"A Bayesian Approach to Identification of Structural VAR Models"

GROUP 6

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I Univariate Model

Consider the following AR(2) model:

$$y_t = c + b_1 y_{t-1} + b_2 y_{t-2} + \epsilon_t \tag{1}$$

where the residuals are serially correlated according to:

$$\epsilon_t = \rho \epsilon_{t-1} + \nu_t, \quad \nu_t \sim N(0, \tau^2) \tag{2}$$

a. Write down the Gibbs sampler for this extended model (on paper). Describe each step, the prior specification, and the conditional posterior distributions including the formulas for their moments.

First we know that roots of $(1 - b_1 L - b_2 L^2) = 0$ lie outside the complex unit circle. Then, the matrix notation of the AR(1) model and the serially correlated residuals are given by

$$\mathbf{Y} = \mathbf{X}\mathbf{b} + \mathbf{e}$$

and

$$\mathbf{e} = \mathbf{E}\rho + \boldsymbol{\nu}, \quad \boldsymbol{\nu} \sim N(0, \tau^2 \mathbf{I}_T),$$

where $\mathbf{Y}$ is a $(T \times 1)$ vector of the endogenous variable, $\mathbf{X}$ is a $(T \times 2)$ matrix of exogenous variables, $\mathbf{e}$ is a $(T \times 1)$ vector of the error term, $\mathbf{E}$ is a $(T \times 1)$ vector of the first lag of $\mathbf{e}$, ρ is the persistence of the serially correlated errors, and $\boldsymbol{\nu}$ is the $(T \times 1)$ vector of uncorrelated vector of errors.

We need the conditional equations for b and ρ:

1. Conditional on b, the problem reduces to making inferences on ρ and τ^2 from the following AR(1) model

   $$e_t^* = \rho e_{t-1}^* + \nu_t, \quad \nu_t \sim N(0, \tau^2),$$

   whose matrix form is given by

   $$\mathbf{e}^* = \mathbf{E}^* \rho + \boldsymbol{\nu}^*, \quad \boldsymbol{\nu}^* \sim N(0, \tau^2 \mathbf{I}_{T-1}),$$

   where $e_t^* = y_t - c - b_1 y_{t-1} - b_2 y_{t-2}$.

2. Conditional on ρ and τ^2, the problem reduces to making inferences on b from the following regression model with known variance:

   $$y_t^* = c^* + b_1^* y_{t-1}^* + b_2^* y_{t-2}^* + \nu_t, \quad \nu_t \sim N(0, \tau^2),$$

   whose matrix form is given by

   $$\mathbf{Y}^* = \mathbf{X}^* \mathbf{b} + \boldsymbol{\nu}^*, \quad \boldsymbol{\nu}^* \sim N(0, \tau^2 \mathbf{I}_{T-1}),$$

   and

   $$\begin{aligned} y_t^* &= y_t - \rho y_{t-1} \\ y_{t-1}^* &= y_{t-1} - \rho y_{t-2} \\ y_{t-2}^* &= y_{t-2} - \rho y_{t-3} \\ c^* &= c(1 - \rho) \end{aligned}$$

Now following Chib (1993), we need to derive the full conditional posterior distributions of the model's parameters. Given the following full conditional posterior distributions of $\rho, \mathbf{b}, \tau^2$, Gibbs sampling can easily be implemented:

A. Conditional distributions of b, given ρ and τ

I) Prior distribution of $\mathbf{b}$

$$\mathbf{b}|\rho, \tau^2 \sim N(\mathbf{b}_0, \Sigma_0)$$

Where $\mathbf{b}_0$ and Σ_0 are known. Prior density can be written as:

$$p(\mathbf{b}|\rho,\tau^2) = (2\pi)^{-\frac{k}{2}}|\Sigma|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(\mathbf{b}-\mathbf{b_0})'\Sigma_0^{-1}(\mathbf{b}-\mathbf{b_0})\right\}$$
$$\propto \exp\left\{-\frac{1}{2}(\mathbf{b}-\mathbf{b_0})'\Sigma_0^{-1}(\mathbf{b}-\mathbf{b_0})\right\}$$

II) Likelihood function

$$\mathcal{L}(\mathbf{b}|\rho,\tau^2,Y) = (2\pi\tau^2)^{-\frac{T}{2}} \exp\left\{-\frac{1}{2\tau^2}(\mathbf{Y}^*-\mathbf{X}^*\mathbf{b})'(\mathbf{Y}^*-\mathbf{X}^*\mathbf{b})\right\}$$
$$\propto \exp\left\{-\frac{1}{2\tau^2}(\mathbf{Y}^*-\mathbf{X}^*\mathbf{b})'(\mathbf{Y}^*-\mathbf{X}^*\mathbf{b})\right\}$$

III) Posterior

$$p(\mathbf{b}|\rho,\tau^2,Y) \propto p(\mathbf{b}|\rho,\tau^2)\mathcal{L}(\mathbf{b}|\rho,\tau^2,Y)$$
$$\propto \exp\left\{-\frac{1}{2}(\mathbf{b}-\mathbf{b_0})'\Sigma_0^{-1}(\mathbf{b}-\mathbf{b_0}) - \frac{1}{2\tau^2}(\mathbf{Y}^*-\mathbf{X}^*\mathbf{b})'(\mathbf{Y}^*-\mathbf{X}^*\mathbf{b})\right\}$$

IV) Moments

Location: $\mathbf{b_1} = (\Sigma_0^{-1} + \tau^{-2}\mathbf{X}^{*\prime}\mathbf{X}^*)^{-1}(\Sigma_0^{-1}\mathbf{b}_0 + \tau^{-2}\mathbf{X}^{*\prime}\mathbf{Y}^*)$

Scale: $\Sigma_1 = (\Sigma_0^{-1} + \tau^{-2}\mathbf{X}^{*\prime}\mathbf{X}^*)^{-1}$

B. Conditional distribution of ρ, given $\mathbf{b}$ and τ^2

I) Prior Distribution of ρ

$$\rho|\mathbf{b},\tau^2 \sim N(c_0, B_0)_{I[s(p)]},$$

where c_0, B_0 are known and $I[s(\rho)]$ is an indicator function used to denote that roots of $\rho(L) = 0$ lie outside the unit circle.
Prior density can be written as:

$$p(\rho|\mathbf{b},\tau^2) = (2\pi)^{-\frac{k}{2}}|B_0|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(\rho-c_0)'B_0^{-1}(\rho-c_0)\right\}$$
$$\propto \exp\left\{-\frac{1}{2}(\rho-c_0)'B_0^{-1}(\rho-c_0)\right\}$$

II) Likelihood function

$$\begin{aligned}\mathcal{L}(\rho|\mathbf{b},\tau^2,Y) &= (2\pi\tau^2)^{-\frac{T}{2}}\exp\left\{-\frac{1}{2\tau^2}(\mathbf{e}^*-\mathbf{E}_t^*\rho)'(\mathbf{e}^*-\mathbf{E}_t^*\rho)\right\}\\ &\propto \exp\left\{-\frac{1}{2\tau^2}(\mathbf{e}^*-\mathbf{E}_t^*\rho)'(\mathbf{e}^*-\mathbf{E}_t^*\rho)\right\}\end{aligned}$$

III) Posterior

$$\begin{aligned}P(\rho|\mathbf{b},\tau^2,Y) &\propto p(\rho|\mathbf{b},\tau^2)\mathcal{L}(\rho|\mathbf{b},\tau^2,Y)\\ &\propto \exp\left\{-\frac{1}{2}(\rho-c_0)'B_0^{-1}(\rho-c_0)-\frac{1}{2\tau^2}(\mathbf{e}^*-\mathbf{E}_t^*\rho)'(\mathbf{e}^*-\mathbf{E}_t^*\rho)\right\}\end{aligned}$$

$$\rho|\mathbf{b},\tau^2,Y \sim N(c_1,B_1)_{I[s(p)]} \tag{3}$$

IV) Moments

$$\begin{aligned}\text{Location:}\quad & c_1 = (B_0^{-1}+\tau^{-2}\mathbf{E}^{*\prime}\mathbf{E}^*)^{-1}(B_0^{-1}c_0+\tau^{-2}\mathbf{E}^{*\prime}\mathbf{e}^*)\\ \text{Scale:}\quad & B_1 = (B_0^{-1}+\tau^{-2}\mathbf{E}^{*\prime}\mathbf{E}^*)^{-1}\end{aligned}$$

C. **Conditional distribution of τ^2, given b and ρ:** First we have that:

$$z_i \overset{iid}{\sim} N\left(0,\frac{1}{\delta}\right)$$

$$W=\sum_{i=1}^{\nu} z_i^2 \sim \Gamma\left(\frac{\nu}{2},\frac{\delta}{2}\right)$$

$$p(W)\propto W^{\frac{t}{2}-1}\exp\left\{-\frac{W\delta}{2}\right\},$$

with $E[W]=\nu\delta^{-1}$ and $\mathrm{Var}(W)=2\nu\delta^{-2}$.

I) Prior distribution of τ^2:

$$\frac{1}{\tau^2}\Big|\mathbf{b},\rho \sim \Gamma\left(\frac{\nu_0}{2},\frac{\delta_0}{2}\right)$$

$$p\left(\frac{1}{\tau^2}\Big|\mathbf{b},\rho\right)\propto\left(\frac{1}{\tau^2}\right)^{\frac{\nu_0}{2}-1}\exp\left\{-\frac{\delta_0}{2\tau^2}\right\},$$

where ν_0 and δ_0 are known.

II) Likelihood function

$$\begin{aligned}\mathcal{L}\left(\frac{1}{\tau^2}\Big|\rho,\mathbf{b},Y\right) &= (2\pi\tau^2)^{-\frac{T}{2}}\exp\left\{-\frac{1}{2\tau^2}(\mathbf{Y}^*-\mathbf{X}^*\mathbf{b})'(\mathbf{Y}^*-\mathbf{X}^*\mathbf{b})\right\}\\ &\propto (\tau^2)^{-\frac{T}{2}}\exp\left\{-\frac{1}{2\tau^2}(\mathbf{Y}^*-\mathbf{X}^*\mathbf{b})'(\mathbf{Y}^*-\mathbf{X}^*\mathbf{b})\right\}\end{aligned}$$

III) Posterior

$$\begin{aligned}
p\left(\frac{1}{\tau^2}\middle|\rho,\mathbf{b},Y\right) &\propto p\left(\frac{1}{\tau^2}\middle|\mathbf{b},\rho\right)\mathcal{L}\left(\frac{1}{\tau^2}\middle|\rho,\mathbf{b},Y\right)\\
&\propto (\tau^2)^{\frac{\nu_0}{2}+\frac{T}{2}-1}\exp\left\{-\frac{1}{2\tau^2}(\delta_0+(\mathbf{Y}^*-\mathbf{X}^*\mathbf{b})'(\mathbf{Y}^*-\mathbf{X}^*\mathbf{b}))\right\}\\
&\propto \left(\frac{1}{\tau^2}\right)^{\frac{\nu_1}{2}-1}\exp\left\{-\frac{\delta_1}{2\tau^2}\right\}
\end{aligned}$$

$$\frac{1}{\tau^2}\middle|\rho,\mathbf{b},Y\sim\Gamma\left(\frac{\nu_1}{2},\frac{\delta_1}{2}\right) \tag{4}$$

IV) Moments

$$\begin{aligned}
\text{Location:}\quad & \nu_1=\nu_0+T\\
\text{Scale:}\quad & \delta_1=\delta_0+(\mathbf{Y}^*-\mathbf{X}^*\mathbf{b})'(\mathbf{Y}^*-\mathbf{X}^*\mathbf{b})
\end{aligned}$$

Now, we can proceed with the Gibbs sampling algorithm. Starting the iteration with an arbitrary starting values, $\rho=\rho^{(0)}$ and $\tau^2=\{\tau^2\}^{(0)}$. Then the following is iterated for Gibbs simulations:

1. Conditional on $\rho=\rho^{(j-1)}$ and $\tau^2=\{\tau^2\}^{(j-1)}$, generate $\mathbf{b}^{(j)}$ from the conditional posterior distribution in (3).

2. Conditional on $\tau^2=\{\tau^2\}^{(j-1)}$ and $\mathbf{b}=\mathbf{b}^{(j)}$, generate $\rho^{(j)}$ from the conditional posterior distribution in (4).

3. Conditional on $\mathbf{b}=\mathbf{b}^{(j)}$ and $\rho=\rho^{(j)}$, generate $\{\tau^2\}^{(j)}$ from the conditional posterior distribution in (5).

4. Set $(j)=(j-1)$, and go to first step of the algorithm until we have $J+B$ draws.

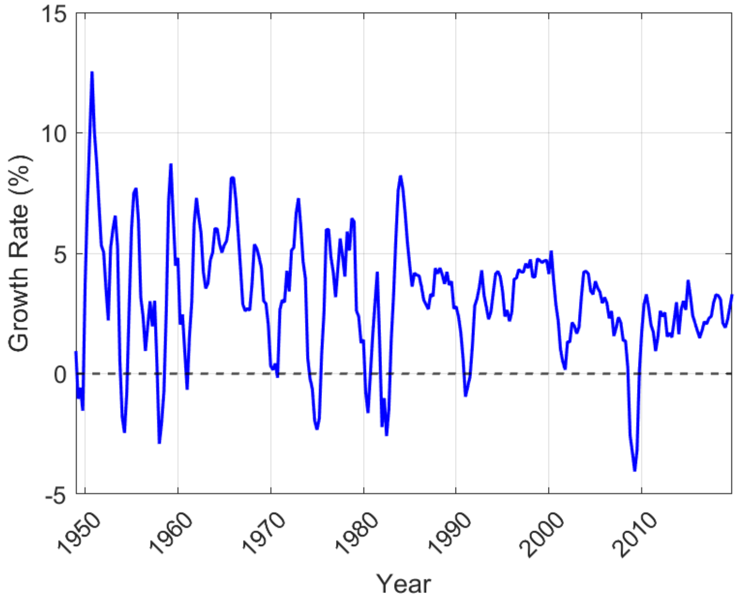
5. Finally we have to discard the firsts B draws, because the influence of the starting values. And we keep the $\{\mathbf{b}^{(j)},\rho^{(j)},\{\tau^2\}^{(j)}\}_{j=B}^{(J+B)}$ draws.

b. Write Matlab code to estimate this model for the annual growth rate of real GDP for the US (FRED code: GDPC1) over the period 1948Q1 to 2019Q4 using the Gibbs sampler described in (a). Provide the following graphs:

i. Plot the annual growth rate of real GDP (in percent).

In Fig. 1 we have the series for the annual growth rate of real GDP.

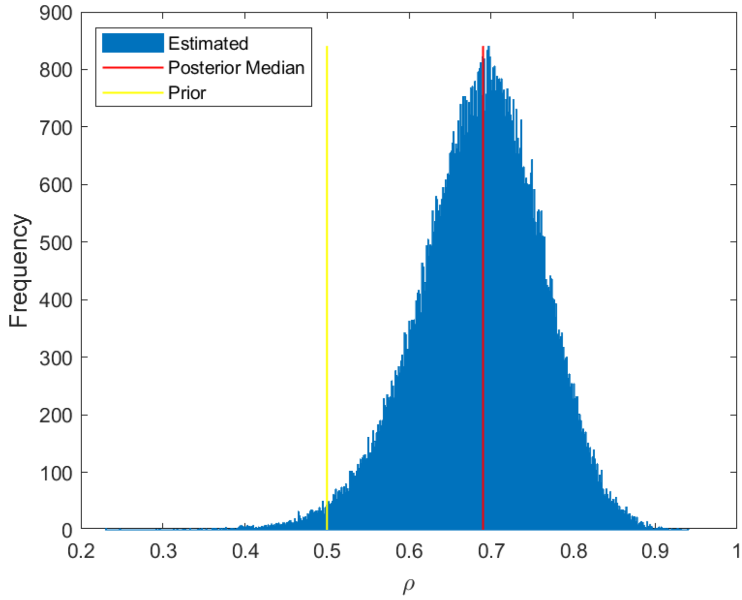
Figure 1: Annual Growth of Real GDP



ii. Plot the histogram of the (marginal) posterior distribution for ρ and report its posterior median.

As observed in Fig. 2, the median of the posterior distribution is close to the prior we used. Additionally, the distribution exhibits a pattern where the highest accumulation occurs at the two extreme values.

Figure 2: Posterior distribution of ρ

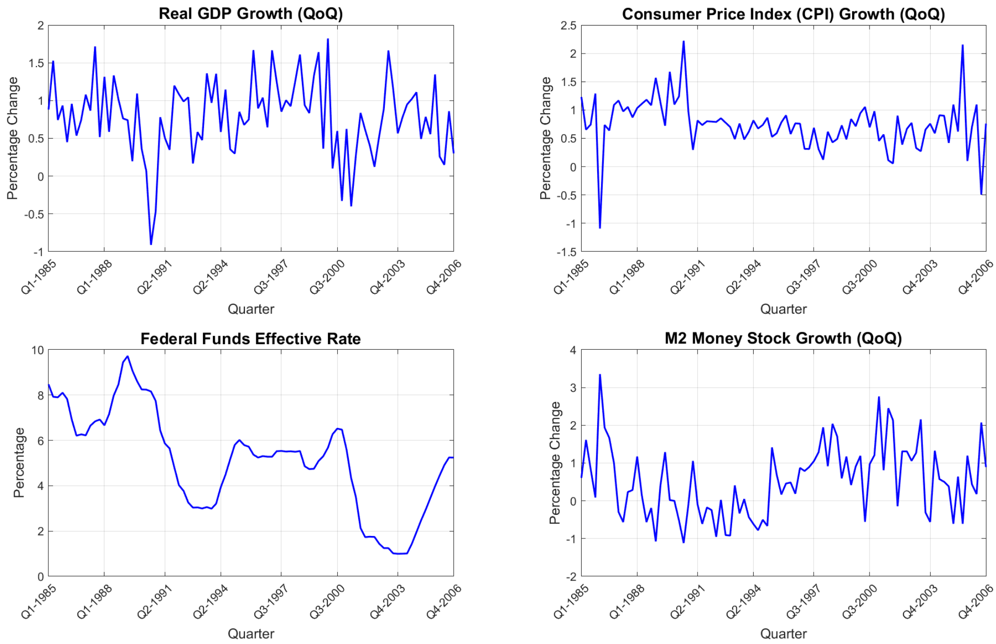


II Multivariate Models

Collect from FRED the following variables for the US economy at quarterly frequency for the period 1985Q1-2006Q4: real GDP, the consumer price index (CPI), the effective federal funds rate, and the M2 money stock.

1. **Transform the data to quarter-on-quarter growth rates where appropriate so that they have a useful economic interpretation. Plot the transformed data with appropriate labels.**

Figure 3: Evolution of variables Quarter on Quarter



We have that Real GDP Growth (Quarter-on-Quarter) because we need a stable model. It can be seen that the GDP growth rate has been fluctuating around 1% in all periods except between the years 1990-1991, where a recession is observed due to the impact of the Gulf War, which generated economic uncertainty. Later, we can also denote the 2001 recession, caused by the collapse of the "dot-com bubble".

In the case of the Federal Funds Effective Rate, we can observe the different scenarios that characterize the period. First, we see high rates to control inflation in the late 1980s. Then, there is a drastic reduction in rates to stimulate the economy due to the 1990-1991 recession. Next, we notice a monetary tightening aimed at preventing economic overheating starting in 1994. Later, in the years 2001-2003, there are drastic cuts due to the dot-com recession. Finally, we see monetary tightening by the FED to prevent inflationary pressures.

In the case of Consumer Price Index (CPI) growth (QoQ), it can be observed that at the beginning of the period, there was significant inflation fluctuation due to rising oil prices and expansionary monetary policies. During the 1990-1991 period, inflation dropped rapidly due to the FED's response. Then, we note a period of stability until a sudden increase in inflation in 2003, driven by economic recovery following the 2001 recession and the dot-com bubble, as well as rising commodity and energy prices.

For the growth of the M2 Money Stock Growth (QoQ), we notice several peaks. By the late 1980s and in 1991, we observe a monetary expansion to counteract the recession as a response by the FED. Later, in the period from 2001 to 2003, another monetary expansion is evident following the dot-com crisis, as it was necessary to stimulate the economy and prevent a confidence crisis.

2. Consider a bivariate VAR model for real GDP growth and CPI inflation.

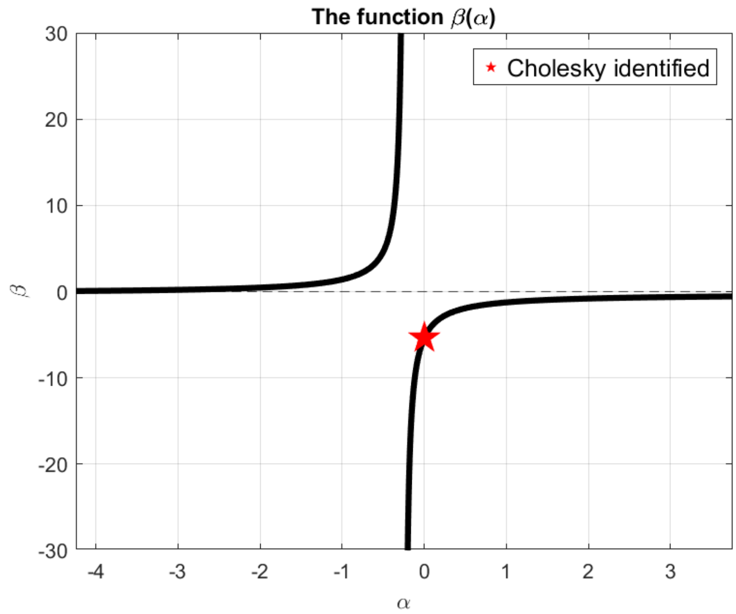
a. Write Matlab code to estimate a reduced-form VAR(4) model with a constant term using OLS. Report the (point) estimates of the reduced-form covariance matrix (omega) and the $(k \times n)$ matrix of reduced-form coefficients where n is the number of endogenous variables

$$\hat{\beta} = \begin{bmatrix} 0.8197 & 0.2399 \\ 0.1555 & 0.0219 \\ -0.0240 & 0.0396 \\ 0.2314 & 0.2008 \\ -0.0486 & 0.2878 \\ -0.1474 & 0.0028 \\ -0.2503 & 0.3347 \\ 0.0948 & -0.0684 \\ -0.0894 & -0.1614 \end{bmatrix}$$

$$\hat{\Omega} = \begin{bmatrix} 0.2076 & -0.0387 \\ -0.0387 & 0.1584 \end{bmatrix}$$

b. Assume that there is a supply-demand model that determines the fluctiations in output and inflation but that you do not know the values of the contemporaneous structural parameters that characterize that model. Plot the identified set for all possible values that are compatible with the observed data.

Figure 4: Set of all possible values compatible with the data



3. Now add the federal funds rate to your bivariate model.

a. Fit a VAR(4) to those data ordered as follows: output, inflation, interest rate. Check whether the model is stable. How can you tell? Show the output you use to determine stability of the system.

After fitting a trivariate VAR(4) with output, inflation, and interest rate, we obtained the following results:

$$A\mathbf{Y}_t = B_1\mathbf{Y}_{t-1} + B_2\mathbf{Y}_{t-2} + B_3\mathbf{Y}_{t-3} + B_4\mathbf{Y}_{t-4} + \varepsilon_t$$

Where:

$$\mathbf{Y}_t = \begin{bmatrix} y_t \\ \pi_t \\ i_t \end{bmatrix}$$

Table 1 shows the eigenvalues.

In VAR models analysis, we verify stability by checking if the eigenvalues lie inside the unit circle, which implies that their absolute values are less than 1 ($|\lambda| < 1$) (Lütkepohl, 2005). In this case, all the absolute values of the eigenvalues are indeed less than 1, indicating that our trivariate VAR(4) model is stable.

Table 1: VAR Eigenvalues

Eigenvalues
-0.5225 + 0.5275i
-0.5225 - 0.5275i
-0.6147 + 0.0000i
-0.2865 + 0.0000i
0.1051 + 0.5478i
0.1051 - 0.5478i
0.8763 + 0.1226i
0.8763 - 0.1226i
0.6213 + 0.3541i
0.6213 - 0.3541i
0.1032 + 0.0000i
0.3706 + 0.0000i

Figure 5: Stability Check: Eigenvalues of the Companion Matrix

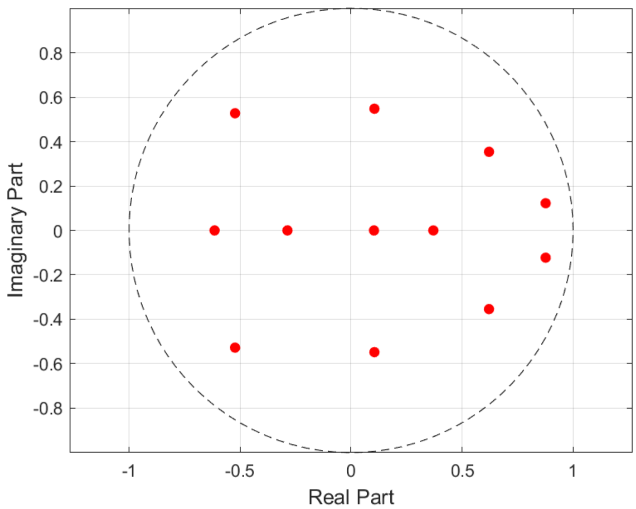


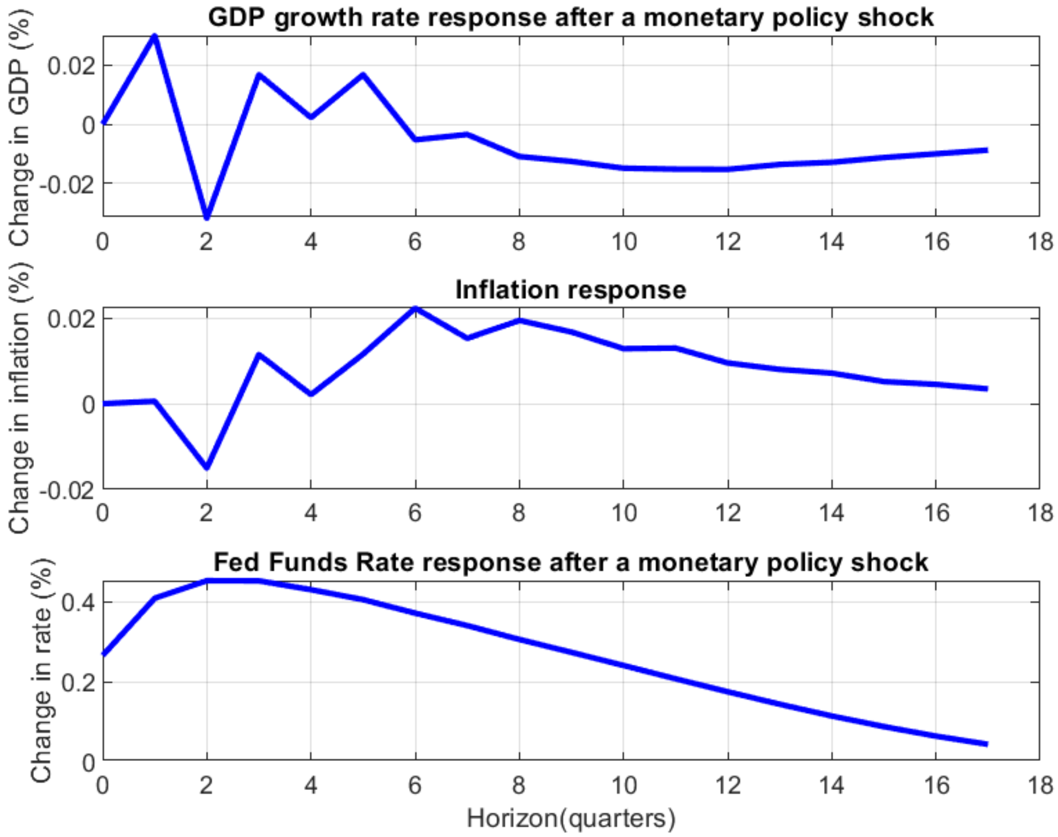
Fig. 7 illustrates this stability condition, showing that all eigenvalues fall within the unit circle. Therefore, we can conclude that, with a stable VAR model, the real GDP growth rate, inflation, and the Federal Funds growth rate do not lead to explosive dynamics in the VAR model.

b. Apply the Choleski decomposition for identification. Plot the impulse responses of the three variables after a monetary policy shock (just the point estimates, without error bands). What do you find? Briefly comment on your results.

Fig. 6 presents all the impulse-response functions after a monetary policy shock.

First, we observe the responses of real GDP growth following a monetary policy shock.

Figure 6: Impulse response after a monetary policy shock



An increase in interest rates negatively impacts real GDP growth, although this effect appears with delays and fluctuations over time. Initially, a positive shock to interest rates leads to a slight increase in GDP during the first period. However, this is followed by a significant decline in the second period, with further fluctuations until the economy reaches equilibrium. The initial increase may reflect the expectations of economic agents, while the changes in subsequent periods are likely due to the tightening of monetary policy, which is expected to dissipate over time.

In second place, we examine the effects of inflation. Monetary policy successfully reduces inflation, but this process involves a degree of inertia and an initial, temporary reaction before stabilization occurs. Initially, inflation does not respond to the shock; it increases during the second period, then decreases in the third period. After the fourth period, we observe another rise in inflation that gradually diminishes over time. This initial response illustrates the traditional monetary transmission mechanism, where an increase in interest rates leads to a reduction in aggregate demand, and consequently, inflation. Following this, we note adjustments in expectations and convergence over the long term.

Finally, we observe the response of the Federal Funds rate. In this context, monetary policy has persistent, but not permanent, effects. Eventually, the interest rate returns to its equilibrium level. In the first two periods, the interest rate rises rapidly after the shock, which is expected as it reflects the immediate impact of monetary policy. Following this, the rate remains elevated for several periods before starting to decline. This suggests that monetary policy maintains a restrictive stance for a while before easing. At the end of the horizon it gradually returns to its original level, indicating that the central bank does not sustain the monetary policy shock permanently.

In conclusion, the results can be attributed to agents' expectations; however, further accuracy is needed. We should consider exploring additional methods to validate these responses and adding confidence bars. In this case, a Bayesian approach could offer a clearer understanding of the impulse response functions (IRF).

c. Compute and report the coefficients on output and inflation in the Taylor rule implied by the estimated model.

In Table 2 we present the estimated coefficients of implied the Taylor rule.

Table 2: Taylor Rule Coefficients on Output and Inflation

Variable	Coefficient
Output	0.0159
Inflation	-0.2649

So, the Taylor Rule will be

$$i_t = 0.0159 y_t - 0.2649 \pi_t$$

4. Now add money to your set of variables. Write down the structural equations that describe these 4 variables. Provide an economic interpretation for each equation and the corresponding contemporaneous structural parameters (A matrix)

Given a SVAR with the following form:

$$A\mathbf{Y}_t = B_1\mathbf{Y}_{t-1} + B_2\mathbf{Y}_{t-2} + B_3\mathbf{Y}_{t-3} + B_4\mathbf{Y}_{t-4} + \varepsilon_t$$

The vector Y_t is composed by:

$$\mathbf{Y}_t = \begin{bmatrix} y_t \\ \pi_t \\ i_t \\ m_t \end{bmatrix}$$

Where y_t is the GDP growth rate, π_t is the inflation rate, i_t is the federal funds rate and m_t is the M2 growth rate.

For the structural equation, we adhere to the traditional IS-LM framework, which assumes that more than 50% of the time, the central bank controls the money supply other than interest rates.

This leave our SVAR composed by 4 structural equations:

$$\begin{aligned} \text{IS equation:} \quad & y_t = \alpha + \alpha^{\pi}\pi_t + \alpha^{i}i_t + \alpha^{m}m_t + \alpha^{X}\mathbf{X_{t-1}} + \mu_t^{y} \\ \text{Phillips curve:} \quad & \pi_t = \beta + \beta^{y}y_t + \beta^{i}i_t + \beta^{m}m_t + \beta^{X}\mathbf{X_{t-1}} + \mu_t^{\pi} \\ \text{Taylor rule:} \quad & i_t = \theta + \theta^{y}y_t + \theta^{\pi}\pi_t + \theta^{m}m_t + \theta^{X}\mathbf{X_{t-1}} + \mu_t^{i} \\ \text{M2 money stock:} \quad & m_t = \gamma + \gamma^{y}y_t + \gamma^{\pi}\pi_t + \gamma^{i}i_t + \gamma^{X}\mathbf{X_{t-1}} + \mu_t^{m} \end{aligned}$$

Where $\mathbf{X}_{t-1}$ are control variables and correspond to lagged values of all included variables. Thus $\mathbf{AY}_t$ matrix looks like.

$$\mathbf{AY}_t = \begin{bmatrix} 1 & -\alpha^{\pi} & -\alpha^{i} & -\alpha^{m} \\ -\beta^{y} & 1 & -\beta^{i} & -\beta^{m} \\ -\theta^{y} & -\theta^{\pi} & 1 & -\theta^{m} \\ -\gamma^{y} & -\gamma^{\pi} & -\gamma^{i} & 1 \end{bmatrix} \begin{bmatrix} y_t \\ \pi_t \\ i_t \\ m_t \end{bmatrix}$$

We can interpret the structural equations following a Cholesky decomposition on the VAR(4) model, so first imposing zero restrictions for α^{π}, α^{i} and α^{m} we have the IS equation that describes GDP is the most exogenous variable and only depends to the lags of the other variables and not of the contemporaneous, this is because y_t doesn't response instantaneously to inflation or monetary policy shocks, we have that Bernanke and Blinder (1992) show that monetary policy effects on economic activity occur with lags.

Then imposing zero restrictions for β^{i} and β^{m} we have the Phillips curve that describes inflation rate only depends on contemporaneous GDP and the lags. This is because economic growth can generate inflationary pressures through increased aggregate demand and intensive use of production factors and the effects of monetary policy on inflation occur with lags Taylor (1993) and money supply does not immediately affect inflation in the short term due to money velocity and liquidity preference (Friedman, 1968). We have that Christiano, Eichenbaum, and Evans (1999) show that shock to GDP growth can immediately impact inflation and shock to interest rates or M2 does not immediately affect inflation in the "lags in monetary policy".

So imposing zero restriction for θ^{m} we have the Taylor rule regime, adjusting its interest rate in response to inflation and GDP fluctuation. We have that Clarida, Galí, and Gertler (2000) provide evidence that central bank react quickly to inflation and economic activity shocks to stabilize the economy.

Finally we have the money supply equation, where m_t is the most endogenous, responding to money and credit demand (Bernanke & Blinder, 1988). The bank credit channel suggest that banks adjust credit and monetary aggregates based on GDP, inflation, or interest rate shocks (Kashyap & Stein, 2000).

Under this assumptions, the model is just identified and we can solve for the structural parameters.

$$\mathbf{AY}_t = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\beta^{y} & 1 & 0 & 0 \\ -\theta^{y} & -\theta^{\pi} & 1 & 0 \\ -\gamma^{y} & -\gamma^{\pi} & -\gamma^{i} & 1 \end{bmatrix} \begin{bmatrix} y_t \\ \pi_t \\ i_t \\ m_t \end{bmatrix}$$

5. Estimate this model using the Baumeister-Hamilton algorithm implementing the Choleski identification in that framework and uninformative priors for B and D.

So we have:

$$\mathbf{b}_i|\mathbf{A}, \mathbf{D} \sim N(m_i, d_{ii}M_i)$$

Where: $M_i^{-1} \to 0$ (Uninformative prior).
And:

$$d_{ii}^{-1}|\mathbf{A} \sim \Gamma(\kappa_i, \tau_i)$$

Where: $\kappa_i, \tau_i \to 0$ (Uniformative prior).

a. Write down (on paper) the prior for each element in A as well as the joint prior for A.

We have that:

$$\mathbf{AY}_t = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\beta^y & 1 & 0 & 0 \\ -\theta^y & -\theta^\pi & 1 & 0 \\ -\gamma^y & -\gamma^\pi & -\gamma^i & 1 \end{bmatrix} \begin{bmatrix} y_t \\ \pi_t \\ i_t \\ m_t \end{bmatrix}$$

We estimate the SVAR using the Choleski identification in a Bayesian framework. Then following with the answer of question 4 we known that the coefficients above the main diagonal are imposed to be 0, so we have to set a dogmatic prior with parameters center at 0 and a zero variance. And for the coefficients under the diagonal, we set an uninformative prior. So for the rest of parameters of the contemporaneous matrix we assume they follow a t student distribution with location parameter $c = 0$, scale parameter $\sigma = 100$ and degrees of freedom $\nu = 3$. Note that we set a high value for the variance which accounts for the fact that we assume we only have "defuse knowledge about the parameters". The consequence of this assumption is that the estimates of these six coefficients are going to be data-driven.

$$p(\beta^y) = \frac{1}{\sigma}\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})}\frac{1}{\sqrt{\nu\pi}}\left(1 + \frac{(\beta^y - c)^2}{\nu\sigma^2}\right)^{-\frac{\nu+1}{2}}$$

$$p(\theta^y) = \frac{1}{\sigma}\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})}\frac{1}{\sqrt{\nu\pi}}\left(1 + \frac{(\theta^y - c)^2}{\nu\sigma^2}\right)^{-\frac{\nu+1}{2}}$$

$$p(\theta^\pi) = \frac{1}{\sigma}\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})}\frac{1}{\sqrt{\nu\pi}}\left(1 + \frac{(\theta^\pi - c)^2}{\nu\sigma^2}\right)^{-\frac{\nu+1}{2}}$$

$$p(\gamma^y) = \frac{1}{\sigma}\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})}\frac{1}{\sqrt{\nu\pi}}\left(1 + \frac{(\gamma^y - c)^2}{\nu\sigma^2}\right)^{-\frac{\nu+1}{2}}$$

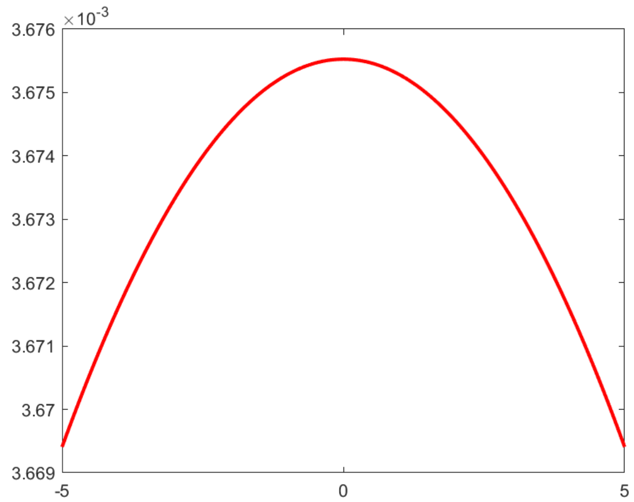
$$p(\gamma^\pi) = \frac{1}{\sigma}\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})}\frac{1}{\sqrt{\nu\pi}}\left(1 + \frac{(\gamma^\pi - c)^2}{\nu\sigma^2}\right)^{-\frac{\nu+1}{2}}$$

$$p(\gamma^i) = \frac{1}{\sigma}\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})}\frac{1}{\sqrt{\nu\pi}}\left(1 + \frac{(\gamma^i - c)^2}{\nu\sigma^2}\right)^{-\frac{\nu+1}{2}}$$

Then, the matrix of contemporaneous **A** has the following prior:

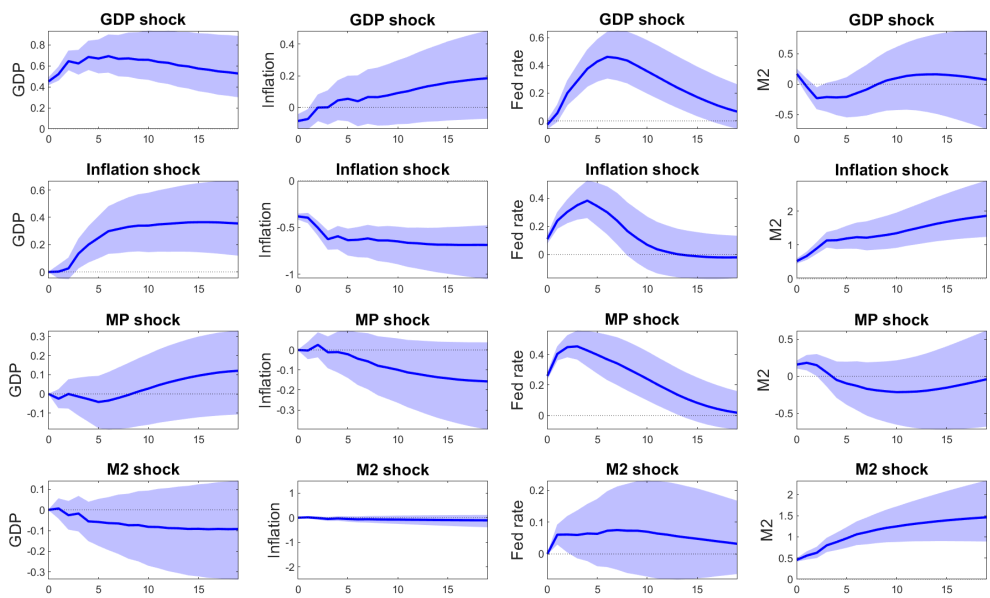
$$p(\mathbf{A}) = p(\beta^y)p(\theta^y)p(\theta^\pi)p(\gamma^y)p(\gamma^\pi)p(\gamma^i)$$

Figure 7: Prior distributions of all coeficients



b. Plot the impulse responses to the one-standard-deviation structural shocks (median together with 16^{th} and 84^{th} percentiles of the posterior distribution) for a horizon of 5 years.

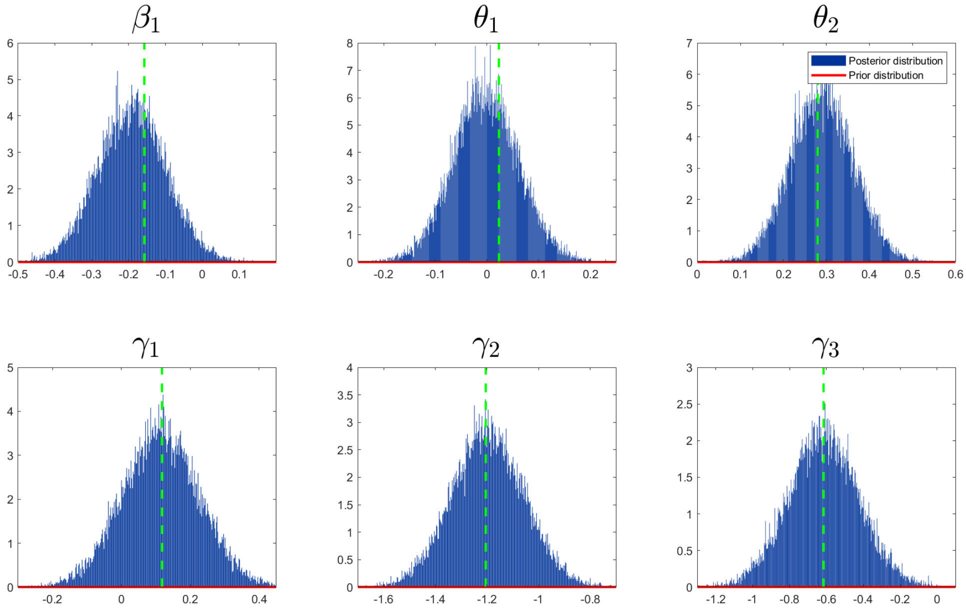
Figure 8: Impulse Responses from various shocks



The impulse responses analysis from the estimated VAR model suggests that, in response to a supply shock (proxied by a shock to the CPI), the central bank should raise its policy interest rate. This result is somewhat controversial, as it challenges the optimal policy response suggested by Clarida et al. (2000). Their framework emphasizes that a central bank should not react aggressively to supply shocks, as doing so could amplify output volatility without necessarily anchoring inflation expectations more effectively. Instead, they argue for a more flexible and forward-looking approach that prioritizes inflation stability over the medium term rather than an immediate tightening of monetary policy. In this scenario, monetary policy responses are more effective when inflation expectations exhibit high volatility. However, since our VAR model does not explicitly account for inflation expectations, this leads to the results observed above. The omission of inflation expectations may introduce bias or imprecision in capturing the full dynamics of monetary policy transmission, highlighting a potential limitation of our approach. Future extensions could incorporate expectation-driven shocks to enhance the model's accuracy in assessing monetary responses under uncertain inflationary environments.

c. Plot the posterior distributions for the contemporaneous structural coefficients. Are the estimates (magnitudes and signs) consistent with the economic interpretation that you provided under (4)?

Figure 9: Posterior distributions of the coeficients

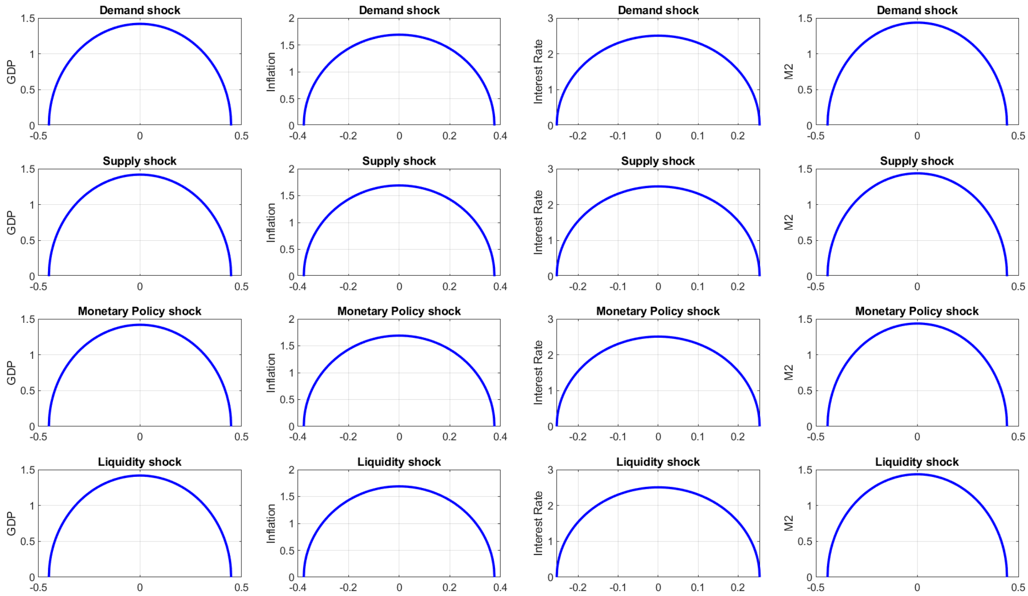


We can see that the posterior distributions of our parameters are analytically similar to their prior distributions. Thus, we also have Student's t-distributions in our posteriors.

6. Suppose you wanted to identify the shocks underlying this 4-variable model using traditional sign-restriction algorithm - but without imposing any signs.

a. Provide a plot for the impact effect of a one-standard deviation shock using the analytical expression for the implicit prior distribution.

Figure 10: Analytical Impact Effect of one-standard deviation shock



The analytical expression of the impact effect of a one-standard deviation is defined in the next way:

First,

$$q_{1i}^2 \sim \text{Beta}\left(\frac{1}{2}, \frac{n-1}{2}\right)$$

and we have that

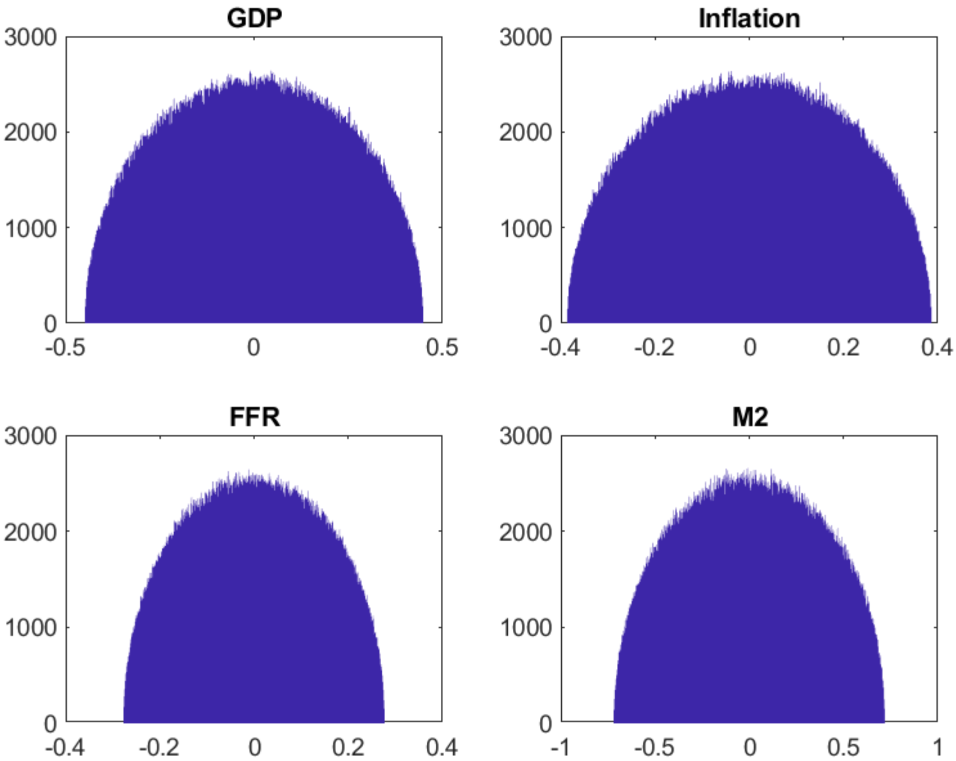
$$p(q_{i1}) = \begin{cases} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}(1-q_{i1}^2)^{\frac{n-3}{2}}, & \text{if } q_{i1} \in [-1, 1] \\ 0, & \text{otherwise} \end{cases}$$

So we can estimate the impact effect of of a one-standard deviation shock with our q_{i1} and our matrix of covariance Ω:

$$h_{11} = p_{11} q_{11} = \sqrt{\omega}\, q_{11}$$

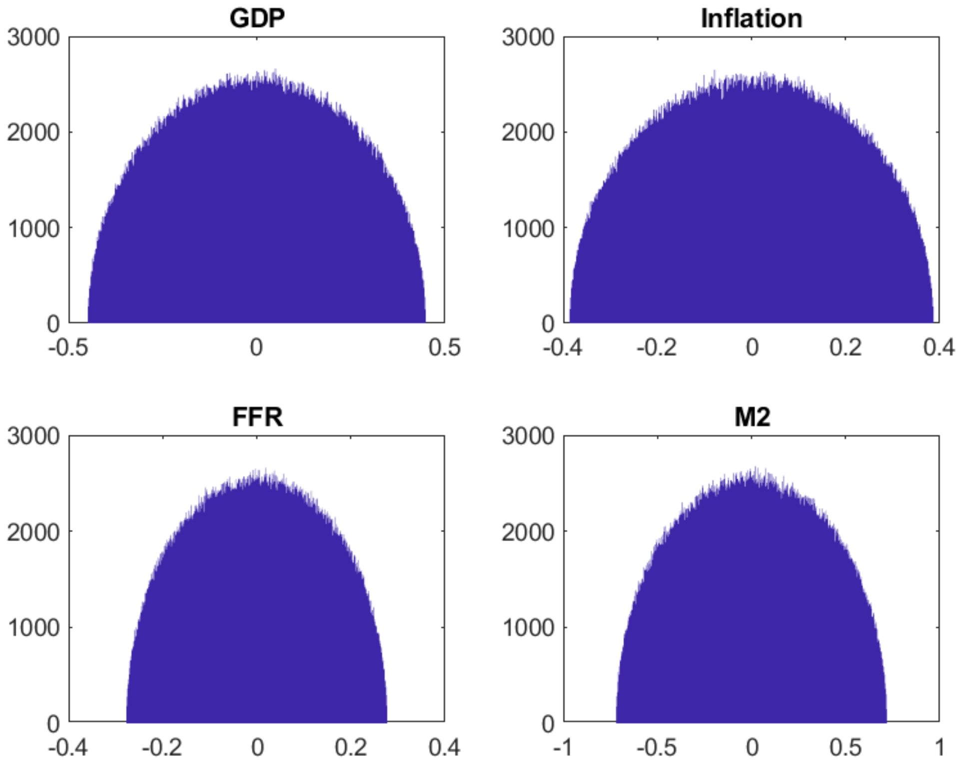
b. Verify empirically what the impact effect for each variable looks like. Report plots of the impact effects and provide the numerical values for the cut-off points.

Figure 11: Impact effects of a demand shock



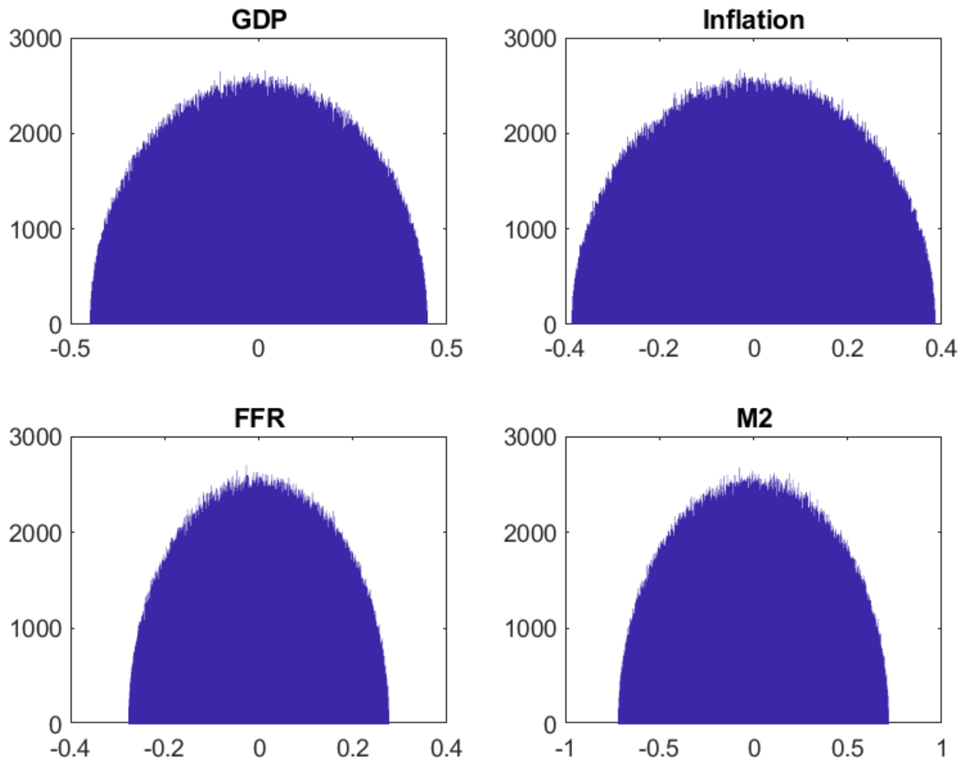
We can see that the impact effect of a one-standard deviation demand shock is very similar to our analytical expression.This is coherent to the theory without signs restrictions.

Figure 12: Impact effects of a monetary policy shock



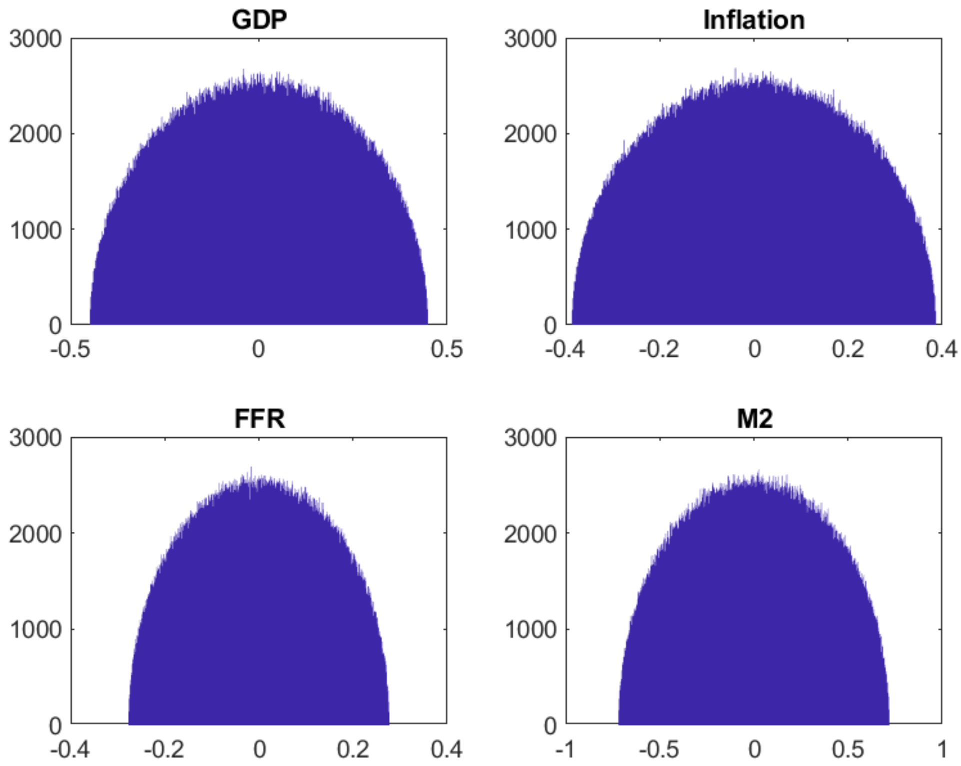
We can see that the impact effect of a one-standard deviation monetary policy shock is very similar to our analytical expression.This is coherent to the theory without signs restrictions.

Figure 13: Impact effects of a liquidity shock



We can see that the impact effect of a one-standard deviation liquidity shock is very similar to our analytical expression.This is coherent to the theory without signs restrictions.

Figure 14: Impact effects of a supply shock



We can see that the impact effect of a one-standard deviation supply shock is very similar to our analytical expression.This is coherent to the theory without signs restrictions.

Table 3: Impact Effect Statistics of Shocks on Economic Variables

Variable	Statistic	Shock Type			
		Demand Shock	Supply Shock	Monetary Policy Shock	Liquidity Shock
GDP	Min	-0.4489	-0.4488	-0.4489	-0.4489
	Max	0.4489	0.4489	0.4489	0.4488
	Median	0.0000	0.0006	-0.0003	-0.0007
Inflation	Min	-0.3866	-0.3866	-0.3866	-0.3866
	Max	0.3866	0.3866	0.3866	0.3866
	Median	0.0008	0.0001	0.0006	0.0006
FFR	Min	-0.2772	-0.2773	-0.2773	-0.2772
	Max	0.2773	0.2773	0.2772	0.2773
	Median	-0.0002	0.0000	0.0007	-0.0001
M2	Min	-0.7197	-0.7196	-0.7197	-0.7197
	Max	0.7197	0.7197	0.7197	0.7197
	Median	-0.0012	0.0000	-0.0012	-0.0009

III References

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