2 An Overview

Exercise 2.1

The fact that $f : \mathbf{R}_+ \to \mathbf{R}_+$ is continuously differentiable, strictly increasing and strictly concave comes directly from the definition of f as

$$f(k) = F(k, 1) + (1 - \delta)k,$$

with $0 < \delta < 1$, and F satisfying the properties mentioned above. In particular, the sum of two strictly increasing functions is strictly increasing, and continuous differentiability is preserved under summation. Finally, the sum of a strictly concave and a linear function is strictly concave.

Also,

$$\begin{aligned}
f(0) &= F(0, 1) = 0, \\
f'(k) &= F_k(k, 1) + (1 - \delta) > 0, \\
\lim_{k \to 0} f'(k) &= \lim_{k \to 0} F_k(k, 1) + \lim_{k \to 0} (1 - \delta) = \infty, \\
\lim_{k \to \infty} f'(k) &= \lim_{k \to \infty} F_k(k, 1) + \lim_{k \to \infty} (1 - \delta) = (1 - \delta).
\end{aligned}$$

Exercise 2.2

a. With the given functional forms for the production and utility function we can write (5) as

$$\frac{\alpha\beta k_t^{\alpha-1}}{k_t^{\alpha} - k_{t+1}} = \frac{1}{k_{t-1}^{\alpha} - k_t},$$

which can be rearranged as

$$\alpha\beta k_t^{\alpha-1}(k_{t-1}^{\alpha} - k_t) = (k_t^{\alpha} - k_{t+1}).$$

Dividing both sides by k_t^{α} and using the change of variable $z_t = k_t/k_{t-1}^{\alpha}$ we obtain

$$\alpha\beta(\frac{1}{z_t} - 1) = 1 - z_{t+1},$$

or

$$z_{t+1} = 1 + \alpha\beta - \frac{\alpha\beta}{z_t},$$

which is the equation represented in Figure 2.1.

Insert Figure 2.1 About Here

As can be seen in the figure, the first-order difference equation has two steady states (that is, z's such that $z_{t+1} = z_t = z$), which are the two solutions to the characteristic equation

$$z^2 - (1+\alpha\beta)z + \alpha\beta = 0.$$

These are given by $z = 1$ and $\alpha\beta$.

b. Using the boundary condition $z_{T+1} = 0$ we can solve for z_T as

$$z_T = \frac{\alpha\beta}{1+\alpha\beta}.$$

Substituting recursively into (**??**) we can solve for z_{T-1} as

$$\begin{aligned} z_{T-1} &= \frac{\alpha\beta}{1+\alpha\beta - z_T} \\ &= \frac{\alpha\beta}{1+\alpha\beta - \frac{\alpha\beta}{1+\alpha\beta}} \\ &= \frac{\alpha\beta(1+\alpha\beta)}{1+\alpha\beta+(\alpha\beta)^2}, \end{aligned}$$

and in general,

$$z_{T-j} = \frac{\alpha\beta[1 + \alpha\beta + ... + (\alpha\beta)^j]}{1 + \alpha\beta + ... + (\alpha\beta)^{j+1}}.$$

Hence for $t = T - j$,

$$\begin{aligned} z_t &= \frac{\alpha\beta[1 + \alpha\beta + ... + (\alpha\beta)^{T-t}]}{1 + \alpha\beta + ... + (\alpha\beta)^{T-t+1}} \\ &= \frac{\alpha\beta s_{T-t}}{s_{T-t+1}} \end{aligned}$$

where $s_i = 1 + \alpha\beta + ... + (\alpha\beta)^i$. In order to solve for the series, take for instance the one in the numerator,

$$s_{T-t} = 1 + \alpha\beta + ... + (\alpha\beta)^{T-t},$$

multiply both sides by $\alpha\beta$ to get

$$\alpha\beta s_{T-t} = \alpha\beta + ... + (\alpha\beta)^{T-t+1},$$

and substract this new expression from the previous one to obtain

$$(1 - \alpha\beta)s_{T-t} = 1 - (\alpha\beta)^{T-t+1}.$$

Hence

$$\begin{aligned} s_{T-t} &= \frac{1 - (\alpha\beta)^{T-t+1}}{1 - \alpha\beta}, \\ s_{T-t+1} &= \frac{1 - (\alpha\beta)^{T-t+2}}{1 - \alpha\beta}, \end{aligned}$$

and therefore

$$z_t = \alpha\beta\frac{1 - (\alpha\beta)^{T-t+1}}{1 - (\alpha\beta)^{T-t+2}},$$

for $t = 1, 2, ..., T + 1$, as in the text. Notice also that

$$\begin{aligned} z_{T+1} &= \alpha\beta\frac{1 - (\alpha\beta)^{T-(T+1)+1}}{1 - (\alpha\beta)^{T-(T+1)+2}} \\ &= 0. \end{aligned}$$

c. Plugging (7) into the right hand side of (5) we get

$$\left(k_{t-1}^{\alpha}-\alpha\beta\frac{\left[1-(\alpha\beta)^{T-t+1}\right]}{\left[1-(\alpha\beta)^{T-t+2}\right]}k_{t-1}^{\alpha}\right)^{-1}=\frac{\left[1-(\alpha\beta)^{T-t+2}\right]}{k_{t-1}^{\alpha}\left(1-\alpha\beta\right)}.$$

Similarly, by plugging (7) into the left hand side of (5) we obtain

$$\begin{aligned}
&\frac{\alpha\beta\left[\alpha\beta\frac{\left[1-(\alpha\beta)^{T-t+1}\right]}{\left[1-(\alpha\beta)^{T-t+2}\right]}k_{t-1}^{\alpha}\right]^{\alpha-1}}{\left[\alpha\beta\frac{\left[1-(\alpha\beta)^{T-t+1}\right]}{\left[1-(\alpha\beta)^{T-t+2}\right]}k_{t-1}^{\alpha}\right]^{\alpha}\left(1-\alpha\beta\frac{\left[1-(\alpha\beta)^{T-t}\right]}{\left[1-(\alpha\beta)^{T-t+1}\right]}\right)}\\
&= \frac{1}{k_{t-1}^{\alpha}}\left(\frac{\left[1-(\alpha\beta)^{T-t+1}\right]-\alpha\beta\left[1-(\alpha\beta)^{T-t}\right]}{\left[1-(\alpha\beta)^{T-t+2}\right]}\right)^{-1}\\
&= \frac{\left[1-(\alpha\beta)^{T-t+2}\right]}{k_{t-1}^{\alpha}\left(1-\alpha\beta\right)}.
\end{aligned}$$

Hence, the law of motion for capital given by (7) satisfies (5).

Evaluating (7) for $t=T$ yields

$$\begin{aligned}
k_{T+1} &= \alpha\beta\frac{1-(\alpha\beta)^{T-T}}{1-(\alpha\beta)^{T-T+1}}k_T^{\alpha}\\
&= 0,
\end{aligned}$$

so (7) satisfies (6) too.

Exercise 2.3

a. We can write the value function using the optimal path for capital given by (8) as

$$\begin{aligned}
\upsilon(k_0) &= \sum_{t=0}^{\infty}\beta^t\log(k_t^{\alpha}-\alpha\beta k_t^{\alpha})\\
&= \frac{\log(1-\alpha\beta)}{(1-\beta)}+\alpha\sum_{t=0}^{\infty}\beta^t\log(k_t).
\end{aligned}$$

The optimal policy function, written (by recursive substitution) as a function of the initial capital stock is (in logs)

$$\log k_t = \left(\sum_{i=0}^{t-1} \alpha^i \right) \log(\alpha\beta) + \alpha^t \log k_0.$$

Using the optimal policy function we can break up the last summation to get

$$\begin{aligned}
\sum_{t=0}^{\infty} \beta^t \log(k_t) &= \frac{\log(k_0)}{(1-\alpha\beta)} + \log(\alpha\beta) \sum_{t=1}^{\infty} \beta^t \left(\sum_{i=0}^{t-1} \alpha^i \right) \\
&= \frac{\log(k_0)}{(1-\alpha\beta)} + \beta \frac{\log(\alpha\beta)}{[(1-\beta)(1-\alpha\beta)]},
\end{aligned}$$

where we have used the fact that the solution to a series of the form $s_t = \sum_{i=0}^{t} \lambda^i$ is $\left(1-\lambda^{t+1}\right) / (1-\lambda)$, as shown in Exercise 2.2b. Hence, we obtain a log linear expression for the value function

$$\upsilon(k_0) = A + B \log(k_0),$$

where

$$A = \left[\log(1-\alpha\beta) + \frac{\alpha\beta \log(\alpha\beta)}{(1-\alpha\beta)} \right] (1-\beta)^{-1},$$

and

$$B = \frac{\alpha}{1-\alpha\beta}.$$

b. We want to verify that

$$\upsilon(k) = A + B \log(k)$$

satisfies (11). For this functional form, the first-order condition of the maximization problem in the right-hand side of (11) is given by

$$g(k) = \frac{\beta B}{1+\beta B} k^{\alpha}.$$

Plugging this policy function into the right hand side of (11) we obtain

$$\begin{aligned}
v(k) &= \log\left(k^{\alpha} - \frac{\beta B}{1+\beta B}k^{\alpha}\right) + \beta\left[A + B\log\left(\frac{\beta B}{1+\beta B}k^{\alpha}\right)\right] \\
&= \alpha\log(k) - \log(1+\beta B) + \beta A \\
&\quad +\beta B\left[\log(\beta B) + \alpha\log(k) - \log(1+\beta B)\right] \\
&= (1+\beta B)\,\alpha\log(k) - (1+\beta B)\log(1+\beta B) \\
&\quad +\beta A + \beta B\log(\beta B).
\end{aligned}$$

Using the expressions for A and B obtained in part a., we get that $(1+\beta B)\,\alpha = B$ and

$$\beta B\log(\beta B) - (1+\beta B)\log(1+\beta B) + \beta A = A,$$

and hence $v(k) = A + B\log(k)$ satisfies (11).

Exercise 2.4

a. The graph of $g(k) = sf(k)$, with $0 < s < 1$, is found in Figure 2.2.

Insert Figure 2.2 About Here.

Since f is strictly concave and continuously differentiable, g will inherit those properties. Also, $g(0) = sf(0) = 0$. In addition,

$$\begin{aligned}
\lim_{k\to 0} g'(k) &= \lim_{k\to 0} sf'(k) \\
&= \lim_{k\to 0} sF_k(k,1) + \lim_{k\to 0} s(1-\delta) = \infty,
\end{aligned}$$

and

$$\begin{aligned}
\lim_{k\to\infty} g'(k) &= \lim_{k\to\infty} sf'(k) \\
&= \lim_{k\to\infty} sF_k(k,1) + \lim_{k\to\infty} s(1-\delta) = s(1-\delta) < 1.
\end{aligned}$$

First, we will prove existence of a non-zero stationary point.

Combining the first limit condition (the one for $k \to 0$) and $g(0) = 0$, we have that for an arbitrary small positive perturbation,

$$0 < \frac{g(0+h) - g(0)}{h}.$$

This term tends to $+\infty$ as $h \to 0$, and hence $g(h)/h \to \infty$. Therefore, there exist an h such that $g(h)/h > 1$, and hence $g(k) > k$ for some k small enough. Similarly, the fact that $g(k) < k$ for k large enough is a direct implication of the second limit condition. Next, define $q(k) = g(k) - k$. By the arguments outlined above, $q(k) > 0$ for k small enough and $q(k) < 0$ for k large enough. By continuity of f, q is also continuous and hence by the Intermediate Value Theorem there exist a k^* such that $g(k^*) = k^*$.

That the stationary point is unique follows from the strict concavity of g. Note that a continuum of stationary points implies that $g'(k) = 1$ contradicting the strict concavity of g. A discrete set of stationary points will imply that one of the stationary points is reached from below, violating again the strict concavity of g. To see this, define $k^* = \min\{k \in \mathbf{R}_+ : q(k) = 0\}$. The limit conditions above, and the fact that g is nondecreasing implies that $g(k^* - \varepsilon) > k^*$, for $\varepsilon > 0$. Define

$$k^m = \min\{k \in \mathbf{R}_+ : q(k) = 0, k > k^* \text{ and } g(k-\varepsilon) - k > 0 \text{ for } \varepsilon > 0\}.$$

Then, by continuity of g, there exist $k \in (k^*, k^m)$ such that $g(k) < k$. Let $\alpha \in (0,1)$ be such that $k = \alpha k^* + (1-\alpha)k^m$. Then,

$$\begin{aligned}
\alpha g(k^*) + (1-\alpha)g(k^m) &= \alpha k^* + (1-\alpha)k^m \\
&= k \\
&> g(k) \\
&= g(\alpha k^* + (1-\alpha)k^m),
\end{aligned}$$

a contradiction.

b. In Figure 2.3, we can see how for any $k_0 > 0$, the sequence $\{k_t\}_{t=0}^{\infty}$ converges to k^* as $t \to \infty$. As can be seen too, this convergence is monotonic, and it does not occur in a finite number of periods if $k_0 \neq k^*$.

Insert Figure 2.3 About Here

Exercise 2.5

Some notation is needed. Let z^t denote the history of shocks up to time t. Equivalently, $z^t = (z^{t-1}, z_t)$, where z_t is the shock in period t.

Consumption and capital are indexed by the history of shocks. They are chosen given the information available at the time the decision is taken, so we represent them by finite sequences of random variables $c = \left\{c_t(z^t)\right\}_{t=0}^{T}$ and $k = \left\{k_t(z^t)\right\}_{t=0}^{T}$.

The pair (k_t, z^t) determines the set of feasible pairs (c_t, k_{t+1}) of current consumption and beginning of next period capital stock. We can define this set as

$$B(k_t, z^t) = \left\{(c_t, k_{t+1}) \in R_+^2 : c_t(z^t) + k_{t+1}(z^t) \leq z_t f[k_t(z^{t-1})]\right\}$$

Because the budget constraint should be satisfied for each t and for every possible state, Lagrange multipliers are also random variables at the time the decisions are taken, and they should also be indexed by the history of shocks, so $\lambda_t(z^{t-1}, z_t)$ is a random variable representing the Lagrange multiplier for the time t constraint.

The objective function

$$U(c_0, c_1, ...) = E\left\{\sum_{t=0}^{\infty} \beta^t u[c_t(z^t)]\right\}$$

can be written as a nested sequence,

$$u(c_0) + \beta \sum_{i=1}^{n} \pi_i \left\{ u[c_1(a_i)] + \beta \sum_{j=1}^{n} \pi_j \left[u(c_2(a_i, a_j) + \beta\,[...]\right] \right\},$$

where π_i stands for the probability that state a_i occurs.

The objects of choice are then the contingent sequences c and k. For instance

$$c=\left\{c_0,c_1(z^1),c_2(z^2),..,c_t(z^t),..c_T(z^T)\right\}.$$

We can see that $c_0\in\mathbf{R}_+$, $c_1\in\mathbf{R}_+^n$, $c_2\in\mathbf{R}_+^{2n}$, and so on, so the sequence c belongs to the obvious cross product of the commodity spaces for each time period t. Similar analysis can be carried out for the capital sequence

$$k=\left\{k_0,k_1(z^1),k_2(z^2),..,k_t(z^t),..k_T(z^T)\right\}.$$

Define this cross product as S. Hence we can define the consumption set as

$$\begin{aligned} C(k_0,z_0) &= \left\{c\in S:\left[c_t(z^t),k_{t+1}(z^t)\right]\in B(k_t,z^t),\right. \\ &\qquad t=0,1,...\text{ for some }k\in S,\ k_0\text{ given.}\} \end{aligned}$$

(Notice that the consumption set, i.e. the set of feasible sequences, is a subset of the Euclidean space defined above.)

The first order conditions for consumption and capital are, respectively, (after cancelling out probabilities on both sides):

$$u'[c_t(z_t,z^{t-1})]=\lambda_t(z_t,z^{t-1})$$

for all $\left(z^{t-1},z_t\right)$ and all t, and

$$\lambda_t(z_t,z^{t-1})=\sum_{i=1}^{n}\pi_i\lambda_t(a_i,z_j^t)f'[k_t(a_i,z^{t-1})]$$

for all $\left(z^{t-1},z_t\right)$ and all t.

Exercise 2.6

As we did before in the deterministic case, we can use the budget constraint to solve for consumption along the optimal path and then write the value function as

$$\begin{aligned}
v(k_0, z_0) &= E_0\left[\sum_{t=0}^{\infty} \beta^t \log(z_t k_t^{\alpha} - \alpha\beta z_t k_t^{\alpha})\right] \\
&= \frac{\log(1-\alpha\beta)}{(1-\beta)} + E_0\left[\sum_{t=0}^{\infty} \beta^t \log(z_t)\right] \\
&\quad +\alpha E_0\left[\sum_{t=0}^{\infty} \beta^t \log(k_t)\right].
\end{aligned}$$

To obtain an expression in terms of the initial capital stock and the initial shock we need to solve for the second and third term above. Denoting $E_0(\log z_t) = \mu$, the second term can be written as

$$\begin{aligned}
E_0\left[\sum_{t=0}^{\infty} \beta^t \log(z_t)\right] &= \log z_0 + \sum_{t=0}^{\infty} \beta^t E_0(\log z_t) \\
&= \log z_0 + \frac{\beta\mu}{1-\beta}.
\end{aligned}$$

In order to solve for the third term, we use the fact that the optimal path for the log of the capital stock can be written as

$$\log k_t = \left(\sum_{i=0}^{t-1} \alpha^i\right) \log(\alpha\beta) + \left(\sum_{i=0}^{t-1} \alpha^{t-1-i}\right) \log(z_i) + \alpha^t \log k_0.$$

Hence

$$\begin{aligned}
\alpha E_0\left[\sum_{t=0}^{\infty} \beta^t \log(k_t)\right] &= \alpha E_0\left[\sum_{t=1}^{\infty} \beta^t \left(\sum_{i=0}^{t-1} \alpha^i\right) \log(\alpha\beta)\right] \\
&\quad +\alpha E_0\left[\sum_{t=1}^{\infty} \beta^t \left(\sum_{i=0}^{t-1} \alpha^{t-1-i} \log(z_i)\right)\right] \\
&\quad +\alpha E_0\left[\sum_{t=1}^{\infty} (\alpha\beta)^t \log(k_0)\right] + \alpha \log k_0.
\end{aligned}$$

Therefore, the next step is to solve for each of the terms above. The first term can be written as

$$\begin{aligned}
\alpha E_0\left[\sum_{t=1}^{\infty}\beta^t\left(\sum_{i=0}^{t-1}\alpha^i\right)\log(\alpha\beta)\right] &= \alpha\log(\alpha\beta)\sum_{t=1}^{\infty}\beta^t\left(\frac{1-\alpha^t}{1-\alpha}\right)\\
&= \frac{\alpha\log(\alpha\beta)}{(1-\alpha)}\left[\frac{\beta}{(1-\beta)}-\frac{\alpha\beta}{(1-\alpha\beta)}\right]\\
&= \frac{\alpha\beta\log(\alpha\beta)}{(1-\beta)(1-\alpha\beta)},
\end{aligned}$$

the second term as

$$\begin{aligned}
& \alpha E_0\left[\sum_{t=1}^{\infty}\beta^t\left(\sum_{i=0}^{t-1}\alpha^{t-1-i}\log(z_i)\right)\right]\\
= \; & \alpha E_0\left[\beta\log(z_0)+\sum_{t=2}^{\infty}\beta^t\left(\sum_{i=0}^{t-1}\alpha^{t-1-i}\log(z_i)\right)\right]\\
= \; & \alpha E_0\left[\beta\log(z_0)+\sum_{t=2}^{\infty}\beta^t\left(\alpha^{t-1}\log(z_0)+\sum_{i=1}^{t-1}\alpha^{t-1-i}\log(z_i)\right)\right]\\
= \; & \frac{\alpha\beta\log(z_0)}{(1-\alpha\beta)}+\alpha\sum_{t=2}^{\infty}\beta^t\left(\sum_{i=1}^{t-1}\alpha^{t-1-i}\mu\right)\\
= \; & \frac{\alpha\beta\log(z_0)}{(1-\alpha\beta)}+\frac{\alpha\mu}{(1-\alpha)}\sum_{t=2}^{\infty}\beta^t(1-\alpha^{t-1})\\
= \; & \frac{\alpha\beta\log(z_0)}{(1-\alpha\beta)}+\frac{\alpha\beta^2\mu}{(1-\beta)(1-\alpha\beta)},
\end{aligned}$$

and finally, the last two terms as

$$\alpha E_0\left[\sum_{t=1}^{\infty}(\alpha\beta)^t\log(k_0)\right]+\alpha\log k_0=\frac{\alpha\log k_0}{(1-\alpha\beta)}.$$

Hence,

$$\begin{aligned}
\alpha E_0\left[\sum_{t=0}^{\infty}\beta^t\log(k_t)\right] &= \frac{\alpha\beta\log(\alpha\beta)}{(1-\beta)(1-\alpha\beta)}+\frac{\alpha\beta\log(z_0)}{(1-\alpha\beta)}\\
&\quad +\frac{\alpha\beta^2\mu}{(1-\beta)(1-\alpha\beta)}+\frac{\alpha\log k_0}{(1-\alpha\beta)},
\end{aligned}$$

and

$$v(k_0, z_0) = A + B\log(k_0) + C\log(z_0)$$

where

$$\begin{aligned} A &= \left[\log(1-\alpha\beta) + \frac{\alpha\beta\log(\alpha\beta)}{(1-\alpha\beta)} + \frac{\beta\mu}{(1-\alpha\beta)}\right](1-\beta)^{-1}, \\ B &= \frac{\alpha}{(1-\alpha\beta)}, \text{ and} \\ C &= \frac{1}{(1-\alpha\beta)}. \end{aligned}$$

Following the same procedure outlined in Exercise 2.3, it can be checked that v satisfies (3) .

Exercise 2.7

a. The sequence of means and variances of the sequence of logs of the capital stocks have a recursive structure. Define μ_t as the mean at time zero of the log of the capital stock in period t. Then

$$\begin{aligned} \mu_t &= E_0[\log k_t] \\ &= E_0[\log(\alpha\beta) + \alpha\log(k_{t-1}) + \log(z_{t-1})] \\ &= \log(\alpha\beta) + \mu + \alpha\mu_{t-1} \\ &= \log(\alpha\beta) + \mu + \alpha\left[\log(\alpha\beta) + \mu\right] + \alpha^2\mu_{t-2} \\ &= \left[\log(\alpha\beta) + \mu\right] + \left[1 + \alpha + ... + \alpha^{t-1}\right] + \alpha^t\mu_0 \\ &= \left[\mu_0 - \frac{\log(\alpha\beta)+\mu}{1-\alpha}\right]\alpha^t + \frac{\log(\alpha\beta)+\mu}{1-\alpha}. \end{aligned}$$

Since $0 < \alpha < 1$,

$$\mu_\infty \equiv \lim_{t\to\infty} \mu_t = \frac{\log(\alpha\beta)+\mu}{1-\alpha}.$$

Similarly, define σ_t as the variance at time zero of the log of the capital stock in period t. Then

$$\begin{aligned} \sigma_t &= Var_0[\log k_t] \\ &= Var_0[\log(\alpha\beta) + \alpha\log(k_{t-1}) + \log(z_{t-1})] \\ &= \alpha^2\sigma_{t-1} + \sigma, \end{aligned}$$

which is also an ordinary differential equation with solution given by

$$\sigma_t = \left[\sigma_0 - \frac{\sigma}{1-\alpha^2}\right]\alpha^{2t} + \frac{\sigma}{1-\alpha^2}.$$

Hence, since $0 < \alpha < 1$,

$$\sigma_\infty \equiv \lim_{t\to\infty} \sigma_t = \frac{\sigma}{1-\alpha^2}.$$

Exercise 2.8

First, we will show that $\left\{c_t^*, k_{t+1}^*\right\}_{t=0}^T$, $k_{T+1}^* = 0$ satisfies the consumer's intertemporal budget constraint. By (19) and the definition of f,

$$f(k_t^*) = F(k_t^*, 1) + (1-\delta)k_t^*.$$

Since F is homogeneous of degree one, using $(20)-(22)$ we have that

$$f(k_t^*) = (r_t^* + 1 - \delta)k_t^* + w_t^* = c_t^* + k_{t+1}^*,$$

and hence the present value budget constraint (12) is satisfied for the proposed allocation when prices are given by $(20)-(22)$.

The feasibility constraint (16) is satisfied by construction. Hence, in equilibrium, the first order conditions for the representative household are (for $k_{t+1}^e > 0$)

$$\begin{aligned} \beta^t U'[f(k_t^e) - k_{t+1}^e] &= \lambda p_t, \\ \lambda[(r_{t+1} + 1 - \delta)p_t - p_t] &= 0, \\ f(k_t^e) - c_t^e - k_{t+1}^e &= 0, \end{aligned}$$

for $t = 0, 1, ..., T$. Combining them and using $(20)-(22)$ we obtain

$$\begin{aligned} U'\left[f\left(k_t^e\right) - k_{t+1}^e\right] &= \beta U'\left[f\left(k_{t+1}^e\right) - k_{t+2}^e\right] f'(k_t^e), \\ f(k_t^e) - c_t^e - k_{t+1}^e &= 0, \end{aligned}$$

for $t = 0, 1, ..., T$, which by construction is satisfied by the proposed sequence $\{k_{t+1}^*\}_{t=0}^T$. Hence $\{(c_t^*, k_{t+1}^*)\}_{t=0}^T$, with $k_{T+1}^* = 0$ and $k_0^* = x_0$ solves the consumer's problem.

Finally, we need to show that $\{k_t^*, n_t^* = 1\}_{t=0}^T$ is a maximizing allocation for the firm. Replacing (21) and (22) in (9) and (10) together with the definition of $f(k)$ and the assumed homogeneity of degree 1 of F, we verify that the proposed sequence of prices and allocations satisfy indeed the first-order conditions of the firm, and that $\pi = 0$.

Exercise 2.9

Under the new setup, the household's decision problem is

$$\max_{\{(c_t,n_t)\}_{t=0}^T} \sum_{t=0}^T \beta^t U(c_t)$$

subject to

$$\sum_{t=0}^T p_t c_t \leq \sum_{t=0}^T p_t w_t n_t + \pi;$$

and

$$0 \leq n_t \leq 1, \quad c_t \geq 0, \quad t = 0, 1, ..., T.$$

Similarly, the firm's problem is

$$\max_{\{(k_t,i_t,n_t)\}_{t=0}^T} \pi = p_0(x_0 - k_0) + \sum_{t=0}^T p_t[y_t - w_t n_t - i_t]$$

subject to

$$\begin{aligned} i_t &= k_{t+1} - (1-\delta)k_t, \quad t = 0, 1, ..., T; \\ y_t &\leq F(k_t, n_t), \quad t = 0, 1, ..., T; \\ k_t &\geq 0, \quad t = 0, 1, ..., T; \\ k_0 &\leq x_0, \ x_0 \text{ given.} \end{aligned}$$

Hence, x_0 can be interpreted as the initial stock of capital and k_0 the stock of capital that is effectively put into production, while k_t for $t \geq 1$ is the capital stock that is chosen one period in advance to be the effective capital allocated into production in period t.

As stated in the text, labor is inelastically supplied by households, prices are strictly positive, and the nonnegativity constraints for consumption are never binding, so equation (14) in the text is the first-order condition for the household.

The first-order conditions for the firm's problem are (after substituting both constraints into the objective function)

$$\begin{aligned} w_t - F_n(k_t, n_t) &= 0, \\ -p_t + p_{t+1}[F_k(k_t, n_t) + (1-\delta)] &\leq 0, \end{aligned}$$

for $t = 0, 1, ..., T$, where the latter holds with equality if $k_{t+1} > 0$.

Evaluating the objective function of the firm's problem using the optimal path for capital and labor, we find that first order conditions are satisfied, and $\pi = p_0 x_0$ so the profits of the firm are given by the value of the initial capital stock.

Next, it rest to verify that the quantities given by (17) − (19) and the prices defined by (20) − (22) constitute a competitive equilibrium. The procedure is exactly as in Exercise 2.8. In equilibrium, combining the first-order conditions for periods t and $t+1$ in the household's problem we obtain

$$\begin{aligned} U'[f(k_t) - k_{t+1}] &= \beta U'[f(k_{t+1}) - k_{t+2}]f'(k_{t+1}), \\ f(k_t) - c_t - k_{t+1} &= 0, \end{aligned}$$

for $t = 1, 2, ...T$, as before. Hence the proposed sequences constitutes a competitive equilibrium.

Exercise 2.10

The firm's decision problem remains as stated in (8) (that is, as a series of one-period maximization problems). Let s_t be the quantity of one period bonds held by the representative household. Its decision problem now is

$$\max_{\{(c_t, k_{t+1}, s_{t+1}, n_t)\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t U(c_t)$$

subject to

$$\begin{aligned} c_t + q_t s_{t+1} + k_{t+1} &\leq r_t k_k + (1-\delta)k_t + w_t n_t, \quad t = 0, 1, ..., T \ ; \\ 0 &\leq n_t \leq 1, \quad c_t \geq 0, \qquad\qquad t = 0, 1, ..., T \ ; \end{aligned}$$

and k_0 given.

We assume, as in the text, that the whole stock of capital is supplied to the market. Now, instead of having one present value budget constraint, we have a sequence of budget constraint, one for each period, and we will denote by $\beta^t \lambda_t$ the corresponding Lagrange multipliers.

In addition, we need to add an additional market clearing condition for the bond market that must be satisfied in the competitive equilibrium. This says that bonds are in zero net supply at the stated prices.

Hence, the first-order conditions that characterize the household's problem are

$$\begin{aligned} U'(c_t) - \lambda_t &= 0, \\ -\lambda_t q_t + \beta \lambda_{t+1} &= 0, \\ -\lambda_t + \beta \lambda_{t+1}[r_{t+1} + 1 - \delta] &\leq 0, \\ \text{with equality for } k_{t+1} &\geq 0, \end{aligned}$$

and the budget constraints, for $t = 0, 1, ..., T$.

We show next that the proposed allocations $\{(c_t^*, k_{t+1}^*)\}_{t=0}^T$ together with the sequence of prices given by (21) $-$ (22) and the pricing equation for the bond, constitute a competitive equilibrium. Combining the first and second equations evaluated at the proposed allocation, we obtain the pricing equation

$$q_t = \beta \frac{U'(c_{t+1}^*)}{U'(c_t^*)}.$$

$>$From the first-order conditions of the firm's problem, and after imposing the equilibrium conditions, $r_t = F_k(k_t^*, 1)$., Combining the first-order conditions for consumption and capital for the household's problem, we obtain

$$f'(k_{t+1}^*)^{-1} = \beta \frac{U'(c_{t+1}^*)}{U'(c_t^*)}.$$

The rest is analogous to the procedure followed in Exercise 2.9. Hence, the sequence of quantities defined by (17) $-$ (19), and the prices defined by (21) $-$ (22) plus the bond price defined in the text indeed define a competitive equilibrium.