Homework 1

1 Exercise 1

1.1

Let's define x_0 as the limit of the net

$$\sum_{a \in A} a \to x_0. \tag{1}$$

Since it converges, it means tha there exist a set $F_0 \subset A$ such that for all $F > F_0$ the following holds

$$\| \sum_{a \in F} a - x_0 \| < \epsilon \tag{2}$$

for any $\epsilon > 0$. Since ϵ is arbitrary, we can choose $F_0' > F_0$ such that for all $F > F_0'$

$$\| \sum_{a \in F} a - x_0 \| < \frac{\epsilon}{|\alpha|}. \tag{3}$$

Then, by properties of the norm we obtain

$$\begin{aligned}
|\alpha| \| \sum_{a \in F} a - x_0 \| &< \frac{\epsilon}{|\alpha|} \alpha = \epsilon \\
\| \alpha \sum_{a \in F} a - \alpha x_0 \| &< \epsilon \\
\| \sum_{a \in F} \alpha a - \alpha x_0 \| &< \epsilon.
\end{aligned}$$

Where in the last step we used the fact that F is finite. This proves that $\alpha \sum_{a \in A} a$ converges to $\alpha x_0 = \alpha \sum_{a \in A} a$.

1.2

The hypothesis that $\sum_{a \in A} a$ and $\sum_{b \in B} b$ implies that there exists an F_0^a and F_0^b such that for every $F^a > F_0^a$ and $F^b > F_0^b$ the following holds

$$\| \sum_{a \in F^a} a - \sum_{a \in A} a \| < \frac{\epsilon}{2} \qquad \| \sum_{b \in F^b} b - \sum_{b \in B} b \| < \frac{\epsilon}{2}. \tag{4}$$

Denote $F_0 = F_0^a \cup F_0^b$, it follows that for every $F > F_0$

$$\begin{aligned}\|\sum_{x\in F} x - \sum_{a\in A} a - \sum_{b\in B} b\| &= \|\sum_{x\in F\cap A} + \sum_{x\in F\cap B} x - \sum_{a\in A} a - \sum_{b\in B} b\| \\ &\leq \|\sum_{x\in F\cap A} x - \sum_{a\in A} a\| + \|\sum_{x\in F\cap B} x - \sum_{b\in B} b\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Because ϵ was arbitrary, we conclude that $\sum_{x\in A\cup B} \to \sum_{a\in A} a + \sum_{b\in B} b$.

1.3

Let's star by proving that any converging net of positive numbers has at most a countable number of non zero elements.

Say that the net converges to M, i.e. $\sum_{a\in A} a = M < \infty$ where for every $a \in A, a > 0$. Consider now the sets $S_n = \{a \in A | a > \frac{1}{n}\}$, then

$$M \geq \sum_{a\in S_n} a \geq \sum_{a\in S_n} \frac{1}{n} = \frac{N}{n}.$$

As $M < \infty$ so is N which is the cardinality of the set S_n. It follows that

$$\#\{a \in A | a > 0\} = \#S = \# \bigcup_{n=\mathbb{N}}^{\infty} S_n \tag{5}$$

We conclude that A has at most countable number of non zero elements as countable union of finite sets.