“A Bayesian Approach to Identification of Structural VAR Models”

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February 08, 2025

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1 PART I: Univariate Model

*Consider the following **AR(2) model**:*

$$y_t = c + b_1 y_{t-1} + b_2 y_{t-2} + \epsilon_t \qquad ...(1)$$

*where the **residuals are serially correlated** according to:*

$$\epsilon_t = \rho \epsilon_{t-1} + \nu_t \qquad ...(2)$$

with $\nu_t \sim N(0, \tau^2)$

1.1 Part I.a

*a. Write down the **Gibbs sampler** for this extended model (on paper). Describe each step, the prior specification, and the conditional posterior distributions including the formulas for their moments.*

First we know that roots of $(1 - b_1 L - b_2 L^2) = 0$ lie outside the complex unit circle. Then we have the matrix notation of the AR model (1) and residuals that are serially correlated (2):

$$\mathbf{Y} = \mathbf{X}\mathbf{b} + \mathbf{e}$$

and

$$\mathbf{e} = \mathbf{E}\rho + \nu$$

Where $\mathbf{X}$ is exogenous and a $(Tx2)$ matrix, $\mathbf{Y}$ and $\mathbf{e}$ are $(Tx1)$ vectors and $\mathbf{E}$ is the matrix of the first lag of $\mathbf{e}$ and a $(Tx1)$ vector. Now, we need the conditional equations for β and ρ: 1. Conditional on β, the problem reduces to making inferences on ρ and τ^2 from the following AR(1) model:

$$e_t^* = \rho e_{t-1}^* + \nu_t, \quad \nu_t \sim N(0, \tau^2)$$

Matrix form:

$$(\mathbf{e}^* = \mathbf{E}_t^* \rho + \nu, \qquad \nu \sim N(0, \tau^2 \mathbf{I}_{T-1})$$

Where $e_t^* = y_t - c - b_1 y_{t-1} - b_2 y_{t-2}$. 2. Conditional on ρ and τ^2, the problem reduces to making inferences on β from the following regression model with known variance:

$$y_t^* = c^* + b_1^* y_{t-1}^* + b_2^* y_{t-2}^* + \nu_t, \qquad \nu_t \sim N(0, \tau^2)$$

Matrix form:

$$(\mathbf{Y}^* = \mathbf{X}^* \mathbf{b} + \nu, \qquad \nu \sim N(0, \tau^2 \mathbf{I}_{T-1}))$$

Where:

$$y_t^* = y_t - \rho y_{t-1}$$

$$y_{t-1}^* = y_{t-1} - \rho y_{t-2}$$

$$y_{t-2}^* = y_{t-2} - \rho y_{t-3}$$

$$c^* = c(1 - \rho)$$

Now following Chib(1993), we need to derive the full conditional posterior distributions of the model's parameters. Given the following full conditional posterior distributions of $\rho, \mathbf{b}, \tau^2$, Gibbs-sampling can easily be implemented:

Conditional Distributions of b, Given ρ and τ^2

I) Prior distribution of **b**:

$$\mathbf{b}|\rho,\tau^2 \sim N(\mathbf{b}_0, \Sigma_0)$$

Where $\mathbf{b}_0$ and Σ_0 are known. Prior density can be written as:

$$p(\mathbf{b}|\rho,\tau^2) = (2\pi)^{-\frac{k}{2}}|\Sigma|^{-\frac{1}{2}}exp\{-\frac{1}{2}(\mathbf{b}-\mathbf{b_0})'\Sigma_0^{-1}(\mathbf{b}-\mathbf{b_0})\}$$

$$p(\mathbf{b}|\rho,\tau^2) \propto exp\{-\frac{1}{2}(\mathbf{b}-\mathbf{b_0})'\Sigma_0^{-1}(\mathbf{b}-\mathbf{b_0})\}$$

II) Likelihood function:

$$L(\mathbf{b}|\rho,\tau^2,Y) = (2\pi\tau^2)^{-\frac{T}{2}}exp\{-\frac{1}{2\tau^2}(\mathbf{Y}^*-\mathbf{X}^*\mathbf{b})'(\mathbf{Y}^*-\mathbf{X}^*\mathbf{b})\}$$

$$L(\mathbf{b}|\rho,\tau^2,Y) \propto exp\{-\frac{1}{2\tau^2}(\mathbf{Y}^*-\mathbf{X}^*\mathbf{b})'(\mathbf{Y}^*-\mathbf{X}^*\mathbf{b})\}$$

III) Posterior:

$$p(\mathbf{b}|\rho,\tau^2,Y) \propto p(\mathbf{b}|\rho,\tau^2)L(\mathbf{b}|\rho,\tau^2,Y)$$

$$p(\mathbf{b}|\rho,\tau^2,Y) \propto exp\{-\frac{1}{2}(\mathbf{b}-\mathbf{b_0})'\Sigma_0^{-1}(\mathbf{b}-\mathbf{b_0}) - \frac{1}{2\tau^2}(\mathbf{Y}^*-\mathbf{X}^*\mathbf{b})'(\mathbf{Y}^*-\mathbf{X}^*\mathbf{b})\}$$

$$\mathbf{b}|\rho,\tau^2,Y \sim N(\mathbf{b}_1,\Sigma_1) \qquad ...(3)$$

IV) Moments:
Location:

$$\mathbf{b_1} = (\Sigma_0^{-1} + \tau^{-2}\mathbf{X}^{*'}\mathbf{X}^*)^{-1}(\Sigma_0^{-1}\mathbf{b}_0 + \tau^{-2}\mathbf{X}^{*'}\mathbf{Y}^*)$$

Scale:

$$\Sigma_1 = (\Sigma_0^{-1} + \tau^{-2}\mathbf{X}^{*'}\mathbf{X}^*)^{-1}$$

Conditional distribution of ρ, given b and τ^2:

I) Prior Distribution of ρ:

$$\rho|\mathbf{b},\tau^2 \sim N(c_0, B_0)_{I[s(p)]}$$

Where c_0, B_0 are known and $I[s(\rho)]$ is an indicator function used to denote that roots of $\rho(L)=0$ lie outside the unit circle. Prior density can be written as:

$$p(\rho|\mathbf{b},\tau^2) = (2\pi)^{-\frac{k}{2}}|B_0|^{-\frac{1}{2}}exp\{-\frac{1}{2}(\rho-c_0)'B_0^{-1}(\rho-c_0)\}$$

$$p(\rho|\mathbf{b},\tau^2) \propto exp\{-\frac{1}{2}(\rho-c_0)'B_0^{-1}(\rho-c_0)\}$$

II) Likelihood function is:

$$L(\rho|\mathbf{b},\tau^2,Y) = (2\pi\tau^2)^{-\frac{T}{2}}exp\{-\frac{1}{2\tau^2}(\mathbf{e}^*-\mathbf{E}_t^*\rho)'(\mathbf{e}^*-\mathbf{E}_t^*\rho)\}$$

$$L(\rho|\mathbf{b},\tau^2,Y) \propto exp\{-\frac{1}{2\tau^2}(\mathbf{e}^*-\mathbf{E}_t^*\rho)'(\mathbf{e}^*-\mathbf{E}_t^*\rho)\}$$

III) Posterior:

$$P(\rho|\mathbf{b},\tau^2,Y) \propto p(\rho|\mathbf{b},\tau^2)L(\rho|\mathbf{b},\tau^2,Y)$$

$$P(\rho|\mathbf{b},\tau^2,Y) \propto exp\{-\frac{1}{2}(\rho-c_0)'B_0^{-1}(\rho-c_0) - \frac{1}{2\tau^2}(\mathbf{e}^*-\mathbf{E}_t^*\rho)'(\mathbf{e}^*-\mathbf{E}_t^*\rho)\}$$

$$\rho|\mathbf{b}, \tau^2, Y \sim N(c_1, B_1)_{I[s(p)]} \qquad ...(4)$$

IV) Moments:
Location:

$$c_1 = (B_0^{-1} + \tau^{-2}\mathbf{E}^{*'}\mathbf{E}^*)^{-1}(B_0^{-1}c_0 + \tau^{-2}\mathbf{E}^{*'}\mathbf{e}^*)$$

Scale:

$$B_1 = (B_0^{-1} + \tau^{-2}\mathbf{E}^{*'}\mathbf{E}^*)^{-1}$$

Conditional distribution of τ^2, given b and ρ:
First we have that:

$$z_i \sim iid\ N(0, \frac{1}{\delta})$$

$$W = \sum_{i=1}^{\nu} z_i^2 \sim \Gamma(\frac{\nu}{2}, \frac{\delta}{2})$$

$$p(W) \propto W^{\frac{t}{2}-1} exp\{-\frac{W\delta}{2}\}$$

With $E(W) = \frac{\nu}{\delta}$ and $Var(W) = 2\frac{\nu}{\delta^2}$
I) Prior distribution of τ^2:

$$\frac{1}{\tau^2}|\mathbf{b}, \rho \sim \Gamma(\frac{\nu_0}{2}, \frac{\delta_0}{2})$$

$$p(\frac{1}{\tau^2}|\mathbf{b}, \rho) \propto (\frac{1}{\tau^2})^{\frac{\nu_0}{2}-1} exp\{-\frac{\delta_0}{2\tau^2}\}$$

Where ν_0 and δ_0 are known.
II) Likelihood function is:

$$L(\frac{1}{\tau^2}|\rho, \mathbf{b}, Y) = (2\pi\tau^2)^{-\frac{T}{2}} exp\{-\frac{1}{2\tau^2}(\mathbf{Y}^* - \mathbf{X}^*\mathbf{b})'(\mathbf{Y}^* - \mathbf{X}^*\mathbf{b})\}$$

$$L(\frac{1}{\tau^2}|\rho, \mathbf{b}, Y) \propto (\tau^2)^{-\frac{T}{2}} exp\{-\frac{1}{2\tau^2}(\mathbf{Y}^* - \mathbf{X}^*\mathbf{b})'(\mathbf{Y}^* - \mathbf{X}^*\mathbf{b})\}$$

III) Posterior:

$$p(\frac{1}{\tau^2}|\rho, \mathbf{b}, Y) \propto p(\frac{1}{\tau^2}|\mathbf{b}, \rho) L(\frac{1}{\tau^2}|\rho, \mathbf{b}, Y)$$

$$p(\frac{1}{\tau^2}|\rho, \mathbf{b}, Y) \propto (\tau^2)^{\frac{\nu_0}{2}+\frac{T}{2}-1} exp\{-\frac{1}{2\tau^2}(\delta_0 + (\mathbf{Y}^* - \mathbf{X}^*\mathbf{b})'(\mathbf{Y}^* - \mathbf{X}^*\mathbf{b}))\}$$

$$p(\frac{1}{\tau^2}|\rho, \mathbf{b}, Y) \propto (\frac{1}{\tau^2})^{\frac{\nu_1}{2}-1} exp\{-\frac{\delta_1}{2\tau^2}\}$$

$$\frac{1}{\tau^2}|\rho, \mathbf{b}, Y \sim \Gamma(\frac{\nu_1}{2}, \frac{\delta_1}{2}) \qquad ...(5)$$

IV) Moments:
Location:

$$\nu_1 = \nu_0 + T$$

Scale:

$$\delta_1 = \delta_0 + (\mathbf{Y}^* - \mathbf{X}^*\mathbf{b})'(\mathbf{Y}^* - \mathbf{X}^*\mathbf{b})$$

Now, we can proceed with the Gibbs-sampling algorithm. Start the iteration with an arbitrary starting values, $\rho = \rho^0$ and $\tau^2 = \{\tau^2\}^0$. Then the following is iterated for Gibbs simulations:

- 1. Conditional on $\rho = \rho^{j-1}$ and $\tau^2 = \{\tau^2\}^{j-1}$, generate $\mathbf{b}^j$ from the conditional posterior distribution in (3).

- 2. Conditional on $\tau^2 = \{\tau^2\}^{j-1}$ and $\mathbf{b} = \mathbf{b}^j$, generate ρ^j from the conditional posterior distribution in (4).

- 3. Conditional on $\mathbf{b} = \mathbf{b}^j$ and $\rho = \rho^j$, generate $\{\tau^2\}^j$ from the conditional posterior distribution in (5).

- 4. Set $j = j - 1$, and go to first step of the algorithm until we have "J+B" draws.

- 5. Finally we have to discard the firsts "B" draws, because the influence of the starting values. And we keep the $\{\mathbf{b}^j, \rho^j, \{\tau^2\}^j\}_{j=B}^{J+B}$ draws.

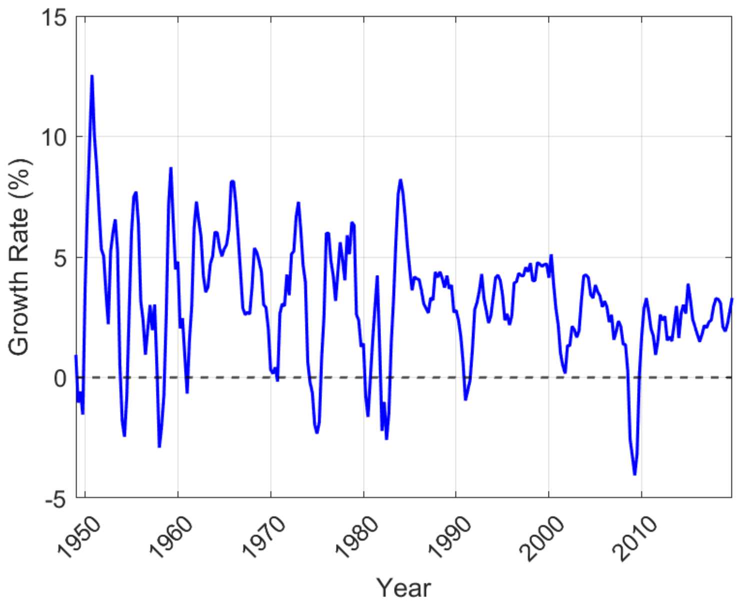
1.2 Part I.b

b. Write Matlab code to estimate this model for the ***annual growth rate of real GDP*** *for the US (FRED code: GDPC1) over the period 1948Q1 to 2019Q4 using the Gibbs sampler described in (a). Provide the following graphs:*

1.2.1 Part I.b.i

i. Plot the annual growth rate of real GDP (in percent).

Figure 1: Annual Growth of Real GDP

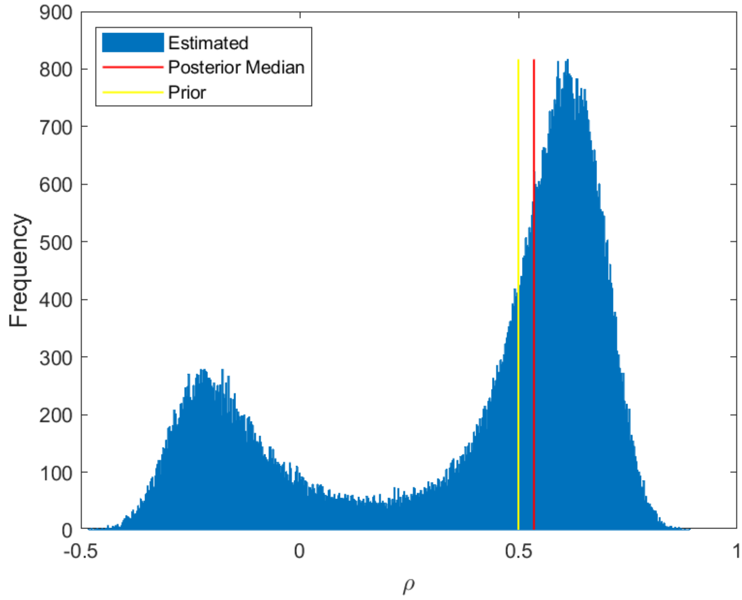


1.2.2 Part I.b.ii

ii. Plot the histogram of the (marginal) posterior distribution for ρ and report its posterior median.

As observed in Figure 2, the median of the posterior distribution is close to the prior we used. Additionally, the distribution exhibits a pattern where the highest accumulation occurs at the two extreme values.

Figure 2: Posterior distribution of ρ



2 PART II: Multivariate Models

Collect from FRED the following variables for the US economy at quarterly frequency for the period 1985Q1-2006Q4: real GDP, the consumer price index (CPI), the effective federal funds rate, and the M2 money stock.

2.1 PART II.1

*1. Transform the data to **quarter-on-quarter growth rates** where appropriate so that they have a useful economic interpretation. Plot the transformed data with appropriate labels.*

2.2 PART II.2

2. Consider a bivariate VAR model for real GDP growth and CPI inflation.

2.2.1 PART II.2.a

a. Write Matlab code to estimate a reduced-form VAR(4) model with a constant term using OLS. Report the (point) estimates of the reduced-form covariance matrix (omega) and the (kxn) matrix of reduced-form coefficients where n is the number of endogenous variables.

$$\Omega = \begin{bmatrix} 0.2076 & -0.0387 \\ -0.0387 & 0.1584 \end{bmatrix}$$

$$\Phi = \begin{bmatrix} 0.8197 & 0.2399 \\ 0.1555 & 0.0219 \\ -0.0240 & 0.0396 \\ 0.2314 & 0.2008 \\ -0.0486 & 0.2878 \\ -0.1474 & 0.0028 \\ -0.2503 & 0.3347 \\ 0.0948 & -0.0684 \\ -0.0894 & -0.1614 \end{bmatrix}$$

2.2.2 PART II.2.b

Assume that there is a supply-demand model that determines the fluctiations in output and inflation but that you do not know the values of the contemporaneous structural parameters that characterize that model. Plot the ***identified set*** *for all possible values that are compatible with the observed data.*

2.3 PART II.3

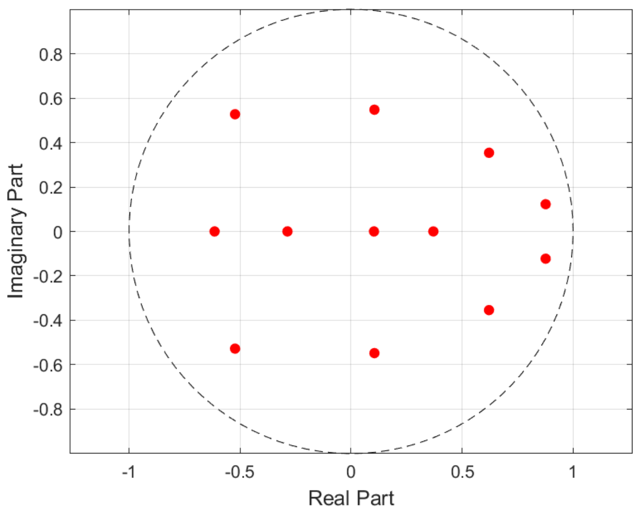
Now add the federal funds rate to your bivariate model.

2.3.1 PART II.3.a

a. Fit a VAR(4) to those data ordered as follows: output, inflation, interest rate. Check whether the model is stable. How can you tell? Show the output you use to determine stability of the system.

Se puede ver en el grafico que todos estan dentro del circulo unitario

Figure 3: Stability Check: Eigenvalues of the Companion Matrix



2.3.2 PART II.3.b

b. Apply the Choleski decomposition for identification. Plot the impulse responses of the three variables after a monetary policy shock (just the point estimates, without error bands). What do you find? Briefly comment on your results.

2.3.3 PART II.3.c

c. Compute and report the coefficients on output and inflation in the Taylor rule implied by the estimated model.

2.4 PART II.4

*4. Now add money to your set of variables. Write down the **structural** equations that describe these 4 variables. Provide an economic interpretation for each equation and the corresponding contemporaneous structural parameters (A matrix)*
Given a SVAR with the following form:

$$A\mathbf{Y}_t = B_1\mathbf{Y}_{t-1} + B_2\mathbf{Y}_{t-2} + B_3\mathbf{Y}_{t-3} + B_4\mathbf{Y}_{t-4} + \varepsilon_t$$

Where:

$$\mathbf{Y}_t = \begin{bmatrix} y_t \\ \pi_t \\ i_t \\ M_t \end{bmatrix}$$

And y_t is the GDP growth rate, π_t is the inflation rate, i_t is the federal funds rate and M_t is the M2 growth rate.
For the structural equation we follows traditional IS-LM model, where at least more than 50% of it belongs to a regime where the central bank controlled money supply other than interest rates. This leave our SVAR composed by 4 structural equation: i) goods supply; ii) goods demand; iii) money supply, which we assume is the central bank instrument and, thus, it is assumed as an exogenous amount, and iv) money demand that follows the Keynesian form. The 4 structural equations are:

- Goods demand

$$y_t = \alpha + \alpha^{\pi}\pi_t + \alpha^{i}r_t + \alpha^{m}m_t + \alpha^{X}\mathbf{X_{t-1}} + \mu_t^{gd}$$

- Goods supply

$$y_t = \beta + \beta^{\pi}\pi_t + \beta^{i}r_t + \beta^{m}m_t + \beta^{X}\mathbf{X_{t-1}} + \mu_t^{gs}$$

- Money demand

$$y_t = \beta + \beta^{\pi}\pi_t + \beta^{i}r_t + \beta^{m}m_t + \beta^{X}\mathbf{X_{t-1}} + \mu_t^{gs}$$

2.5 PART II.5

*5. Estimate this model using the Baumeister-Hamilton algorithm implementing the Choleski identification in that framework and uninformative priors for **B** and **D**.*

2.5.1 PART II.5.a

a. Write down (on paper) the prior for each element in $\boldsymbol{A}$ *as well as the joint prior for* $\boldsymbol{A}$.

2.5.2 PART II.5.b

b. Plot the impulse responses to the one-standard-deviation structural shocks (median together with 16^{th} *and* 84^{th} *percentiles of the posterior distribution) for a horizon of 5 years.*

2.5.3 PART II.5.c

c. Plot the posterior distributions for the contemporaneous structural coefficients. Are the estimates (magnitudes and signs) consistent with the economic interpretation that you provided under (4)?

2.6 PART II.6

6. Suppose you wanted to identify the shocks underlying this 4-variable model using traditional sign-restriction algorithm - but ***without*** *imposing any signs.*

2.6.1 PART II.6.a

a. Provide a plot for the impact effect of a one-standard deviation shock using the analytical expression for the implicit prior distribution.

2.6.2 PART II.6.b

b. Verify empirically what the impact effect for each variable looks like. Report plots of the impact effects and provide the numerical values for the cut-off points.

3 References

Chib, S. (1993). Bayes regression with autoregressive errors: A Gibbs sampling approach. Journal of Econometrics, 58(3), 275-294.

Kim, C.-J., & Nelson, C. R. (1999). State-space models with regime switching: Classical and Gibbs-sampling approaches with applications. MIT Press.