Homework 1

1 Exercise 1

In order to prove the following equivalence

$$\int_0^t W(s)dS = \int_0^t (t-s)dW(s)$$

We define the following stochastic function

$$f(t, W(t)) = tW(t) \tag{1}$$

We use now Ito's Lemma to derive df. From simple calculation we compute the following derivatives

$$\frac{\partial f}{\partial t} = W(t) \quad \frac{\partial f}{\partial W} = t \quad \frac{\partial^2 f}{\partial^2 W} = 0 \tag{2}$$

From Ito's lemma, we get the following Stochastic differential equation

$$df(t, W(t)) = W(t)dt + tdW(t) \tag{3}$$

Next, we apply Ito's integral on both sides

$$\int_0^t df(t, W(t)) = \int_0^t W(t)dt + \int_0^t sdW(s) \tag{4}$$

Looking at the right hand side of the equation

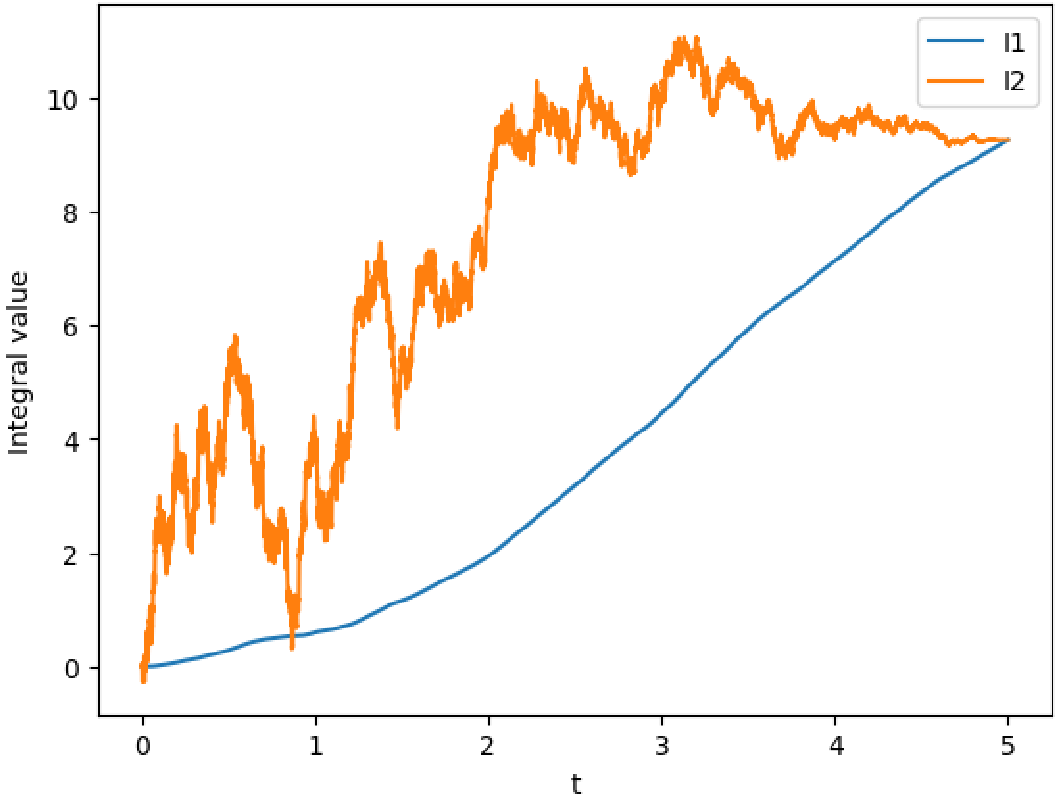
$$\int_0^t df(t, W(t)) = dW(t) - 0 = dW(t) \tag{5}$$

Finally, plugging it in back in Equation 4, the following holds

$$\begin{aligned}\int_0^t W(s)ds &= tW(t) - \int_0^t sdW(s) \\ &= \int_0^t dW(s) - \int_0^t sdW(s) \\ &= \int_0^t (t-s)dW(s)\end{aligned}$$

Numerical analysis

Numerically we compute the integrals of the right hand side and left and side. We plot the values of such integral with respect to the time. Therefore, for the assumption to be true, it must hold that the values of the two integral coincide at the final value of $t = 5$ which is indeed the case.



2 Exercise 2

2.1 a

Having the stochastic differential equation for the geometric brownian motion

$$\frac{\partial S(t)}{S(t)} = \mu dt + \sigma dW(t) \tag{6}$$

We are interested in finding the dynamics of $Y_1(t) = \mu S^2(t)$.
In order to apply Ito's lemma, we compute the following derivatives

$$\frac{\partial Y}{\partial t} = 0 \quad \frac{\partial Y}{\partial S} = 2\mu S(t) \quad \frac{\partial Y}{\partial S^2} = 2\mu \tag{7}$$

We plug them into the Ito's Lemma formula and obtain

$$dY(t) = 0dt + 2\mu S(t)dS + \mu(dS)^2 \tag{8}$$

Using Ito's table, we compute

$$(dS)^2 = \sigma^2 S^2(t)dt \tag{9}$$

Plugging it back into equation 8 we obtain

$$\begin{aligned} dY(t) &= 2\mu S(t)(\mu S(t)dt + \sigma S(t)dW(t)) + \mu(\sigma^2 S^2 dt) \\ &= 2\mu^2 S^2(t)dt + 2\mu\sigma S^2 dW(t) + \mu\sigma^2 S^2 dt \end{aligned}$$

2.2 b

Similarly, we are interested in finding the dynamics of

$$Y_2(t) = e^{W(t)} \tag{10}$$

Again, we need to first compute the following derivatives in order to apply the Ito's Lemma

$$\frac{\partial Y}{\partial t} = 0 \quad \frac{\partial Y}{\partial S} = \frac{\partial Y}{\partial W} = e^{W(t)} \quad \frac{\partial Y}{\partial W} = e^{W(t)} \tag{11}$$

Thus, plugging them into the Ito's formula, using the fact that $\overline{\sigma} = 1$ and $\overline{\mu} = 0$ we get

$$\begin{aligned} dY_2 &= \left(e^{W(t)} + \frac{1}{2}e^{W(t)}\right) dt + e^{W(t)} dW(t) \\ &= e^{W(t)} \left(\frac{3}{2}dt + dW(t)\right) \\ &= Y_2 \left(\frac{3}{2}dt + dW(t)\right) \end{aligned}$$

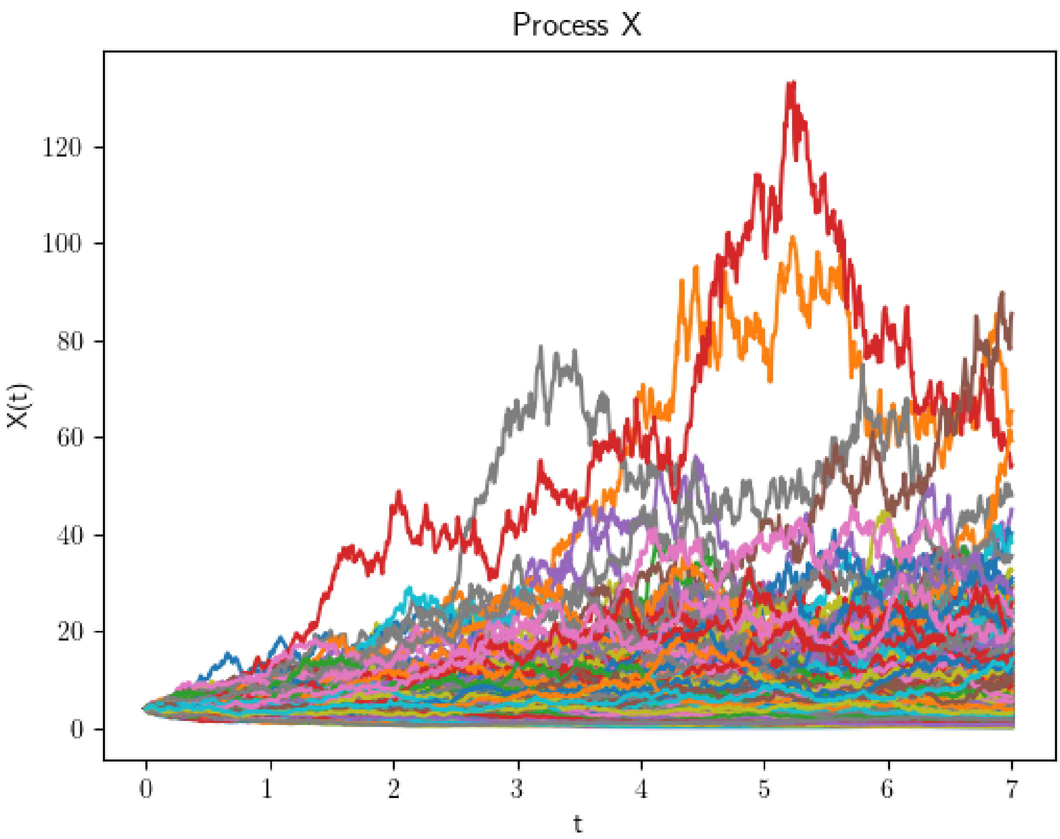
Finally, we can note that Equation **??** is not a Martingale because $dY_2 \neq \sigma W(t)dt$ because $\frac{3}{2}e^{W(t)} \neq$, thus Y_2 is not a Martingale.

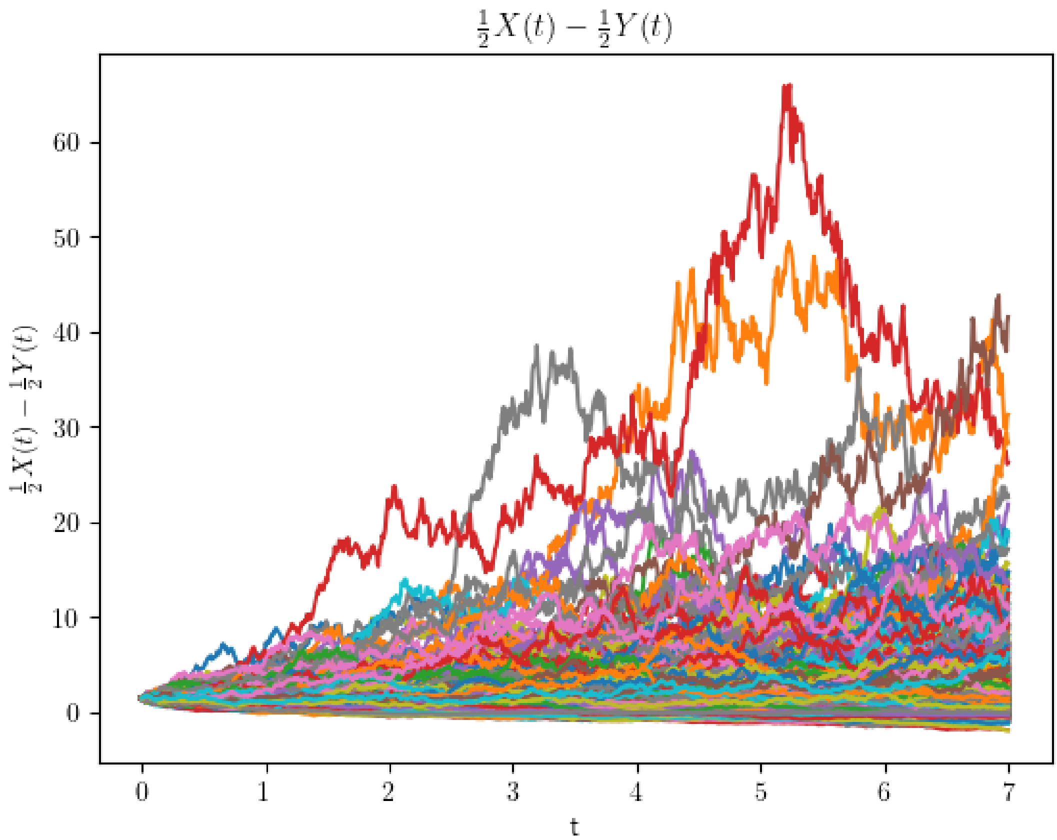
3 Exercise 3

In order to compute the expected value of

$$\frac{1}{M(t)} \max(\frac{1}{2}X(T) - \frac{1}{2}(T), K) \tag{12}$$

Under the $\mathbb{Q}$ measure, we started by simulating $\frac{1}{2}X(T) - \frac{1}{2}(T)$ under the interest rate r. Let's begin by simulating the Geometric Brownian Motion. In the image 3 you can see some of the path generated by $X(t)$ given by $dX(t) = 0.04X(t)dt + \sigma X(tdW^{\mathcal{P}}(t))$ with $X(t_0) = 4$.
We simulated in similar fashion $Y(t)$ given by $dY(t) = \beta Y(t)dt + 0.15Y(t)dW^{\mathcal{P}}(t)$ with $Y(t_0) = 1$. Next, we combine the two as it can be seen in figure 3
Finally, we compute $V(t)$ by taking the average over the paths at the last time $t = 7.0$. We repeat the process for different values of K and plot it in the image 3.


Process X
X(t)
0
20
40
60
80
100
120
t
0
1
2
3
4
5
6
7


$\frac{1}{2}X(t) - \frac{1}{2}Y(t)$
$\frac{1}{2}X(t) - \frac{1}{2}Y(t)$
0
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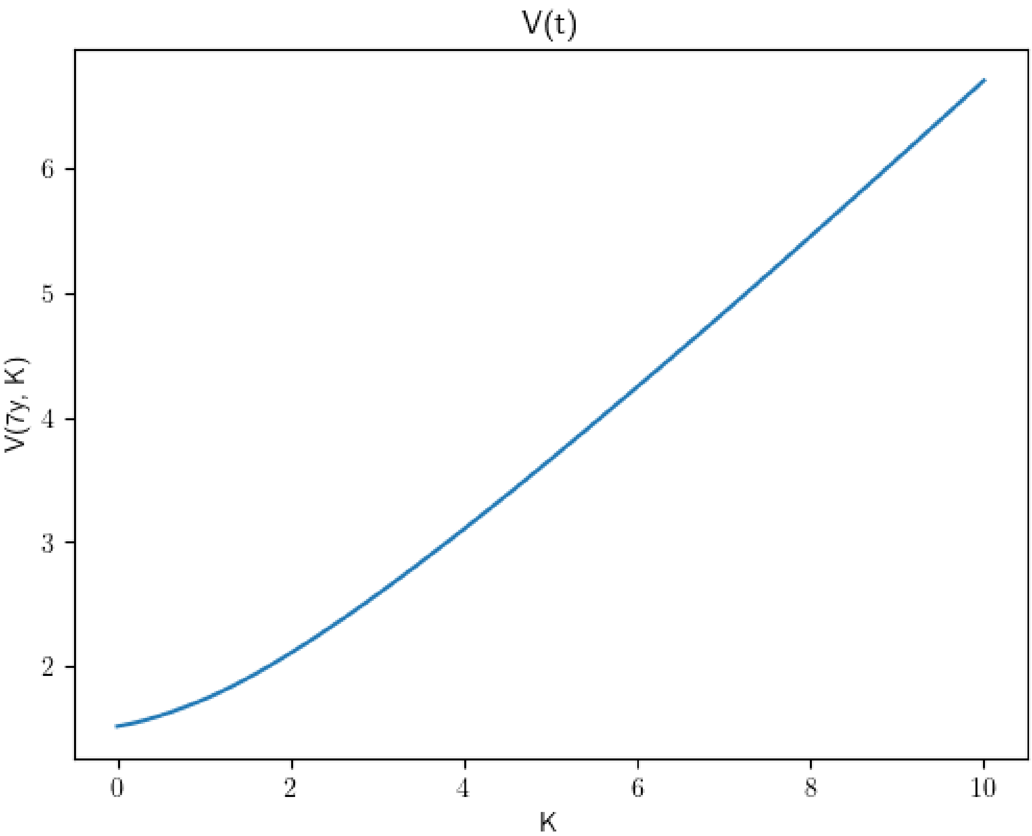


4 Exercise 4

Given the following Wiener process

$$X(t) = W(t) - \frac{t}{T}W(T-t) \quad 0 \leq t \leq T \tag{13}$$

We compute the variance as follows

$$\begin{aligned}
\text{Var}[X] &= \text{Var}\left[W(t) - \frac{t}{T}W(T-t)\right] \\
&= \mathbb{E}[X^2] \\
&= \mathbb{E}[W^2(t) + \left(\frac{t}{T}\right)^2 W^2(T-t) - 2\frac{t}{T}W(t)W(T-t)] \\
&= \mathbb{E}[W^2(t)] + \left(\frac{t}{T}\right)^2 \mathbb{E}[W^2(T-t)] - \frac{2t}{T}\mathbb{E}[W(t)W(T-t)]
\end{aligned}$$

Let's now analyze each of the components of the right hand side. The first part:

$$\mathbb{E}[N(0,t)^2] = \text{Var}[N(0,t)] = t \tag{14}$$

The second part:

$$\mathbb{E}[W^2(T-t)] = \mathbb{E}[N(0,T-t)^2] = T-t \tag{15}$$

In order to compute the third part, let's start by noting that

$$\text{Cov}\,[X,Y] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \tag{16}$$

In the case of W, $\mathbb{E}[X]=0$ thus $\text{Cov}\,[W,W] = \mathbb{E}[WW]$. Putting it all together we get

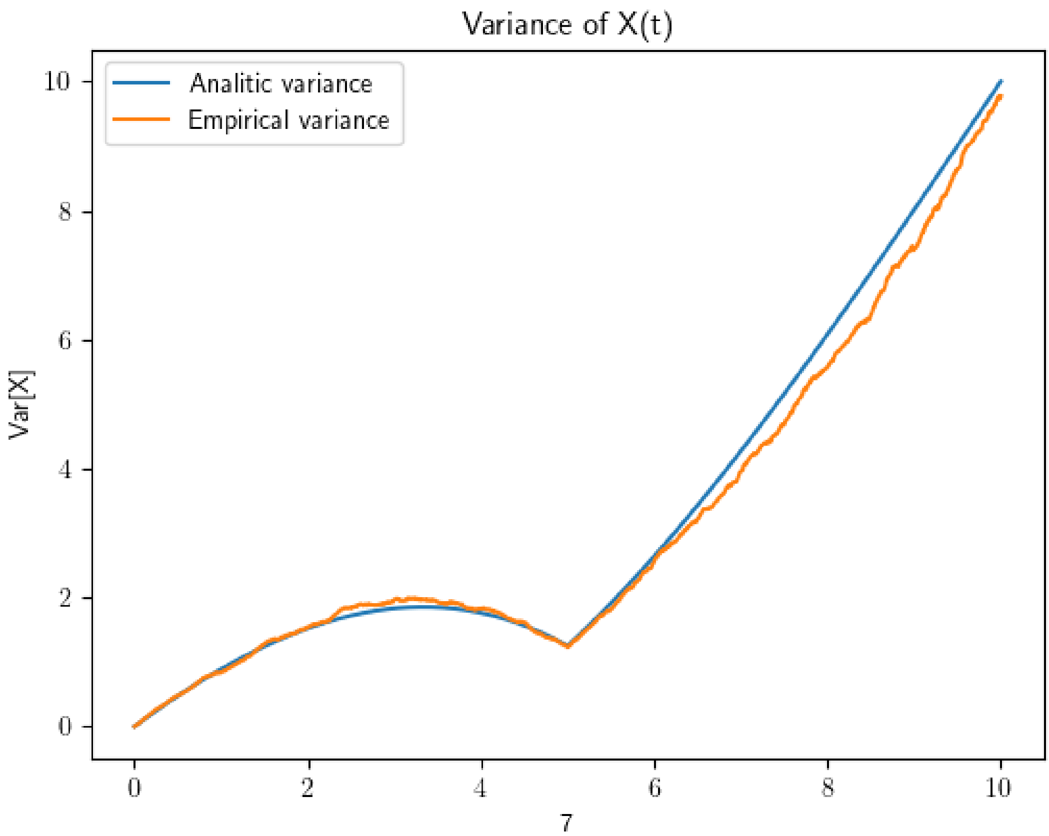
$$\mathbb{E}[W(t)W(T-t)] = \text{Cov}\,[W(t),W(T-t)] = \min(t,T-t)$$

we can then finalize the derivation

$$\text{Var}\,[X] = \mathbb{E}[X^2] = t + \frac{t^2}{T^2}(T-t) - \frac{2t}{T}\min(t,T-t) \tag{17}$$

$$= t\left(1 + \frac{t}{T^2}(T-t) - \frac{2}{T}\min(t,T-t)\right) \tag{18}$$

In order to numerically compute the variance of $X(t)$ with $T=0$ we started by simulation the Wiener process. Figure 4 shows an example of simulated paths.



Next, we compute the variance of the paths at each time t. In the image **??** we compare the analitical variance as represented in equation 17 as well as the empirical variance.