51) Vamos a ir desglosando las transiciones de estados que se producen en el circuito

Estado inicial: Dado que los 2 primeros qubits son $|0\rangle$. El estado inicial se define como:

$$|0\rangle \otimes |0\rangle \otimes |\psi\rangle$$

Aplicamos H:

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$H|0\rangle \otimes H|0\rangle \otimes |\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |\psi\rangle$$

$$= \frac{1}{2}(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \otimes |\psi\rangle =$$

$$= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \otimes |\psi\rangle$$

Aplicamos CCNot (Toffoli)

$$\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \otimes |\psi\rangle =$$

$$= \frac{1}{2}(|00\rangle \otimes |\psi\rangle + |01\rangle \otimes |\psi\rangle + |10\rangle \otimes |\psi\rangle + |11\rangle \otimes |\psi\rangle)$$

→ Aplicación CNot

$$\frac{1}{2}(|00\rangle \otimes |\psi\rangle + |01\rangle \otimes |\psi\rangle + |10\rangle \otimes |\psi\rangle + |11\rangle \otimes T|\psi\rangle)$$

Aplicamos S

$$\frac{1}{2}((|00\rangle + |01\rangle + |10\rangle) \otimes S|\psi\rangle + |11\rangle \otimes S(T|\psi\rangle)) -$$

Aplicamos CCNot (toffoli)

$$\frac{1}{2}[(|00\rangle + |01\rangle + |10\rangle) \otimes S|\psi\rangle + |11\rangle \otimes T(S(T|\psi\rangle))]$$

Aplicamos Hadamard

$$\frac{1}{2}\left[(|00\rangle + |01\rangle + |10\rangle) \otimes S|\psi\rangle + |11\rangle \otimes T(S(T|\psi\rangle))\right]$$

$\downarrow$ Hadamard

$$\frac{1}{2}\left[(H\otimes H|00\rangle + H\otimes H|01\rangle + H\otimes H|10\rangle) \otimes S|\psi\rangle + H\otimes H|11\rangle \otimes T(S(T|\psi\rangle))\right]$$

$$H\otimes H|00\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

$$H\otimes H|01\rangle = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle)$$

$$H\otimes H|10\rangle = \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle - |11\rangle)$$

$$\frac{1}{2}(3|00\rangle + |01 + |10\rangle - |11\rangle)$$

$$H\otimes H|11\rangle = \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle)$$

Agrupamos los resultados

$$\frac{1}{2}\left[\frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle) \otimes S|\psi\rangle + \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle) \otimes T(S(T|\psi\rangle))\right] =$$

$$= \frac{1}{4}\left[(|00\rangle - |01\rangle - |10\rangle + |11\rangle) \otimes S|\psi\rangle + (|00\rangle - |01 - |10\rangle + |11\rangle) \otimes T(S(T|\psi\rangle))\right]$$

Si al medir, los 2 primeros qubits valen 0 el estado es:

$$\frac{1}{4}(S|\psi\rangle \otimes T(S(T|\psi\rangle))$$

$q_0=0$ $\quad q_1=0$ Debemos demostrar que

$$\frac{1}{4}\left(S|\psi\rangle \otimes TST|\psi\rangle\right) = R_z(\phi)|\psi\rangle \quad , \text{con } \cos\phi = 3/5$$

Para el resto de los casos hay que demostrar que

$$\frac{1}{4}\left[(-|01\rangle - |10\rangle + |11\rangle) \otimes S|\psi\rangle + (-|01\rangle - |10\rangle + |11\rangle) \otimes TST|\psi\rangle\right] =$$

$$= Z|\psi\rangle$$

$$\frac{1}{4}\left[(-|01\rangle - |10\rangle + |11) \otimes (S|\psi\rangle + TST|\psi\rangle)\right] = Z|\psi\rangle$$

$$\frac{1}{4}\left[(-|01\rangle - |10\rangle + |11\rangle) \otimes ((S+TST)|\psi\rangle)\right] = Z|\psi\rangle$$