Macro, Money, and Finance:

a continuous-time approach

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2025

PUC Rio

Introduction

History of Macro and Finance - one slide overview

Integrated view: Fisher, Keynes, Hayek, ...
Verbal Reasoning, qualitative approach

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Macro

- Growth theory
  - Dynamic (cts time)
  - Deterministic

  $\Rightarrow$ Solow
- Introduce stochasticity
  - Discrete time

  $\Rightarrow$ Brock-Mirman (stochastic growth model), Stokey-Lucas
  $\Rightarrow$ DSGE models

Finance

- Portfolio theory
  - Static
  - Stochastic

  $\Rightarrow$ Markowitz portfolio theory
- Introduce dynamics
  - Continuous time

  $\Rightarrow$ Options (Black -Scholes)
  $\Rightarrow$ Term structure (CIR)
  $\Rightarrow$ Agency theory (Sannikov)

$\searrow$ $\swarrow$

Continuous-time macro with financial frictions

What is Macro-Finance?

- **Macro**: resource allocation
  - real frictions (e.g., adjustment costs for investment, search and matching in labor markets, etc.)
- **Finance**: risk allocation
  - financial / contracting frictions (e.g., equity-issuance constraint, collateral constraint, absence of insurance markets, etc.)

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  - inside money creation

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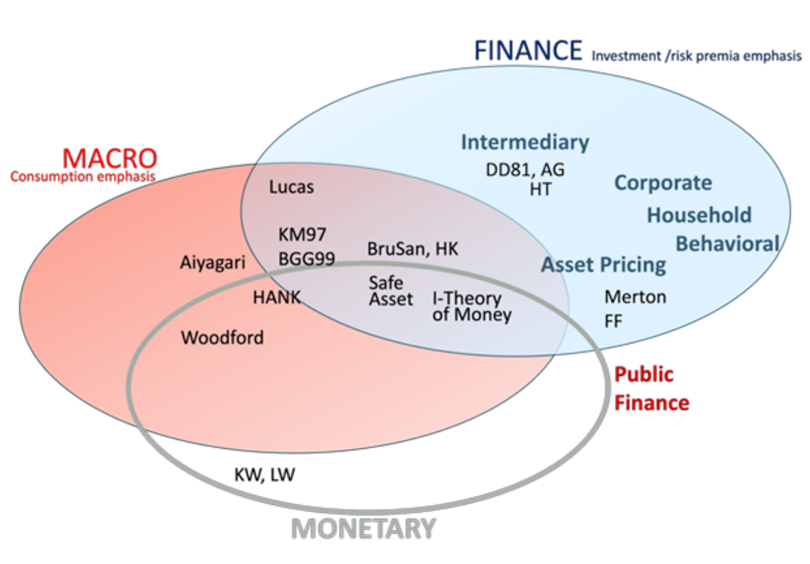
Key: How to design the financial sector / markets (e.g. government bond market) to achieve optimal resource and risk allocation

Topics

- Amplification, percolation, persistence of shocks, resilience, financial cycle.
- Financial stability, spillovers, systemic risk measures.
- (Un)conventional monetary policy, central bank policy, balance sheet management, CBDC
- Capital flows

Macro-Finance



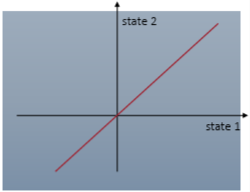
Financial frictions and heterogeneity

- **Lending-borrowing / insuring** because agents are different ...
  - More productive *vs* less productive
  - Impatient *vs* patient
  - Risk-tolerant *vs* Risk-averse
  - Optimist *vs* Pessimist
  - Rich *vs* poor

  ... but lending-borrowing / insuring is limited due to **financial frictions**.
- Financial frictions: **different prices / SDFs / MRSs** differ (after transactions) $\Rightarrow$ **wealth distribution** among subgroups matters!
- **Financial sector** is **not a veil**. It *reduces the frictions but not perfectly*.

Financial Frictions and Distortions

- **Incomplete markets**
  - "Natural" leverage constraint (BruSan)
  - Costly state verification (BGG)
- **+ Leverage constraint**
  - Exogenous limit (Bewley/ Ayagari)
  - Collateral constraints
    - Current price: $D_t \leq q_t k_t$
    - Next period's price (KM): $D_t \leq q_{t+1} k_t$
    - Next period's VaR (BruPed): $D_t \leq VaR_t(q_{t+1}) k_t$
- **Search friction** (DGP, LW)
- **Belief distortions**



Financial Sector

- **Financial sector** helps to: (i) overcome financing frictions; (ii) channels resources; (iii) creates money
- but ... when the financial sector becomes **impaired** (e.g., limited net worth)
  - Credit crunch due to **adverse feedback loops & liquidity spirals**
  - **Non-linear dynamics**
- New insights about financial stability and monetary policy.

Macro: Consumer vs Finance focused

Consumption decision – traditional macro

- Demand management
  - Sticky prices make output demand determined.
  - Interest rate affect consumption (Euler equation).
  - Liquidity trap (ZLB): alternatives to push aggregate demand.
- Expectations — *reduced/no role for risk premia*
  - Expectations hypothesis: all returns equalized
  - Uncovered interest parity (UIP) in international models
- Heterogeneity: wealth distribution across consumers (het. MPCs key)

Investment and portfolio decision – macrofinance

- Risk-free rate and **time-varying risk-premia**

$$\textit{risk premia} = \textit{price of risk} \times (\textit{exogeneous risk} + \textit{endogenous risk})$$

  - Endo. risk: runs, sudden stops, spirals (*Non-linearities, multiplicity*)
  - $\Delta \textit{asset price} = f(\Delta \mathbb{E}[\textit{future cash flows}], \Delta \textit{risk premia}, \Delta \textit{risk-free rate})$
- Heterogeneity: wealth distribution across investors.
- Financial frictions: w/ complete market, aggregation theorems kick in.

MacroFinance vs HANK

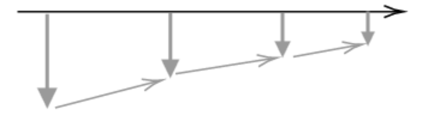
Agents	**Heterogenous investor focus** - Net worth distribution (often discrete)	**Heterogenous consumer focus** - Net worth distribution (often cts.)
Tradition:	Finance (Merton) *Portfolio and consumption choice* - Full/global dynamical system - Focused on non-linearities away from steady state (crisis ...) - Length of recession is stochastic	DSGE (Woodford) *Consumption choice* - Zero probability shock - Deterministic transition dynamics back to steady state - Length of recession deterministic
Risk	Risk and Financial Frictions	No aggregate risk (in HANK paper)
Price of risk:	Idiosyncratic and aggregate risk	N/A
Assets:	Capital, money, bonds with different risk profile - Risk-return trade-off - Liquidity-return trade-off - Flight-to-safety	All assets are risk free - No risk-return trade-off - Liquidity-return trade-off
Money:	Risk and Financial Frictions	Price stickiness

Introduction

Seminal contributions: persistent, resilience, and (dynamic) amplification

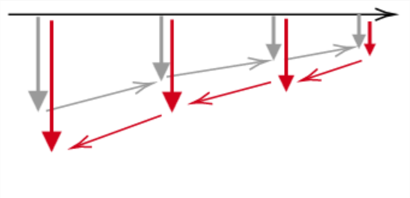
Persistence and Resilience

- Even in standard real business cycle models, temporary adverse shocks can have long-lasting effects.
- Due to feedback effects, **persistence** is much stronger in models with financial frictions.
  - Bernanke Gertler (1989)
  - Carlstrom Fuerst (1997)
- Negative shocks to net worth exacerbate frictions and lead to lower capital, investment and net worth in future periods.



Persistence leads to dynamic amplification

- **Static amplification** occurs because fire-sales of capital from productive sector to less productive sector depress asset prices.
  - Importance of *market liquidity* of physical capital
- **Dynamic amplification** occurs because a temporary shock translates into a persistent decline (in output and asset prices) and asset prices are forward looking.
  - Forward in time: grow net worth via retained earnings
  - Backward in time: asset pricing $\rightarrow$ tightens constraints
  - Kiyotaki Moore (1997), Bernanke, Gertler, Gilchrist (1999)



Introduction

“Single Shock" critique and *second generation* models

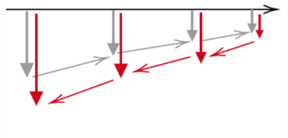
"Single Shock" Critique

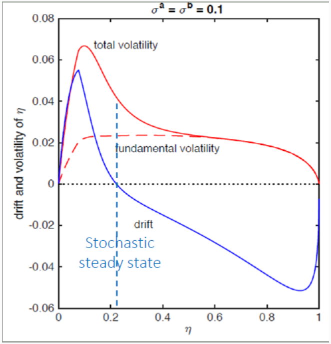
- **Critique**: After the shock all agents in the economy know that the economy will deterministically return to the steady state.
  - Seminal models: length of slump is deterministic (and commonly known)
    - *No safety cushion* needed
  - Reality: adverse shock may be followed by additional adverse shocks
    - *Build-up extra safety cushion* for an additional shock in a crisis
- Impulse response vs. volatility dynamics
  - This course provides a toolkit to have volatility and risk price dynamics.

Endogenous Volatility

- Endogenous risk / volatility dynamics (BruSan)

beyond impulse responses ... $\Rightarrow$





- Precautionary savings $\Rightarrow$ Role for money / safe assets
- Nonlinearities in crisis $\Rightarrow$ endogenous fait tails, skewness
- Volatility Paradox: Low exogenous (measured) volatility leads to high build-up of (hidden) endogenous volatility (Minksy)

Introduction

Why continuous time?

Why continuous time modeling?

- **Analytic advantages**
  - Further characterization of equilibrium dynamics, in particular, of time-varying second moments
- **Computational advantages**
  - Global solution of equilibrium dynamics via solving PDEs
  - Key to study rare situations (e.g., financial crises)
- **Time aggregation**
  - Data come in different frequency (GDP quarterly, High frequency financial data)
- **Consumption IES**
  - Discrete time: IES/RA within period $= \infty$, but across periods $= 1/\gamma$.
  - Cts time: same IES within and across periods
- **Optimal Stopping problems** – no integer issues

Analytic advantage: characterization of dynamics

- Ito diffusion processes... fully characterized by drift and volatility

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dZ_t$$

- Characterization for **full volatility dynamics** on Prob.-space
  - Discrete time: Probability-loading on states. Conditional expectations difficult to handle.
  - Cts time: Loading on a Brownian Motion captured by σ.
- Normal distribution for dt, yet with skewness for $\Delta t > 0$.
  - If σ_t is time-varying, we can get skewness over a period of positive length. More generally, you can approximate any distribution.

Analytic advantage: characterization of dynamics

- Consumption choice
  - Nice process: consumption/wealth ratio is constant for log-utility
  - Beginning = end of period net worth/wealth
- Evolution of wealth (shares)/distribution
  - Evolution of distributions (e.g. wealth distribution) characterized by **Kolmogorov Forward Equation** (Fokker-Planck equation)

Continuity of Ito Processes

Continuous path (assumption)

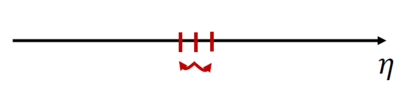
- Information arrives continuously "smoothly" – not in lumps.
- Implicit assumption: can react continuously to continuous info flow
- Never jumps over a specific point / threshold, e.g. insolvency point
- No default risk
  - Can continuously delever as wealth declines (might embolden investors ex-ante)
- Collateral constraint (short term debt)
  - Discrete time: $b_t R_{t,t+1} \leq \min_\omega q_{t+1}(\omega) k_t$
  - Cts time: $b_t \leq (p_t + \underbrace{dp_t}_{\to 0}) k_t$
- Levy processes... with jumps

Conditional Expectations for Ito processes

- discrete time: e.g. $E[V(\eta)]$
  - Need function $V(\eta)$ across **all states** η
  - Simulate η to obtain probability weights
- continuous time:

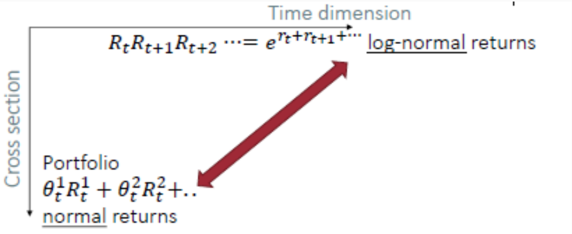
$$\mathbb{E}[dV(\eta)] = V'(\eta)\mu^{\eta}dt + 0.5V''(\eta)(\sigma^{\eta})^2 dt$$

  - derivatives approximated with neighboring grid points
  - continuous paths: only to the right or left in a short interval.



  - Similar for price function $q(\eta)$.
    $\rightarrow$ requires only slope of price function to determine amplification instead of whole price function across all η in discrete time.

- Portfolio choice – tension in discrete time



  - Linearize: kills σ-term, all assets are equivalent
  - 2nd order approximation: kills time-varying σ
  - Log-linearize a la Campbell-Shiller

- As $\Delta t \to 0$ (log) returns converge to normal distribution
  - Constantly adjust the approximation point
  - Nice formula for portfolio choice for Ito process

Computational Advantages relative to Discrete Time

- Borrowing constraints only show up in boundary conditions
  - FOCs always hold with "$=$"
- "Tomorrow is today"
  - FOCs are "static", compute by hand: $c^{-\gamma} = v_a(a, y)$.
- Sparsity
  - solving Bellman, distribution $=$ inverting matrix
  - but matrices very sparse ("tridiagonal")
  - reason: continuous time $\Rightarrow$ one step left or one step right
- Two birds with one stone
  - tight link between solving (HJB) and (KF) for distribution
  - matrix in discrete (KF) is transpose of matrix in discrete (HJB)
  - reason: diff. operator in (KF) is adjoint of operator in (HJB)

Stochastic calculus basics

Measure theory: basic concepts

- **Uncertainty** represented by countable states of the world
  - $\Omega = \{\omega_1, \omega_2, \omega_3 ...\}$
  - $F =$ sigma algebra: collection of subsets of Ω such that
    - $\emptyset \in F$
    - $A \in F \Rightarrow A^c := \Omega - A \in F$
    - $A, B \in F \Rightarrow A \cup B \in F$
  - elements of F are called events
  - measurable space: (Ω, F)

- **Measure**
  - Measure: function $\mu : F \to \mathbb{R} \cup \{\infty\}$ s.t.
    - Non-negativity: $\mu(A) \geq 0$
    - Null empty set: $\mu(\emptyset) = 0$
    - Countable additivity: If $A_i \in F$ are disjoint sets for $i \in \mathbb{N}$ then

$$\mu\left(\cup_i A_i\right) = \sum_i \mu\left(A_i\right)$$

  - measure space: (Ω, F, μ)

Probability theory: basic concepts

- **Probability measure**
  - Function $P : F \to [0, 1]$ satisfying
    - $P(\emptyset) = 0$
    - $P(\Omega) = 1$
    - A, B disjoint, i.e., $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$
  - Probability space: (Ω, F, P)

- **Random variable**
  - Probability space (Ω, F, P) and measure space $(\mathbb{R}, \mathcal{B})$
  - Random variable $X : \Omega \to \mathbb{R}$ such that

$$\forall B \in \mathcal{B} : \{\omega \in \Omega : X(\omega) \in B\} \in F$$

I.e., X is F-measurable.

- Sigma-algebra **generated by a collection of subsets**: add subsets necessary to satisfy the definition of a sigma-algebra.
- Sigma-algebra **generated by a random variable**
    - Sigma-algebra generated by the pre-images of random variable
    - Smallest sigma-algebra respect to which the random variable is measurable (basically, by definition)
    - Notation: F^X
- Sigma-algebra as a **measure of information**
    - If $F^X \subset F^Y$, then Y is "more informative" than X.
    - Simple example: $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, $X(\omega_1) = X(\omega_2) = 10$, $X(\omega_3) = X(\omega_4) = 20$, $Y(\omega_1) = 1$, $Y(\omega_2) = 2$, $Y(\omega_3) = 3$, $Y(\omega_4) = 4$.

Probability theory: basic concepts

Dynamics: how is uncertainty revealed over time?

- (Ω, F, P) probability space, F is sigma algebra
- **Filtration** = sequence of increasing sigma algebras

  $$\mathcal{F} = \{F_0, F_1, F_2, ...\}$$

  such that

  - no memory loss: $F_t \subset F_s$ for $s \geq t$
  - F_0 trivial: $A \in F_0 \Rightarrow P(A) \in \{0, 1\}$

  $F_t \subset F$ is a set of events that are known at time t

  Notation: Later, we will also use $\{\mathcal{F}_t\}$ to denote the filtration, and $\mathcal{F}_t$ will denote the sigma-algebra at t.

- **Adapted process** X is a sequence $\{X_0, X_1, ...\}$ such that X_t is a random variable with respect to (Ω, F_t).
- A **filtration generated by the random process** is the sequence of sigma-algebras generated by each random variable of the stochastic process $\mathcal{F}^X$.
- Filtered probability space: $(\Omega, F, \mathcal{F}, P)$

Stochastic processes: basic concepts

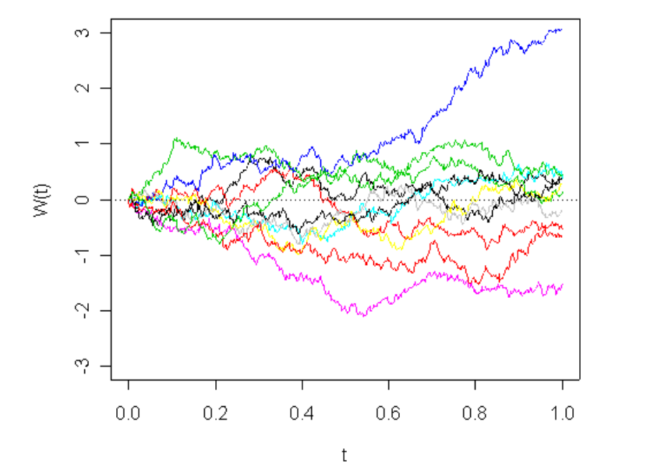
- **Stochastic / random process**
  - Parametrized collection of random variables

    $$\{X_t\}_{t \in T}$$

    defined on probability space (Ω, F, P) and assuming values in $\mathbb{R}^n$.
  - usually t indexes time and $T = [0, \infty)$
  - For each t, $X_t(\omega) : \Omega \to \mathbb{R}^n$ is a random variable.
  - For each ω, $X_t(\omega) : T \to \mathbb{R}^n$ is called a path.

- **Brownian Motion**
  - Stochastic process Z_t with $t \in [0, \infty)$ is a Brownian motion on filtration $\{\mathcal{F}_t\}$ if
    - $Z_0 = 0$
    - $Z_{t+s} - Z_t \sim N(0, s)$
  - Filtration generated by a Brownian motion $\{\mathcal{F}^Z\}$

Stochastic processes: Standard Brownian motion



- Mean zero $\mathbb{E}[Z_t] = 0$ but variance blows up $\mathbb{V}ar(Z_t) = t$.

- **Brownian Motion**: some **properties**
  - Continuous sample paths.
  - These paths are highly irregular, and not differentiable.
  - During the time interval of length s the Brownian motion moves by a distance on the order of $\sqrt{s}$. When s small, $\sqrt{s}$ is disproportionately large in comparison.
  - Thus, a Brownian motion oscillates very wildly on a small scale and has a positive quadratic variation ($[Z]_t = t$). Quadratic variation of a random process X is

$$[X]_t := \lim \sum_{k=1}^{n} (X_{t_k} - X_{t_{k-1}})^2$$

    as the partitions $0 = t_0 < t_1 < ... t_n = t$ of the interval $[0, t]$ get finer and finer.
  - If $Z_t = x$; then it passes through x infinitely many times from time t to time $t + \epsilon$.

- **Martingales**
  - A stochastic process M_t is a martingale on $\{\mathcal{F}_t\}$ if for all $t \geq 0$ and $s \geq 0$,

    $$\mathbb{E}[M_{t+s}|\mathcal{F}_t] = M_t$$

    where the expectation conditions on all the information known at time t.
  - **Super-martingale**: process that goes down in expectation

    $$\mathbb{E}[M_{t+s}|\mathcal{F}_t] \leq M_t$$

  - **Sub-martingale**: process that goes up in expectation

    $$\mathbb{E}[M_{t+s}|\mathcal{F}_t] \geq M_t$$

Stochastic integral

- If $\{\beta_t\}_{t\geq 0}$ is a random process, then the stochastic integral

$$M_t = \int_0^t \beta_s dZ_s$$

  is also a stochastic process. Recall $\beta : \Omega \times T \to \mathbb{R}$.
- How do we define this integral? If we divide the the interval $[0, 1]$ into a fine partition $0 = t_0 < t_1 < ...t_n = t$, then

$$\int_0^t \beta_s dZ_s \approx \beta_{t_0}(Z_{t_1} - Z_{t_0}) + \beta_{t_1}(Z_{t_2} - Z_{t_1}) + ... + \beta_{t_N}(Z_{t_N} - Z_{t_{N-1}})$$

  provided that the process β_t is sufficiently "nice".
- The **stochastic integral adds up the oscillations of the Brownian motion** Z, by multiplying the increment at time s with a **weight** β_s.
- Formally, we define the stochastic integral for simple processes (i.e, those that are constant within intervals of the partition) using the last equation. Then, definition is extended to the space of processes that satisfy $\mathbb{E}[\int_0^t \beta_s^2 ds] < \infty$.

Stochastic integral

- Most used notation: M_0 given and

$$dM_t = \beta_t dZ_t$$

Ito processes

- An Ito process X_t satisfies

$$X_t = X_0 + \int_0^t \mu(\omega, s) ds + \int_0^t \beta(\omega, s) dZ_s$$

  where the first integral should be interpreted as an ordinary, non-stochastic integral.

- Differential notation: X_0 given and

$$dX_t = \mu(\omega, t) dt + \beta(\omega, t) dZ_t$$

- $\mathbb{E}_t(dX_t) = \mu(\omega, t) dt$ follows from the fact that $dZ_t \sim N(0, dt)$

Ito diffusion

- Ito diffusion satisfies

$$dX_t = \mu(X_t, t)dt + \beta(X_t, t)dZ_t$$

where μ and β are sufficiently "nice".

Time-homogeneous Ito diffusion

- Ito diffusion satisfies

$$dX_t = \mu(X_t)dt + \beta(X_t)dZ_t$$

where μ and β are sufficiently "nice".

- **Markov property**: All information until t summarized by X_t.
- **Approximation results**: By choosing functions μ and σ, you can get pretty much any continuous stochastic process you want (except jumps).
- We will be dealing with these processes most of the class (except for the section with jumps)

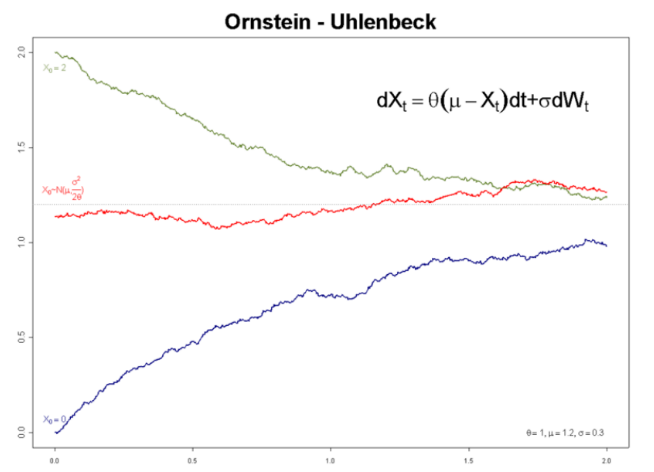
Ornstein-Uhlenbeck Process

- Brownian motion (with a constant drift) $dX_t = \mu dt + \sigma dZ_t$ is not stationary.
- But we can choose a $\mu(.)$ to make it stationary. For example,

$$dX_t = \theta(\bar{x} - X_t)dt + \sigma dZ_t$$

- Analogue of AR(1) process $X_{t+1} = \theta\bar{x} + (1-\theta)X_t + \sigma\varepsilon_t$
- Autocorrelation $e^{\theta} \approx 1 - \theta$

Stochastic processes: Examples



- Stationary distribution: $N(\bar{x}, \sigma^2/(2\theta))$

"Moll" Process

- Design a process that stays in the interval $[0,1]$ and mean-reverts around $1/2$.
- Diffusion goes to zero at boundaries $\sigma(0) = \sigma(1) = 0$ and mean-reverts $\Rightarrow$ always stay in $[0,1]$
- How can we be sure it never escapes $[0,1]$?

Geometric Brownian motion

$$dX_t = X_t \mu dt + X_t \sigma dZ_t$$

- $X \in [0,\infty)$, no stationary distribution but we can write

$$\log X_t \sim N((\mu - \sigma^2/2)t, \sigma^2 t)$$

Feller square root process (finance: "Cox-Ingersoll-Ross")

$$dX_t = \theta(\bar{x} - X_t)dt + \sigma\sqrt{X_t}dZ_t$$

- $X \in [0, \infty)$, stationary distribution is $\Gamma(\gamma, 1/\beta)$, i.e.,

$$g(X) \propto e^{-\beta}X^{\gamma-1}$$

  where $\beta = 2\theta/\sigma^2$, $\gamma = 2\theta\bar{x}/\sigma^2$

Other processes in Wong (1964), "The Construction of a Class of Stationary Markoff Processes"

One-dimensional Ito formula

- Let X_t be an Ito process and $g(t, x) \in C^2[T \times \mathbb{R}]$, then
  - $Y_t = g(t, X_t)$ is an Ito process and

$$dY_t = \frac{\partial g}{\partial t} dt + \frac{\partial g}{\partial x}(dX_t) + 0.5\frac{\partial^2 g}{\partial x^2}(dX_t)^2$$

- Multiplication "rules": $dt^2 = 0$, $dZ_t dt = 0$, $(dZ_t)^2 = dt$.

General Ito formula

- Let $X_t \in \mathbb{R}^n$ be a n-dimensional Ito process associated with a m-dimensional Brownian motion $Z_t \in \mathbb{R}^m$, and $g : T \times \mathbb{R}^n \to \mathbb{R}^p$ be C^2, then
  - $Y_t = g(t, X_t)$ is a p-dimensional Ito process and its k-th entry follows

$$dY_{k,t} = \frac{\partial g_k}{\partial t} dt + \sum_{i=1}^{n} \frac{\partial g_k}{\partial x_i}(dX_{i,t}) + 0.5 \sum_{i,j=1}^{n} \frac{\partial^2 g_k}{\partial x_i \partial x_j}(dX_{i,t})(dX_{j,t})$$

  for $k = \{1, 2.., p\}$, where $dZ_{u,t} dZ_{v,t} = 0$ for $u, v \in \{1, 2, ..., m\}$ and $u \neq v$.

General Ito formula

- Example: Let $g : T \times \mathbb{R}^2 \to \mathbb{R}$, then

$$dY_t = \frac{\partial g}{\partial t} dt + \frac{\partial g}{\partial x_1}(dX_{1,t}) + \frac{\partial g}{\partial x_2}(dX_{2,t}) + 0.5\frac{\partial^2 g}{\partial x_1^2}(dX_{1,t})^2 + 0.5\frac{\partial^2 g}{\partial x_2^2}(dX_{2,t})^2 \dots$$
$$\dots + \frac{\partial^2 g}{\partial x_1 \partial x_2}(dX_{1,t})(dX_{2,t})$$

- Special case: product rule. If $g(x_1, x_2) = x_1 x_2$, then

$$dY_t = (dX_{1,t})X_{2,t} + X_{1,t}(dX_{2,t}) + (dX_{1,t})(dX_{2,t})$$

Stochastic processes: Generator of an Ito diffusion

- The **infinitesimal generator** $\mathcal{A}$ of an Ito diffusion $X_t \in \mathbb{R}^n$ is defined by

$$\mathcal{A}f(x,t) := \lim_{s \searrow 0} \frac{\mathbb{E}_t[f(X_{t+s}, t+s) | X_t = x] - f(x,t)}{s}$$

- If $f \in C^2$ and $X_t \in \mathbb{R}$

$$\mathcal{A}f(x,t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\mu(x,t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma(x,t)^2$$

- Note that

$$\mathcal{A}f(X_t,t) = \frac{\mathbb{E}_t[df(X_t,t)]}{dt}$$

so we can get it by applying Ito's formula and taking expectation (which drops that stochastic terms).

Consider the following control problem

$$\max_{\{A_t\}} \mathbb{E}\left[\int_0^\infty e^{-rt} g(X_t, A_t) dt\right]$$

s.t.

$$dX_t = \mu\left(X_t, A_t\right) dt + \sigma(X_t, A_t) dZ_t$$

where X_t is a vector of n_x states, A_t is a vector of n_a controls, and dZ_t a vector of n_z Brownian motions. X_0 is given.

Stochastic HJB Equation

Look for $V(X)$ that solves the following functional equation

$$0 = \max_A g(X, A) + \frac{\mathbb{E}[dV(X)]}{dt} - rV(X)$$

and the associated policy functions $A(X)$.

Stochastic optimization: dynamic programming

Stochastic HJB Equation

- For the single state case, we can write (using Ito formula)

$$0 = \max_{A} g(X, A) + \mu(X, A)V'(X) + 0.5\sigma(X, A)^2 V''(X) - rV(X)$$

- Formally, the value and policy functions found by solving the HJB equation are a solution candidate.
- We still need to verify that the candidate policy is indeed optimal and the guess of the continuation value process is correct. How?

Verification argument

- Define auxiliary process

$$G_t = [\text{PV of payoff received until } t] + e^{-rt}[\text{continuation value at time } t]$$

- G_t is the value at time 0 of following a given policy until t and then go to the (candidate to) optimal one (and therefore getting the associated continuation value).
- Key step: check that the process G_t is a martingale under the conjectured optimal policy, and a super-martingale (i.e. goes down in expectation) under any alternative policy.

Stochastic optimization: dynamic programming

Verification argument

- Intuition: It is not necessary to optimize the control over the entire time interval $[0, \infty)$ at once, we can partition the time interval into smaller chunks and optimize individually. Most powerful: partition size goes to 0 (taking differential).
- For our problem,

$$G_t = \int_0^t e^{-rs} g(X_s, A_s) ds + e^{-rt} V(X_t)$$

then

$$dG_t = e^{-rt}\left(g(X_t, A_t) + \frac{\mathbb{E}[dV(X_t)]}{dt} - rV(X_t)\right) dt + e^{-rt} V'(X_t)\sigma(X_t, A_t) dZ_t$$

- Given that we found our candidate policy and associated value function from the HJB, we have that G_t is a martingale ($\mathbb{E}[dG_t] = 0$) under this policy and a super-martingale ($\mathbb{E}[dG_t] \leq 0$) under alternative ones.
- We cannot improve total payoff by local deviations.

Verification argument

- Knowing that G_t is a martingale under the candidate policy, we can elaborate a simple verification argument.
- Note that $G_0 = V(X_0)$, then we have

$$V(X_0) = G(X_0) \geq \mathbb{E}[G_t] = \mathbb{E}\left[\int_0^t e^{-rs} g(X_s, A_s) ds + e^{-rt} V(X_t)\right]$$

  under any admissible policy and with equality under the candidate optimal policy.
- We would like to take the limit as $t \to \infty$, and claim that

$$V(X_0) \geq \mathbb{E}\left[\int_0^\infty e^{-rs} g(X_s, A_s) ds\right]$$

  under any admissible policy and with equality under the candidate optimal policy. This proofs that the candidate policy achieves the associated value function and that it is a solution to our problem.

Verification argument

- The latter is true if

$$\lim_{t\to\infty} \mathbb{E}[e^{-rt}V(X_t)] = 0$$
$$\lim_{t\to\infty} \mathbb{E}\left[\int_t^{\infty} e^{-rs} g(X_s, A_s) ds\right] = 0$$

  The first condition is known as the transversality condition (TVC).

- What can go wrong with verification?
  - See examples of HJB solutions that are not solutions to the original problem in notes by Yuliy Sannikov.
  - If value function is bounded, then TVC holds.

- We know that

$$dX_t = \sigma_t dZ_t$$

is martingale. Turns out the the converse is also true.

Martingale Representation Theorem

- Suppose that M_t is a martingale on the filtration $\{\mathcal{F}^Z\}$, then there exists a random process β_t such that

$$dM_t = \beta_t dZ_t$$

  - This theorem says that whenever M_t is known from the path $Z_s, s \in [0, t]$, it can be represented as a stochastic integral with respect to Z.
  - Implication: any martingale measurable with respect to a Brownian filtration cannot have jumps.

Stochastic processes: Girsanov Theorem

- Imagine all possible sample paths that Z may take. The defining properties of the Brownian motion imply a certain probability measure over these paths.
- **Intuition**: if we alter the probability measure by making some paths more likely and other paths less likely, then we can endow Z with any drift.

Girsanov's Theorem

- Let Z be a Brownian motion under the probability measure P. Suppose that θ is a random process (that satisfies appropriate technical conditions) and define

$$\xi_0 = 1, \qquad d\xi_t = \xi_t \theta_t dZ_t$$

Define probability measure Q as

$$Q[A] = \mathbb{E}^P[\xi_t 1_A] \quad \text{or equivalently} \quad dQ(A) = \xi_t dP(A)$$

for any $\mathcal{F}_t$-measurable set of paths A. Then, ...

- … Then, under measure Q, process Z has drift θ and volatility 1. That is,

$$Z_t^Q = Z_t - \int_0^t \theta_s ds$$

is a standard Brownian motion under Q.

Interpretation

- Any given path $Z_s, s \in [0, t]$ is more likely to arise under Q than under P by a factor of ξ_t.
- Under Q, the process Z has drift θ.
- The theorem tells us that ξ exists and how to construct it (given θ).
- Conditions that define ξ_t: (i) martingale under P, (ii) $\xi_t Z_t^Q$ is a martingale.

Stochastic processes: Kolmogorov Forward Equation

How does the density of an Ito diffusion evolves?

- **Kolmogorov Forward Equation (KFE)** or Fokker-Planck Equation

$$g_t(x,t) = -\frac{\partial}{\partial x}[\mu(x,t)g(x,t)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[\sigma(x,t)^2 g(x,t)] =: \mathcal{A}^* g(x,t)$$

  where $g(x,t)$ is the distribution of Ito diffusion x at time t. Hence, its derivative w.r.t. time delivers how this distribution evolves.
- Example: a continuum of agents whose wealth evolve as Ito process x (e.g., HANK model), then KFE describes how the distribution evolves.
- Example: wealth share of a group of agents evolve as Ito process x (e.g., BruSan model), then we can use the KFE to do a distributional impulse response (i.e., with confidence intervals that capture uncertainty). Start from degenerate distribution.

Stochastic processes: Kolmogorov Forward Equation

Stationary distribution

- The stationary distribution $g(x)$ of a time-homogeneous Ito diffusion must satisfy (set the time derivative to zero)

$$0 = -\frac{\partial}{\partial x}[\mu(x)g(x)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[\sigma(x)^2 g(x)]$$

- Integrating w.r.t. x and assuming the constant of integration is zero, we have a simple ODE for $D(x) := \sigma(x)^2 g(x)$

$$D'(x) = 2\frac{\mu(x)}{\sigma(x)^2}D(x) \;\Rightarrow\; D(x) = K \exp\left(\int 2\frac{\mu(y)}{\sigma(y)^2}dy\right)$$

  where the constant is pinned down so that the density integrates to one.

- Example: the stationary distribution of the Ornstein-Uhlenbeck process

$$dX_t = \theta(\bar{x} - X_t)dt + \sigma dZ_t$$

  is $N(\bar{x}, \sigma^2/(2\theta))$.

Derivation of KFE

- Consider the expectation

$$\mathbb{E}[f(X_t)] = \int f(x)g(x,t)dx$$

  for an arbitrary $f \in C^2$.
- We can find the derivative of this expectation w.r.t. time $d\mathbb{E}[f(X_t)]/dt$ in two ways:

$$\int f(x)g_t(x,t)dx$$

$$\begin{aligned}\int \frac{\mathbb{E}[df(x)]}{dt} g(x,t)dx &= \int [f'(x)\mu(x,t) + 0.5f''(x)\sigma(x,t)^2]g(x,t)dx \\ &= \int f(x)\left(-\frac{\partial}{\partial x}[\mu(x,t)g(x,t)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[\sigma(x,t)^2 g(x,t)]\right)dx\end{aligned}$$

  Details on integration by parts: click here, minutes 7:50-12:00.
- Since the latter equation holds for any $f \in C^2$, the KFE follows

Adjoint operators

- Adjoint operators mimic the behavior of the transpose matrix on real Euclidean space. Recall that the transpose A^T of a $m \times n$ matrix A satisfies

$$\langle Ax, y \rangle = \langle x, A^T y \rangle$$

  for all $x \in R^n$ and $y \in R^m$, where $\langle ., . \rangle$ is the Euclidean inner product.
- Adjoint operators, generalize this idea. $\mathcal{A}$ and $\mathcal{A}^*$ are adjoint operators if

$$\langle \mathcal{A} f, g \rangle = \langle f, \mathcal{A}^* g \rangle$$

  for all functions f, g, where $\langle f, g \rangle := \int f(x) g(x) dx$.
- Note that when we integrated by parts in the proof of the KFE, we showed that the infinitesimal generator of a diffusion and the "KFE operator" are adjoint operators $\Rightarrow$ useful for numerical purposes!

Stochastic processes: Backward stochastic differential equations (BSDE)

- Consider the filtered probability space $(\Omega, F, \mathcal{F}^Z, P)$, we are interested in

$$Y_t = \xi + \int_t^T f(s, \omega, Y_s, \sigma_s) ds - \int_t^T \sigma_s dZ_s$$

  where ξ is a $\mathcal{F}_T$-measurable random variable. f is called the generator of the BSDE.

- A solution is a process (Y, σ) where Y is continuous and adapted and σ is predictable.
- At time t, (Y_t, σ_t) is $\mathcal{F}_t$-measurable, so it does not know the terminal condition yet (which is $\mathcal{F}_T$-measurable).
- Equivalent formulation of BSDE

$$-dY_t = f(t, \omega, Y_t, \sigma_t) dt - \sigma_t dZ_t, \qquad Y_T = \xi$$

  We solve it backward in time from a terminal condition.

- BSDE capture how the expectation of a random variable evolves (when $f \equiv 0$)

$$Y_t = \mathbb{E}_t \left[\xi + \int_t^T f(s, \omega, Y_s, \sigma_s) ds \right]$$

  Of course, it captures the evolution of more general processes for the cases where $f \neq 0$.

- A BSDE makes no sense if we want to consider it as a forward equation (with some initial condition). Think of the case $f \equiv 0$, of course we can choose any σ independently of initial condition and we would have infinitely many solutions.

Stochastic processes: Forward-Backward Stochastic Differential Equation (FBSDE)

- Forward-Backward stochastic system (uncoupled)

$$
\begin{aligned}
dX_t &= a(X,t)dt + b(X_t,t)dZ_t & X_0 &= x \\
-dY_t &= f(t, X_t, Y_t, \sigma_t)dt - \sigma_t dZ_t & Y_T &= \varphi(X_T)
\end{aligned}
$$

- (Generalized) Feyman-Kac Formula
  Assume $u \in C^2$ solves

$$
\begin{aligned}
0 &= \mathcal{A}u(x,t) + f(t, x, u(x,t), u_x(x,t)b(x,t)) \\
u(x,T) &= \varphi(X_T)
\end{aligned}
$$

  then

$$
\begin{aligned}
Y_t &= u(X_t, t) \\
\sigma_t &= u_x(X_t, t)b(X_t, t)
\end{aligned}
$$

  for all $t \in [0, T]$

Stochastic Optimization: Stochastic Maximum Principle

- Consider the following control problem

$$\max_{\{A_t\}} \mathbb{E}\left[\int_0^T g(t, X_t, A_t) + G(X_T)\right]$$

s.t.

$$dX_t = \mu\left(X_t, A_t\right) dt + \sigma(X_t, A_t) dZ_t$$

where $X_t \in \mathbb{R}^{n_x}$ are states, $A_t \in \mathbb{R}^{n_a}$ controls, and $dZ_t \in \mathbb{R}^{n_z}$ Brownian motions.

- Let p_t be the dynamic Lagrange multiplier that evolves as

$$dp_t = \mu_t^p dt + \sigma_t^p dZ_t$$

- Define the Hamiltonian

$$H = g(t, X, A) + p \cdot \mu(X, A) + \operatorname{tr}\left(\left(\sigma^p\right)^T \sigma(X, A)\right)$$

- Let

$$\mu_t^p = -H_X(t, X_t, A_t, p_t, \sigma_t^p)$$
$$A_t \in \arg\max_A H(t, X_t, A_t, p_t, \sigma_t^p)$$
$$p_T = G'(X_T)$$

  Then, we have that p_t follows a BSDE.

- **Theorem**: Suppose
  - (p_t, σ_t^p) solve the BSDE for controls A_t that maximize H
  - H is jointly concave in the state X and control A for all pairs (p_t, σ_t^p) along the solution path
  - G is concave in X_T.

  Then, we have an optimal policy.

Stochastic Optimization: Stochastic Maximum Principle

- Even if the Hamiltonian is not concave in X and A, the stochastic maximum principle still provides necessary conditions for the optimal policy. Once a candidate policy has been found, we can always try to construct a martingale verification argument to prove that it is optimal.
- The stochastic maximum principle is complementary to dynamic programming. It is useful particularly in non-stationary settings, in which the state space may be high dimensional or difficult to define: even in those settings it can provide conditions that are necessary and sometimes sufficient, for optimization.

We are almost done with the math....

Later in the course ....

- Boundaries and limit behavior
- Repeating some topics with jumps!

Optimization: Consumption and Portfolio Choice

Single-agent consumption-portfolio choice

- Problem

$$\max_{\{c_t, x_t\}} \mathbb{E}\left[\int_0^\infty e^{-\rho t} u(c_t) dt\right]$$

s.t

$$\begin{aligned} dn_t &= -c_t dt + n_t[(1 - x \cdot \mathbb{1}) r_t dt + x \cdot dr_t^x] \\ dr_t^x &= (r_t \mathbb{1} + \varphi_t^x) dt + \sigma_t^x dZ_t \\ n_t &\geq 0 \end{aligned}$$

where $x \in \mathbb{R}^{n_x}$, $Z_t \in \mathbb{R}^{n_z}$, and $\sigma_t^x \in \mathbb{R}^{n_x \times n_z}$, and initial condition n_0 is taken as given. We assume that Brownian motions Z_t are independent. It is an straightforward extension to deal with the correlated case.

- The solvency constraint $n_t \geq 0$ can be imposed or derived from a "no Ponzi condition."
- Reasonably general set-up for portfolio choice.

Single-agent consumption-portfolio choice

- Individual state: wealth n_t
- Aggregate states:
  - Let $Y_t \in \mathbb{R}^{n_y}$ drive all time-varying objects beyond the control of the agent: prices $\{r(Y_t), \varphi^x(Y_t)\}$ and volatility of risky return $\sigma^x(Y_t)$.
  - Assume that Y_t is a (time- homogeneous) Ito diffusion

$$dY_t = \mu_Y(Y_t)dt + \sigma_Y(Y_t)dZ_t$$

Stochastic control methods

We will use three approaches to characterize the solution

1. Dynamic Programming: HJB equation
2. Stochastic Maximum Principle: the Hamiltonian
3. Martingale Method

Dynamic Programming: HJB equation

- HJB equation

$$\rho V(n, Y) = \max_{\{c,x\}} u(c) + \frac{\mathbb{E}[dV(n, Y)]}{dt}$$

- Budget constraint notation (dropping time subscripts, fully recursive)

$$dn = (n\mu^n - c)dt + (n\sigma^n)dZ_t$$

- Applying Ito's formula and taking expectation

$$\frac{\mathbb{E}[dV(n, Y)]}{dt} = V_n(n\mu^n - c) + \frac{1}{2}V_{nn}||n\sigma^n||^2 + V_{nY} \cdot [(\sigma_Y)(n\sigma^n)^T]...$$
$$... + V_Y \cdot \mu_Y + \frac{1}{2}\sum_{i,j} V_{Y_i Y_j}\sigma_{Y_i} \cdot \sigma_{Y_j}$$

- The HJB is a non-linear PDE for $V(n, Y)$ (non-linearity enters through the max operator)

Dynamic Programming: HJB equation

- We can re-arrange the HJB as

$$\rho V = \max_c \{u(c) - V_n c\} \dots$$
$$\dots + \max_x \left\{ V_n(n\mu^n) + \frac{1}{2} V_{nn}||n\sigma^n||^2 + V_{nY} \cdot [(\sigma_Y)(n\sigma^n)^T] \right\} \dots$$
$$\dots + V_Y \cdot \mu_Y + \frac{1}{2} \sum_{i,j} V_{Y_i Y_j} \sigma_{Y_i} \cdot \sigma_{Y_j}$$

- "Separation" between the consumption and the portfolio decisions
- Aggregate state dynamics only influence portfolio choice when they are correlated with individual state.

Consumption decision

- Marginal utility of consumption equal to marginal value of wealth

$$u'(c) = V_n$$

Portfolio decision

$$\max_x (V_n n)\mu^n + \frac{1}{2}(V_{nn}n^2)||\sigma^n||^2 + (V_{nY}n) \cdot [(\sigma_Y)(\sigma^n)^T]$$

- From the budget constraint of the agent, we have

$$\mu^n = r_t + \varphi^x \cdot x$$
$$\sigma^n = (\sigma^x)^T x$$

- FOC for portfolio share x

$$(V_n n)\varphi^x + (V_{nn}n^2)\sigma^x(\sigma^x)^T x + \sigma^x(\sigma_Y)^T(V_{nY}n) = 0$$

which we can re-arrange as

$$\underbrace{\varphi^x}_{\substack{\text{risk}\\ \text{premium}}} = \underbrace{\sigma^x}_{\substack{\text{risk}\\ \text{quantity}}} \underbrace{\left(-\frac{V_{nn}n}{V_n}\sigma^n + (\sigma_Y)^T\left(-\frac{V_{nY}}{V_n}\right)\right)}_{\substack{\text{risk}\\ \text{price}}}$$

Risk premium

$$\underbrace{\varphi^x}_{\text{risk premium}} = \underbrace{\sigma^x}_{\text{risk quantity}} \underbrace{\left(\underbrace{-\frac{V_{nn}n}{V_n}\sigma^n}_{\text{corr. indiv. state}} + \underbrace{(\sigma_Y)^T \left(-\frac{V_{nY}}{V_n}\right)}_{\text{corr. aggregate state}} \right)}_{\text{risk price}}$$

- Risk is priced because it correlates with the relevant states for the agent.

Optimal portfolio

- Recall $\sigma^n = (\sigma^x)^T x$, then we can re-arrange optimal portfolio decision as

$$x = \underbrace{\left(-\frac{V_{nn}n}{V_n}\right)^{-1} (\sigma^x(\sigma^x)^T)^{-1}\varphi^x}_{\text{Myopic dm}} + \underbrace{(\sigma^x(\sigma^x)^T)^{-1}\sigma^x\sigma_y \left(-\frac{V_{nY}}{V_{nn}n}\right)}_{\text{Hedging dm}}$$

- Myopic demand: trade-off between expected excess return and asset risk
- (Inter-temporal) Hedging demand: determined by the covariance of the risky asset and the value function (future investment opportunities).

Dynamic Programming: HJB equation

- Once we replace optimal consumption and optimal portfolio into the RHS of the HJB, we have a highly non-linear PDE for $V(n, Y)$.
- We can solve the mentioned PDE numerically $\Rightarrow$ challenging!
  Even w/ one aggregate state, we need to solve a PDE (2 states).
- Use a reasonable guess to further simplify the functional equation we will take to the computer $\Rightarrow$ drop n as state (also allows further analytical characterization)

$$V(n, Y) = \frac{u(\zeta(Y)n)}{\rho}$$

  we call ζ investment opportunities. It transforms "wealth" into "effective wealth".
- We will work with CRRA but later in the course we generalize to Recursive Preferences that disentangle risk aversion and the IES.

$$u(c) = \frac{c^{1-\gamma} - 1}{1 - \gamma}$$

  Note that $\lim_{\gamma \to 1} u(c) = \log(c)$.

Dynamic Programming: HJB equation

- We need to verify our guess, i.e., use it and show that the HJB holds for some $\zeta(Y)$ process.

Consumption decision

$$c = \rho^{\frac{1}{\gamma}} \zeta^{1-\frac{1}{\gamma}} n$$

- Reaction of c/n to investment opportunities ζ depends on IES $= \gamma^{-1}$
  - $\gamma^{-1} < 1$ better investment opportunities $\Rightarrow$ consumption $\uparrow$, savings $\downarrow$
  - $\gamma^{-1} = 1$ consumption-wealth ratio independent of inv. opportunities
  - $\gamma^{-1} > 1$ better investment opportunities $\Rightarrow$ consumption $\downarrow$, savings $\uparrow$
- Two effects of better investment opportunities:
  - Income effect: makes the agent effectively richer $\Rightarrow$ consume more
  - Substitution effect: makes saving more attractive $\Rightarrow$ consume less

Issue:
$\gamma^{-1} < 1$ substitution effect weak
$\gamma^{-1} > 1$ investor's risk aversion is low

Given that $\zeta(Y)$, it follows

$$\frac{d\zeta_t}{\zeta_t} = \mu^\zeta dt + \sigma^\zeta dZ_t$$

and we can use Ito formula to get $\zeta\sigma^\zeta = \sigma_Y^T \zeta_y$

Risk premium

$$\underbrace{\varphi^x}_{\substack{\textbf{risk}\\\textbf{premium}}} = \underbrace{\sigma^x}_{\substack{\textbf{risk}\\\textbf{quantity}}} \underbrace{\left(\underbrace{\gamma\sigma^n}_{\substack{\text{corr.}\\\text{indiv. state}}} + \underbrace{(\gamma-1)\sigma^\zeta}_{\substack{\text{corr.}\\\text{aggregate state}}} \right)}_{\textbf{risk price}}$$

- LHS is the **market risk premium**, the RHS is the **risk premium required** for the agent to hold the asset. If he has an unconstrained position, these two must be equal.
- Does the agent prefers to have more wealth when investment opportunities are high or low? Depends on risk aversion, i.e., $\gamma > 1$ or $\gamma < 1$.

Dynamic Programming: HJB equation

- Two opposing forces:
  - Prefer high returns when investment opportunities are good: greater **"bang for their buck"**
  - Prefer high returns when investment opportunities are bad: provides **insurance or hedging**.
- The second force becomes stronger with higher risk aversion.
  - If $\gamma = 1$, both forces cancel out and correlation between investment opportunities and the asset return (i.e., $\sigma^x \sigma_\zeta$) is irrelevant for (required) risk premia
  - If $\gamma > 1$, insurance motive prevails and $\sigma^x \sigma_\zeta > 0$ increases the (required) risk premia. Agent demands additional compensation for taking on additional risk!
  - If $\gamma < 1$, "bang for their buck" motive prevails and $\sigma^x \sigma_\zeta > 0$ reduces the (required) risk premia. Agent willing to accept lower risk premia because the higher payoff occur when investment opportunities are good.
- When $\gamma \to 0$, agent only cares on getting the greatest "bang for their buck". When $\gamma \to \infty$, agent only cares about insurance / hedging.

Optimal portfolio

$$x = \underbrace{\frac{1}{\gamma}(\sigma^x(\sigma^x)^T)^{-1}\varphi^x}_{\textbf{Myopic dm}} + \underbrace{\frac{1-\gamma}{\gamma}(\sigma^x(\sigma^x)^T)^{-1}\sigma^x\sigma^\zeta}_{\textbf{Hedging dm}}$$

- Myopic demand: trade-off between expected excess return and asset risk
- (Inter-temporal) Hedging demand:
  - **Insurance** dominates ($\gamma > 1$): $\sigma^x\sigma^\zeta > 0$ reduces the demand for risky asset x.
  - **"Bang for the buck"** dominates ($\gamma < 1$): $\sigma^x\sigma^\zeta > 0$ increases the demand for risky asset x.

Dynamic Programming: HJB equation

Verifying the conjecture about the value function

- We need to replace consumption and portfolio decisions into RHS of the HJB and verify that it holds, i.e., that we can find a process for ζ_t independent of n that makes HJB hold.
- After some algebra ...

$$\frac{\rho}{1-\gamma} = \frac{\gamma}{1-\gamma}\rho^{\frac{1}{\gamma}}\zeta^{1-\frac{1}{\gamma}} + \mu^n(x^*) - \frac{\gamma}{2}||\sigma^n(x^*)||^2 - (\gamma-1)\sigma^n(x^*)\cdot\sigma^\zeta \dots$$
$$\dots + \mu^\zeta - \frac{\gamma}{2}||\sigma^\zeta||^2$$

  where we use Ito's formula to get σ^ζ and μ^ζ.
  - Portfolio solution x^* does not depend on n
- Since this is a PDE for $\zeta(Y)$ independent of n, the conjecture is verified.
- We can solve this functional equation numerically (later in the course).
- Once we have $\zeta(Y)$ (which implies we have $\sigma^\zeta(Y)$ and $\mu^\zeta(Y)$ through Ito's formula), we can turn to our solutions to get $\frac{c}{n}(Y)$ and $x(Y)$.
- We are done!

Dynamic Programming: HJB equation

An easier route ...

- We have done this backwards (for pedagogical purposes)
- Easier to first make the conjecture, then write the HJB and find optimal policies. This approach starts from

$$\frac{\rho}{1-\gamma} = \max_{c} \left\{ \frac{\rho}{1-\gamma} \left(\frac{c/n}{\zeta} \right)^{1-\gamma} - \frac{c}{n} \right\} \dots$$

$$\dots + \max_{x} \left\{ \mu^{n}(x) - \frac{\gamma}{2} ||\sigma^{n}(x)||^{2} - (\gamma - 1)\sigma^{n}(x) \cdot \sigma^{\zeta} \right\} \dots$$

$$\dots + \mu^{\zeta} - \frac{\gamma}{2} ||\sigma^{\zeta}||^{2}$$

- This leads more directly to the FOCs shown in previous slides.

Dynamic Programming: Example

Single risky asset, single shock and single aggregate state

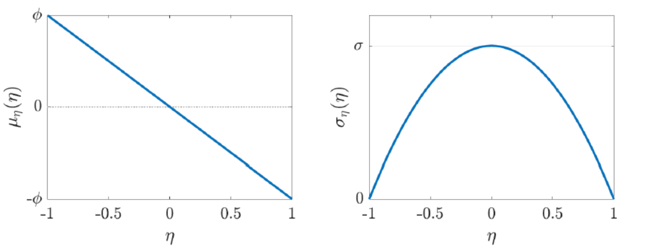
Consider CRRA utility function, a single risky asset and a single Brownian shock driving uncertainty

$$\mu^n n = n(r + x\varphi^x)$$
$$\sigma^n n = n(x\sigma^x)$$

Also, consider a single aggregate state η that evolves according to

$$\mu^\eta \eta = -\phi\eta$$
$$\sigma^\eta \eta = \sigma(1-\eta^2)$$

which drives $\sigma^x(\eta), \varphi^x(\eta), r(\eta)$.



Dynamic Programming: Example

Solution

- Guess for value function

$$V(\eta, n) = \frac{1}{\rho} \frac{(\zeta(\eta) n)^{1-\gamma} - 1}{1-\gamma}$$

- Given that ζ is a function of η, we have

$$d\zeta = \zeta(\mu^{\zeta} dt + \sigma^{\zeta} dZ_t)$$

  and we can find its drift and volatility using Ito's formula.

- We can express the HJB as

$$\frac{\rho}{1-\gamma} = \max_c \left\{ \frac{\rho}{1-\gamma} \left(\frac{c/n}{\zeta} \right)^{1-\gamma} - \frac{c}{n} \right\} \ldots$$
$$\ldots + \max_x \left\{ \mu^n(x) - \frac{\gamma}{2} \sigma^n(x)^2 - (\gamma - 1) \sigma^n(x) \sigma^{\zeta} \right\} \ldots$$
$$\ldots + \mu^{\zeta} - \frac{\gamma}{2} (\sigma^{\zeta})^2$$

Solution

- We will find consumption and portfolio decisions as functions of investment opportunities ζ.
- Then, we will solve for $\zeta(\eta)$ from the HJB numerically. In some cases, we will actually solve for a "less kinky" transformation.
- Optimal consumption

$$\frac{c}{n} = \rho^{\frac{1}{\gamma}} \zeta^{1-\frac{1}{\gamma}}$$

- Optimal portfolio (risk premium)

$$\underbrace{\varphi^x}_{\textbf{risk premium}} = \underbrace{\sigma^x}_{\textbf{risk quantity}} \underbrace{\left(\underbrace{\gamma\sigma^n}_{\text{corr. indiv. state}} + \underbrace{(\gamma-1)\sigma^\zeta}_{\text{corr. aggregate state}} \right)}_{\textbf{risk price}}$$

Dynamic Programming: Example

- Optimal portfolio

$$x^* = \underbrace{\frac{1}{\gamma}\frac{\varphi^x}{(\sigma^x)^2}}_{\textbf{Myopic dm}} + \underbrace{\frac{1-\gamma}{\gamma}\frac{\sigma^\zeta}{\sigma^x}}_{\textbf{Hedging dm}}$$

- Replacing into the HJB

$$\frac{\rho}{1-\gamma} = \frac{\gamma}{1-\gamma}\rho^{\frac{1}{\gamma}}\zeta^{1-\frac{1}{\gamma}} + \mu^n(x^*) - \frac{\gamma}{2}\sigma^n(x^*)^2 - (\gamma-1)\sigma^n(x^*)\sigma^\zeta ...$$
$$... + \mu^\zeta - \frac{\gamma}{2}(\sigma^\zeta)^2$$

where

$$\mu^n(x^*) = r + x^*\varphi^x$$
$$\sigma^n(x^*) = \frac{1}{\gamma}\frac{\varphi^x}{\sigma^x} + \frac{1-\gamma}{\gamma}\sigma^\zeta$$

and μ^ζ and σ^ζ come from Ito's formula.

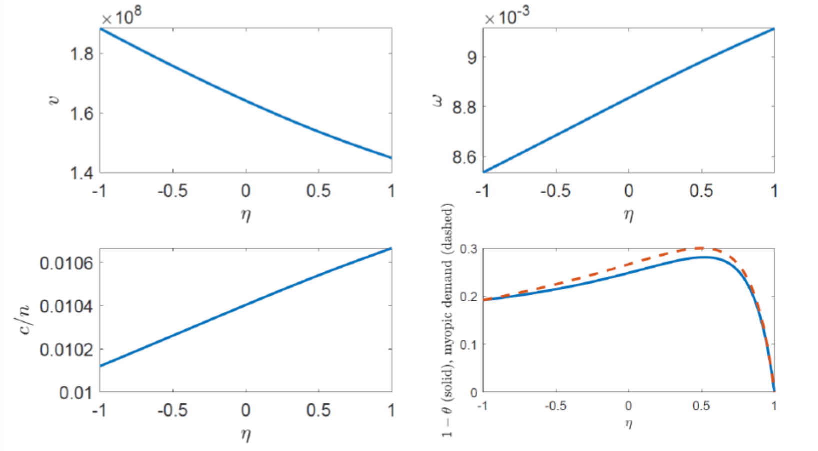
Dynamic Programming: Example

- Using Ito's formula

$$\mu^{\zeta}\zeta = \zeta_{\eta}(\mu^{\eta}\eta) + \frac{1}{2}\zeta_{\eta\eta}(\sigma^{\eta}\eta)^2$$
$$\sigma^{\zeta}\zeta = \zeta_{\eta}(\sigma^{\eta}\eta)$$

- Hence, we have a second-order ODE for $\zeta(\eta)$. We can solve this numerically (later in the course).

Dynamic Programming: Example



- Functional forms: $r(\eta) = r^0 + r^1\eta$, $\varphi^x(\eta) = \varphi^0 - \varphi^1\eta$, and $\sigma^x(\eta) = \sigma^0 - \sigma^1\eta$
- Parameters: $\rho = 0.02$, $\gamma = 5$, $\varphi = 0.2$, $\sigma = 0.01$, $\sigma^0 = 0.15$, $\sigma^1 = 0.1$, $r^0 = 0.02$, $r^1 = 0.01$, $\varphi^0 = 0.3$, $\varphi^1 = 0.03$.
- Notation on axis: $1 - \theta = x$, $\nu = \zeta^{1-\gamma}$, $\omega = \zeta$

Dynamic Programming: Example

Example with fixed prices and volatility

- Let $r(\eta) = r^0$, $\varphi^x(\eta) = \varphi^0$, and $\sigma^x = \sigma^0$, which implies that there is no aggregate state.
- In this case, we guess that $\zeta(\eta) = \zeta$. To verify this, we need to check that a constant value of ζ solves the HJB.
- Replacing this conjecture in our HJB

$$\frac{\rho}{1-\gamma} = \frac{\gamma}{1-\gamma}\rho^{\frac{1}{\gamma}}\zeta^{1-\frac{1}{\gamma}} + \mu^n(x^*) - \frac{\gamma}{2}\sigma^n(x^*)^2$$

  where

$$\mu^n(x^*) = r + \frac{1}{\gamma}\frac{(\varphi^x)^2}{(\sigma^x)^2}, \qquad \sigma^n(x^*) = \frac{1}{\gamma}\frac{\varphi^x}{\sigma^x}$$

- Note that in this cases φ^x, r and σ^x are just constants, so the conjecture is verified.
- Some algebra delivers

$$\zeta = \rho\left[1 + \frac{1}{\rho}\left(\frac{\gamma-1}{\gamma}\right)\left(r - \rho + \frac{1}{2\gamma}\frac{(\varphi^x)^2}{(\sigma^x)^2}\right)\right]^{\frac{\gamma}{\gamma-1}}$$

  consumption-wealth ratio is constant when IES:$= 1/\gamma = 1$.

Stochastic Maximum Principle: refresher

- Consider the following control problem

$$\max_{\{A_t\}} \mathbb{E}\left[\int_0^T g(t, X_t, A_t) + G(X_T)\right]$$

s.t.

$$dX_t = \mu\left(X_t, A_t\right) dt + \sigma(X_t, A_t) dZ_t$$

where $X_t \in \mathbb{R}^{n_x}$ are states, $A_t \in \mathbb{R}^{n_a}$ controls, and $dZ_t \in \mathbb{R}^{n_z}$ Brownian motions.

- Let p_t be the dynamic Lagrange multiplier (also called co-state) that evolves as

$$dp_t = \mu_t^p dt + \sigma_t^p dZ_t$$

- Define the Hamiltonian

$$H = g(t, X, A) + p \cdot \mu(X, A) + \operatorname{tr}\left(\left(\sigma^p\right)^T \sigma(X, A)\right)$$

Stochastic Maximum Principle: refresher

- Let

$$\mu_t^p = -H_X(t, X_t, A_t, p_t, \sigma_t^p)$$
$$A_t \in \arg\max_A H(t, X_t, A_t, p_t, \sigma_t^p)$$
$$p_T = G'(X_T)$$

Then, we have that p_t follows a BSDE.

- **Theorem**: Suppose
  - (p_t, σ_t^p) solve the BSDE for controls A_t that maximize H
  - H is jointly concave in the state X and control A for all pairs (p_t, σ_t^p) along the solution path
  - G is concave in X_T.

Then, we have an optimal policy.

(find reference for the following statement)

- For infinite-horizon problems, we replace the terminal condition for the dynamic Lagrange multiplier with

$$\lim_{T\to\infty} p_T X_T = 0$$

Stochastic Maximum Principle: back to our problem

- Problem

$$\max_{\{c_t, x_t\}} \mathbb{E}\left[\int_0^\infty e^{-\rho t} u(c_t) dt\right]$$

s.t

$$\begin{aligned} dn_t &= -c_t dt + n_t[(1 - x \cdot \mathbb{1}) r_t dt + x \cdot dr_t^x] \\ dr_t^x &= (r_t \mathbb{1} + \varphi_t^x) dt + \sigma_t^x dZ_t \\ n_t &\geq 0 \end{aligned}$$

where $x \in \mathbb{R}^{n_x}$, $Z_t \in \mathbb{R}^{n_z}$, and $\sigma_t^x \in \mathbb{R}^{n_x \times n_z}$, and initial condition n_0 is taken as given.

- Let $Y_t \in \mathbb{R}^{n_y}$ drive all time-varying objects beyond the control of the agent: prices $\{r(Y_t), \varphi^x(Y_t)\}$ and volatility of risky return $\sigma^x(Y_t)$. Assume that Y_t is a (time- homogeneous) Ito diffusion

$$dY_t = \mu_Y(Y_t) dt + \sigma_Y(Y_t) dZ_t$$

Stochastic Maximum Principle: the Hamiltonian

- We will have one Lagrange multiplier (co-state) for each state variable.
- Let p^n be the Lagrange multiplier associated with individual wealth and $p^y \in \mathbb{R}^{n_y}$ the ones associated with aggregate states. Denote

$$\begin{aligned} dp_t^n &= p_t^n(\mu_t^{p,n}dt + \sigma_t^{p,n} \cdot dZ_t) \\ dp_t^y &= p_t^y(\mu_t^{p,y}dt + \sigma_t^{p,y}dZ_t) \end{aligned}$$

- Hamiltonian

$$H = \underbrace{e^{-\rho t}u(c) + p^n(n\mu^n) + tr((p^n\sigma^{p,n})^T\sigma^n n)}_{\equiv H^n(n,Y,c,x,t)} + \underbrace{p^y\mu^y + tr((p^y\sigma^{p,y})^T\sigma^y)}_{\equiv H^y(Y)}$$

- Note that the component associated with aggregate states dynamics H^y does not depend on controls or the individual state.
- Then, we can find optimal controls by maximizing H^n and $H_n^n = H_n$.

Stochastic Maximum Principle: the Hamiltonian

- Replacing the law of motion of wealth into H^n

$$H^n = e^{-\rho t} u(c_t) + p_t^n n_t (-c_t / n_t + r_t + x_t \cdot \varphi_t^x) + tr((p_t^n \sigma_t^{p,n})(x_t^T \sigma_t^x n_t))$$

- FOC for controls

$$\begin{aligned} e^{-\rho t} u'(c_t) &= p_t^n \\ \varphi_t^x &= \sigma_t^x (-\sigma_t^{p,n}) \end{aligned}$$

The co-state for individual wealth is the marginal utility of consumption (at time 0) and minus its (geometric) volatility is the price of risk.

- Conditions for co-states drifts

$$\begin{aligned} \mu_t^{p,n} p_t^n = -H_n^n &= -p_t^n (r_t + x_t \cdot \varphi_t^x) - p_t^n tr((\sigma_t^{p,n})(x_t^T \sigma_t^x)) \\ &= -p_t^n (r_t - x_t \cdot [\sigma_t^x \sigma_t^{p,n}]) - p_t^n tr((\sigma_t^{p,n})(x_t^T \sigma_t^x)) \\ &= -p_t^n r_t \end{aligned}$$

where the last line follows from a trace operator property: for two vectors, the trace of the outer product is equivalent to the inner product.

Stochastic Maximum Principle: the Hamiltonian

- Most of the time, we will only care about getting the FOCs for controls and switch to an alternative method to characterize p_t^n, which corresponds to the Stochastic Discount Factor (SDF) of the agent.
- Connection to dynamic programming approach:

$$p_t^n = e^{-\rho t} u'(c) = e^{-\rho t} V_n$$

Martingale approach

- Let ξ_t be the stochastic discount factor SDF of an agent that evolves according to

$$d\xi_t = \xi_t(\mu_t^\xi dt + \sigma_t^\xi dZ_t)$$

- We can show that

$$\mu_t^\xi = -r_t, \qquad \sigma_t^\xi = -\varsigma_t$$

where r_t denotes the risk-free rate and ς_t the risk price the agent requires, i.e., that for any asset with return

$$dr_t^x = \mu_t^x dt + \sigma_t^x dZ_t$$

we must have

$$\mu^x = r_t + \varsigma_t \sigma_t^x$$

Martingale approach

- Proof: Already done! Look at the derivation of the portfolio problem through the Stochastic Maximum Principle. We showed

$$\begin{aligned} p_t^n &= e^{-\rho t} u'(c_t) \\ \mu_t^{p,n} &= -r_t \\ r_t + \varphi_t^x &= r_t + (-\sigma_t^{p,n})\sigma_t^x \end{aligned}$$

- Martingale method: Given an SDF which evolution we can write as

$$d\xi_t = \xi_t(-r_t dt - \varsigma_t dZ_t)$$

we can price assets as

$$\mu^x = r_t + \varsigma_t \sigma_t^x$$

Martingale approach

- Why "martingale" method?
  The optimal condition for holding an asset is that the SDF times the value of a strategy that invest in the asset and reinvest any dividends is a martingale.
- Let v_t be the value of the strategy described above for asset x_t, then

$$\frac{dv_t}{v_t} = dr_t^x$$

  and the optimal condition for holding an asset is $\xi_t v_t$ is a martingale. Then,

$$\mathbb{E}[d(\xi_t v_t)] = 0$$

  which applying Ito's formula delivers

$$\mu^x - r_t - \varsigma_t \sigma_t^x = 0$$

A Simple Real Macro Model with Heterogeneous Agents

Simple Two Sector Model: environment

Basak Cuoco (1998)

- **Agents:** experts and households, a continuum of each, $i \in \mathbb{I}$ for experts and $j \in \mathbb{J}$ for households
- **Preferences:** symmetric

$$\mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} u(c_t) dt \right]$$

  where $u(c) = \log(c)$
- **Technology** (only available to experts)
  - Linear production technology in capital: $y_t = ak_t$
  - Investment with adjustment costs:
    - Invest $\iota_t k_t$ units of final good $\Rightarrow$ get $\Phi(\iota_t)k_t$ units of capital.
    - Φ is a concave function ($\Phi' > 0, \Phi'' < 0$).
  - Capital holdings subject to aggregate "quality" shocks. Individual capital holdings evolve as

$$dk_{i,t} = k_{i,t}(\Phi(\iota_{i,t}) - \delta)dt + k_{i,t}\sigma dZ_t + d\Delta_{i,t}$$

    where $d\Delta_{i,t}$ correspond to net capital purchases. We call σ fundamental volatility.

Simple Two Sector Model: environment

- **Assets**
  - The only physical asset is capital. Traded among experts at price q_t.
  - **Market incompleteness**: only financial asset is risk-free debt. Traded at interest rate r_t.
  - Recall the numeraire is the consumption good.
- **Information**: all uncertainty is encoded in Z_t.
- **Initial condition**
  - Initial allocation of capital $\{k_{i,0}, k_{j,0}\}$, which we can add up to get K_0.
  - Since households cannot hold it, they immediately sell it. It only pins down initial distribution of wealth (not aggregate wealth, which depends on q_0 and is endogenous).
  - Alternative initial condition
    - Aggregate capital K_0 and a probability measure g over $\mathbb{I} \cup \mathbb{J}$ such that $n_{z,0} = g(z) N_0$ for all $z \in \{\mathbb{I} \cup \mathbb{J}\}$.
    - N_0 is aggregate wealth and is endogenous. In particular, $N_0 = q_0 K_0$.

Simple Two Sector Model: agents' problems

Expert problem

$$\max_{c_t, k_t, \iota_t} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} u(c_t) dt \right]$$

s.t

$$\begin{aligned} dn_t &= -c_t dt + (n_t - q_t k_t) r_t dt + q_t k_t dr_t^k \\ n_t &\geq 0 \end{aligned}$$

where

$$dr_t^k \equiv \frac{d(q_t k_t)}{q_t k_t} + \frac{a - \iota_t}{q_t}$$

Note: the law of motion dk_t in the return definition does not consider net purchases. They do not generate returns instantaneously.

Simple Two Sector Model: agents' problems

Household problem

$$\max_{c_t} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} u(c_t) dt \right]$$

s.t

$$dn_t = -c_t dt + n_t r_t dt$$
$$n_t \geq 0$$

Simple Two Sector Model: market clearing

Market clearing

- Goods

$$\int_{i\in\mathbb{I}}(a-\iota_t)k_{i,t}di = \int_{i\in\mathbb{I}} c_{i,t}di + \int_{j\in\mathbb{J}} c_{j,t}dj$$

- Capital

$$\int_{i\in\mathbb{I}} k_{i,t}di = K_t$$

  where aggregate capital evolves as

$$dK_t = \left(\int_{i\in\mathbb{I}}(\Phi(\iota_{i,t})-\delta)k_{i,t}di\right)dt + \sigma K_t dZ_t$$

  with initial condition K_0.

- Bonds: Walras Law.

Simple Two Sector Model: equilibrium

Equilibrium definition

Given initial capital endowments $\{k_{i,0}, k_{j,0} : i \in \mathbb{I}, j \in \mathbb{J}\}$, an equilibrium consists of stochastic processes — adapted to the filtered probability space generated by $\{Z_t : t \geq 0\}$ — for

- prices: capital price q_t, interest rate r_t
- allocations: capital $\{k_{i,t}\}$, consumption $\{c_{i,t}, c_{j,t}\}$, investment $\{\iota_{i,t}\}$ for agents $i \in \mathbb{I}$ and $j \in \mathbb{J}$
- net worth $\{n_{i,t}, n_{j,t}\}$ for agents $i \in \mathbb{I}$ and $j \in \mathbb{J}$

such that

- initial net worths satisfy $n_{i,0} = q_0 k_{i,0}$ and $n_{j,0} = q_0 k_{j,0}$ for $i \in \mathbb{I}$ and $j \in \mathbb{J}$
- taking prices as given, each expert i and household j solves his problem
- markets clear at all dates

Simple Two Sector Model: equilibrium characterization

- Conjecture a process for capital price

$$dq_t = q_t(\mu_t^q dt + \sigma_t^q dZ_t)$$

- Then, the return of capital (using Ito's formula)

$$dr_t^k = \left(\underbrace{\frac{a - \iota_t}{q_t}}_{\text{dividend yield}} + \underbrace{\Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q}_{\mathbb{E}\left[\frac{d(q_t k_t)}{q_t k_t}\right]} \right) dt + (\sigma + \sigma_t^q) dZ_t$$

- Now, we are ready to solve individual problems

Simple Two Sector Model: equilibrium characterization

Experts

- It is straightforward to cast the expert's problem into the individual problem solved last class ...

$$\mu_t^n = r_t + x_t \left(\frac{a - \iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q - r_t \right)$$
$$\sigma_t^n = x_t(\sigma + \sigma_t^q)$$

where $x_t \equiv q_t k_t / n_t$ is the portfolio share invested in capital.

- Given the logarithmic utility ($\gamma = 1$), the consumption decision is

$$c_t = \rho n_t$$

and the portfolio problem reduces to

$$\max_{x,\iota} \mu^n(x) - \frac{1}{2}(\sigma^n(x))^2$$

Note that investment opportunities do not influence the portfolio choice with logarithmic preferences.

Experts

- Optimal investment

$$\frac{1}{q_t} = \Phi'(\iota_t)$$

This defines an investment function $\iota(q_t)$.

- Optimal portfolio condition

$$\frac{a-\iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q - r_t = (\sigma + \sigma_t^q)\sigma_t^n$$

or equivalently

$$x = \frac{\frac{a-\iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q - r_t}{(\sigma + \sigma_t^q)^2}$$

The risk price is σ^n.

- Due to log preferences, we don't need to solve for investment opportunities to get the optimal portfolio choice.

Simple Two Sector Model: equilibrium characterization

Household

- Consumption

$$c_t = \rho n_t$$

Market clearing

- Goods

$$\rho q_t K_t = (a - \iota(q_t))K_t$$

- Capital

$$x_t N_{e,t} = q_t K_t$$

$$\eta_t x_t = 1$$

where $N_{e,t} = \int_{i \in \mathbb{I}} n_{i,t} di$, $N_{h,t} = \int_{j \in \mathbb{J}} n_{j,t} dj$ and

$$\eta_t \equiv \frac{N_{e,t}}{N_{e,t} + N_{h,t}}$$

(Fundamental) state

- The only (fundamental) state of the (scaled version of the) economy is η_t. Consumption and investment scale with aggregate capital K_t.

Simple Two Sector Model: equilibrium characterization

Aggregate states

- In general, the distribution of individual states is (part of) the aggregate states of the economy.
- In this case, since all decisions are linear in wealth. Only the wealth distribution among groups matter, i.e., one number η_t.
- How do I "see" this? Only η_t appears in the market clearing conditions (once we scaled the economy by aggregate capital K_t).

Simple Two Sector Model: equilibrium characterization

Useful derivation

- Drop time subscripts and let

$$\frac{dn_j}{n_j} = \left(\mu^{n,j} - \frac{c_j}{n_j}\right) dt + \sigma^{n,j} \cdot dZ_t$$

  for $j = A, B$.

- Define

$$\eta = \frac{n_A}{n_A + n_B}$$

- Then, the law of motion follows (Ito's formula)

$$d\eta = \eta\mu^{\eta} dt + \eta\sigma^{\eta} \cdot dZ_t$$

  where

$$\eta\mu^{\eta} = \eta(1-\eta)\left(\mu_{n,A} - \mu_{n,B} + \frac{c_B}{n_B} - \frac{c_A}{n_A}\right) - \eta\sigma_{\eta}(\eta\sigma_{n,A} + (1-\eta)\sigma_{n,B})$$
$$\eta\sigma^{\eta} = \eta(1-\eta)(\sigma_{n,A} - \sigma_{n,B})$$

Law of motion for η

$$d\eta = \eta\mu^{\eta} dt + \eta\sigma^{\eta} dZ_t$$

where

$$\eta\mu^{\eta} = \eta(1-\eta)\left(\mu_{n,e} - \mu_{n,h} + \frac{c_h}{n_h} - \frac{c_e}{n_e}\right) - \eta\sigma_{\eta}(\eta\sigma_{n,e})$$

$$\eta\sigma^{\eta} = \eta(1-\eta)\sigma_{n,e}$$

Equilibrium characterization

Given $\eta_0 \in (0,1)$, consider a process $(\eta_t, q_t, r_t)_{t\geq 0}$ with dynamics for q_t and η_t described by

$$dq_t = q_t(\mu_t^q dt + \sigma_t^q dZ_t)$$
$$d\eta_t = \frac{(1-\eta)^2}{\eta}(\sigma + \sigma_t^q)^2 dt + (1-\eta)(\sigma + \sigma_t^q) dZ_t$$

If $\eta_t \in [0,1]$, and equations

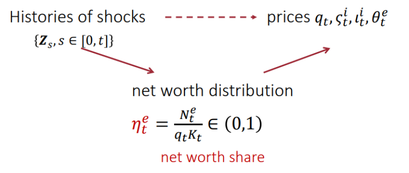
$$\frac{a - \iota(q_t)}{q_t} + \Phi(\iota(q_t)) - \delta + \mu_t^q + \sigma\sigma_t^q - r_t = \frac{(\sigma + \sigma_t^q)^2}{\eta}$$
$$\rho q_t = (a - \iota(q_t))$$

hold for all $t \geq 0$, then $(\eta_t, q_t, r_t)_{t\geq 0}$ corresponds to an equilibrium.

Simple Two Sector Model: equilibrium characterization

Recursive solution

- We will restrict our attention to equilibria in which equilibrium objects are functions of η_t (in general this is an assumption but most models will require this)
- These functions plus the (endogenous) law of motion for the aggregate state, deliver the stochastic processes for equilibrium objects.



Consistency of dynamics (Ito conditions)

- Given that we are imposing $q(\eta)$, dynamics of q must be consistent with dynamics of η. This consistency is captured by Ito's formula

$$q\mu^q = q_\eta \mu_\eta + \frac{1}{2} q_\eta (\eta\sigma^\eta)^2$$

$$q\sigma^q = q_\eta (\eta\sigma_\eta)$$

Simple Two Sector Model: equilibrium characterization

Given initial conditon η_0, a Recursive Equilibrium is a set of functions of η for allocations $\{\frac{c_e}{n_e}, \frac{c_h}{n_h}, x\}$, prices $\{r, q, \sigma^q, \mu^q\}$, and dynamics for aggregate state $\{\sigma^\eta, \mu^\eta\}$ such that

- Agents optimize (FOCs for $x, c_e/n_e, c_h/n_h$)
- Markets clear (equations for goods and capital markets)
- Aggregate state η evolves consistently with individual states
- Consistency conditions for dynamics are satisfied

Simple Two Sector Model: equilibrium characterization

Solution to our simple model....

- From logarithmic utility, we know that $c_e/n_e = c_h/n_h = \rho$.
- From goods market clearing

$$\rho q_t K_t = (a - \iota(q_t))K_t$$

- We get a solution for q_t in terms of parameters, i.e. a constant q^*. This implies $\mu_t^q = \sigma_t^q = 0$.
- From capital market clearing, we know $x = 1/\eta_t$.
- Optimal portfolio choice for experts implies

$$\frac{a - \iota(q^*)}{q^*} + \Phi(\iota(q^*)) - \delta - r_t = \frac{\sigma^2}{\eta_t}$$

delivers the equilibrium interest rate. Re-arranging and using market clearing for goods

$$r_t = \rho + \Phi(\iota(q^*)) - \delta - \frac{\sigma^2}{\eta_t}$$

Simple Two Sector Model: equilibrium characterization

- We can use FOC for portfolio condition to write

$$\mu_{n,e} = r + \sigma_{n,e}\sigma_{n,e}$$

where the last term corresponds to the quantity of risk $\sigma_{n,e}$ times its price $\sigma_{n,e}$.

- We know $\sigma_{n,e} = \sigma/\eta$. Also, $\mu_{n,h} = r$ and $\sigma_{n,h} = 0$.
- Then, using derivations for LoM of share η_t (few slides above)

$$\frac{d\eta_t}{\eta_t} = \left(\frac{1-\eta_t}{\eta_t}\right)^2 \sigma^2 dt + \frac{1-\eta_t}{\eta_t}\sigma dZ_t$$

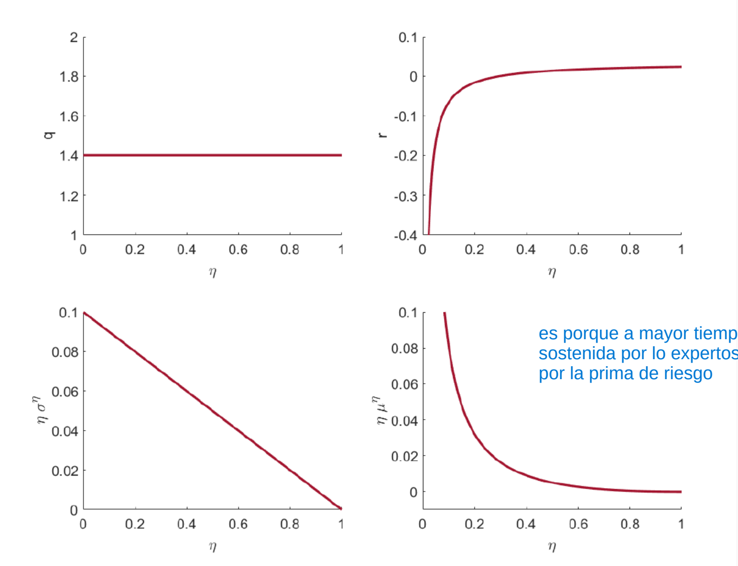
Simple Two Sector Model: Numerical Example

- Let $\Phi(\iota) = \phi^{-1}\log(\phi\iota + 1)$. Then,

$$q^* = \frac{1+\phi a}{1+\phi\rho}$$
$$\Phi(\iota(q^*)) = \phi^{-1}\log\left(q^*\right)$$

- Parameters: $a = 0.11, \rho = 0.05, \sigma = 0.1, \phi = 10$.

Simple Two Sector Model: Numerical Example



es porque a mayor tiempo la riqueza sostenida por lo expertos aumenta por la prima de riesgo

Simple Two Sector Model: observations

- η_t fluctuates with macro shocks, since experts are levered.
- Price of risk (for experts)

$$\sigma_{n,e} = x\sigma = \frac{\sigma}{\eta}$$

  - it goes to ∞ as $\eta_t \to 0$
  - achieved via risk-free interest rate

$$r_t = \rho + \Phi(\iota(q^*)) - \delta - \frac{\sigma^2}{\eta_t} \to -\infty$$

  - Rather than depressing price of risky asset $(q = q^*)$
- No endogenous risk $\sigma_q = 0$
  - no amplification, no endogenous response of volatility
- In the long run, HH-net worth share vanishes
  - positive drift for η_t always $\eta\mu_\eta = (1-\eta_t)^2\eta_t > 0$
  - we could address this with... (i) different discount rates [KM], (ii) switching types [BGG], (iii) perpetual youth

Desired Model Properties

- Normal regime: stable around steady state
  - If experts well capitalized, they can absorb macro shock
- Endogenous risk and price of risk
  - Fire-sales, liquidity spirals, fat tails
  - Spillovers across assets and agents
  - Volatility paradox
  - Financial innovation $\Rightarrow$ less stable economy
- “Net worth trap" double-humped stationary distribution

Endogenous Risk Dynamics in Real Macro Model with Heterogeneous Agents

Model with Endogenous Risk Dynamics: Environment

Brunnermeier Sannikov (2016) – Handbook chapter

- **Agents:** experts and households, a continuum of each, $i \in \mathbb{I}$ for experts and $j \in \mathbb{J}$ for households
- **Preferences:** for $z \in \{e, h\}$

$$\mathbb{E}_0 \left[\int_0^\infty e^{-\rho^z t} u(c_t) dt \right]$$

  where $u(c) = (c^{1-\gamma} - 1)/(1-\gamma)$.
  Experts are more impatient than HHs: $\rho^e \geq \rho^h$
- **Technology**
  - Linear production technology in capital: $y_t = a^z k_t$
  - Experts more productive than HHs: $a^e > a^h$
  - Investment with adjustment costs:
    - Invest $\iota_t k_t$ units of final good $\Rightarrow$ get $\Phi(\iota_t) k_t$ units of capital.
    - Φ is a concave function ($\Phi' > 0, \Phi'' < 0$).

Model with Endogenous Risk Dynamics: Environment

- **Technology**
  - Capital holdings subject to aggregate "quality" shocks. Individual capital holdings evolve as

$$dk_{i,t} = k_{i,t}(\Phi(\iota_{i,t}) - \delta)dt + k_{i,t}\sigma dZ_t + d\Delta_{i,t}$$

    where $d\Delta_{i,t}$ correspond to net capital purchases.
- **Assets**
  - The only physical asset is capital. Freely traded at price q_t.
  - **Financial assets**:
    - Risk-free debt traded at interest rate r_t.
  - Recall the numeraire is the consumption good.
- **Information**: all uncertainty is encoded in Z_t.
- **Initial condition**: Initial allocation of capital $\{k_{i,0}, k_{j,0}\}$, which we can add up to get K_0.

Model with Endogenous Risk Dynamics: Agents' problem

Expert problem

$$\max_{c_t, k_t, \iota_t} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho^e t} u(c_t) dt \right]$$

s.t

$$\begin{aligned} dn_t &= -c_t dt + (n_t - q_t k_t) r_t dt + q_t k_t dr_t^{k,e} \\ n_t &\geq 0 \end{aligned}$$

where

$$dr_t^{k,e} \equiv \mu_t^{r,e} dt + \sigma_t^r dZ_t \equiv \frac{d(q_t k_t)}{q_t k_t} + \frac{a^e - \iota_t}{q_t} dt$$

Note 1: the law of motion dk_t in the return definition does not consider net purchases. They do not generate returns instantaneously.

Model with Endogenous Risk Dynamics: Agents' problem

Household problem

$$\max_{c_t, k_t, \iota_t} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho^e t} u(c_t) dt \right]$$

s.t

$$dn_t = -c_t dt + (n_t - q_t k_t) r_t dt + q_t k_t dr_t^{k,h}$$

$$k_t \geq 0$$

$$n_t \geq 0$$

where

$$dr_t^{k,h} \equiv \mu_t^{r,h} dt + \sigma_t^r dZ_t \equiv \frac{d(q_t k_t)}{q_t k_t} + \frac{a^h - \iota_t}{q_t} dt$$

Market clearing

- Goods

$$\int_{i\in\mathbb{I}}(a^e-\iota_{i,t})k_{i,t}di+\int_{j\in\mathbb{J}}(a^h-\iota_{j,t})k_{j,t}dj=\int_{i\in\mathbb{I}}c_{i,t}di+\int_{j\in\mathbb{J}}c_{j,t}dj$$

- Capital

$$\int_{i\in\mathbb{I}}k_{i,t}di+\int_{i\in\mathbb{J}}k_{j,t}dj=K_t$$

where aggregate capital evolves as

$$dK_t=\left(\int_{i\in\mathbb{I}}(\Phi(\iota_{i,t})-\delta)k_{i,t}di+\int_{j\in\mathbb{J}}(\Phi(\iota_{j,t})-\delta)k_{j,t}dj\right)dt+\sigma K_t dZ_t$$

with initial condition K_0.

- Bonds: Walras Law.

Equilibrium definition

Given initial capital endowments $\{k_{i,0}, k_{j,0} : i \in \mathbb{I}, j \in \mathbb{J}\}$, an equilibrium consists of stochastic processes — adapted to the filtered probability space generated by $\{Z_t : t \geq 0\}$ — for

- prices: capital price q_t, interest rate r_t, outside equity price π_t
- allocations: capital $\{k_{i,t}, k_{j,t}\}$, consumption $\{c_{i,t}, c_{j,t}\}$, investment $\{\iota_{i,t}, \iota_{j,t}\}$, for agents $i \in \mathbb{I}$ and $j \in \mathbb{J}$
- net worth $\{n_{i,t}, n_{j,t}\}$ for agents $i \in \mathbb{I}$ and $j \in \mathbb{J}$

such that

- initial net worths satisfy $n_{i,0} = q_0 k_{i,0}$ and $n_{j,0} = q_0 k_{j,0}$ for $i \in \mathbb{I}$ and $j \in \mathbb{J}$
- taking prices as given, each expert i and household j solves his problem
- markets clear at all dates

Model with Endogenous Risk Dynamics: equilibrium characterization

- Conjecture a process for capital price

$$dq_t = q_t(\mu_t^q dt + \sigma_t^q dZ_t) \tag{1}$$

- Then, the return of capital (using Ito's formula) for experts

$$dr_t^{k,e}(\iota) = \left(\underbrace{\frac{a^e - \iota}{q_t}}_{\text{dividend yield}} + \underbrace{\Phi(\iota) - \delta + \mu_t^q + \sigma\sigma_t^q}_{\mathbb{E}\left[\frac{d(q_t k_t)}{q_t k_t}\right]} \right) dt + (\sigma + \sigma_t^q) dZ_t$$

- The return can differ across experts if they chose different investment rates ι, however, in equilibrium they all chose the same.
- Analogous expression for $dr_t^{k,h}(\iota)$ but with productivity a^h.

Experts

- We can re-write the problem as

$$\mu_t^n = r_t + x_t^k \left(\frac{a^e - \iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q - r_t \right)$$

$$\sigma_t^n = x_t^k (\sigma + \sigma_t^q)$$

where $x_t^k \equiv q_t k_t / n_t$ is the portfolio share invested in capital

Households

- We can re-write the problem as

$$\mu_t^n = r_t + x_t^k \left(\frac{a^h - \iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q - r_t \right)$$
$$\sigma_t^n = x_t^k(\sigma + \sigma_t^q)$$
$$x_t^k \geq 0$$

where $x_t^k \equiv q_t k_t / n_t$ is the portfolio share invested in capital

Model with Endogenous Risk Dynamics: equilibrium characterization

Solution — both agents

- Guess for value function

$$V(n, Y) = \frac{1}{\rho} \frac{(\zeta(Y)n)^{1-\gamma} - 1}{1-\gamma}$$

  where Y is a placeholder for aggregate states.

- Conjecture

$$d\zeta = \zeta(\mu^{\zeta} dt + \sigma^{\zeta} dZ_t)$$

- We can express the HJB as

$$\frac{\rho}{1-\gamma} = \max_{c} \left\{ \frac{\rho}{1-\gamma} \left(\frac{c/n}{\zeta} \right)^{1-\gamma} - \frac{c}{n} \right\} \ldots$$

$$\ldots + \max_{x^k, \iota} \left\{ \mu^n(x) - \frac{\gamma}{2} \sigma^n(x)^2 - (\gamma - 1)\sigma^n(x)\sigma^{\zeta} \right\} \ldots$$

$$\ldots + \mu^{\zeta} - \frac{\gamma}{2} (\sigma^{\zeta})^2$$

  where the portfolio maximization is subject to the relevant constraints.

Experts

- Optimal consumption

$$\frac{c^e}{n^e} = \rho^{\frac{1}{\gamma}} (\zeta^e)^{1-\frac{1}{\gamma}}$$

- Optimal investment

$$q^{-1} = \Phi'(\iota)$$

which defines function $\iota(q)$. Define

$$\mu^{r,e} := \frac{a^e - \iota(q)}{q} + \Phi(\iota(q)) - \delta + \mu^q + \sigma\sigma^q$$

- Optimal portfolio

$$[x^k] : \qquad \mu^{r,e} - r = (\gamma\sigma^{n,e} + (\gamma - 1)\sigma^{\zeta,e})(\sigma + \sigma^q)$$

- To complete the characterization of the experts' problem, we need to replace solutions into the HJB. Intuitively, the HJB disciplines ζ^e.

Model with Endogenous Risk Dynamics: equilibrium characterization

Households

- Optimal consumption

$$\frac{c^h}{n^h} = \rho^{\frac{1}{\gamma}} (\zeta^h)^{1-\frac{1}{\gamma}}$$

- Optimal investment $q^{-1} = \Phi'(\iota)$ which defines function $\iota(q)$. Define

$$\mu^{r,h} := \frac{a^h - \iota(q)}{q} + \Phi(\iota(q)) - \delta + \mu^q + \sigma\sigma^q$$

- Optimal portfolio

$$\begin{aligned} [x^k]: \quad & \mu^{r,h} - r = (\gamma\sigma^{n,h} + (\gamma-1)\sigma^{\zeta,h})(\sigma + \sigma^q) - \lambda_k \\ \text{Slackness}: \quad & 0 = \lambda_k x^k \text{ and } \lambda_k, x^k \geq 0 \end{aligned}$$

or in a more compact way

$$0 = \min\left\{x^k, (\gamma\sigma^{n,h} + (\gamma-1)\sigma^{\zeta,h})(\sigma + \sigma^q) - (\mu^{r,h} - r)\right\}$$

- To complete the characterization of the HHs' problem, we need to replace solutions into the HJB. Intuitively, the HJB disciplines ζ^h.

Market clearing

- Goods

$$[\eta\rho_e^{\frac{1}{\gamma}}(\zeta^h)^{1-\frac{1}{\gamma}} + (1-\eta)\rho_h^{\frac{1}{\gamma}}(\zeta^h)^{1-\frac{1}{\gamma}}]q + \iota(q) = a^e\kappa^e + a^h\kappa^h$$

- Capital

$$\underbrace{\eta x^{k,e}}_{\equiv\kappa^e} + \underbrace{(1-\eta)x^{k,h}}_{\equiv\kappa^h} = 1$$

- Bonds : Walras law

Note: Market clearing conditions imply

$$\eta\sigma^{n,e} + (1-\eta)\sigma^{n,h} = \sigma + \sigma^q$$

You can even replace capital or outside equity market clearing condition with this one.

Law of motion for η

$$d\eta = \eta\mu^{\eta} dt + \eta\sigma^{\eta} dZ_t$$

where

$$\eta\mu^{\eta} = \eta(1-\eta)\left(\mu^{n,e} - \mu^{n,h} + \frac{c_h}{n_h} - \frac{c_e}{n_e}\right) - \eta\sigma_{\eta}(\eta\sigma^{n,e} + (1-\eta)\sigma^{n,h})$$

$$\eta\sigma^{\eta} = \eta(1-\eta)(\sigma^{n,e} - \sigma^{n,e})$$

Model with Endogenous Risk Dynamics: equilibrium characterization

For convenience, let us denote risk prices as

$$\varsigma_e \equiv \gamma\sigma^{n,e} + (\gamma - 1)\sigma^{\zeta,e}$$
$$\varsigma_h \equiv \gamma\sigma^{n,h} + (\gamma - 1)\sigma^{\zeta,h}$$

Evolution of net worth

- Agents are compensated by the risk they hold (according to their risk prices). This implies that the drift of their net worth can be written as

$$\mu^{n,e} = r + \varsigma_e\sigma^{n,e}$$
$$\mu^{n,h} = r + \varsigma_h\sigma^{n,h}$$

- This holds whether constraint is binding or not.

Model with Endogenous Risk Dynamics: Log preferences

- With log-preferences there is no role for investment opportunities. Portfolios are chose in a myopic way.
- Goods market clearing (Price-Output)

$$(a^e - a^h)\kappa^e + a^h = \iota(q) + q \underbrace{\left[\eta\rho_e + (1-\eta)\rho_h\right]}_{\equiv\bar{\rho}(\eta)} \qquad \text{(PO)}$$

- From portfolio optimal decisions (Risk balance)

$$0 = \min\left\{1-\kappa^e, \frac{a^e - a^h}{q} - \left[\frac{\kappa^e - \eta}{\eta(1-\eta)}\right](\sigma + \sigma_q)^2\right\} \qquad \text{(RB)}$$

- From portfolio decision (risk free rate)

$$\mu_q + \Phi(\iota(q)) + \sigma\sigma_q + \frac{\kappa^e a_e + (1-\kappa^e)a_h}{q} - r = \left(\frac{(\kappa^e)^2}{\eta} + \frac{(1-\kappa^e)^2}{1-\eta}\right)(\sigma + \sigma_q)^2 \qquad (2)$$

Model with Endogenous Risk Dynamics: Log preferences

Law of motion for η

$$d\eta_t = \eta_t \mu_{\eta,t} dt + \eta_t \sigma_{\eta,t} dZ_t, \quad \text{given} \quad \eta_0, \tag{3}$$

where

$$\eta\mu_\eta = \eta(1-\eta)(\rho_h - \rho_e) + (\kappa^e - 2\eta\kappa^e + \eta^2)\frac{\kappa^e - \eta}{\eta(1-\eta)}(\sigma + \sigma_q)^2 \tag{4}$$

$$\eta\sigma_\eta = (\kappa - \eta)(\sigma + \sigma_q)^2. \tag{5}$$

Equilibrium characterization

Given $\eta_0 \in (0,1)$, consider a process $(\eta_t, q_t, \kappa_t, r_t)_{t\geq 0}$ with dynamics for q_t and η_t described by (1) and (3), respectively. If $\eta_t \in [0,1]$, $\kappa_t \in [0,1]$, and equations (PO), (RB), (2), (4) and (5) hold for all $t \geq 0$, then $(\eta_t, q_t, \kappa_t, r_t)_{t\geq 0}$ corresponds to an equilibrium. Moreover, any distinct pair of such processes corresponds to distinct equilibria.

Simple Two Sector Model: equilibrium characterization

Recursive solution

- We will restrict our attention to equilibria in which equilibrium objects are functions of η_t (in general this is an assumption but most models will require this)
- These functions plus the (endogenous) law of motion for the aggregate state, deliver the stochastic processes for equilibrium objects.

Consistency of dynamics (Ito conditions)

- Given that we are imposing $q(\eta)$, dynamics of q must be consistent with dynamics of η. This consistency is captured by Ito's formula

$$q\mu^q = q_\eta \mu_\eta + \frac{1}{2} q_\eta (\eta \sigma^\eta)^2$$
$$q\sigma^q = q_\eta (\eta \sigma_\eta)$$

Model with Endogenous Risk Dynamics: equilibrium characterization

Risk concentration

- Volatility of net worth

$$\sigma^{n,e} = x^{k,e}(\sigma + \sigma^q) = \frac{\kappa^e}{\eta}(\sigma + \sigma^q)$$
$$\sigma^{n,h} = x^{k,h}(\sigma + \sigma^q) = \frac{1-\kappa^e}{1-\eta}(\sigma + \sigma^q)$$

- Volatility of experts' wealth share

$$\eta\sigma^\eta = (\kappa^e - \eta)(\sigma + \sigma^q)$$

- Consistency of dynamics

$$q\sigma^q = q_\eta(\eta\sigma_\eta)$$

- Two-way feedback loop

Risk concentration

- Solving the two-way feedback loop

$$\sigma_\eta = \left(\frac{\kappa^e}{\eta} - 1\right) \frac{\sigma}{1 - \left(\frac{\kappa^e}{\eta} - 1\right)\frac{\eta q_\eta}{q}}$$

$$\sigma + \sigma^q = \frac{\sigma}{1 - \left(\frac{\kappa^e}{\eta} - 1\right)\frac{\eta q_\eta}{q}}$$

  where χ^e/η is a measure of leverage and $\eta q_\eta / q$ is the elasticity of capital w.r.t. η

- Loss spiral (market illiquidity): A one percent drop in capital value generates a $\varphi \equiv \left(\frac{\kappa^e}{\eta} - 1\right)\frac{\eta q_\eta}{q}$ percent drop in experts' wealth share because experts are more exposed to capital price fluctuations. Second-round drop: φ^2 of percent. Total drop: $1/(1-\varphi)$ percent.
- Solving for risk concentration is usually a key step. It allows to solve for the volatility of variables of interest (i.e., capital price and investment opportunities) using the consistency condition for dynamics.

Model with Endogenous Risk Dynamics: Log preferences

Equilibrium

- The following system defines an ODE for $q(\eta)$. Input q into the system and you can solve for q_η.

$$(a^e - a^h)\kappa^e + a^h = \iota(q) + q\left[\eta\rho_e + (1-\eta)\rho_h\right]$$

$$0 = \left(\frac{a^e - a^h}{q} - \left[\frac{\kappa^e - \eta}{\eta(1-\eta)}\right](\sigma + \sigma_q)^2\right)(1-\kappa^e)$$

$$\sigma + \sigma^q = \frac{\sigma}{1 - \left(\frac{\kappa^e}{\eta} - 1\right)\frac{\eta q_\eta}{q}}$$

Numerical strategy

- In fact, this is an explicit first-order ODE

$$0 = \left(\frac{a^e - a^h}{q} - \left[\frac{\kappa^e(q,\eta) - \eta}{\eta(1-\eta)} \right] \left(\frac{\sigma}{1 - \left(\frac{\kappa^e(q,\eta)}{\eta} - 1 \right) \frac{\eta q_\eta}{q}} \right)^2 \right) (1 - \kappa^e(q,\eta))$$

where

$$\kappa^e(q,\eta) \equiv \frac{\iota(q) + q\left[\eta\rho_e + (1-\eta)\rho_h\right] - a^h}{a^e - a^h}$$

- The equation is an ODE for q when $\kappa^e < 1$, otherwise $q(\eta) = q^e(\eta)$ where the latter is defined by $1 = \kappa^e(q^e, \eta)$.
- Since q_η is squared, there are two solutions for the ODE. One associated to $\sigma + \sigma^q > 0$ and another one associated with $\sigma + \sigma^q < 0$.
- This is a glimpse of the potential multiplicity in (general equilibrium) models with financial frictions and heterogeneity in productivity.
- For now, assume $\sigma + \sigma^q > 0$.

Model with Endogenous Risk Dynamics: Log preferences

Numerical strategy

- We use $q(0) = q^h$ as boundary condition where $a^h = \iota(q^h) + q^h \rho_h$ which implies $\kappa^e = 0$.
- Solve the ODE from $\eta = 0$ until $\kappa^e(\eta) = 1$, then set $\kappa^e(\eta) = 1$, $q(\eta) = q^e(\eta)$.

Thinking about multiplicity

- Other equilibria recursive in η? Khorrami Mendo (2025a) "Dynamic Self-Fulfilling Fire Sales".
  - Does the solution with $\sigma + \sigma^q < 0$ make sense?
  - Is $\kappa(0) = 0$ the only possible boundary condition?
- What about equilibria that are not recursive in η? Khorrami Mendo (2025b) "Rational Sentiments and Financial Frictions".

Recursive solution

- Consistency conditions.
  - Given that we are looking for functions $q(\eta)$, $\zeta^e(\eta)$, and $\zeta^h(\eta)$, dynamics of these variables must be consistent with dynamics of η. This consistency is captured by Ito's formula

$$y\mu^y = y_\eta(\eta\mu_\eta) + \frac{1}{2}y_{\eta\eta}(\eta\sigma^\eta)^2$$
$$y\sigma^y = y_\eta(\eta\sigma_\eta)$$

for $y \in \{q, \zeta^e, \zeta^h\}$

Recursive equilibrium

Given initial conditon η_0, a Recursive Equilibrium is a set of functions of η for allocations $\{\frac{c_e}{n_e}, \frac{c_h}{n_h}, x^{k,e}, x^{k,h}\}$, value functions $\{\zeta^e, \zeta^h, \sigma^{\zeta,e}, \sigma^{\zeta,h}, \mu^{\zeta,e}, \mu^{\zeta,h}\}$, prices $\{r, q, \sigma^q, \mu^q\}$, and dynamics for aggregate state $\{\sigma^\eta, \mu^\eta\}$ such that

- Agents optimize (FOCs for $x^{k,e}, x^{k,h}, c_e/n_e, c_h/n_h$ and HJB for ζ^e, ζ^h)
- Markets clear (equations for goods, and capital)
- Aggregate state η evolves consistently with individual states
- Consistency conditions for dynamics are satisfied

Model with Endogenous Risk Dynamics: equilibrium characterization

Then, we can re-write optimal portfolio decisions as ...

- Experts:

$$\mu^{r,e} - r = \varsigma_e(\sigma + \sigma^q)$$

- Households:

$$0 = \min\{1 - \kappa_e, \varsigma_h(\sigma + \sigma_q) - (\mu^{r,h} - r)\}$$

- Risk balance

$$0 = \min\left\{1 - \kappa^e, \frac{a_e - a_h}{q} - (\varsigma_e - \varsigma_h)(\sigma + \sigma^q)\right\}$$

We replace the FOC condition for HH with the latter equation

Model with Endogenous Risk Dynamics: equilibrium characterization

Equilibrium system and numerical strategy

- Conjecture investment opportunities functions $\zeta^e(\eta), \zeta^h(\eta)$.
- Solve for $q(\eta)$ from the ODE defined by

$$(a^e - a^h)\kappa^e + a^h = \iota(q) + q\left[\eta\rho_e^{\frac{1}{\gamma}}(\zeta^e)^{1-\frac{1}{\gamma}} + (1-\eta)\rho_h^{\frac{1}{\gamma}}(\zeta^h)^{1-\frac{1}{\gamma}}\right]$$

$$0 = \min\left\{\frac{a^e - a^h}{q} - (\varsigma_e - \varsigma_h)(\sigma + \sigma_q), 1 - \kappa^e\right\}$$

  where $\sigma + \sigma^q$ correspond to the expression above and

$$\varsigma_e - \varsigma_h = (\kappa^e - \eta)\left[\frac{\gamma}{\eta(1-\eta)} + (\gamma - 1)\left(\frac{\zeta^e_\eta}{\zeta^e} - \frac{\zeta^h_\eta}{\zeta^h}\right)\right](\sigma + \sigma_q)$$

- We need a boundary condition for this ODE, we can use the solution for $q(0)$ from an economy with only households, i.e., the value q_h that solves

$$a^h = \iota(q_h) + q_h\left[\rho_h^{\frac{1}{\gamma}}(\zeta^h(0))^{1-\frac{1}{\gamma}}\right]$$

- We have now solutions to $\{q, \kappa^e, \sigma^q\}$. From these, we can immediately solve for $\{\sigma^\eta, \sigma^{\zeta,e}, \sigma^{\zeta,h}, \sigma^{n,e}, \sigma^{n,h}, \mu^{n,e} - r, \mu^{n,h} - r\}$

Equilibrium system and numerical strategy

- We can solve for the drift of the state from

$$\eta\mu^{\eta} = \eta(1-\eta)\left([\mu^{n,e} - r] - [\mu^{n,h} - r] - \rho_e^{\frac{1}{\gamma}}(\zeta^e)^{1-\frac{1}{\gamma}} + \rho_h^{\frac{1}{\gamma}}(\zeta^h)^{1-\frac{1}{\gamma}}\right) \dots$$

$$\dots - \eta\sigma_{\eta}(\eta\sigma^{n,e} + (1-\eta)\sigma^{n,h})$$

- Recover $\mu_q - r$ from experts' FOC for capital

$$\mu^{r,e} - r = \varsigma_e(\sigma + \sigma^q)$$

and we can get μ^q (and therefore r) from consistency of dynamics

$$q\mu^q = q_{\eta}(\mu^{\eta}\eta) + \frac{1}{2}q_{\eta\eta}(\sigma_{\eta}\eta)^2$$

Equilibrium system and numerical strategy

- Finally, we can update the investment opportunities using the HJBs and assuming that ζ^z depends directly on time

$$\frac{\rho}{1-\gamma} = \frac{\gamma}{1-\gamma}\rho^{1/\gamma}(\zeta^z)^{1-\frac{1}{\gamma}} + \mu^{n,z} - \frac{\gamma}{2}(\sigma^{n,z})^2 - (\gamma-1)\sigma^{n,z}\sigma^{\zeta,z}...$$
$$... + \frac{\partial\zeta^z/\partial t}{\zeta^z} + \mu^{\zeta,z} - \frac{\gamma}{2}(\sigma^{\zeta,z})^2$$

for $z \in \{e, h\}$

Question for the following problem set: derive (check!) the set of equations presented, i.e., the system that characterizes the equilibrium.

Potential issues ....

- Non-differentiability of investment opportunities ζ^e, ζ^h at some points... but Barles-Souganidis (1991) state that there is not problem with non-differentiable points
- Does the scheme strictly satisfies the Barles-Souganidis conditions?
- Is this doomed to fail? Not sure!
- Probably better chances if we solve for something smooth (plus we get to avoid involving $q_{\eta\eta}$ in the loop).
- Brunnermeier and Sannikov propose

$$V(n(\eta, K), \eta) = \frac{v(\eta)K^{1-\gamma}}{1-\gamma} - (\rho(1-\gamma))^{-1}$$

- But before, let us solve the case with logarithmic preferences.

Macro, Money, and Finance: a continuous-time approach

- Numerical methods: finite difference

Fernando Mendo

2023

PUC Rio

Numerical methods: finite difference

Slides by Yuliy Sannikov

Macro, Money, and Finance: a continuous-time approach

- Endogenous Risk Dynamics in Real Macro Model with Heterogeneous Agents

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Endogenous Risk Dynamics in Real Macro Model with Heterogeneous Agents

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- Non-differentiability of investment opportunities ζ^e, ζ^h at some points... but Barles-Souganidis (1991) state that there is not problem with non-differentiable points
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- Brunnermeier and Sannikov propose

$$V(n(\eta, K), \eta) = \frac{v(\eta)K^{1-\gamma}}{1-\gamma} - (\rho(1-\gamma))^{-1}$$

- Equating our previous and the new guess for the value function

$$V(n, \eta) = \frac{1}{\rho} \frac{(\zeta(\eta) n)^{1-\gamma} - 1}{1-\gamma} = \frac{v(\eta) K^{1-\gamma}}{1-\gamma} - \frac{1}{\rho(1-\gamma)}$$

- Note that

$$n^{z,i} = s^i N^z = s^i \eta^z q K$$

where $s^i \equiv n^{z,i} / N^z$ is the share of agent i of the aggregate net worth of sector $z \in \{e, h\}$. Useful notation: $\eta^e \equiv \eta$ and $\eta^h \equiv 1 - \eta$.

- As long as the share s^i remains constant (as is the case in this model), we can normalize it to one WLOG.
    - What about with a perpetual youth (stochastic death) structure?
- Hence, for $z \in \{e, h\}$

$$\zeta^z = \frac{(\rho v^z)^{\frac{1}{1-\gamma}}}{\eta^z q}$$

Model with Endogenous Risk Dynamics: Change of variable

- Consistency of dynamics imply

$$\sigma^{\zeta,e} = \frac{1}{1-\gamma}\sigma^{v,e} - \sigma^{\eta} - \sigma^{q}$$

$$\sigma^{\zeta,h} = \frac{1}{1-\gamma}\sigma^{v,h} + \frac{\eta}{1-\eta}\sigma^{\eta} - \sigma^{q}$$

where we used $\sigma^{\eta^h} = -\frac{\eta}{1-\eta}\sigma^{\eta}$

- From the expression in the previous slide for individual wealth, we have

$$\sigma^{n,e} = \sigma^{\eta} + \sigma^{q} + \sigma$$

$$\sigma^{n,h} = -\frac{\eta}{1-\eta}\sigma^{\eta} + \sigma^{q} + \sigma$$

- Hence, prices of risk become

$$\varsigma_e = \sigma^{\eta} + \sigma^{q} + \gamma\sigma - \sigma^{v,e}$$

$$\varsigma_h = -\frac{\eta}{1-\eta}\sigma^{\eta} + \sigma^{q} + \gamma\sigma - \sigma^{v,h}$$

Model with Endogenous Risk Dynamics: Change of variable

- Consumption-wealth ratio and consumption-capital ratio (normalizing share to one)

$$\frac{c_z}{n_z} = \frac{(q\eta^z)^{\frac{1}{\gamma}-1}}{(v^z)^{\frac{1}{\gamma}}}, \qquad \frac{c_z}{K} = \left(\frac{q\eta^z}{v^z}\right)^{\frac{1}{\gamma}}$$

- We could find an expression for $\mu^{\zeta,z}$ to replace in the corresponding HJB (just as we did with the volatilities).
- It is more efficient to derive the HJB directly with the new conjecture for the value function

$$\rho_z \frac{v^z K^{1-\gamma}}{1-\gamma} - \frac{1}{1-\gamma} = \frac{(c_z)^{1-\gamma}}{1-\gamma} - \frac{1}{1-\gamma} + \frac{1}{1-\gamma}\mathbb{E}[d\left(v^z K^{1-\gamma}\right)]$$

- Re-arranging and using the consumption capital ratio

$$\rho_z = \frac{(q\eta^z)^{\frac{1}{\gamma}-1}}{(v^z)^{\frac{1}{\gamma}}} + \mathbb{E}\left[\frac{d\left(v^z K^{1-\gamma}\right)}{v^z K^{1-\gamma}}\right]$$

Model with Endogenous Risk Dynamics: Change of variable

- Using LoM for aggregate capital

$$\rho_z = \frac{(q\eta^z)^{\frac{1}{\gamma}-1}}{(v^z)^{\frac{1}{\gamma}}} + \mu^{v,z} + (1-\gamma)(\Phi(\iota(q)) - \delta) - \frac{\gamma(1-\gamma)}{2}\sigma^2 + (1-\gamma)\sigma^{v,z}\sigma$$

for $z \in \{e, h\}$

- Conjecture investment opportunities functions $v^e(\eta)$ and $v^h(\eta)$.
- **Static step**: similar to solution for log-preferences
    - Use as boundary conditions for $q(0)$, the value if there were only households, i.e., the value q_h that solves $a^h = \iota(q_h) + \left(\frac{q_h}{v^h(0)}\right)^{\frac{1}{\gamma}}$
    - Solve for the function $q(\eta)$ from

$$\chi^e = \max\{\alpha\kappa^e, \eta\}$$

$$0 = \iota(q) + \left(\frac{q\eta}{v^e}\right)^{\frac{1}{\gamma}} + \left(\frac{q(1-\eta)}{v^h}\right)^{\frac{1}{\gamma}} - (a^e - a^h)\kappa^e - a^h$$

$$0 = \left(\frac{a^e - a^h}{q} - \alpha\left(\frac{1}{\eta(1-\eta)} - \frac{v^e_\eta}{v^e} + \frac{v^h_\eta}{v^h}\right)(\chi^e - \eta)(\sigma + \sigma_q)^2\right)(1-\kappa^e)$$

$$0 = \sigma + \sigma^q - \frac{\sigma}{1 - \left(\frac{\chi^e}{\eta} - 1\right)\frac{\eta q_\eta}{q}}$$

recall that as long as $\kappa^e < 1$ (for the ODE part), we have $\chi^e = \alpha\kappa^e$.

Model with Endogenous Risk Dynamics: Equilibrium and numerical strategy

- **Time step**: updating value function
  - From static step, we have solutions to $\{q, \kappa^e, \chi^e, \sigma^q\}$. From these, we can immediately solve for $\{\sigma^\eta, \sigma^{v,e}, \sigma^{v,h}, \sigma^{n,e}, \sigma^{n,h}, \mu^{n,e} - r, \mu^{n,h} - r, \mu_q - r\}$
  - Solve for the drift of η from

$$\eta\mu^\eta = \eta(1-\eta)\left([\mu^{n,e} - r] - [\mu^{n,h} - r] - \frac{(q\eta)^{\frac{1}{\gamma}-1}}{(v^e)^{\frac{1}{\gamma}}} + \frac{(q(1-\eta))^{\frac{1}{\gamma}-1}}{(v^h)^{\frac{1}{\gamma}}}\right)\ldots$$

$$\ldots - \eta\sigma_\eta(\eta\sigma^{n,e} + (1-\eta)\sigma^{n,h})$$

  - Solve for the drift $\mu^{v,z}$ from HJB conditions for $z \in \{e, h\}$
  - Update value function from consistency conditions

$$v^z\mu^{v,z} = v^z_\eta(\eta\mu_\eta) + \frac{1}{2}v^z_{\eta\eta}(\eta\sigma^\eta)^2 + \frac{\partial v^z}{\partial t}$$

    for $z \in \{e, h\}$.
- Once updated, go to static step and iterate until convergence.

- **Rest of objects**: disentangling μ^q and r
  - We can get μ^q (and therefore r) from consistency of dynamics

    $$q\mu^q = q_\eta(\eta\mu^\eta) + \frac{1}{2}q_{\eta\eta}(\sigma_\eta\eta)^2$$

    We don't need to do this second order calculations while solving the model.

Some details about the numerical strategy

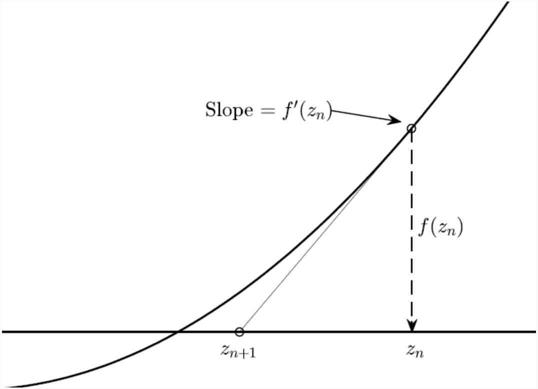
- The code Lec3Main.m implements the described numerical strategy using finite differences: the Newton's method for the static step and value function iteration for the time step.
- The grid is bounded away from 0 and 1 to avoid singularities (division by zero)
- For the **static step**, instead of actually solving for $q(\eta_i), \kappa^e(\eta_i), \sigma_q(\eta_i)$ given $q(\eta_{i-1})$, we just take one step using Newton's method.
- It saves time and potentially avoids going to much in the wrong direction but there is no theorem behind this ..... so we just hope that this together with the time step lead to convergence.
- We could take an arbitrary number of steps.

Overview Newton Method

- Find the root of equation system $F(z_n) = 0$ via iterative method

$$z_{n+1} = z_n - J_n^{-1} F(z_n)$$

where J_n^{-1} is the Jacobian, i.e., $J_{ij} - \partial f_i(z)/\partial z_j$.



Overview Newton Method

- The vector $z = (q_i, \kappa_i^e, \sigma + \sigma_i^q)$ and q_{i-1} is given. We also take as given the conjecture for functions v^e, v^h and
- The error function $F(z)$ is

$$0 = \iota(q) + \left(\frac{q\eta}{v^e}\right)^{\frac{1}{\gamma}} + \left(\frac{q(1-\eta)}{v^h}\right)^{\frac{1}{\gamma}} - (a^e - a^h)\kappa^e - a^h$$

$$0 = \frac{a^e - a^h}{q} - \alpha\left(\frac{1}{\eta(1-\eta)} - \frac{v_\eta^e}{v^e} + \frac{v_\eta^h}{v^h}\right)(\alpha\kappa^e - \eta)(\sigma + \sigma_q)^2$$

$$0 = \sigma + \sigma^q - \frac{\sigma}{1 - \left(\frac{\chi^e}{\eta} - 1\right)\frac{\eta q_\eta}{q}}$$

- Solve from $\eta = 0$ until $\kappa^e > 1$. At that point, we stop and set $\kappa^e = 1$ for larger values of η.

Model with Endogenous Risk Dynamics: numerical solution

Some details about the numerical strategy

- For the **dynamic step**, we have

$$0 = -v^z \mu^{v,z} + v^z_\eta(\eta\mu_\eta) + \frac{1}{2} v^z_{\eta\eta}(\eta\sigma^\eta)^2 + \frac{\partial v^z}{\partial t}$$

  which can be written as we can write as

$$0 = -\Lambda(g(\boldsymbol{v_{t+\Delta t}}))\boldsymbol{v_t} + M(\boldsymbol{v_{t+\Delta t}})\boldsymbol{v_t} + \frac{\boldsymbol{v_{t+\Delta t}} - \boldsymbol{v_t}}{\Delta t}$$

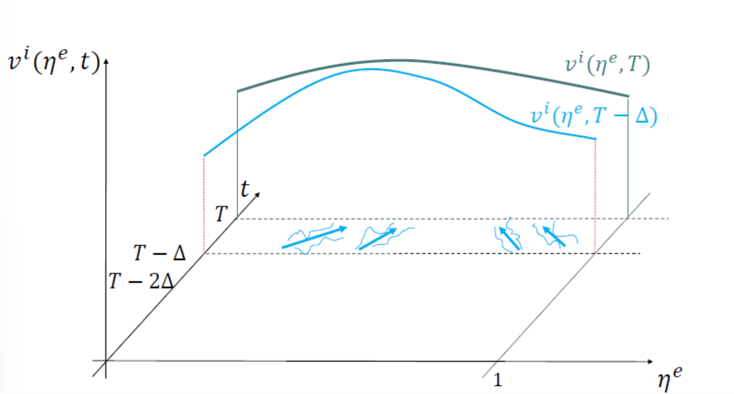
  where $\Lambda(x)$ is the square matrix that has vector x as main diagonal and zeros everywhere else. Vector $g(v_{t+\Delta t})$ is the drift $\mu^{v,z}$ solved from HJB equation.

- Implicit step

$$\boldsymbol{v_t} = \left(\boldsymbol{I} + \Lambda(g(\boldsymbol{v_{t+\Delta t}}))\Delta t - \Delta t M(\boldsymbol{v_{t+\Delta t}})\right)^{-1} \boldsymbol{v_{t+\Delta t}}$$

- Similar to the approach studied but with a time-varying discount rate $g(\boldsymbol{v_{t+\Delta t}})$. Is $g(\boldsymbol{v_{t+\Delta t}}) > 0$? What about be the exact condition for the scheme monotone?

- **Dynamic step**



Missing in this version of slides

- Results: Graphs with equilibrium objects (prices, allocation) and comparative statics. See here slides 68 onward.
- Set of slides with examples on how to use KFE. Including the stationary distribution. Slides here.
- Toolbox: Price taking social planner
- Toolbox: Change of numeraire

Macro, Money, and Finance: a continuous-time approach

- Stochastic Stability, Multiplicity and Sentiments

Fernando Mendo

2023

PUC Rio

Stochastic Stability, Multiplicity and Sentiments

Khorrami Mendo (2023), Rational Sentiments and Financial Frictions. See both, paper and slides.

- Equilibrium definition(as stochastic processes)
- Equilibrium characterization (set of equations that characterize equilibrium)
- Intuition behind multiplicity
- Recursive equilibrium in fundamental variables
- Useful lemma in Stochastic Stability Theory
- Equilibria with sentiments
- Solcing puzzles with sentiment

Macro, Money, and Finance: a continuous-time approach

- Recursive preferences and Jumps

Fernando Mendo

2023

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Recursive preferences and Jumps

Motivation

- Macro-finance: decouple IES and risk aversion.
- Asset pricing: make the SDF (marginal utility of consumption) depend on more than just current consumption. Potentially: low volatility of consumption with high volatility of SDF.

Time-additive preferences / Expected Utility

- Widespread use based on mathematical convenience.
- **Drawbacks**
  1. EIS is the reciprocal of relative risk aversion (RRA).
  2. Indifference to the timing of the resolution of uncertainty.
  3. Expected utility admits no distinction between risk and uncertainty.

Time-additive utility: IES and RRA

$$U = \mathbb{E}\left[\int_0^\infty e^{-\rho t} u(c_t) dt\right]$$

- **RRA**: measure of attitude towards risk (curvature of marginal utility).

$$RRA = -\frac{\partial u'(c_t)}{\partial c_t}\frac{c_t}{u'(c_t)} = -u''(c_t)\frac{c_t}{u'(c_t)}$$

- **IES**: the percentage change in consumption growth in response to a 1 percent increase in the interest rate (or in the growth rate of marginal utility of consumption)

$$EIS = -\frac{\partial(\dot{c}/c_t)}{\partial(\dot{u}'(c_t)/u'(c_t))} = \frac{\partial(\dot{c}/c_t)}{\partial r_t} = \frac{1}{RRA}$$

- Why should the willingness to substitute across dates be related to the willingness to substitute across states? **No good reason**.
- The restriction is a fundamental cause of Equity Premium/Risk-Free Rate Puzzle. We need a high RRA to explain the mean return on risky assets, but then this implies a very low EIS, which generates a very high risk-free rate.

Time-additive utility: Resolution of uncertainty

- Time-additive: indifference to the timing of the resolution of uncertainty.
- Consider the following 2 lotteries Formally, fix a small gap between coin tosses.
  - At time 0 a single coin is tossed. If it is Heads, then $c_t = H$ for all t. If it is Tails, then $c_t = T$ for all t.
  - At each date t a coin will be tossed. If it is Heads then $c_t = H$. If Tails, then $c_t = T$.
- An agent with time-additive/expected-utility preferences will be indifferent to these 2 lotteries, since the expected utility of both is

$$\int_0^\infty e^{-\rho t}\left[\frac{1}{2}u(H) + \frac{1}{2}u(T)\right] dt$$

- Reasonable? Maybe. In the first lottery, uncertainty is resolved earlier.
- A basic axiom of expected utility theory is that agents reduce compound lotteries. Questionable assumption, especially in a dynamic setting. A preference for early/late resolution reflects an unwillingness to reduce (intertemporal) compound lotteries.
- Recursive preferences can impose expected utility w.r.t. to **atemporal** lotteries, but relax the expected utility axioms w.r.t. **intertemporal** lotteries.

- Expected utility admits no distinction between risk (**known** probabilities) and uncertainty (**unknown** probabilities).
- Many have argued (e.g., Knight, Keynes) that this distinction is especially important in financial markets.
- Recursive preferences can be reinterpreted in a way that allows this distinction to be operationalized.

Ellsberg paradox

- Urn A contains 50 red and 50 black
- Urn B contains an unknown mix of red and black balls.
  - Bet 1A: \$1 if red is drawn from urn A, 0 otherwise
  - Bet 2A: \$1 if black is drawn from urn A, 0 otherwise
  - Bet 1B: \$1 if red is drawn from urn B, 0 otherwise
  - Bet 2B: \$1 if black is drawn from urn B, 0 otherwise
- Typically, participants indifferent between bet 1A and bet 2A (consistent with expected utility theory) but were seen to **strictly prefer Bet 1A to Bet 1B and Bet 2A to 2B**.
- **No probabilities about balls in Urn B can rationalize this under expected utility!**
- Interpreted to be a consequence of ambiguity / uncertainty aversion: people dislike situations where they cannot attach probabilities to outcomes.

We will not explore this topic (risk vs. uncertainty) further in this class.

Recursive Preferences

- Time-additive CRRA utility does have a couple of desirable features, which we want to retain:
  1. **Dynamic Consistency**: allows us to use Dynamic Programming to characterize optimal behavior.
  2. **Scale Invariance**: CRRA utility has the attractive property that interest rates and risk premia are stationary in the presence of growth.
- Kreps and Porteus (1978) show that dynamic consistency can be preserved while relaxing time-additivity. They define current utility recursively, using two distinct functions:

$$V_t = W(c_t, \mathcal{R}_t(V_{t+1}))$$

  - **Time Aggregator** $W(\cdot)$ combines current consumption and the certainty equivalent of future utility into a measure of current utility. It is increasing and captures **inter-temporal substitution**.
  - **Certainty Equivalent operator** translates random future utility into consumption units.

$$\mathcal{R}_t(x) \equiv \phi^{-1}(\mathbb{E}_t[\phi(x)])$$

    for some auxiliary increasing function $\phi(\cdot)$. Captures **risk aversion**.

Recursive Preferences: IES and RRA

- Recursive preferences

$$V_t = W(c_t, \mathcal{R}_t(V_{t+1})), \qquad \mathcal{R}_t(x) \equiv \phi^{-1}(\mathbb{E}_t[\phi(x)])$$

- Useful definitions:

$$U(c, \mathcal{R}) \equiv \phi(W(c, \mathcal{R}))$$
$$\nu(c, E) \equiv \phi(W(c, \phi^{-1}(E)))$$

- Assuming U is an homothetic function, we have

$$RRA = -\frac{\phi''(\mathcal{R})\mathcal{R}}{\phi'(\mathcal{R})}$$
$$IES = \frac{\partial \log(\mathcal{R}/c)}{\partial IMRS} = \frac{U_c}{\mathcal{R}(U_{c\mathcal{R}} - U_c U_{\mathcal{R}\mathcal{R}}/U_{\mathcal{R}})}$$

- Early (late) resolution of uncertainty is preferred if and only if $\nu(c, E)$ is convex (concave) in E. If $\nu(c, E)$ is affine in E, the decision maker is indifferent to the timing of uncertainty resolution.

Reference: Brown Kim (Management Science 2014)

Recursive Preferences: special cases

- Of course, additive preferences are recursive too, and correspond to

$$W(x, y) = u(x) + \beta y, \qquad \mathcal{R}_t(z) = \mathbb{E}_t(z)$$

- Epstein and Zin (1989) develop a special case of Kreps-Porteus preferences, where both the Time Aggregator and Certainty Equivalent are CES functions:

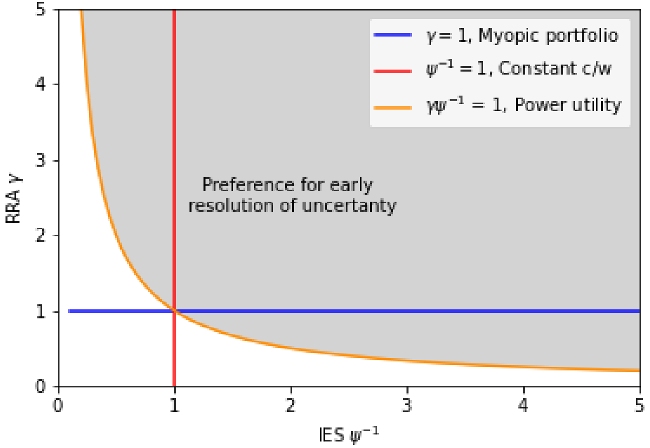
$$W(x, y) = [(1-\beta)x^{1-\psi} + \beta y^{1-\psi}]^{\frac{1}{1-\psi}}, \qquad \mathcal{R}_t(z) = [\mathbb{E}_t(z^{1-\gamma})]^{\frac{1}{1-\gamma}}$$

- Epstein Zin preferences

$$V_t = \left[(1-\beta)c_t^{1-\psi} + \beta[\mathbb{E}_t(V_{t+1}^{1-\gamma})]^{\frac{1-\psi}{1-\gamma}}\right]^{\frac{1}{1-\psi}}$$

- Properties
  - Homothetic $\rightarrow$ scale invariant
  - $RRA = \gamma$ and $IES = \psi^{-1}$
  - Early resolution of uncertainty is preferred if $\gamma > \psi$, i.e., when RRA and/or EIS are sufficiently large.

Epstein Zin Preferences



Recursive Preferences: Stochastic Discount Factor

- Recursive preferences

$$V_t = W(c_t, \mathcal{R}_t(V_{t+1})), \qquad \mathcal{R}_t(x) \equiv \phi^{-1}(\mathbb{E}_t[\phi(x)])$$

- SDF

$$\frac{M_{t+1}}{M_t} = \frac{\frac{\partial V_t}{\partial c_{t+1}}}{\frac{\partial V_t}{\partial c_t}} = \frac{\frac{\partial V_t}{\partial V_{t+1}}\frac{\partial V_{t+1}}{\partial c_{t+1}}}{\frac{\partial V_t}{\partial c_t}}$$

$$= W_{\mathcal{R}}(c_t, \mathcal{R}_t)\mathcal{R}'(V_{t+1})\frac{W_c(c_{t+1}, \mathcal{R}_{t+1})}{W_c(c_t, \mathcal{R}_t)}$$

- EZ preferences

$$V_t = \left[(1-\beta)c_t^{1-\psi} + \beta[\mathbb{E}_t(V_{t+1}^{1-\gamma})]^{\frac{1-\psi}{1-\gamma}}\right]^{\frac{1}{1-\psi}}$$

- EZ SDF

$$\frac{M_{t+1}}{M_t} = \beta\left(\frac{c_{t+1}}{c_t}\right)^{-\psi}\left(\frac{V_{t+1}}{\mathcal{R}_t(V_{t+1})}\right)^{\psi-\gamma}$$

The SDF responds to innovations in anticipated future utility. This injects an additional source of volatility into the SDF, which can help to fit the data.

EZ preferences in continuous time

- The continuous-time version of EZ preferences is known as Stochastic Differential Utility (Duffie and Epstein (1992)).
- Since the Certainty Equivalent, $\mathcal{R}(x) = \left(\mathbb{E}[x^{1-\gamma}]\right)^{\frac{1}{1-\gamma}}$ is nonlinear, Ito's lemma implies that in the continuous-time limit a variance term will appear.
- DE show that this variance term can be avoided by using the ordinally equivalent utility index. Let $\mathcal{V}_t \equiv \frac{V_t^{1-\gamma}}{1-\gamma}$, then

$$V_t = \left[(1-\beta)c_t^{1-\psi} + \beta[\mathbb{E}_t(V_{t+1}^{1-\gamma})]^{\frac{1-\psi}{1-\gamma}}\right]^{\frac{1}{1-\psi}}$$

$$\mathcal{V}_t = \frac{1}{1-\gamma}\left[(1-\beta)c_t^{1-\psi} + \beta[(1-\gamma)\mathbb{E}_t(\mathcal{V}_{t+1})]^{\frac{1-\psi}{1-\gamma}}\right]^{\frac{1-\gamma}{1-\psi}}$$

EZ preferences: continuous-time limit

- Instead of a unit time step, consider an arbitrarily time step Δt, i.e., consider $e^{-\rho\Delta t}$ instead of β and $U_{t+\Delta_t}$ instead of U_{t+1}. Then, we can re-arrange recursive formulation to write

$$\mathbb{E}_t[\mathcal{V}_{t+\Delta t}] = \frac{1}{1-\gamma}\left[e^{\rho\Delta t}[(1-\gamma)U_t]^{\frac{1-\psi}{1-\gamma}} - (e^{\rho\Delta t}-1)c_t^{1-\psi}\right]^{\frac{1-\gamma}{1-\psi}}$$

- Taking the limit as limit $\Delta t \to 0$ and using L'Hopital rule

$$\mathbb{E}_t\left[\frac{d\mathcal{V}_t}{dt}\right] = \lim_{\Delta t\to 0}\frac{\mathbb{E}_t[\mathcal{V}_{t+\Delta t}]-\mathcal{V}_t}{\Delta t}$$
$$= -\frac{1}{1-\psi}\left([(1-\gamma)\mathcal{V}_t]^{\frac{\psi-\gamma}{1-\gamma}}\rho c_t^{1-\psi} - (1-\gamma)\rho\mathcal{V}_t\right) \equiv -f(c_t,\mathcal{V}_t)$$

  where $f(c_t,\mathcal{V}_t)$ is the continuous-time (normalized) aggregator.
- Missing step? Conclude that $d\mathcal{V}_t/dt = -f(c_t,\mathcal{V}_t)$. The stochastic variable c_t only appears inside a integral. [Not sure]
- Assuming the transversality condition $\lim_{T\to\infty}$ holds, we have

$$\mathcal{V}_t = \mathbb{E}_t\left[\int_t^\infty f(c_s,\mathcal{V}_t ds)\right]$$

EZ preferences in continuous-time

- EZ in continuous-time

$$\mathcal{V}_t = \mathbb{E}_t\left[\int_t^{\infty} f(c_s, \mathcal{V}_t) ds\right]$$

$$f(c, \mathcal{V}) = \frac{1}{1-\psi}\left([(1-\gamma)\mathcal{V}]^{\frac{\psi-\gamma}{1-\gamma}}\rho c^{1-\psi} - (1-\gamma)\rho\mathcal{V}\right)$$

- Consider the CRRA case ($\gamma = \psi$), then we have

$$\mathcal{V}_t = \rho\mathbb{E}_t\left[\int_t^{\infty} e^{-\rho(s-t)}\frac{c_s^{1-\gamma}}{1-\gamma} ds)\right]$$

- Some authors work with the transformation $U_t = \mathcal{V}_t/\rho$. We recover that equivalent representation by using this change of variable on $d\mathcal{V}_t/dt = -f(c_t, \mathcal{V}_t)$.
- EZ in continuous-time

$$U_t = \mathbb{E}_t\left[\int_t^{\infty} \varphi(c_s, U_s)\, ds\right]$$

$$\varphi(c, U) = \frac{1}{1-\psi}\left\{[(1-\gamma)\rho U]^{\frac{\psi-\gamma}{1-\gamma}} c^{1-\psi} - (1-\gamma)\rho U\right\}$$

EZ preferences in continuous-time

- EZ preferences

$$U_t = \mathbb{E}_t\left[\int_t^\infty \varphi\left(c_s, U_s\right) ds\right]$$

$$\varphi(c, U) = \frac{1}{1-\psi}\left\{\left[(1-\gamma)\rho U\right]^{\frac{\psi-\gamma}{1-\gamma}} c^{1-\psi} - (1-\gamma)\,\rho U\right\}$$

where ψ^{-1} is the Intertemporal Elasticity of Substitution (IES) and γ is the Relative Risk Aversion.

- CRRA: $\psi = \gamma \neq 1$

$$\varphi(c, U) = \frac{c^{1-\gamma}}{1-\gamma} - \rho U \Rightarrow U_t = \mathbb{E}_t\left[\int_t^\infty e^{-\rho(s-t)}\frac{c^{1-\gamma}}{1-\gamma} ds\right]$$

- Unitary IES: $\psi^{-1} \to 1$

$$\varphi(c, U) = (1-\gamma)\rho U\left(\log c - \frac{1}{1-\gamma}\log((1-\gamma)U)\right)$$

- For convergence to logarithmic preferences, would need to subtract the appropriate scaled constant and revisit $dU_t/dt = -\varphi(c_t, U_t)$. In particular, set $\mathcal{U}_t \equiv U_t - [\rho(1-\gamma)]^{-1}$.

EZ preferences in continuous-time

For completeness... $\mathcal{U}_t \equiv U_t - [\rho(1-\gamma)]^{-1}$

- EZ preferences

$$\mathcal{U}_t = \mathbb{E}_t\left[\int_t^\infty g\left(c_s, \mathcal{U}_s\right) ds\right]$$

$$g(c, \mathcal{U}) = \frac{1}{1-\psi}\left\{\left[(1-\gamma)\rho\mathcal{U}+1\right]^{\frac{\psi-\gamma}{1-\gamma}} c^{1-\psi} - \left[(1-\gamma)\,\rho\mathcal{U}+1\right]\right\}$$

  where ψ^{-1} is the IES and γ is the RRA.

- CRRA: $\psi = \gamma \neq 1$

$$g(c, \mathcal{U}) = \frac{c^{1-\gamma}-1}{1-\gamma} - \rho\mathcal{U} \Rightarrow \mathcal{U}_t = \mathbb{E}_t\left[\int_t^\infty e^{-\rho(s-t)}\left(\frac{c^{1-\gamma}-1}{1-\gamma}\right) ds\right]$$

- Unitary IES: $\psi^{-1} \to 1$

$$g(c, \mathcal{U}) = \left[(1-\gamma)\rho\mathcal{U}+1\right]\left(\log c - \frac{1}{1-\gamma}\log((1-\gamma)\mathcal{U}+1)\right)$$

- Logarithmic preferences: $\psi = \gamma = 1$

$$g(c, \mathcal{U}) = \log(c) - \rho\mathcal{U} \Rightarrow \mathcal{U}_t = \mathbb{E}_t\left[\int_t^\infty e^{-\rho(s-t)}\log(c_s) ds\right]$$

EZ preferences: the SDF

- We could take continuous-time limits of the discrete-time SDF, but it is easier (and perhaps more illuminating) to consider a classic perturbation/ variational argument.
- Equilibrium price of an asset = change in the expected present value of utility that buying the asset generates.

$$M_t P_t = \mathbb{E}_t \left[\sum_{s=t+1}^{\infty} M_s X_s \right]$$

where $\{X_s\}$ represent the cash flow stream that the asset generates.

- To extend this intuition to continuous time with recursive preferences, consider the following perturbed value function

$$V_t(\eta) = \mathbb{E}_t \left[\int_t^{\infty} f\left(c_s + \eta X_s, V_s(\eta)\right) ds \right]$$

where η is the perturbation parameter.

EZ preferences: the SDF

- Consider the marginal change in value generated by the perturbation

$$\frac{\partial V_t(\eta)}{\partial \eta} \equiv \lim_{\eta \to 0} \frac{V_t(\eta) - V_t(0)}{\eta}$$
$$= \mathbb{E}_t \left[\int_t^\infty \lim_{\eta \to 0} \frac{1}{\eta} \left\{ f\left(c_s + \eta X_s, V_s(\eta)\right) - f\left(c_s, V_s(0)\right) \right\} ds \right]$$
$$= \mathbb{E}_t \left[\int_t^\infty f_c(c_s, V_s) X_s + f_v(c_s, V_s) \frac{\partial V_s(\eta)}{\partial \eta} ds \right]$$

Solving the latter functional equation (derive solution w.r.t. t to verify)

$$\frac{\partial V_t(\eta)}{\partial \eta} = \mathbb{E}_t \left[\int_t^\infty \exp\left(\int_t^s f_v(c_u, V_u) du \right) f_c(c_s, V_s) X_s ds \right]$$

- Drawing the parallel with asset pricing condition

$$\frac{\partial V_t(\eta)}{\partial \eta} = f_c(c_t, V_t) P_t \quad \text{and} \quad P_t = \mathbb{E}_t \left[\int_t^\infty \frac{M_s}{M_t} X_s ds \right]$$

- Therefore

$$M_t = \exp\left(\int_0^t f_v(c_u, V_u) du \right) f_c(c_t, V_t)$$

- SDF

$$M_t = \exp\left(\int_0^t f_v(c_u, V_u)du\right) f_c(c_t, V_t)$$

The partial derivative of the aggregator w.r.t. continuation utility plays the role of the rate of time preference.

- Special case: $f(c, V) = u(c) - \rho V$

$$M_t = e^{-\rho t} u'(c_t)$$

Poisson Process

- Let τ be an exponential random variable with Poisson density

$$F(\tau) = 1 - \exp(-\lambda\tau)$$
$$f(\tau) = \lambda \exp(-\lambda\tau)$$

  for $\tau \geq 0$.
- Let $\{\tau_i\}_{i\in\mathbb{N}}$ be a sequence of independent exponential random variables with $\mathbb{E}(\tau_i) = 1/\lambda$.
- Arrival times: $S_n = \sum_{k=1}^{n} \tau_k$
- Poisson process: $N_t = \#$ of events until $t = \max\{n \geq 0 : S_n \leq t\}$
- Compensated Poisson $N_t - t\lambda$ is a martingale.
- Properties
  - $Pr(N_t = k) = \frac{(\lambda t)^k}{k!} \exp(-\lambda t)$ for $k = 0, 1, 2....$
  - Increments are stationary and independent. Let $t_0 < t_1 < t_2 < ...$, then for $t_j > t_i$, $Pr(N_{t_j} - N_{t_i} = k) = \frac{(\lambda \Delta t)^k}{k!} \exp(-\lambda \Delta t)$ where $\Delta t = t_j - t_i$.
  - Moments: $\mathbb{E}(N_t - N_s) = (t-s)\lambda$ and $\mathbb{V}(N_t - N_s) = (t-s)\lambda$

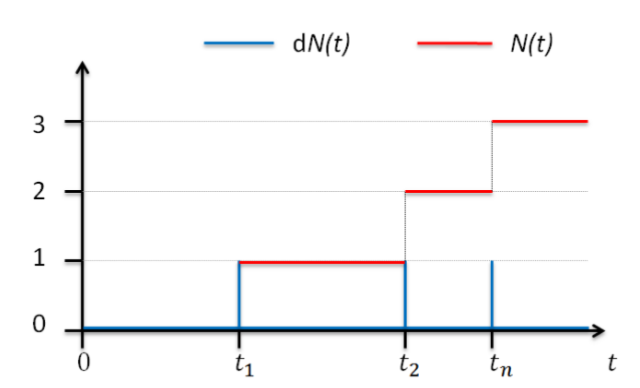


Figure 1: Poisson Process

Compound Poisson process

- Poisson process with random jump size
- Let $\{Y_i\}_{i\in\mathbb{N}}$ be a sequence of iid random variables with $\mathbb{E}(Y_i) = \mu_y$.
- $Q_t = \sum_{i=1}^{N_t} Y_i$ for $t \geq 0$ is a compound Poisson process.
- $Q_t - \lambda\mu_y$ is a compensated Poisson process.
- Increments are also stationary.
- Note that $N_t \in \mathbb{N}$ and $Q_t \in \mathbb{R}$.

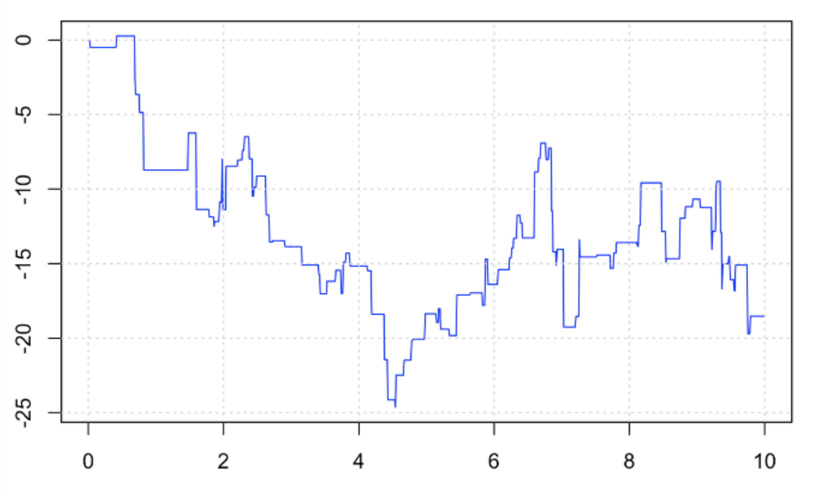


Figure 2: Compound Poisson Process

Basic stochastic calculus with jumps: Ito's Lemma

- Let

$$dx_t = \nu(x_{t^-})dN_t$$

where the intensity of the Poisson process is λ and

$$x_{t^-} \equiv \lim_{s \nearrow t} x_s$$

- Then

$$df(x_t) = [f(x_{t^-} + \nu(x_{t^-})) - f(x_{t^-})]dN_t$$

and

$$\mathbb{E}_t[df(x_t)] \equiv \lim_{s \nearrow t} \mathbb{E}_s[df(x_t)] = [f(x_{t^-} + \nu(x_{t^-})) - f(x_{t^-})]\lambda dt$$

- We can have $\lambda(x_{t^-})$.
Need to double-check references.

Basic stochastic calculus with jumps: Ito's Lemma

- Let

$$dx_t = \mu(x_t)dt + \sigma(x_t)dZ_t + \nu(x_t)dN_t$$

where the intensity of the Poisson process is λ.

- **Notation for this class**: drift $\mu(.)$, diffusion $\sigma(.)$, and sensitivity to jumps $\nu(.)$ are always function of the left limit x_{t^-} but we will omit the minus.
- In fact, for the solution to the SDE x_t is the same whether $\mu(\cdot)$ and $\sigma(\cdot)$ are evaluated at the left limit or not, because the measure of jump times is zero (countable many jumps).
- Then

$$\begin{aligned} df(x_t) &= [\mu(x_t)f'(x_t) + 0.5\sigma(x_t)f''(x_t)]dt + \sigma(x_t)dZ_t \ldots \\ &\quad \ldots + [f(x_t + \nu(x_t)) - f(x_t)]dN_t \\ \mathbb{E}_t[df(x_t)] &= [\mu(x_t)f'(x_t) + 0.5\sigma(x_t)f''(x_t)]dt + [f(x_t + \nu(x_t)) - f(x_t)]\lambda dt \end{aligned}$$

- Easier to remember:

$$df(x_t) = f'(x_t)dx_t + 0.5f''(x_t)(dx_t)^2 + [f(x_t + \nu(x_t)) - f(x_t)]dN_t$$

Dynamic programming with jumps

Consider the following control problem

$$\max_{\{A_t\}} \mathbb{E}\left[\int_0^{\infty} e^{-rt} g(X_t, A_t)\, dt\right]$$

s.t.

$$dX_t = \mu\left(X_t, A_t\right) dt + \sigma(X_t, A_t) dZ_t + \nu(X_t, A_t) dJ_t$$

where X_t is a vector of n_x states, A_t is a vector of n_a controls, and Z_t a vector of n_z Brownian motions and J_t is a vector of n_j Poisson processes with intensities $\lambda \in \mathbb{R}^{n_j}$. X_0 is given.

Stochastic HJB Equation

Look for $V(X)$ that solves the following functional equation

$$0 = \max_A g(X, A) + \frac{\mathbb{E}[dV(X)]}{dt} - rV(X)$$

and the associated policy functions $A(X)$. Same HJB?

Dynamic programming with jumps

- Same HJB? Yes, but dynamics of V now include jumps ..

$$0 = \max_A g(X, A) + \frac{\mathbb{E}[\boldsymbol{dV(X)}]}{\boldsymbol{dt}} - rV(X)$$

$$0 = \max_A g(X, A) + \frac{\mathbb{E}[dV(X)|dJ = 0]}{dt} + \sum_j \lambda_j (V(X + \nu_j(X)) - V(X)) - rV(X)$$

- We can consider the dynamics without jumps (red) extend it by adding the dynamics associated to jumps (blue).

Single-agent consumption-portfolio choice: jumps and EZ preferences

- Problem

$$\max_{\{c_t, x_t\}} U_0$$

s.t

$$dn_t = -c_t dt + n_t[(1 - x \cdot \mathbb{1})r_t + x \cdot dr_t^x]$$
$$dr_t^x = (r_t \mathbb{1} + \varphi_t^x)dt + \sigma_t^x dZ_t + \nu_t^x dJ_t$$
$$n_t \geq 0$$

where $x \in \mathbb{R}^{n_x}$, $Z_t \in \mathbb{R}^{n_z}$, $\sigma_t^x \in \mathbb{R}^{n_x \times n_z}$, $J_t \in \mathbb{R}^{n_j}$, $\nu_t^x \in \mathbb{R}^{n_x \times n_j}$ and initial condition n_0 is taken as given. Z_t is a vector of independent Brownian motions, J_t is a vector of independent Poisson processes and $\lambda \in \mathbb{R}^{n_j}$ represent the arrival rates of J_t.

- Reference? We could extend the set-up to allow for endogenous arrival rates $\lambda(x)$.
- The solvency constraint $n_t \geq 0$ can be imposed or derived from a "no Ponzi condition."

Single-agent consumption-portfolio choice: jumps and EZ preferences

- Individual state: wealth n_t
- Aggregate states:
  - Let $Y_t \in \mathbb{R}^{n_y}$ drive all time-varying objects beyond the control of the agent: prices $\{r(Y_t), \varphi^x(Y_t), \}$, volatility of risky return $\sigma^x(Y_t)$, and sensitivity to jumps $\nu^x(Y_t)$.
  - Assume that Y_t is a (time- homogeneous) Ito diffusion

$$dY_t = \mu_Y(Y_t)dt + \sigma_Y(Y_t)dZ_t + \nu_Y(Y_t)dJ_t$$

- EZ Preferences

$$U_t = \mathbb{E}_t \left[\int_t^\infty g\left(c_s, U_s\right) ds \right]$$

$$g(c, U) = \frac{1}{1-\psi} \left\{ \left[(1-\gamma)\rho U + 1\right]^{\frac{\psi-\gamma}{1-\gamma}} c^{1-\psi} - \left[(1-\gamma)\rho U + 1\right] \right\}$$

where ψ^{-1} is the IES and γ is the RRA.

Single-agent consumption-portfolio choice: jumps and EZ preferences

- Budget constraint notation (dropping time subscripts, fully recursive)

$$dn = (n\mu^n - c)dt + (n\sigma^n)dZ_t + n\nu^n \cdot dJ_t$$

where

$$\mu^n = r_t + \varphi^x \cdot x$$
$$\sigma^n = (\sigma^x)^T x$$
$$\nu^n = (\nu^x)^T x$$

- HJB for $U(n; Y)$

$$0 = \max_{c,x} g(c, U) + \frac{\mathbb{E}[dU(n; Y)|_{dJ=0}]}{dt} + \sum_j \lambda_j (U(n + n\nu_j^n, Y + \nu_{Y,j}) - U(n; Y))$$

The discounting is inside payoff function $g(c, U)$

- Guess for value function (same as in CRRA case)

$$U(Y,n) = \frac{1}{\rho}\frac{(\zeta(Y)n)^{1-\gamma}-1}{1-\gamma}$$

- Given that ζ is a function of Y, we have

$$d\zeta = \zeta(\mu^{\zeta}dt + \sigma^{\zeta}dZ_t + \nu^{\zeta}\cdot dJ_t)$$

and we can find its drift, volatility and sensitivity to jumps as function of Y using Ito's formula.

Jumps and EZ preferences: HJB

- We can express the HJB as

$$0 = \max_c \left\{ \frac{\rho}{1-\psi}\left(\frac{c/n}{\zeta}\right)^{1-\psi} - \frac{c}{n} \right\} \dots$$

$$\dots + \max_x \left\{ \mu^n(x) - \frac{\gamma}{2}||\sigma^n(x)||^2 - (\gamma-1)\sigma^n(x)\cdot\sigma^\zeta + \sum_{j=1}^{n_j}\frac{\lambda_j}{1-\gamma}\left(\frac{\tilde{\zeta}^j\tilde{n}^j}{\zeta n}\right)^{1-\gamma} \right\} \dots$$

$$\dots + \mu^\zeta - \frac{\gamma}{2}||\sigma^\zeta||^2 - \frac{\rho}{1-\psi} - \sum_{j=1}^{n_j}\frac{\lambda_j}{1-\gamma}$$

where $\tilde{z}^j$ represents the value of the variable after the realization of Poisson shock j, i.e.,

$$\tilde{z}^j = z(1+\nu_j^z(z))$$

- The second line is our extended portfolio problem.
- We need to verify our guess, i.e., use it and show that the HJB holds for some $\zeta(Y)$ process.

Jumps and EZ preferences: consumption decision

Consumption decision

$$c = \rho^{\frac{1}{\psi}} \zeta^{1-\frac{1}{\psi}} n$$

- Reaction of c/n to investment opportunities ζ depends on IES ψ^{-1}
  - $\psi^{-1} < 1$ better investment opportunities $\Rightarrow$ consumption $\uparrow$, savings $\downarrow$
  - $\psi^{-1} = 1$ consumption-wealth ratio independent of inv. opportunities
  - $\psi^{-1} > 1$ better investment opportunities $\Rightarrow$ consumption $\downarrow$, savings $\uparrow$
- Two effects of better investment opportunities:
  - Income effect: makes the agent effectively richer $\Rightarrow$ consume more
  - Substitution effect: makes saving more attractive $\Rightarrow$ consume less

Now, we can potentially have a strong substitution effect $\psi^{-1} > 1$ and high investor's risk aversion $\gamma > 1$

Portfolio decision

$$\varphi^x = \sigma^x \pi + (-\nu^x)\alpha$$

where $\pi \in \mathbb{R}^{n_z}$, $\alpha \in \mathbb{R}^{n_j}$, and

$$\pi = -\sigma_{SDF} = \gamma \sigma^n + (\gamma - 1)\sigma^\zeta$$

$$\alpha_j = \lambda_j \frac{SDF_{jump}}{SDF_{no\ jump}} = \lambda_j \left(\frac{\tilde{n}^j}{n}\right)^{-\gamma} \left(\frac{\tilde{\zeta}^j}{\zeta}\right)^{1-\gamma} = \lambda_j (1+\nu_j^n)^{-\gamma}(1+\nu_j^\zeta)^{1-\gamma}$$

- LHS is the **market excess return absent shocks**, the RHS is the **required compensation** for the agent to hold the asset.
- Recall $SDF = Discount * \frac{\partial g(c,U)}{\partial c} = Discount * \frac{\partial U}{\partial n} = Discount * \zeta^{1-\gamma} n^{-\gamma}$
- Jump risk:
  - Quantity of risk $(-\nu^x)$ = losses when the jump is realized.
  - Price of risk α = prob. of shock $\lambda \times$ jump in the SDF
    - The SDF increases if the jump generates wealth losses $(-\nu_j^n)$. Greater wealth losses, greater risk price α.
    - The effect of ζ over the SDF depends on γ. Why?

Jumps and EZ preferences: portfolio decision

- Two opposing forces:
  - Prefer high returns when investment opportunities are good: greater **"bang for their buck"**
  - Prefer high returns when investment opportunities are bad: provides **insurance or hedging**.
- The second force becomes stronger with higher risk aversion.
  - If $\gamma = 1$, both forces cancel out and the jump of investment opportunities are irrelevant for risk price.
  - If $\gamma > 1$, insurance motive prevails and agents requires larger compensation when investment opportunities are bad after the jump. Inv. opp. are not offering insurance after the loss (due to the inv. in the asset).
  - If $\gamma < 1$, "bang for their buck" motive prevails and agents requires larger compensation when investment opportunities are good after the jump. Inv. opp. are great after the loss (due to the inv. in the asset) so the loss is more costly.
- When $\gamma \to 0$, agent only cares on getting the greatest "bang for their buck". When $\gamma \to \infty$, agent only cares about insurance / hedging.

Optimal portfolio

- Even conditional on investment opportunities the FOC is non-linear in x

Verification

- TBC [replace and verify the conjecture – HJB does not depend on wealth level – should be straightforward]

Jumps: versions of risk price

- Risk premium:

$$\mathbb{E}_t[dr_t^x / dt] - r_t = \varphi^x + \nu^x \lambda$$

- Re-write asset pricing condition

$$\varphi^x + \nu^x \lambda = \sigma^x \pi + (-\nu^x)(\alpha - \lambda)$$

  - Price of risk α:
    price of potential loss = expected loss + uncertainty around realization.
  - Price of risk $\hat{\alpha} \equiv \alpha - \lambda$:
    price of uncertainty around realization.
    - On average, the loss is $-\lambda \nu^x$ percent but there is uncertainty.
      - If $dJ_t = 0$, then there is no loss.
      - If $dJ_t = 1$, then the loss is $(-\nu^x)$.
      - The probability of $dJ_t = 1$ over a period dt is λdt.
    - This price in terms of the SDF

$$\alpha - \lambda = \lambda \left(\frac{SDF_{jump}}{SDF_{no\ jump}} - 1 \right)$$

No variation in SDF $\iff$ zero price for the uncertainty.

Jumps: Martingale Approach

- Let the SDF evolve as

$$\frac{d\xi_t}{\xi_t} = \mu_t^{\xi} dt + \sigma_t^{\xi} dZ_t + \nu_t^{\xi} dJ_t$$

  and consider the value process v_t associated with asset x_t

$$\frac{dv_t}{v_t} = dr_t^x = \mu_t^x dt + \sigma_t^x dZ_t + \nu_t^x dJ_t$$

  Denote the arrival rate of the Poisson shock as λ.

- Then, the asset pricing condition is

$$\mathbb{E}[d(v_t \xi_t)] = 0$$

- Using Ito's formula

$$\mu_t^x + \mu_t^{\xi} + \sigma_t^x \sigma_t^{\xi} + \lambda[(1 + \nu_t^x)(1 + \nu_t^{\xi}) - 1] = 0$$

Jumps: Martingale Approach

- Asset pricing condition

$$\mu_t^x + \mu_t^\xi + \sigma_t^x \sigma_t^\xi + \lambda[(1+\nu_t^x)(1+\nu_t^\xi) - 1] = 0$$

- Consider the following assets
  - Risk free rate

$$dr_t^x = r_t^f dt$$
$$0 = r_t^f + \mu_t^\xi + \lambda \nu_t^\xi$$

  - Brownian-risk derivative

$$dr_t^x = (\pi_t + r_t^f)dt + dZ_t$$
$$0 = \pi_t + r_t^f + \mu_t^\xi + \sigma_t^\xi + \lambda \nu_t^\xi$$

  - Poisson-risk derivative

$$dr_t^x = (-\alpha_t + r_t^f)dt + dJ_t$$
$$0 = -\alpha_t + r_t^f + \mu_t^\xi + \lambda(1 + 2\nu_t^\xi)$$

Jumps: Martingale Approach

- Solving for the dynamics of the SDF in terms of the risk-free rate and risk prices

$$\begin{aligned} \mu_t^{\xi} &= -r_t^f - \alpha_t + \lambda \\ \sigma_t^{\xi} &= -\pi_t \\ \nu_t^{\xi} &= -\left(1 - \frac{\alpha_t}{\lambda}\right) \end{aligned}$$

which we can write as

$$\frac{d\xi_t}{\xi_t} = -r_t^f dt - \pi_t dZ_t - \left(\frac{\lambda - \alpha_t}{\lambda}\right)(dJ_t - \lambda dt)$$

- Asset pricing condition becomes

$$\boldsymbol{\mu_t^x = r_t^f + \pi_t \sigma^x + \alpha_t(-\nu_t^x)}$$

Jumps: Martingale Approach

Asset pricing condition

$$\mu_t^x = r_t^f + \pi_t \sigma^x + \alpha_t(-\nu_t^x)$$

Some transformations of the condition

- Consider the price of only the uncertainty around the jump $\hat{\alpha} = \alpha - \lambda$, then

$$\mu_t^x + \lambda \nu_t^x = r_t^f + \pi_t \sigma^x + \hat{\alpha}_t(-\nu_t^x)$$

- Consider just the negative of the coefficient of the SDF $\beta_t \equiv -\nu_t^\xi$. Note that $\nu_t^\xi = -\hat{\alpha}/\lambda$, so

$$\mu_t^x + \lambda \nu_t^x = r_t^f + \pi_t \sigma^x + \lambda \beta_t \nu_t^x$$

Can we ensure the sign of α? Yes! $\lambda SDF_{jump}/SDF_{no\ jump} > 0$
What about the sign of $\hat{\alpha}$ or β? Not really, depends on SDF_{jump} vs. $SDF_{no\ jump}$

Takeaway

- Let ξ_t be the stochastic discount factor SDF of an agent. We can write its dynamics as

$$\frac{d\xi_t}{\xi_t} = -r_t^f dt - \pi_t dZ_t - \beta_t(dJ_t - \lambda dt)$$

where J_t is a Poisson process with arrival rate λ. Define α_t from

$$\beta_t = \frac{\lambda - \alpha_t}{\lambda}$$

- Consider an asset x with total return

$$dr_t^x = \mu_t^x dt + \sigma_t^x dZ_t + \nu_t^x dJ_t$$

- Then, the asset pricing condition for the asset is

$$\mu_t^x = r_t^f + \pi_t \sigma^x + \alpha_t(-\nu_t^x)$$

or equivalently

$$\mu_t^x + \lambda \nu_t^x = r_t^f + \pi_t \sigma^x + \lambda \beta_t \nu_t^x$$

Jumps in General Equilibrium: a simple example

Simplified Brunnermeier Sannikov (2016) + jumps

- **Agents:** experts and households, a continuum of each, $i \in \mathbb{I}$ for experts and $j \in \mathbb{J}$ for households
- **Preferences:** for $z \in \{e, h\}$

$$\mathbb{E}_0 \left[\int_0^\infty e^{-\rho^z t} \log(c_t) dt \right]$$

  Experts are more impatient than HHs: $\rho^e \geq \rho^h$
- **Technology**
  - Linear production technology in capital: $y_t = a^z k_t$
  - Experts more productive than HHs: $a^e > a^h$
  - Investment with adjustment costs:
    - Invest $\iota_t k_t$ units of final good $\Rightarrow$ get $\Phi(\iota_t) k_t$ units of capital.
    - Φ is a concave function ($\Phi' > 0, \Phi'' < 0$).

Jumps in General Equilibrium: a simple example

- **Technology**
  - Capital holdings are subject to two types of “capital quality" shocks: Brownian and Poisson. They Poisson shock destroys a fraction of capital $\nu^k \in (0,1)$. Individual capital holdings evolve as

$$dk_{i,t} = k_{i,t}(\Phi(\iota_{i,t}) - \delta)dt + k_{i,t}\sigma dZ_t - k_{i,t}\nu^k dJ_t + d\Delta_{i,t}$$

    where $d\Delta_{i,t}$ correspond to net capital purchases.
- **Assets**
  - The only physical asset is capital. Freely traded at price q_t.
  - **Financial assets**:
    - Risk-free debt traded at interest rate r_t.
    - No equity issuance
- **Information**: all uncertainty is encoded in Z_t.
- **Initial condition**: Initial allocation of capital $\{k_{i,0}, k_{j,0}\}$, which we can add up to get K_0.

Expert problem

$$\max_{c_t, k_t, \iota_t} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho^e t} u(c_t) dt \right]$$

s.t

$$dn_t = -c_t dt + (n_t - q_t k_t) r_t dt + q_t k_t dr_t^{k,e}$$
$$n_t \geq 0$$

where

$$dr_t^{k,e} \equiv \mu_t^{r,e} dt + \sigma_t^r dZ_t - \nu_t^r dJ_t \equiv \frac{d(q_t k_t)}{q_t k_t} + \frac{a^e - \iota_t}{q_t} dt$$

Note 1: the law of motion dk_t in the return definition does not consider net purchases. They do not generate returns instantaneously.

Model with Endogenous Risk Dynamics: Agents' problem

Household problem

$$\max_{c_t, k_t, \iota_t} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho^e t} u(c_t) dt \right]$$

s.t

$$dn_t = -c_t dt + (n_t - q_t k_t) r_t dt + q_t k_t dr_t^{k,h}$$
$$k_t \geq 0$$
$$n_t \geq 0$$

where

$$dr_t^{k,e} \equiv \mu_t^{r,e} dt + \sigma_t^r dZ_t - \nu_t^r dJ_t \equiv \frac{d(q_t k_t)}{q_t k_t} + \frac{a^h - \iota_t}{q_t} dt$$

Market clearing

- Goods

$$\int_{i\in\mathbb{I}}(a^e-\iota_{i,t})k_{i,t}di+\int_{j\in\mathbb{J}}(a^h-\iota_{j,t})k_{j,t}dj=\int_{i\in\mathbb{I}}c_{i,t}di+\int_{j\in\mathbb{J}}c_{j,t}dj$$

- Capital

$$\int_{i\in\mathbb{I}}k_{i,t}di+\int_{i\in\mathbb{J}}k_{j,t}dj=K_t$$

where aggregate capital evolves as

$$dK_t=\left(\int_{i\in\mathbb{I}}(\Phi(\iota_{i,t})-\delta)k_{i,t}di+\int_{j\in\mathbb{J}}(\Phi(\iota_{j,t})-\delta)k_{j,t}dj\right)dt+\sigma K_t dZ_t-\nu^k dJ_t$$

with initial condition K_0.

- Bonds: Walras Law.

Equilibrium definition

Given initial capital endowments $\{k_{i,0}, k_{j,0} : i \in \mathbb{I}, j \in \mathbb{J}\}$, an equilibrium consists of stochastic processes — adapted to the filtered probability space generated by $\{Z_t, J_t : t \geq 0\}$ — for

- prices: capital price q_t, interest rate r_t, outside equity price π_t
- allocations: capital $\{k_{i,t}, k_{j,t}\}$, consumption $\{c_{i,t}, c_{j,t}\}$, investment $\{\iota_{i,t}, \iota_{j,t}\}$, for agents $i \in \mathbb{I}$ and $j \in \mathbb{J}$
- net worth $\{n_{i,t}, n_{j,t}\}$ for agents $i \in \mathbb{I}$ and $j \in \mathbb{J}$

such that

- initial net worths satisfy $n_{i,0} = q_0 k_{i,0}$ and $n_{j,0} = q_0 k_{j,0}$ for $i \in \mathbb{I}$ and $j \in \mathbb{J}$
- taking prices as given, each expert i and household j solves his problem
- markets clear at all dates

Jumps in General Equilibrium: equilibrium characterization

- Conjecture a process for capital price

$$dq_t = q_t(\mu_t^q dt + \sigma_t^q dZ_t - \nu^q dJ_t)$$

- Then, the return of capital (using Ito's formula) for experts

$$dr_t^{k,e}(\iota) = \left(\frac{a^e - \iota}{q_t} + \Phi(\iota) - \delta + \mu_t^q + \sigma\sigma_t^q\right) dt + (\sigma + \sigma_t^q) dZ_t$$
$$\ldots - [1 - (1 - \nu_t^q)(1 - \nu^k)] dJ_t$$

- The return can differ across experts if they chose different investment rates ι, however, in equilibrium they all chose the same.
- Analogous expression for $dr_t^{k,h}(\iota)$ but with productivity a^h.

Jumps in General Equilibrium: equilibrium characterization

Experts

- We can re-write the problem using

$$\mu_t^n = r_t + x_t^k \left(\frac{a^e - \iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q - r_t \right)$$
$$\sigma_t^n = x_t^k (\sigma + \sigma_t^q)$$
$$\nu_t^n = x_t^k [1 - (1 - \nu_t^q)(1 - \nu^k)]$$

where $x_t^k \equiv q_t k_t / n_t$ is the portfolio share invested in capital. ν^n represent losses, i.e.,

$$\frac{dn}{n} = \mu^n dt + \sigma^n dZ_t - \nu^n dJ_t$$

Households

- Face the exact same problem but with their own productivity a^h. Also, the no short-sale constraint for capital might be binding (omitted in experts' problem):

$$x_t^k \geq 0$$

Jumps in General Equilibrium: equilibrium characterization

Experts

- Optimal consumption and investment

$$c^e = \rho n^e, \qquad q^{-1} = \Phi'(\iota)$$

Investment defines function $\iota(q)$. Define

$$\mu^{r,e} := \frac{a^e - \iota(q)}{q} + \Phi(\iota(q)) - \delta + \mu^q + \sigma\sigma^q$$

- Portfolio problem

$$\max_x \mu^n - \frac{1}{2}(\sigma^n)^2 + \lambda \log\left(\frac{\tilde{n}}{n}\right)$$

- Optimal portfolio

$$[x^k]: \quad \mu^{r,e} - r = \sigma^n(\sigma + \sigma^q) + \lambda(1 - \nu^n)^{-1}[1 - (1 - \nu^q_t)(1 - \nu^k)]$$

In our risk price notation $\alpha = \lambda SDF_{jump} / SDF_{no\ jump} = \lambda(1 - \nu^n)^{-1}$

Households

- Optimal consumption $c^h = \rho n^h$ and investment $q^{-1} = \Phi'(\iota)$. The latter defines function $\iota(q)$. Define

$$\mu^{r,h} := \frac{a^h - \iota(q)}{q} + \Phi(\iota(q)) - \delta + \mu^q + \sigma\sigma^q$$

- Portfolio problem

$$\max_x \mu^n - \frac{1}{2}(\sigma^n)^2 + \lambda \log\left(\frac{\tilde{n}}{n}\right) \text{ s.t. } x \geq 0$$

- Optimal portfolio

$$\mu^{r,h} - r \leq \sigma^n(\sigma + \sigma^q) + \lambda(1 - \nu^n)^{-1}[1 - (1 - \nu_t^q)(1 - \nu^k)]$$

or more compactly

$$0 = \min\left\{\sigma^n(\sigma + \sigma^q) + \lambda(1 - \nu^n)^{-1}[1 - (1 - \nu_t^q)(1 - \nu^k)] - (\mu^{r,h} - r), 0\right\}$$

In our risk price notation: $\alpha = \lambda SDF_{jump}/SDF_{no\ jump} = \lambda(1 - \nu^n)^{-1}$

Market clearing

- Goods

$$\bar{\rho}(\eta)q + \iota(q) = a^e\kappa + a^h(1-\kappa)$$

where $\bar{\rho}(\eta) \equiv \eta\rho^e + (1-\eta)\rho^h$ and $\kappa \equiv \eta x^e$, and capital market clearing imply $1-\kappa = (1-\eta)x^h$.

- Bonds : Walras law

Jumps in General Equilibrium: equilibrium characterization

Recursive solution

- Law of motion for this model

$$d\eta = \eta\mu^{\eta} dt + \eta\sigma^{\eta} dZ_t - \nu^{\eta} dJ_t$$

where

$$\eta\mu^{\eta} = \eta(1-\eta)\left(\mu^{n,e} - \mu^{n,h} + \frac{c_h}{n_h} - \frac{c_e}{n_e}\right) - \eta\sigma_{\eta}(\eta\sigma^{n,e} + (1-\eta)\sigma^{n,h})$$

$$\eta\sigma^{\eta} = \eta(1-\eta)(\sigma^{n,e} - \sigma^{n,e})$$

$$\eta\nu^{\eta} = \eta\left[1 - \frac{1-\nu^{n.e}}{1-\eta\nu^{n.e} - (1-\eta)\nu^{n.h}}\right]$$

- Consistency conditions.

$$\mu^{q} q = q_{\eta}(\eta\mu_{\eta}) + \frac{1}{2} q_{\eta\eta}(\eta\sigma^{\eta})^2$$

$$\sigma^{q} q = q_{\eta}(\eta\sigma_{\eta})$$

$$(1-\nu^{q})q = q_{(\eta(1-\nu^{\eta}))}$$

The last term is not a multiplication, it is the function $q(\cdot)$ evaluated at $\eta(1-\nu^{\eta})$. This will a challenge when looking for the numerical solution.

Recursive equilibrium

Given initial conditon η_0, a Recursive Equilibrium is a set of functions of η for allocations $\{\frac{c_e}{n_e}, \frac{c_h}{n_h}, x^{k,e}, x^{k,h}\}$, prices $\{r, \pi, \alpha, q, \sigma^q, \mu^q, \nu^q\}$, and dynamics for aggregate state $\{\sigma^\eta, \mu^\eta, \nu^\eta\}$ such that

- Agents optimize (FOCs for $x^{k,e}, x^{k,h}, c_e/n_e, c_h/n_h$)
- Markets clear (equations for goods and capital)
- Aggregate state η evolves consistently with individual states
- Consistency conditions for dynamics are satisfied

- The following system defines a functional equation for $q(\eta)$.

$$0 = (a^e - a^h)\kappa + a^h - \iota(q) - q\bar{\rho}(\eta)$$

$$0 = \min\left\{\frac{a^e - a^h}{q} - \left[\frac{\kappa - \eta}{\eta(1-\eta)}\right](\sigma + \sigma_q)^2 - (\Delta\alpha_{(\kappa,\nu^q)})\nu^{qk}_{(\kappa,\nu^q)}, 1 - \kappa\right\}$$

$$0 = \sigma + \sigma^q - \frac{\sigma}{1 - \left(\frac{\kappa}{\eta} - 1\right)\frac{\eta q_\eta}{q}}$$

$$0 = (1 - \nu^q)q - q_{(\eta(1-\nu^\eta))}$$

where

$$\Delta\alpha_{(\kappa,\nu^q)} = \alpha^e - \alpha^h = \lambda\left[(1 - \nu^{n,e}_{(\kappa,\nu^q)})^{-1} - [(1 - \nu^{n,h}_{(\kappa,\nu^q)})^{-1}\right]$$

$$\nu^\eta_{(\kappa,\nu^q)} = 1 - \frac{1 - \nu^{n,e}_{(\kappa,\nu^q)}}{1 - v^{qk}_{(\kappa,\nu^q)}}$$

$$\nu^{n,e}_{(\kappa,\nu^q)} = \frac{\kappa}{\eta}\nu^{qk}_{(\kappa,\nu^q)}, \qquad \nu^{n,h}_{(\kappa,\nu^q)} = \left(\frac{1-\kappa}{1-\eta}\right)\nu^{qk}_{(\kappa,\nu^q)}$$

$$\nu^{qk}_{(\kappa,\nu^q)} = 1 - (1 - \nu^q)(1 - \nu^k)$$

- The variables of the system are $\{\kappa, \sigma^q, \nu^q, q\}$.

Jumps in General Equilibrium: equilibrium characterization

- Equilibrium system

$$0 = (a^e - a^h)\kappa + a^h - \iota(q) - q\bar{\rho}(\eta)$$

$$0 = \min\left\{\frac{a^e - a^h}{q} - \left[\frac{\kappa - \eta}{\eta(1-\eta)}\right](\sigma + \sigma_q)^2 - (\Delta\alpha_{(\kappa,\nu^q)})\nu^{qk}_{(\kappa,\nu^q)}, 1 - \kappa\right\}$$

$$0 = \sigma + \sigma^q - \frac{\sigma}{1 - \left(\frac{\kappa}{\eta} - 1\right)\frac{\eta q_\eta}{q}}$$

$$0 = (1 - \nu^q)q - q_{(\eta(1-\nu^\eta))}$$

The variables of the system are $\{\kappa, \sigma^q, \nu^q, q\}$.

- The system does NOT reduces to a differential equation, NOT EVEN to a delayed differential equation. The size of the jump in q and η is endogenous.
- To the best of my knowledge, there is no general numerical strategy to address this system.
- There are applications in the literature that develop numerical procedures particular to their models. The closest to the model presented is a working paper by Li(2021). Click here.

Jumps in General Equilibrium: Modeling Financial Panics

Go to slides of Mendo (2020), "Risky low-volatility environments and the stability paradox."

Macro, Money, and Finance: a continuous-time approach

- Introduction to Monetary Models

Fernando Mendo

2023

PUC Rio

Introduction to Monetary Models

Intro to Monetary Models

Go to slides by Markus Brunnermeier

Macro, Money, and Finance: a continuous-time approach

- One Sector Monetary Model with Heterogeneous Agents

Fernando Mendo

2023

PUC Rio

One Sector Monetary Model with Heterogeneous Agents

Single-agent consumption-portfolio choice: idiosyncratic risk

- Problem

$$\max_{\{c_t, x_t\}} \mathbb{E}\left[\int_0^\infty e^{-\rho t} u(c_t) dt\right]$$

s.t

$$\begin{aligned} dn_t &= -c_t dt + n_t[(1 - x \cdot \mathbb{1}) r_t + x \cdot dr_t^x] \\ dr_t^x &= (r_t \mathbb{1} + \varphi_t^x) dt + \sigma_t^x dZ_t + \tilde{\sigma}^x d\tilde{Z}_t^i \\ n_t &\geq 0 \end{aligned}$$

where $x \in \mathbb{R}^{n_x}$, $Z_t \in \mathbb{R}^{n_z}$, and $\sigma_t^x \in \mathbb{R}^{n_x \times n_z}$, and initial condition n_0 is taken as given. We assume that Brownian motions Z_t are independent. It is an straightforward extension to deal with the correlated case.

- The solvency constraint $n_t \geq 0$ can be imposed or derived from a "no Ponzi condition."
- Reasonably general set-up for portfolio choice.

Single-agent consumption-portfolio choice: idiosyncratic risk

- Individual state: wealth n_t
- Aggregate states:
  - Let $Y_t \in \mathbb{R}^{n_y}$ drive all time-varying objects beyond the control of the agent: prices $\{r(Y_t), \varphi^x(Y_t)\}$ and volatility of risky return $\sigma^x(Y_t)$.
  - Assume that Y_t is a (time- homogeneous) Ito diffusion

$$dY_t = \mu_Y(Y_t)dt + \sigma_Y(Y_t)dZ_t$$

Aggregate states (and therefore investment opportunities later) will not be affected by idiosyncratic risk. Why?

Single-agent consumption-portfolio choice: idiosyncratic risk

- Budget constraint notation (dropping time subscripts, fully recursive)

$$dn = (n\mu^n - c)dt + (n\sigma^n)dZ_t + n\tilde{\sigma}^n d\tilde{Z}_t^i$$

where

$$\mu^n = r_t + \varphi^x \cdot x$$
$$\sigma^n = (\sigma^x)^T x$$
$$\tilde{\sigma}^n = (\tilde{\sigma}^x)x$$

- Same steps as before and we can write the HJB as

$$\frac{\rho}{1-\gamma} = \max_c \left\{ \frac{\rho}{1-\gamma} \left(\frac{c/n}{\zeta} \right)^{1-\gamma} - \frac{c}{n} \right\} ...$$
$$... + \max_x \left\{ \mu^n(x) - \frac{\gamma}{2}\sigma^n(x)^2 - (\gamma-1)\sigma^n(x)\sigma^\zeta - \frac{\gamma}{2}(\tilde{\sigma}^n(x))^2 \right\} ...$$
$$... + \mu^\zeta - \frac{\gamma}{2}(\sigma^\zeta)^2$$

- Portfolio optimization

$$\varphi^x = \sigma^x \pi + \gamma \tilde{\sigma}^n \tilde{\sigma}^x$$

- Price of risk: $\gamma \tilde{\sigma}^n$
- Quantity of risk: $\tilde{\sigma}^x$

One Sector Monetary Model: Environment

Brunnermeier Merkel Sannikov (2022)

- **Agents:** continuum of agents $i \in \mathbb{I}$
- **Preferences:**

$$\mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \left(u(c_t) + f(\mathcal{G}_t K_t) \right) dt \right]$$

  where $u(c) = (c^{1-\gamma} - 1)/(1-\gamma)$.
- **Technology**
  - Linear production technology in capital: $y_t = a k_t$
  - Investment with adjustment costs:
    - Invest $\iota_t k_t$ units of final good $\Rightarrow$ get $\Phi(\iota_t) k_t$ units of capital.
    - Φ is a concave function ($\Phi' > 0, \Phi'' < 0$).
  - Capital holdings subject to idiosyncratic "quality" shocks. Individual capital holdings evolve as

$$dk_{i,t} = k_{i,t}(\Phi(\iota_{i,t}) - \delta) dt + \tilde{\sigma} d\tilde{Z}_t^i + d\Delta_{i,t}$$

    where $d\Delta_{i,t}$ correspond to net capital purchases.

One Sector Monetary Model: Environment

- **Assets**
  - Capital: traded at price q_t^k.
  - Money or Government bonds: each dollar is traded at price $q_t^{\mathcal{B}} K_t / \mathcal{B}_t =: 1/\mathcal{P}_t$
  - Financial friction: no asset to hedge idiosyncratic risk
  - Recall the numeraire is the consumption good.
- **Government**:
  - Government spending $\mathcal{G}_t K_t$
  - Proportional output tax τ_t (total is $\tau_t a K_t$)
  - Nominal value of total government debt supply $d\mathcal{B}_t = \mu_t^{\mathcal{B}} \mathcal{B}_t dt$
  - Floating nominal interest rate i_t on outstanding bonds
  - Budget constraint

$$(\underbrace{\mu_t^{\mathcal{B}} - i_t}_{=:\check{\mu}_t^{\mathcal{B}}})\mathcal{B}_t + \mathcal{P}_t K_t (\underbrace{\tau_t a - \mathcal{G}_t}_{=:s_t}) = 0$$

  - Policy instruments: $\mu_t^{\mathcal{B}}, i_t, \mathcal{G}_t, \tau_t$. Government can choose 3 and the other must adjust to satisfy the budget constraint. We assume that τ is endogenous.

One Sector Monetary Model: Environment

- **Information**: no aggregate uncertainty
- **Initial condition**: Initial endowments of capital and nominal debt $\{k_{i,0}, \mathcal{B}_{i,0}\}$ for $i \in \mathbb{I}$

One Sector Monetary Model: Environment: Agents' problem

$$\max_{c_t, k_t, \iota_t} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \left(u(c_t) + f(\mathcal{G}_t K_t) \right) dt \right]$$

s.t

$$dn_t = -c_t dt + (n_t - q_t k_t) dr_t^{\mathcal{B}} + q_t k_t dr_t^k$$
$$n_t \geq 0$$

where

$$dr_t^{k,i} \equiv \mu_t^r dt + \sigma_t^r d\tilde{Z}_t^i \equiv \frac{d(q_t k_{i,t})}{q_t k_{i,t}} + \frac{(1-\tau)a - \iota_t}{q_t} dt$$

$$dr_t^{\mathcal{B}} \equiv \mu_t^{r,\mathcal{B}} dt = i_t dt + \underbrace{\frac{d(q_t^{\mathcal{B}} K_t / \mathcal{B}_t)}{q_t^{\mathcal{B}} K_t / \mathcal{B}_t}}_{\text{-inflation}} = -\check{\mu}_t^{\mathcal{B}} dt + \frac{d(q_t^{\mathcal{B}} K_t)}{q_t^{\mathcal{B}} K_t}$$

One Sector Monetary Model: Market clearing

Market clearing

- Goods

$$\int_{i \in \mathbb{I}} (a^e - \iota_{i,t}) k_{i,t} di = \int_{i \in \mathbb{I}} c_{i,t} di + \mathcal{G} K_t$$

- Capital

$$\int_{i \in \mathbb{I}} k_{i,t} di = K_t$$

where aggregate capital evolves as

$$dK_t = \left(\int_{i \in \mathbb{I}} (\Phi(\iota_{i,t}) - \delta) k_{i,t} di \right) dt$$

with initial condition K_0.

- Bonds or Money: Walras Law.

One Sector Monetary Model: Equilibrium

Given initial capital endowments $\{k_{i,0}, \mathcal{B}_{i,0} : i \in \mathbb{I}\}$, and policy choices $\{i_t, \mu_t^{\mathcal{B}}, \mathcal{G}_t\}$, an equilibrium consists of stochastic processes — agent i variables will be adapted to the filtered probability space generated by $\{\tilde{Z}_t^i, : t \geq 0\}$ – for

- prices: capital price q_t^K, bond value per unit of capital $q_t^{\mathcal{B}}$
- allocations: capital $\{k_{i,t}\}$, consumption $\{c_{i,t}\}$, investment $\{\iota_{i,t}\}$, for agents $i \in \mathbb{I}$
- net worth $\{n_{i,t}\}$ for agents $i \in \mathbb{I}$
- tax rate τ_t

such that

- initial net worths satisfy $n_{i,0} = q_0^K k_{i,0} + \mathcal{B}_{i,0} \frac{q_0^{\mathcal{B}} K_0}{\mathcal{B}_0}$ for $i \in \mathbb{I}$
- taking prices as given, each agent i solves his problem
- markets clear at all dates
- the government budget constraint is satisfied at all dates

One Sector Monetary Model: Equilibrium characterization

- Conjecture a process for prices

$$
\begin{aligned}
dq_t^K &= q_t^K \mu_t^{q,K} dt \\
dq_t^{\mathcal{B}} &= q_t^{\mathcal{B}} \mu_t^{q,\mathcal{B}} dt
\end{aligned}
$$

- Capital return

$$
\begin{aligned}
dr_t^k(\iota) &= \left(\frac{(1-\tau_t)a - \iota}{q_t^K} + \Phi(\iota) - \delta + \mu_t^{q,K} \right) dt + \tilde{\sigma} d\tilde{Z}_t^i \\
&= \left(\frac{a - \mathcal{G}_t - \iota}{q_t^K} + \frac{q_t^{\mathcal{B}}}{q_t^K} \check{\mu}_t^{\mathcal{B}} + \Phi(\iota) - \delta + \mu_t^{q,K} \right) dt + \tilde{\sigma} d\tilde{Z}_t^i
\end{aligned}
$$

where the second expression replaces τ_t using the government budget constraint

$$
(\mu_t^{\mathcal{B}} - i_t) q_t^{\mathcal{B}} + (\tau_t a - \mathcal{G}_t) = 0
$$

- Bond return

$$
\begin{aligned}
dr_t^{\mathcal{B}} = i_t dt + \frac{d(q_t^{\mathcal{B}} K_t / \mathcal{B}_t)}{q_t^{\mathcal{B}} K_t / \mathcal{B}_t} &= -\check{\mu}_t^{\mathcal{B}} dt + \frac{d(q_t^{\mathcal{B}} K_t)}{q_t^{\mathcal{B}} K_t} \\
&= -\hat{\mu}_t^{\mathcal{B}} + \mu_t^{q,\mathcal{B}} + \Phi(\iota) - \delta
\end{aligned}
$$

The investment rate that appears in this return corresponds to the evolution of aggregate capital, i.e., it is not a control for the agent.

One Sector Monetary Model: Equilibrium characterization

- Wealth evolution

$$\frac{dn}{n} = \mu^n dt + \tilde{\sigma}^n d\tilde{Z}_t^i$$

where

$$\begin{aligned}
\mu_t^n &= x_t^k \left(\frac{a - \mathcal{G}_t - \iota}{q_t^K} + \frac{q_t^{\mathcal{B}}}{q_t^K} \check{\mu}_t^{\mathcal{B}} + \Phi(\iota) - \delta + \mu_t^{q,K} \right) \ldots \\
&\quad \ldots + (1 - x_t^k)(-\hat{\mu}_t^{\mathcal{B}} + \mu_t^{q,\mathcal{B}} + \Phi(\iota) - \delta) \\
\tilde{\sigma}_t^n &= x_t^k \tilde{\sigma}
\end{aligned}$$

and $x_t^k \equiv q_t^K k_t / n_t$ is the portfolio share invested in capital.

One Sector Monetary Model: equilibrium characterization

Solution

- Guess for value function

$$V(n, Y) = \frac{1}{\rho} \frac{(\zeta(Y)n)^{1-\gamma} - 1}{1-\gamma}$$

where Y is a placeholder for aggregate states.

- Conjecture

$$d\zeta = \zeta(\mu^{\zeta} dt + \sigma^{\zeta} dZ_t)$$

- We can express the HJB as

$$\frac{\rho}{1-\gamma} = \max_{c} \left\{ \frac{\rho}{1-\gamma} \left(\frac{c/n}{\zeta} \right)^{1-\gamma} - \frac{c}{n} \right\} \ldots$$

$$\ldots + \max_{x^k, \iota} \left\{ \mu^n(x) - \frac{\gamma}{2}\sigma^n(x)^2 - (\gamma - 1)\sigma^n(x)\sigma^{\zeta} - \frac{\gamma}{2}\tilde{\sigma}^n(x)^2 \right\} \ldots$$

$$\ldots + \mu^{\zeta} - \frac{\gamma}{2}(\sigma^{\zeta})^2$$

where the portfolio maximization is subject to the relevant constraints.

Agents optimal conditions

- Optimal consumption

$$\frac{c_t}{n_t} = \rho^{\frac{1}{\gamma}} \zeta_t^{1-\frac{1}{\gamma}}$$

- Optimal investment

$$(q^K)^{-1} = \Phi'(\iota)$$

- Optimal portfolio

$$\frac{a - \mathcal{G}_t - \iota}{q_t^K} + \left(\frac{q_t^{\mathcal{B}} + q_t^K}{q_t^K} \right) \check{\mu}_t^{\mathcal{B}} + \mu_t^{q,K} - \mu_t^{q,\mathcal{B}} = \gamma \tilde{\sigma}^n \tilde{\sigma}$$

- HJB that disciplines investment opportunities ξ_t (includes μ^{ζ} term).

One Sector Monetary Model: equilibrium characterization

Market clearing

- Goods

$$a - \mathcal{G}_t - \iota = (q_t^{\mathcal{B}} + q_t^K)\frac{c_t}{n_t}$$

- Capital

$$x^k = \frac{q_t^K}{q_t^K + q_t^{\mathcal{B}}}$$

Equilibrium

- If we look for a stationary equilibria, we would impose that variables are constant (since the economy has no fundamental states). Then, the system above delivers $q^K, q^{\mathcal{B}}, \zeta, c/n, x^k, \iota$. This requires constant policy choices $\mathcal{G}, \breve{\mu}^{\mathcal{B}}$.
- We will look for a non-stationary equilibria which allows for time-varying policy choices.

Non-stationary equilibrium

- We have to also solve for dynamics of certain variables, i.e., $\mu_t^{q,K}, \mu_t^{q,B}, \mu_t^{\zeta}$.
- Turns out that in this model, equilibrium conditions do discipline the share of value of bonds relative total wealth

$$\vartheta := \frac{q_t^{\mathcal{B}}}{q_t^{\mathcal{B}} + q_t^K}$$

- After some algebra and using Ito's lemma for μ_t^{ϑ}, we have

$$\mu_t^{\vartheta} = \frac{c_t}{n_t} + \check{\mu}_t^{\mathcal{B}} - \gamma(1-\vartheta_t)^2\tilde{\sigma}^2$$

The money valuation equation

Agenda

- Perhaps for next semester I could add a numerical solution for the single agent problem with jumps.
- Next semester or homework:
- Model with endogenous risk: EZ preferences. just verify that nothing changes. At least before the change of variable this should be true.
- I should do the 5 steps approach. Inside or before this 5-steps approach I can do the 2 useful toolboxes (price taking planner and change of numeraire). I can do this here so I do it for EZ or I can do it after the model w/ CRRA. Focus on the link to SDF. Check my old solution manual.
- GE applications of jumps: Maybe a quick discussion of the Li paper. Mendo paper: see summary slides by Markus, use my slides. Revisit multiplicity to use jumps (maybe not so relevant)
- Homework
  - with simulations!!! Two state model?