Unit 9, Part 1

Binary Trees and Huffman Encoding
Binary Search Trees

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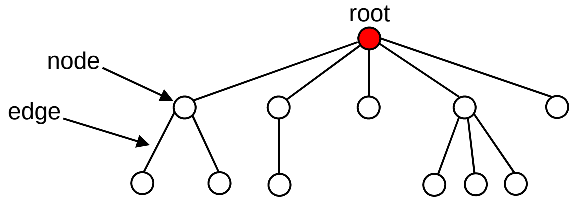
Motivation: Maintaining a Sorted Collection of Data

- A *data dictionary* is a sorted collection of data with the following key operations:
  - *search* for an item (and possibly delete it)
  - *insert* a new item
- If we use a list to implement a data dictionary, efficiency = $O(n)$.

data structure	***searching for an item***	***inserting an item***
a list implemented using an array		
a list implemented using a linked list		

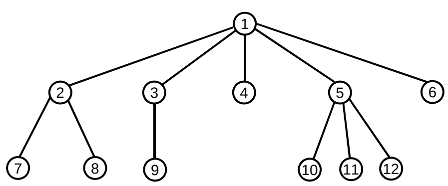
- In the next few lectures, we'll look at data structures (trees and hash tables) that can be used for a more efficient data dictionary.
- We'll also look at other applications of trees.

What Is a Tree?



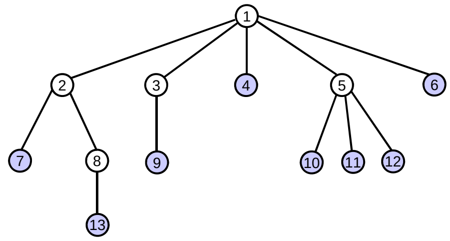
- A tree consists of:
  - a set of *nodes*
  - a set of *edges*, each of which connects a pair of nodes
- Each node may have one or more *data items.*
  - each data item consists of one or more fields
  - *key field* = the field used when searching for a data item
  - multiple data items with the same key are referred to as *duplicates*
- The node at the "top" of the tree is called the *root* of the tree.

Relationships Between Nodes



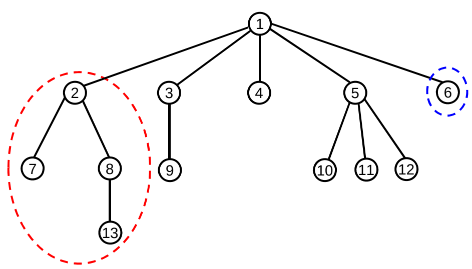
- If a node N is connected to other nodes that are directly below it in the tree, N is referred to as their *parent* and they are referred to as its *children*.
  - example: node 5 is the parent of nodes 10, 11, and 12
- Each node is the child of *at most one* parent.
- Other family-related terms are also used:
  - nodes with the same parent are *siblings*
  - a node's *ancestors* are its parent, its parent's parent, etc.
    - example: node 9's ancestors are 3 and 1
  - a node's *descendants* are its children, their children, etc.
    - example: node 1's descendants are *all* of the other nodes

Types of Nodes



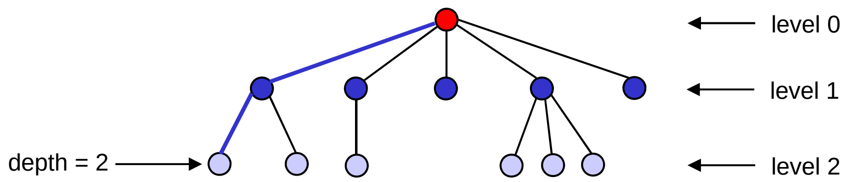
- A *leaf node* is a node without children.
- An *interior node* is a node with one or more children.

A Tree is a Recursive Data Structure



- Each node in the tree is the root of a smaller tree!
  - refer to such trees as *subtrees* to distinguish them from the tree as a whole
  - example: node 2 is the root of the subtree circled above
  - example: node 6 is the root of a subtree with only one node
- We'll see that tree algorithms often lend themselves to recursive implementations.

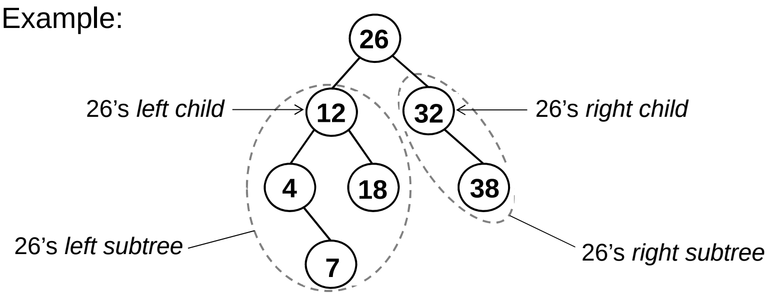
Path, Depth, Level, and Height



- There is exactly one *path* (one sequence of edges) connecting each node to the root.
- *depth* of a node = # of edges on the path from it to the root
- Nodes with the same depth form a *level* of the tree.
- The *height* of a tree is the maximum depth of its nodes.
  - example: the tree above has a height of 2

Binary Trees

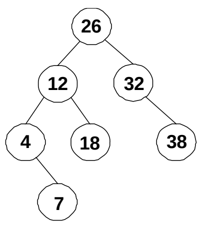
- In a *binary tree*, nodes have *at most two* children.
- Recursive definition: a binary tree is either:
  1) empty, or
  2) a node (the root of the tree) that has
     - one or more data fields
     - a *left child*, which is itself the root of a binary tree
     - a *right child*, which is itself the root of a binary tree
- Example:

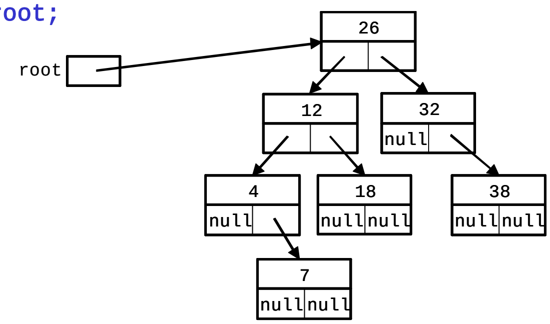


- How are the edges of the tree represented?

Representing a Binary Tree Using Linked Nodes

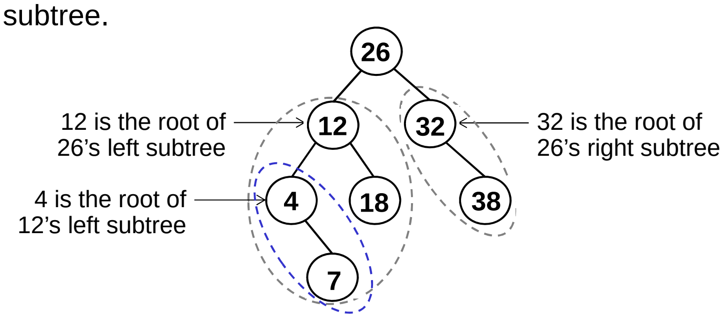
```
public class LinkedTree {
    private class Node {
        private int key;
        private LLList data;   // list of data for that key
        private Node left;     // reference to left child
        private Node right;    // reference to right child
        …
    }

    private Node root;
    …
}
```





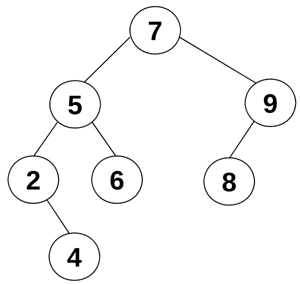
Traversing a Binary Tree

- Traversing a tree involves *visiting* all of the nodes in the tree.
  - visiting a node = processing its data in some way
    - example: print the key
- We will look at four types of traversals. Each of them visits the nodes in a different order.
- To understand traversals, it helps to remember the recursive definition of a binary tree, in which every node is the root of a subtree.



Preorder Traversal

- preorder traversal of the tree whose root is N:
  1) visit the root, N
  2) recursively perform a preorder traversal of N's left subtree
  3) recursively perform a preorder traversal of N's right subtree



- Preorder traversal of the tree above:

  7 5 2 4 6 9 8

Implementing Preorder Traversal

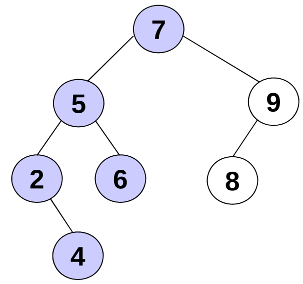
```
public class LinkedTree {
    …
    private Node root;

    public void preorderPrint() {
        if (root != null)
            preorderPrintTree(root);
    }

    private static void preorderPrintTree(Node root) {
        System.out.print(root.key + "  ");
        if (root.left != null)
            preorderPrintTree(root.left);
        if (root.right != null)
            preorderPrintTree(root.right);
    }
}
```

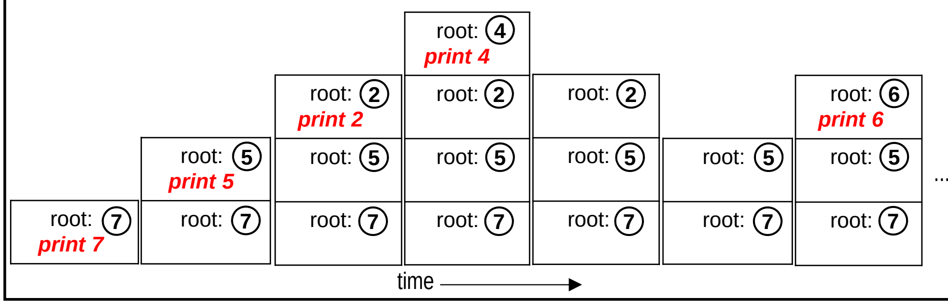
Not always the same as the root of the entire tree.

- `preorderPrintTree()` is a static, recursive method that takes as a parameter the root of the tree/subtree that you want to print.
- `preorderPrint()` is a non-static method that makes the initial call. It passes in the root of the entire tree as the parameter.

Tracing Preorder Traversal

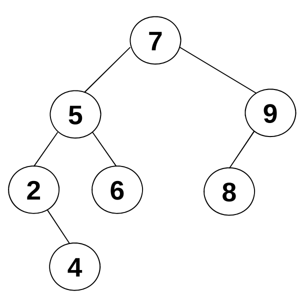
```
void preorderPrintTree(Node root) {
   System.out.print(root.key + “  ”);
   if (root.left != null)
       preorderPrintTree(root.left);
   if (root.right != null)
       preorderPrintTree(root.right);
}
```





Postorder Traversal

- postorder traversal of the tree whose root is N:
  1) recursively perform a postorder traversal of N’s left subtree
  2) recursively perform a postorder traversal of N’s right subtree
  3) visit the root, N



- Postorder traversal of the tree above:

  4 2 6 5 8 9 7

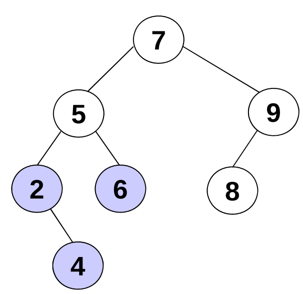
Implementing Postorder Traversal

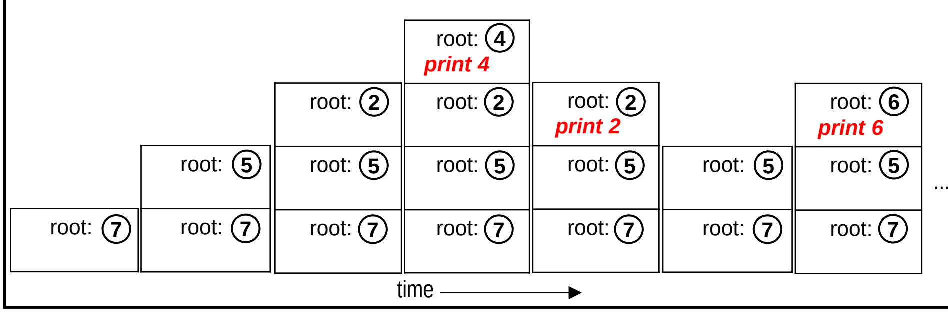
```
public class LinkedTree {
    …
    private Node root;

    public void postorderPrint() {
        if (root != null)
            postorderPrintTree(root);
    }

    private static void postorderPrintTree(Node root) {
        if (root.left != null)
            postorderPrintTree(root.left);
        if (root.right != null)
            postorderPrintTree(root.right);
        System.out.print(root.key + "  ");
    }
}
```

- Note that the root is printed *after* the two recursive calls.

Tracing Postorder Traversal

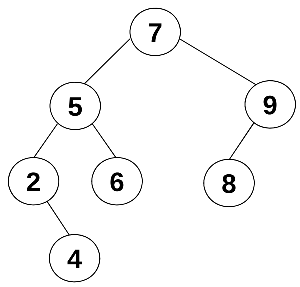
```
void postorderPrintTree(Node root) {
   if (root.left != null)
       postorderPrintTree(root.left);
   if (root.right != null)
       postorderPrintTree(root.right);
   System.out.print(root.key + "  ");
}
```





Inorder Traversal

- inorder traversal of the tree whose root is N:
  1) recursively perform an inorder traversal of N's left subtree
  2) visit the root, N
  3) recursively perform an inorder traversal of N's right subtree



- Inorder traversal of the tree above:

  2 4 5 6 7 8 9

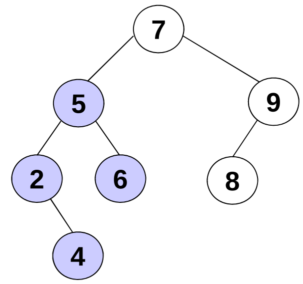
Implementing Inorder Traversal

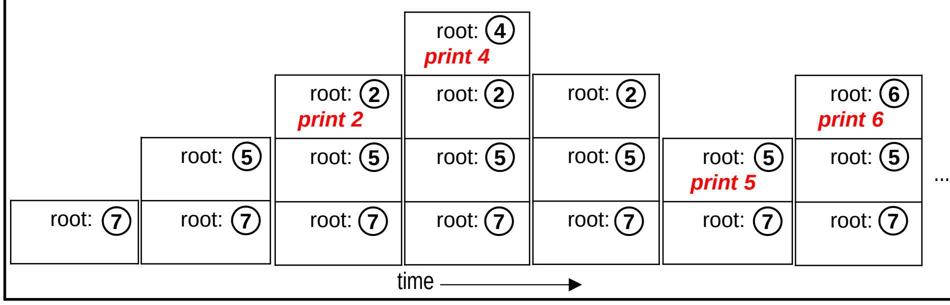
```
public class LinkedTree {
    …
    private Node root;

    public void inorderPrint() {
        if (root != null)
            inorderPrintTree(root);
    }

    private static void inorderPrintTree(Node root) {
        if (root.left != null)
            inorderPrintTree(root.left);
        System.out.print(root.key + "  ");
        if (root.right != null)
            inorderPrintTree(root.right);
    }
}
```

- Note that the root is printed *between* the two recursive calls.

Tracing Inorder Traversal

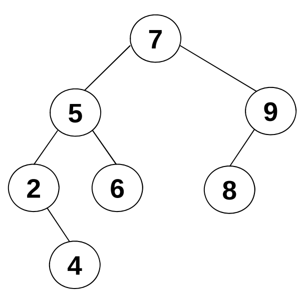
```
void inorderPrintTree(Node root) {
   if (root.left != null)
       inorderPrintTree(root.left);
   System.out.print(root.key + "  ");
   if (root.right != null)
       inorderPrintTree(root.right);
}
```





Level-Order Traversal

- Visit the nodes one level at a time, from top to bottom and left to right.

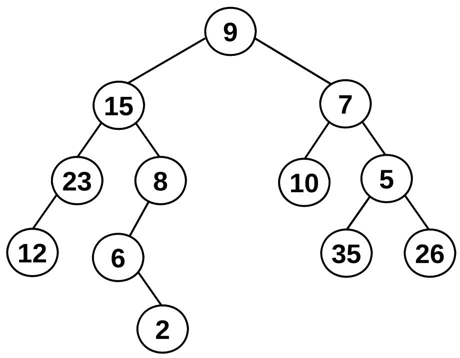


- Level-order traversal of the tree above: **7 5 9 2 6 8 4**
- How could we implement this type of traversal?

Tree-Traversal Summary

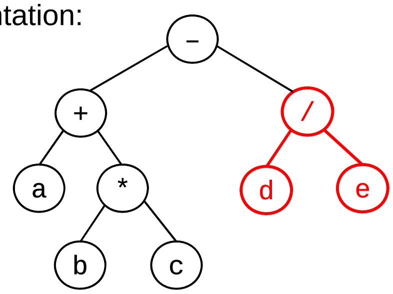
preorder: root, left subtree, right subtree
postorder: left subtree, right subtree, root
inorder: left subtree, root, right subtree
level-order: top to bottom, left to right

- Perform each type of traversal on the tree below:



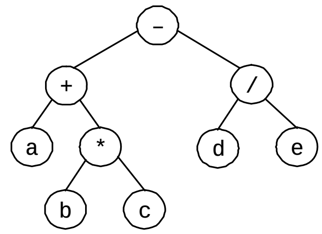
Using a Binary Tree for an Algebraic Expression

- We'll restrict ourselves to fully parenthesized expressions and to the following binary operators: `+`, –, `*`, `/`
- Example expression: `((a + (b * c)) – (d / e))`
- Tree representation:



- Leaf nodes are variables or constants; interior nodes are operators.
- Because the operators are binary, either a node has two children or it has none.

Traversing an Algebraic-Expression Tree



- Inorder gives conventional algebraic notation.
  - print '(' before the recursive call on the left subtree
  - print ')' after the recursive call on the right subtree
  - for tree at right: `((a + (b * c)) – (d / e))`
- Preorder gives functional notation.
  - print '('s and ')'s as for inorder, and commas after the recursive call on the left subtree
  - for tree above: `subtr(add(a, mult(b, c)), divide(d, e))`
- Postorder gives the order in which the computation must be carried out on a stack/RPN calculator.
  - for tree above: `push a, push b, push c, multiply, add,…`

Fixed-Length Character Encodings

- A character encoding maps each character to a number.
- Computers usually use fixed-length character encodings.
  - ASCII (American Standard Code for Information Interchange) uses 8 bits per character.

char	dec	binary
a	97	01100001
b	98	01100010
c	99	01100011
…	…	…

example: "`bat`" is stored in a text file as the following sequence of bits:

`01100010 01100001 01110100`

  - Unicode uses 16 bits per character to accommodate foreign-language characters. (ASCII codes are a subset.)
- Fixed-length encodings are simple, because
  - all character encodings have the same length
  - a given character always has the same encoding

Variable-Length Character Encodings

- Problem: fixed-length encodings waste space.
- Solution: use a variable-length encoding.
  - use encodings of different lengths for different characters
  - assign shorter encodings to frequently occurring characters
- Example:

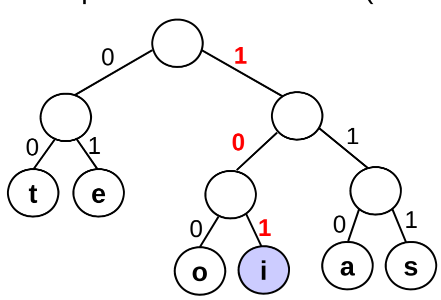
e	01
o	100
s	111
t	00

"test" would be encoded as
`00 01 111 00` → `000111100`

- Challenge: when decoding/decompressing an encoded document, how do we determine the boundaries between characters?
  - example: for the above encoding, how do we know whether the next character is 2 bits or 3 bits?
- One requirement: no character's encoding can be the prefix of another character's encoding (e.g., couldn't have `00` and `001`).

Huffman Encoding

- Huffman encoding is a type of variable-length encoding that is based on the actual character frequencies in a given document.
- Huffman encoding uses a binary tree:
  - to determine the encoding of each character
  - to decode an encoded file – i.e., to decompress a compressed file, putting it back into ASCII
- Example of a Huffman tree (for a text with only six chars):



Leaf nodes are characters.

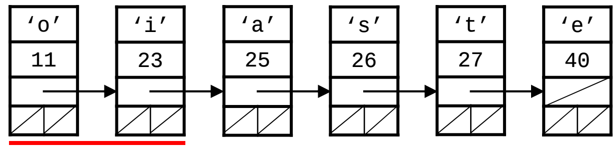
Left branches are labeled with a 0, and right branches are labeled with a 1.

If you follow a path from root to leaf, you get the encoding of the character in the leaf
example: 101 = 'i'

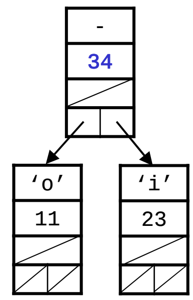
Building a Huffman Tree

1) Begin by reading through the text to determine the frequencies.

2) Create a list of nodes that contain (character, frequency) pairs for each character that appears in the text.

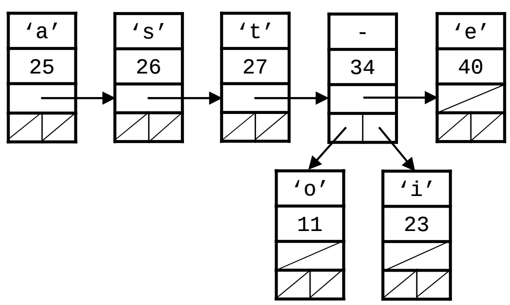


3) Remove and "merge" the nodes with the two lowest frequencies, forming a new node that is their parent.
   - left child = lowest frequency node
   - right child = the other node
   - frequency of parent = sum of the frequencies of its children
     - in this case, 11 + 23 = 34



Building a Huffman Tree (cont.)

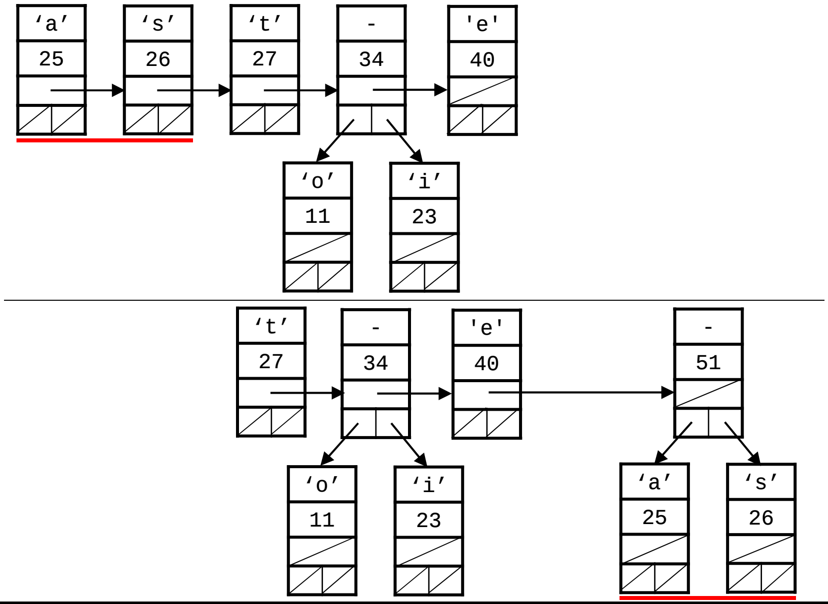
4) Add the parent to the list of nodes (maintaining sorted order):



5) Repeat steps 3 and 4 until there is only a single node in the list, which will be the root of the Huffman tree.

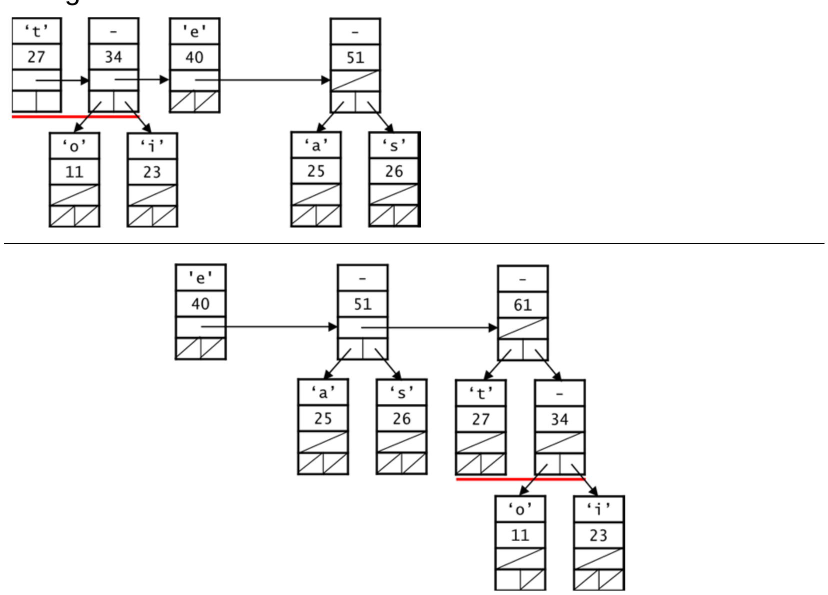
Completing the Huffman Tree Example I

- Merge the two remaining nodes with the lowest frequencies:



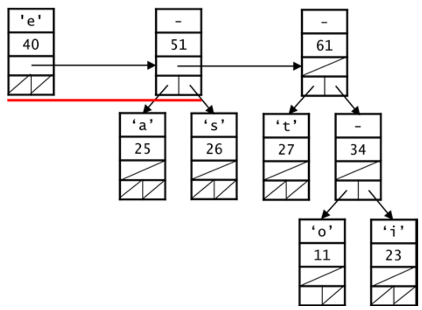
Completing the Huffman Tree Example II

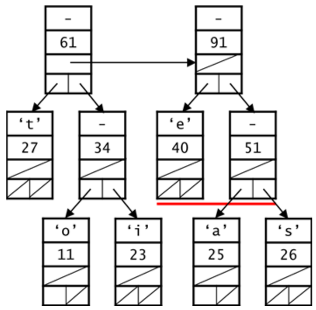
- Merge the next two nodes:



Completing the Huffman Tree Example II

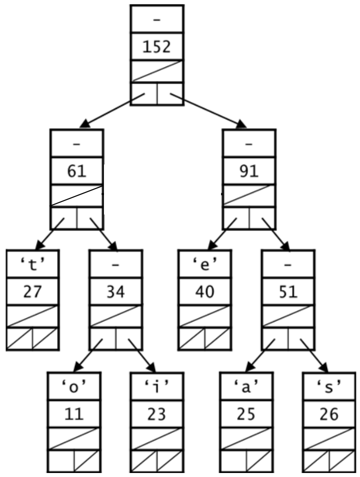
- Merge again:

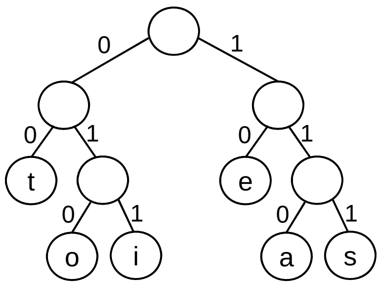




Completing the Huffman Tree Example IV

- The next merge creates the final tree:

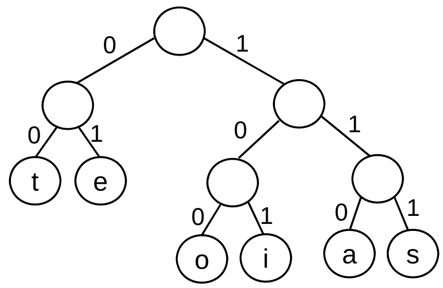




- Characters that appear more frequently end up higher in the tree, and thus their encodings are shorter.

The Shape of the Huffman Tree

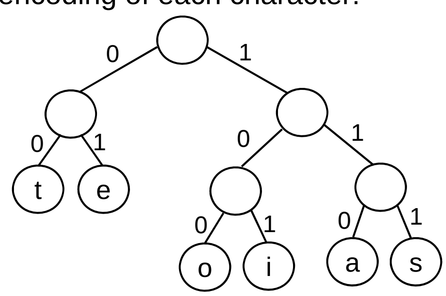
- The tree on the last slide is fairly symmetric.
- This won't always be the case!
  - depends on the frequencies of the characters in the document being compressed
- For example, changing the frequency of `'o'` from 11 to 21 would produce the tree shown below:



- This is the tree that we'll use in the remaining slides.

Using Huffman Encoding to Compress a File

1) Read through the input file and build its Huffman tree.
2) Write a file header for the output file.
   – include an array containing the frequencies so that the tree can be rebuilt when the file is decompressed.
3) Traverse the Huffman tree to create a table containing the encoding of each character:



a	**?**
e	**?**
i	101
o	100
s	111
t	00

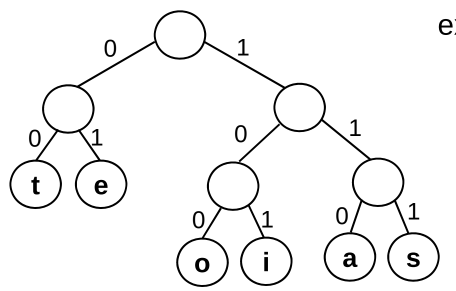
4) Read through the input file a second time, and write the Huffman code for each character to the output file.

Using Huffman Decoding to Decompress a File

1) Read the frequency table from the header and rebuild the tree.
2) Read one bit at a time and traverse the tree, starting from the root:

    when you read a bit of 1, go to the right child
    when you read a bit of 0, go to the left child
    when you reach a leaf node, record the character,
        return to the root, and continue reading bits

The tree allows us to easily overcome the challenge of determining the character boundaries!

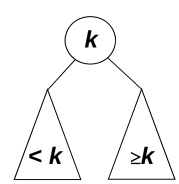


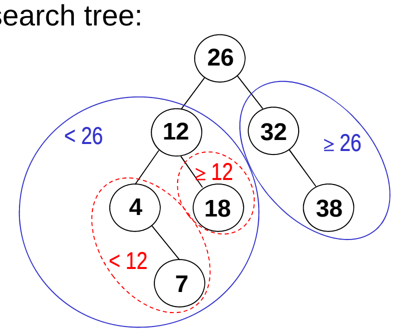
example: 101111110000111100

101 = right,left,right = i
111 = right,right,right= s
110 = right,right,left = a
00 = left,left = t
01 = left,right = e
111 = right,right,right= s
00 = left,left = t

Binary *Search* Trees

- Search-tree property: for each node *k:*
  - all nodes in *k*'s left subtree are < *k*
  - all nodes in *k*'s right subtree are >= *k*
- Our earlier binary-tree example is a search tree:



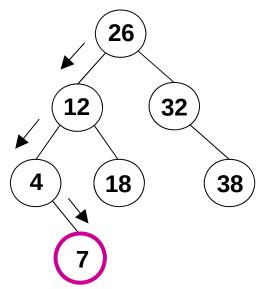


Searching for an Item in a Binary Search Tree

- Algorithm for searching for an item with a key *k*:

  if *k* == the root node's key, you're done
  else if *k* < the root node's key, search the left subtree
  else search the right subtree

- Example: search for 7

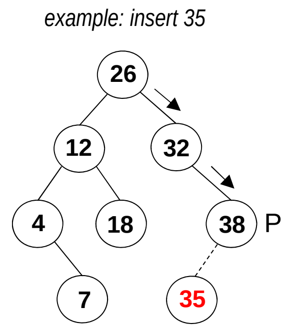


Implementing Binary-Tree Search

```
public class LinkedTree {    // Nodes have keys that are ints
    …
    private Node root;

    public LLList search(int key) {
        Node n = searchTree(root, key);
        if (n == null)
            return null;     // no such key
        else
            return n.data;   // return list of values for key
    }

    private static Node searchTree(Node root, int key) {
        // write together





    }
}
```

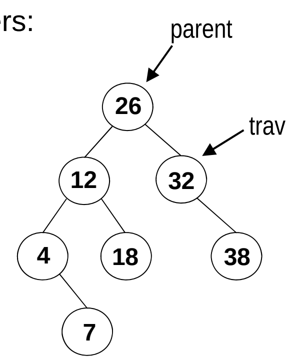
Inserting an Item in a Binary Search Tree

- We want to insert an item whose key is *k*.
- We traverse the tree as if we were searching for *k.*
- If we find a node with key *k*, we add the data item to the list of items for that node.
- If we don't find it, the last node we encounter will be the parent P of the new node.
  - if *k* `<` P's key, make the new node P's left child
  - else make the node P's right child
- *Special case:* if the tree is empty, make the new node the root of the tree.
- The resulting tree is still a search tree.



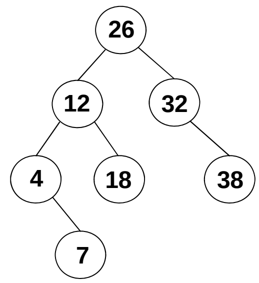
Implementing Binary-Tree Insertion

- We'll implement part of the `insert()` method together.
- We'll use iteration rather than recursion.
- Our method will use two references/pointers:
  - `trav`: performs the traversal down to the point of insertion
  - `parent`: stays one behind `trav`
    - like the `trail` reference that we sometimes use when traversing a linked list



Implementing Binary-Tree Insertion

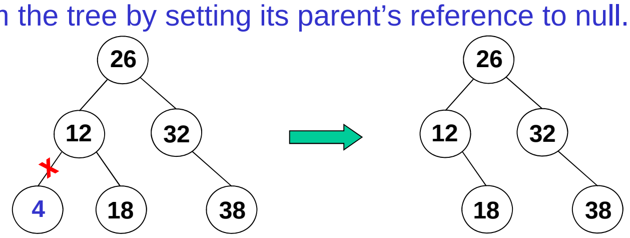
```
public void insert(int key, Object data) {
    Node parent = null;
    Node trav = root;
    while (trav != null) {
        if (trav.key == key) {
            trav.data.addItem(data, 0);
            return;
        }



    }
    Node newNode = new Node(key, data);
    if (root == null)    // the tree was empty
        root = newNode;
    else if (key < parent.key)
        parent.left = newNode;
    else
        parent.right = newNode;
}
```



Deleting Items from a Binary Search Tree

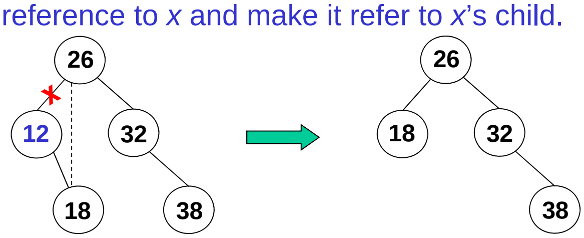
- Three cases for deleting a node *x*
- **Case 1:** *x* has no children.
  Remove *x* from the tree by setting its parent's reference to null.

ex: delete 4



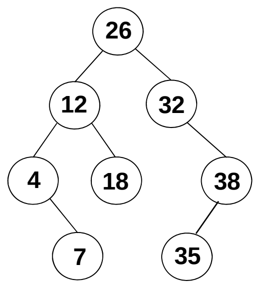
- **Case 2:** *x* has one child.
  Take the parent's reference to *x* and make it refer to *x*'s child.

ex: delete 12



Deleting Items from a Binary Search Tree (cont.)

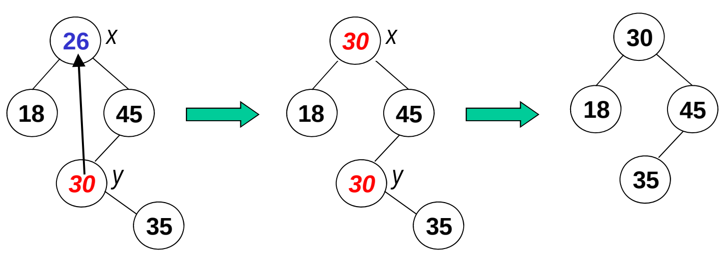
- **Case 3:** *x* has two children
  - we can't just delete *x*. why?
  - instead, we replace *x* with a node from elsewhere in the tree
  - to maintain the search-tree property, we must choose the replacement carefully
    - example: what nodes could replace 26 below?



Deleting Items from a Binary Search Tree (cont.)

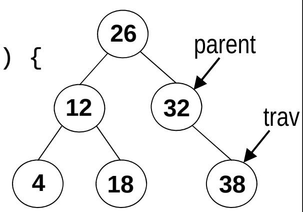
- **Case 3:** *x* has two children (continued):
  - replace *x* with the smallest node in *x*'s right subtree—call it *y*
  - *y* will either be a leaf node or will have one right child. why?
- After copying *y*'s item into *x*, we delete *y* using case 1 or 2.

ex:
delete 26



Implementing Binary-Tree Deletion

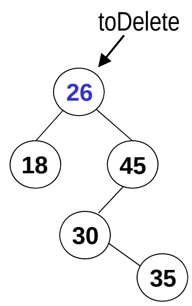
```
public LLList delete(int key) {
    // Find the node and its parent.
    Node parent = null;
    Node trav = root;
    while (trav != null && trav.key != key) {
        parent = trav;
        if (key < trav.key)
            trav = trav.left;
        else
            trav = trav.right;
    }
    // Delete the node (if any) and return the removed items.
    if (trav == null)    // no such key
        return null;
    else {
        LLList removedData = trav.data;
        deleteNode(trav, parent);
        return removedData;
    }
}
```



- This method uses a helper method to delete the node.

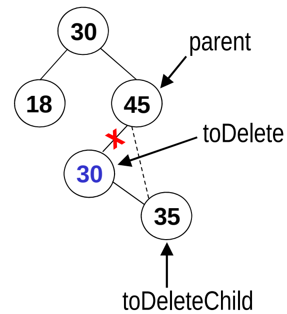
Implementing Case 3

```
private void deleteNode(Node toDelete, Node parent) {
    if (toDelete.left != null && toDelete.right != null) {
        // Find a replacement – and
        // the replacement's parent.
        Node replaceParent = toDelete;
        // Get the smallest item
        // in the right subtree.
        Node replace = toDelete.right;
        // What should go here?



        // Replace toDelete's key and data
        // with those of the replacement item.
        toDelete.key = replace.key;
        toDelete.data = replace.data;
        // Recursively delete the replacement
        // item's old node. It has at most one
        // child, so we don't have to
        // worry about infinite recursion.
        deleteNode(replace, replaceParent);
    } else {
        ...
}
```



Implementing Cases 1 and 2

```
private void deleteNode(Node toDelete, Node parent) {
    if (toDelete.left != null && toDelete.right != null) {
        ...
    } else {
        Node toDeleteChild;
        if (toDelete.left != null)
            toDeleteChild = toDelete.left;
        else
            toDeleteChild = toDelete.right;
        // Note: in case 1, toDeleteChild
        // will have a value of null.

        if (toDelete == root)
            root = toDeleteChild;
        else if (toDelete.key < parent.key)
            parent.left = toDeleteChild;
        else
            parent.right = toDeleteChild;
    }
}
```

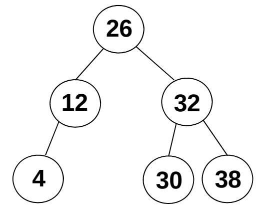


Efficiency of a Binary Search Tree

- The three key operations (search, insert, and delete) all have the same time complexity.
  - insert and delete both involve a search followed by a constant number of additional operations
- Time complexity of searching a binary search tree:
  - best case: $O(1)$
  - worst case: $O(h)$, where h is the height of the tree
  - average case: $O(h)$
- What is the height of a tree containing n items?
  - it depends! why?

Balanced Trees

- A tree is *balanced* if, for each node, the node's subtrees have the same height or have heights that differ by 1.
- For a balanced tree with n nodes:
  - height = $O(\log_2 n)$.
  - gives a worst-case time complexity that is logarithmic ($O(\log_2 n)$)
    - the best worst-case time complexity for a binary tree



What If the Tree Isn't Balanced?

- Extreme case: the tree is equivalent to a linked list
  - height = n - 1
  - worst-case time complexity = $O(n)$
- We'll look next at search-tree variants that take special measures to ensure balance.