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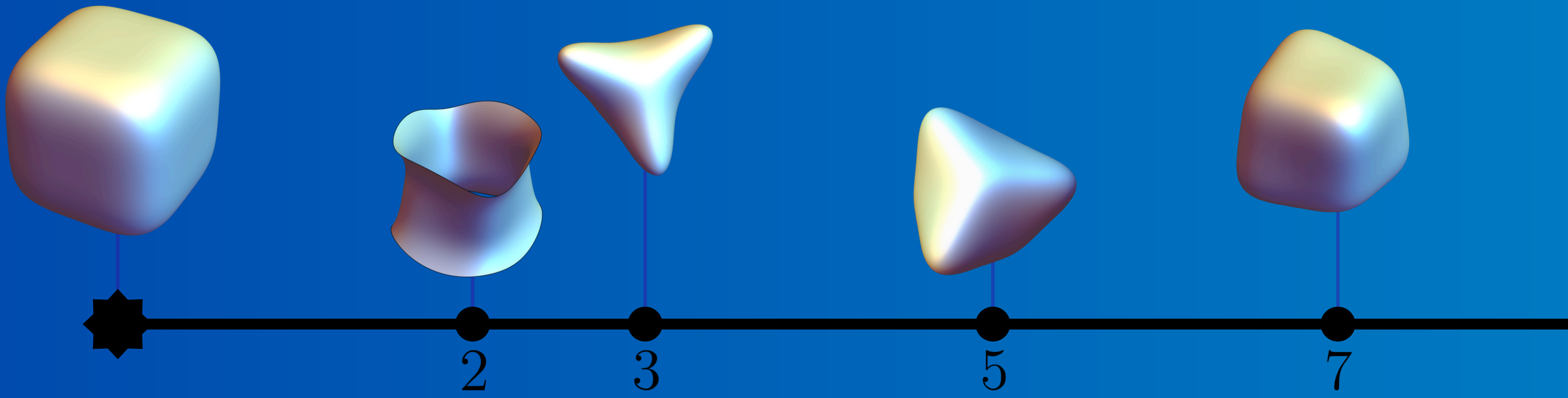
K3 surfaces with everywhere good reduction



Back to our first course on elliptic curves

$$E/\mathbb{Q}: \; y^2 = x^3 - \frac{3888}{625}x + \frac{221616}{15625}$$

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$$\begin{cases} x \mapsto \frac{1}{5^2}x \\ y \mapsto \frac{1}{5^3}y \end{cases}$$

$$E/\mathbb{Z}: \; y^2 = x^3 - 3888x + 221616$$

Back to our first course on elliptic curves

$$E/\mathbb{Z}: \quad y^2 = x^3 - 3888x + 221616$$

$$\Big\downarrow \text{ Choose your favourite prime p (mine is 7!)}$$

$$y^2 = x^3 - 3x + 3$$

Back to our first course on elliptic curves

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$$\Big\downarrow \text{Choose your favourite prime p (mine is 7!)}$$

$$7 \nmid \Delta$$

$$\Delta(E) = -2^{12} \cdot 3^{18} \cdot 11$$

$$E/\mathbb{F}_7: \quad y^2 = x^3 - 3x + 3$$

All the primes not dividing

$$\Delta(E) = -2^{12} \cdot 3^{18} \cdot 11$$

are primes of good reduction of

$$E/\mathbb{Z}: \quad y^2 = x^3 - 3888x + 221616$$

But, is this all?

At the prime 2

Any model of elliptic curve of the form

$$y^2 = x^3 + Ax + B$$

will reduce to a singular curve.

So in order to study the reduction at 2, we would need to either find a better model for the elliptic curve, or use other methods.

At the prime 3

Over $\mathbb{Q}(\sqrt{3})$

$$E/\mathbb{Z}[\sqrt{3}]: \quad y^2 = x^3 - 3888x + 221616$$

$$\begin{cases} x \mapsto \frac{1}{108}x + \frac{1}{3} \\ y \mapsto \frac{1}{648\sqrt{3}}y - \frac{1}{2} \end{cases}$$

$$E'/\mathbb{Z}[\sqrt{3}]: \quad y^2 + y = x^3 - x^2$$

$$\Delta(E') = -11$$

So, the curve acquires good reduction at 2 and 3 over a field extension.

At the prime 11

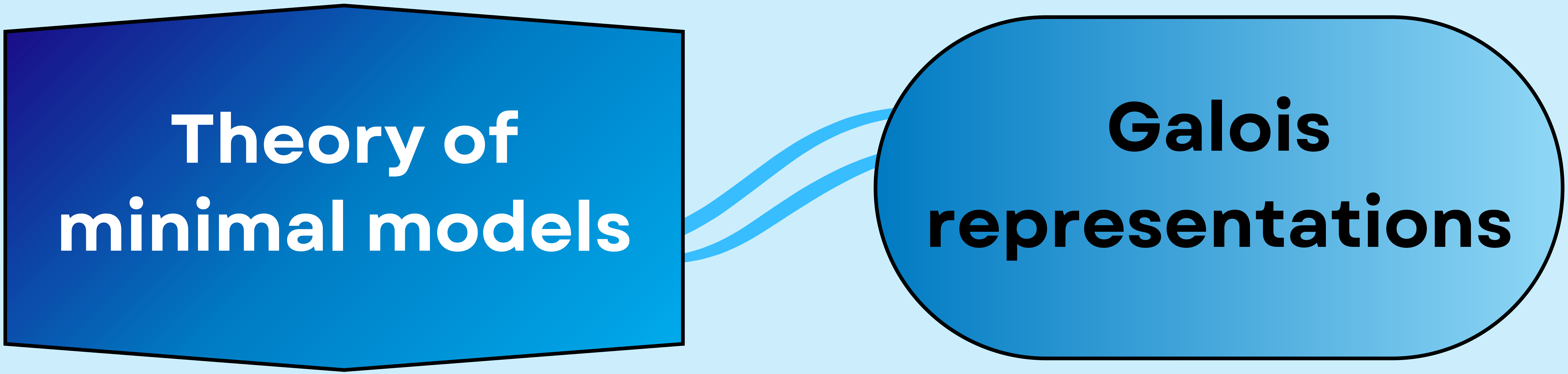
The curve has genuinely bad (multiplicative) reduction:

$$E/\mathbb{F}_{11} : \quad y^2 = (x-3)^2(x-5)$$

This can be seen as 11 divides the denominator of the j-invariant!

$$j(E) = -\frac{4096}{11}$$

For abelian varieties



Serre and **Tate** provided the fundations for studying the reduction of abelian varieties in their landmark paper titled

Good reduction of abelian varieties

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Is there an abelian variety defined over the rationals with good reduction at all primes?

No!

Fontaine and Abrashkin proved independently that there are no abelian varieties with everywhere good reduction defined over the rationals.

What about other number fields?

There are still none over $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-3})$, **or** $\mathbb{Q}(\sqrt{5})$.

But there is over $\mathbb{Q}(\sqrt{29})$:

$$y^2 + xy + \left(\frac{5+\sqrt{29}}{2}\right)^2 y = x^3$$

Varieties with everywhere good reduction over the rationals

- <u>**Curves**</u>
    1. **Genus 0**

Varieties with everywhere good reduction over the rationals

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     for instance, $\mathbb{P}^1$

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  1. **Rational surfaces**

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     for instance, $\mathbb{P}^2$

Varieties with everywhere good reduction over the rationals

- **Curves**
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  2. **Higher genus** ❌
- **Surfaces** **(maybe?)**
  1. **Rational surfaces** ✅
  2. **Abelian surfaces** ❌

Varieties with everywhere good reduction over the rationals

- **Curves**
  1. **Genus 0** ✅
  2. **Higher genus** ❌

- **Surfaces (maybe?)**
  1. **Rational surfaces** ✅
  2. **Abelian surfaces** ❌
  3. **What else is there?**

The next simplest kind of surface are K3 surfaces

K3 surfaces

A **K3 surface** X is a simply connected surface with trivial canonical bundle, meaning

$$h^1(X, \mathcal{O}_X) = 0,$$
$$\omega_X = \wedge^2 \Omega_X \simeq \mathcal{O}_X.$$

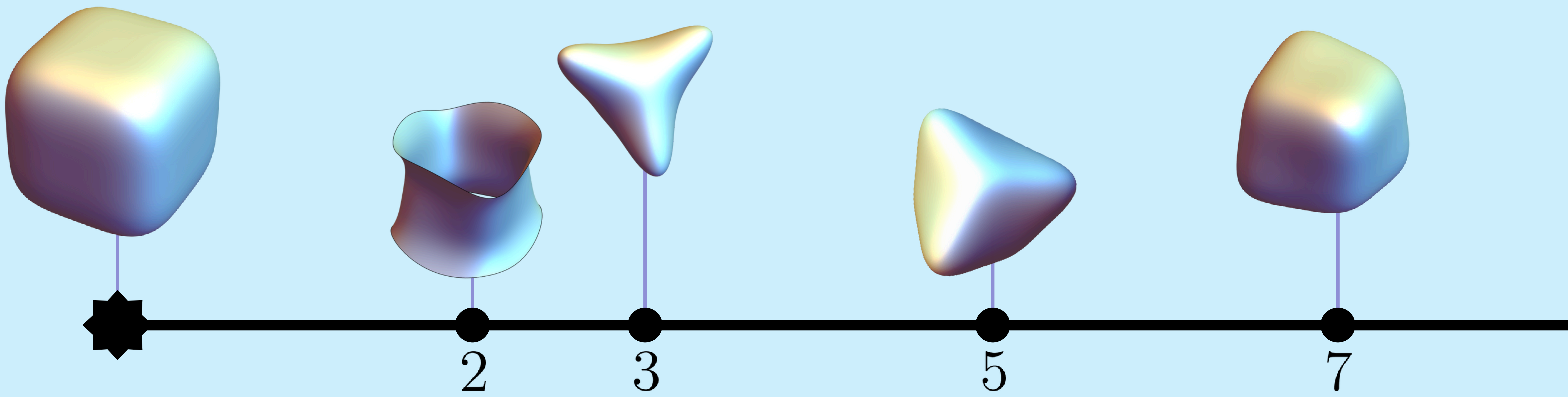
Examples of K3 surfaces

- **Smooth quartics in $\mathbb{P}^3$.**
- **Smooth complete intersections of three conics in $\mathbb{P}^5$.**
- **Kummer surfaces:**

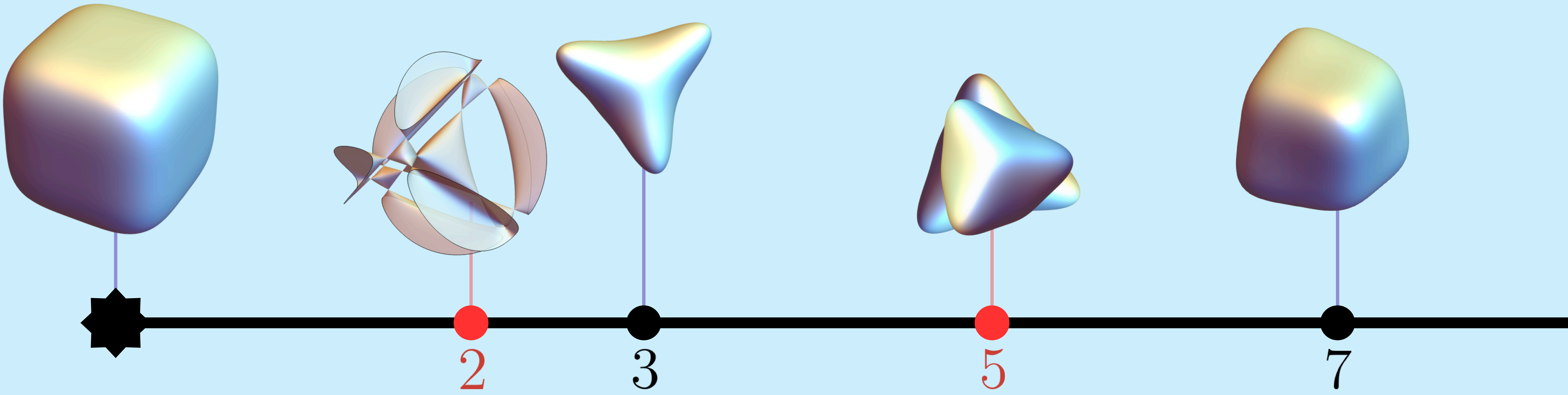
Kummer surfaces

Let $\mathcal{A}$ be an Abelian surface and let ι be the involution in $\mathcal{A}$ that sends an element to its inverse. Then, the **Kummer surface** associated to $\mathcal{A}$, $\mathrm{Kum}(\mathcal{A})$ is the quotient variety $\mathcal{A}/\iota$.

So, are there examples of K3 surfaces with everywhere good reduction over the rationals?

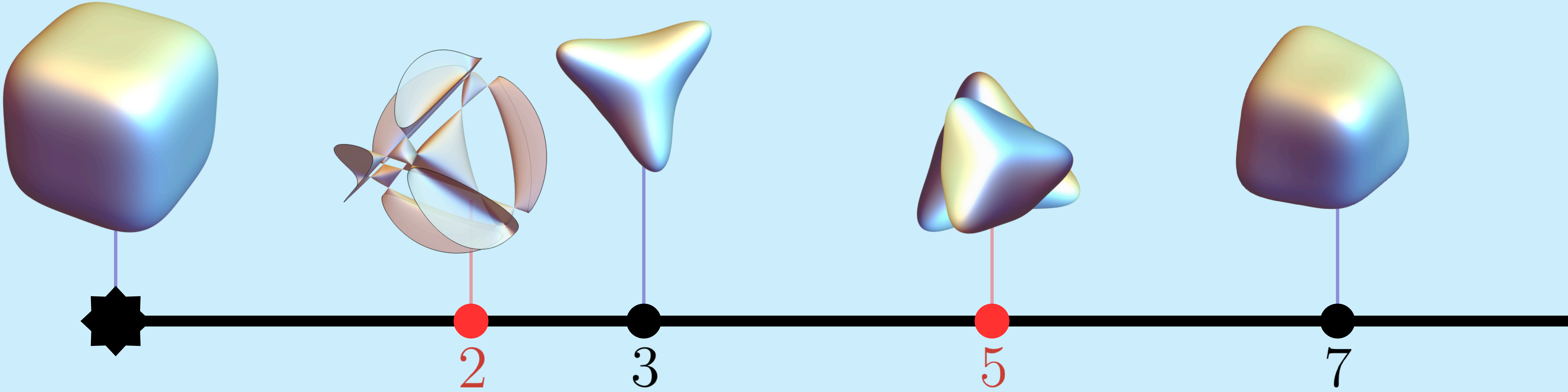


No!



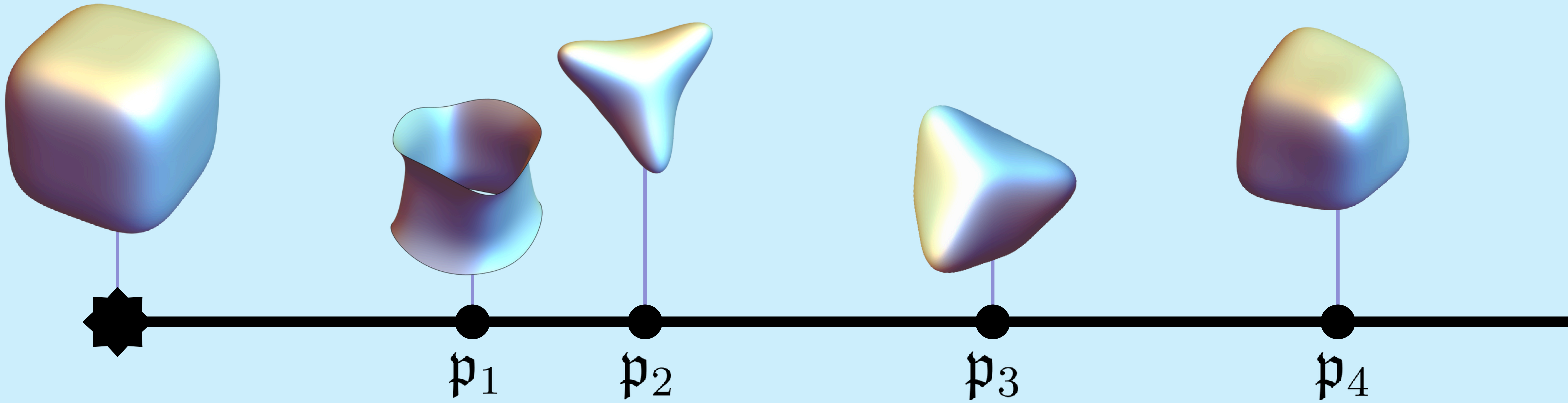
No!

Abrashkin and **Fontaine** proved that there are no K3 surfaces with everywhere good reduction over the rationals.



So, are there examples of K3 surfaces with everywhere good reduction over ~~the rationals~~?

a number field



Yes!

Theorem (G.)

There exists a K3 surface Z defined over $\mathbb{Z}[\frac{1+\sqrt{353}}{2}] = \mathcal{O}_{\mathbb{Q}(\sqrt{353})}$ that has good reduction at all primes.

Furthermore, a model for this surface can be found as the blow-up of 4 lines of a smooth K3 surface $X \subset \mathbb{P}^5$.

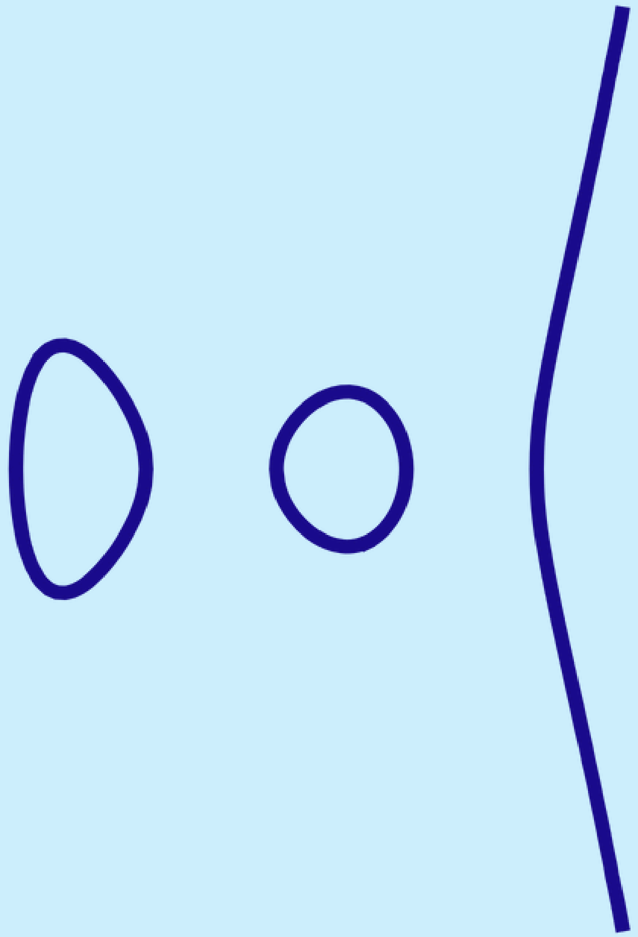
$$X \subset \mathbb{P}^5_{\{b_1,b_2,b_3,b_4,b_5,b_6\}}$$ **where** $$w = \frac{1+\sqrt{353}}{2}.$$

$$\begin{gathered}
(1452709374948488612467\,5 + 163333184882590475654\,9w)b_1^2 + (528456860823969752651\,6 + 594162491410199983308w)b_1b_2 + (210020451530755182687 + 23613332322725819433w)b_2^2 \\
+(541844804782179013876 + 60921502403257318668w)b_1b_3 + (193642224640643963\,97 + 21771871114570259\,31w)b_2b_3 + (212993331163719105 + 23947583555008119w)b_3^2 \\
+(5742287842316765151 + 645625462306548969w)b_1b_4 + (50397155427910737 + 5666328067412295w)b_2b_4 + (-1787709049454413 - 200998367413419w)b_1b_5 + (16252032444984 + 1827272726280w)b_2b_5 \\
+(581492979 + 65379285w)b_3b_5 + (-1162985958 - 130758570w)b_4b_5 + (21268042436 + 2391240236w)b_1b_6 + (-464255322 - 52197846w)b_2b_6 + (5915097 + 665055w)b_3b_6 = 0,
\end{gathered}$$

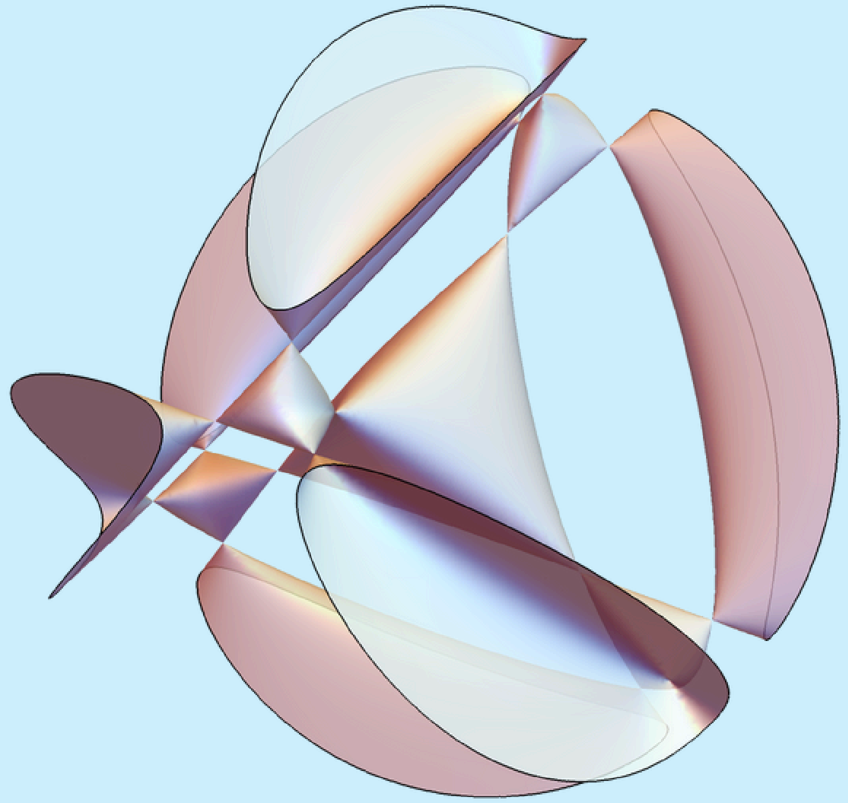
$$\begin{gathered}
(484131085002349914903\,7 + 544325474714457885803w)b_1^2 + (1761130964023435839706 + 198010100512701953062w)b_1b_2 + (69989236158867286071 + 7869134079025894065w)b_2^2 \\
+(18057487665569288966\,9 + 20302663576464557411w)b_1b_3 + (64529457957597812\,70 + 725527216160176794w)b_2b_3 + (70977140699228925 + 7980207634218555w)b_3^2 \\
+(19136714213216460\,39 + 215160756482583681w)b_1b_4 + (16791472437285390 + 1887923847223266w)b_2b_4 + (193830993 + 21793095w)b_4^2 \\
+(-595770851724353 - 66984596059223w)b_1b_5 + (5417342176629 + 609090687075w)b_2b_5 + (172351014 + 19378026w)b_3b_5 \\
+(-387661986 - 43586190w)b_4b_5 + (7087775171 + 796903293w)b_1b_6 + (-154731717 - 17397027w)b_2b_6 + (1971699 + 221685w)b_3b_6 = 0,
\end{gathered}$$

$$\begin{gathered}
(484236458091975404041\,9 + 544443949356573127509w)b_1^2 + (1761522868556347211720 + 198054163707057391280w)b_1b_2 + (70006817133038293201 + 7871110769308346919w)b_2^2 \\
+(180614934839802590428 + 20307167457904344812w)b_1b_3 + (6454740816314215349 + 725729036585611587w)b_2b_3 + (70997776977378135 + 7982527842990081w)b_3^2 \\
+(1914095946510647319 + 215208487331147193w)b_1b_4 + (16799051780893503 + 1888776019276521w)b_2b_4 + (-361974 - 40698w)b_3b_4 + (-595903016195803 - 66999455771981w)b_1b_5 \\
+(5417344152010 + 609090909174w)b_2b_5 + (193830993 + 21793095w)b_3b_5 + (-387661986 - 43586190w)b_4b_5 \\
+(7089347478 + 797080078w)b_1b_6 + (-154751774 - 17399282w)b_2b_6 + (1971699 + 221685w)b_3b_6 = 0
\end{gathered}$$

How did I compute it?



Jacobian of a genus two curve with good ordinary reduction at 2



Its associated Kummer surface (which is a K3)

Sketch of the steps of the proof

1. Find abelian surfaces with everywhere good reduction

3. Construct an explicit scheme model for the example

2. Check the reduction of its associated Kummer surface

1. Find abelian surfaces with everywhere good reduction over a number field

Examples over a quadratic field have been computed by Dembélé and Kumar (2016) and Dembélé (2020).

$g(x)$	$f(x)$	w
wx^3+wx^2+w+1	$-4x^6+(w-17)x^5+(12w-27)x^4+(5w-122)x^3$ $+(45w-25)x^2+(-9w-137)x+14w+9$	$\frac{1+\sqrt{53}}{2}$
x^3+x+1	$(w-5)x^6+(3w-14)x^5+(3w-19)x^4$ $+(4w-3)x^3-(3w+16)x^2+(3w+11)x-(w+4)$	$\frac{1+\sqrt{73}}{2}$
$w\left(x^3+1\right)$	$-2(4414w+43089)x^6+(31147w+303963)x^5$ $-10(4522w+44133)x^4+2(17290w+168687)x^3$ $-18(816w+7967)x^2+27(122w+1189)x-(304w+3003)$	$\frac{1+\sqrt{421}}{2}$
x^3+x^2+1	$-2x^6+(-3w+1)x^5-219x^4+(-83w+41)x^3-1806x^2$ $+(-204w+102)x-977$	$\frac{1+\sqrt{409}}{2}$
x^3+x+1	$-134x^6-(146w-73)x^5-13427x^4-(3255w-1627)x^3$ $-89746x^2-(6523w-3261)x-39941$	$\frac{1+\sqrt{809}}{2}$
x^3+x+1	$23x^6+(90w-45)x^5+33601x^4+(28707w-14354)x^3$ $+3192149x^2+(811953w-405977)x+19904990$	$\frac{1+\sqrt{929}}{2}$

2. Check the reduction of its associated Kummer surface

A good property of Kummer surfaces is that good reduction is preserved at all primes that do not lie above 2.

At the primes lying above 2, a result by Lazda and Skorobogatov ensures that if the abelian surface has good ordinary reduction, then the Kummer surface has <u>potential good reduction.</u>

2. Check the reduction of its associated Kummer surface at 2

Theorem (Lazda, Skorobogatov)

Let $A = \mathrm{Jac}(\mathcal{C})$ be an abelian surface with good ordinary reduction at 2, let K be a discretely valued field with perfect residue field k of characteristic 2, and let $\mathcal{A}/\mathcal{O}_K$ be the Néron model of A/K, which is an abelian scheme with generic fiber $\mathcal{A}_K \cong A$. Let us fix an algebraic closure $\bar{K}$ of K, with residue field $\bar{k}$, and let Γ_K denote the Galois group of $\bar{K}/K$. Then, we have the exact sequence of Γ_K-modules:

$$0 \longrightarrow \mathcal{A}[2]^{\circ}(\bar{K}) \longrightarrow \mathcal{A}[2](\bar{K}) \longrightarrow \mathcal{A}[2](\bar{k}) \longrightarrow 0$$

where $\mathcal{A}[2]^{\circ}$ is the connected component of the identity of the 2-torsion subscheme $\mathcal{A}[2] \subseteq \mathcal{A}$.

Then, the Kummer surface associated to A has good reduction over K if and only if the previous exact sequence of Γ_K-modules split.

3. Construct an explicit smooth scheme model for the example

Characteristic zero

$\mathrm{Kum}(\mathcal{C}) = X_4 \subset \mathbb{P}^3$

Model with 16 A_1 singularities

Blow-up of singular locus

$X_{2,2,2} \subset \mathbb{P}^5$

Smooth model

Blow-up of 4 lines

Smooth model with four (-1) curves

mod 2

Characteristic 2 ordinary case

$\mathrm{Kum}(\mathcal{C}) = X_4 \subset \mathbb{P}^3$

Model with 4 D_4^1 singularities

Blow-up of singular locus

$X_{2,2,2} \subset \mathbb{P}^5$

Model with 12 A_1 singularities

Blow-up of 4 lines

Smooth model

Thank you!