Balanced Search Trees

Computer Science S-111
Harvard University

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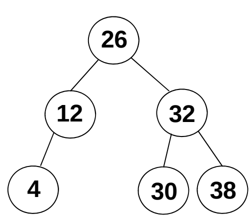
Review: Balanced Trees

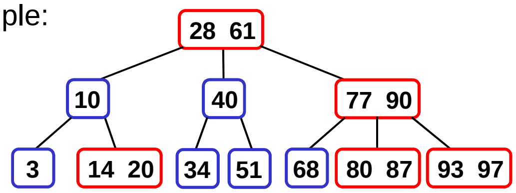
- A tree is *balanced* if, for each node, the node's subtrees have the same height or have heights that differ by 1.
- For a balanced tree with n nodes:
  - height = $O(\log_2 n)$.
  - gives a worst-case time complexity that is logarithmic ($O(\log_2 n)$)
    - the best worst-case time complexity for a binary search tree

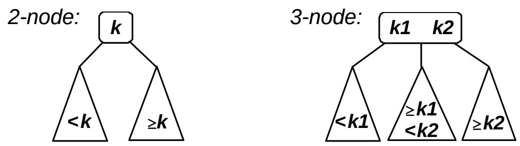


- With a binary search tree, there's no way to ensure that the tree remains balanced.
  - can degenerate to $O(n)$ time

2-3 Trees

- A 2-3 tree is a balanced tree in which:
  - *all* nodes have equal-height subtrees (perfect balance)
  - each node is either
    - a **2-node**, which contains one data item and 0 or 2 children
    - a **3-node**, which contains two data items and 0 or 3 children
  - the keys form a search tree
- Example:

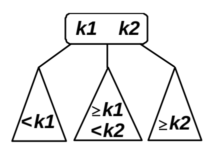




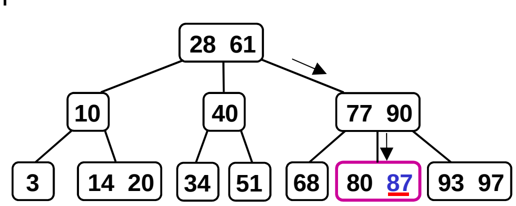
Search in 2-3 Trees

- Algorithm for searching for an item with a key *k*:

  if *k* == one of the root node's keys, you're done
  else if *k* < the root node's first key
      search the left subtree
  else if the root is a 3-node and k < its second key
      search the middle subtree
  else
      search the right subtree



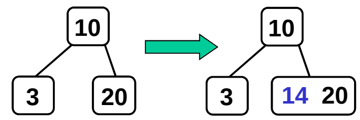
- Example: search for 87



Insertion in 2-3 Trees

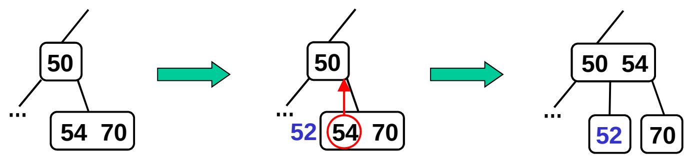
- Algorithm for inserting an item with a key *k*:

  search for *k,* but don't stop until you hit a leaf node
  let L be the leaf node at the end of the search
  if L is a 2-node
  add *k* to L, making it a 3-node



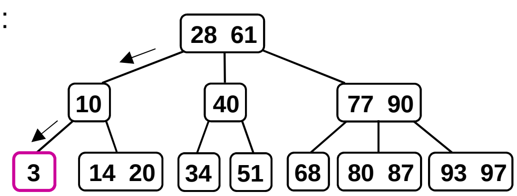
  else if L is a 3-node
  split L into two 2-nodes containing the items with the smallest and largest of: *k,* L's 1st key, L's 2nd key
  the middle item is "sent up" and inserted in L's parent

  example: add 52

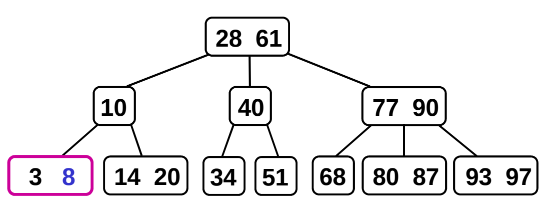


Example 1: Insert 8

- Search for 8:

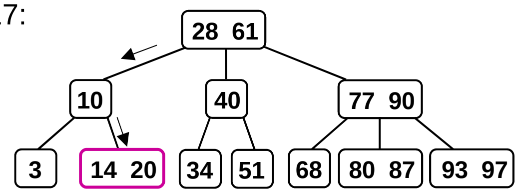


- Add 8 to the leaf node, making it a 3-node:

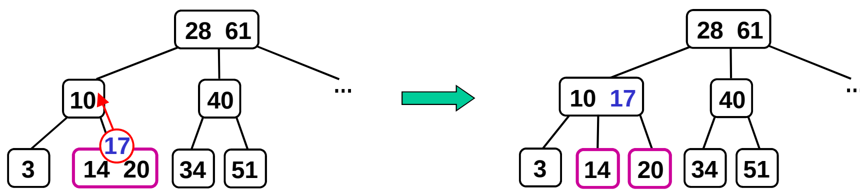


Example 2: Insert 17

- Search for 17:

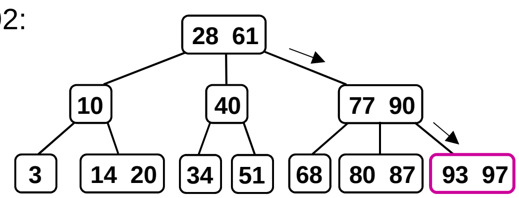


- Split the leaf node, and send up the middle of 14, 17, 20 and insert it the leaf node's parent:

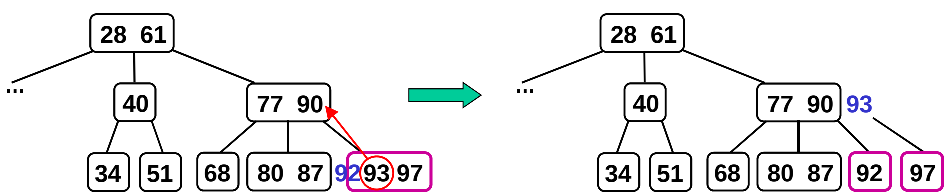


Example 3: Insert 92

- Search for 92:



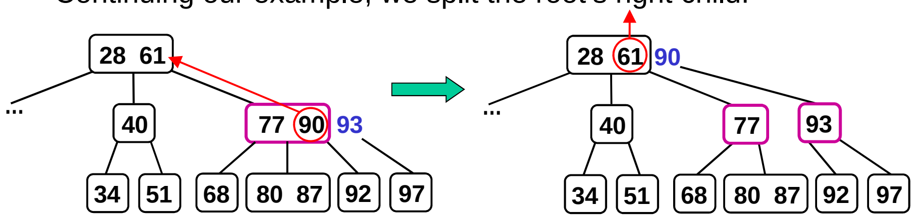
- Split the leaf node, and send up the middle of 92, 93, 97 and insert it the leaf node's parent:



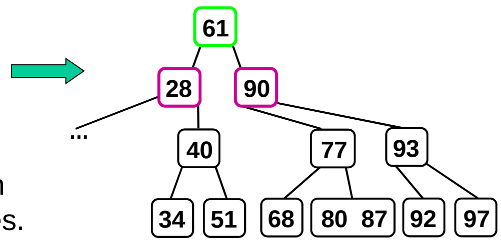
- In this case, the leaf node's parent is also a 3-node, so we need to split is as well…

Splitting the Root Node

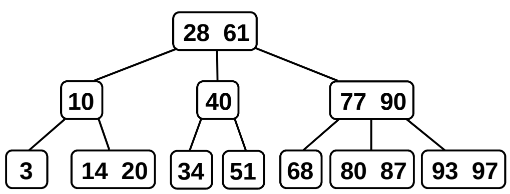
- If an item propagates up to the root node, and the root is a 3-node, we split the root node and create a new, 2-node root containing the middle of the three items.
- Continuing our example, we split the root's right child:



- Then we split the root, which increases the tree's height by 1, but the tree is still balanced.
- This is only case in which the tree's height increases.



Efficiency of 2-3 Trees



- A 2-3 tree containing n items has a height <= $\log_2 n$.
- Thus, search and insertion are both $O(\log n)$.
  - a search visits at most $\log_2 n$ nodes
  - an insertion begins with a search; in the worst case, it goes all the way back up to the root performing splits, so it visits at most $2\log_2 n$ nodes
- Deletion is tricky – you may need to coalesce nodes! However, it also has a time complexity of $O(\log n)$.
- Thus, we can use 2-3 trees for a $O(\log n)$-time data dictionary.

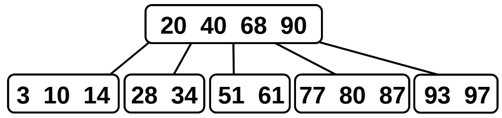
External Storage

- The balanced trees that we've covered don't work well if you want to store the data dictionary externally – i.e., on disk.
- Key facts about disks:
  - data is transferred to and from disk in units called *blocks,* which are typically 4 or 8 KB in size
  - disk accesses are slow!
    - reading a block takes ~10 milliseconds (10^{-3} sec)
    - vs. reading from memory, which takes ~10 nanoseconds
    - in 10 ms, a modern CPU can perform millions of operations!

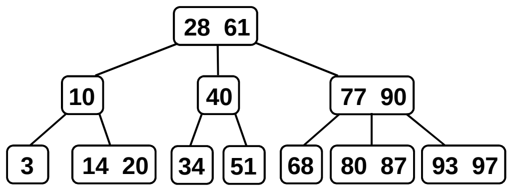
B-Trees

- A B-tree of order *m* is a tree in which each node has:
  - at most 2*m* entries (and, for internal nodes, 2*m* + 1 children)
  - at least *m* entries (and, for internal nodes, *m* + 1 children)
  - exception: the root node may have as few as 1 entry
  - a 2-3 tree is essentially a B-tree of order 1
- To minimize the number of disk accesses, we make *m* as large as possible.
  - each disk read brings in more items
  - the tree will be shorter (each level has more nodes), and thus searching for an item requires fewer disk reads
- A large value of *m* doesn’t make sense for a memory-only tree, because it leads to many key comparisons per node.
- These comparisons are less expensive than accessing the disk, so large values of *m* make sense for on-disk trees.

Example: a B-Tree of Order 2



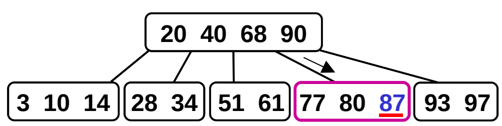
- Order 2: at most 4 data items per node (and at most 5 children)
- The above tree holds the same keys as one of our earlier 2-3 trees, which is shown again below:



- We used the same order of insertion to create both trees:
  51, 3, 40, 77, 20, 10, 34, 28, 61, 80, 68, 93, 90, 97, 87, 14
- For extra practice, see if you can reproduce the trees!

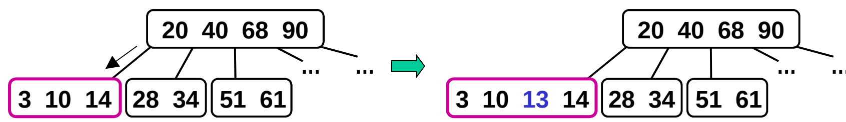
Search in B-Trees

- Similar to search in a 2-3 tree.
- Example: search for 87



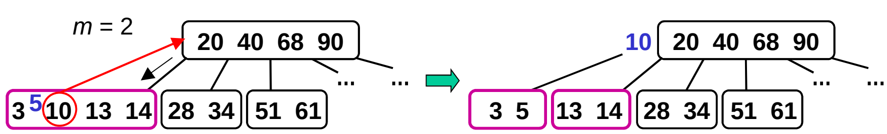
Insertion in B-Trees

- Similar to insertion in a 2-3 tree:
  search for the key until you reach a leaf node
  if a leaf node has fewer than 2*m* items, add the item to the leaf node
  else split the node, dividing up the 2*m* + 1 items:
    the smallest *m* items remain in the original node
    the largest *m* items go in a new node
    send the middle entry up and insert it (and a pointer to the new node) in the parent
- Example of an insertion without a split: insert 13

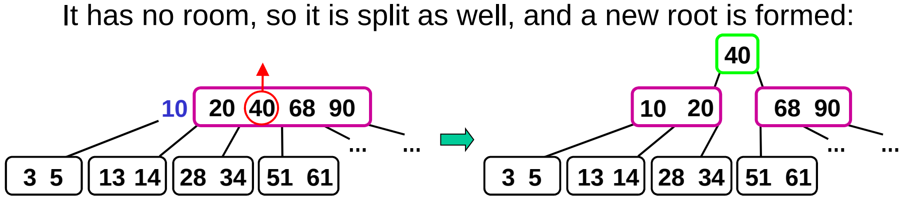


Splits in B-Trees

- Insert 5 into the result of the previous insertion:

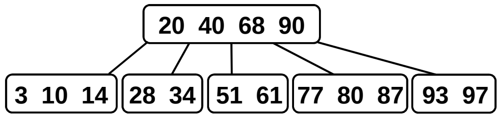


- The middle item (the 10) was sent up to the root.
  It has no room, so it is split as well, and a new root is formed:



- Splitting the root increases the tree's height by 1, but the tree is still balanced. This is only way that the tree's height increases.
- When an internal node is split, its 2*m* + 2 pointers are split evenly between the original node and the new node.

Analysis of B-Trees



- All internal nodes have at least *m* children (actually, at least *m*+1).
- Thus, a B-tree with n items has a height <= $\log_m n$, and search and insertion are both $O(\log_m n)$.
- As with 2-3 trees, deletion is tricky, but it's still logarithmic.

Search Trees: Conclusions

- Binary search trees can be $O(\log n)$, but they can degenerate to $O(n)$ running time if they are out of balance.
- 2-3 trees and B-trees are *balanced* search trees that guarantee $O(\log n)$ performance.
- When data is stored on disk, the most important performance consideration is reducing the number of disk accesses.
- B-trees offer improved performance for on-disk data dictionaries.