3 *Mathematical Preliminaries*

Exercise 3.1

Given $k_0 = k$, the lifetime utility given by the sequence $\{k_t\}_{t=1}^{\infty}$ in which $k_{t+1} = g_0(k_t)$ is

$$\begin{aligned} w_0(k) &= \sum_{t=0}^{\infty} \beta^t u[f(k_t) - g_0(k_t)] \\ &= u[f(k) - g_0(k)] + \beta \sum_{t=1}^{\infty} \beta^{t-1} u[f(k_t) - g_0(k_t)]. \end{aligned}$$

But

$$\begin{aligned} \sum_{t=1}^{\infty} \beta^{t-1} u[f(k_t) - g_0(k_t)] &= \sum_{t=0}^{\infty} \beta^t u[f(k_{t+1}) - g_0(k_{t+1})] \\ &= w_0(k_1) \\ &= w_0[g_0(k)]. \end{aligned}$$

Hence

$$w_0(k) = u[f(k) - g_0(k)] + \beta w_0[g_0(k)]$$

for all $k \geq 0$.

Exercise 3.2

a. The idea of the proof is to show that any finite dimensional Euclidean space $\mathbf{R}^l$ satisfies the definition of a real vector space, using the fact that the real numbers form a field.

Take any three arbitrary vectors $x = (x_1, ..., x_l)$, $y = (y_1, ..., y_l)$ and $z = (z_1, ..., z_l)$ in $\mathbf{R}^l$. and any two real numbers α and $\beta \in \mathbf{R}$. Define a zero vector $\theta = (0, ..., 0) \in \mathbf{R}^l$.

Define the addition of two vectors as the element by element sum, and a scalar multiplication by the multiplication of each element of the vector by a scalar. That any finite $\mathbf{R}^l$ space satisfies those properties is trivial.

a :

$$\begin{aligned} x + y &= (x_1 + y_1, x_2 + y_2, ..., x_l + y_l) \\ &= (y_1 + x_1, y_2 + x_2, ..., y_l + x_l) = y + x \in \mathbf{R}^l. \end{aligned}$$

b :

$$\begin{aligned} (x + y) + z &= (x_1 + y_1, ..., x_l + y_l) + (z_1, ..., z_l) \\ &= (x_1 + y_1 + z_1, ..., x_l + y_l + z_l) \\ &= (x_1, ..., x_l) + (y_1 + z_1, ..., y_l + z_l) \\ &= x + (y + z) \in \mathbf{R}^l. \end{aligned}$$

c :

$$\begin{aligned} \alpha(x + y) &= \alpha(x_1 + y_1, ..., x_l + y_l) \\ &= (\alpha x_1 + \alpha y_1, ..., \alpha x_l + \alpha y_l) \\ &= (\alpha x_1, ..., \alpha x_l) + (\alpha y_1, ..., \alpha y_l) = \alpha x + \alpha y \in \mathbf{R}^l. \end{aligned}$$

d :

$$\begin{aligned} (\alpha + \beta)x &= ((\alpha + \beta)x_1, ...(\alpha + \beta)x_l) \\ &= (\alpha x_1 + \beta x_1, ..., \alpha x_l + \beta x_l) \\ &= \alpha x + \beta x \in \mathbf{R}^l. \end{aligned}$$

e :

$$\begin{aligned} (\alpha\beta)x &= (\alpha\beta x_1, ..., \alpha\beta x_l) \\ &= \alpha(\beta x_1, ..., \beta x_l) = \alpha(\beta x) \in \mathbf{R}^l. \end{aligned}$$

f :

$$\begin{aligned} x+\theta &= (x_1+0,...,x_l+0) \\ &= (x_1,...,x_l) = x \in \mathbf{R}^l. \end{aligned}$$

g :

$$\begin{aligned} 0x &= (0x_1,...,0x_l) \\ &= (0,...,0) = \theta \in \mathbf{R}^l. \end{aligned}$$

h :

$$\begin{aligned} 1x &= (1x_1,...,1x_l) \\ &= (x_1,...,x_l) = x \in \mathbf{R}^l. \end{aligned}$$

b. Straightforward extension of the result in part a.

c. Define the addition of two sequences as the element by element addition, and scalar multiplication as the multiplication of each element of the sequence by a real number. Then proceed as in part a. with the element by element operations. For example, take property *c*. Consider a pair of sequences $x = (x_0, x_1, x_2, ...) \in X = \mathbf{R} \times \mathbf{R} \times \mathbf{R}...$ and $y = (y_0, y_1, y_2, ...) \in X = \mathbf{R} \times \mathbf{R} \times \mathbf{R}...$ and $\alpha \in \mathbf{R}$, we just add and multiply element by element, so

$$\begin{aligned} \alpha(x+y) &= (\alpha(x_0+y_0), \alpha(x_1+y_1), \alpha(x_2+y_2), ...) \\ &= (\alpha x_0 + \alpha y_0, \alpha x_1 + \alpha y_1, \alpha x_2 + \alpha y_2, ...) \\ &= \alpha x + \alpha y \in X. \end{aligned}$$

The proof of the remaining properties is analogous.

d. Take $f, g : [a,b] \to \mathbf{R}$ and $\alpha \in \mathbf{R}$. Let $\theta(x) = 0$. Define the addition of functions by $(f+g)(x) = f(x) + g(x)$, and scalar multiplication by $(\alpha f)(x) = \alpha f(x)$. A function f is continuous if

$x_n \to x$ implies that $f(x_n) \to f(x)$. To see that $f+g$ is continuous, take a sequence $x_n \to x$ in $[a,b]$. Then

$$\begin{aligned} \lim_{x_n \to x} (f+g)(x_n) &= \lim_{x_x \to x} [f(x_x) + g(x_n)] \\ &= \lim_{x_n \to x} f(x_n) + \lim_{x_n \to x} g(x_n) \\ &= f(x) + g(x) \\ &= (f+g)(x). \end{aligned}$$

Note that a function defines an infinite sequence of real numbers, so we can proceed as in part c. to check that each of the properties are satisfied.

e. Take the vectors $(0,1)$ and $(1,0)$. Then $(1,0)+(0,1) = (1,1)$ which is not an element of the unit circle.

f. Choose $\alpha \in (0,1)$. Then $1 \in I$ but $\alpha 1 \notin I$, which violates the definition of a real vector space.

g. Let $f : [a,b] \to \mathbf{R}_+$, and $\alpha < 0$, then $\alpha f \le 0$, which does not belong to the set of nonnegative functions on $[a,b]$.

Exercise 3.3

a. Clearly, the absolute value is real valued and well defined on $S \times S$. Take three different arbitrary integers x, y, z. The nonnegativity property holds trivially by the definition of absolute value. Also,

$$\rho(x,y) = |x-y| = |y-x| = \rho(y,x)$$

by the properties of the absolute value, so the commutative property holds.

Finally,

$$\begin{aligned} \rho(x,z) &= |x-z| \\ &= |x-y+y-z| \\ &\le |x-y| + |y-z| \\ &= \rho(x,y) + \rho(y,z), \end{aligned}$$

so the triangle inequality holds.

c. Take three arbitrary functions $x, y, z \in S$. As before, the first two properties are immediate from the definition of absolute value. Note also that as x and y are continuous on $[a,b]$, they are bounded, and the proposed metric is real valued (and not extended real valued). To prove that the triangle inequality holds, notice that

$$\begin{aligned}
\rho(x,z) &= \max_{a\leq t\leq b} |x(t)-z(t)| \\
&= \max_{a\leq t\leq b} |x(t)-y(t)+y(t)-z(t)| \\
&\leq \max_{a\leq t\leq b} (|x(t)-y(t)|+|y(t)-z(t)|) \\
&\leq \max_{a\leq t\leq b} |x(t)-y(t)| + \max_{a\leq t\leq b} |y(t)-z(t)| \\
&= \rho(x,y)+\rho(y,z).
\end{aligned}$$

f. The first two properties follow by definition of absolute value as before, plus the fact that $f(0)=0$, so $x=y$ implies $\rho(x,y)=0$. In order to prove the last property, notice that

$$\begin{aligned}
\rho(x,y) &= f(|x-y|) = f(|x-z+z-y|) \\
&\leq f(|x-z|+|z-y|) \\
&\leq f(|x-z|)+f(|z-y|) \\
&= \rho(x,z)+\rho(z,y),
\end{aligned}$$

where the first inequality comes from the fact that f is strictly increasing and the second one from the concavity of f. To see the last point, without loss of generality, define $|x-z|=a$ and $|z-y|=b$, with $a<b$ and let $\mu=a/b$. By the strict concavity of f,

$$f(b) > \mu f(a) + (1-\mu)f(a+b),$$

and hence

$$\begin{aligned}
f(a+b) &< \frac{b}{(b-a)}f(b) - \frac{a}{(b-a)}f(a) \\
&< f(b)+f(a).
\end{aligned}$$

Exercise 3.4

a. The first property in the definition of a normed vector space is evidently satisfied for the standard Euclidean norm, given that it is just the sum of squared numbers, where each component of the sum is an element of an arbitrary vector $x \in \mathbf{R}^l$. It is zero if and only if each component is zero. To prove the second property, notice that

$$\|\alpha x\|^2 = \sum_{i=1}^{l} (\alpha x_i)^2 = \alpha^2 \sum_{i=1}^{l} {x_i}^2 = \alpha^2 \|x\|^2 ,$$

which implies that

$$\|\alpha x\| = |\alpha| \, \|x\| ,$$

by property *a*. To prove the triangle inequality, we make use of the Cauchy-Schwarz inequality, which says that given two arbitrary vectors x and y,

$$\left(\sum_{i=1}^{l} x_i y_i \right)^2 \leq \sum_{i=1}^{l} x_i^2 \sum_{i=1}^{l} y_i^2 .$$

Hence,

$$\begin{aligned}
\|x+y\|^2 &= \sum_{i=1}^{l} (x_i + y_i)^2 \\
&\leq \sum_{i=1}^{l} x_i^2 + 2 \sum_{i=1}^{l} x_i y_i + \sum_{i=1}^{l} y_i^2 \\
&\leq \sum_{i=1}^{l} x_i^2 + 2 \left(\sum_{i=1}^{l} x_i^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{l} y_i^2 \right)^{\frac{1}{2}} + \sum_{i=1}^{l} y_i^2 \\
&= \|x\|^2 + 2 \|x\| \, \|y\| + \|y\|^2 \\
&= (\|x\| + \|y\|)^2 .
\end{aligned}$$

d. As we consider only bounded sequences, the propsed norm is real valued (and not extended real valued). To see that the first property holds, note that since $|x_k| \geq 0$, all k, $\|x\| = \sup_k |x_k| \geq 0$, and if $x_k = 0$, all k, $\|x\| = \sup_k |x_k| = 0$. The second property holds because

$$\begin{aligned}
\|\alpha x\| &= \sup_k |\alpha x_k| \\
&= \sup_k |\alpha| \, |x_k| \\
&= |\alpha| \sup_k |x_k| \\
&= |\alpha| \, \|x\| \, .
\end{aligned}$$

To see that the triangle inequality holds notice that

$$\begin{aligned}
\|x+y\| &= \sup_k |x_k + y_k| \\
&\leq \sup_k (|x_k| + |y_k|) \\
&\leq \sup_k |x_k| + \sup_k |y_k| \\
&= \|x\| + \|y\| \, .
\end{aligned}$$

e. We prove already that $C[a,b]$ is a vector space (see Exercise 3.2 d.). To see that property a. is satisfied, let $x \in C[a,b]$. Then $|x(t)| \geq 0$ for all $t \in [a,b]$. Hence $\sup_{a \leq t \leq b} |x(t)| \geq 0$, and if $x(t) = 0$ for all $t \in [a,b]$, then $\sup_{a \leq t \leq b} |x(t)| = 0$. To check that the remaining properties are satisfied, we proceed as in part d.

Exercise 3.5

a. If $x_n \to x$, for each $\varepsilon_x > 0$, there exist N_{ε_x} such that $\rho(x_n, x) < \varepsilon_x$, for all $n \geq N_{\varepsilon_x}$. Similarly, if $x_n \to y$, for each $\varepsilon_y > 0$, there exist N_{ε_y} such that $\rho(x_n, y) < \varepsilon_y$, for all $n \geq N_{\varepsilon_y}$. Choose $\varepsilon_x = \varepsilon_y = \varepsilon/2$. Then, by the triangle inequality,

$$\rho(x,y) \leq \rho(x_n, x) + \rho(x_n, y) < \varepsilon$$

for all $n \geq \max\left\{N_{\varepsilon_x}, N_{\varepsilon_y}\right\}$. As ε was arbitrary, this implies $\rho(x,y) = 0$ which implies, since ρ is a metric, that $x = y$.

b. Suppose $\{x_n\}$ converges to a limit x. Then, given any $\varepsilon > 0$, there exist an integer N_ε such that $\rho(x_n, x) < \varepsilon/2$ for all $n > N_\varepsilon$. But then $\rho(x_n, x_m) \leq \rho(x_n, x) + \rho(x_m, x) < \varepsilon$ for all $n, m > N_\varepsilon$.

c. Let $\{x_n\}$ be a Cauchy sequence and let $\varepsilon = 1$. Then, $\exists\, N$ such that for all $n, m \geq N$,

$$\rho(x_m, x_n) < 1.$$

Hence, by the triangle inequality,

$$\begin{aligned} \rho(x_n, 0) &\leq \rho(x_m, x_n) + \rho(x_m, 0) \\ &< 1 + \rho(x_m, 0), \end{aligned}$$

and therefore $\rho(x_n, 0) \leq 1 + \rho(x_N, 0)$ for $n \geq N$. Let

$$M = 1 + \max\{\rho(x_m, 0),\ m = 1, 2, ..., N\} + 1,$$

then $\rho(x_m, 0) \leq M$ for all n, so the Cauchy sequence $\{x_n\}$ is bounded.

d. Suppose that every subsequence of $\{x_n\}$ converges to x. We will prove the contrapositive. That is, if x_n does not converge to x, there exist a subsequence that does not converge. If x_n does not converge to x, there exist $\varepsilon > 0$ such that for all N, there exist $n > N$ with $|x_n - x| > \varepsilon$. Using this repeatedly, we can construct a sequence $\{x_{n_k}\}$ such that $|x_{n_k} - x| > \varepsilon$ for all n_k.

Conversely, suppose $x_n \to x$. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ with $n_1 < n_2 < n_3 < ....$ Then, since $\rho(x_n, x) < \varepsilon$ for all $n \geq N_\varepsilon$, it holds that $\rho(x_{n_i}, x) < \varepsilon$ for all $n_i \geq N_\varepsilon$.

Exercise 3.6

a. The metric space in 3.3a. is complete. Let $\{x_n\}$ be a Cauchy sequence, with $x_n \in S$ for all n. Choose $0 < \varepsilon < 1$, then there exist N_ε such that $|x_m - x_n| < \varepsilon < 1$ for all $n, m \geq N_\varepsilon$. Hence, $x_m = x_n \equiv x \in S$ for all $n, m \geq N_\varepsilon$.

The metric space in 3.3b. is complete. Let $\{x_n\}$ be a Cauchy sequence, with $x_n \in S$ for all n. Choose $0 < \varepsilon < 1$, then there exist

N_ε such that $\rho(x_m, x_n) < \varepsilon < 1$ for all $n, m \geq N_\varepsilon$. By the functional form of the metric used $\rho(x_m, x_n) < 1$ implies that $x_m = x_n \equiv x \in S$ for all $n, m \geq N_\varepsilon$.

The normed vector space in 3.4a. is complete. Let $\{x_n\}$ be a Cauchy sequence, with $x_n \in S$ for all n, and let x_n^k be the k^{th} entry of the n^{th} element of the sequence. Then

$$\begin{aligned} \|x_m - x_n\| &= \left(\sum_{k=1}^{l}(x_m^k - x_n^k)^2\right)^{\frac{1}{2}} \\ &\leq \left(l \max_k (x_m^k - x_n^k)^2\right)^{\frac{1}{2}} \\ &\leq l \max_k \left|x_m^k - x_n^k\right| \end{aligned}$$

for $k = 1, ..., l$, and hence $\left\{x_n^k\right\}$ is a Cauchy sequence for all k. As shown in Exercise 3.5 b., $\left\{x_n^k\right\}$ is bounded for all k, and by the Bolzano-Weierstrass Theorem, every bounded sequence in $\mathbf{R}$ has a convergent subsequence. Hence, using the result proved in Exercise 3.5 d., we can conclude that a sequence in $\mathbf{R}$ converges if and only if it is a Cauchy sequence. Define $x^k = \lim_{n\to\infty} x_n^k$, for all k. Since $\mathbf{R}$ is a closed set, clearly $x = (x^1, ..., x^l) \in S$. To show that $\|x_n - x\| \to 0$ as $n \to \infty$, note that $\|x_m - x\| \leq l \max_k \left|x_n^k - x^k\right| \to 0$ which completes the proof.

The normed vector spaces in 3.4b. and 3.4c. are complete. The proof is the same as that outlined in the paragraph above, with the obvious modifications to the norm.

The normed vector space in 3.4d. is complete. Let $\{x_n\}$ be a Cauchy sequence, with $x_n \in S$ for all n. Note that x_n is a bounded sequence and hence $\{x_n\}$ is a sequence of bounded sequences. Denote by x_n^k the k^{th} element of the bounded sequence x_n. Then $\|x_m - x_n\| = \sup_k \left|x_m^k - x_n^k\right| \geq \left|x_m^k - x_n^k\right|$ for all k. Hence $\|x_m - x_n\| \to 0$ implies $\left|x_m^k - x_n^k\right| \to 0$ for all k and so the sequences of real numbers $\left\{x_n^k\right\}$ are Cauchy sequences. Then, by the completeness of the real numbers, for each k there exist a real number x^k such that

$x_n^k \to x^k$. Since $\{x_n\}$ is bounded, so is $\{x_n^k\}$ for all k. Hence $x = (x^1, x^2, ...) \in S$. To show that $x_n \to x$, by the triangle inequality, $\left|x_n^k - x^k\right| \leq \left|x_n^k - x_m^k\right| + \left|x_m^k - x^k\right|$ for all k. Pick N_ε such that for all $n, m \geq N$, $\left|x_n^k - x_m^k\right| < \varepsilon/2$ for all k. Hence for m large enough $\left|x_m^k - x^k\right| < \varepsilon/2$ and so $\left|x_n^k - x^k\right| < \varepsilon$ implies $\sup_k \left|x_n^k - x^k\right| < \varepsilon$.

The normed vector space in 3.4e. is complete. Let $\{x_n\}$ be a Cauchy sequence of continuous functions in $C[a,b]$ and fix $t \in [a,b]$. Then

$$\begin{aligned} |x_n(t) - x_m(t)| &\leq \sup_{a \leq t \leq b} |x_n(t) - x_m(t)| \\ &= \|x_n - x_m\| \end{aligned}$$

and therefore the sequence of real numbers $\{x_n(t)\}$ satisfies the Cauchy criterion. By the completeness of the real numbers $x\,(t) \to x(t) \in \mathbf{R}$. The limiting values define a function $x : [a,b] \to \mathbf{R}$, which is taken as our candidate function.

To show that $x_n \to x$, pick an arbitrary t, then

$$\begin{aligned} |x_n(t) - x(t)| &\leq |x_n(t) - x_m(t)| + |x_m(t) - x(t)| \\ &\leq \|x_n - x_m\| + |x_m(t) - x(t)|\,. \end{aligned}$$

Since $\{x_n\}$ is a Cauchy sequence, there exist N such that for all $n, m \geq N$, $\|x_n - x_m\| < \varepsilon/2$ and $|x_m(t) - x(t)| < \varepsilon/2$. Therefore, $|x_n(t) - x(t)| < \varepsilon$. Because t was arbitrary, it holds for all $t \in [a,b]$. Hence $\sup_{a \leq t \leq b} |x_n(t) - x(t)| < \varepsilon$ and so $x_n \to x$.

It remains to be shown that x is a continuous function. A function $x(t)$ is continuous in t if for all ε, there exist a δ such that $|t - t'| < \delta$ implies $|x(t) - x(t')| < \varepsilon$. By the triangle inequality,

$$\left|x(t) - x(t')\right| \leq |x(t) - x_n(t)| + \left|x_n(t) - x_n(t')\right| + \left|x_n(t') - x(t')\right|$$

for any $t, t' \in [a,b]$. Fix $\varepsilon > 0$, since $x_n \to x$ there exist N such that

$$|x(t) - x_n(t)| < \varepsilon/3$$

for all $n \geq N$, and N' such that

$$\left|x(t') - x_n(t')\right| < \varepsilon/3$$

for all $n \geq N'$. Since x_n is continuous, there exist δ such that for all t, t', $|t - t'| < \delta$,

$$\left| x_n(t) - x_n(t') \right| < \varepsilon/3.$$

Hence $|x(t) - x(t')| < \varepsilon$.

The metric space in 3.3c. is not complete. To prove this, it is enough to find a sequence of continuous, strictly increasing functions that converges to a function that is not in S. Consider the sequence of functions

$$x_n(t) = 1 + \frac{t}{n},$$

for $t \in [a, b]$. Pick any arbitrary m. Then

$$\begin{aligned} \rho(x_n, x_m) &= \max_{a \leq t \leq b} \left| \frac{t}{n} - \frac{t}{m} \right| \\ &= \max_{a \leq t \leq b} \left| \frac{t(m-n)}{nm} \right| \\ &= \left| \frac{b(n-m)}{nm} \right| \\ &\leq \frac{1}{\min\{n, m\}}. \end{aligned}$$

Notice that $\rho(x_n, x_m) \to 0$ as $n, m \to \infty$. But clearly $x_n(t) \to x(t) = 1$, a constant function.

From the proof it is obvious that this counterexample does not work for the weaker requirement of nondecreasing functions.

The metric space in 3.3d. is not complete. The proof similar to 3.3c. and the same counter-example works in this case, with obvious modifications for the distance function.

The metric space in 3.3e. is not complete. The set of rational number is defined as

$$Q = \left\{ \frac{p}{r} : p, r \in Z, r \neq 0 \right\}$$

where Z is the set of integers. Let

$$x_n = 1 + \sum_{i=1}^{n} \frac{1}{i!}.$$

Clearly x_n is a rational number, however $x_n \to e \notin Q$.

The metric space in 3.4f. is not complete. Take the function

$$x_n(t) = \left(\frac{t-a}{b-a}\right)^n.$$

First, assume $a = 0$, $b = 1$, and $m > n$. Hence

$$\begin{aligned}\|x_n(t) - x_m(t)\| &= \int_0^1 (t^n - t^m)\, dt \\ &= \int_0^1 t^n \left(1 - t^{n-m}\right) dt \\ &\leq \int_0^1 t^n dt \to 0\end{aligned}$$

But the sequence of functions $x_n(t) \to 0$ for $0 \leq t < 1$, and 1 for $t = 1$, a discontinuous function at 1.

In order to show that the space in 3.3c. is complete if "strictly increasing" is replaced by "nondecreasing", we can prove the existence of a limit sequence as we did before. It is left to prove that the limit sequence is nondecreasing. The proof is by contradiction. Take a Cauchy sequence f_n of nondecreasing functions converging to f, and contrary to the statement, suppose $f(t) - f(t') > \varepsilon$ for $t' > t$. Hence,

$$0 < \varepsilon < f(t) - f(t') = f(t) - f_n(t) + f_n(t) - f_n(t') + f_n(t') - f(t').$$

Using the fact that for every t, $\{f_n(t)\}$ converges to $f(t)$,

$$0 < \varepsilon < 2\,\|f_n - f\| + f_n(t) - f_n(t').$$

Choosing N_ε such that for all $n \geq N_\varepsilon$, $\|f_n - f\| \leq \varepsilon/2$, we get

$$0 < \varepsilon < f_n(t) - f_n(t'),$$

a contradiction.

b. Since $S' \subseteq S$ is closed, any convergent sequence in S' converges to a point in S'. Take the set of Cauchy sequences in S.

They all converge to points in S since S is complete. Take the subset of those sequences that belong to S', then by the argument above they converge to a point in S', so S' is complete.

Exercise 3.7

a. First we have to prove that $C^1[a,b]$ is a normed vector space. By definition of absolute value, the non-negativity property is clearly satisfied,

$$\|f\| = \sup_{x\in X}\left\{|f(x)| + \left|f'(x)\right|\right\} \geq 0.$$

To see that the second property is satisfied, note that

$$\begin{aligned}
\|\alpha f\| &= \sup_{x\in X}\left\{|\alpha f(x)| + \left|\alpha f'(x)\right|\right\} \\
&= \sup_{x\in X}\left\{|\alpha|\left[|f(x)| + \left|f'(x)\right|\right]\right\} \\
&= |\alpha| \sup_{x\in X}\left\{|f(x)| + \left|f'(x)\right|\right\} \\
&= |\alpha|\,\|f\|.
\end{aligned}$$

The triangle inequality is satisfied, since

$$\begin{aligned}
\|f+g\| &= \sup_{x\in X}\left\{|f(x)+g(x)| + \left|f'(x)+g'(x)\right|\right\} \\
&\leq \sup_{x\in X}\left\{|f(x)| + |g(x)| + \left|f'(x)\right| + \left|g'(x)\right|\right\} \\
&\leq \sup_{x\in X}\left\{|f(x)| + \left|f'(x)\right|\right\} + \sup_{x\in X}\left\{|g(x)| + \left|g'(x)\right|\right\} \\
&= \|f\| + \|g\|.
\end{aligned}$$

Hence, $C^1[a,b]$ is a normed vector space.

Let $\{f_n\}$ be a Cauchy sequence of functions in $C^1[a,b]$. Fix x, then

$$|f_n(x) - f_m(x)| + \left|f_n'(x) - f_m'(x)\right| \leq \|f_n - f_m\|,$$

and

$$\max\left\{\sup_{x\in X}|f_n(x) - f_m(x)|, \sup_{x\in X}\left|f_n'(x) - f_m'(x)\right|\right\} \leq \|f_n - f_m\|,$$

therefore the sequences of numbers $\{f_n(x)\}$ and $\{f'_m(x)\}$ converge and the limit values define the functions $f : X \to \mathbf{R}$ and $f' : X \to \mathbf{R}$. The proof is similar to the one outlined in Theorem 3.1, and repeatedly used in Exercise 3.6. It follows that f' is continuous. Our candidate for the limit is the function f defined by

$$f(a) = \lim_{n\to\infty} f_n(a),$$

and

$$f(x) = f(a) + \int_0^x f'(z)dz.$$

It is clear that f is continuously differentiable, so that $f \in C^1$.

To see that $\|f_n - f\| \to 0$ note that

$$\begin{aligned}
\|f_n - f\| &\leq \sup_{x\in X} |f_n(x) - f(x)| + \sup_{x\in X} \left|f'_n(x) - f'(x)\right| \\
&\leq \sup_{x\in X} \left| f_n(a) + \int_0^x f'(z)dz - f(a) - \int_0^x f'(z)dz \right| \\
&\quad + \sup_{x\in X} \left|f'_n(x) - f'(x)\right| \\
&\leq |f_n(a) - f(a)| + \int_0^b \left|f'_n(z) - f'(z)\right| dz \\
&\quad + \sup_{x\in X} \left|f'_n(x) - f'(x)\right| \\
&\leq |f_n(a) - f(a)| + (b+1) \sup_{x\in X} \left|f'_n(x) - f'(x)\right|.
\end{aligned}$$

Since $\{f_n(a)\} \to f(a)$, and $\{f'_n\} \to f'$ uniformly, both terms go to zero as $n \to \infty$.

b. See part c.

c. Consider $C^k[a,b]$, the space of k times continuously differentiable functions on $[a,b]$, with the norm given in the text. Clearly $\alpha_i \geq 0$ is needed for the norm to be well defined.

If $\alpha_i > 0$, all i, then the space is complete. The proof is a trivial adaptation of the one presented in a. However, if $\alpha_j = 0$, for any j, then the space is not complete. To see this choose a function $h : [a,b] \to [a,b]$ that is continuous, satisfies $h(a) = a$

and $h(b) = b$,and is $(k-j)$ times continuously differentiable, with $h^i(a) = h^i(b) = 0$, $i = 1, 2, ..., k-j$.

Then consider the following sequence of functions

$$f_n^j(x) = \begin{cases} a & \text{if } x < \frac{a}{n} \\ h(nx) & \text{if } \frac{a}{n} \leq x \leq \frac{b}{n} \\ b & \text{if } x > \frac{b}{n} \end{cases}$$

and

$$f_n^{i-1}(x) = \int_0^x f_n^i(z)dz, \quad i = 1, ..., j$$

Each function f_n is k times continuously differentiable. However, the limiting function f has a discontinuous j-th derivative.

So an example to be applied to part b. would be, for instance, $X = [-1, 1]$ and

$$f_n'(x) = \begin{cases} -1 & \text{if} \quad x < -\frac{1}{n} \\ nx & \text{if} \quad -\frac{1}{n} \leq x \leq \frac{1}{n} \\ 1 & \text{if} \quad x > \frac{1}{n}. \end{cases}$$

Hence

$$f_n(x) = \begin{cases} -x & \text{if} \quad x < -\frac{1}{n} \\ \frac{1}{2n} + \frac{n}{2}x^2 & \text{if} \quad -\frac{1}{n} \leq x \leq \frac{1}{n} \\ x & \text{if} \quad x > \frac{1}{n}. \end{cases}$$

This sequence is clearly not Cauchy in the norm of part a.

Exercise 3.8

The function $T : S \to S$ is uniformly continuous if for every $\varepsilon > 0$ there exist a $\delta > 0$ such that for all x and y in S with $|x - y| < \delta$ we have that $|Tx - Ty| < \varepsilon$.

If T is a contraction, then for some $\beta \in (0, 1)$

$$\frac{|Tx - Ty|}{|x - y|} \leq \beta < 1 \qquad \text{all } x, y \in S \text{ with } x \neq y.$$

Hence to prove that T is uniformly continuous in S, let $\delta \equiv \varepsilon/\beta$, then for any arbitrary $\varepsilon > 0$, if $|x - y| < \delta$ then

$$|Tx - Ty| \leq \beta\,|x - y| < \beta\delta = \varepsilon.$$

Hence T is uniformly continuous.

Exercise 3.9

Observe that

$$\begin{aligned}\rho(T^n v_0, v) &\leq \rho(T^n v_0, T^{n+1} v_0) + \rho(T^{n+1} v_0, v) \\ &= \rho(T^n v_0, T^{n+1} v_0) + \rho(T^{n+1} v_0, Tv) \\ &\leq \rho(T^n v_0, T^{n+1} v_0) + \beta \rho(T^n v_0, v),\end{aligned}$$

where the first line uses the triangle inequality, the second the fact that v is the fixed point of T, and the third line follows from the Contraction Mapping Theorem. Rearranging terms this implies that

$$\rho(T^n v_0, v) \leq \frac{1}{1-\beta} \rho(T^n v_0, T^{n+1} v_0).$$

Exercise 3.10

a. Since v is bounded, the continuous function f is bounded, say by M, on $[-\|v\|, +\|v\|]$. Hence

$$|(Tv)(s)| \leq |c| + sM,$$

so Tv is bounded on $[0, t]$. Since

$$\int_0^s f\left[v\left(z\right)\right] dz$$

is continuous for all f, Tv is continuous.

b. Let $w, v \in C\left(0, t\right)$,and let B be their common bound. Note that

$$\begin{aligned}|Tv(s) - Tw(s)| &\leq \int_0^s |f(v(z)) - f(w(z))| \, dz \\ &\leq \int_0^s B \, |v(z) - w(z)| \, dz \\ &\leq Bs \, \|v - w\| \, .\end{aligned}$$

Choose $\tau = \beta/B$, where $0 < \beta < 1$, then $0 \leq s \leq \tau$ implies that $Bs \|v - w\| \leq \beta \|v - w\|$.

c. The fixed point is $x \in C[0,\tau]$, such that

$$x(s) = c + \int_0^s f\ [x(z)]dz.$$

Hence, for $0 \leq s,\ s' \leq \tau$,

$$\begin{aligned} x(s) - x(s') &= \int_{s'}^{s} f\ [x(z)]dz \\ &= f\ [x(\hat{z})](s - s'), \quad \text{for some } \hat{z} \in [s, s']. \end{aligned}$$

Therefore

$$\frac{x(s) - x(s')}{s - s'} = f\ [x(\hat{z})].$$

Let $s' \to s$, then $\hat{z} \to s$, and so $x'(s) = f\ [x(s)]$.

Exercise 3.11

a. We have to prove that Γ is lower hemi-continuous (l.h.c.) and then the result follows by the definition of a continuous correspondence. Towards a contradiction, assume Γ is not lower hemi-continuous. Then, for all $\varepsilon > 0$, and any N, $\exists\ n > N$ such that $|y_n - y| > \varepsilon$. Construct a subsequence $\{y_{n_k}\}$ from these and consider the corresponding subsequence$\{x_{n_k}\}$ where $y_{n_k} \in \Gamma(x_{n_k})$. As $x_n \to x$, $\{x_{n_k}\} \to x$. But as Γ is upper hemi-continuous (u.h.c.), there exist $y_{n_{k_j}} \to y$, a contradiction.

c. That Γ is compact comes from the fact that a finite union of compact sets is compact. To show that Γ is u.h.c., fix x and pick any arbitrary $x_n \to x$ and $\{y_n\}$ such that $y_n \in \Gamma(x_n)$. Hence $y_n \in \phi(x_n)$ or $y_n \in \psi(x_n)$, and therefore there is a subsequence of $\{y_n\}$ whose elements belong to $\phi(x_n)$ and/or a subsequence of $\{y_n\}$ whose elements belong to $\psi(x_n)$. Call them $\left\{y_{n_k}^{\phi}\right\}$ and $\left\{y_{n_k}^{\psi}\right\}$ respectively. By ϕ and ψ u.h.c., those sequences have a convergent subsequence that converges to $y \in \phi(x)$ or $\psi(x)$ respectively. By construction, those subsequences of $\left\{y_{n_k}^{\phi}\right\}$ and $\left\{y_{n_k}^{\psi}\right\}$ are convergent subsequences of $\{y_n\}$ that converge to $y \in \Gamma(x)$, which completes the proof.

e. For each x, the set of feasible y's is compact. Similarly, for each y, the set of feasible z's is compact. Hence, for each x, Γ is a finite union of compact sets, which is compact.

To see that Γ is u.h.c., pick any arbitrary $x_n \to x$ and $(\{z_n\}, \{y_n\})$ such that $z_n \in \psi(y_n)$ for $y_n \in \phi(x_n)$. By ϕ u.h.c. there is a convergent subsequence of $\{y_n\}$ whose limit point is in $\phi(x)$.

Take this convergent subsequence of $\{y_n\}$. Call it $\{y_{n_k}\}$. By ψ u.h.c. any sequence $\{z_{n_k}\}$ with $z_{n_k} \in \psi(y_{n_k})$ has a convergent subsequence that converges to $z \in \psi(y)$.

Hence, $\{z_{n_k}\}$ is a convergent subsequence of $\{z_n\}$ that converges to $z \in \Gamma(x)$.

Exercise 3.12

a. If Γ is l.h.c. and single valued, then Γ is nonempty and for every $y \in \Gamma(x)$ and every sequence $x_n \to x$, the sequence $\{y_n\}$ with $y_n = \Gamma(x_n)$ converges to y. Hence Γ is a continuous function.

c. Fix x. Clearly $\Gamma(x)$ is nonempty if ϕ or ψ are l.h.c. To show that Γ is l.h.c., pick any arbitrary $y \in \Gamma(x)$ and a sequence $x_n \to x$. By definition, either $y \in \phi(x)$, or $y \in \psi(x)$ or both. Hence, by ϕ and ψ l.h.c., there exist $N \geq 1$ and a sequence $\{y_n\}$ such that $y_n \in \phi(x_n)$ or $y_n \in \psi(x_n)$ for all $n \geq N$, so $\{y_n\}$ is a sequence such that $y_n \in \Gamma(x_n)$ and $y_n \to y$ for all $n \geq N$. Hence, $\Gamma(x)$ is l.h.c. at x. Because x was arbitrary chosen, the proof is complete.

e. It is clear that Γ is nonempty if ϕ and ψ are nonempty. Pick any $z \in \Gamma(x)$ and a sequence $x_n \to x$. The objective is to find $N \geq 1$ and a sequence $\{z_n\}_{n=N}^{\infty} \to z$ such that $z_n \in \Gamma(x_n)$. To construct such a sequence, note that if $z \in \Gamma(x)$, then $z \in \psi(y)$ for some $y \in \phi(x)$. So pick any $y \in \phi(x)$ such that $z \in \psi(y)$.

By ϕ l.h.c. there exist $N_1 \geq 1$ and $\{y_n\}$ such that $y_n \to y$ and $y_n \in \phi(x_n)$ for all $n \geq N_1$. Call this sequence $\left\{y_n^\phi\right\}$.

By ψ l.h.c., for $\left\{y_n^\phi\right\} \to y$, there exist $N_2 \geq 1$ and $\{z_n\}$ such that $z_n \to z$ and $z_n \in \phi(y_n^\phi)$ for all $n \geq N_2$. Take $N = \max\{N_1, N_2\}$. Hence, $\Gamma(x)$ is l.h.c. at x. Because x was arbitrary chosen, the proof is complete.

Exercise 3.13

a. Same as part b. with $f(x) = x$.

b. Choose any x. Since $0 \in \Gamma(x)$, $\Gamma(x)$ is nonempty. Choose any $y \in \Gamma(x)$ and consider the sequence $x_n \to x$. Let $\gamma \equiv y/f(x) \leq 1$ and $y_n = \gamma f(x_n)$. Then $y_n \in \Gamma(x_n)$, all $n \geq 1$, and using the continuity of f

$$\lim y_n = \gamma \lim f(x_n) = \gamma f(x) = y.$$

Hence Γ is l.h.c. at x.

Given x, $[0, f(x)]$ is compact and hence $\Gamma(x)$ is compact-valued. Take arbitrary sequences $x_n \to x$ and $y_n \in \Gamma(x_n)$. Define $\epsilon = \sup_{x_n} \|x_n - x\|$ and let $N(x, \epsilon)$ denote the closed $\epsilon-$neighborhood of x. Since the set

$$\{z : z \in \left[0, \overline{f}\right], \overline{f} = \max_{x' \in N(x,\epsilon)} f(x')\},$$

is compact, there exists a convergent subsequence of y_n call it y_{n_k} with $\lim y_{n_k} \equiv y$. Since $y_{n_k} \leq f(x_{n_k})$ all k, we know that $y \leq f(x)$ by the continuity of f and standard properties of the limit. Hence $y \in \Gamma(x)$ and Γ is u.h.c. at x.

Since x was chosen arbitrarily, Γ is a continuous correspondence.

c. Since the set

$$\left\{\left(x^1, ..., x^l\right) : \sum_{i=1}^{l} x^i \leq x\right\},$$

is compact, fix $(x^1, ..., x^l)$ and proceed coordinate by coordinate using the proof in b. with $f(x) = f_i(x^i, z)$.

Exercise 3.14

a. Same as part b., with the following exceptions. Suppose $x \neq 0$; let 0 play the role of $\hat{y}$ (since $H(x, 0) > H(0, 0) = 0$), and use monotonicity rather than concavity to establish all the necessary inequalities. For $x = 0$, use monotonicity and the fact that the sequence $\{x_n\}$ must converge to $x = 0$ from above.

b. We prove first that Γ is l.h.c. Fix x. Choose $y \in \Gamma(x)$ and $\{x_n\} \to x$. We must find a sequence $\{y_n\} \to y$ such that $y_n \in \Gamma(x_n)$, all n.

Suppose that $H(x, y) > 0$. Since H is continuous, it follows that for some N, $H(x_n, y) > 0$, all $n \geq N$. Then the sequence $\{y_n\}_{n=N}^{\infty}$ with $y_n = y$, $n \geq N$, has the desired property.

Suppose that $H(x, y) = 0$. By hypothesis there exists some $\hat{y}$ such that $H(x, \hat{y}) > 0$. Since H is continuous, there exists some N such that $H(x_n, \hat{y}) > 0$, all $n \geq N$. Define $y^\lambda = (1 - \lambda)y + \lambda\hat{y}$, $\lambda \in [0, 1]$. Then for each $n \geq N$, define

$$\lambda_n = \min\left\{\lambda \in [0, 1] : H(x_n, y^\lambda) \geq 0\right\}.$$

Since $H(x_n, y^1) = H(x_n, \hat{y}) > 0$, the set on the right is nonempty; clearly it is compact. Hence the minimum is attained.

Next note that $\{\lambda_n\} \to 0$. To see this, notice that by the concavity of H,

$$H(x, y^\zeta) \geq (1 - \zeta)H(x, y) + \zeta H(x, \hat{y}) > 0, \quad \text{all } \zeta \in (0, 1].$$

Hence, for any ζ, there exist N_ζ such that $H(x, y^\zeta) \geq 0$, all $n \geq N_\zeta$. Therefore $\lambda_n \leq \zeta$, for all $n \geq N_\zeta$. Hence $\{\lambda_n\} \to 0$. Therefore, the sequence $y_n = y^{\lambda_n}$, $n \geq N$, has the desired properties. By construction, $H(x_n, y_n) \geq 0$, all n, so $y_n \in \Gamma(x_n)$, all n, and since $\{\lambda_n\} \to 0$, it follows that $\{y_n\} \to y$.

Next, we prove that Γ is u.h.c. Choose $\{x_n\} \to x$ and $\{y_n\}$ such that $y_n \in \Gamma(x_n)$, all n. We must show that there exist a convergent subsequence of $\{y_n\}$ whose limit point y is in $\Gamma(x)$. It suffices to show that the sequence $\{y_n\}$ is bounded. For if it is, then it has a convergent subsequence, call it $\{y_{n_k}\}$, with limit y. Then, since $H(x_{n_k}, y_{n_k}) \geq 0$, all k, $\{(x_{n_k}, y_{n_k})\} \to (x, y)$, and H is continuous, it follows that $H(x, y) \geq 0$.

Let $\|\cdot\|$ denote the Euclidean norm in $\mathbf{R}^m$.Choose $M < \infty$ such that $\|y\| < M$, all $y \in \Gamma(x)$. Since $\Gamma(x)$ is compact, this is possible. Suppose $\{y_n\}$ is not bounded. Then, there exist a subsequence $\{y_{n_k}\}$ such that $N < n_1 < n_2 ...$ and $\|y\| > M + k$, all k. Define $S = \{y \in \mathbf{R}^m : \|y\| = M + 1\}$, which is clearly a compact set. Since $\|\hat{y}\| < M$, and $\|y_{n_k}\| > M + k$, all k, for any element in the sequence $\{y_{n_k}\}$, there exists a unique value $\lambda \in (0, 1)$ such that

$$\|\tilde{y}_{n_k}\| = \|\lambda y_{n_k} + (1 - \lambda)\hat{y}\| + M + 1.$$

Moreover, since $H(x_{n_k}, \hat{y}) > 0$ and $H(x_{n_k}, y_{n_k}) \geq 0$, it follows from the concavity of H that $H(x_{n_k}, \tilde{y}_{n_k}) > 0$, all k. Since by construction the sequence $\{\tilde{y}_{n_k}\}$ lies in the compact set S, it has a convergent subsequence; call this subsequence $\{\tilde{y}_j\}$ and call its limit point $\tilde{y}$. Note that since $\tilde{y} \in S$, $\|\tilde{y}\| = M + 1$. Along the chosen subsequence, $H(x_j, \tilde{y}_j) > 0$, all j; and $\{(x_j, \tilde{y}_j)\} \to (x, \tilde{y})$. Since H is continuous, this implies that $H(x, \tilde{y}) \geq 0$. But then $\|\tilde{y}\| = M + 1$, a contradiction.

c. The correspondence can be written in this case as

$$\begin{aligned} \Gamma(x) &= \{y \in \mathbf{R} : H(x, y) \geq 0\} \\ &= \{y \in \mathbf{R} : 1 - \max\{|x|, |y|\} \geq 0\}. \end{aligned}$$

It can be checked that, at $x = 1$, $\Gamma(x)$ is not lower hemi-continuous. Notice that $\Gamma(1) = [-1, +1]$, which is compact and has a nonempty interior, but $\Gamma(1 + 1/n) = 0$, for all $n > 0$.

Exercise 3.15

Let $\{x_n, y_n\}$ be a sequence in A. We need to show that this sequence has a convergent subsequence. Because X is compact, the sequence $\{x_n\}$ has a convergent subsequence, say $\{x_{n_k}\}$ converging to $x \in X$. Because Γ is u.h.c., every sequence $x_n \to x \in X$, has an associated sequence $\{y_n\}$ such that $y_n \in \Gamma(x_n)$, all n, with a convergent subsequence, say $\{y_{n_k}\}$ whose limit point $y \in \Gamma(x)$. Then, $\{(x_n, y_n)\} \in A$, has a convergent subsequence with limit (x, y).

Exercise 3.16

a. The correspondence G is

$$\begin{aligned} G(x) &= \left\{ y \in [-1,1] : xy^2 = \max_{\tilde{y} \in [-1,1]} x\tilde{y}^2 \right\} \\ &= \{ y \in [-1,1] : y = 0 \text{ for } x < 0, \\ &\quad\ y \in [-1,1] \text{ for } x = 0,\ y = \pm 1 \text{ for } x > 0 \}. \end{aligned}$$

Thus, $G(x)$ can be drawn as shown in Figure 3.1.

Insert Figure 3.1 About Here.

Then, $G(x)$ is nonempty, and it is clearly compact valued. Furthermore, A, the graph of G, is closed in $\mathbf{R}^2$ since it is a finite union of closed sets. Hence, by Theorem 3.4, $G(x)$ is u.h.c.

To see that $G(x)$ is not l.h.c. at $x = 0$, choose an increasing sequence $\{x_n\} \to x = 0$ and $y = 1/2 \in G(0)$. In this case, any $\{y_n\}$ such that $y_n \in G(x_n)$ implies that $y_n = 0$, so $\{y_n\} \to 0 \neq 1/2$.

b. Let be $X = \mathbf{R}$.

Then,

$$\begin{aligned} h(x) &= \max_{y \in [0,4]} \left\{ \max\left\{ 2 - (y-1)^2, x + 1 - (y-2)^2 \right\} \right\} \\ &= \max \left\{ \max_{y \in [0,4]} [2 - (y-1)^2], \max_{y \in [0,4]} [x + 1 - (y-2)^2] \right\} \\ &= \max\{2, x+1\} \end{aligned}$$

Hence,

$$h(x) = \begin{cases} 2 & \text{if } x \leq 1 \\ x+1 & \text{if } x > 1 \end{cases}$$

Then,

$$\begin{aligned} G(x) &= \{y \in [0,4] : y = 1 \text{ for } x < 1, \\ &\quad y \in \{1,2\} \text{ for } x = 1,\ y = 2 \text{ for } x > 1\}, \end{aligned}$$

which is represented in Figure 3.2.

Insert Figure 3.2 About Here

Evidently, $G(x)$ is nonempty and compact valued. Further, its graph is closed in $\mathbf{R}^2$ since the graph is given by the union of closed sets. Thus, $G(x)$ is u.h.c.

However, $G(x)$ is not l.h.c. at $x = 1$. To see this, let $\{x_n\} \to 1$ for $x_n > 1$ and $y = 1 \in G(1)$. It is clear that any sequence $\{y_n\}$ such that $y_n \in G(x_n)$ converges to $2 \neq 1$.

c. Here,

$$h(x) = \max_{-x \leq y \leq x} \{\cos(y)\} = 1$$

and hence

$$G(x) = \{y \in [-x, x] : \cos(y) = 1\}.$$

Then, since $\cos(y) = 1$ for $y = \pm 2n\pi$, where $n = 0, 1, 2, ...$, the correspondence $G(x)$ can be depicted as in Figure 3.3.

Insert Figure 3.3 About Here

The argument to show that $G(x)$ is u.h.c. is the same outlined in b. However, $G(x)$ is not l.h.c. at $x = \pm 2n\pi$, where $n = 0, 1, 2, ...$; which can be proved using the same kind of construction of sequences developed before.