Eco529: Modern Macro, Money, and International Finance

Lecture 11: One Sector Monetary Model with Time-Varying Idiosyncratic Risk and $\beta < 0$ Safe Asset

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Course Overview

Real Macro-Finance Models with Heterogeneous Agents

1. A Simple Real Macro-finance Model
2. Endogenous (Price of) Risk Dynamics
3. A Model with Jumps due to Sudden Stops/Runs

Money Models

1. A Simple Money Model
2. Cashless vs. Cash Economy and "The I Theory of Money"
3. Welfare Analysis & Optimal Policy
   - Fiscal, Monetary, and Macroprudential Policy

International Macro-Finance Models

1. International Financial Architecture

Digital Money

Overview Across Lectures 10-13

- Store of Value Monetary Model with One Sector and No Aggregate Risk
  - Safe Asset and Service Flows
  - Bubble (mining) or not
  - 2 Different Asset Pricing Perspectives/SDFs
- Store of Value Monetary Model with Time-varying Idiosyncratic Risk
  - Safe asset, Flight-to-Safety and negative CAPM-β
  - Flight-to-Safety and Equity Excess Volatility
  - Debt valuation puzzle, Debt Laffer Curve,
  - Safe Asset and Bubble Complementarity
  - Policies to Maintain Safe Asset Privilege on Gov. Bond
- Medium of Exchange Role and Different "Monetary Theories"

Overview

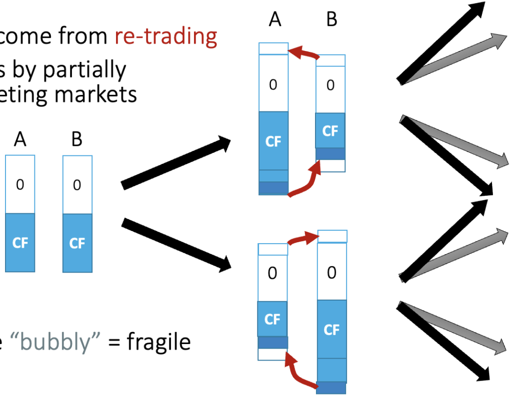
- Aggregate Risk in Form of Time-varying Idiosyncratic Risk, N_t-Numeraire Analysis
- Stationary Monetary Equilibrium with Bubble on Bonds
  - Safe asset, Flight-to-Safety and Negative CAPM-β
  - Exorbitant Privilege, Laffer Curve
- Non-uniqueness of Equilibrium
  - Safe Asset $\neq$ Bubble but Complementarity
  - Loss of Safe Asset Status
    - Bubble Bursts or Jumps to Other Asset, Which? (Ponzi-Right-Assignment)
- How to Ensure Uniqueness
  - Elimination Non-stationary Equilibria
  - Elimination of Bubbles on Other Assets
- Modern Debt Sustainability Analysis
  - Debt Valuation Puzzles
  - Off-equilibrium Fiscal Capacity

Safety $\neq$ risk free
$\neq$ liquidity
$\neq$ bubble

What's a Safe Asset? What is its Service Flow?

$$\frac{\mathcal{B}_t}{\mathcal{P}_t} = \mathbb{E}_t[PV_{\xi**}(\text{cash flow})] + \mathbb{E}_t[PV_{\xi**}(\text{service flow})]$$

- Value come from re-trading
- Insures by partially completing markets



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In recessions:

Risk is higher

- Service flow is more valuable
- Cash flows are lower
  (depends on fiscal policy)

...

...

- Can be "bubbly" = fragile

What's a Safe Asset?

- In incomplete markets setting (Bewley, Aiyagari, BruSan, ...)
- Good friend analogy (Brunnermeier Haddad, 2012)
  - When one needs funds, one can sell at stable price ... since others buy
    - **Idiosyncratic shock**: *Partial insurance through* **retrading**, *low bid-ask spread*
    - **Aggregate (volatility) shock**: *Appreciate in value in times of crises*

- **Safe asset definition**
  - Tradeable: no asymmetric info – info insensitive
  - $\beta < 0$ relative to individual net worth:

$$Cov_t\left[\mathrm{d}\xi_t^i/\xi_t^i, \mathrm{d}r_t^{safe} - \mathrm{d}r_t^{n^i}\right] > 0 \Leftrightarrow \beta_t^i = -\frac{Cov_t\left[\mathrm{d}\xi_t^i/\xi_t^i, \mathrm{d}r_t^{safe} - \mathrm{d}r_t^{n^i}\right]}{Var_t\left[\mathrm{d}\xi_t^i/\xi_t^i\right]} < 0,$$

where ξ_t^i is SDF of agent i.

Note: $-Cov_t[\mathrm{d}\xi_t^i/\xi_t^i, \mathrm{d}r_t] = \varsigma_t^i\sigma_t^r + \tilde{\varsigma}_t^i\tilde{\sigma}_t^{r,i}$, where $\mathrm{d}\xi_t^i/\xi_t^i = -r_t^f\mathrm{d}t - \varsigma_t^i\mathrm{d}Z_t - \tilde{\varsigma}_t^i\mathrm{d}\tilde{Z}_t^{\tilde{i}}$

Model with Capital + Safe Asset

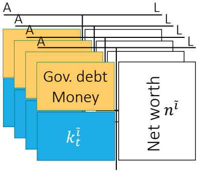
- Each heterogenous citizen $\tilde{i} \in [0,1]$:

$$\mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \left(\frac{(c_t^{\tilde{i}})^{1-\gamma}}{1-\gamma} + f(\mathcal{G}_t K_t) \right) \mathrm{d}t \right] \text{ where } K_t := \int k_t^{\tilde{i}} \mathrm{d}\tilde{i}, \text{ and } \sigma^K = 0$$

$$s.t. \frac{\mathrm{d}n_t^{\tilde{i}}}{n_t^{\tilde{i}}} = -\frac{c_t^{\tilde{i}}}{n_t^{\tilde{i}}} \mathrm{d}t + \mathrm{d}r_t^{\mathcal{B}} + (1-\theta_t^{\tilde{i}})(\mathrm{d}r_t^{K,\tilde{i}}(\iota_t^{\tilde{i}}) - \mathrm{d}r_t^{\mathcal{B}}) \text{ \& No Ponzi}$$

- Each citizen operates physical capital $k_t^{\tilde{i}}$
  - Output (net investment): $y_t^{\tilde{i}} \mathrm{d}t = (a_t k_t^{\tilde{i}} - \iota_t^{\tilde{i}} k_t^{\tilde{i}}) \mathrm{d}t$
  - $\frac{\mathrm{d}k_t^{\tilde{i}}}{k_t^{\tilde{i}}} = \left(\Phi(\iota_t^{\tilde{i}}) - \delta \right) \mathrm{d}t + \tilde{\sigma}_t \mathrm{d}\tilde{Z}_t^{\tilde{i}} + \mathrm{d}\Delta_t^{k,\tilde{i}},$

    ($\mathrm{d}\tilde{Z}_t^{\tilde{i}}$ idiosyncratic Brownian)
  - Output tax $\tau a_t k_t^{\tilde{i}} \mathrm{d}t$



- Financial Friction: Incomplete markets: no $\mathrm{d}\tilde{Z}_t^{\tilde{i}}$ claims
- Aggregate risk $\tilde{\sigma}_t, a_t, \mathcal{G}_t$ exogenous process by aggregate Brownian $\mathrm{d}Z_t$
  - E.g. Heston model: $\mathrm{d}\tilde{\sigma}_t^2 = -\psi(\tilde{\sigma}_t^2 - (\tilde{\sigma}^0)^2) - \sigma\tilde{\sigma}_t \mathrm{d}Z_t$ CIR-ensures $\tilde{\sigma}_t$ stays positive
  - $a_t = a(\tilde{\sigma}_t), \mathcal{G}_t = \mathcal{G}(\tilde{\sigma}_t)$
- Money/bond issuing policy: $\mathrm{d}\mathcal{B}_t/\mathcal{B}_t = \mu_t^{\mathcal{B}} \mathrm{d}t + \sigma_t^{\mathcal{B}} \mathrm{d}Z_t$

Government: Taxes, Bond/Money Supply, Gov. Budget

- $d\mathcal{B}_t/\mathcal{B}_t = \mu_t^{\mathcal{B}} dt + \sigma_t^{\mathcal{B}} dZ_t$, initial debt $\mathcal{B}_0$ (state variable)
- $\sigma_t^{\mathcal{B}} \neq 0$ leads to stochastic (state contingent) "seigniorage revenue"
- Relabel tax revenue process to: $\frac{d\tau_t}{\tau_t} = \mu_t^{\tau} dt + \sigma_t^{\tau} dZ_t$
  - we can also label s_t (primary surplus) as a process
- Government budget constraint (BC)

$$d\mathcal{B}_t - i_t \mathcal{B}_t + \mathcal{P}_t K_t (d\tau_t a - \mathcal{G} dt) = 0$$

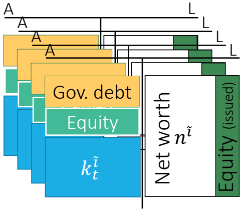
- Return on Gov. Bond/Money: in output/consumption numeraire

$$dr_t^{\mathcal{B}} = i_t dt + \underbrace{\frac{d(1/\mathcal{P}_t)}{1/\mathcal{P}_t}}_{-inflation} = i_t dt + \underbrace{\frac{d(q_t^{\mathcal{B}} K_t/\mathcal{B}_t)}{q_t^{\mathcal{B}} K_t/\mathcal{B}_t}}_{-inflation}$$

$$= \frac{d(q_t^{\mathcal{B}} K_t)}{q_t^{\mathcal{B}} K_t} - \check{\mu}_t^{\mathcal{B}} dt - \sigma_t^{\mathcal{B}} dZ_t + \sigma_t^{\mathcal{B}} (\sigma_t^{\mathcal{B}} - \cancel{\sigma^K} - \sigma_t^{q,\mathcal{B}}) dt$$

Introduce Outside Equity and Mutual Funds

- Equity Market
  - Each citizen $\tilde{i}$ can sell off a fraction $(1-\bar{\chi})$ of capital risk to outside equity holders
  - Return $\mathrm{d}r_t^{E,\tilde{i}}$
    - Same risk as $\mathrm{d}r_t^{K,\tilde{i}}$
    - But $\mathbb{E}[\mathrm{d}r_t^{E,\tilde{i}}] < \mathbb{E}[\mathrm{d}r_t^{K,\tilde{i}}]$... due to insider premium
- Prop.: Model equations as before but replace $\tilde{\sigma}$ with $\bar{\chi}\tilde{\sigma}$



Equilibrium (before solving for portfolio choice)

Equilibrium:

$$\boxed{\begin{aligned} q_t^{\mathcal{B}} &= \vartheta_t \frac{1+\phi\check{a}}{(1-\vartheta_t)+\phi\check{\rho}_t} \\ q_t^K &= (1-\vartheta_t)\frac{1+\phi\check{a}}{(1-\vartheta_t)+\phi\check{\rho}_t} \\ \iota_t &= \frac{(1-\vartheta_t)\check{a}-\check{\rho}_t}{(1-\vartheta_t)+\phi\check{\rho}_t} \end{aligned}}$$

$\check{a} = a - \mathscr{G}$

For log utility

$\check{\rho}_t = \rho$

- Moneyless equilibrium with $q_t^{\mathcal{B}} = 0 \Rightarrow \vartheta_t = 0$
- Next, determine portfolio choice.

1. Portfolio choice θ: Bond Evaluation/FTPL Equation

- Recall martingale method
  - Excess expected return of risky asset A to risky asset B:

$$\boxed{\mu_t^A - \mu_t^B = \varsigma_t^i(\sigma_t^A - \sigma_t^B) + \tilde{\varsigma}_t^i(\tilde{\sigma}_t^A - \tilde{\sigma}_t^B)}$$

- 4 alternative derivations:
  - In consumption numeraire

    i. Expected excess return of capital w.r.t. bond return

    Note: With $\sigma_t^{\mathcal{B}} \neq 0$ seigniorage is stochastic. As it lowers capital taxes it complicates capital return to:

$$\mathrm{d}r_t^{K,\tilde{i}} = \left(\frac{a_t - \mu_t^{\mathcal{G}} - \iota_t^{\tilde{i}}}{q_t^K} + \Phi(\iota_t^{\tilde{i}}) - \delta + \mu_t^{q^K} + \frac{q_t^{\mathcal{B}}}{q_t^K}(\check{\mu}_t^{\mathcal{B}} + (\sigma + \sigma_t^{q^{\mathcal{B}}} - \sigma_t^{\mathcal{B}})\sigma_t^{\mathcal{B}})\right)\mathrm{d}t + \left(\sigma + \sigma_t^{q^K} + \frac{q_t^{\mathcal{B}}\sigma_t^{\mathcal{B}} - \sigma_t^{\mathcal{G}}}{q_t^K}\right)\mathrm{d}Z_t + \tilde{\sigma}_t \mathrm{d}\tilde{Z}_t^{\tilde{i}}$$

    ii. Expected excess return of net worth (portfolio) w.r.t. bond return

  - In total net worth numeraire

    iii. Expected excess return of capital w.r.t. bond return

    iv. Expected excess return of individual net worth (=net worth share) w.r.t. bond return (per bond)

    Note: even with $\sigma_t^{\mathcal{B}} \neq 0$ equation stay tractable

1. Portfolio choice θ (N_t-numeraire, N_t to single bond/coin)

- Asset pricing equation (martingale method)

$$\frac{\mathbb{E}[\mathrm{d}r_t^{\tilde{\eta}^{\tilde{i}}}]}{\mathrm{d}t} = \check{\rho}_t = \left(r_t^f - (\Phi(\iota_t) - \delta) - \mu_t^{q^K+q^{\mathcal{B}}} - \sigma\sigma_t^{q^K+q^{\mathcal{B}}} + \varsigma_t\sigma_t^{q^K+q^{\mathcal{B}}} \right) + (\varsigma_t - \sigma_t^N)0 + \tilde{\varsigma}_t(1-\theta_t)\bar{\chi}\tilde{\sigma}$$

$$\frac{\mathbb{E}[\mathrm{d}r_t^{\vartheta/\mathcal{B}}]}{\mathrm{d}t} = \mu_t^{\vartheta/\mathcal{B}} = \underbrace{\left(r_t^f - (\Phi(\iota_t) - \delta) - \mu_t^{q^K+q^{\mathcal{B}}} - \sigma\sigma_t^{q^K+q^{\mathcal{B}}} + \varsigma_t\sigma_t^{q^K+q^{\mathcal{B}}} \right)}_{\text{risk-free rate in } N_t\text{-numeraire}} + \underbrace{(\varsigma_t - \sigma_t^N)\sigma_t^{\vartheta/\mathcal{B}}}_{\textit{price of risk in } N_t \textit{ numeraire}}$$

$$\check{\rho}_t - \mu_t^{\vartheta/\mathcal{B}} = -(\varsigma_t - \sigma_t^N)\sigma_t^{\vartheta/\mathcal{B}} + \tilde{\varsigma}_t(1-\theta_t)\tilde{\sigma}\bar{\chi}$$

1. Portfolio choice θ (N_t-numeraire, N_t to single bond/coin)

- Asset pricing equation (martingale method)

$$\frac{\mathbb{E}[\mathrm{d}r_t^{\tilde{\eta}^{\tilde{i}}}]}{\mathrm{d}t} = \check{\rho}_t = \left(r_t^f - (\Phi(\iota_t) - \delta) - \mu_t^{q^K+q^{\mathcal{B}}} - \sigma\sigma_t^{q^K+q^{\mathcal{B}}} + \varsigma_t\sigma_t^{q^K+q^{\mathcal{B}}}\right) + (\varsigma_t - \sigma_t^N)0 + \tilde{\varsigma}_t(1-\theta_t)\bar{\chi}\tilde{\sigma}$$

$$\frac{\mathbb{E}[\mathrm{d}r_t^{\vartheta/\mathcal{B}}]}{\mathrm{d}t} = \mu_t^{\vartheta/\mathcal{B}} = \underbrace{\left(r_t^f - (\Phi(\iota_t) - \delta) - \mu_t^{q^K+q^{\mathcal{B}}} - \sigma\sigma_t^{q^K+q^{\mathcal{B}}} + \varsigma_t\sigma_t^{q^K+q^{\mathcal{B}}}\right)}_{\text{risk-free rate in } N_t\text{-numeraire}} + \underbrace{(\varsigma_t - \sigma_t^N)\sigma_t^{\vartheta/\mathcal{B}}}_{\textit{price of risk in } N_t \textit{ numeraire}}$$

$$\check{\rho}_t - \mu_t^{\vartheta/\mathcal{B}} = -(\varsigma_t - \sigma_t^N)\sigma_t^{\vartheta/\mathcal{B}} + \tilde{\varsigma}_t(1-\theta_t)\tilde{\sigma}\bar{\chi}$$

- Remark:
  - Value of a single bond/coin in N_t -numeraire

$$\frac{\mathrm{d}(\vartheta_t/\mathcal{B}_t)}{\vartheta_t/\mathcal{B}_t} = \mu_t^{\vartheta}\mathrm{d}t + \sigma_t^{\vartheta}\mathrm{d}Z_t - \mu_t^{\mathcal{B}}\mathrm{d}t - \sigma_t^{\mathcal{B}}\mathrm{d}Z_t + \sigma_t^{\mathcal{B}}(\sigma_t^{\mathcal{B}} - \sigma_t^{\vartheta})\mathrm{d}t$$

$$= \mu_t^{\vartheta/\mathcal{B}}\mathrm{d}t + \sigma_t^{\vartheta/\mathcal{B}}\mathrm{d}Z_t \text{ (defining return-drift and volatility)}$$

  - Terms are shifted into risk-free rate in N_t-numeraire, which drop out when differencing

1. Portfolio choice θ (N_t-numeraire, N_t to single bond/coin)

- Asset pricing equation (martingale method)

$$\frac{\mathbb{E}[\mathrm{d}r_t^{\tilde{\eta}^{\tilde{i}}}]}{\mathrm{d}t} = \check{\rho}_t = \left(r_t^f - (\Phi(\iota_t) - \delta) - \mu_t^{q^K+q^{\mathcal{B}}} - \sigma\sigma_t^{q^K+q^{\mathcal{B}}} + \varsigma_t\sigma_t^{q^K+q^{\mathcal{B}}}\right) + (\varsigma_t - \sigma_t^N)0 + \tilde{\varsigma}_t(1-\theta_t)\bar{\chi}\tilde{\sigma}$$

$$\frac{\mathbb{E}[\mathrm{d}r_t^{\vartheta/\mathcal{B}}]}{\mathrm{d}t} = \mu_t^{\vartheta/\mathcal{B}} = \underbrace{\left(r_t^f - (\Phi(\iota_t) - \delta) - \mu_t^{q^K+q^{\mathcal{B}}} - \sigma\sigma_t^{q^K+q^{\mathcal{B}}} + \varsigma_t\sigma_t^{q^K+q^{\mathcal{B}}}\right)}_{\text{risk-free rate in } N_t\text{-numeraire}} + \underbrace{(\varsigma_t - \sigma_t^N)\sigma_t^{\vartheta/\mathcal{B}}}_{\textit{price of risk in } N_t \textit{ numeraire}}$$

$$\check{\rho}_t - \mu_t^{\vartheta/\mathcal{B}} = -(\varsigma_t - \sigma_t^N)\sigma_t^{\vartheta/\mathcal{B}} + \tilde{\varsigma}_t(1-\theta_t)\tilde{\sigma}\bar{\chi}$$

  - Price of risk: ς_t = Step 3

  - Capital market clearing: $1 - \theta = 1 - \vartheta$

Recall:

$\mu_t^{\vartheta/\mathcal{B}} = \mu_t^{\vartheta} - \mu_t^{\mathcal{B}} + \sigma_t^{\mathcal{B}}(\sigma_t^{\mathcal{B}} - \sigma_t^{\vartheta})$

$\sigma_t^{\vartheta/\mathcal{B}} = \sigma_t^{\vartheta} - \sigma_t^{\mathcal{B}}$

3. Deriving price of idiosyncratic risk $\tilde{\varsigma}^{\tilde{i}}$ and C/N-ratio $\check{\rho}$

- Recall CRRA-value function:

$$V_t^{\tilde{i}} = \frac{1}{\rho}\frac{(\omega_t^i \tilde{n}_t^{\tilde{i}})^{1-\gamma}}{1-\gamma} = \underbrace{\frac{1}{\rho}\left(\omega_t^i n_t^i / K_t\right)^{1-\gamma}}_{v_t^i :=} \underbrace{\left(\frac{\tilde{n}_t^{\tilde{i}}}{n_t^i}\right)^{1-\gamma}}_{(\tilde{\eta}_t^{\tilde{i}})^{1-\gamma}} \frac{K_t^{(1-\gamma)}}{1-\gamma}$$

- Recall value function envelop condition

$$\frac{\partial V_t^{\tilde{i}}}{\partial \tilde{n}_t^{\tilde{i}}} = \overbrace{\frac{1}{\rho}(\omega_t^i)^{1-\gamma}}^{=v_t^i\left(K_t/n_t^i\right)^{1-\gamma}} \overbrace{(\tilde{n}_t^{\tilde{i}})^{-\gamma}}^{=(\tilde{\eta}_t^{\tilde{i}})^{-\gamma}(n_t^i)^{-\gamma}} = (c_t^{\tilde{i}})^{-\gamma} = \frac{\partial u}{\partial c_t^{\tilde{i}}}$$

$$= v_t^i K_t^{1-\gamma}(n_t^i)^{-1}(\tilde{\eta}_t^{\tilde{i}})^{-\gamma}$$

$$= v_t^i K_t^{-\gamma}(q_t^{\mathcal{B}} + q_t^K)(\tilde{\eta}_t^{\tilde{i}})^{-\gamma} \qquad \text{(after noting that } n_t^i = N_t = (q_t^B + q_t^K)K_t\text{)}$$

- for aggregate price of risk (recall: $\sigma_t^K = 0$, no aggregate risk for K)
  - $\sigma_t^v - \sigma_t^{q^B+q^K} - \cancel{\gamma\sigma} = -\gamma\sigma_t^{c^i} = -\varsigma_t$
- for idiosyncratic price of risk (recall: $\sigma_t^{\tilde{\eta}^{\tilde{i}}} = \sigma_t^{\tilde{n}^{\tilde{i}}}$)
  - $\tilde{\varsigma}_t^{\tilde{i}} = \gamma\tilde{\sigma}_t^{n^{\tilde{i}}} = \gamma(1-\vartheta)\tilde{\sigma}_t$
- For log utility $\gamma = 1$: $\varsigma_t = \sigma_t^{n^i}, \tilde{\varsigma}_t^{\tilde{i}} = \tilde{\sigma}_t^{n^{\tilde{i}}}$, $\check{\rho} = \rho$

1. Portfolio choice θ (N_t-numeraire, N_t to single bond/coin)

- Asset pricing equation (martingale method)

$$\frac{\mathbb{E}[\mathrm{d}r_t^{\tilde{\eta}^{\tilde{i}}}]}{\mathrm{d}t} = \check{\rho}_t = \left(r_t^f - (\Phi(\iota_t) - \delta) - \mu_t^{q^K+q^{\mathcal{B}}} - \sigma\sigma_t^{q^K+q^{\mathcal{B}}} + \varsigma_t \sigma_t^{q^K+q^{\mathcal{B}}}\right) + (\varsigma_t - \sigma_t^N)0 + \tilde{\varsigma}_t(1-\theta_t)\bar{\chi}\tilde{\sigma}$$

$$\frac{\mathbb{E}[\mathrm{d}r_t^{\vartheta/\mathcal{B}}]}{\mathrm{d}t} = \mu_t^{\vartheta/\mathcal{B}} = \underbrace{\left(r_t^f - (\Phi(\iota_t) - \delta) - \mu_t^{q^K+q^{\mathcal{B}}} - \sigma\sigma_t^{q^K+q^{\mathcal{B}}} + \varsigma_t \sigma_t^{q^K+q^{\mathcal{B}}}\right)}_{\text{risk-free rate in } N_t\text{-numeraire}} + \underbrace{(\varsigma_t - \sigma_t^N)\sigma_t^{\vartheta/\mathcal{B}}}_{\textit{price of risk in } N_t \textit{ numeraire}}$$

$$\check{\rho}_t - \mu_t^{\vartheta/\mathcal{B}} = -(\varsigma_t - \sigma_t^N)\sigma_t^{\vartheta/\mathcal{B}} + \tilde{\varsigma}_t(1-\theta_t)\tilde{\sigma}\bar{\chi}$$

- Price of risk: $\varsigma_t = -\sigma_t^v + \sigma_t^{q^B+q^K} + \gamma\sigma$, $\tilde{\varsigma}_t = \gamma\tilde{\sigma}_t^n = \gamma(1-\theta_t)\bar{\chi}\tilde{\sigma}$

$$\check{\rho}_t - \mu_t^{\frac{\vartheta}{\mathcal{B}}} = (\sigma_t^v - (\gamma-1)\sigma)\sigma_t^{\frac{\vartheta}{\mathcal{B}}} + \gamma(1-\theta_t)^2\bar{\chi}^2\tilde{\sigma}^2$$

- Capital market clearing: $1-\theta = 1-\vartheta$

Recall:

$\mu_t^{\vartheta/\mathcal{B}} = \mu_t^{\vartheta} - \mu_t^{\mathcal{B}} + \sigma_t^{\mathcal{B}}(\sigma_t^{\mathcal{B}} - \sigma_t^{\vartheta})$

$\sigma_t^{\vartheta/\mathcal{B}} = \sigma_t^{\vartheta} - \sigma_t^{\mathcal{B}}$

3. BSDE for v_t^i

$$\frac{\mathrm{d}V_t^{\tilde{i}}}{V_t^{\tilde{i}}} = \frac{\mathrm{d}\left(v_t^i(\tilde{\eta}_t^{\tilde{i}})^{1-\gamma}K_t^{1-\gamma}\right)}{v_t^i(\tilde{\eta}_t^{\tilde{i}})^{1-\gamma}K_t^{1-\gamma}}$$

- By Itô's product rule:

$$= \left[\mu_t^v + (1-\gamma)(\Phi(\iota_t)-\delta) - \frac{1}{2}\gamma(1-\gamma)\left(\sigma^2 + (\tilde{\sigma}^{n^{\tilde{i}}})^2\right) + (1-\gamma)\sigma\sigma_t^v\right]\mathrm{d}t + \text{volatility terms}$$

- Recall by consumption optimality $\frac{dV_t^{\tilde{i}}}{V_t^{\tilde{i}}} - \rho\mathrm{d}t + \frac{c_t^{\tilde{i}}}{n_t^{\tilde{i}}}$ follows a martingale
    - Hence, drift above $= \rho - \frac{c_t^{\tilde{i}}}{n_t^{\tilde{i}}}$
- Equate drift terms to obtain BSDE:

$$\mu_t^v + (1-\gamma)(\Phi(\iota_t)-\delta) - \frac{1}{2}\gamma(1-\gamma)\left(\sigma^2 + (\tilde{\sigma}^{n^{\tilde{i}}})^2\right) + (1-\gamma)\sigma\sigma_t^v = \rho - \frac{c_t^{\tilde{i}}}{n_t^{\tilde{i}}}$$

4. Numerical Steps for Log utility $\gamma = 1$

- Drift of ϑ from the model = drift of $\vartheta(\tilde{\sigma})$ from Ito's Lemma
  $-\mu_t^{\frac{\vartheta}{\mathcal{B}}} = -\check{\rho}_t + (\sigma_t^v - (\gamma - 1)\sigma)\sigma_t^{\frac{\vartheta}{\mathcal{B}}} + \gamma(1 - \theta_t)^2 \bar{\chi}^2 \tilde{\sigma}^2$
- Recall: $\mu_t^{\vartheta/\mathcal{B}} = \mu_t^{\vartheta} - \mu_t^{\mathcal{B}} + \sigma_t^{\mathcal{B}}(\sigma_t^{\mathcal{B}} - \sigma_t^{\vartheta}), \sigma_t^{\vartheta/\mathcal{B}} = \sigma_t^{\vartheta} - \sigma_t^{\mathcal{B}}$
  - $\mu_t^{\vartheta} = \mu_t^{\mathcal{B}} - \sigma_t^{\mathcal{B}}(\sigma_t^{\mathcal{B}} - \sigma_t^{\vartheta}) + \check{\rho}_t - \cancel{(\sigma_t^v - (\gamma - 1)\sigma)(\sigma_t^{\vartheta} - \sigma_t^{\mathcal{B}})} - \cancel{\gamma}(1 - \theta_t)^2 \bar{\chi}^2 \tilde{\sigma}^2$
  - $\rho\vartheta(\tilde{\sigma}) = (1 - \vartheta(\tilde{\sigma}))^2 \bar{\chi}^2 \tilde{\sigma}^2 \vartheta(\tilde{\sigma}) + \mu_t^{\vartheta}\vartheta(\tilde{\sigma}) - \mu_t^{\mathcal{B}}\vartheta(\tilde{\sigma}) + \sigma_t^{\mathcal{B}}(\sigma_t^{\mathcal{B}} - \sigma_t^{\vartheta})\vartheta(\tilde{\sigma})$
- Drift $\vartheta(\eta)$ from Itô's Lemma:
  $\mu_t^{\vartheta}\vartheta_t = \partial\vartheta(\tilde{\sigma})\mu_t^{\tilde{\sigma}}\tilde{\sigma}_t + \frac{1}{2}\partial_{\tilde{\sigma}\tilde{\sigma}}\vartheta(\tilde{\sigma})\tilde{\sigma}^2$
- Equate drift and add time-derivative

$$\rho\vartheta(\tilde{\sigma}) = [(1 - \vartheta(\tilde{\sigma}))^2 \bar{\chi}^2 \tilde{\sigma}^2 - \check{\mu}^{\mathcal{B}}]\vartheta(\tilde{\sigma}) + b(\tilde{\sigma}^{ss} - \tilde{\sigma})\vartheta'(\tilde{\sigma}) + \frac{\nu^2 \tilde{\sigma}}{2}\vartheta''(\tilde{\sigma})$$

$$\rho\vartheta_t(\tilde{\sigma}) = \partial_t \vartheta_t(\tilde{\sigma}) + \underbrace{[(1 - \vartheta(\tilde{\sigma}))^2 \bar{\chi}^2 \tilde{\sigma}^2 \vartheta(\tilde{\sigma}) - \check{\mu}^{\mathcal{B}}]}_{\boldsymbol{u\vartheta}} + \underbrace{b(\tilde{\sigma}^{ss} - \tilde{\sigma})\vartheta'(\tilde{\sigma}) + \frac{\nu^2 \tilde{\sigma}}{2}\vartheta''(\tilde{\sigma})}_{\mathbf{M}\boldsymbol{\vartheta}}$$

- solve for $\vartheta(\tilde{\sigma})$ with iteration

4. Numerical Steps for CRRA utility

1. Generalize PDE for ϑ: (now with $\gamma \neq 1$ and σ_t^v)
2. Derive PDE for v:
   - $\mu_t^v + (1-\gamma)(\Phi(\iota_t) - \delta) - \frac{1}{2}\gamma(1-\gamma)\left(\sigma^2 + (\tilde{\sigma}^{n^{\tilde{i}}})^2\right) + (1-\gamma)\sigma\sigma_t^v = \rho - \frac{c_t^{\tilde{i}}}{n_t^{\tilde{i}}}$
   - Itô's Lemma for $v(\tilde{\sigma})$:

$$dv(\tilde{\sigma}) = \underbrace{\left(b(\tilde{\sigma}^{ss} - \tilde{\sigma})\partial_{\tilde{\sigma}} v + \frac{1}{2}\nu^2 \tilde{\sigma} \partial_{\tilde{\sigma}\tilde{\sigma}} v\right)}_{= v\mu_t^v} \mathrm{d}t + \underbrace{\nu\sqrt{\tilde{\sigma}}\partial_{\tilde{\sigma}} v}_{= v\sigma_t^v} \mathrm{d}Z_t$$

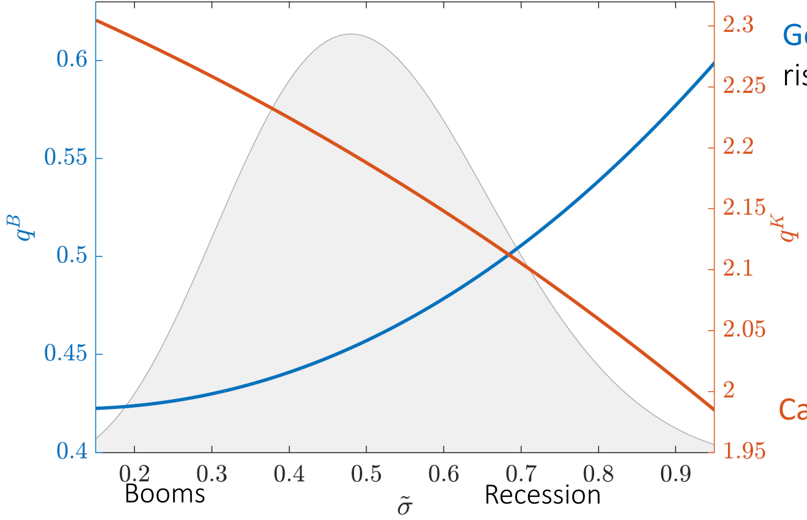
   - PDE for v:

$$\rho v(\tilde{\sigma}) = \partial_t v(\tilde{\sigma}) + \overbrace{\left(\frac{c_t^{\tilde{i}}}{n_t^{\tilde{i}}} + (1-\gamma)(\Phi(\iota_t) - \delta) - \frac{1}{2}\gamma(1-\gamma)\left(\sigma^2 + (\tilde{\sigma}^{n^{\tilde{i}}})^2\right) + (1-\gamma)\sigma\sigma_t^v\right) v}^{= \boldsymbol{u v}}$$
$$+ \underbrace{b(\tilde{\sigma}^{ss} - \tilde{\sigma}) v'(\tilde{\sigma}) + \frac{\nu^2 \tilde{\sigma}}{2} v''(\tilde{\sigma})}_{\mathbf{M}\boldsymbol{v}}$$

- 2 PDEs: Solve both by iterating simultaneously (outer loop)
  - No inner loop since trivial (since κ, χ within sector have no macro-implications)

Bond and Capital Value for time-varying idiosyncratic risk $\tilde{\sigma}$

- Comparative static w.r.t. idiosyncratic risk $\tilde{\sigma}$



Gov. debt value rises in recessions

Capital price

FTPL Equation with Bubble: 2 Perspectives

- Agent $\tilde{i}$'s SDF, $\xi_t^{\tilde{i}}$: $\mathrm{d}\xi_t^{\tilde{i}}/\xi_t^{\tilde{i}} = -r_t^f \mathrm{d}t - \varsigma_t \mathrm{d}Z_t - \tilde{\varsigma}_t^{\tilde{i}} \mathrm{d}\tilde{Z}_t^{\tilde{i}}$
- Buy and Hold Perspective:

$$\frac{\mathcal{B}_0}{\mathcal{P}_0} = \lim_{T\to\infty} \left(\mathbb{E}\left[\int_0^T \xi_t^{\tilde{i}} s_t K_t \mathrm{d}t \right] + \mathbb{E}\left[\xi_T^{\tilde{i}} \frac{\mathcal{B}_T}{\mathcal{P}_T} \right] \right)$$

Bubble is possible: $\lim_{T\to\infty} \mathbb{E}[\bar{\xi}_t \frac{\mathcal{B}_T}{\mathcal{P}_T}] > 0$ if $r_t^f + \varsigma_t \sigma_t^{q,\mathcal{B}} \leqslant g_t$ (on average)

$g - \breve{\mu}^{\mathcal{B}} =$ discount rate

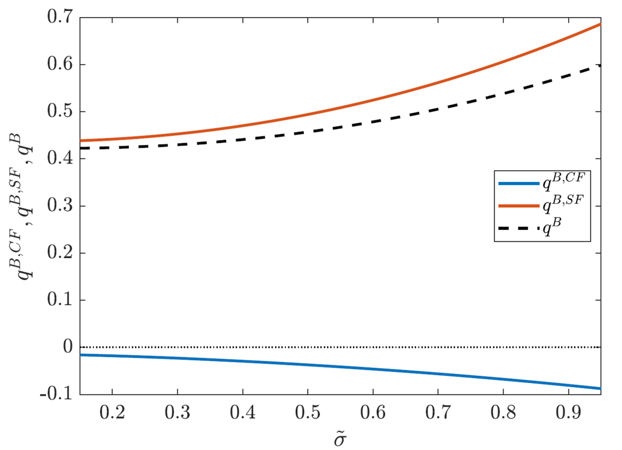
- Dynamic Trading Perspective:
  - Value cash flow from individual bond portfolios, including trading cash flows
  - Integrate over citizens weighted by net worth share $\eta_t^{\tilde{i}}$
  - Bond as part of a dynamic trading strategy

$$\frac{\mathcal{B}_0}{\mathcal{P}_0} = \mathbb{E}\left[\int_0^\infty \underbrace{\left(\int \xi_t^{\tilde{i}} \eta_t^{\tilde{i}} d\tilde{i} \right)}_{\xi_t^{**}} s_t K_t \mathrm{d}t \right] + \mathbb{E}\left[\int_0^\infty \underbrace{\left(\int \xi_t^{\tilde{i}} \eta_t^{\tilde{i}} d\tilde{i} \right)}_{\xi_t^{**}} \gamma(\tilde{\sigma}_t^c)^2 \frac{\mathcal{B}_t}{\mathcal{P}_t} \mathrm{d}t \right]$$

  - Discount rate: $\mathbb{E}[dr_t^\eta]/\mathrm{d}t = r^f + \tilde{\varsigma}\tilde{\sigma}$

Dynamic Trading Perspective Decomposition

$$\frac{\mathcal{B}_0}{\mathcal{P}_0} = \overbrace{\mathbb{E}\left[\int_0^\infty \xi_t^{**} s_t K_t \mathrm{d}t\right]}^{\mathbb{E}\text{PV(cash flow)}} + \overbrace{\mathbb{E}\left[\int_0^\infty \xi_t^{**} \gamma (\tilde{\sigma}_t^c)^2 \frac{\mathcal{B}_t}{\mathcal{P}_t} \mathrm{d}t\right]}^{\mathbb{E}\text{PV(service flow)}}$$



Excess Stock Market Volatility due to Flight to Safety

- "Aggregate Intertemporal Budget Constraint

$$\underbrace{q_t^K K_t + q_t^{\mathcal{B}} K_t}_{\text{total (net) wealth}} = \mathbb{E}_t \left[\int_t^\infty \frac{\int \xi_s^i \eta_s^i di}{\int \xi_t^i \eta_t^i di} C_s \mathrm{d}s \right] \qquad (*)$$

- Lucas-type models: $q^{\mathcal{B}} = 0$ (also $C_t = Y_t$, no idiosyncratic risk)
  - Value of equity (Lucas tree) = PV of consumption claim
  - Volatility equity values require volatile RHS of (*)
- This model: even for constant RHS of (*), $q_t^K K_t$ can be volatile due to flight to safety: $(Cov[q^K, q^{\mathcal{B}}] < 0)$
  - increase in $\tilde{\sigma}_t \Rightarrow$ Portfolio reallocation from capital to bonds, $q_t^K K_t \downarrow, \mathcal{B}_t/\mathcal{P}_t \uparrow$,
- Outside equity is linked to $q_t^K K_t$ and even more volatile due to countercyclical insider equity premium. (see below)
- Quantitatively relevant? Yes
  Excess return volatility
  - 2.9% in equivalent bondless model ($s = 0$ and no bubble)
  - 12.9% in out model

Calibration

- Exogenous processes:
  Recessions feature high idiosyncratic risk and low consumption
  - $\tilde{\sigma}_t$: Heston (1993) model of stochastic volatility:

$$\mathrm{d}\tilde{\sigma}_t^2 = -\psi(\tilde{\sigma}_t^2 - (\tilde{\sigma}^0)^2)\mathrm{d}t - \sigma\tilde{\sigma}_t \mathrm{d}Z_t$$

  - a_t: $a_t = a(\tilde{\sigma}_t)$:

$$a_t(\tilde{\sigma}) = a^0 - \alpha^a(\tilde{\sigma}_t - \tilde{\sigma}^0)$$

  - $\mathscr{G}_t = 0$
- Government (bubble-mining policy)

$$\check{\mu}_t^{\mathcal{B}} = \check{\mu}_t^{\mathcal{B},0} + \alpha^{\mathcal{B}}(\tilde{\sigma} - \tilde{\sigma}^0)$$

- Calibration to US data (1970-2019, period length is one year)

Calibration: Parameters

parameter	description	value	parameter	description	value
$\tilde{\sigma}^0$	$\tilde{\sigma}_t^2$ stoch. steady state	0.54	$\mathfrak{g}$	gov. expenditures	0.138
ψ	$\tilde{\sigma}_t^2$ mean reversion	0.67	$\breve{\mu}^{\mathcal{B},0}$	$\breve{\mu}_t^{\mathcal{B}}$ stoch. steady state	0.0026
σ	$\tilde{\sigma}_t^2$ volatility	0.4	α^a	a_t slope	0.072
$\bar{\chi}$	undiversifiable risk	0.3	$\alpha^{\mathcal{B}}$	$\breve{\mu}_t^{\mathcal{B}}$ slope	0.12
γ	risk aversion	6	ϕ	capital adj. cost	8.1
ρ	time preference	0.138	ι^0	capital adj. intercept	-0.022
a^0	a_t stoch. steady state	0.625	δ	depreciation rate	0.055

Quantitative Model Fit

moment symbol	description	model	data
	(1) targeted moments		
$\sigma(Y)$	output volatility	1.3%	1.3%
$\sigma(C)/\sigma(Y)$	relative consumption volatility	0.61	0.64
$\sigma(I)/\sigma(Y)$	relative investment volatility	3.35	3.38
$\sigma(S/Y)$	surplus volatility	1.1%	1.1%
$\mathbb{E}[C/Y]$	average consumption-output ratio	0.58	0.56
$\mathbb{E}[G/Y]$	average government expenditures-output ratio	0.22	0.22
$\mathbb{E}[S/Y]$	average surplus-output ratio	-0.0005	-0.0005
$\mathbb{E}[I/K]$	average investment rate	0.12	0.12
$\mathbb{E}[q^K K/Y]$	average capital-output ratio	3.48	3.73
$\mathbb{E}[q^B K/Y]$	average debt-output ratio	0.74	0.71
$\mathbb{E}[d\bar{r}^E - dr^B]$	average (unlevered) equity premium	3.59%	3.40%
$\frac{\mathbb{E}[dr^E - dr^B]}{\sigma(dr^E - dr^B)}$	equity sharpe ratio	0.31	0.31
	(2) untargeted moments		
$\rho(Y,C)$	correlation of output and consumption	0.98	0.92
$\rho(Y,I)$	correlation of output and investment	0.99	0.94
$\rho(Y,S/Y)$	correlation of output and surpluses	0.98	0.60
$\sigma(q^B K/Y)$	volatility of debt-output ratio	4.8%	2.0%
$\mathbb{E}[r^f]$	average risk-free rate	5.18%	0.64%
$\sigma(r^f)$	volatility of risk-free rate	5.47%	2.25%

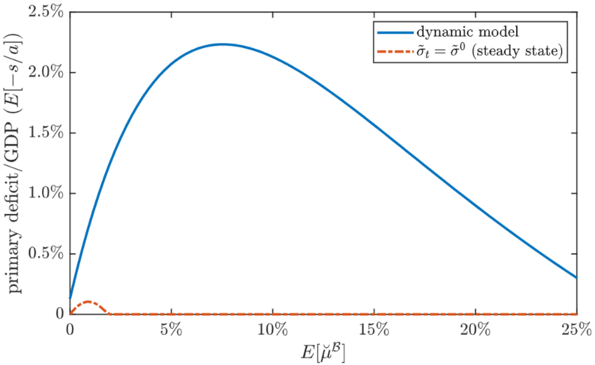
Two Debt Valuation Puzzles

- Properties of US primary surpluses
  - Average surplus ≈ 0
  - Procyclical surplus (> 0 in booms, < 0 in recessions)
- Two valuation puzzles from standard perspective:
  (Jiang, Lustig, van Nieuwerburgh, Xiaolan, 2019, 2020)
  1. "Public Debt Valuation Puzzle"
     - Empirical $\mathbb{E}[PV(\mathit{surpluses})] < 0$, yet $\mathcal{B}/\mathcal{P} > 0$
     - Our model: bubble/service flow component overturns results
  2. "Gov. Debt Risk Premium Puzzle"
     - Debt should be positive β asset, but market don't price it this way
     - Our model: can be rationalized with countercyclical bubble/service flow

Debt Laffer Curve

- Issue bonds at a faster rate $\breve{\mu}^{\mathcal{B}}$ (esp. in recessions)
  - $\Rightarrow$ tax precautionary self insurance $\Rightarrow$ tax rate $\uparrow$
  - $\Rightarrow$ real value of bonds: $\mathcal{B}/\mathcal{P} \downarrow \Rightarrow$ "tax base" $\downarrow$

  - Less so in recession due to flight-to-safety



Sizeable revenue only if Gov. debt has negative β

Roadmap

- Stationary Monetary Equilibrium with Bubble on Bonds
  - Safe asset, Flight-to-Safety and Negative CAPM-β
  - Exorbitant Privilege, Laffer Curve
- Non-uniqueness of Equilibrium
  - Safe Asset $\neq$ Bubble but Complementarity
  - Loss of Safe Asset Status
    - Bubble Bursts or Jumps to Other Asset, Which? (Ponzi-Right-Assignment)
- How to Ensure Uniqueness
  - Elimination Non-stationary Equilibria
  - Elimination of Bubbles on Other Assets
- Modern Debt Sustainability Analysis
  - Debt Valuation Puzzles
  - Off-equilibrium Fiscal Capacity

Safety $\neq$ risk free
$\neq$ liquidity
$\neq$ bubble

Non-uniqueness of Equilibria

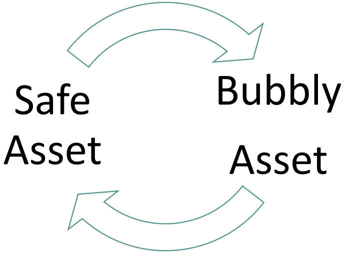
- Among the stationary monetary equilibria with bubble on government bonds analyzed equilibrium is unique
  - See Appendix A.2 of Safe Asset Paper BruMerSan (2023)
- However, there might be
  - Non-bubble equilibrium
  - Bubble can be Associated with/jump to a Different Asset Than Gov. bond
  - Non-stationary Equilibria, in which Bubble Decays over time
- Safe Asset is related to concepts of Bubbles and Liquidity

Recall Safe Asset Definition (time and individual specific)

- Re-tradable
- Good friend: $Cov_t\left[\mathrm{d}\xi_t^i/\xi_t^i, \mathrm{d}r_t^{safe} - \mathrm{d}r_t^{n^i}\right] > 0 \Leftrightarrow \beta_t^i = -\frac{Cov_t\left[\mathrm{d}\xi_t^i/\xi_t^i, \mathrm{d}r_t^{safe} - \mathrm{d}r_t^{n^i}\right]}{Var_t\left[\mathrm{d}\xi_t^i/\xi_t^i\right]} < 0$
  w.r.t. idiosyncratic & aggregate risk (relative to own net worth return $\mathrm{d}r_t^{n^{\tilde{i}}}$)
  - Safe asset return : $\mathrm{d}r_t^{safe} = \mu_t^{safe}\mathrm{d}t + \sigma_t^{safe}\mathrm{d}Z_t + \int \tilde{\sigma}_t^{safe,\tilde{i}}\mathrm{d}\tilde{Z}_t^{\tilde{i}}$
  - Net worth return of agent $\tilde{i} \in i$: $\mathrm{d}r_t^{n^{\tilde{i}}} = \mu_t^{n^{\tilde{i}}}\mathrm{d}t + \sigma_t^{n^{\tilde{i}}}\mathrm{d}Z_t + \tilde{\sigma}_t^{n^{\tilde{i}},\tilde{i}}\mathrm{d}\tilde{Z}_t^{\tilde{i}}$
  - SDF of agent $\tilde{i} \in i$: $\frac{\mathrm{d}\xi_t^{\tilde{i}}}{\xi_t^{\tilde{i}}} = -r_t^f\mathrm{d}t - \varsigma_t^{\tilde{i}}\mathrm{d}Z_t - \tilde{\varsigma}_t^i\mathrm{d}\tilde{Z}_t^{\tilde{i}}$
  - Typical safe asset doesn't load on $\mathrm{d}\tilde{Z}_t^{\tilde{i}}$
  - $Cov_t\left[\mathrm{d}\xi_t^i/\xi_t^i, \mathrm{d}r_t^{safe} - \mathrm{d}r_t^{n^i}\right] = \varsigma_t^i\left(-\sigma_t^{safe} + \sigma_t^{n^{\tilde{i}}}\right) > 0$

Complementarity btw Safe Asset & Bubble

- bubble $\Rightarrow$ easier to satisfy safe asset definition: negative β
  - Bubble raises $q_t^{\mathcal{B}}$ by keeping $q_t^{\mathcal{B},cash\ flow}$ unaffected $\Rightarrow \alpha_t^{cf}$ declines $\Rightarrow \beta$ is lower (partial equilibrium argument)
- safe asset $\Rightarrow$ bubble condition easier: $\lim_{T\to\infty} \mathbb{E}\left[\xi_T \frac{\mathcal{B}_T}{\mathcal{P}_t}\right] > 0 \Leftrightarrow r^{safe} < g$
  - Negative β (or $\sigma^{safe} < 0$) $\Rightarrow r^{safe}$ is lower



Complementarity btw Safe Asset & Bubble

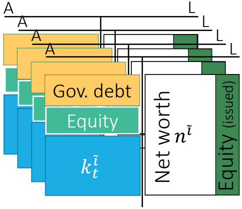
- bubble $\Rightarrow$ easier to satisfy safe asset definition: negative β
  - Bubble raises $q_t^{\mathcal{B}}$ by keeping $q_t^{\mathcal{B},cash\ flow}$ unaffected $\Rightarrow$ α_t^{cf} declines $\Rightarrow$ β is lower (partial equilibrium argument)
- safe asset $\Rightarrow$ bubble condition easier: $\lim_{T\to\infty} \mathbb{E}\left[\xi_T \frac{\mathcal{B}_T}{\mathcal{P}_t}\right] > 0 \Leftrightarrow r^{safe} < g$
  - Negative β (or $\sigma^{safe} < 0$) $\Rightarrow$ r^{safe} is lower
- Split up aggregate risk (covariance with $\mathrm{d}Z_t$)

$$\sigma_t^{safe} = \alpha_t^{cf}\sigma_t^{cash\ flows} + (1-\alpha_t^{cf})\overbrace{\sigma_t^{service\ flows}}^{<0}, \text{ where } \alpha_t^{cf} = \frac{q_t^{\mathcal{B},cash\ flow}}{q_t^{\mathcal{B}}}$$

$$r_t^{safe} = r_t^f + \varsigma_t \underbrace{\sigma^{safe}}_{<0} r^{safe} = r_t^f + \underbrace{\beta_t^{safe,C}}_{=\frac{\sigma_t^{safe}\sigma_t^C}{\sigma_t^C\sigma_t^C}} \underbrace{(r_t^C - r_t^f)}_{=\varsigma_t\sigma_t^C} \ (C = \text{aggr. consumption claim})$$

- Note, for $\sigma_t^{cash\ flows}$ sufficiently high, bubble condition is violated

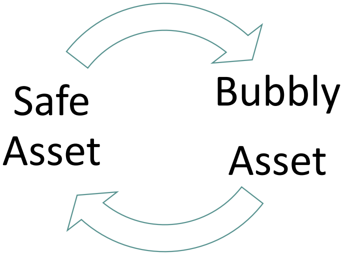
Safe Asset-Bubble on Gov Debt or Equity Mutual Fund



- For Gov. Debt:
  $\sigma_t^{cash\ flows} = \sigma_t^{primary\ surplus}$
  - < 0 if procyclical (austerity) fiscal policy
  - > 0 if countercyclical (stimulus) fiscal policy
- For Equity Mutual Fund:
  $\tilde{\sigma}_t^{cash\ flows}$ diversified stock portfolio is free of idiosyncratic risk
  - can also self-insure idiosyncratic risk
  - $\Rightarrow$ Good friend in idiosyncratically bad times
- $\sigma_t^{cash\ flows} > 0$ poor hedge against aggregate risk, losses value in recessions
  - $\Rightarrow$ Bad friend in aggregate bad times
  - Equity are claims to capital, but marginal capital holder is insider
  - Insider bears idiosyncratic risk, must be compensated
  - $\tilde{\sigma}_t \uparrow \Rightarrow$ insider premium $\mathbb{E}_t[\mathrm{d}r_t^K] - \mathbb{E}_t[\mathrm{d}r_t^E] \uparrow \Rightarrow$ payouts to outside stockholders fall

Aside: Complementarity btw Safety and Liquidity (but $\neq$)

- If high market liquidity (low bid-ask spread) $\Rightarrow$ better safe asset
- Safe asset $\Rightarrow$ high trading volume and better market liquidity



- Simply assumed in our model
  – all assets perfect liquidity (highlights safety $\neq$ liquidity)
- How to maintain safe asset status?
  Central Banks as Market Maker of Last Resort
  - Example: 10 year US Treasury in March 2020

Overview

- Stationary Monetary Equilibrium with Bubble on Bonds
  - Safe asset, Flight-to-Safety and Negative CAPM-β
  - Exorbitant Privilege, Laffer Curve
- Non-uniqueness of Equilibrium
  - Safe Asset $\neq$ Bubble but Complementarity
  - Loss of Safe Asset Status
    - Bubble Bursts or Jumps to Other Asset, Which? (Ponzi-Right-Assignment)
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  - Debt Valuation Puzzles
  - Off-equilibrium Fiscal Capacity

Safety $\neq$ risk free
$\neq$ liquidity
$\neq$ bubble

Policies to Prevent Loss of Safe Asset Status

- Are there policies to prevent a loss of safe asset status?
  - Raise (positive) surpluses to generate safe cash flow component $q_t^{B,CF}$ and possibly also $\sigma_t^{cash\ flows} < 0$
  - If surpluses always exceed a (positive) fraction of total output, no bubble
  - But: gives up revenues from bubble mining

- Off-equilibrium tax backing
  - Sufficient to (credibly) promise policy above off-equilibrium
  - See "FTPL with a Bubble" paper

Uniqueness of Stationary Bubble Equilibria

- Assume bubble is possibly only on gov. debt

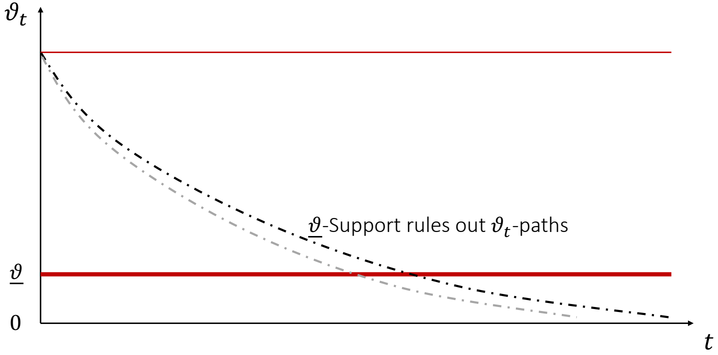
$$\underbrace{\vartheta_t \mu_t^{\vartheta}}_{\dot{\vartheta}_t} = \vartheta_t(\rho - (1-\vartheta_t)^2\tilde{\sigma}^2 + \check{\mu}_t^{\mathcal{B}})$$

1. ϑ^* steady state level
2. $\vartheta = 0$
3. Continuum of $\vartheta_0 \in [0, \vartheta^*]$ equilibria, in which ϑ_t converges to 0.

- Fiscal rule: whenever ϑ_t drops below $\underline{\vartheta}$,
  support $\underline{\vartheta}$ by switching to positive surplus $s > 0$
  - Off-equilibrium backing eliminates No-bubble and decaying bubble equilibria (2. and 3.)
    $\Rightarrow$ equilibrium is stationary (Obstfeld-Rogoff Analogy)
    - Uniqueness among stationary equilibria (see "Safe Asset Paper", Appendix A.2)

Uniqueness of Stationary Bubble Equilibria

- So far, unique equilibrium among stationary bubble equilibria (Safe Asset Paper)
- Rule out declining bubble and no-bubble equilibrium



Uniqueness of Stationary Bubble Equilibria on Gov. Debt

- Bubble can be on other assets, e.g. crypto asset
  Who can issue bubble assets in increasing quantity, run a Ponzi scheme?
  Who owns "Exorbitant privilege" *is an equilibrium selection*
  - $q_t^{\mathcal{C}} K_t :=$ real value of crypto coin
  - $\hat{\vartheta}_t = \vartheta_t^{\mathcal{B}} + \vartheta_t^{\mathcal{C}}$
  - Two ODEs: one for $\hat{\vartheta}_t$ and one for $\vartheta_t^{\mathcal{B}}$.
  - As before $\vartheta_t^{\mathcal{B}} \geqslant \underline{\vartheta}$ all the time, where $\underline{\vartheta}$ is supported by off-equilibrium $s > 0$.
- If $\underline{\vartheta} < \vartheta^*$
  - If $\breve{\mu}_t^{\mathcal{C}} > \breve{\mu}_t^{\mathcal{B}}$ crypto bubble can't exist
    (as crypto dilution rate exceeds gov debt dilution rate, $\vartheta_t^{\mathcal{B}}$ must shrink,
    but sum of bubbles is bounded)
  - If $\breve{\mu}_t^{\mathcal{C}} \leqslant \breve{\mu}_t^{\mathcal{B}}$ gov. debt and cryptocoin bubble can co-exist
    - But government can
      - Impose solvency law, taxes ...

Other Policy Tools to Keep Bubble on Gov. Debt

- If $\check{\mu}_t^{\mathcal{C}} \leqslant \check{\mu}_t^{\mathcal{B}}$ gov. debt and cryptocoin bubble can co-exist
- But government can
  - Impose solvency law $\Rightarrow$ private institutions cannot run Ponzi scheme
  - Impose taxes on crypto holdings $\Rightarrow$ same as increasing $\check{\mu}_t^{\mathcal{C}}$
  - Impose trading restrictions
  - Financial repression

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$\neq$ liquidity
$\neq$ bubble

Fiscal Debt Sustainability (DSA): A Modern Perspective

- Debt valuation/FTPL equation with a bubble
  with service flow using representative agent SDF ξ^{**}
- Exorbitant privilege
  - Debt Laffer Curve (tax on self-insurance) – only sizable with negative β
  - Ponzi scheme/mining the bubble
- Credible Off-equilibrium Fiscal Capacity
  - Bubble (incl. possibly safe asset status) can
    - Burst
    - Jump to foreign safe asset
    Jump to crypto asset

Overview Across Lectures 10-13

- Store of Value Monetary Model with One Sector and No Aggregate Risk
  - Safe Asset and Service Flows
  - Bubble (mining) or not
  - 2 Different Asset Pricing Perspectives/SDFs

- Store of Value Monetary Model with Time-varying Idiosyncratic Risk
  - Safe asset, Flight-to-Safety and negative CAPM-β
  - Flight-to-Safety and Equity Excess Volatility
  - Debt valuation puzzle, Debt Laffer Curve,
  - Safe Asset and Bubble Complementarity
  - Policies to Maintain Safe Asset Privilege on Gov. Bond

- Medium of Exchange Role and Different "Monetary Theories"