ML HW10

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Problem 1: PF-ODE Derivation

1. Consider a forward SDE

$$\mathrm{d}x_t = f(x_t, t)\,\mathrm{d}t + g(x_t, t)\,\mathrm{d}W_t,$$

show that the corresponding probability flow ODE is written as

$$\mathrm{d}x_t = \left[f(x_t, t) - \frac{\partial}{\partial x} g^2(x_t, t) - \frac{g^2(x_t, t)}{2} \frac{\partial}{\partial x} \log p(x_t, t)\right] \mathrm{d}t.$$

Solution:

Before demonstration, we need some setup and lemmas:

Definition 1: Variation

For processes ϕ, $\psi : \mathbb{R} \to \mathbb{R}$, we define the d-variation of ϕ on $[0, T]$ by:

$$[\phi]_d := \lim_{\|\Pi\| \to 0} \sum_{i=0}^{n-1} |\phi(t_{i+1}) - \phi(t_i)|^d$$

And co-variation is defined as:

$$[\phi, \psi]_d = \sum_{i=0}^{n-1} |\phi(t_{1+1}) - \phi(t_i)| \cdot |\psi(t_{i+1}) - \psi(t_i)|,$$

where for all $n \in \mathbb{Z}^+$, $0 = t_0 \leq t_1 \leq \cdots \leq t_n = T$, Π is the partition.

Corollary 1

A continuous process has finite variation must have zero as its quadratic variation.

Proof.

$$0 \leq [\phi]_2 = \lim_{\|\Pi\| \to 0} \sum_{i=0}^{n-1} |\phi(t_{i+1}) - \phi(t_i)|^2 \leq \limsup_{\|\Pi\| \to 0} |\phi(t_{i+1}) - \phi(t_i)| \cdot [\phi]_1 = 0 \cdot L = 0$$

Hence $[\phi]_2 = 0$ is proved. □

Corollary 2

A continuous process and a finite variation process have zero as their co-variation.

Proof. Trivial, analogous to the proof to corollary 1. □

Now we consider variations on two main process:

- **Identity map:** $\mathbf{id} : [0,T] \to \mathbb{R};\ t \mapsto t$

$$[\mathbf{id}]_1 = \lim_{\|\Pi\|\to 0} \sum_{i=0}^{n-1} |\mathbf{id}(t_{i+1}) - \mathbf{id}(t_i)| = \lim_{\|\Pi\|\to 0} \sum_{i=0}^{n-1} |t_{i+1} - t_i| = \lim_{\|\Pi\|\to 0} \sum_{i=0}^{n-1} (t_{i+1} - t_i) = T$$

  Hence

$$[\mathbf{id}]_1 = T$$

- **Wiener process:** W_t on $[0,T]$.
  For an equal length partition Π on $[0,T]$, we have

$$[W_t]_1 = \lim_{\|\Pi\|\to 0} \sum_{i=0}^{n-1} |W_{t_{i+1}} - W_{t_i}| = \lim_{\|\Pi\|\to 0} \sum_{i=0}^{n-1} \left| \sqrt{\frac{T}{n}} \varepsilon \right|, \qquad \varepsilon \sim \mathcal{N}(0,1)$$

  Since $|\varepsilon|$ is a standard half-normal distribution, meaning that:

$$\mathbb{E}|\varepsilon| = 2\int_0^{+\infty} \frac{1}{\sqrt{2\pi}} z e^{-\frac{1}{2}z^2} \mathrm{d}z = -\frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\Big|_0^{+\infty} = \sqrt{\frac{2}{\pi}}$$

  By taking expected value on $[W_t]$, we obtain:

$$\mathbb{E}([W_t]) = \lim_{\|\Pi\|\to 0} \sum_{i=0}^{n-1} \sqrt{\frac{T}{n}} \mathbb{E}\,|\varepsilon| = \lim_{n\to\infty} \left(n \cdot \sqrt{\frac{T}{n}} \cdot \sqrt{\frac{2}{\pi}} \right) = \sqrt{\frac{2T}{\pi}} \lim_{n\to\infty} \sqrt{n} = +\infty$$

  Hence we conclude that

$$[W_t]_1 = +\infty$$

Also, by corollary 1 and 2, we have:

$$[\mathbf{id}]_2 = 0 \quad \text{and} \quad [\mathbf{id}, W_t] = 0$$

Lemma 1: Variation on SDE

For time t and the Browning motion/Wiener process W_t, we have:

$$(\mathrm{d}t)^2 = 0; \qquad \mathrm{d}t\mathrm{d}W_t = 0; \qquad (\mathrm{d}t)^m(\mathrm{d}W_t)^n = 0 \quad \text{for } m,n \in \mathbb{Z}^+$$

Proof. By definition,

$$(\mathrm{d}t)^2 \equiv \mathrm{d}[\mathbf{id}, \mathbf{id}] = \mathrm{d}[\mathbf{id}]_2 = \mathrm{d}(0) = 0$$

and

$$\mathrm{d}t\mathrm{d}W_t \equiv \mathrm{d}[\mathbf{id}, W_t] = \mathrm{d}(0) = 0.$$

Also apply double induction, we can get:

$$(\mathrm{d}t)^m(\mathrm{d}W_t)^n = 0 \quad \text{for } m,n \in \mathbb{Z}^+$$

□

Lemma 2: Quadratic Variation of Wiener Process

For Wiener Process W_t defined on $[0, T]$, we have:

$$(\mathrm{d}W_t)^2 = \mathrm{d}t$$

Proof. Recall that for $0 \leq s \leq t \leq T$, we have:

$$W_t - W_s \sim \mathcal{N}(0, t-s) \qquad \text{or} \qquad W_t - W_s \sim \sqrt{t-s}\varepsilon, \quad \varepsilon \sim \mathcal{N}(0,1)$$

So, we take the expected value on quadratic variation on W_t, we get:

$$\mathbb{E}([W_t]_2) = \lim_{\|\Pi\|\to 0} \sum_{i=0}^{n-1} \mathbb{E}[(\sqrt{t_{i+1}-t_i}\varepsilon)^2] = \lim_{\|\Pi\|\to 0} \sum_{i=0}^{n-1} (t_{i+1}-t_i)\mathbb{E}(\varepsilon^2) \tag{1}$$

Also, by definition of variance:

$$1 = \mathrm{Var}(\varepsilon) = \mathbb{E}(\varepsilon^2) - [\mathbb{E}(\varepsilon)]^2 = \mathbb{E}(\varepsilon^2) - 0^2 \implies \mathbb{E}(\varepsilon^2) = 1 \tag{2}$$

With equation (1) and (2), we obtain:

$$\mathbb{E}([W_t]_2) = \lim_{\|\Pi\|\to 0} \sum_{i=0}^{n-1} (t_{i+1}-t_i) = T$$

Consider the variance of the quadratic variation on W_t, we have:

$$\mathrm{Var}([W_t]_2) = \lim_{\|\Pi\|\to 0} \sum_{i=0}^{n-1} \mathrm{Var}((t_{i+1}-t_i)\varepsilon^2) = \lim_{\|\Pi\|\to 0} \sum_{i=0}^{n-1} (t_{i+1}-t_i)^2 \mathrm{Var}(\varepsilon^2)$$

We will use 4th moment and 2nd moment to solve for $\mathrm{Var}(\varepsilon^2)$:

$$\mathrm{Var}(\varepsilon^2) = \mathbb{E}(\varepsilon^4) - \big(\mathbb{E}(\varepsilon^2)\big)^2 = 3 - 1^2 = 2$$

Therefore

$$\mathrm{Var}([W_t]_2) = 2 \lim_{\|\Pi\|\to 0} \sum_{i=0}^{n-1} (t_{i+1}-t_i)^2 = 2[\mathbf{id}]_2 = 0$$

Zero variance implies the random variable is a constant, hence

$$(\mathrm{d}W_t)^2 = \mathrm{d}[W_t]_2 = \mathrm{d}t$$

$\square$

The two lemmas have shown the calculation law in Itô's calculus, one difference is that the general chain rule **does not hold** in Itô's calculus.

$$\mathrm{d}\phi(x,t) \neq \frac{\partial \phi}{\partial x}\mathrm{d}x + \frac{\partial \phi}{\partial t}\mathrm{d}t$$

Theorem 1: Itô's Lemma

for a 1-dimension Itô process $\phi(x,t) \in \mathcal{C}^2(\mathbb{R}\times\mathbb{R})$, we have:

$$\mathrm{d}\phi = \frac{\partial \phi}{\partial t}\mathrm{d}t + \frac{\partial \phi}{\partial x}\mathrm{d}x + \frac{1}{2}\frac{\partial^2 \phi}{\partial x^2}(\mathrm{d}x)^2$$

More generally, for an Itô process $\Phi(X,t) \in \mathcal{C}^2(\mathcal{X}\times\mathbb{R})$, $\mathcal{X} \subseteq \mathbb{R}^d$, we have:

$$\mathrm{d}\Phi = \frac{\partial \Phi}{\partial t}\,\mathrm{d}t + \nabla\Phi\cdot\mathrm{d}X + \frac{1}{2}\sum_{i=1}^{d}\sum_{k=1}^{d}\frac{\partial^2\Phi}{\partial X^i\partial X^k}\,\mathrm{d}X^i\mathrm{d}X^k$$

Proof. By **Taylor theorem** and lemma 1 and 2,

$$\begin{aligned}\mathrm{d}\phi &= \frac{\partial \phi}{\partial x}\mathrm{d}x + \frac{\partial \phi}{\partial t}\mathrm{d}t + \frac{1}{2}\frac{\partial^2\phi}{\partial x^2}(\mathrm{d}x)^2 + \frac{1}{2}\frac{\partial^2\phi}{\partial t^2}(\mathrm{d}t)^2 + \frac{\partial^2\phi}{\partial t\partial x}\mathrm{d}t\mathrm{d}x + \cdots \\ &= \frac{\partial \phi}{\partial x}\mathrm{d}x + \frac{\partial \phi}{\partial t}\mathrm{d}t + \frac{1}{2}\frac{\partial^2\phi}{\partial x^2}(\mathrm{d}x)^2 + 0\end{aligned}$$

□

Consider $p(x,t) \in \mathcal{C}^2(\mathbb{R}\times\mathbb{R})$, and let

$$\mathrm{d}x_t = f(x,t)\mathrm{d}t + g(x,t)\mathrm{d}W_t$$

We derive the **Fokker Planck equation** first. For **arbitrary** smooth function with compact support $V(x) \in \mathcal{C}_c^\infty(\mathbb{R})$, then by Itô's lemma,

$$\begin{aligned}\mathrm{d}V(x) &= \frac{\partial V}{\partial t}\mathrm{d}t + \frac{\partial V}{\partial x}\mathrm{d}x + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(\mathrm{d}x)^2 = 0 + \frac{\partial V}{\partial x}\left(f(x,t)\mathrm{d}t + g(x,t)\mathrm{d}W_t\right) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}g^2(x,t)\mathrm{d}t \\ &= \left(\frac{\partial V}{\partial x}f(x,t) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}g^2(x,t)\right)\mathrm{d}t + \frac{\partial V}{\partial x}g(x,t)\mathrm{d}W_t\end{aligned}$$

Hence

$$\mathbb{E}(V(x)) = \mathbb{E}\left(\int \mathrm{d}V\right) = \mathbb{E}\left(\int \frac{\partial V}{\partial x}f(x,t) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}g^2(x,t)\mathrm{d}t\right) + \cancelto{0}{\mathbb{E}\left(\int \cdots \mathrm{d}W_t\right)}$$

Then take the derivative with respect to t,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbb{E}(V(x))\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbb{E}\left(\int \frac{\partial V}{\partial x}f(x,t) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}g^2(x,t)\mathrm{d}t\right)\right) \\ &= \mathbb{E}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\int \frac{\partial V}{\partial x}f(x,t) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}g^2(x,t)\mathrm{d}t\right)\right) \\ &= \mathbb{E}\left(\frac{\partial V}{\partial x}f(x,t) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}g^2(x,t)\right) \\ &= \int_{\mathbb{R}}\left(\frac{\partial V}{\partial x}f(x,t) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}g^2(x,t)\right)p(x,t)\mathrm{d}x\end{aligned}$$

Recall the original definition of the expected value, which is:

$$\mathbb{E}(V(x)) = \int_{\mathbb{R}} V(x)p(x,t)\mathrm{d}x$$

Taking the derivative with respect to t in this expression yields:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbb{E}(V(x))\right) = \int_{\mathbb{R}} V(x)\frac{\partial p(x,t)}{\partial t}\mathrm{d}x =$$

Two expressions must be equivalent, so:

$$\begin{aligned}\int_{\mathbb{R}} V(x)\frac{\partial p(x,t)}{\partial t}\mathrm{d}x &= \int_{\mathbb{R}} \left(\frac{\partial V}{\partial x} f(x,t) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2} g^2(x,t)\right) p(x,t)\mathrm{d}x \\ &= \underbrace{\int_{\mathbb{R}} \frac{\partial V}{\partial x} f(x,t)p(x,t)\mathrm{d}x}_{\text{(i)}} + \frac{1}{2}\underbrace{\int_{\mathbb{R}} \frac{\partial^2 V}{\partial x^2} g^2(x,t)p(x,t)\mathrm{d}x}_{\text{(ii)}}\end{aligned}$$

Remember that $V(x)$ has compact support, so

$$\lim_{x\to\pm\infty} V(x) = 0; \qquad \lim_{x\to\pm\infty} \frac{\partial V}{\partial x} = 0$$

We aim to factor out all the $V(x)$, so we do the integration by parts on (i) and (ii).

(i)

$$\int_{\mathbb{R}} \frac{\partial V}{\partial x} f(x,t)p(x,t)\mathrm{d}x = \cancelto{0}{V(x)f(x,t)p(x,t)\Big|_{-\infty}^{+\infty}} - \int_{\mathbb{R}} V(x)\frac{\partial(f(x,t)p(x,t))}{\partial x}\mathrm{d}x$$

(ii)

$$\begin{aligned}\int_{\mathbb{R}} \frac{\partial^2 V}{\partial x^2} g^2(x,t)p(x,t)\mathrm{d}x &= \cancelto{0}{\frac{\partial V}{\partial x} g^2(x,t)p(x,t)\Big|_{-\infty}^{+\infty}} - \int_{\mathbb{R}} \frac{\partial V}{\partial x}\frac{\partial(g^2(x,t)p(x,t))}{\partial x}\mathrm{d}x \\ &= \cancelto{0}{-V(x)\frac{\partial(g^2(x,t)p(x,t))}{\partial x}\Big|_{-\infty}^{\infty}} + \int_{\mathbb{R}} V\frac{\partial^2(g(x,t)p(x,t))}{\partial x^2}\mathrm{d}x\end{aligned}$$

Bring back the result in (i) and (ii),

$$\int_{\mathbb{R}} V(x)\frac{\partial p(x,t)}{\partial t}\mathrm{d}x = \int_{\mathbb{R}} V(x)\left(-\frac{\partial(f(x,t)p(x,t))}{\partial x} + \frac{1}{2}\frac{\partial^2(g^2(x,t)p(x,t))}{\partial x^2}\right)\mathrm{d}x$$

Since our function $V(x)$ is **arbitrary**, so

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial(f(x,t)p(x,t))}{\partial x} + \frac{1}{2}\frac{\partial^2(g^2(x,t)p(x,t))}{\partial x^2}, \tag{1}$$

whcih is the **Fokker Planck equation**.

Theorem 2: Continuity Equation (Physics)

For the density $\rho(x,t)$ of a conserved quantity at position time x and t, $J(x,t)$ be the flux, we have the following relation:

$$\frac{\partial \rho(x,t)}{\partial t} = -\frac{\partial J(x,t)}{\partial x}$$

An intuitive example to the theorem is

> "Imagine a single line of people moving into a stadium through a narrow turnstile."

- State Space (x): The position along the line.
- Density (ρ): The number of people per meter in the line at position x.
- Flux (J): The number of people passing a point x per second.

Does our $p(x,t)$ fit in the equation?

$$\textbf{Yes}\text{, because } p(x,t) \text{ is conservative } \left(\int_{\mathbb{R}} p(x,t)\mathrm{d}x = 1 \quad \forall t\right)$$

So let's assume some $u(x,t)$ *s.t.* $u(x,t)p(x,t) = J(x,t)$ and satisfy the continuity equation:

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial J(x,t)}{\partial x} = -\frac{\partial (u(x,t)p(x,t))}{\partial x} \tag{2}$$

We bring equation (2) to equation (1) to solve for $u(x,t)$:

$$-\frac{\partial (u(x,t)p(x,t))}{\partial x} = \frac{\partial p(x,t)}{\partial t} = -\frac{\partial (f(x,t)p(x,t))}{\partial x} + \frac{\partial}{\partial x}\left(\frac{1}{2}\frac{\partial (g^2(x,t)p(x,t))}{\partial x}\right) \tag{3}$$

Integrate the equation (3) with respect to x, we get:

$$\begin{aligned}
-u(x,t)p(x,t) &= -f(x,t)p(x,t) + \frac{1}{2}\frac{\partial (g^2(x,t)p(x,t))}{\partial x} \\
&= -f(x,t)p(x,t) + \frac{1}{2}\frac{\partial g^2(x,t)}{\partial x}p(x,t) + \frac{1}{2}g^2(x,t)\frac{\partial p(x,t)}{\partial x} \\
&= -f(x,t)p(x,t) + \frac{1}{2}\frac{\partial g^2(x,t)}{\partial x}p(x,t) + \frac{1}{2}g^2(x,t)\frac{\partial \log p(x,t)}{\partial x}p(x,t)
\end{aligned} \tag{4}$$

We cancel out $p(x,t)$ at both sides, getting:

$$u(x,t) = f(x,t) - \frac{1}{2}\frac{\partial g^2(x,t)}{\partial x} - \frac{g^2(x,t)}{2}\frac{\partial \log p(x,t)}{\partial x}$$

Look back the definition of $u(x,t)$ from the equation (2), we know that $u(x,t)$ stands for the speed of flow, which can also be defined as $\frac{\mathrm{d}x_t}{\mathrm{d}t}$, so

$$\mathrm{d}x_t = u(x_t,t)\mathrm{d}t = \left(f(x_t,t) - \frac{1}{2}\frac{\partial g^2(x_t,t)}{\partial x} - \frac{g^2(x_t,t)}{2}\frac{\partial \log p(x_t,t)}{\partial x}\right)\mathrm{d}t,$$

which completes the demonstration.