Macro, Money and Finance: A Continuous-Time Approach

Problem Set

Professor: Fernando Mendo

Due Date: Sunday March 2nd, 10 pm. Please send your solution to the academic coordinator of the summer course. Please include your codes.

1. Consider an infinitely-lived household with logarithmic preferences over consumption $\{c_t\}_{t\geq 0}$,

$$U_0 = \mathbb{E}\left[\int_0^\infty e^{-\rho t}\log(c_t)dt\right]$$

The household has initial wealth $n_0 > 0$ and does not receive any endowment or labor income. Wealth can be invested into two assets. A risk-free bond with (instantaneous) return $r^b dt$ and a risky stock with return $r^s dt + \sigma dZ_t$, where Z_t is a Brownian motion. Here, r^b, r^s, and σ are constant parameters.

The household's net worth evolution is

$$dn_t = -c_t dt + n_t[(1-\theta_t^s)r^b dt + \theta_t^s(r^s dt + \sigma dZ_t)]$$

where θ_t^s denotes the fraction of wealth invested into the stock. The household chooses consumption $\{c_t\}_{t\geq 0}$ and portfolio shares $\theta_t^s\}_{t\geq 0}$ to maximize utility U_0 subject to the net worth evolution (and a solvency constraint $n_t \geq 0$).

   (a) In this part, you will solve the consumption-portfolio choice problem using the Hamilton-Jacobi-Bellman (HJB) equation. The state space of this decision problem is one-dimensional with state variable n_t, so you can denote the household's value function by $V(n)$.

      i. Write down the (deterministic) HJB equation for the value function $V(n)$.

      ii. Take first-order conditions with respect to all choice variables.

      iii. Let's make a guess that optimal consumption is proportional to net worth, $c(n) = an$ with some constant $a > 0$ (to be determined below). Use the first-order condition for consumption derived in part (b) to turn this into a guess for the value function $V(n)$. *Hint:* Don't forget to add an integration constant (call it b) when moving from $V'(n)$ to $V(n)$; $V(n)$ is the sum of two terms

      iv. Use your guess for $V(n)$ to simplify the first-order condition for θ_t^s and solve the resulting equation for θ_t^s.

      v. Substitute the optimal choices and the guess for $V(n)$ into the HJB equation to eliminate $V(n)$, $V'(n)$, $V''(n)$, c, θ^s and the max operator.

      vi. The resulting equation in step (e) has to hold for all $n > 0$ (if it does not, the previous guess was incorrect). Show that this is indeed possible if we choose a and b appropriately. What are the required values for a and b?

   (b) Now consider the same decision problem as before but approach it with the stochastic maximum principle instead of the HJB equation.

i. Denote by ξ_t the costate for net worth n_t and by σ_t^ξ its (arithmetic) volatility loading (that is $d\xi_t = \mu_t^\xi dt + \sigma_t^\xi dZt$ with some drift μ_t^ξ). Write down the Hamiltonian of the problem.

ii. The choice variables have to maximize the Hamiltonian at all times. Take the first-order conditions in this maximization problem.

iii. Let's again make the guess $c_t = an_t$ with an unknown constant $a > 0$. Use the first-order condition for consumption derived in part (b) to turn this into a guess for the costate ξ_t. Also determine the implied costate volatility σ_t^ξ.

iv. Determine the optimal solution for θ_t^s.

v. Write down the costate equation for ξ_t and substitute in your guess for c_t, the implied guesses for ξ_t and σ_t^ξ, and the implied optimal solution for θ_t^s. Show that the costate equation is indeed satisfied (and hence the guess was correct) if you choose a suitably. Which value(s) for a work?

vi. Verify that the optimal solution coincides with the one you obtained from the HJB approach. Also show that $\xi_t = e^{-\rho t}V'(n_t)$, where V is the value function determined previously.

2. In this exercise, you will solve BruSan (2014) numerically, under the assumption of log utility. Our goal is to construct functions $q(\eta)$, $\iota(\eta)$, $\kappa(\eta)$ and $\sigma^q(\eta)$ on the $[0,1]$ grid. Slides 133-135 describe the set of equations and the algorithm. The parameter values are $\rho_e = 0.06$, $\rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\delta = 0.05$, $\phi = 10$, $\alpha = 0.5$, $\sigma = 0.1$ where $\Phi(\iota) = (1/\phi)\log(1+\phi\iota)$

(a) Solve the model at the boundaries: for $\eta = 0$ and $\eta = 1$

(b) Create a uniform grid for $\eta \in [0.0001, 0.9999]$

(c) Solve the ODE for $q(\eta)$ assuming $\kappa(0) = 0$ as boundary condition. Stop once you reach $\kappa \geq 1$. From here on, set $\kappa = 1$, solve for q and σ^q.

(d) Verify your solution by plotting $q(\eta)$ and $\sigma^q(\eta)$.Also plot $\iota(\eta)$, $\kappa(\eta)$.

(e) An alternative derivation for the drift and volatility of η in the general case is given by:

$$\mu_t^\eta = (1-\eta_t)\left[(\varsigma_t^e - \sigma - \sigma_t^q)(\sigma_t^\eta + \sigma + \sigma_t^q) - (\varsigma_t^h - \sigma - \sigma_t^q)\left(-\frac{\eta_t}{1-\eta_t}\sigma_t^\eta + \sigma + \sigma_t^q\right) - \left(\frac{C_t^e}{N_t^e} - \frac{C_t^h}{N_t^h}\right)\right]$$

$$\sigma_t^\eta = \frac{\kappa_t - \eta_t}{\eta_t}(\sigma + \sigma_t^q)$$

where ς_e and ς_h are risk prices. Show these expressions are equivalent to the ones derived in the slides (you can assume logarithmic preferences).

(f) Plot $\eta\mu^\eta(\eta)$ and $\eta\sigma^\eta(\eta)$

(g) Plot $r(\eta)$. Note that you will need to approximate a second order derivative.

3. Consider the first monetary model studied in class with log utility and without government policy ($\mu_B = i = \sigma_B = G = \tau = 0$). There can still be a constant supply of bonds $B_t \neq 0$. In this problem, we add stochastic volatility to the model. Suppose idiosyncratic risk $\tilde{\sigma}$ evolves according to the exogenous stochastic process

$$d\tilde{\sigma}_t = b(\tilde{\sigma}_{ss} - \tilde{\sigma}_t)dt + \nu\sqrt{\tilde{\sigma}_t}dZ_t$$

where $\tilde{\sigma}_{ss}$, b, and ν are constants.

(a) Use goods market clearing and optimal investment to express q_K, q_B, and ι in terms of $\vartheta := \frac{q_B}{q_B+q_K}$.

(b) Derive the "money valuation equation", i.e., an expression of the form $\mu_{\vartheta t} = f(\vartheta_t, \tilde{\sigma}_t)$ (drift of ϑ) where f only depends on parameters of the model.

> Feel free to use the following suggestion or an alternative procedure
>
> (a) Postulate a Geometric Brownian motion for ϑ, q_B and q_K.
>
> (b) Use the definition of ϑ to find the law of motion $d\vartheta_t$ using Ito's Lemma
>
> (c) Use the martingale pricing condition to simplify the expression for $\mu_{\vartheta t}$
>
> $$\frac{\mathbb{E}\left[dr_t^{K,\tilde{i}}\right]}{dt} - \frac{\mathbb{E}\left[dr_t^{B}\right]}{dt} = \zeta_t\left(\sigma_t^{K,\tilde{i}} - \sigma_t^{B}\right) + \tilde{\zeta}_t\left(\tilde{\sigma}_t^{K,\tilde{i}} - \tilde{\sigma}_t^{B}\right)$$
>
> where ζ is the price of risk.
>
> (d) Find the price of risk and replace in the expression for $\mu_{\vartheta t}$

(c) Suppose that $\sigma_{\sigma,t} = 0$ and the economy is at the steady state with $\tilde{\sigma}_t = \tilde{\sigma}^{ss}$ for some $\tilde{\sigma}^{ss} > 0$.

i. Derive expressions for q^B, q^K and ϑ in the monetary and non-monetary equilibria.

ii. What is the smallest value of $\tilde{\sigma}^{ss}$ that allows for a monetary equilibrium? Denote this value by $\tilde{\sigma}^{ss}_{\min}$.

iii. Suppose that $\tilde{\sigma}^{ss} > \tilde{\sigma}^{ss}_{\min}$, what happens to q^B, q^K and ϑ as $\tilde{\sigma}^{ss}$ rises?

iv. Suppose that $0 < \tilde{\sigma}^{ss} < \tilde{\sigma}^{ss}_{\min}$, what happens to q^B, q^K and ϑ as $\tilde{\sigma}^{ss}$ falls?

4. Solving the previous model numerically.

(a) Set $a = 0.2$, $\phi = 1$, $\delta = 0.05$, $\rho = 0.01$, $\tilde{\sigma}^{ss} = 0.2$, $b = 0.05$, $\nu = 0.02$.

(b) Apply Ito's lemma to $\vartheta_t = \vartheta(\tilde{\sigma}_t)$, and equate the drift term with $\vartheta_t \mu_t^{\vartheta}$, where μ_t^{ϑ} is given by the Cox-Ingersol-Ross process above. This gives you an HJB-looking equation for $\vartheta(\tilde{\sigma})$.

(c) Solve the model using value function iteration:

i. Suggest a grid for $\tilde{\sigma}$ and construct the M matrix using `build_M.m`.

ii. Rewrite the money valuation equation such that in the discretized form you get:

$$\rho\vartheta = u(\vartheta) + M\vartheta$$

iii. Write a loop that updates $\vartheta(\tilde{\sigma})$ with the implicit method:

$$\vartheta_{t-\Delta t} = \left((1+\rho\Delta t)I - \Delta t M\right)^{-1}\left(\Delta t u(\vartheta_t) + \vartheta_t\right)$$

iv. Iterate over $\vartheta(\tilde{\sigma})$ until convergence

(d) Plot $\vartheta, q^B, q^K, r^f, \varsigma, \tilde{\xi}$ as functions of $\tilde{\sigma}$. Explain the dependence of the variables on $\tilde{\sigma}$.