

# Boundedness and exponential stability for nonautonomous RCNNs with distributed delays

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## Abstract

Some sufficient conditions for the ultimate boundedness and global exponential stability of a class of nonautonomous reaction–diffusion cellular neural networks (RCNNs) with distributed delays are obtained by means of the Lyapunov functional method. Without assuming that the activation functions  $f_{ijl}(\cdot)$  are bounded, the results extend and improve the earlier publications. © 2007 Elsevier Ltd. All rights reserved.

**Keywords:** Cellular neural networks; Boundedness; Global exponential stability; Reaction–diffusion; Distributed delays; Lyapunov functional

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## 1. Introduction

Cellular neural networks (CNNs) first introduced by Chua et al. [1,2], have found many important applications in motion-related areas such as classification of patterns, processing of moving images and recognition of moving objects [3,4]. In [5–18], under the assumption that response functions are bounded and satisfy the Lipschitz conditions, by constructing suitable Lyapunov functionals, the authors established the sufficient conditions for global exponential stability of the equilibrium of CNNs and the existence of periodic solutions. However, we see, for the nonautonomous CNNs with delay, up until now, the study works are very few [19,20].

However, neural networks usually have a spatial extent due to the presence of a multitude of parallel pathways with a variety of axon sizes and lengths. Thus there will be a distribution of conduction velocities along these pathways and a distribution of propagation delays. In these circumstances, the signal propagation is not instantaneous and cannot be modelled with discrete delays and a more appropriate way is to incorporate continuously distributed delays.

Moreover, strictly speaking, diffusion effect cannot be avoided in the neural networks model when electrons are moving in an asymmetric electromagnetic field, so we must consider the space is varying with the time. Refs. [19,21,22] have considered the stability of neural networks with diffusion terms, which are expressed by partial differential equations. It is also common to consider the diffusion effect in biological systems (such as immigration) [23–25]. To the best of our knowledge, few authors have considered the boundedness and the global exponential stability for the nonautonomous RCNNs with distributed delays.

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In this paper, we will derive some criteria of the ultimate boundedness and global exponential stability for the RCNNs by constructing suitable Lyapunov functional and using inequality techniques. The activation functions are only supposed to be globally Lipschitz continuous, similar to the conditions  $A$  in [26], which are more general than the usual bounded, differentiable or non-decreasing ones as in [11–15]; also we do not assume that the considered model has any equilibriums. We will see the obtained results improve and extend the main results on the stability for the neural networks with distributed delays given by researchers in [22,24,27,28].

## 2. System description

In this paper, we will consider the following nonautonomous RCNNs with distributed delays:

$$\begin{aligned} \frac{\partial u_i(t, x)}{\partial t} = & \sum_{k=1}^p \frac{\partial}{\partial x_k} \left( D_{ik} \frac{\partial u_i(t, x)}{\partial x_k} \right) - a_i(t) u_i(t, x) \\ & + \sum_{l=1}^m \sum_{j=1}^n b_{ijl}(t) \int_{-\infty}^t K_{ijl}(t-s) f_{jl}(u_j(s, x)) ds + I_i(t) \end{aligned} \quad (1)$$

for  $i \in \{1, 2, \dots, n\}$ ,  $t > 0$ , where  $x = (x_1, x_2, \dots, x_p)^T \in \Omega \subset R^p$ ,  $\Omega$  is a bounded compact set with smooth boundary  $\partial\Omega$  and  $\text{mes}\Omega > 0$  in space  $R^p$ ;  $u_i(t, x)$  is the state of the  $i$ th unit at time  $t$ ; the integer  $n$  corresponds to the number of units in a neural network;  $f_{jl}(\cdot)$  denotes the output of the  $j$ th unit at time  $t$ ;  $a_i(t) > 0$  represents the rate with which the  $i$ th neuron will reset its potential to the resting state in isolation when disconnected from the network and external inputs at time  $t$ ;  $b_{ijl}(t)$  denotes the strength of the  $j$ th unit on the  $i$ th unit at time  $t$ ;  $I_i(t)$  denotes the external bias on the  $i$ th unit at time  $t$ . Smooth functions  $D_{ik} = D_{ik}(t, x, u) \geq 0$  correspond to the transmission diffusion operators along the  $i$ th neurons.

The delay kernel  $K_{ijl}(\cdot)$  ( $i, j = 1, 2, \dots, n, l = 1, 2, \dots, m$ ) is assumed to satisfy the following conditions simultaneously:

- (i)  $K_{ijl} : [0, \infty) \rightarrow [0, \infty)$ ;
- (ii)  $K_{ijl}$  is bounded, i.e.  $K_{ijl} \leq \mathcal{M}$ , and continuous on  $[0, \infty)$ ;
- (iii)  $\int_0^\infty K_{ijl}(s) ds = 1$ ;
- (iv) there exists a positive number  $\varepsilon$  such that  $\int_0^\infty K_{ijl}(s) e^{\varepsilon s} ds < \infty$ .

The literature [29] has given some examples to meet the above conditions.

To obtain our results, we first give the following assumptions:

- (H<sub>1</sub>)  $a_i(t)$ ,  $b_{ijl}(t)$  and  $I_i(t)$  ( $i, j = 1, 2, \dots, n, l = 1, 2, \dots, m$ ) are bounded and continuous functions defined on  $t \in R^+ = [0, +\infty)$ .
- (H<sub>2</sub>) The neurons activation functions  $f_{jl}(\cdot)$  ( $j = 1, 2, \dots, n, l = 1, 2, \dots, m$ ) satisfy the Lipschitz-continuous, that is, there exist constants  $L_{jl} > 0$  such that

$$|f_{jl}(\xi_1) - f_{jl}(\xi_2)| \leq L_{jl} |\xi_1 - \xi_2|$$

for all  $\xi_1, \xi_2 \in R$ .

The boundary conditions are given by

$$\frac{\partial u_i}{\partial n} := \left( \frac{\partial u_i}{\partial x_1}, \frac{\partial u_i}{\partial x_2}, \dots, \frac{\partial u_i}{\partial x_p} \right)^T = 0, \quad i = 1, 2, \dots, n. \quad (2)$$

We denoted by  $C^n(-\infty, 0]$  the Banach space of continuous functions  $\phi(\theta, x) = (\phi_1(\theta, x), \phi_2(\theta, x), \dots, \phi_n(\theta, x)) : (-\infty, 0] \rightarrow R^n$  with the norm  $\|\phi\|_2 = [\int_\Omega |\phi(\theta, x)|^2 dx]^{1/2}$ , where  $|\phi(\theta, x)| = (\sum_{i=1}^n \phi_i^2(\theta, x))^{1/2}$ . In this paper, we always assume that all solutions of system (1) satisfy the following initial conditions

$$u_i(s, x) = \phi_{ui}(s, x), \quad s \in (-\infty, 0], \quad i = 1, 2, \dots, n, \quad (3)$$

where  $\phi_{ui}(s, x) \in C^n(-\infty, 0]$  ( $i = 1, 2, \dots, n$ ). It is well known that by the fundamental theory of functional differential equations [30], system (1) has a unique solution satisfying the initial conditions (3).

**Definition 1.** System (1) is said to be ultimate boundedness, if there is a constant  $M > 0$  such that for any solution  $u(t, x) = (u_1(t, x), u_2(t, x), \dots, u_n(t, x))^T$  of system (1), one has

$$\lim_{t \rightarrow \infty} \int_{\Omega} \sup |u_i(t, x)| dx \leq M, \quad i = 1, 2, \dots, n. \quad (4)$$

**Definition 2.** System (1) is said to be globally exponentially stable, if there are constants  $\varepsilon > 0$  and  $\tilde{M} \geq 1$  such that for any two solutions  $u(t, x)$  and  $v(t, x)$  with the initial functions  $\phi \in C^n(-\infty, 0]$  and  $\varphi \in C^n(-\infty, 0]$ , respectively, one has

$$\|u - v\|_2^2 \leq \tilde{M} \|\phi - \varphi\|_2^2 e^{-\varepsilon t}, \quad \forall t \geq 0, \quad (5)$$

where

$$\|\phi - \varphi\|_2^2 = \sup_{s \in (-\infty, 0]} \|\phi - \varphi\|_2^2.$$

### 3. Boundedness

We first introduce the following result on the boundedness of solution for general functional partial differential equations. We consider the following equation

$$\frac{\partial u(t, x)}{\partial t} = F(t, u(t, x)), \quad (6)$$

where  $F(t, \varphi) : R^+ \times C^n(-\infty, 0] \rightarrow R^n$  is continuous with respect to  $(t, \varphi)$  and satisfied the local Lipschitz condition with respect to  $\varphi$ . Let the functions  $W_i(r) : R^+ \rightarrow R^+$  ( $i = 1, 2, 3, 4$ ) be continuous and increasing,  $W_i(0) = 0$  and  $W_i(r) \rightarrow \infty$  as  $r \rightarrow \infty$ . Let further the functional  $V(t, \varphi) : R^+ \times C^n(-\infty, 0] \rightarrow R$  is continuous with respect to  $(t, \varphi)$  and satisfied the local Lipschitz condition with respect to  $\varphi$ . We have the following lemma.

**Lemma 1.** If there are functional  $V(t)$  and functions  $W_i(u(t, x))$  ( $i = 1, 2, 3, 4$ ) such that

$$(a) \quad \int_{\Omega} W_1(|u(t, x)|) dx \leq V(t) \leq \int_{\Omega} W_2(|u(t, x)|) dx + \int_{\Omega} W_3 \left( \int_0^\infty \int_{t-s}^t W_4(|u(w)|) dw ds \right) dx$$

and

$$(b) \quad \left. \frac{dV}{dt} \right|_{(6)} \leq - \int_{\Omega} W_4(|u(t, x)|) dx + Z^*$$

for some constant  $Z^* > 0$ , then solutions of Eq. (6) are uniform ultimate bounded.

As we can see, the above result on partial differential equations extends the bounded lemma (see [30, Theorem 4.2.10]).

On the boundedness of solutions of the system (1), we have the following result.

**Theorem 1.** Assume that  $(H_1)$ – $(H_2)$  hold and the delay kernels  $K(\cdot)$  satisfy (i)–(iv). There are constants  $\lambda_i > 0$  and  $\rho > 0$  such that

$$\lambda_i a_i(t) - \frac{\lambda_i}{2} \sum_{l=1}^m \sum_{j=1}^n |b_{ijl}(t)|^{2\alpha_{ijl}} L_{jl}^{2\beta_{jl}} - \frac{1}{2} \sum_{l=1}^m \sum_{j=1}^n \lambda_j L_{il}^{2\beta'_{il}} |b_{jil}(t)|^{2\alpha'_{jil}} > \rho, \quad (7)$$

for all  $t \geq 0$  and  $i = 1, 2, \dots, n$ , where  $\alpha_{ijl} + \alpha'_{jil} = 1$ ,  $\beta_{il} + \beta'_{il} = 1$ ,  $i, j = 1, 2, \dots, n$ ,  $l = 1, 2, \dots, m$ , then all solutions of the system (1) are ultimately bounded.

**Proof.** Let  $u(t, x) = (u_1(t, x), u_2(t, x), \dots, u_n(t, x))^T$  be any solution of the system (1). And in (6), set

$$F(t, u(t, x)) = \sum_{k=1}^p \frac{\partial}{\partial x_k} \left( D_{ik} \frac{\partial u_i(t, x)}{\partial x_k} \right) - a_i(t) u_i(t, x) \\ + \sum_{l=1}^m \sum_{j=1}^n b_{ijl}(t) \int_{-\infty}^t K_{ijl}(t-s) f_{jl}(u_j(s, x)) ds + I_i(t).$$

Now consider the Lyapunov functional as follows:

$$V(t) = \frac{1}{2} \int_{\Omega} \sum_{i=1}^n \left[ \lambda_i u_i^2(t, x) + \lambda_i \sum_{l=1}^m \sum_{j=1}^n L_{jl}^{2\beta'_{jl}} \int_0^{\infty} K_{ijl}(s) \int_{t-s}^t |b_{ijl}(t)|^{2\alpha'_{ijl}} u_j^2(w, x) dw ds \right] dx. \quad (8)$$

Then we let

$$\underline{\lambda} = \min_{1 \leq i \leq n} \lambda_i, \quad \bar{\lambda} = \max_{1 \leq i \leq n} \lambda_i, \quad Q_{ijl} = |b_{ijl}(t)|^{2\alpha'_{ijl}},$$

and  $Q = \max\{Q_{jil} L_{il}^{2\beta'_{il}} : i, j = 1, 2, \dots, n, l = 1, 2, \dots, m\}$ , we further choose the functions  $W_i(r) : R^+ \rightarrow R^+$  ( $i = 1, 2, 3, 4$ ) as follows  $W_1(r) = \frac{1}{2} \underline{\lambda} r^2$ ,  $W_2(r) = \frac{1}{2} \bar{\lambda} r^2$ ,  $W_3(r) = \frac{\bar{\lambda} n m}{\rho} M Q r$ , and  $W_4(r) = \frac{1}{2} \rho r^2$ .

So from (8), we can directly obtain

$$V(t) \geq \int_{\Omega} \frac{\underline{\lambda}}{2} \sum_{i=1}^n u_i^2(t, x) dx = \int_{\Omega} \frac{\underline{\lambda}}{2} |u(t, x)|^2 dx = \int_{\Omega} W_1(|u(t, x)|) dx \quad (9)$$

and

$$V(t) \leq \int_{\Omega} \left[ \frac{\bar{\lambda}}{2} \sum_{i=1}^n u_i^2(t, x) + \frac{\bar{\lambda}}{2} M Q n m \sum_{i=1}^n \int_0^{\infty} \int_{t-s}^t u_i^2(w, x) dw ds \right] dx \\ \leq \int_{\Omega} \left[ W_2(|u(t)|) + W_3 \left( \int_0^{\infty} \int_{t-s}^t W_4(|u(w)|) dw ds \right) \right] dx. \quad (10)$$

Thus, it follows from (9) and (10) that

$$\int_{\Omega} W_1(|u(t, x)|) dx \leq V(t) \leq \int_{\Omega} W_2(|u(t, x)|) dx + \int_{\Omega} W_3 \left( \int_{t-\sigma}^t W_4(|u(s, x)|) ds \right) dx. \quad (11)$$

By calculating the derivative  $\frac{dV}{dt}$  of  $V(t)$  along the solutions of the system (6), it follows that

$$\frac{dV}{dt} = \frac{1}{2} \int_{\Omega} \sum_{i=1}^n \lambda_i \left[ 2u_i(t, x) \frac{\partial u_i(t, x)}{\partial t} + \lambda_i \sum_{l=1}^m \sum_{j=1}^n L_{jl}^{2\beta'_{jl}} \int_0^{\infty} K_{ijl}(s) |b_{ijl}(t)|^{2\alpha'_{ijl}} u_j^2(t, x) ds \right. \\ \left. - \lambda_i \sum_{l=1}^m \sum_{j=1}^n L_{jl}^{2\beta'_{jl}} \int_0^{\infty} K_{ijl}(s) |b_{ijl}(t)|^{2\alpha'_{ijl}} u_j^2(t-s, x) ds \right] dx \\ = \frac{1}{2} \int_{\Omega} \sum_{i=1}^n \lambda_i \left\{ 2u_i(t, x) \left[ \sum_{k=1}^p \frac{\partial}{\partial x_k} \left( D_{ik} \frac{\partial u_i(t, x)}{\partial x_k} \right) - a_i(t) u_i(t, x) \right. \right. \\ \left. \left. + \sum_{l=1}^m \sum_{j=1}^n b_{ijl}(t) \int_{-\infty}^t K_{ijl}(t-s) f_{jl}(u_j(s, x)) ds + I_i(t) \right] + \lambda_i \sum_{l=1}^m \sum_{j=1}^n L_{jl}^{2\beta'_{jl}} |b_{ijl}(t)|^{2\alpha'_{ijl}} u_j^2(t, x) \right. \\ \left. - \lambda_i \sum_{l=1}^m \sum_{j=1}^n L_{jl}^{2\beta'_{jl}} \int_{-\infty}^t K_{ijl}(t-s) |b_{ijl}(t)|^{2\alpha'_{ijl}} u_j^2(s, x) ds \right\} dx \\ \leq \frac{1}{2} \int_{\Omega} \sum_{i=1}^n \lambda_i \left\{ 2u_i(t, x) \sum_{k=1}^p \frac{\partial}{\partial x_k} \left( D_{ik} \frac{\partial u_i(t, x)}{\partial x_k} \right) - 2a_i(t) u_i^2(t, x) \right.$$

$$\begin{aligned}
& + \sum_{l=1}^m \sum_{j=1}^n 2|u_i(t, x)| |b_{ijl}(t)| L_{jl} \int_{-\infty}^t K_{ijl}(t-s) |u_j(s, x)| ds \\
& + 2|u_i(t, x)| \left( |I_i(t)| + \sum_{l=1}^m \sum_{j=1}^n |b_{ijl}(t)| |f_{jl}(0)| \right) + \lambda_i \sum_{l=1}^m \sum_{j=1}^n L_{jl}^{2\beta'_{jl}} |b_{ijl}(t)|^{2\alpha'_{ijl}} u_j^2(t, x) \\
& - \lambda_i \sum_{l=1}^m \sum_{j=1}^n L_{jl}^{2\beta'_{jl}} \int_{-\infty}^t K_{ijl}(t-s) |b_{ijl}(t)|^{2\alpha'_{ijl}} u_j^2(s, x) ds \Big\} dx.
\end{aligned} \tag{12}$$

From the boundary condition (2), we get

$$\begin{aligned}
\sum_{k=1}^p \int_{\Omega} u_i(t, x) \frac{\partial}{\partial x_k} \left( D_{ik} \frac{\partial u_i(t, x)}{\partial x_k} \right) dx &= \int_{\Omega} u_i(t, x) \nabla \cdot \left( D_{ik} \frac{\partial u_i(t, x)}{\partial x_k} \right)_{k=1}^p dx \\
&= \int_{\Omega} \nabla \cdot \left( u_i(t, x) D_{ik} \frac{\partial u_i(t, x)}{\partial x_k} \right)_{k=1}^p dx \\
&\quad - \int_{\Omega} \left( D_{ik} \frac{\partial u_i(t, x)}{\partial x_k} \right)_{k=1}^p \nabla \cdot u_i(t, x) dx \\
&= \int_{\partial\Omega} \left( u_i(t, x) D_{ik} \frac{\partial u_i(t, x)}{\partial x_k} \right)_{k=1}^p dx \\
&\quad - \sum_{k=1}^p \int_{\Omega} D_{ik} \left( \frac{\partial u_i(t, x)}{\partial x_k} \right)^2 dx \\
&= - \sum_{k=1}^p \int_{\Omega} D_{ik} \left( \frac{\partial u_i(t, x)}{\partial x_k} \right)^2 dx,
\end{aligned} \tag{13}$$

in which  $\nabla = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_p} \right)^T$  is the gradient operator, and

$$\left( D_{ik} \frac{\partial u_i(t, x)}{\partial x_k} \right)_{k=1}^p = \left( D_{i1} \frac{\partial u_i(t, x)}{\partial x_1}, D_{i2} \frac{\partial u_i(t, x)}{\partial x_2}, \dots, D_{ip} \frac{\partial u_i(t, x)}{\partial x_p} \right)^T.$$

In addition, we have

$$\begin{aligned}
& 2|u_i(t, x)| |b_{ijl}(t)| L_{jl} \int_{-\infty}^t K_{ijl}(t-s) |u_j(s, x)| ds \\
&= 2 \int_{-\infty}^t K_{ijl}(t-s) \left( |b_{ijl}(t)|^{\alpha'_{ijl}} L_{jl}^{\beta_{jl}} |u_i(t, x)| \right) \left( |b_{ijl}(t)|^{\alpha'_{ijl}} L_{jl}^{\beta'_{jl}} |u_j(s, x)| \right) ds \\
&\leq |b_{ijl}(t)|^{2\alpha'_{ijl}} L_{jl}^{2\beta_{jl}} u_i^2(t, x) + \int_{-\infty}^t K_{ijl}(t-s) |b_{ijl}(t)|^{2\alpha'_{ijl}} L_{jl}^{2\beta'_{jl}} u_j^2(s, x) ds.
\end{aligned} \tag{14}$$

Then from (12)–(14), we obtain

$$\begin{aligned}
\frac{dV}{dt} &\leq \sum_{i=1}^n \left[ -\lambda_i a_i(t) + \frac{\lambda_i}{2} \sum_{l=1}^m \sum_{j=1}^n |b_{ijl}(t)|^{2\alpha'_{ijl}} L_{jl}^{2\beta_{jl}} + \frac{1}{2} \sum_{l=1}^m \sum_{j=1}^n \lambda_j L_{il}^{2\beta'_{il}} |b_{jil}(t)|^{2\alpha'_{jil}} \right] \int_{\Omega} u_i^2(t, x) dx \\
&\quad + \sum_{i=1}^n \lambda_i \left( |I_i(t)| + \sum_{l=1}^m \sum_{j=1}^n |b_{ijl}(t)| |f_{jl}(0)| \right) \int_{\Omega} |u_i(t, x)| dx \\
&\leq -\rho \int_{\Omega} \sum_{i=1}^n u_i^2(t, x) dx + Z \int_{\Omega} \sum_{i=1}^n |u_i(t, x)| dx
\end{aligned} \tag{15}$$

where

$$Z = \sup_{t \in [0, \infty)} \sum_{i=1}^n \lambda_i \left( |I_i(t)| + \sum_{l=1}^m \sum_{j=1}^n |b_{ijl}(t)| |f_{jl}(0)| \right) > 0.$$

For any  $u(t, x) = (u_1(t, x), u_2(t, x), \dots, u_n(t, x))^T \in R^n$ , we let  $|u(t, x)|_1 = \sum_{i=1}^n |u_i(t, x)|$ ,  $|u(t, x)| = \sqrt{\sum_{i=1}^n u_i^2(t, x)}$ . By the equivalent of the norms in  $R^n$ , there is a constant  $\kappa > 0$ , such that  $\kappa |u(t, x)| \geq |u(t, x)|_1$  for all  $x \in R^n$ . From this and by (15), we further have

$$\begin{aligned} \frac{dV}{dt} &\leq -\rho \int_{\Omega} |u(t, x)|^2 dx + Z \int_{\Omega} |u(t, x)|_1 dx \\ &\leq -\frac{1}{2} \rho \int_{\Omega} |u(t, x)|^2 dx - \frac{1}{2} \rho \int_{\Omega} |u(t, x)|^2 dx + Z \rho \int_{\Omega} |u(t, x)| dx \\ &\leq -\frac{1}{2} \rho \int_{\Omega} |u(t, x)|^2 dx + Z^* = - \int_{\Omega} W_4(|u(t, x)|) dx + Z^*, \end{aligned} \quad (16)$$

where

$$\begin{aligned} Z^* &= \sup_{t \geq 0} \int_{\Omega} \left( Z \rho |u(t, x)| - \frac{1}{2} \rho |u(t, x)|^2 \right) dx \\ &= \sup_{t \geq 0} \int_{\Omega} \rho \left[ -\frac{1}{2} (|u(t, x)| - Z)^2 + \frac{1}{2} Z^2 \right] dx. \end{aligned}$$

Therefore, there must have some constant  $Z^* > 0$ . In addition,  $V(t)$  satisfies the condition (b) of Lemma 1. So from (9)–(11) and (16) and Lemma 1, we know all solutions of the system (1) are defined on  $[0, \infty)$  and are ultimately bounded. This completes the proof of Theorem 1.  $\square$

**Remark 1.** Obviously, in Theorem 1 we do not require the boundedness of the activation functions  $f_{jl}(\cdot)$  on  $R$ . However, in [11–16, 22, 24, 27, 28] we see, in order to obtain the boundedness of all solutions of the systems the authors assumed that all the activation functions are bounded.

Further, as consequence of Theorem 1 we have the following a series of corollaries.

**Corollary 1.** Assume that  $(H_1)$ – $(H_2)$  hold and the delay kernels  $K(\cdot)$  satisfy (i)–(iv). There are constants  $\lambda_i > 0$  and  $\rho > 0$  such that one of the following conditions holds

$$\lambda_i a_i(t) - \frac{\lambda_i}{2} \sum_{l=1}^m \sum_{j=1}^n |b_{ijl}(t)| L_{jl} - \frac{1}{2} \sum_{l=1}^m \sum_{j=1}^n \lambda_j L_{il} |b_{jil}(t)| > \rho, \quad (17)$$

$$\lambda_i a_i(t) - \frac{\lambda_i}{2} \sum_{l=1}^m \sum_{j=1}^n |b_{ijl}(t)|^{-1} L_{jl}^3 - \frac{1}{2} \sum_{l=1}^m \sum_{j=1}^n \lambda_j L_{il}^{-1} |b_{jil}(t)|^3 > \rho, \quad (18)$$

and

$$\lambda_i a_i(t) - \frac{1}{2} \sum_{l=1}^m \sum_{j=1}^n \lambda_j L_{il}^2 |b_{jil}(t)|^2 > \rho, \quad (19)$$

for all  $t \geq 0$  and  $i = 1, 2, \dots, n$ , then all solutions of the system (1) are ultimately bounded.

In fact, conditions (17)–(19) are the special cases of the condition (7) as  $\alpha_{ijl} = \alpha'_{ijl} = \frac{1}{2}$ ,  $\beta_{il} = \beta'_{il} = \frac{1}{2}$ ;  $\alpha_{ijl} = -\frac{1}{2}$ ,  $\alpha'_{ijl} = \frac{3}{2}$ ,  $\beta_{il} = \frac{3}{2}$ ,  $\beta'_{il} = -\frac{1}{2}$  and  $\alpha_{ijl} = 0$ ,  $\alpha'_{ijl} = 1$ ,  $\beta_{il} = 0$ ,  $\beta'_{il} = 1$ , respectively. Therefore, by Theorem 1 we see that Corollary 1 is true.

**Remark 2.** In fact, from Corollary 1 we see that as long as in the condition (7), the constants  $\alpha_{ijl}$ ,  $\alpha'_{ijl}$ ,  $\beta_{il}$ ,  $\beta'_{il}$  are chosen specifically, then from Theorem 1 we obtain right away a concrete sufficient condition to determine the ultimate boundedness of all solutions for the system (1).

#### 4. Global exponential stability

In this section, the global exponential stability for the RCNNs with distributed delays is discussed.

**Theorem 2.** Assume that (H<sub>1</sub>)–(H<sub>2</sub>) hold and the delay kernels  $K(\cdot)$  satisfy (i)–(iv). There are constants  $\lambda_i > 0$ ,  $\rho > 0$ , and  $\alpha_{ijl} + \alpha'_{ijl} = 1$ ,  $\beta_{il} + \beta'_{il} = 1$ ,  $j = 1, 2, \dots, n$ ,  $l = 1, 2, \dots, p$ , such that

$$\lambda_i a_i(t) - \frac{\lambda_i}{2} \sum_{l=1}^m \sum_{j=1}^n |b_{ijl}(t)|^{2\alpha_{ijl}} L_{jl}^{2\beta_{jl}} - \sum_{l=1}^m \sum_{j=1}^n L_{il}^{2\beta'_{il}} |b_{jil}(t)|^{2\alpha'_{jil}} \int_0^\infty K_{jil}(s) e^{\varepsilon s} ds > \rho, \quad (20)$$

then the system (1) is globally exponentially stable.

**Proof.**  $z_1(t, x) = (z_{11}(t, x), z_{12}(t, x), \dots, z_{1n}(t, x))^T$  and  $z_2(t, x) = (z_{21}(t, x), z_{22}(t, x), \dots, z_{2n}(t, x))^T$  are any two solutions of the system (1) satisfying the initial conditions  $z_1(\theta, x) = \phi_1(\theta, x)$  and  $z_2(\theta, x) = \phi_2(\theta, x)$  for all  $\theta \in (-\infty, 0]$ , respectively. Let  $y_i(t, x) = z_{1i}(t, x) - z_{2i}(t, x)$ , then by (H<sub>2</sub>), we have

$$\frac{\partial y_i(t, x)}{\partial t} \leq \sum_{k=1}^p \frac{\partial}{\partial x_k} \left( D_{ik} \frac{\partial y_i(t, x)}{\partial x_k} \right) - a_i(t) y_i(t, x) + \sum_{l=1}^m \sum_{j=1}^n |b_{ijl}(t)| L_{jl} \int_{-\infty}^t K_{ijl}(t-s) |y_j(s, x)| ds. \quad (21)$$

From (20), we can choose a constant  $\varepsilon > 0$  such that

$$\lambda_i a_i(t) - \frac{\lambda_i}{2} \sum_{l=1}^m \sum_{j=1}^n |b_{ijl}(t)|^{2\alpha_{ijl}} L_{jl}^{2\beta_{jl}} - \frac{\lambda_i}{2} \varepsilon - \sum_{l=1}^m \sum_{j=1}^n L_{il}^{2\beta'_{il}} |b_{jil}(t)|^{2\alpha'_{jil}} \int_0^\infty K_{jil}(s) e^{\varepsilon s} ds > \frac{1}{2} \rho \quad (22)$$

for all  $t \geq 0$ . Now we choose the Lyapunov functional as follows:

$$V(t) = \frac{1}{2} \int_{\Omega} \lambda_i \sum_{i=1}^n \left[ y_i^2(t, x) e^{\varepsilon t} + \sum_{l=1}^m \sum_{j=1}^n L_{jl}^{2\beta'_{jl}} \int_0^\infty K_{ijl}(s) \int_{t-s}^t |b_{ijl}(t)|^{2\alpha'_{ijl}} y_j^2(w, x) e^{\varepsilon(s+w)} dw ds \right] dx. \quad (23)$$

By calculating the derivative  $\frac{dV}{dt}$  of  $V(t)$  along the solutions of the system (21), it follows that

$$\begin{aligned} \frac{dV}{dt} &= \frac{1}{2} \int_{\Omega} \sum_{i=1}^n \lambda_i \left[ 2y_i(t, x) \frac{\partial y_i(t, x)}{\partial t} e^{\varepsilon t} + \varepsilon y_i^2(t, x) e^{\varepsilon t} \right. \\ &\quad + \sum_{l=1}^m \sum_{j=1}^n L_j^{2\beta'_j} |b_{ij}(t)|^{2\alpha'_{ij}} \int_0^\infty K_{ij}(s) y_j^2(t, x) e^{\varepsilon(s+t)} ds \\ &\quad \left. - \sum_{l=1}^m \sum_{j=1}^n L_j^{2\beta'_j} |b_{ij}(t)|^{2\alpha'_{ij}} \int_0^\infty K_{ij}(s) y_j^2(t-s, x) e^{\varepsilon t} ds \right] dx \\ &\leq \frac{1}{2} e^{\varepsilon t} \int_{\Omega} \sum_{i=1}^n \lambda_i \left\{ 2y_i(t, x) \sum_{k=1}^p \frac{\partial}{\partial x_k} \left( D_{ik} \frac{\partial y_i(t, x)}{\partial x_k} \right) - 2a_i(t) y_i^2(t, x) + \varepsilon y_i^2(t, x) \right. \\ &\quad + \sum_{l=1}^m \sum_{j=1}^n 2|y_i(t, x)| |b_{ij}(t)| L_j \int_{-\infty}^t K_{ij}(t-s) |y_j(s, x)| ds \\ &\quad + \sum_{l=1}^m \sum_{j=1}^n L_j^{2\beta'_j} |b_{ij}(t)|^{2\alpha'_{ij}} \int_0^\infty K_{ij}(s) y_j^2(t, x) e^{\varepsilon s} ds \\ &\quad \left. - \sum_{l=1}^m \sum_{j=1}^n \left[ L_j^{2\beta'_j} |b_{ij}(t)|^{2\alpha'_{ij}} \int_0^\infty K_{ij}(s) y_j^2(t-s, x) ds \right] dx \right\} \\ &\leq \sum_{i=1}^n \left[ -\lambda_i a_i(t) + \frac{\lambda_i}{2} \sum_{l=1}^m \sum_{j=1}^n |b_{ij}(t)|^{2\alpha_{ij}} L_j^{2\beta_j} \right. \\ &\quad \left. + \frac{\lambda_i}{2} \varepsilon + \sum_{l=1}^m \sum_{j=1}^n L_i^{2\beta'_{il}} |b_{jil}(t)|^{2\alpha'_{jil}} \int_0^\infty K_{jil}(s) e^{\varepsilon s} ds \right] \int_{\Omega} y_i^2(t, x) dx \Big\} \quad (24) \end{aligned}$$

for all  $t \geq 0$ . Therefore, we have

$$V(t) \leq V(0), \quad t \geq 0. \quad (25)$$

Obviously, it can be get from (23)

$$V(t) \geq \frac{1}{2} e^{\varepsilon t} \min_{1 \leq i \leq n} (\lambda_i) \int_{\Omega} \sum_{i=1}^n y_i^2(t, x) dx \quad (26)$$

and

$$\begin{aligned} V(0) &= \frac{1}{2} \int_{\Omega} \sum_{i=1}^n \lambda_i \left[ y_i^2(0, x) + \lambda_i \sum_{l=1}^m \sum_{j=1}^n L_{jl}^{2\beta'_{jl}} \int_0^{\infty} K_{ijl}(s) \int_{-s}^0 |b_{ijl}(0)|^{2\alpha'_{ijl}} e^{\varepsilon(s+w)} y_j^2(w, x) dw ds \right] dx \\ &\leq \frac{1}{2} \max_{1 \leq i \leq n} (\lambda_i) \sum_{i=1}^n \left[ \sup_{s \in (-\infty, 0]} \|\phi_{1i}(s) - \phi_{2i}(s)\|_2^2 + \sum_{l=1}^m \sum_{j=1}^n L_{jl}^{2\beta'_{jl}} Q_{ijl} \sup_{s \in (-\infty, 0]} \|\phi_{1j}(s) - \phi_{2j}(s)\|_2^2 \right] \\ &\leq \frac{1}{2} \max_{1 \leq i \leq n} (\lambda_i) \sum_{i=1}^n \left[ 1 + \sum_{l=1}^m \sum_{j=1}^n L_{il}^{2\beta'_{il}} Q_{jil} \right] \|\phi_{1m} - \phi_{2m}\|_2^2, \end{aligned} \quad (27)$$

where

$$Q_{ijl} = \sup_{s \in (-\infty, 0]} \left\{ \int_0^{\infty} K_{ijl}(s) \int_{-s}^0 |b_{ijl}(0)|^{2\alpha'_{ijl}} e^{\varepsilon(s+w)} dw ds \right\} > 0,$$

$$\|\phi_{1m} - \phi_{2m}\|_2^2 = \sup_{s \in (-\infty, 0]} \sum_{i=1}^n \|\phi_1(s, x) - \phi_2(s, x)\|_2^2.$$

Hence, by (25)–(27) we further obtain

$$\int_{\Omega} \sum_{i=1}^n y_i^2(t, x) dx \leq M_0 \|\phi_{1m} - \phi_{2m}\|_2^2 e^{-\varepsilon t} \quad (28)$$

for all  $t \geq 0$ , where  $M_0 \geq 1$  is a constant and independent of any solution of the system (1). From (28), we finally derive

$$\|z_1 - z_2\|_2^2 \leq M_0 \|\phi_{1m} - \phi_{2m}\|_2^2 e^{-\varepsilon t}, \quad \forall t \geq 0, \quad (29)$$

where

$$\|\phi_{1m} - \phi_{2m}\|_2^2 = \sup_{s \in (-\infty, 0]} \|\phi_1 - \phi_2\|_2^2 \quad (30)$$

for all  $t \geq 0$ . This shows that solutions of the system (1) are globally exponentially stable. This completes the proof of Theorem 2.

As a special case of the system (1), we assume that system (1) has a unique equilibrium  $u^* = (u_1^*, u_2^*, \dots, u_n^*)^T$ , by Theorem 2 we can obtain the following result on the global exponential stability of the equilibrium  $u^*$  of the system (1).  $\square$

**Corollary 2.** Assume that (H<sub>1</sub>)–(H<sub>2</sub>) hold and the system (1) has a unique equilibrium  $u^*$ . There are constants  $\lambda_i > 0$ ,  $\rho > 0$ , and  $\alpha_{ijl} + \alpha'_{ijl} = 1$ ,  $\beta_{il} + \beta'_{il} = 1$ ,  $i, j = 1, 2, \dots, n$ ,  $l = 1, 2, \dots, p$ , satisfy (20), then the equilibrium  $u^*$  is globally exponentially stable.

The following corollary are similar to Corollary 1 and can be easily proved.

**Corollary 3.** Assume that (H<sub>1</sub>)–(H<sub>2</sub>) hold and the delay kernels  $K(\cdot)$  satisfy (i)–(iv). There are constants  $\lambda_i > 0$  and  $\rho > 0$  such that one of the following conditions holds

$$\lambda_i a_i(t) - \frac{\lambda_i}{2} \sum_{l=1}^m \sum_{j=1}^n |b_{ijl}(t)| L_{jl} - \frac{1}{2} \sum_{l=1}^m \sum_{j=1}^n \lambda_j L_{il} |b_{jil}(t)| \int_0^{\infty} K_{jil}(s) e^{\varepsilon s} ds > \rho, \quad (31)$$



$$\lambda_i a_i(t) - \frac{\lambda_i}{2} \sum_{l=1}^m \sum_{j=1}^n |b_{ijl}(t)|^{-1} L_{jl}^3 - \frac{1}{2} \sum_{l=1}^m \sum_{j=1}^n \lambda_j L_{il}^{-1} |b_{jil}(t)|^3 \int_0^\infty K_{jil}(s) e^{\varepsilon s} ds > \rho, \quad (32)$$

and

$$\lambda_i a_i(t) - \frac{1}{2} \sum_{l=1}^m \sum_{j=1}^n \lambda_j L_{il}^2 |b_{jil}(t)|^2 \int_0^\infty K_{jil}(s) e^{\varepsilon s} ds > \rho, \quad (33)$$

for all  $t \geq 0$  and  $i = 1, 2, \dots, n$ , then all solutions of the system (1) are globally exponentially stable.

**Remark 3.** From Corollary 3 we see that as long as in the condition (20), the constants  $\alpha_{ijl}$ ,  $\alpha'_{ijl}$ ,  $\beta_{il}$ ,  $\beta'_{il}$  and  $\lambda_i > 0$  are chosen specifically, then from Theorem 2 we obtain right away a concrete sufficient condition to determine the global exponential stability of all solutions for the system (1).

**Remark 4.** As we can see, when  $D_{ik}(t, x, u) = 0$  ( $i = 1, 2, \dots, n$ ) and the distributed delays become discrete delays, the system (1) becomes some models which are studied in [20]. Since the studies of the model with distributed delays have more important significance than the ones of model with discrete delays, the system studied in [20] is a special case of the system (1) in this paper.

**Remark 5.** Obviously, from Theorem 2, we see that the results obtained in the section improve and extend the main results on the global exponential stability of the solutions for some autonomous neural networks with distributed delays given by Cui et al. in [22,24,27,28].

## 5. Conclusions

In this paper, we have derived some simple sufficient conditions in term of systems parameters for the boundedness and global exponential stability for the RCNNs with distributed delays. The results presented here improve and extend those in the literature.

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