Theta_+ + \Theta_-)) = \{\text{even functions}\} + \{\text{odd functions}\}$$

↳ the reduction of these still give a partial desingularisation of the Kummer in char 2!

Thank you!