4 Dynamic Programming under Certainty

Exercise 4.1

a. The original problem was

$$\max_{\{c_t,k_{t+1}\}_{t+0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$\begin{aligned} c_t + k_{t+1} &\leq f(k_t), \\ c_t, k_{t+1} &\geq 0, \end{aligned}$$

for all $t = 0, 1, ...$ with k_0 given. This can be equivalently written, after substituting the budget constraint into the objective function, as

$$\max_{\{k_{t+1}\}_{t+0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u[f(k_t) - k_{t+1}]$$

subject to

$$0 \leq k_{t+1} \leq f(k_t),$$

for all $t = 0, 1, ...$ with k_0 given. Hence, defining

$$F(k_t, k_{t+1}) = u[f(k_t) - k_{t+1}],$$

and

$$\Gamma(k_t) = \{k_{t+1} \in \mathbf{R}_+ : 0 \leq k_{t+1} \leq f(k_t)\},$$

we obtain the (SP) formulation given in the text.

b. Note that in this case $c_t \in \mathbf{R}_+^l$ for all $t = 0, 1, ...$ and we cannot simply substitute for consumption in the objective function. Instead, define

$$\Gamma(k_t) := \left\{ k_{t+1} \in \mathbf{R}_+^l : (k_{t+1} + c_t, k_t) \in Y \subseteq \mathbf{R}_+^{2l}, c_t \in \mathbf{R}_+^l \right\},$$

and

$$\Phi\left(k_t, k_{t+1}\right) := \left\{ c_t \in \mathbf{R}_+^l : (k_{t+1} + c_t, k_t) \in Y \subseteq \mathbf{R}_+^{2l} \right\}.$$

Then, let

$$F\left(k_t, k_{t+1}\right) = \sup_{c_t \in \Phi(k_t, k_{t+1})} u\left(c_t\right),$$

and the problem is in the form of the (SP).

Exercise 4.2

a. Define x_{it} as the ith component of the l dimensional vector x_t. Hence,

$$\max_i x_{it} \leq \theta^t \left\| x_0 \right\|.$$

Let $e = (1, ..., 1, ...1)$ be an l dimensional vector of ones. Hence, the fact that F is increasing in its first l arguments and decreasing in its last l arguments implies that for all θ

$$F(x_1, x_2) \leq F(x_1, 0) \leq F(\theta \left\| x_0 \right\| e, 0)$$

Then, if $\theta \leq 1$, $F(\theta^t \left\| x_0 \right\| e, 0) \leq F(\left\| x_0 \right\| e, 0)$ and

$$\begin{aligned} \lim_{n\to\infty} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) &\leq \lim_{n\to\infty} \sum_{t=0}^{\infty} \beta^t F(\left\| x_0 \right\| e, 0) \\ &= \frac{F(\left\| x_0 \right\| e, 0)}{(1-\beta)}, \end{aligned}$$

as $\beta < 1$. Otherwise, if $\theta > 1$, $F(\theta^t \left\| x_0 \right\| e, 0) \leq \theta^t F(\left\| x_0 \right\| e, 0)$ by the concavity of F and

$$\begin{aligned} \lim_{n\to\infty} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) &\leq \lim_{n\to\infty} \sum_{t=0}^{\infty} (\theta\beta)^t F(\left\| x_0 \right\| e, 0) \\ &= \frac{F(\left\| x_0 \right\| e, 0)}{(1-\theta\beta)}, \end{aligned}$$

as $\beta\theta < 1$. Hence the limit exists.

b. By assumption, for all $x_0 \in X$, $F(x_1, 0) \leq \theta F(x_0, 0)$. Hence

$$F(x_t, x_{t+1}) \leq F(x_t, 0) \leq \theta F(x_{t-1}, 0) \leq ... \leq \theta^t F(x_0, 0).$$

Then,

$$\begin{aligned}\lim_{n\to\infty} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) &\leq \lim_{n\to\infty} \sum_{t=0}^{\infty} (\theta\beta)^t F(x_0, 0) \\ &= \frac{F(x_0, 0)}{1 - \theta\beta}.\end{aligned}$$

Therefore, the limit exists.

Exercise 4.3

a. Let $\upsilon(x_0)$ be finite. Since υ satisfies (FE), as shown in the proof of Theorem 4.3, for every $x_0 \in X$ and every $\varepsilon > 0$, there exists $\underset{\sim}{x} \in \Pi(x_0)$ such that

$$\upsilon(x_0) \leq u_n(\underset{\sim}{x}) + \beta^{n+1}\upsilon(x_{n+1}) + \frac{\varepsilon}{2}.$$

Taking the limit as $n \to \infty$ gives

$$\begin{aligned}\upsilon(x_0) &\leq u(\underset{\sim}{x}) + \lim\sup_{n\to\infty} \beta^{n+1}\upsilon(x_{n+1}) + \frac{\varepsilon}{2} \\ &\leq u(\underset{\sim}{x}) + \frac{\varepsilon}{2}.\end{aligned}$$

Since

$$u(\underset{\sim}{x}) \leq \upsilon^*(x_0),$$

for all $\underset{\sim}{x} \in \Pi(x_0)$, this gives

$$\upsilon(x_0) \leq \upsilon^*(x_0) + \frac{\varepsilon}{2},$$

for all $\varepsilon > 0$. Hence,

$$\upsilon(x_0) \leq \upsilon^*(x_0),$$

for all $x_0 \in X$.

If $\upsilon(x_0) = -\infty$, the result follows immediately. If $\upsilon(x_0) = +\infty$, the proof goes along the lines of the last part of Theorem 4.3. Hence $\upsilon(x_0) \leq \upsilon^*(x_0)$, all $x_0 \in X$.

b. Since υ satisfies FE, by the argument of Theorem 4.3, for all $x_0 \in X$ and $\underset{\sim}{x} \in \Pi(x_0)$

$$\upsilon(x_0) \geq u_n(\underset{\sim}{x}) + \beta^{n+1}\upsilon(x_{n+1}).$$

In particular, for $\underset{\sim}{x}$ and $\underset{\sim}{x}'$as described,

$$\begin{aligned}\upsilon(x_0) &\geq \lim_{n\to\infty} u_n(\underset{\sim}{x}') + \lim_{n\to\infty} \beta^n \upsilon(x'_{n+1}) \\ &= u(\underset{\sim}{x}') \\ &\geq u(\underset{\sim}{x})\end{aligned}$$

all $\underset{\sim}{x} \in \Pi(x_0)$. Hence

$$\upsilon(x_0) \geq \upsilon^*(x_0) = \sup_{\underset{\sim}{x} \in \Pi(x_0)} u(\underset{\sim}{x}),$$

and in combination with the result proved in part a., the desired result follows.

Exercise 4.4

a. Let K be a bound on F and M be a bound on f. Then

$$(Tf)(x) \leq K + \beta M, \qquad \text{for all } x \in X.$$

Hence $T : B(X) \to B(X)$.

In order to show that T has a unique fixed point $\upsilon \in B(X)$ we will use the Contraction Mapping Theorem. Note that $(B(X), \rho)$ is a complete metric space, where ρ is the metric induced by the sup norm.

We will use Blackwell's sufficient conditions to show that T is a contraction. To prove monotonicity, let $f, g \in B(X)$, with $f(x) \leq g(x)$ for all $x \in X$. Then

$$\begin{aligned}
(Tf)(x) &= \max_{y \in \Gamma(x)} \{F(x,y) + \beta f(y)\} \\
&= F(x, y^*) + \beta f(y^*) \\
&\leq F(x, y^*) + \beta g(y^*) \\
&\leq \max_{y \in \Gamma(x)} \{F(x,y) + \beta g(y)\} = (Tg)(x),
\end{aligned}$$

where

$$y^* = \arg \max_{y \in \Gamma(x)} \{F(x,y) + \beta f(y)\}.$$

For discounting, let $a \in \mathbf{R}$. Then

$$\begin{aligned}
T(f+a)(x) &= \max_{y \in \Gamma(x)} \{F(x,y) + \beta [f(y) + a]\} \\
&= \max_{y \in \Gamma(x)} \{F(x,y) + \beta f(y)\} + \beta a \\
&= (Tf)(x) + \beta a.
\end{aligned}$$

Hence by the Contraction Mapping Theorem, T has a unique fixed point $v \in B(X)$, and for any $v_0 \in B(X)$,

$$\|T^n v_0 - v\| \leq \beta^n \|v_0 - v\|.$$

That the optimal policy correspondence $G : X \to X$, where

$$G(x) = \{y \in \Gamma(x) : v(x) = F(x,y) + \beta v(y)\},$$

is nonempty is immediate from the fact that Γ is nonempty and finite valued for all x. Hence, the maximum is always attained.

b. Note that as F and f are bounded, $T_h f$ is bounded. Hence $T_h : B(X) \to B(X)$. That T_h satisfies Blackwell's sufficient conditions for a contraction can be proven following the same steps as in part a. with the corresponding adaptations. Hence, T_h is a contraction and by the Contraction Mapping Theorem it has a unique fixed point $w \in B(X)$.

c. First, note that

$$\begin{aligned} w_n(x) &= (T_{h_n} w_n)(x) \\ &= F\left[x, h_n(x)\right] + \beta w_n\left[h_n(x)\right] \\ &\leq \max_{y \in \Gamma(x)} \{F(x,y) + \beta w_n(y)\} \\ &= (T w_n)(x) \\ &= (T_{h_{n+1}} w_n)(x). \end{aligned}$$

Hence for all $n = 0, 1, ...$ $w_n \leq T w_n$. Applying the operator $T_{h_{n+1}}$ to both sides of this inequality and using monotonicity gives

$$T w_n = T_{h_{n+1}} w_n \leq \left(T_{h_{n+1}}\right)(T w_n) = T^2_{h_{n+1}} w_n.$$

Iterating on this operator gives

$$T w_n \leq T^N_{h_{n+1}} w_n.$$

But $w_{n+1} = \lim_{N \to \infty} T^N_{h_{n+1}} w_n$, for $w_n \in B(X)$. Hence $T w_n \leq w_{n+1}$ and

$$w_0 \leq T w_0 \leq w_1 \leq T w_1 \leq ... \leq T w_{n-1} \leq T w_n \leq v.$$

By the Contraction Mapping Theorem,

$$\left\| T^N w_n - v \right\| \leq \beta^N \left\| w_n - v \right\|.$$

Then,

$$\begin{aligned} \|w_n - v\| &\leq \|T w_{n-1} - v\| \leq \beta \|w_{n-1} - v\| \\ &\leq \beta \|T w_{n-2} - v\| \leq \beta^2 \|w_{n-2} - v\| \leq ... \\ &\leq \beta^n \|w_0 - v\| \end{aligned}$$

and hence $w_n \to v$ as $n \to \infty$.

Exercise 4.5

First, we prove that $g(x)$ is strictly increasing. Towards a contradiction, suppose that there exists $x, x' \in X$ with $x < x'$ such that $g(x) \geq g(x')$. Then as f is increasing, using the first-order condition (5)

$$\begin{aligned} \beta v'[g(x')] &= U'[f(x') - g(x')] \\ &< U'[f(x) - g(x)] = \beta v'[g(x)] \end{aligned}$$

which contradicts v strictly concave.

We prove next that $0 < g(x') - g(x) < f(x') - f(x)$, if $x' > x$. Let $x' > x$. As $g(x)$ is strictly increasing, using the first-order condition we have

$$\begin{aligned} U'[f(x) - g(x)] &= \beta v'[g(x)] \\ &> \beta v'[g(x')] = U'[f(x') - g(x')]. \end{aligned}$$

The result follows from U strictly concave.

Exercise 4.6

a. By Assumption 4.10 $\|x_t\|_E \leq \alpha \|x_{t-1}\|_E$ for all t. Hence

$$\alpha \|x_{t-1}\|_E \leq \alpha^2 \|x_{t-2}\|_E$$

and

$$\|x_t\|_E \leq \alpha^2 \|x_{t-2}\|_E$$

The desired result follows by induction.

b. By Assumption 4.10 $\Gamma : X \to X$ is nonempty.
Combining Assumptions 4.10 and 4.11,

$$\begin{aligned} |F(x_t, x_{t+1})| &\leq B\left(\|x_t\|_E + \|x_{t+1}\|_E\right) \\ &\leq B(1+\alpha)\|x_t\|_E \\ &\leq B(1+\alpha)\alpha^t \|x_0\|_E \end{aligned}$$

for $\alpha \in (0, \beta^{-1})$ and $0 < \beta < 1$. So, by Exercise 4.2, Assumption 4.2 is satisfied.

c. By Assumption 4.11 F is homogeneous of degree one, so

$$F(\lambda x_t, \lambda x_{t+1}) = \lambda F(x_t, x_{t+1}).$$

Then

$$\begin{aligned} u(\lambda \underset{\sim}{x}) &= \lim_{n\to\infty} \sum_{t=0}^{n} \beta^t F(\lambda x_t, \lambda x_{t+1}) \\ &= \lambda \lim_{n\to\infty} \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1}) = \lambda u(\underset{\sim}{x}). \end{aligned}$$

By Assumption 4.10 the correspondence Γ displays constant returns to scale. Then clearly $\underset{\sim}{x} \in \Pi(x_0)$ if and only if $\lambda \underset{\sim}{x} \in \Pi(\lambda x_0)$. Hence

$$\begin{aligned} v^*(\lambda x_0) &= \sup_{\lambda \underset{\sim}{x} \in \Pi(\lambda x_0)} u(\lambda \underset{\sim}{x}) \\ &= \lambda \sup_{\underset{\sim}{x} \in \Pi(x_0)} u(\underset{\sim}{x}) \\ &= \lambda v^*(x_0). \end{aligned}$$

By Assumption 4.11,

$$\begin{aligned} |F(x_t, x_{t+1})| &\leq B\left(\|x_t\|_E + \|x_{t+1}\|_E\right) \\ &\leq B(1+\alpha)\|x_t\|_E \leq B(1+\alpha)\alpha^t \|x_0\|_E . \end{aligned}$$

Hence

$$\begin{aligned} |v^*(x_0)| &= \left| \sup_{\underset{\sim}{x} \in \Pi(x_0)} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \right| \\ &\leq \sup_{\underset{\sim}{x} \in \Pi(x_0)} \sum_{t=0}^{\infty} \beta^t |F(x_t, x_{t+1})| \\ &\leq \sum_{t=0}^{\infty} B(1+\alpha)(\alpha\beta)^t \|x_0\|_E \\ &= \frac{B(1+\alpha)}{1-\alpha\beta} \|x_0\|_E . \end{aligned}$$

Therefore $v^*(x_0) \leq c \|x_0\|_E$, all $x_0 \in X$, where

$$c = \frac{B(1+\alpha)}{1-\alpha\beta}.$$

Exercise 4.7

a. Take f and g homogeneous of degree one, and $\alpha \in \mathbf{R}$, then $f+g$ and αf are homogeneous of degree one, and clearly $\|\cdot\|$ is a norm, so H is a normed vector space. We hence turn to the proof that H is complete. Let $\{f_n\}$ be a Cauchy sequence in H. Then $\{f_n\}$ converges pointwise to a limit function f. We need to show that $f_n \to f \in H$ where the convergence is in the norm of H. The proof of convergence, and that f is continuous, are analogous to the proof of Theorem 3.1. To see that f is homogeneous of degree one, note that for any $x \in X$ and any $\lambda \geq 0$

$$f(\lambda x) = \lim_{n\to\infty} f_n(\lambda x) = \lim_{n\to\infty} \lambda f_n(x) = \lambda f(x).$$

b. Take $f \in H(X)$. Tf is continuous by the Theorem of the Maximum. To show that Tf is homogeneous of degree one, notice that

$$\begin{aligned}(Tf)(\lambda x) &= \sup_{\lambda y \in \Gamma(\lambda x)} \{F(\lambda x, \lambda y) + \beta f(\lambda y)\} \\ &= \sup_{y \in \Gamma(x)} \lambda \{F(x,y) + \beta f(y)\} \\ &= \lambda (Tf)(x),\end{aligned}$$

where the second line follows from Assumption 4.10.

Exercise 4.8

In order to prove the results, we need to add the restriction that f is non-negative, and strictly positive on $\mathbf{R}^l_{++}$. As a counterexample without this extra assumption, let $X = \mathbf{R}^2_+$ and consider the function

$$f(x) = \begin{cases} x_1^{1/2} x_2^{1/2} & \text{if } x_2 \geq x_1 \\ 0 & \text{otherwise.} \end{cases}$$

This function is clearly not concave, but is homogeneous of degree one and quasi-concave. To see homogeneity, let $\lambda \in [0, \infty)$ and note that

$$\begin{aligned} f(\lambda x) &= \begin{cases} \lambda x_1^{1/2} x_2^{1/2} & \text{if } x_2 \geq x_1 \\ 0 & \text{otherwise.} \end{cases} \\ &= \lambda f(x). \end{aligned}$$

To see quasi-concavity, let x, $x' \in X$ with $f(x) \geq f(x')$. If $f(x') = 0$ the result follows from f non-negative. If $f(x') > 0$, then $x_2 \geq x_1$ and $x_2' \geq x_1'$, and as $f(x)$ is Cobb-Douglas in this range, it is quasi-concave.

a. Pick two arbitrary vectors $x, x' \in X$, and assume that f is non-negative, and strictly positive on $\mathbf{R}^l_{++}$. We have to consider four cases:

i) $x = x'$
ii) $x = \alpha x'$ for any $\alpha \in \mathbf{R}$, and $x \neq 0$, $x' \neq 0$
iii) $x \neq \alpha x'$ for any $\alpha \in \mathbf{R}$, and $x \neq 0$, $x' \neq 0$
iv) $x \neq x'$ and $x = 0$ or $x' = 0$

i) and ii) are trivial.

iii) Suppose $f(x) \geq f(x')$, then f being homogeneous of degree one and non-negative implies that $f(x)/f(x') \geq 1$, so there exist a number $\gamma \in (0, 1)$ such that $\gamma f(x) = f(\gamma x) = f(x')$. Hence for any $\lambda \in (0, 1)$ we may write

$$f(\gamma x) = \lambda f(\gamma x) + (1 - \lambda) f(x') = f(x').$$

By the assumed quasi-concavity of f, for any $\omega \in (0, 1)$,

$$f[\omega\gamma x + (1 - \omega)x'] \geq \omega f(\gamma x) + (1 - \omega) f(x').$$

Then, for any $t > 0$,

$$\begin{aligned} f[t\omega\gamma x + t(1 - \omega)x'] &= tf[\omega\gamma x + (1 - \omega)x'] \\ &\geq t\omega f(\gamma x) + t(1 - \omega) f(x') \\ &= t\omega\gamma f(x) + t(1 - \omega) f(x'). \end{aligned}$$

For any $\gamma \in (0,1)$, $[1-\omega(1-\gamma)]^{-1} > 0$, hence choosing

$$t = [1-\omega(1-\gamma)]^{-1}$$

we get

$$\begin{aligned} & f\left[\frac{\omega\gamma}{1-\omega(1-\gamma)}x + \frac{(1-\omega)}{1-\omega(1-\gamma)}x'\right] \\ \geq\ & \frac{\omega\gamma}{1-\omega(1-\gamma)}f(x) + \frac{(1-\omega)}{1-\omega(1-\gamma)}f(x'), \end{aligned}$$

so if we let $\theta = \omega\gamma/\left[1-\omega(1-\gamma)\right]$, we obtain

$$f[\theta x + (1-\theta)x'] \geq \theta f(x) + (1-\theta)f(x').$$

In order to see that the above expression holds for any θ, define $g_\gamma : (0,1) \to (0,1)$ by

$$g_\gamma(\omega) = \frac{\omega\gamma}{1-\omega\,(1-\gamma)},$$

which is continuous and strictly increasing. Hence the proof is complete.

iv) Suppose $x' = 0$. Then, $f(x') > 0$ or $f(x') = 0$. In the former case the proof of iii) applies without change. In the latter case $f(x') = 0$, so for any $x \neq 0$ and $\theta \in (0,1)$,

$$f[\theta x + (1-\theta)x'] = f(\theta x) = \theta f(x) + (1-\theta)f(x').$$

b. Same proof as in case iii) in part a. assuming $f(x) \geq f(x')$ and replacing $\geq$ by $>$ everywhere else.

c. In order to prove that the fixed point v of the operator T defined in (2) is strictly quasi-concave, we need X, Γ, F and β to satisfy Assumptions 4.10 and 4.11. In addition, we need F to be strictly quasi-concave (see part b.). To show this, let $H'(X) \subset H(X)$ be the set of functions on X that are continuous, homogeneous of degree one, quasi-concave and bounded in the norm in (1), and let $H''(X)$ be the set of strictly quasi-concave functions. Since $H'(X)$ is a closed subset of the complete metric space $H(X)$, by Theorem 4.6 and Corollary 1 to the Contraction Mapping Theorem, it is sufficient to show that $T[H'(X)] \subseteq H''(X)$.

To verify that this is so, let $f \in H'(X)$ and let

$$x_0 \neq x_1, \qquad \theta \in (0,1), \qquad \text{and} \quad x_\theta = \theta x_0 + (1-\theta)x_1.$$

Let $y_i \in \Gamma(x_i)$ attain $(Tf)(x_i)$, for $i = 0,1$, and let $F(x_0,y_0) > F(x_1,y_1)$. Then by Assumption 4.10, $y_\theta = \theta y_0 + (1-\theta)y_1 \in \Gamma(x_\theta)$. It follows that

$$\begin{aligned} (Tf)(x_\theta) &\geq F(x_\theta, y_\theta) + \beta f(y_\theta) \\ &> F(x_1,y_1) + \beta f(y_1) \\ &= (Tf)(x_1), \end{aligned}$$

where the first line uses (3) and the fact that $y_\theta \in \Gamma(x_\theta)$; the second uses the hypothesis that f is quasi-concave and the quasi-concavity restriction on F; and the last follows from the way y_0 and y_1 were selected. Since x_0 and x_1 were arbitrary, it follows that Tf is strictly quasi-concave, and since f was arbitrary, that $T[H'(X)] \subseteq H''(X)$. Hence the unique fixed point v is strictly quasi-concave.

d. We need X, Γ, F, and β to satisfy Assumptions 4.9, 4.10 and 4.11, and in addition F to be strictly quasi-concave. Considering x, $x' \in X$ with $x \neq \alpha x'$ for any $\alpha \in \mathbf{R}$, Theorem 4.10 applies.

Exercise 4.9

Construct the sequence $\{k_t^*\}_{t=0}^{\infty}$ using

$$k_{t+1} = g(k_t) = \alpha\beta k_t^{\alpha},$$

given some $k_0 \in X$. If $k_0 = 0$ we have that $k_t^* = 0$ for all $t = 0, 1, ...$ which is the only feasible policy and is hence optimal. If $k_0 > 0$, then for all $t = 0, 1, ...$ we have that $k_{t+1}^* \in int\Gamma(k_t^*)$ as $\alpha\beta \in (0,1)$.

Let

$$E(x_t, x_{t+1}) := F_y(x_t, x_{t+1}) + \beta F_x(x_t, x_{t+1}).$$

Then for all $t = 0, 1, 2, ...$ we have that

$$\begin{aligned} E\left(k_t^*, k_{t+1}^*\right) &= \beta \frac{\alpha k_t^{*\alpha-1}}{k_t^{*\alpha} - k_{t+1}^*} - \frac{1}{k_{t-1}^{*\alpha} - k_t^*} \\ &= \beta \frac{\alpha k_t^{*\alpha-1}}{k_t^{*\alpha} - \alpha\beta k_t^{*\alpha}} - \frac{1}{k_{t-1}^{*\alpha} - \alpha\beta k_{t-1}^{*\alpha}} \\ &= \frac{\alpha\beta}{k_t^*(1-\alpha\beta)} - \frac{1}{k_{t-1}^{*\alpha}(1-\alpha\beta)} \\ &= \frac{1}{k_{t-1}^{*\alpha}(1-\alpha\beta)} - \frac{1}{k_{t-1}^{*\alpha}(1-\alpha\beta)} = 0, \end{aligned}$$

from repeated substitution of the policy function. Hence the Euler equation holds for all $t = 0, 1, ...$

To see that the transversality condition holds, let

$$T\left(x_t, x_{t+1}\right) = \lim_{t\to\infty} \beta^t F_x\left(x_t, x_{t+1}\right) \cdot x_t.$$

Then,

$$\begin{aligned} T\left(k_t^*, k_{t+1}^*\right) &= \lim_{t\to\infty} \beta^t \frac{\alpha k_t^{*\alpha-1}}{k_t^{*\alpha} - k_{t+1}^*} k_t \\ &= \lim_{t\to\infty} \beta^t \frac{\alpha k_t^{*\alpha}}{k_t^{*\alpha} - \alpha\beta k_t^{*\alpha}} \\ &= \lim_{t\to\infty} \beta^t \frac{\alpha}{(1-\alpha\beta)} = 0, \end{aligned}$$

where the result comes from the fact that $0 < \beta < 1$.