Homework 1

1 Exercise 1

1.1

Let's define x_0 as the limit of the net

$$\sum_{a \in A} a \to x_0. \tag{1}$$

Since it converges, it means that there exists a set $F_0 \subset A$ such that for all $F > F_0$ the following holds

$$\| \sum_{a \in F} a - x_0 \| < \epsilon \tag{2}$$

for a given $\epsilon > 0$. Since ϵ is arbitrary, we can choose $F_0' > F_0$ such that for all $F > F_0'$

$$\| \sum_{a \in F} a - x_0 \| < \frac{\epsilon}{|\alpha|}. \tag{3}$$

Then, by properties of the norm, we obtain

$$\begin{aligned}
|\alpha| \| \sum_{a \in F} a - x_0 \| &< \frac{\epsilon}{|\alpha|} \alpha = \epsilon \\
\| \alpha \sum_{a \in F} a - \alpha x_0 \| &< \epsilon \\
\| \sum_{a \in F} \alpha a - \alpha x_0 \| &< \epsilon.
\end{aligned}$$

Where in the last step we used the fact that F is finite. This proves that $\alpha \sum_{a \in A} a$ converges to $\alpha x_0 = \alpha \sum_{a \in A} a$.

1.2

The hypothesis that $\sum_{a \in A} a$ and $\sum_{b \in B} b$ implies that there exists an F_0^a and F_0^b such that for every $F^a > F_0^a$ and $F^b > F_0^b$ the following holds

$$\| \sum_{a \in F^a} a - \sum_{a \in A} a \| < \frac{\epsilon}{2} \qquad \| \sum_{b \in F^b} b - \sum_{b \in B} b \| < \frac{\epsilon}{2}. \tag{4}$$

Denote $F_0 = F_0^a \cup F_0^b$, it follows that for every $F > F_0$

$$\begin{aligned}\|\sum_{x\in F} x - \sum_{a\in A} a - \sum_{b\in B} b\| &= \|\sum_{x\in F\cap A} + \sum_{x\in F\cap B} x - \sum_{a\in A} a - \sum_{b\in B} b\| \\ &\leq \|\sum_{x\in F\cap A} x - \sum_{a\in A} a\| + \|\sum_{x\in F\cap B} x - \sum_{b\in B} b\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Because ϵ was arbitrary, we conclude that $\sum_{x\in A\cup B} \to \sum_{a\in A} a + \sum_{b\in B} b$.

1.3

Let's start by proving the following proposition:

Proposition 1. *Any converging $\sum_{a\in A} a \to x$ of positive numbers has at most a countable number of non-zero elements. That is, $\sum_{a\in A} x = \sum_{i\in\mathbb{N}} a_i$, meaning that they converge to the same value x.*

Proof. Say that the net converges to M, i.e. $\sum_{a\in A} a = M < \infty$ where for every $a \in A, a > 0$. Consider now the sets $S_n = \{a \in A | a > \frac{1}{n}\}$, then

$$M \geq \sum_{a\in S_n} a \geq \sum_{a\in S_n} \frac{1}{n} = \frac{N}{n}.$$

As $M < \infty$ so is N which is the cardinality of the set S_n. It follows that

$$\#\{a \in A | a > 0\} = \#S = \# \bigcup_{n=\mathbb{N}}^{\infty} S_n \tag{5}$$

We conclude that A has at most countable number of non-zero elements as a countable union of finite sets. □

Let's now prove the ($\implies$) direction. Given the previously proven statement, we can rewrite the net as a countable sum and thus define a corresponding sequence $x_n = \sum_{i=0}^n a_i$ where w.log. we associated every non zero element a of A to an index i so that $a_i = a$. From standard analysis we obtain that every converging increasing sequence is bounded from above, i.e. there exists $N \in R$ so that $x_n < N$ for every n. It follows that for every finite $F \subset I$

$$\sum_{a\in F} a \leq \sum_{i\in\mathbb{N}} a_i \leq N. \tag{6}$$

We now prove the opposite implication ($\impliedby$). Assume that $\sup\left\{\sum_{a\in F} a : F \in \mathcal{F}\right\} = x_0$. We proceed now by contradiction, suppose that $\sum_{a\in A} a \to x_0 + t$ for an arbitrary $t > 0$. Let's define now what it means for a net to increase. Given a set F_0, it holds that for every $F > F_0$ we have that $\sum_{a\in F} a \geq \sum_{a\in F_0} a$.

By defining net convergence for standard set inclusion as order, we obtain that the net $\sum_{a\in A} a$ is increasing. It follows that for every F, $\sum_{a\in F} \leq x_0 + t$. Moreover, since $\sum_{a\in A} a \to x_0 + t$, there exists and F_t such that $\forall F > F_t$ we have $\| \sum_{a\in F} a - x_0 - t\| < \frac{t}{2}$. It follows

$$\begin{aligned}
&\| \sum_{a\in F} a - x_0 - t\| \leq 0 \\
&\| \sum_{a\in F} a - x_0 - t\| = \sum_{a\in F} a - x_0 - t < \frac{t}{2} \\
&\sum_{a\in F} a > x_0 + \frac{t}{2}
\end{aligned}$$

Which is a contradiction. As this holds for any arbitrary t, $\sum_{a\in A} a \leq x_0$.
Let's conclude the proof by showing that

$$\sup\left\{\sum_{a\in F} a : F \in \mathcal{F}\right\} = \sum_{a\in A} a = x_0. \tag{7}$$

Take an arbitrary $x < x_0$. By contradiction, assume that $\sup\left\{\sum_{a\in F} a : F \in \mathcal{F}\right\} = x$. Again, by convergence of the net, there exists a F_0 such that $\forall F > F_0$ we have $\| \sum_{a\in F} a - x_0\| < |x - x_0|$. As both quantities are negative, it follows that $\sum_{a\in F} a - x_0 > x - x_0$ which leads to the contradiction $\sum_{a\in F} a > x$. Combining it with the previous point we get the desired equality.

2 Exercise 2

2.1

Let's define $H' = \bigvee \mathcal{F}$. By theorem 4.13 we have that $\forall h \in H'$, h can be written as $h = \sum_{e\in F} \langle h, e\rangle e$ as F is the basis for H' again by theorem 4.13.
Moreover, for every $x \in H$, we have that by definition $P_F x \in H'$. We define the operator Q acting on x as such

$$Qx \coloneqq \sum_{e\in F} \langle x, e\rangle e. \tag{8}$$

For Q to be equal to P_F, Qx has to be the unique elements in H' such that $x - Qx \perp H'$. We proceed by taking an orthogonal basis of H such that $\mathcal{E} \subset B$, this basis is guaranteed to exist given proposition 4.2. By theorem 4.13 again, by trivially noting that $\bigvee H = H$,

x can be represented as $x = \sum_{e \in B} \langle x, e \rangle e$. It follows that

$$
\begin{aligned}
x - Qx &= \sum_{e \in B} \langle x, e \rangle e - \sum_{e \in F} \langle x, e \rangle e \\
&= \sum_{e \in F} \langle x, e \rangle e + \sum_{e \in B \setminus F} \langle x, e \rangle e - \sum_{e \in F} \langle x, e \rangle e \\
&= \sum_{e \in B \setminus F} \langle x, e \rangle e
\end{aligned}
$$

We conclude that since B is orthogonal, it follows that $x - Qx \perp H' = \bigvee F$.

2.2

By the previous point, we can write $P_G x = \sum_{e \in G} \langle x, e \rangle e$ for every $G \subset \mathcal{G}$. It follows that

$$
\begin{aligned}
P_F P_G x &= P_F \left(\sum_{e \in G} \langle x, e \rangle e \right) \\
&= \sum_{e \in G} P_F \langle x, e \rangle e. \\
&= \sum_{e \in G} \sum_{e' \in F} \langle \langle e, x \rangle e, e' \rangle e' \\
&= \sum_{e \in G} \sum_{e' \in F} \langle e, x \rangle \langle e, e' \rangle e' \\
&= \sum_{e \in G \cap F} \langle e, x \rangle e.
\end{aligned}
$$

Where we used the orthogonality of P_F and the fact that the set F, G are orthonormal.

2.3

Given an arbitrary finite family of disjoint orthonormal basis $(F_n)_{n \in \mathbb{N}}$.

$$
P_{\bigcup_{i=1}^{n} F_i} = P_{\bigcup_{i=1}^{n-1} \cup F_n} = P_{F_j \cup F_n}
$$

Where $F_j := \bigcup_{i=1}^{n-1} F_i$. Thus, we just have to prove the above statement for two sets. This follows directly from the linearity of the sum, i.e.

$$
P_{F_i} x + P_{F_j} x = \sum_{e \in F_i} \langle x, e \rangle e + \sum_{e \in F_j} \langle x, e \rangle e = \sum_{e \in F_i \cup F_j} \langle x, e \rangle e = P_{F_i \cup F_j}.
$$

For the convergence, we have that for a fixed $\epsilon > 0$, for all $n > n_0$ the following holds true:

$$\|P_{\bigcup_{i=1}^{\infty} F_i} x - \sum_{e \in \bigcup_{i=1}^{n} F_i} \langle x, e \rangle e\| = \|P_{\bigcup_{i=1}^{\infty} F_i} x - \sum_{i=1}^{n} \sum_{e \in F_i} \langle x, e \rangle e\| < \epsilon. \tag{9}$$

Equivalently, for the same ϵ, there exists a n_0' such that for all $n > n_0$

$$\|\sum_{i}^{\infty} P_{F_i} x - \sum_{i}^{n} P_{F_i} x\| = \|\sum_{i}^{\infty} P_{F_i} x - \sum_{i}^{n} \sum_{e \in F_i} \langle e, x \rangle x\| < \epsilon \tag{10}$$

By taking $\overline{n} = \max(n_0, n_0')$ we get that both limits converge to the same value. By uniqueness of the limit we have that

$$P_{\bigcup_{i=1}^{\infty} F_i} x = \sum_{i=1}^{\infty} P_{F_i} x. \tag{11}$$

2.4

We prove by giving a counterexample that $\sum_i^{\infty} P_{F_i} x$ does not converge in the operator norm. Let's define $F_i = e_i$ where $e_i \cap e_j = \varnothing$ and $\|e_i\| = 1$ for every i. Take now $\epsilon = \frac{1}{2}$, then, for for every $n > N$

$$\|\sum_{i=0}^{\infty} P_{F_i} x - \sum_{i=0}^{n} P_{F_i} x\| = \|\sum_{i=n}^{\infty} P_{F_i} x\| = 1 > \frac{1}{2}$$

Therefore, for $\epsilon = \frac{1}{2}$ there exists no $n > 0$ such that the the difference of the norm converges.

3 Exercise 3

3.1

3.1.1 Inner product

We start by proving that H is indeed an inner product space given the definition of the inner product $\langle x, y \rangle = \sum_{i \in I} \langle x(i), y(i) \rangle_i, \quad x, y \in H$.
The first property we prove is the conjugate symmetry. For any $x, y \in H$ and assume the net converges $\langle x, y \rangle \to x_0$. Then, for a fixed $\epsilon > 0$, there exists an F_0 such that for all

$F > F_0$

$$\begin{aligned}
\|\overline{\langle x, y\rangle}_F - \overline{x_0}\| &< \epsilon \\
\|\overline{\sum_{i\in F}\langle x_i, y_i\rangle} - \overline{x_0}\| &< \epsilon \\
\|\sum_{i\in F}\overline{\langle x_i, y_i\rangle} - \overline{x_0}\| &< \epsilon \\
\|\sum_{i\in F}\langle y_i, x_i\rangle - \overline{x_0}\| &< \epsilon.
\end{aligned}$$

we conclude that $\overline{\langle x, y\rangle} \to \overline{x_0}$.

3.1.2 Linearity in the first argument

We approach the proof in the same fashion as in the previous part. Assume $\langle x, y\rangle$ converges in H to x_0 . Then, for every $F > F_0$

$$\begin{aligned}
\|\langle x, y\rangle_F - x_0\| &< \frac{\epsilon}{\lambda} \\
|\lambda|\|\langle x, y\rangle_F - x_0\| &< \frac{\epsilon}{|\lambda|}|\lambda| = \epsilon \\
\|\lambda\langle x, y\rangle_F - \lambda x_0\| &< \epsilon\| < \lambda x, y >_F - \lambda x_0\| && < \epsilon
\end{aligned}$$

Where in the last equality we used the linearity of the norm in a finite-dimensional setting.

3.1.3 Positive semi-definiteness

This part is trivial as $\langle x_i, x_i\rangle$ for any $i \in I$. It follows that

$$\langle x, x\rangle = \sum_{i\in I}\langle x_i, x_i\rangle > 0. \tag{12}$$

Then the so-defined inner product satisfies the inner product properties.

3.1.4 Well definiteness

Here we show that for every $x, y \in H$, $\langle x, y\rangle \in \mathbb{K}$.

$$\begin{aligned}\langle x, y\rangle &= \sum_{i\in I}\langle x_i, y_i\rangle \\ &\leq \sum_{i\in I} |\langle x_i, y_i\rangle| \\ &\leq \sum_{i\in I} \|x_i\|\|y_i\| \\ &\leq \sum_{i\in I} \|x_i\|^2 + \sum_{i\in I} \|y_i\|^2 < \infty.\end{aligned}$$

Where in the first inequality we used the Cauchy-Schwarz inequality.

3.1.5 Completeness

We now proceed to prove the completeness of H. Let $(h_n)_{n\in\mathbb{N}}$ be a Cauchy sequence, that is, for a certain $N > 0$, then for all $n, m > N$ $\|h_n - h_m\| < \epsilon$. Therefore, $\|h_n(i) - h_m(i)\| < \epsilon$ is also a Cauchy sequence in H_i, thus it converges to the value say $h(i)$. Consider then $\lim_{n\to\infty} h_n(i) = h(i)$ the candidate element for the Cauchy sequence to converge to.
Set N_0 so that for every $n, m > N_0$ $\|h_n - h_m\| < \frac{\epsilon^2}{2}$. Thus, simply by the definition of h we get that for every $i \in I$ the following holds true.

$$\lim_{m\to\infty} \|h_n(i) - h_m(i)\| = \|h_n(i) - h(i)\|$$

.

Let $G \subset I$ finite. Then

$$\begin{aligned}\sum_{i\in G} \|h_n(i) - h(i)\|^2 &= \sum_{i\in G} \lim_{m\to\infty} \|h_n(i) - h_m(i)\| \\ &= \lim_{m\to\infty} \sum_{i\in G} \|h_n(i) - h_m(i)\| \\ &< \lim_{m\to\infty} \sum_{i\in I} \|h_n(i) - h_m(i)\| \\ &= \lim_{m\to\infty} \|h_n(i) - h_m(i)\| < \frac{\epsilon^2}{2} < \epsilon^2\end{aligned}$$

By the triangle inequality, we prove that $h \in H$.

$$\begin{aligned}
\|h\|^2 &= \sum_{i\in I} \|h(i)\|^2 \\
&\leq \sum_{i\in I} (\|h_n(i) - h(i)\| + \|h_n(i)\|)^2 \\
&= \sum_{i\in I} \|h_n(i) - h(i)\|^2 + \|h_n(i)\|^2 + \|h_n(i) - h(i)\|\|h_n(i)\| \\
&\leq \|h_n - h\| + \|h_n\| + \|h_n - h\|^2 + \|h_n\|^2 < \infty
\end{aligned}$$

3.2

Let $x_i = \sum_k \mathcal{E}_i^k \alpha_i^k$ for every $x_i \in H_i$ where the subscript k is used as second index instead of power. By definition, $x \in H \implies \bigcup_{i\in I}^{\infty} x_i = \sum_{i\in I} \sum_k \mathcal{E}_i^k \alpha_i^k$. This is sufficient to claim that $\bigcup_{i\in I} \mathcal{E}_i$ is a basis in H.

3.3

Let's start by proving the first implication.
Given that each H_i is countable, we take a countable basis $B_i \subset H_i$. By the previous point, $B := \bigcup_{i=1}^{\infty} B_i$ is a basis for H. Given that each B_I is countable and we take a countable union, it implies that B is also countable. Given that H has a countable basis B it implies that H is itself countable.
Let's now prove the other implication. For every $H_i \in H$ take $V_i \in H_i$ open. Thus, $(V_i)_{i\in I}$ is a collection of disjoint open sets and it is uncountable, therefore H is not second countable, which is equivalent to not being separable as H is a metrizable space.

3.4

Assume $\sup_{i\in I} A_i = k < \infty$, and assume that $\|A\|_{op} = k_i$. Then, for every $x \in H$ such that $\|x\| \leq 1$ the following holds true.

$$\|Ax\|^2 = \sum_{i\in I} \|A_i x_i\|^2 \leq \|k_i x_i\|^2. \tag{13}$$

Then, $\sup \|Ax\|^2 = \sup \sum_{i\in I} k_i^2 x_i^2 \leq k \sum_{i\in I} x_i < \infty$ where the supremum is taken over the elements x with $\|x\| \leq 1$.
For the converse ($\implies$), assume $\exists j \in I$ such that $\|A_k\| = \infty$, i.e $\forall N > 0, \exists h_n$ such that $\|h_n\| \leq 1$ and $\|A_j h_n\| > N$. Take now $\overline{x} \in H$ such that $\overline{x}(j) = h_n$ and $\overline{x}(i) = 0$ for all $i \neq j$. It follows that $\|x\| \leq 1$ and that

$$\|A\| = \sup\{\|Ax\|, \|x\| \leq 1\} \geq \|A\overline{x}\| = A_j x = \infty \tag{14}$$

In order to prove the equality, we note from the first part that

$$\sup \|Ax\|^2 \leq \sup_i \|A_i\| \sum_i \|x_i\|^2 = \sup \|A_i\| \|x\|. \tag{15}$$

We now prove the other inequality.

$$\|A_i x_i\|_{H_i}^2 \leq \sum_{i_I} \|A_i x_i\|^2 = \sum_{i \in I} \|Ax\|_{H_i}^2 = \|Ax\|_H^2 \leq \|A\|^2. \tag{16}$$

Thus $\sup_i \|Ai\|^2 \leq \|A\|^2 < \infty$. Both inequalities conclude the desired equality.

4 Exercise 4

4.1

Let's start noting how the operator S affects the inner product.

$$\langle Sx, y \rangle = \langle (0, x_1, \dots), y \rangle = \sum_{i=1}^{\infty} x_i y_{i+1}$$

We can see that the inner product with x and the adjoint on y must be equal $\langle x, S^* y \rangle = \sum_{i=1}^{\infty} x_i y_{i+1}$ which is the left-shift operator

$$S^*(x_1, x_2, \dots) = (x_2, x_3, \dots)$$

4.2

Let's compute now the concatenation of S and its adjoint.

$$SS^*(x_1, x_2, \dots) = S(x_2, x_3, \dots) = (0, x_2, x_3, \dots)$$

Conversely

$$S^*S(x_1, x_2, \dots) = S(0, x_1, x_2, x_3, \dots) = (x_1, x_2, \dots) = x$$

4.3

And we can extend this to S^n and $(S^*)^n$

$$S^n (S^*)^n x = (0^n, x_{n+1}, n_{n+2}, \dots)$$

and similarly

$$(S^*)^n S^n = x$$