Numerical Methods for Continuous-Time Models in Economics and Finance

ECO 529: Financial and Monetary Economics

Past, present, future

$$\xrightarrow[\textbf{Distributions}]{\textbf{Forward Equation:}} \quad \textbf{Today's Decision} \quad \xrightarrow[\textbf{Value Functions / Prices}]{\textbf{Backward Equation:}}$$

The equation of interest

$$F(x, f, Df, D^2 f) = 0$$

weakly decreasing

weakly increasing

For mxm matrices X and Y, if $X \leq Y$
(i.e., $Y - X$ is positive semidefinite)
$F(x, f, p, X) \leq F(x, f, p, Y)$

Example: valuation equation

mxd in R^d

State $\mathbf{X} \in \mathbb{R}^m$ follows

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dZ_t$$

Firm pays dividend g(X_t), cost of equity is ρ

Equity value

$$f(X_0) = E_0\left[\int_0^\infty e^{-\rho t} g(X_t)\,dt \mid X_0\right]$$

f solves

$$0 = \overbrace{g(x) + \mu(x)Df(x) + \frac{1}{2}\,\mathrm{tr}\left[\sigma^T(x)D^2 f(x)\sigma(x)\right]}^{\text{dividend plus expected appreciation}} - \rho f(x)$$

increasing decreasing

Example: HJB

mxd in R^d

Given control a_t, state $X_t \in \mathbb{R}^m$ follows

$$dX_t = \mu(X_t, a_t)\, dt + \sigma(X_t, a_t)\, dZ_t$$

Payoff flow $g(X_t, a_t)$, discount rate ρ

Value Function

$$f(X_0) = \max_{\{a\}} E_0 \left[\int_0^\infty e^{-\rho t} g(X_t, a_t)\, dt \,\middle|\, X_0 \right]$$

f solves

$$0 = \max_a g(x, a) + \mu(x, a) Df(x) + \frac{1}{2} \operatorname{tr} \left[\sigma^T(x, a) D^2 f(x) \sigma(x, a) \right] - \rho f(x)$$

increasing decreasing

Forward and backward equations

Functions f(t, x) over time and space, Df is the space derivative

Forward	$f_t = F(x, f, Df, D^2 f)$
Backward	$f_t + F(x, f, Df, D^2 f) = 0$

Example: HJB

mxd   in R^d

$$dX_t = \mu(X_t, a_t)\,dt + \sigma(X_t, a_t)\,dZ_t$$

Payoff flow $g(X_t, a_t)$ **over [0, T],** $G(X_T)$ **at time T, discount rate** ρ

$$f(0, X_0) = \max_{\{a\}} E_0 \left[\int_0^T e^{-\rho t} g(X_t, a_t)\,dt + e^{-\rho T} G(X_T) \,\middle|\, X_0 \right]$$

$$0 = f_t + \max_a g(x, a) + \mu(x, a)^T Df + \frac{1}{2}\mathrm{tr}\left[\sigma^T(x, a) D^2 f\, \sigma(x, a)\right] - \rho f(t, x)$$

$$f(T, x) = G(x)$$

increasing   decreasing

boundary condition

Example: KFE

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dZ_t$$

Density $f(0,x)$ **at time 0. After** $f(t,x)$ **satisfies the KFE**

$$f_t(t,x) = -\frac{\partial}{\partial x}\left(\mu(x)f(t,x)\right) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\sigma(x)^2 f(t,x)\right)$$

The equations of interest

For mxm matrices X and Y, if X $\leq$ Y
(i.e. Y - X is positive semidefinite)
$F(x, f, p, X) \leq F(x, f, p, Y)$

$$F(x, f, Df, D^2 f) = 0$$

weakly decreasing   weakly increasing

Forward $\quad f_t = F(x, f, Df, D^2 f)$

Backward $\quad f_t + F(x, f, Df, D^2 f) = 0$

Barles and Souganidis (1991)

A numerical finite difference scheme converges to the viscosity solution as long as it is

- **consistent**
- **stable**
- **monotone**

derivative approximations converge to true derivatives

doesn't explode

new values depend positively on all known values

Example

Income $dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dZ_t$ on $[0, \bar{X}]$

with $\sigma(0) = \sigma(\bar{X}) = 0$, $\mu(0) \geq 0$, $\mu(\bar{X}) \leq 0$

boundary condition

Utility $u(x_t)$**, payoff** $U(x_T)$ **at time T, discount rate** ρ

$$0 = f_t + u(x) + \mu(x) f_x + \frac{1}{2}\sigma(x)^2 f_{xx} - \rho f(x,t), \quad f(x,T) = U(x)$$

Grid (with space Δx**)** $[0 = x_0, x_1, x_2, \ldots, x_N = \bar{X}]$

Time step Δt

Derivative approximations

Left: $Df(t, x_n) = \frac{f(t,x_n)-f(t,x_{n-1})}{\Delta x}$

Right: $Df(t, x_n) = \frac{f(t,x_{n+1})-f(t,x_n)}{\Delta x}$

Both are consistent... so is central.

$$Df(t, x_n) = \frac{f(t,x_{n+1})-f(t,x_{n-1})}{2\Delta x}$$

2nd derivative: $D^2 f(t, x_n) = \frac{f(t,x_{n+1})+f(t,x_{n-1})-2f(t,x_n)}{\Delta x^2}$

Time: $f_t(t, x_n) = \frac{f(t,x_n)-f(t-\Delta t,x_n)}{\Delta t}$

Valuation equation

$$0 = f_t + u(x) + \mu(x) f_x + \frac{1}{2}\sigma(x)^2 f_{xx} - \rho f(x,t), \quad f(x,T) = U(x)$$

For reasons that'll be clear later, let's represent this differential operator with weights ≥ 0 away from x.

f_x: the upwind method

$$M = \begin{bmatrix} - & + & 0 & 0 & 0 & 0 \\ + & - & + & 0 & 0 & 0 \\ 0 & + & - & + & 0 & 0 \\ 0 & 0 & + & - & + & 0 \\ 0 & 0 & 0 & + & - & + \\ 0 & 0 & 0 & 0 & + & - \end{bmatrix}$$

rows add to 0

Direction corresponds to the sign of $\mu(x_n)$

$$\mu(x_n)_- \frac{f(t,x_n)-f(t,x_{n-1})}{\Delta x} + \mu(x_n)_+ \frac{f(t,x_{n+1})-f(t,x_n)}{\Delta x} + \frac{\sigma(x_n)^2}{2} \frac{f(t,x_{n+1})+f(t,x_{n-1})-2f(t,x_n)}{\Delta x^2}$$

In matrix form, Mf

$$\ldots M_{n,n-1},\ M_{nn},\ M_{n,n+1},\cdots = \underbrace{-\frac{\mu(x_n)_-}{\Delta x} + \frac{\sigma(x_n)^2}{2\Delta x^2}}_{\text{weight on } f(t,x_{n-1})},\ \underbrace{\frac{\mu(x_n)_-}{\Delta x} - \frac{\mu(x_n)_-}{\Delta x} - \frac{\sigma(x_n)^2}{2\Delta x^2}}_{\text{weight on } f(t,x_n)},\ \underbrace{\frac{\mu(x_n)_+}{\Delta x} + \frac{\sigma(x_n)^2}{2\Delta x^2}}_{\text{weight on } f(t,x_{n+1})}$$

Matrix M

$$M = \begin{bmatrix} - & + & 0 & 0 & 0 & 0 \\ + & - & + & 0 & 0 & 0 \\ 0 & + & - & + & 0 & 0 \\ 0 & 0 & + & - & + & 0 \\ 0 & 0 & 0 & + & - & + \\ 0 & 0 & 0 & 0 & + & - \end{bmatrix}$$

Approximate

$$\mu(x) Df(x) + \frac{1}{2} \operatorname{tr}\left[\sigma^T(x) D^2 f(x) \sigma(x)\right]$$

as Mf, with coefficients ≥ 0 off the main diagonal, rows add up to 0

In one dimension,

$$\dots M_{n,n-1},\ M_{nn},\ M_{n,n+1}, \dots = -\frac{\mu(x_n)_-}{\Delta x} + \frac{\sigma(x_n)^2}{2\Delta x^2}, \frac{\mu(x_n)_-}{\Delta x} - \frac{\mu(x_n)_+}{\Delta x} - \frac{\sigma(x_n)^2}{2\Delta x^2}, \frac{\mu(x_n)_+}{\Delta x} + \frac{\sigma(x_n)^2}{2\Delta x^2}$$

Explicit scheme: left time derivative

$$0 = \frac{f(t,\cdot) - f(t-\Delta t,\cdot)}{\Delta t} + u + Mf(t,\cdot) - \rho f(t,\cdot)$$

Explicit scheme: left time derivative

$$0 = \frac{f(t,\cdot) - f(t-\Delta t,\cdot)}{\Delta t} + u + Mf(t,\cdot) - \rho f(t,\cdot)$$

Find $f(t-\Delta t, x_n)$ **from** $f(t,\cdot)$

$$f(t-\Delta t,\cdot) = \Delta t\, u(\cdot) + \underbrace{((1-\rho\Delta t)I + \Delta t\, M)}_{\text{all coefficients must be } \geq 0} f(t,\cdot)$$

Monotone if Δt **is small enough,** $O(\Delta x^2)$

$$M = \begin{bmatrix} - & + & 0 & 0 & 0 & 0 \\ + & - & + & 0 & 0 & 0 \\ 0 & + & - & + & 0 & 0 \\ 0 & 0 & + & - & + & 0 \\ 0 & 0 & 0 & + & - & + \\ 0 & 0 & 0 & 0 & + & - \end{bmatrix}$$

Implicit scheme: right time derivative

$$0 = \frac{f(t+\Delta t,\cdot) - f(t,\cdot)}{\Delta t} + u + Mf(t,\cdot) - \rho f(t,\cdot)$$

Hence,

$$\underbrace{(I(1+\rho\Delta t) - \Delta t M)}\, f(t,\cdot) = \Delta t\, u(\cdot) + f(t+\Delta t,\cdot)$$

$$M = \begin{bmatrix} - & + & 0 & 0 & 0 & 0 \\ + & - & + & 0 & 0 & 0 \\ 0 & + & - & + & 0 & 0 \\ 0 & 0 & + & - & + & 0 \\ 0 & 0 & 0 & + & - & + \\ 0 & 0 & 0 & 0 & + & - \end{bmatrix}$$

sums of rows > 0

$$f(t,\cdot) = \underbrace{(I(1+\rho\Delta t) - \Delta t M)^{-1}}_{\textbf{all entries}\ \geq 0} (\Delta t\, u(\cdot) + f(t+\Delta t,\cdot))$$

Matrix M

$$M = \begin{bmatrix} - & + & 0 & 0 & 0 & 0 \\ + & - & + & 0 & 0 & 0 \\ 0 & + & - & + & 0 & 0 \\ 0 & 0 & + & - & + & 0 \\ 0 & 0 & 0 & + & - & + \\ 0 & 0 & 0 & 0 & + & - \end{bmatrix}$$

Approximate

$$\mu(x) Df(x) + \frac{1}{2} \operatorname{tr}\left[\sigma^T(x) D^2 f(x) \sigma(x)\right]$$

as Mf, with coefficients ≥ 0 off the main diagonal, rows add up to 0

In one dimension,

$$\ldots M_{n,n-1},\ M_{nn},\ M_{n,n+1}, \cdots = -\frac{\mu(x_n)_-}{\Delta x} + \frac{\sigma(x_n)^2}{2\Delta x^2}, \frac{\mu(x_n)_-}{\Delta x} - \frac{\mu(x_n)_+}{\Delta x} - \frac{\sigma(x_n)^2}{2\Delta x^2}, \frac{\mu(x_n)_+}{\Delta x} + \frac{\sigma(x_n)^2}{2\Delta x^2}$$

Fastforward to the end...

Value function for a stationary problem

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dZ_t \quad \text{on } [0, \bar{X}]$$

Payoff flow $u(X_t)$ to time ∞

$$0 = f_t + u(x) + \underbrace{\mu(x) f_x + \frac{1}{2}\sigma(x)^2 f_{xx}}_{\mathbf{Mf}} - \rho f(x,t)$$

Hence,

$$f = (\rho I - M)^{-1} u$$

Example: KFE

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dZ_t$$

Density $f(0,x)$ at time 0. After $f(t,x)$ satisfies the KFE

$$f_t(t,x) = \underbrace{-\frac{\partial}{\partial x}\left(\mu(x) f(t,x)\right) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\sigma(x)^2 f(t,x)\right)}_{\text{M'f(t,.)}}$$

Hence, with implicit method

$$f(t,\cdot) - f(t-\Delta t,\cdot) = \Delta t\, M' f(t,\cdot)$$

$$f(t,\cdot) = \left(I - \Delta t\, M'\right)^{-1} f(t-\Delta t,\cdot)$$

Solving HJB: An Example

Income $\frac{dy_t}{y_t} = \mu^y \, dt + \sigma^y \, dZ_t$

Savings $ds_t = (rs_t + y_t - c_t) \, dt$

Utility $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ **with discount rate** ρ

Let $\nu_t = \frac{s_t}{s_t + y_t} \in [0, 1]$

Conjecture value function of the form $f(\nu_t) y_t^{1-\gamma}$

HJB: An Example

$$\nu_t = \frac{s_t}{s_t + y_t} \in [0,1]$$

$$\frac{dy_t}{y_t} = \mu^y \, dt + \sigma^y \, dZ_t$$

$$d\nu_t = \nu_t(1-\nu_t)\left(r + \frac{y_t}{s_t} - \frac{c_t}{s_t} - \mu^y + (1-\nu_t)(\sigma^y)^2\right) dt$$
$$-\nu_t(1-\nu_t)\sigma^y \, dZ$$

Value function $\quad f(\nu_t) y_t^{1-\gamma} \quad \frac{dy_t^{1-\gamma}}{y_t^{1-\gamma}} = (1-\gamma)(\mu^y - \gamma(\sigma^y)^2/2)\, dt + (1-\gamma)\sigma^y \, dZ_t$

HJB $\quad \rho f(\nu) y^{1-\gamma} = \max_c \frac{c^{1-\gamma}}{1-\gamma} + f'(\nu)\nu(1-\nu)(r + y/s - c/s - \mu^y + (1-\nu)(\sigma^y)^2) y^{1-\gamma}$

$$\frac{f''(\nu)}{2}\nu^2(1-\nu^2)(\sigma^y)^2 y^{1-\gamma} + f(\nu)(1-\gamma)(\mu^y - \gamma(\sigma^y)^2/2) y^{1-\gamma} - f'(\nu)\nu(1-\nu)\sigma^y(1-\gamma)\sigma^y y^{1-\gamma}$$

HJB: An Example $\nu_t = \frac{s_t}{s_t + y_t} \in [0,1] \quad \Rightarrow \quad \frac{y}{s} = \frac{1-\nu}{\nu}$

$$\frac{dy_t}{y_t} = \mu^y \, dt + \sigma^y \, dZ_t$$

$$d\nu_t = \nu_t(1-\nu_t)\left(r + \frac{y_t}{s_t} - \frac{c_t}{s_t} - \mu^y + (1-\nu_t)(\sigma^y)^2\right) dt$$
$$-\nu_t(1-\nu_t)\sigma^y dZ$$

Value function $\quad f(\nu_t) y_t^{1-\gamma} \quad \frac{dy_t^{1-\gamma}}{y_t^{1-\gamma}} = (1-\gamma)(\mu^y - \gamma(\sigma^y)^2/2)\, dt + (1-\gamma)\sigma^y dZ_t$

HJB $\quad \rho f(\nu) y^{1-\gamma} = \max_c \frac{c^{1-\gamma}}{1-\gamma} + f'(\nu)\nu(1-\nu)(r + y/s - c/s - \mu^y + (1-\nu)(\sigma^y)^2) y^{1-\gamma}$

$$\frac{f''(\nu)}{2}\nu^2(1-\nu^2)(\sigma^y)^2 y^{1-\gamma} + f(\nu)(1-\gamma)(\mu^y - \gamma(\sigma^y)^2/2) y^{1-\gamma} - f'(\nu)\nu(1-\nu)\sigma^y(1-\gamma)\sigma^y y^{1-\gamma}$$

HJB: An Example

$\nu_t = \frac{s_t}{s_t + y_t} \in [0,1] \quad \Rightarrow \quad \frac{y}{s} = \frac{1-\nu}{\nu}$

$\frac{dy_t}{y_t} = \mu^y \, dt + \sigma^y \, dZ_t$

$$d\nu_t = \nu_t(1-\nu_t)\left(r + \frac{y_t}{s_t} - \frac{c_t}{s_t} - \mu^y + (1-\nu_t)(\sigma^y)^2\right) dt$$
$$-\nu_t(1-\nu_t)\sigma^y \, dZ$$

Value function $\quad f(\nu_t)\frac{y_t^{1-\gamma}}{1-\gamma} \quad \frac{dy_t^{1-\gamma}}{y_t^{1-\gamma}} = (1-\gamma)(\mu^y - \gamma(\sigma^y)^2/2)\, dt + (1-\gamma)\sigma^y \, dZ_t$

HJB $\quad \left(\rho - (1-\gamma)(\mu^y - \gamma(\sigma^y)^2/2)\right) f(\nu) =$

$$\max_{\zeta = c/y} \frac{\zeta^{1-\gamma}}{1-\gamma} + f'(\nu)\nu(1-\nu)\left(r + \frac{1-\nu}{\nu}(1-\zeta) - \mu^y + (\gamma - \nu)(\sigma^y)^2\right) + \frac{f''(\nu)\nu^2(1-\nu^2)(\sigma^y)^2}{2}$$

HJB: more general form

$$0 = \max_a g(x, a) + f_x(t, x)\mu(x, a) + f_{xx}(t, x)\frac{\sigma(x, a)^2}{2} - \rho(x, a)f(t, x)$$

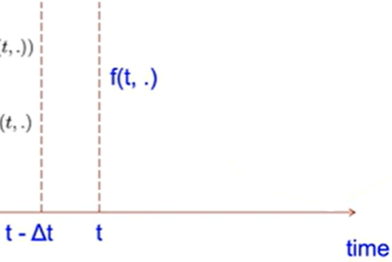
HJB

Determine policy $a(x_n)$ over $[t - \Delta t, t]$, construct matrix M_a

$$0 = f_t(t,x) + \max_a \; g(x,a) + f_x(t,x)\mu(x,a) + f_{xx}(t,x)\frac{\sigma(x,a)^2}{2} - \rho(x,a)f(t,x)$$

$$f(t-\Delta t,.) = (I(1+\rho_a\,\Delta t) - \Delta t\, M_a)^{-1}(\Delta t\, g_a + f(t,.))$$

$$f(t-\Delta t,.) = \Delta t\, g_a + ((1-\rho_a\,\Delta t)I + \Delta t\, M_a)f(t,.)$$



Determining a(x_n)

$$\max_a \; g(x_n, a) + \mu_-(x_n, a)\frac{f(t,x_n)-f(t,x_{n-1})}{\Delta x} + \mu_+(x_n, a)\frac{f(t,x_{n+1})-f(t,x_n)}{\Delta x} +$$

$$\frac{f(t,x_{n+1})-2f(t,x_n)+f(t,x_{n-1})}{\Delta x^2}\frac{\sigma(x_n,a)^2}{2} \; -\rho(x_n, a)f(t, x_n)$$

Barles and Souganidis (1991)

A numerical finite difference scheme converges to the viscosity solution as long as it is

- **consistent** ← derivative approximations converge to true derivatives
- **stable** ← doesn't explode
- **monotone** ← new values depend positively on all known values

From Thursday class:

- **m-dimensional valuation equation (let's even allow jumps)**

$$\underbrace{\rho f(x) = g(x) + \mu(x)Df(x) + \frac{tr[\sigma(x)\sigma(x)^T D^2 f(x)]}{2} + \int \phi(y \mid x)(f(y) - f(x))\, dy}_{\text{differential/jump operator}}$$

- As long as we can discretize the operator as Mf, where M
  - has entries ≤ 0 on the main diagonal, ≥ 0 off
  - rows add up to 0
- the resulting implicit numerical scheme is monotone and stable

What goes wrong if the scheme is not monotone

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dZ_t \quad \text{on } [0, \bar{X}]$$

$$\sigma(x) = 0, \quad \mu(x) = 1 \quad \text{for } x \neq 0, \bar{X}, \quad \mu(0) = \mu(\bar{X}) = 0$$

Upwind scheme: right derivative. What if we use the left derivative?

$$f(t-\Delta t,\cdot) = \Delta t\, g + \underbrace{((1-\rho\Delta t)I + \Delta t M)}_{\textbf{eigenvalue } 1-\rho\Delta t+\Delta t/\Delta x > 1 \textbf{ (if } \Delta x \textbf{ is small)}} f(t,\cdot)$$

$$M = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ -1/\Delta x & 1/\Delta x & 0 & 0 & \cdots \\ 0 & -1/\Delta x & 1/\Delta x & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}}_{\textbf{Eigenvalues } 0,\ 1/\Delta x}$$

$$f(t,\cdot) = \underbrace{(I(1+\rho\Delta t) - \Delta t M)^{-1}}_{\textbf{eigenvalue } (1+\rho\Delta t-\Delta t/\Delta x)^{-1}} (\Delta t\, g + f(t+\Delta t,\cdot))$$

What goes wrong if the scheme is not monotone

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dZ_t \quad \text{on } [0, \bar{X}]$$

$$\mu(x) = 0, \quad \sigma(x) = 1 \quad \text{for } x \neq 0, \bar{X}, \quad \sigma(0) = \sigma(\bar{X}) = 0$$

Suppose M has the desired sign pattern but Δt is too large

$$M = \frac{1}{2}\underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1/\Delta x^2 & -2/\Delta x^2 & 1/\Delta x^2 & 0 & \cdots \\ 0 & 1/\Delta x^2 & -2/\Delta x^2 & 1/\Delta x^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}}_{\textbf{Negative eigenvalues, } O(-1/\Delta x^2)}$$

$$f(t-\Delta t, \cdot) = \Delta t\, g + \underbrace{((1-\rho\Delta t)I + \Delta t M)}_{\substack{\textbf{eigenvalue } 1-\rho\Delta t + \Delta t O(-1/\Delta x^2) < -1 \\ \textbf{if } \Delta t \textbf{ larger than } O(\Delta x^2)}} f(t, \cdot)$$

m-dimensional state space: the valuation equation

State $X \in [0, \bar{X}]^m$ follows

$$dX_t = \mu(X_t)\, dt + \sigma(X_t)\, dZ_t$$

mxd in R^d

Payoff flow $g(X_t)$, discount rate ρ

$$f(X_0) = E_0 \left[\int_0^\infty e^{-\rho t} g(X_t)\, dt \mid X_0 \right]$$

f solves $$0 = f_t + g(x) + \mu(x) Df(x) + \frac{1}{2} tr[\sigma(x)\sigma(x)^T D^2 f(x)] - \rho f(x)$$

Matrix M to represent this, with entries ≥ 0 off the main diagonal

1^{st}-order term: generalized upwind scheme

Grid step Δx (along each dimension)

Drift $\mu(x) = (\mu_1(x), \mu_2(x))$

Represent

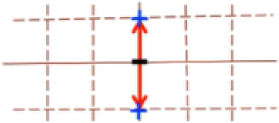
$$\mu(x) Df(x) = \mu_1(x)_- \frac{f(x) - f(x - (\Delta x, 0))}{\Delta x} + \mu_1(x)_+ \frac{f(x + (\Delta x, 0)) - f(x)}{\Delta x}$$

$$+\mu_2(x)_- \frac{f(x) - f(x - (0, \Delta x))}{\Delta x} + \mu_2(x)_+ \frac{f(x + (0, \Delta x)) - f(x)}{\Delta x}$$

This generalizes to any number of dimensions

The problematic second-order term $tr[\sigma^T(x,a)D^2f\sigma(x,a)]$

$$f_{11} = \frac{f(x-(\Delta x,0))-2f(x)+f(x+(\Delta x,0))}{\Delta x^2}$$



$$f_{22} = \frac{f(x-(0,\Delta x))-2f(x)+f(x+(0,\Delta x))}{\Delta x^2}$$

$$f_{12} = \frac{f(x+(\Delta x,\Delta x))-f(x+(\Delta x,0))-f(x+(0,\Delta x))+f(x)}{\Delta x^2}$$

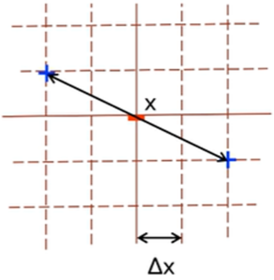
2^{nd}-order term: central idea

To evaluate $\underbrace{tr[\sigma(x)\sigma(x)^T}_{\text{positive semidefinite}} Df(x)]$ we seek representation

$$\sigma(x)\sigma(x)^T = \eta_1 \xi_1 \xi_1^T + \eta_2 \xi_2 \xi_2^T + \dots$$

where ξ_k are vectors of integers, so

$$\underbrace{Df_{\xi\xi} = \frac{f(x-\xi\Delta x) - 2f(x) + f(x+\xi\Delta x)}{\Delta x^2}}_{\text{we can evaluate on the grid}}$$



and $tr[\sigma(x)\sigma(x)^T Df(x)] = \eta_1 f_{\xi_1\xi_1} + \eta_2 f_{\xi_2\xi_2} + \dots$

Convex positive semidefinite cone

X and Y are positive semidefinite, then so is $a_x X + a_y Y$, for $a_x, a_y \geq 0$

Extreme rays pass through $\zeta\zeta^T$, where ζ is a vector

...any positive semidefinite X can be decomposed as

$$X = Q\Lambda Q^T = \lambda_1\zeta_1\zeta_1^T + \lambda_2\zeta_2\zeta_2^T + \cdots + \lambda_m\zeta_m\zeta_m^T$$

orthogonal matrix of eigenvectors nonnegative eigenvalues orthogonal eigenvectors

We can approximate any ζ by an integer vector ξ (times a constant)

$\eta\xi\xi^T$ are a dense subset of extreme points

To sum up / Caratheodory's

Any X in a convex set in $\mathbb{R}^k$ is a convex combination of K+1 extreme points

Any m×m positive-semidefinite (symmetric) matrix X has

representation: $X = \lambda_1 \zeta_1 \zeta_1^T + \lambda_2 \zeta_2 \zeta_2^T + \ldots$

approximation: $\eta_1 \xi_1 \xi_1^T + \eta_2 \xi_2 \xi_2^T + \ldots$

nonnegative numbers **integer vectors**

... but what's a good way to find these η and ξ?

It has nothing to do with eigenvectors...

$m = 2$

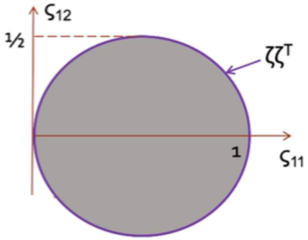
Positive semidefinite cone for 2 by 2 matrices $\begin{bmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{12} & \zeta_{22} \end{bmatrix}$

Cross-section with trace, $\zeta_{11} + \zeta_{22} = 1$**:**

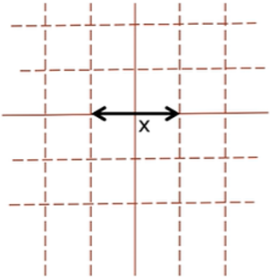
$$\zeta_{12}^2 \le \zeta_{11}(1 - \zeta_{11}) \quad \text{or} \quad \zeta_{12}^2 + (\zeta_{11} - 1/2)^2 \le 1/4$$

Integer $\xi = [z_1, z_2]$ **corresponds to**

$$\frac{1}{|\xi|^2} \begin{bmatrix} z_1^2 & z_1 z_2 \\ z_1 z_2 & z_2^2 \end{bmatrix} \longrightarrow$$

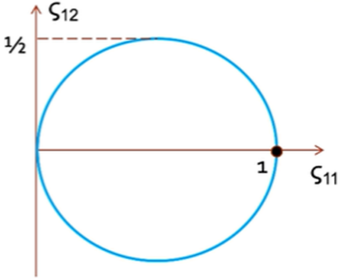


Integer ξ and points in the circle

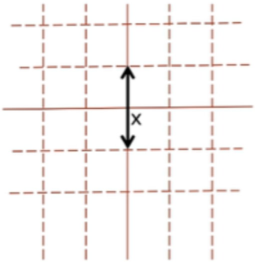


$$\xi = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\frac{\xi\xi^T}{|\xi|^2} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

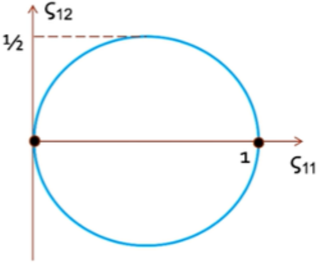


Integer ξ and points in the circle

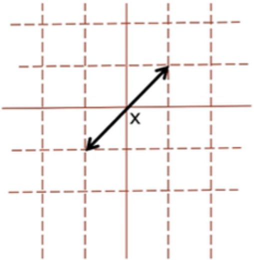


$$\xi = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\frac{\xi\xi^T}{|\xi|^2} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

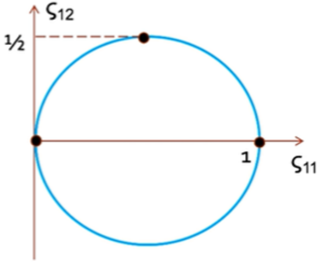


Integer ξ and points in the circle

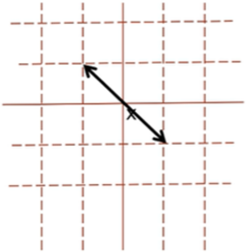


$$\xi = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\frac{\xi\xi^T}{|\xi|^2} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

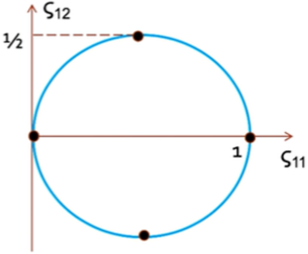


Integer ξ and points in the circle

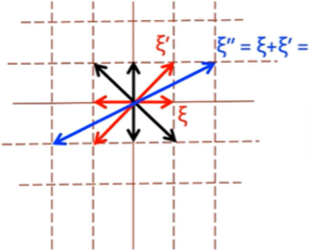


$$\xi = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\frac{\xi\xi^T}{|\xi|^2} = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

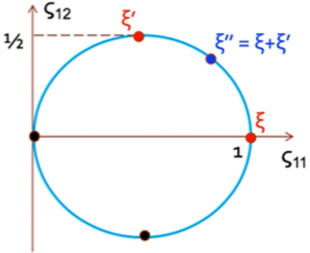


Integer ξ and points in the circle

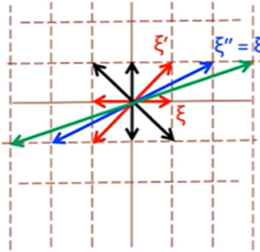


$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 4/5 & 2/5 \\ 2/5 & 1/5 \end{bmatrix}$$

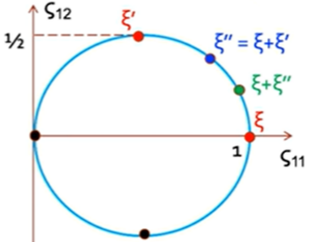


Integer ξ and points in the circle



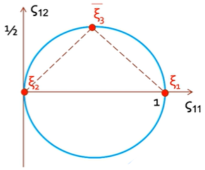
$$\xi'' = \xi + \xi' = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 4/5 & 2/5 \\ 2/5 & 1/5 \end{bmatrix}$$



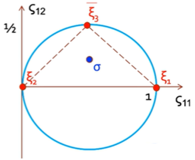
Algorithm to represent any $\sigma\sigma^T = \begin{bmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{12} & \zeta_{22} \end{bmatrix}$

- **Let** $\xi_1 = [1\ 0]^T,\ \xi_2 = [0\ 1]^T$
- $\xi_3 = [1\ 1]^T$ **if** $\zeta_{12} \geq 0$, $\xi_3 = [1\ \ -1]^T$ **if** $\zeta_{12} < 0$
- **(*) Solve** $\begin{bmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{12} & \zeta_{22} \end{bmatrix} = \eta_1\xi_1\xi_1^T + \eta_2\xi_2\xi_2^T + \eta_3\xi_3\xi_3^T$



Algorithm to represent any $\sigma\sigma^T = \begin{bmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{12} & \zeta_{22} \end{bmatrix}$

- **Let** $\xi_1 = [1\ 0]^T,\ \xi_2 = [0\ 1]^T$
- $\xi_3 = [1\ 1]^T$ **if** $\zeta_{12} \geq 0$**,** $\xi_3 = [1\ \ -1]^T$ **if** $\zeta_{12} < 0$
- **(*) Solve** $\begin{bmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{12} & \zeta_{22} \end{bmatrix} = \eta_1 \xi_1 \xi_1^T + \eta_2 \xi_2 \xi_2^T + \eta_3 \xi_3 \xi_3^T$
- **If** $\eta_1, \eta_2, \eta_3 \geq 0$**, we are done.**



Algorithm to represent any $\sigma\sigma^T = \begin{bmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{12} & \zeta_{22} \end{bmatrix}$

- **Let** $\xi_1 = [1\ 0]^T,\ \xi_2 = [0\ 1]^T$
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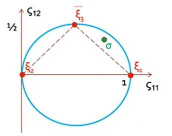
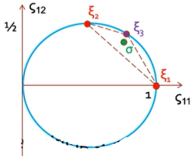
- **(*) Solve** $\begin{bmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{12} & \zeta_{22} \end{bmatrix} = \eta_1\xi_1\xi_1^T + \eta_2\xi_2\xi_2^T + \eta_3\xi_3\xi_3^T$



- **If** $\eta_1, \eta_2, \eta_3 \geq 0$**, we are done.**
- **Else,** $\eta_i < 0$ **for** $i = 1$ **or** 2 **(not 3).Remove** ξ_i**, relabel remaining** ξ_1 **and** ξ_2**.**

Algorithm to represent any $\sigma\sigma^T = \begin{bmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{12} & \zeta_{22} \end{bmatrix}$

- **Let** $\xi_1 = [1\ 0]^T,\ \xi_2 = [0\ 1]^T$
- $\xi_3 = [1\ 1]^T$ **if** $\zeta_{12} \geq 0$**,** $\xi_3 = [1\ \ -1]^T$ **if** $\zeta_{12} < 0$
- **(*) Solve** $\begin{bmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{12} & \zeta_{22} \end{bmatrix} = \eta_1 \xi_1 \xi_1^T + \eta_2 \xi_2 \xi_2^T + \eta_3 \xi_3 \xi_3^T$



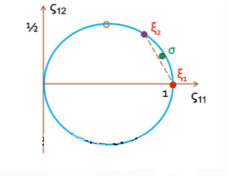
- **If** $\eta_1, \eta_2, \eta_3 \geq 0$**, we are done.**
- **Else,** $\eta_i < 0$ **for** $i = 1$ **or** 2 **(not 3). Remove** ξ_i**, relabel remaining** ξ_1 **and** ξ_2**.**
- **Take** $\xi_3 = \xi_1 + \xi_2$**, repeat from (*)**

Algorithm to represent any $\sigma\sigma^T = \begin{bmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{12} & \zeta_{22} \end{bmatrix}$

- **Let** $\xi_1 = [1\ 0]^T,\ \xi_2 = [0\ 1]^T$

- $\xi_3 = [1\ 1]^T$ **if** $\zeta_{12} \geq 0$, $\xi_3 = [1\ \ -1]^T$ **if** $\zeta_{12} < 0$

- **(*) Solve** $\begin{bmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{12} & \zeta_{22} \end{bmatrix} = \eta_1 \xi_1 \xi_1^T + \eta_2 \xi_2 \xi_2^T + \eta_3 \xi_3 \xi_3^T$

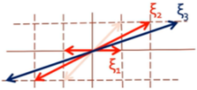


- **If** $\eta_1, \eta_2, \eta_3 \geq 0$**, we are done.**

- **Else,** $\eta_i < 0$ **for** $i = 1$ **or** 2 **(not 3).Remove** ξ_i**, relabel remaining** ξ_1 **and** ξ_2**.**

- **Take** $\xi_3 = \xi_1 + \xi_2$**, repeat from (*)**

- **Unless** $|\xi_3| > K$**, then approximate** $\begin{bmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{12} & \zeta_{22} \end{bmatrix} \approx \eta_1 \xi_1 \xi_1^T + \eta_2 \xi_2 \xi_2^T$



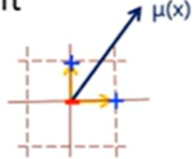
Assembling M

Discrete state space $[0 = x_0, x_1, x_2, \dots, x_N = \bar{X}]^2$

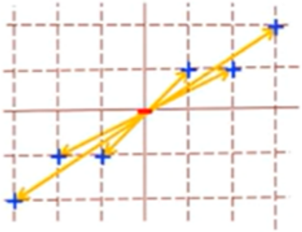
Represent

$$0 = f_t + g(x) + \mu(x) Df(x) + \frac{1}{2} tr\left[\sigma(x)\sigma(x)^T D^2 f(x)\right] - \rho f(x)$$

Drift



volatility



To sum up...

From the in-class exercise, if M

- **has entries ≤ 0 on the main diagonal, ≥ 0 off**
- **rows add up to 0**

then the implicit scheme is monotone and stable

We know how to construct M for 1$^{\text{st}}$-order (and jump) terms

- **for $m = 2$, we have an algorithm for 2$^{\text{nd}}$-order terms**
  - **requires using non-neighbor grid points**

- **for $m > 2$, monotone representation of the 2$^{\text{nd}}$-order term always possible (Caratheodory), but I don't know a good algorithm to find it**

HJB in m dimensions

$$0 = f_t + \max_a g(x,a) + \mu(x,a)^T Df + \frac{1}{2} tr[\sigma^T(x,a) D^2 f \sigma(x,a)] - \rho f(t,x)$$

$\mathbf{D^+f}$ and $\mathbf{D^-f}$: vectors of right and left derivatives

$\mu^+(x,a)$ **and** $\mu^-(x,a)$**: vectors of positive and negative parts of** $\mu(x,a)$

$$\sigma(x,a)\sigma^T(x,a) = \eta_1(x,a)\xi_1(x,a)\xi_1(x,a)^T + \eta_2(x,a)\xi_2(x,a)\xi_2(x,a)^T + \dots$$

Choose a to maximize

$$\mu^+(x,a)^T D^+ f + \mu^-(x,a)^T D^- f + \frac{1}{2}\sum_k \eta_k(x,a) f_{\xi_k(x,a)\xi_k(x,a)}$$

... then solve backward over Δt