Macro, Money and Finance: A Continuous-Time Approach

Problem Set

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1. Consider an infinitely-lived household with logarithmic preferences over consumption $\{c_t\}_{t\geq 0}$,

$$U_0 = \mathbb{E}\left[\int_0^\infty e^{-\rho t}\log(c_t)dt\right] \tag{1}$$

The household has initial wealth $n_0 > 0$ and does not receive any endowment or labor income. Wealth can be invested into two assets:

- A risk-free bond with (instantaneous) return $r^b dt$.
- A risky stock with return $r^s dt + \sigma dZ_t$, where Z_t is a Brownian motion.

Here, r^b, r^s, and σ are constant parameters.

The household's net worth evolution is given by:

$$dn_t = -c_t dt + n_t\left[(1-\theta_t^s)r^b dt + \theta_t^s(r^s dt + \sigma dZ_t)\right] \tag{2}$$

where θ_t^s denotes the fraction of wealth invested into the stock. The household chooses consumption $\{c_t\}_{t\geq 0}$ and portfolio shares $\{\theta_t^s\}_{t\geq 0}$ to maximize utility U_0 subject to the net worth evolution and a solvency constraint $n_t \geq 0$.

(a) In this part, you will solve the consumption-portfolio choice problem using the Hamilton-Jacobi-Bellman (HJB) equation. The state space of this decision problem is one-dimensional with state variable n_t, so you can denote the household's value function by $V(n)$.

i. Write down the (deterministic) HJB equation for the value function $V(n)$.
La ecuación principal es:

$$\rho V(n) = \log(c) + E[V(n)]$$

Para resolver esto, tomamos la derivada temporal esperada:

$$\frac{E[V(n)]}{dt} = \frac{V_n\, dn + \frac{1}{2}V_{nn}\,(dn)^2}{dt}$$

$$V_n\, dn = V_n\left((-c + n[(1-\theta^s)r^b + r^s\theta^s])\, dt + n\sigma\theta^s\, dZ\right)$$

Y el término cuadrático:

$$V_{nn}\,(dn)^2 = V_{nn}\,(n\sigma\theta^s)^2\, dt$$

Combinando todo,

$$\frac{E[V(n)]}{dt} = \frac{V_n(-c + n[(1-\theta^s)r^b + r^s\theta^s])dt + \frac{1}{2}V_{nn}\,(n\sigma\theta^s)^2\, dt}{dt}$$

Simplificando:

$$\frac{E[V(n)]}{dt} = V_n\left(-c + n[(1-\theta^s)r^b + r^s\theta^s]\right) + \frac{1}{2}V_{nn}(n\sigma\theta^s)^2$$

Finalmente, resolviendo para $V(n)$:

$$\rho V(n) = \log(c) + V_n\left(-c + n[(1-\theta^s)r^b + r^s\theta^s]\right) + \frac{1}{2}V_{nn}(n\sigma\theta^s)^2$$

ii. Take first-order conditions with respect to all choice variables.

$$\rho V(n) = \max_c\{\log(c) - V_n c\} + \max_{\theta^s}\left\{V_n\, n\,(1-\theta^s)r^b + r^s\theta^s + \frac{1}{2}V_{nn}\,(n\sigma\theta^s)^2\right\}$$

Condición de Óptimo de Consumo
Para c^*, tenemos:

$$\frac{\partial\rho V(n)}{\partial c} = \frac{1}{c} - V_n = 0$$

De donde se obtiene:

$$\frac{1}{c} = V_n \quad \Rightarrow \quad c = \frac{1}{V_n}$$

Condición de Óptimo para θ^s
La ecuación de primer orden con respecto a θ^s es:

$$\frac{\partial \rho V(n)}{\partial \theta^s} = V_n(-nr^b + r^s n) + V_{nn}(n\sigma)^s \theta^s = 0$$

Despejando θ^s:

$$\theta^s = \frac{nr^b - r^s n^s}{(n\sigma)^2} \frac{V_n}{V_{nn}}$$

iii. Assume optimal consumption is proportional to net worth, $c(n) = an$, with some constant $a > 0$ (to be determined below). Use the first-order condition for consumption derived in part (b) to turn this into a guess for the value function $V(n)$. *Hint: Don't forget to add an integration constant (call it b) when moving from $V'(n)$ to $V(n)$; $V(n)$ is the sum of two terms.* Asumimos que en el óptimo se cumple:

$$\frac{1}{c} = V_n$$

Despejando:

$$c = \frac{1}{V_n}$$

Para la función de valor $V(n)$, integramos V_n:

$$V = \int V_n \, dn = \int \frac{1}{an} \, dn$$

Resolviendo la integral:

$$V(n) = \frac{1}{a} \log(n) + b$$

donde b es una constante de integración.
**Condición de Primer Orden (CPO)
De la ecuación de la CPO:

$$\theta^s = \frac{nr^b - r^s n^s}{(n\sigma)^2 V_{nn}}$$

Usamos las derivadas de $V(n)$:

$$V_n = \frac{1}{an}, \quad V_{nn} = \frac{-1}{an^2}$$

Sustituyendo en la ecuación de la CPO:

$$\theta^s = \frac{n(r^b - r^s)}{(n\sigma)^2} \left(\frac{V_n}{V_{nn}} \right)$$

Reemplazando V_n y V_{nn}:

$$\theta^s = \frac{n(r^b - r^s)}{(n\sigma)^2} \left(\frac{\frac{1}{an}}{\frac{-1}{an^2}} \right)$$

Simplificando:

$$\theta^s = \frac{(r^s - r^b)}{\sigma^2}$$

iv. Use your guess for $V(n)$ to simplify the first-order condition for θ_t^s and solve the resulting equation for θ_t^s.
Tenemos

$$\theta^s = \frac{n(r^b - r^s)}{(n\sigma)^2}\left(\frac{V_n}{V_{nn}}\right)$$

Reemplazando V_n y V_{nn}:

$$\theta^s = \frac{n(r^b - r^s)}{(n\sigma)^2}\left(\frac{\frac{1}{an}}{\frac{-1}{an^2}}\right)$$

Simplificando:

$$\theta^s = \frac{(r^s - r^b)}{\sigma^2}$$

v. Substitute the optimal choices and the guess for $V(n)$ into the HJB equation to eliminate $V(n)$, $V'(n)$, $V''(n)$, c, θ^s and the max operator.

$$\rho\left(\frac{1}{a}\log(n) + b\right) = \log(an) - \frac{an}{an} + \frac{1}{an}n\left[(1-\theta^s)r^b + r^s\theta^s\right] - \frac{1}{2an^2}(n\sigma\theta^s)^2$$

Expandiendo términos:

$$\rho\left(\frac{1}{a}\log(n) + b\right) = \log(a) + \log(n) - 1 + \frac{1}{a}\left[(r^s - r^b)\theta^s\right] - \frac{1}{2an^2}(n\sigma\theta^s)^2 + \frac{r^b}{a}$$

Recordemos que:

$$\theta^s = \frac{(r^s - r^b)}{\sigma^2}$$

Sustituyendo θ^s:

$$\rho\left(\frac{1}{a}\log(n) + b\right) = \log(a) + \log(n) - 1 + \frac{1}{a}\left[(r^s - r^b)\frac{(r^s - r^b)}{\sigma^2}\right] - \frac{1}{2a}\left(\sigma\frac{(r^s - r^b)}{\sigma^2}\right)^2 + \frac{r^b}{a}$$

Finalmente, despejando ρb:

$$\rho b = \log a + \log n\left(n - \frac{\rho}{a}\right) - 1 + \frac{1}{2a}\frac{(r^b - r^s)^2}{\sigma^2} + \frac{r^b}{a}$$

vi. The resulting equation in step (e) has to hold for all $n > 0$ (if it does not, the previous guess was incorrect). Show that this is indeed possible if we choose a and b appropriately. What are the required values for a and b?
Para eliminar la dependencia en n, necesitamos que:

$$\log(n) \cdot \left(1 - \frac{\rho}{a}\right) = 0$$

De donde se sigue que:

$$1 - \frac{\rho}{a} = 0$$

Despejando a:

$$a = \rho$$

Ahora, considerando b, tenemos la ecuación:

$$\rho b = \log\left(\frac{1}{\rho}\right) - 1 + \frac{1}{2\rho}\frac{(r^b - r^s)^2}{\sigma^2} + \frac{r^b}{\rho}$$

Despejamos b:

$$b = \frac{1}{\rho}\left(\log\left(\frac{1}{\rho}\right) - 1 + \frac{1}{2\rho}\frac{(r^b - r^s)^2}{\sigma^2} + \frac{r^b}{\rho}\right)$$

(b) Now consider the same decision problem as before but approach it with the stochastic maximum principle instead of the HJB equation.

i. Denote by ξ_t the costate for net worth n_t and by σ_t^ξ its (arithmetic) volatility loading (that is $d\xi_t = \mu_t^\xi dt + \sigma_t^\xi dZ_t$ with some drift μ_t^ξ). Write down the Hamiltonian of the problem.

$$H = e^{-\rho t} \ln u(c) + n\xi_t \mu^n + \operatorname{tr}\left((\sigma_\xi)^T \sigma^n\right)$$

Recordar

$$dn = n\left(\frac{-c}{n} + \left[(1-\theta^s)r^b + r^s\theta^s\right]\right)dt + n\sigma\theta^s\, dZ)$$

Definiendo μ^n y σ^n:

$$\mu^n = \left(\frac{-c}{n} + \left[(1-\theta^s)r^b + r^s\theta^s\right]\right)$$

$$\sigma^n = \sigma\theta^s$$

Entonces, la ecuación diferencial se puede reescribir como:

$$dn = n\mu^n dt + n\sigma^n dZ$$

Una reformulación de la Hamiltoniana:

$$H = e^{-\rho t} \ln u(c) + n\xi_t\left(\frac{-c}{n} + \left[(1-\theta^s)r^b + r^s\theta^s\right]\right) + \operatorname{tr}\left((\sigma^\xi)^T \sigma\theta^s n\right)$$

ii. The choice variables have to maximize the Hamiltonian at all times. Take the first-order conditions in this maximization problem.
Calculamos las CPO: CPO c:

$$\frac{\partial H}{\partial c} = -\xi_t + \frac{e^{\rho t}}{c} = 0$$

De aquí, despejamos c^*:

$$c^* = \frac{e^{-\rho t}}{\xi_t}$$

CPO θ^s

$$\frac{\partial H}{\partial \theta^s} = n\xi_t(-r^b + r^s) + \left((\sigma^\xi)^T n\sigma\right) = 0$$

$$(r^s - r^b) = -\frac{\left((\sigma^\xi)\sigma\right)}{\xi_t}$$

iii. Let's again make the guess $c_t = an_t$ with an unknown constant $a > 0$. Use the first-order condition for consumption derived in part (b) to turn this into a guess for the costate ξ_t. Also determine the implied costate volatility σ_t^ξ.
Partimos de la condición óptima para c^*:

$$c^* = \frac{e^{-\rho t}}{\xi_t}$$

Dado nuestro **guess** $c = an$, igualamos ambas expresiones:

$$an = \frac{e^{-\rho t}}{\xi_t}$$

Despejamos ξ_t:

$$\xi_t = \frac{e^{-\rho t}}{an}$$

Partimos de la ecuación diferencial para ξ_t:

$$d\xi_t = \mu^\xi \, dt + \sigma^\xi dZ$$

Sustituyendo $\xi_t = \frac{e^{-\rho t}}{an}$, obtenemos:

$$d\xi_t = -\frac{\rho e^{-\rho t}}{an} dt + \frac{e^{-\rho t}}{an^3}(dn)^2 - \frac{e^{-\rho t}}{an^2} dn$$

Recordando que:

$$dn = n\left(r^b(1-\theta^s) + r^s\theta^s - \frac{c}{n}\right) dt + n\sigma\theta^s dZ$$

Al elevar al cuadrado dn:

$$(dn)^2 = n^2(\sigma\theta^s)^2 dZ^2$$

Dado que $dZ^2 = dt$, sustituimos:

$$(dn)^2 = n^2(\sigma\theta^s)^2 dt$$

Sustituyendo en $d\xi_t$:

$$d\xi_t = -\frac{\rho e^{-\rho t}}{an} dt - \frac{e^{-\rho t}}{an^2} dn + \frac{e^{-\rho t}}{an^3} n^2(\sigma\theta^s)^2 dt$$

Simplificando:

$$d\xi_t = -\frac{\rho e^{-\rho t}}{an} dt - \frac{e^{-\rho t}}{an^2} dn + \frac{e^{-\rho t}}{an}(\sigma\theta^s)^2 dt$$

También tenemos que:

$$\sigma^\xi = \frac{-e^{-\rho t}}{an}\sigma\theta^s$$

$$\sigma^\xi = -\xi_t\sigma\theta^s$$

iv. Determine the optimal solution for θ_t^s.
Usando

$$(r^s - r^b) = -\frac{\left((\sigma^\xi)\sigma\right)}{\xi_t}$$

$$(r^s - r^b) = \frac{((\xi_t\sigma\theta^s)\sigma)}{\xi_t}$$

$$\frac{(r^s - r^b)}{\sigma^2} = \theta^s$$

v. Write down the costate equation for ξ_t and substitute in your guess for c_t, the implied guesses for ξ_t and σ_t^ξ, and the implied optimal solution for θ_t^s. Show that the costate equation is indeed satisfied (and hence the guess was correct) if you choose a suitably. Which value(s) for a work?
Partimos de la ecuación diferencial:

$$d\xi_t = \mu^\xi \, dt + \sigma^\xi dZ$$

Sustituyendo su expresión:

$$d\xi_t = -\frac{\rho e^{-\rho t}}{an} dt + \frac{e^{-\rho t}}{an^3}(dn)^2 - \frac{e^{-\rho t}}{an^2} dn$$

Recordamos que:

$$dn = n\left(r^b(1-\theta^s) + r^s\theta^s - \frac{c}{n}\right) dt + n\sigma\theta^s dZ$$

Elevando al cuadrado:

$$(dn)^2 = n^2(\sigma\theta^s)^2 dt$$

Sustituyendo en la ecuación de $d\xi_t$:

$$d\xi_t = -\frac{\rho e^{-\rho t}}{an}dt + \frac{e^{-\rho t}}{an}(\sigma\theta^s)^2 dt - \frac{e^{-\rho t}}{an}\left(\left(r^b(1-\theta^s) + r^s\theta^s - \frac{c}{n}\right)dt + \sigma\theta^s dZ\right)$$

Recordamos que:

$$\mu^\xi = \frac{e^{-\rho t}}{an}\left(\rho + ((\sigma\theta^s)^2 + r^b(1-\theta^s) + r^s\theta^s - \frac{c}{n})\right)$$

Y utilizando las relaciones:

$$c = an, \quad \theta^s = \frac{r^s - r^b}{\sigma^2}$$

Sustituyendo estos valores en la ecuación final de μ^ξ:

$$\mu^\xi = -\frac{e^{-\rho t}}{an}\left(\rho + \left(\frac{r^s - r^b}{\sigma}\right)^2 + r^b + \frac{(r^s - r^b)^2}{\sigma^2} - a\right)$$

Simplificamos:

$$\mu^\xi = -\frac{e^{-\rho t}}{an}\left(\rho + r^b - a\right)$$

Calculo de H_n^n Dado que:

$$\xi_t = \frac{e^{-\rho t}}{an}$$

La función Hamiltoniana es:

$$H = e^{-\rho t}\ln u(c) + n\xi_t\left(\frac{-c}{n} + \left[(1-\theta^s)r^b + r^s\theta^s\right]\right) + \operatorname{tr}\left((\sigma^\xi)^T\sigma\theta^s n\right)$$

Calculamos H_n^n:

$$H_n^n = -r^b\theta^s\xi_t + r^b\xi_t + r^s\theta^s\xi_t + \sigma(-\xi_t\sigma\theta^s)\theta^s$$

Sustituyendo $\xi_t = \frac{e^{-\rho t}}{an}$ en **todas** las instancias:

$$H_n^n = -r^b\theta^s\frac{e^{-\rho t}}{an} + r^b\frac{e^{-\rho t}}{an} + r^s\theta^s\frac{e^{-\rho t}}{an} + \sigma\left(-\frac{e^{-\rho t}}{an}\sigma\theta^s\right)\theta^s$$

Factorizamos $\frac{e^{-\rho t}}{an}$:

$$H_n^n = \frac{e^{-\rho t}}{an}\left(-r^b\theta^s + r^b + r^s\theta^s - \sigma^2\theta^s\theta^s\right)$$

Sustituyendo $\theta^s = \frac{r^s - r^b}{\sigma^2}$:

$$H_n^n = \frac{e^{-\rho t}}{an}\left(r^b + (r^s - r^b)\frac{r^s - r^b}{\sigma^2} - (r^s - r^b)\sigma^2\left(\frac{r^s - r^b}{\sigma^2}\right)^2\right)$$

$$H_n^n = \frac{e^{-\rho t}}{an}\left(r^b + \frac{(r^s - r^b)^2}{\sigma^2} - \frac{(r^s - r^b)^2}{\sigma^2}\right)$$

$$H_n^n = \frac{e^{-\rho t}}{an}r^b$$

Por lo tanto, la expresión final es:

$$H_n^n = \frac{e^{-\rho t}r^b}{an}$$

vi. Verify that the optimal solution coincides with the one you obtained from the HJB approach. Also show that $\xi_t = e^{-\rho t}V'(n_t)$, where V is the value function determined previously.

2. In this exercise, you will solve BruSan (2014) numerically, under the assumption of log utility. Our goal is to construct functions $q(\eta)$, $\iota(\eta)$, $\kappa(\eta)$ and $\sigma^q(\eta)$ on the $[0,1]$ grid. Slides 133-135 describe the set of equations and the algorithm. The parameter values are:

$$\rho_e = 0.06, \quad \rho_h = 0.05, \quad a_e = 0.11, \quad a_h = 0.03, \quad \delta = 0.05, \quad \phi = 10, \quad \alpha = 0.5, \quad \sigma = 0.1$$

where $\Phi(\iota) = (1/\phi)\log(1+\phi\iota)$.

(a) Solve the model at the boundaries: for $\eta = 0$ and $\eta = 1$.

(b) Create a uniform grid for $\eta \in [0.0001, 0.9999]$.

(c) Solve the ODE for $q(\eta)$ assuming $\kappa(0) = 0$ as boundary condition. Stop once you reach $\kappa \geq 1$. From here on, set $\kappa = 1$, solve for q and σ^q.

(d) Verify your solution by plotting $q(\eta)$ and $\sigma^q(\eta)$. Also plot $\iota(\eta)$, $\kappa(\eta)$.

(e) An alternative derivation for the drift and volatility of η in the general case is given by:

$$\mu_t^\eta = (1-\eta_t)\left[(\varsigma_t^e - \sigma - \sigma_t^q)(\sigma_t^\eta + \sigma + \sigma_t^q) - (\varsigma_t^h - \sigma - \sigma_t^q)\left(\frac{-\eta_t}{1-\eta_t}\sigma_t^\eta + \sigma + \sigma_t^q\right)\right] - \left(\frac{C_t^e}{N_t^e} - \frac{C_t^h}{N_t^h}\right) \tag{3}$$

$$\sigma_t^\eta = \frac{\kappa_t - \eta_t}{\eta_t}(\sigma + \sigma_t^q) \tag{4}$$

where ς_e and ς_h are risk prices. Show these expressions are equivalent to the ones derived in the slides (you can assume logarithmic preferences).

(f) Plot $\eta\mu^\eta(\eta)$ and $\eta\sigma^\eta(\eta)$.

(g) Plot $r(\eta)$. Note that you will need to approximate a second order derivative.

3. Consider the first monetary model studied in class with log utility and without government policy ($\mu_B = i = \sigma_B = G = \tau = 0$). There can still be a constant supply of bonds $B_t \neq 0$. In this problem, we add stochastic volatility to the model. Suppose idiosyncratic risk $\bar{\sigma}$ evolves according to the exogenous stochastic process

$$d\bar{\sigma}_t = b(\bar{\sigma}_{ss} - \bar{\sigma}_t)dt + \nu\sqrt{\bar{\sigma}_t}dZ_t \tag{5}$$

where $\bar{\sigma}_{ss}$, b, and ν are constants.

(a) Use goods market clearing and optimal investment to express q_K, q_B, and ι in terms of

$$\vartheta := \frac{q_B}{q_B + q_K}.$$

(b) Derive the "money valuation equation", i.e., an expression of the form $\mu_{\vartheta t} = f(\vartheta_t, \bar{\sigma}_t)$ (drift of ϑ) where f only depends on parameters of the model.

Feel free to use the following suggestion or an alternative procedure:

(a) Postulate a Geometric Brownian motion for ϑ, q_B, and q_K.

(b) Use the definition of ϑ to find the law of motion $d\vartheta_t$ using Itô's Lemma.

(c) Use the martingale pricing condition to simplify the expression for $\mu_{\vartheta t}$:

$$\mathbb{E}\left[\frac{dr_t^{K,i}}{dt}\right] - \mathbb{E}\left[\frac{dr_t^B}{dt}\right] = \zeta_t\left(\sigma_t^{K,i} - \sigma_t^B\right) + \tilde{\zeta}_t\left(\bar{\sigma}_t^{K,i} - \bar{\sigma}_t^B\right) \tag{6}$$

where ζ is the price of risk.

(d) Find the price of risk and replace it in the expression for $\mu_{\vartheta t}$.

Tenemos lo siguiente Partimos de la ecuación:

$$\vartheta = \frac{q_B}{q_B + q_K}$$

Las derivadas parciales son:

$$\vartheta_{q_B} = \frac{q_K}{(q_B + q_K)^2}$$

$$\vartheta_{q_K} = -\frac{q_B}{(q_B + q_K)^2}$$

Las segundas derivadas parciales son:

$$\vartheta_{q_B q_B} = -\frac{2q_K}{(q_B + q_K)^3}$$

$$\vartheta_{q_K q_K} = \frac{2q_B}{(q_B + q_K)^3}$$

$$\vartheta_{q_B q_K} = \frac{q_B - q_K}{(q_B + q_K)^3}$$

Dado un proceso estocástico q_t, aplicamos el Lema de Itô para calcular la diferencial de ϑ:

$$d\vartheta = \vartheta_{q_B} dq_B + \vartheta_{q_K} dq_K + \frac{1}{2}\left(\vartheta_{q_B q_B}(dq_B)^2 + 2\vartheta_{q_B q_K} dq_B dq_K + \vartheta_{q_K q_K}(dq_K)^2\right)$$

Sabemos que:

$$dq_B = q_B \mu^{q_B} dt + q_B \sigma^{q_B} dZ$$

$$dq_K = q_K \mu^{q_K} dt + q_K \sigma^{q_K} dZ$$

Sustituyendo en la ecuación de $d\vartheta$:

$$d\vartheta = \vartheta_{q_B}(q_B \mu^{q_B} dt + q_B \sigma^{q_B} dZ) + \vartheta_{q_K}(q_K \mu^{q_K} dt + q_K \sigma^{q_K} dZ)$$

$$+\frac{1}{2}\left(\vartheta_{q_B q_B}(q_B \sigma^{q_B} dZ)^2 + 2\vartheta_{q_B q_K}(q_B \sigma^{q_B} dZ)(q_K \sigma^{q_K} dZ) + \vartheta_{q_K q_K}(q_K \sigma^{q_K} dZ)^2\right)$$

Usando la propiedad de **diferenciales estocásticas**:

$$(dZ)^2 = dt$$

Se obtiene:

$$d\vartheta = \left[\vartheta_{q_B} q_B \mu^{q_B} + \vartheta_{q_K} q_K \mu^{q_K} + \frac{1}{2}\left(\vartheta_{q_B q_B} q_B^2 (\sigma^{q_B})^2 + 2\vartheta_{q_B q_K} q_B q_K \sigma^{q_B} \sigma^{q_K} + \vartheta_{q_K q_K} q_K^2 (\sigma^{q_K})^2\right)\right] dt$$

$$+ \left[\vartheta_{q_B} q_B \sigma^{q_B} + \vartheta_{q_K} q_K \sigma^{q_K}\right] dZ$$

Sustituyendo las derivadas:

$$\vartheta_{q_B} = \frac{q_K}{(q_B + q_K)^2}, \quad \vartheta_{q_K} = -\frac{q_B}{(q_B + q_K)^2}$$

$$\vartheta_{q_B q_B} = -\frac{2q_K}{(q_B + q_K)^3}, \quad \vartheta_{q_K q_K} = \frac{2q_B}{(q_B + q_K)^3}, \quad \vartheta_{q_B q_K} = \frac{q_B - q_K}{(q_B + q_K)^3}$$

Sustituyendo en la ecuación de $d\vartheta$:

$$d\vartheta = \left[\frac{q_K}{(q_B+q_K)^2} q_B \mu^{q_B} - \frac{q_B}{(q_B+q_K)^2} q_K \mu^{q_K}\right] dt$$

$$+\frac{1}{2}\left[\left(-\frac{2q_K}{(q_B+q_K)^3}\right) q_B^2 (\sigma^{q_B})^2 + 2\left(\frac{q_B-q_K}{(q_B+q_K)^3}\right) q_B q_K \sigma^{q_B}\sigma^{q_K} + \left(\frac{2q_B}{(q_B+q_K)^3}\right) q_K^2 (\sigma^{q_K})^2\right] dt$$

$$+\left[\frac{q_K}{(q_B+q_K)^2} q_B \sigma^{q_B} - \frac{q_B}{(q_B+q_K)^2} q_K \sigma^{q_K}\right] dZ$$

Dado que:

$$\vartheta = \frac{q_B}{q_B+q_K}, \quad 1-\vartheta = \frac{q_K}{q_B+q_K}$$

Sustituyéndolo en la ecuación diferencial de $d\vartheta$:

$$d\vartheta = \left[\vartheta(1-\vartheta)(\mu^{q_B}-\mu^{q_K})\right] dt$$

$$+\frac{1}{2}\left[-2\vartheta^2(1-\vartheta)q_B(\sigma^{q_B})^2 + 2\vartheta(1-\vartheta)(\vartheta-(1-\vartheta))\sigma^{q_B}\sigma^{q_K} + 2\vartheta(1-\vartheta)^2 q_K(\sigma^{q_K})^2\right] \frac{dt}{q_B+q_K}$$

$$+\left[\vartheta(1-\vartheta)(\sigma^{q_B}-\sigma^{q_K})\right] dZ$$

Simplifico

$$d\vartheta = \vartheta(1-\vartheta)[[(\mu^{q_B}-\mu^{q_K})] dt$$

$$\left[-\vartheta q_B(\sigma^{q_B})^2 + (\vartheta-(1-\vartheta))\sigma^{q_B}\sigma^{q_K} + (1-\vartheta)q_K(\sigma^{q_K})^2\right] dt$$

$$+\left[(\sigma^{q_B}-\sigma^{q_K})\right] dZ]$$

La ecuación diferencial de ϑ está dada por:

$$d\vartheta = \vartheta(1-\vartheta)((\mu^{q_B}-\mu^{q_K})dt$$

$$+(\sigma^{q_K}-\sigma^{q_B})\left[(1-\vartheta)\sigma^{q_K}+\vartheta\sigma^{q_B}\right] dt$$

$$+(\sigma^{q_B}-\sigma^{q_K}))dZ$$

Recordar que

$$dk_{i,t} = k_{i,t}((\Phi(u_{i,t})-\delta)\, dt + \tilde{\sigma} d\tilde{z})$$

$$dK = \int (\Phi(u_{i,t})-\delta) k_{i,t}\, dt$$

Tengo también

$$d(q_t^k k_t) = k_t dq_t^k + q_t^k dk_t$$

Sustituyendo las expresiones diferenciales:

$$d(q_t^K k_t) = k_t q_t^k\,(\mu^{q_k} dt + \sigma^{q_K} dZ) + q_t^K k_t((\Phi(\iota)-\delta)\, dt + \tilde{\sigma} d\tilde{z})$$

Por lo tanto, la expresión final para dr_t^k queda:
Dado que:

$$d(q_t^k k_t) = k_t dq_t^k + q_t^k dk_t$$

Sustituyendo las expresiones diferenciales:

$$d(q_t^k k_t) = k_t q_t^k \left(\mu^{q_k} dt + \sigma^{q_K} dZ\right) + q_t^k k_t \left((\Phi(\iota) - \delta)dt + \tilde{\sigma} d\tilde{z}\right)$$

Por lo tanto, la expresión final para dr_t^k queda:

$$dr_t^k = \left(\frac{a-\iota}{q}\right) dt + \mu^{q_K} dt + \sigma^{q_K} dZ + (\Phi(\iota) - \delta)dt + \tilde{\sigma} d\tilde{z}$$

Para dr_t^B
La ecuación diferencial está dada por:

$$dr_t^B = \frac{d\left(\frac{q_t^B K_t}{B_t}\right)}{\frac{q_t^B K_t}{B_t}}$$

Dado que:

$$dB_t = \mu^B B_t dt$$

La diferencial de la fracción se obtiene aplicando la regla de Itô:

$$d\left(\frac{q_t^B K_t}{B_t}\right) = \frac{K_t}{B_t} dq_t^B + \frac{q_t^B}{B_t} dK_t - \frac{q_t^B K_t}{B_t^2} dB_t$$

Sustituyendo las ecuaciones diferenciales de q_t^B, K_t y B_t:

$$d\left(\frac{q_t^B K_t}{B_t}\right) = \frac{K_t}{B_t} \left((q_t^B) \left(\mu^{q_B} dt + \sigma^{q_B} dZ\right)\right)$$

$$+\frac{q_t^B}{B_t} K_t \left((\Phi(\iota) - \delta)dt\right) - \frac{q_t^B K_t}{B_t^2} B_t \mu^B dt$$

Por lo tanto, la ecuación final para dr_t^B queda:

$$dr_t^B = \mu^{q_B} dt + \sigma^{q_B} dZ + (\Phi(\iota) - \delta)dt - \mu^B dt$$

Con esta información puedo plantear el problema de maximización

$$\max\left(\int_0^\infty e^{\rho t} c_t \, dt\right)$$

Sujeto a la ecuación diferencial:

$$dn_t = -c_t dt + \left(n_t - q_t^K k_t\right) dr_t^B + q_t^K k_t dr_t^K$$

Reemplazando en dn_t:

$$\begin{aligned} dn_t = & -c_t dt + \left(n_t - q_t^K k_t\right) \left[(\mu^{q_B} dt + \sigma^{q_B} dZ) + (\Phi(\iota) - \delta)dt - \mu^B dt\right] \\ & + q_t^K k_t \left[\left(\frac{a-\iota}{q}\right) dt + \mu^{q_K} dt + \sigma^{q_K} dZ + (\Phi(\iota) - \delta)dt + \tilde{\sigma} d\tilde{z}\right]. \end{aligned}$$

$$\frac{dn}{n} = -\frac{c}{n} dt + \mu^n dt + \tilde{\sigma}^n d\tilde{Z} + \sigma^n dZ$$

$$\mu^n = (1-x_t)\Big(\mu^{q_B}dt + (\Phi(\iota)-\delta)dt - \mu^B dt\Big) + x_t\left(\frac{a-\iota}{q}dt + \mu^{q_K}dt + (\Phi(\iota)-\delta)dt + \tilde{\sigma}d\tilde{z}\right)$$

$$\sigma^n = (1-x_t)\sigma^{q_B} + x_t\sigma^{q_K}$$

$$\tilde{\sigma}^n = x_t\tilde{\sigma}$$

Donde

$$x_t = \frac{q_t^K k_t}{n}$$

Con esta información puedo plantear la HJB Tenemos que

$$u = \log(c_t)$$

Suponemos que la función valor tiene esta forma

$$V(\xi n) = \frac{1}{\rho}\log(\xi n)$$

Recordemos que la HJB

$$\rho(\frac{1}{\rho})\log(\xi n) = log(c_t) + \frac{E(\frac{1}{\rho}\log(\xi n))}{dt}$$

$$\frac{E(\frac{1}{\rho}\log(\xi n))}{dt} = \frac{1}{\rho dt}((\frac{1}{\xi}d\xi + \frac{1}{n}(-\frac{c}{n}dt + \mu^n dt) - \frac{1}{\xi^2}(d\xi)^2 - \frac{1}{n^2}(dn)^2)$$

$$d\xi = \mu^\xi \xi dt + \sigma^\xi \xi dZ$$

$$\frac{E(\frac{1}{\rho}\log(\xi n))}{dt} = \frac{1}{\rho}((\mu^\xi) + (-\frac{c}{n} + \mu^n) - (\sigma^\xi)^2 - (\tilde{\sigma}^n)^2 - (\sigma^n)^2$$

Planteamos la ecuación de HJB:

$$\log(\xi n) = \log(c_t) + \frac{1}{\rho}\left(\mu^\xi + \left(-\frac{c}{n} + \mu^n\right) - (\sigma^\xi)^2 - (\tilde{\sigma}^n)^2 - (\sigma^n)^2\right)$$

El problema de maximización:

$$\log(\xi n) = \max_c\left\{\log(c_t) - \frac{c}{\rho n}\right\} + \max_{x,i}\left\{\frac{1}{\rho}\left(\mu^n - (\tilde{\sigma}^n)^2 - (\sigma^n)^2\right)\right\} + \frac{1}{\rho}\left(\mu^\xi - (\sigma^\xi)^2\right)$$

Condiciones de Primer Orden (CPO):

Para c:

$$\frac{1}{c} = \frac{1}{\rho n} \Rightarrow c = \rho n$$

Para x:

$$\mu^{q_K} - \mu^{q_K\beta} - \mu^\beta + \frac{a-\iota}{q^K} + x\left(\sigma^{q_K} - \sigma^{q_B}\right)^2 + \sigma^{q_K}\sigma^{q_B} - (\sigma^{q_B})^2 + \tilde{\sigma}^2(x) = 0$$

Para ι:

$$\Phi(\iota) = (q^K)^{-1}$$

Para encontrar $d\vartheta$

Usamos la CPO de x

$$\mu^{q_K} - \mu^{q_B} - \mu^{\beta} + \frac{a-\iota}{q^K} + x\left(\sigma^{q_K} - \sigma^{q_B}\right)^2 + \sigma^{q_K}\sigma^{q_B} - \left(\sigma^{q_B}\right)^2 + \tilde{\sigma}^2(x) = 0$$

$$\mu^{q_K} - \mu^{q_B} - \mu^{\beta} + \frac{a-\iota}{q^K} + \left(\sigma^{q_K} - \sigma^{q_B}\right)\left(x\sigma^{q_K} + (1-x)\sigma^{q_B}\right) + \tilde{\sigma}^2(x) = 0$$

recordar que por clearing market condition

$$x = \frac{q_t^k}{q_t^k + q_t^B}$$

$$x = 1 - \vartheta$$

$$\mu^{q_K} - \mu^{q_B} - \mu^{\beta} + \frac{a-\iota}{q^K} + \left(\sigma^{q_K} - \sigma^{q_B}\right)\left(x\sigma^{q_K} + (1-x)\sigma^{q_B}\right) + \tilde{\sigma}^2(x) = 0$$

(c) Suppose that $\sigma_{\sigma,t} = 0$ and the economy is at the steady state with $\bar{\sigma}_t = \bar{\sigma}^{ss}$ for some $\bar{\sigma}^{ss} > 0$.

(i) Derive expressions for q^B, q^K and ϑ in the monetary and non-monetary equilibria.

(ii) What is the smallest value of $\bar{\sigma}^{ss}$ that allows for a monetary equilibrium? Denote this value by $\bar{\sigma}^{ss}_{\min}$.

(iii) Suppose that $\bar{\sigma}^{ss} > \bar{\sigma}^{ss}_{\min}$, what happens to q^B, q^K and ϑ as $\bar{\sigma}^{ss}$ rises?

(iv) Suppose that $0 < \bar{\sigma}^{ss} < \bar{\sigma}^{ss}_{\min}$, what happens to q^B, q^K and ϑ as $\bar{\sigma}^{ss}$ falls?

4. Solving the previous model numerically

(a) Set $a = 0.2$, $\phi = 1$, $\delta = 0.05$, $\rho = 0.01$, $\bar{\sigma}^{ss} = 0.2$, $b = 0.05$, $\nu = 0.02$.

(b) Apply Itô's lemma to $\vartheta = \vartheta(\bar{\sigma}_t)$, and equate the drift term with $\vartheta_t\mu_t^{\vartheta}$, where μ_t^{ϑ} is given by the Cox-Ingersol-Ross process above. This gives you an HJB-looking equation for $\vartheta(\bar{\sigma})$.

(c) Solve the model using value function iteration:

(i) Suggest a grid for $\bar{\sigma}$ and construct the M matrix using `buildM.m`.

(ii) Rewrite the money valuation equation such that in the discretized form you get:

$$\rho\vartheta = u(\vartheta) + M\vartheta \tag{7}$$

(iii) Write a loop that updates $\vartheta(\bar{\sigma})$ with the implicit method:

$$\vartheta_{t-\Delta t} = \left((1+\rho\Delta t)I - \Delta t M\right)^{-1}\left(\Delta t u(\vartheta_t) + \vartheta_t\right) \tag{8}$$

(iv) Iterate over $\vartheta(\bar{\sigma})$ until convergence.

(d) Plot ϑ, q^B, q^K, r^f, ς, $\tilde{\xi}$ as functions of $\bar{\sigma}$. Explain the dependence of the variables on $\bar{\sigma}$.