Homework 1 Rework

1 Exercise 1

The so defined time inversion process

$$B(t,\alpha) = \begin{cases} t^\alpha W(1/t) & t > 0 \\ 0 & t = 0 \end{cases} \tag{1}$$

in order to be a Brownian Motion has to satisfy the following properties:

1. with probability 1, the mapping $t \mapsto W(t)$ is continuous and $W(0) = 0$;
2. if $0 = t_0 < t_1 < \cdots < t_N = T$, then the increments

$$W(t_N) - W(t_{N-1}), \ldots, W(t_1) - W(t_0)$$

   are *independent*; and
3. for all $t > s$ the increment $W(t) - W(s)$ has the *normal* distribution, with $E[W(t) - W(s)] = 0$ and $E[(W(t) - W(s))^2] = t - s$

We begin by verifying property (1). It's clear from the property of the Brownian Motion W that $B(t,\alpha)$ is continuous in $(0,\infty)$. The only point that we need to check the continuity is in fact at $t = 0$.
As the hint suggests, we start by computing the following for an arbitrary $h > 0, t > 0$.

$$\mathrm{Cov}\left[B(t+h,\alpha), B(t,\alpha)\right] = \mathbb{E}\left[(t+h)^\alpha t^\alpha W\left(\frac{1}{t}\right) W\left(\frac{1}{t+h}\right)\right] - \mathbb{E}\left[B(t+h,\alpha)\right]\mathbb{E}\left[B(t,\alpha)\right].$$

We note that $\mathbb{E}\left[B(t,\alpha)\right] = t^\alpha \mathbb{E}\left[W\left(\frac{1}{t}\right)\right] = 0$. Thus, the right hand side can be simplified as follows.

$$\begin{aligned} \mathrm{Cov}\left[B(t+h,\alpha), B(t,\alpha)\right] &= \mathbb{E}\left[(t+h)^\alpha t^\alpha W\left(\frac{1}{t}\right) W\left(\frac{1}{t+h}\right)\right] \\ &= (t+h)^\alpha t^\alpha \frac{1}{t+h} \\ &= (t+h)^{\alpha-1} t^\alpha \end{aligned}$$

Where in the one to last equality we used the fact that $\mathbb{E}\left[W_s W_t\right] = \min(s,t)$. This gives a necessary condition for $B(t,\alpha)$ to be a Brownian Motion, that is $\alpha = 1$. We will proceed in showing the other properties of $B(t,\alpha)$ assuming $\alpha = 1$.

For a mesh $\Pi = t_1 < t_2 < \cdots < t_n$ we write the process B_t as follows.

$$\begin{bmatrix} B(t_1) \\ \vdots \\ B(t_n) \end{bmatrix} = A \begin{bmatrix} W\left(\frac{1}{t_1}\right) \\ \vdots \\ W\left(\frac{1}{t_n}\right) \end{bmatrix} \tag{2}$$

Where just by the definition of the process B_t, we have $A = \operatorname{diag}(t_1, \ldots, t_n)$. Moreover, we can write the matrix on the right hand side as follows

$$\begin{bmatrix} W\left(\frac{1}{t_1}\right) \\ \vdots \\ W\left(\frac{1}{t_n}\right) \end{bmatrix} = 0^n + \begin{bmatrix} \frac{1}{\sqrt{t_1}} \\ \vdots \\ \frac{1}{\sqrt{t_n}} \end{bmatrix} \begin{bmatrix} \mathcal{N}(0,1) \\ \vdots \\ \mathcal{N}(0,1) \end{bmatrix}. \tag{3}$$

Upon defining the matrix D^n as follows

$$D^n = \begin{bmatrix} 0 & \cdots & 0 \\ & I_{n-1} & \end{bmatrix}. \tag{4}$$

We can then write the matrix of the increments with the ingredients we have prepared so far.

$$B' = \begin{bmatrix} B_{t_1} \\ \vdots \\ B_{t_n} \end{bmatrix} - \begin{bmatrix} 0 \\ B_1 \\ \vdots \\ B_{t_{n-1}} \end{bmatrix} \tag{5}$$

It follows that $B' = A\mathcal{W} - D^n A\mathcal{W} = (I - D^n)A\mathcal{W}$, where $\mathcal{W} = \begin{bmatrix} W\left(\frac{1}{t_1}\right) \\ \vdots \\ W\left(\frac{1}{t_n}\right) \end{bmatrix}$. From this representation we get that B' is multivariate Gaussian distributed, implying that each marginal is also Gaussian. Moreover, we have that $\mathbb{E}\left[B_{t_n} - B_{t_{n-1}} = 0\right]$. Moreover, we have that

$$\begin{aligned} \operatorname{Cov}\left[B'_i, B'_j\right] &= \mathbb{E}\left[B'_i B'_j\right] \\ &= \mathbb{E}\left[\left(W\left(\frac{1}{t_i}\right) - W\left(\frac{1}{t_{i-1}}\right)\right)\left(W\left(\frac{1}{t_j}\right) - W\left(\frac{1}{t_{j-1}}\right)\right)\right] \\ &= \mathbb{E}\left[W\left(\frac{1}{t_i}\right) W\left(\frac{1}{t_j}\right)\right] - \mathbb{E}\left[W\left(\frac{1}{t_i}\right) W\left(\frac{1}{t_{j-1}}\right)\right] - \\ &\quad - \mathbb{E}\left[W\left(\frac{1}{t_{i-1}}\right) W\left(\frac{1}{t_j}\right)\right] + \mathbb{E}\left[W\left(\frac{1}{t_{i-1}}\right) W\left(\frac{1}{t_{j-1}}\right)\right] \\ &= 0 \end{aligned}$$

From the previous point, by taking the limit of the mesh $\mathcal{P}$, we have that for every a_t and $t \in [0,\infty) \cap \mathbb{Q}$, the following holds true: $P(\{B(t) < a_t, a_t \in \mathbb{R}, t \in [0,\infty) \cap \mathbb{Q}\}) = P(\{B(t) < a_t, a_t \in \mathbb{R}, t \in [0,\infty) \cap \mathbb{Q}\})$.
To conclude the proof that B is continuous at $t = 0$ look at the quantity $P[\lim_{t\to 0} B_t = 0]$. Let Q be the measure of the process B. We have that

$$P[\lim_{t\to 0} B_t = 0] = Q(x \in C^0(0,1] : \lim_{t\to 0} x(t) = 0). \tag{6}$$

Moreover, we can write the event in the following way

$$\{x \in C^0(0,1] : \lim_{t\to 0} x(t) = 0\} = \bigcup_{t\in Q\cap(0,\infty)}^{\infty} \bigcap_{\epsilon\in Q\cap(0,\infty)}^{\infty} \{x : |x(t)| < \epsilon\}. \tag{7}$$

Since the Borel sigma-algebra on $C([],1]$ is generated by the evaluation map $e_t(x) = x(t)$, thus by the form $e_t^{-1}(B)$ for B in the Borel of $\mathbb{R}$. Therefore, $A_{t,\epsilon} := e_t^{-1}((-\epsilon,\epsilon))$ is measurable in $B(C[0,1])$. Therefore, each $\{|x(t)| < \epsilon\}$ is measurable. Taking countable union and intersection we get that the event is measurable. Using now the fact that the two measures agree on the rationals, we can rewrite it as

$$Q(x \in C^0(0,1] : \lim_{t\to 0} x(t) = 0) = \overline{W}(x \in C^0(0,1] : \lim_{t\to 0} x(t) = 0) = P[\lim_{t\to 0} B_t = 0] = 1. \tag{8}$$

Where $\overline{W}$ is the Wiener measure. It follows that the process B is continuous at $t = 0$.

2 Exercise 2

2.1

In this exercise we will prove the following equality:

$$\int_0^T t\, dW(t) = TW(T) - \int_0^T W(t)\, dt \tag{9}$$

Looking at the left hand side, by taking it's forward Euler we obtain

$$I_1 = \int_0^T t dW(t) = \sum_{n=0}^{N-1} t_n (W(t_{n+1} - W(t_n))). \tag{10}$$

Applying the hint, i.e. using the Abel's summation by parts we get

$$\begin{aligned}\sum_{n=0}^{N-1} t_n(W(t_{n+1} - W(t_n))) &= t_N W(t_N) - t_0 W(t_0) - \sum_{n=1}^{N-1} W(t_n)(t_n - t_{n-1}) \\ &= TW(T) - \sum_{n=1}^{N-1} W(t_n)(t_n - t_{n-1}).\end{aligned}$$

What is left to prove is the convergence in L_2 of the right hand side, i.e. the following equation.

$$\lim_{n\to\infty} \mathbb{E}\left[\left(\int_0^T W(t)dt - \sum_{n=1}^{N-1} W(t_n)(t_n - t_{n-1})\right)^2\right] = 0. \tag{11}$$

Using the linearity of the integral, we can rewrite it as follows

$$\begin{aligned}\mathbb{E}\left[\left(\sum_{n=0}^{N-1}\int_{t_n}^{t_n+1} W(t)dt - \sum_{n=1}^{N-1} W(t_n)(t_n - t_{n-1})\right)^2\right] &= \mathbb{E}\left[\left(\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} W(t) - W(t_n)dt\right)^2\right] \\ &= \sum_{n=0}^{N-1}\mathbb{E}\left[\left(\int_{t_n}^{t_{n+1}} W(t) - W(t_n)dt\right)^2\right] \\ &= \sum_{n=0}^{N-1}\mathbb{E}\left[e_n^2\right].\end{aligned}$$

Where e_n is n-th error term. Moreover, in the last equation we dropped the cross terms since $\mathbb{E}\left[e_i e_j\right] = 0$ by the properties of the Brownian motion.
Next, we compute $\mathbb{E}\left[e_n^2\right]$ for every $n \geq 0$.

$$\begin{aligned}\mathbb{E}\left[e_n^2\right] &= \mathbb{E}\left[\left(\int_{t_n}^{t_{n+1}} W(t) - W(t_n)dt\right)^2\right] \\ &= \mathbb{E}\left[\int_{t_n}^{t_{n+1}}\int_{t_n}^{t_{n+1}} \left(W(t) - W(t_n)dt\right)\left(W(s) - W(t_n)ds\right)\right] \\ &= \mathbb{E}\left[\int_{t_n}^{t_{n+1}}\int_{t_n}^{t_{n+1}} \left(W(t) - W(t_n)\right)\left(W(s) - W(t_n)\right)dtds\right].\end{aligned}$$

Where we used twice Fubini's Theorem as the integrands are squared integrable functions since the Brownian Motion is. We also use Fubini for the following equality as the expectation is finite for arbitrary Brownian motion.

$$\mathbb{E}\left[\int_{t_n}^{t_{n+1}}\int_{t_n}^{t_{n+1}}\left(W(t)-W(t_n)\right)\left(W(s)-W(t_n)\right)dtds\right]=$$
$$=\int_{t_n}^{t_{n+1}}\int_{t_n}^{t_{n+1}}\mathbb{E}\left[\left(W(t)-W(t_n)\right)\left(W(s)-W(t_n)\right)dtds\right]$$
$$=\int_{t_n}^{t_{n+1}}\int_{t_n}^{t_{n+1}}\min(t,s)-t_k dtds$$
$$=\frac{1}{3}(\Delta t_{\max})^3.$$

Finally, by summing all the values, we get

$$\sum_{n=0}^{T}\mathbb{E}\left[e_n\right]=\sum_{n=0}^{t}\frac{1}{3}(\Delta t_{\max})^3<\frac{T^3}{n}. \tag{12}$$

It follows that by letting n go to infinity, the Forward Euler converges in L_2.

2.2

As in the hint, we now prove the following

$$\sum_{n=0}^{N-1}W(t_n)\big(W(t_{n+1})-W(t_n)\big)=\sum_{n=0}^{N-1}\frac{W(t_{n+1})^2-W(t_n)^2}{2}-\frac{\big(W(t_{n+1})-W(t_n)\big)^2}{2} \tag{13}$$

We start by looking at the right hand side.

$$\sum_{n=0}^{N-1}\frac{W(t_{n+1})^2-W(t_n)^2}{2}-\frac{\big(W(t_{n+1})-W(t_n)\big)^2}{2}=$$
$$=\sum_{n=0}^{N-1}\frac{W(t_{n+1})^2-W(t_n)^2}{2}-\frac{W(t_{n+1})^2-2W_{t_{n+1}}W(t_n)+W_{t_n}^2}{2}$$
$$=\sum_{n=0}^{N-1}\frac{2W_{t_n}W_{t_{n+1}}-2W(t_n)}{2}=\sum_{n=0}^{N-1}W(t_n)\big(W(t_{n+1})-W(t_n)\big)$$

Note that the

$$\sum_{n=0}^{N-1}\frac{W(t_{n+1})^2-W(t_n)^2}{2}=\frac{W(T)}{2} \tag{14}$$

as it is a telescopic sum. To finalize the proof, we need to show that the second part converges to $\frac{T}{2}$ or in other words that

$$\mathbb{E}\left[\left(\sum_{n=0}^{N-1}\left[W(t_{n+1})-W(t_n)\right]^2-T\right)^2\right]\to 0. \tag{15}$$

We define the summation $S_N \coloneqq \sum_{n=0}^{N-1} \left[W(t_{n+1}) - W(t_n)\right]^2$. It follows that

$$\mathbb{E}\left[(S_N - T)^2\right] = \mathbb{E}\left[S_N^2\right] - T^2 - 2T\mathbb{E}\left[S_N\right]. \tag{16}$$

We focus now on $\mathbb{E}\left[S_N\right]$ as follows.

$$\begin{aligned}
\mathbb{E}\left[S_N\right] &= \sum_{n=0}^{N-1} \mathbb{E}\left[W_{t_{n-1}}^2 + W_{t_n}^2 - 2W_{t_{n+1}}W_{t_n}\right] \\
&= \sum_{n=0}^{N-1} t_{n+1} + t_n - 2t_n = \sum_{n=0}^{N-1} t_{n+1} - t_n = T.
\end{aligned}$$

Where we used the fact that $\mathbb{E}\left[W_s W_t\right] = \min(s,t)$. It follows that $\mathbb{E}\left[(S_N - T)^2\right] = \mathbb{E}\left[S_N^2\right] - T^2 = \mathbb{E}\left[S_n^2\right] - \mathbb{E}\left[S_N\right]^2 = \mathrm{Var}\left[S_n\right]$.
The problem reduces then to compute $\mathbb{E}\left[S_N^2\right]$.

$$\mathbb{E}\left[S_N^2\right] = \sum_{n}^{N-1} \sum_{m}^{N_1} \mathbb{E}\left[\Delta W_n^2 \Delta W_m^2\right]$$

For $m = n$ we get $\mathbb{E}\left[\Delta W_m^4\right] = 3\Delta t^2$ by using the forth moment of the standard normal. On the other hand, for $n \neq m$ we get the following

$$\mathbb{E}\left[\Delta W_m^w \Delta W_n^2\right] = \mathbb{E}\left[\Delta W_m^2\right] \mathbb{E}\left[\Delta W_n^w\right] = (\Delta t)^2. \tag{17}$$

Putting everything together we obtain

$$\begin{aligned}
\mathbb{E}\left[S_N^2\right] &= 3N(\Delta t)^2 + N(N-1)(\Delta t)^2 \\
&= \frac{2T^2}{N} - T^2 N \\
\implies \mathbb{E}\left[(S_N - T)^2\right] &= \mathrm{Var}\left[S_N\right] = \frac{2T^2}{N}
\end{aligned}$$

Which clearly converges to 0 as $n \to \infty$.

3 Exercise 3

3.1 Mean and Variance

Given the stochastic process

$$X(t) = x_\infty + e^{-at}(x_0 - x_\infty) + b \int_0^t e^{-a(t-s)}\, dW(s). \tag{18}$$

We compute its expectation and variance as follows.

$$\mathbb{E}\left[X_t\right] = x_\infty + e^{-at}(x_0 - x_\infty) + b\mathbb{E}\left[(f(t) \cdot W_s)_t\right]$$

.

Using the fact that the ito integral of an adapted (since it is deterministic) process with respect to the Brownian Motion has 0 expectation, it further simplifies to.

$$\mathbb{E}\left[X_t\right] = x_\infty + e^{-at}(x_0 - x_\infty). \tag{19}$$

We now take its limit for $t \to \infty$ we obtain

$$\begin{aligned}\lim_{t\to\infty} \mathbb{E}\left[X_t\right] &= \lim_{t\to\infty} x_\infty + e^{-at}(x_{0-x_\infty}) \\ &= x_\infty.\end{aligned}$$

We now proceed onto computing the variance of the process and it's limit as $t \to \infty$.

$$\begin{aligned}\mathrm{Var}\left[X_t\right] = \mathbb{E}\left[(X_t - \mathbb{E}\left[X_t\right])^2\right] &= \mathbb{E}\left[X_t^2\right] - \mathbb{E}\left[X_t\right]^2 \\ &= b^2\mathbb{E}\left[\left(\int_0^t e^{-a(t-s)}dW_s\right)\right].\end{aligned}$$

In the last equality we simply used the fact that the Ito integral has again zero expectation, therefore it annihilates all the terms that are multiplied to as they are bounded. We shall now use the Ito isometry.

$$\begin{aligned}b^2\mathbb{E}\left[\left(\int_0^t e^{-a(t-s)}dW_s\right)\right] &= b^2\mathbb{E}\left[\int_0^t e^{-2a(t-s)}ds\right] \\ &= b^2e^{-2at}\mathbb{E}\left[\int_0^t e^{-2as}ds\right] \\ &= \frac{b^2e^{-2at}}{2a}(e^{2at}+1) \\ &= \frac{b^2}{2a}(1-e^{-2at}).\end{aligned}$$

Thus

$$\lim_{t\to\infty} \mathrm{Var}\left[X_t\right] = \lim_{t\to\infty} \frac{b^2}{2a}(1-e^{-2at}) = \frac{b^2}{2a}. \tag{20}$$

3.2 Covariance

For a more general settings, we compute the variance between X_t and X_s where w.log. we assume $t > s$.

$$
\begin{aligned}
\operatorname{Cov}\left[X_s, X_t\right] &= b^2 \operatorname{Cov}\left[\int_0^s e^{a(t-u)} dW_u, \int_0^t e^{-a(t-u)} dW_u\right] \\
&= \mathbb{E}\left[\int_0^s e^{-a(t-u)} dW_u \int_0^t e^{-a(t-u)} dW_u\right] \\
&= b^2 e^{-a(s+t)} \mathbb{E}\left[\int_0^s e^{au} dW_u \int_0^t e^{au} dW_u\right] \\
&= b^2 e^{-a(s+t)} \mathbb{E}\left[\int_0^s e^{au} dW_u \left(\int_0^s e^{au} dW_u + t_s^t e^{au} dW_u\right)\right]
\end{aligned}
$$

By using the independence of the increments of Ito integral, we have the expectation of the product as the product of the expectation as follows.

$$
\begin{aligned}
&b^2 e^{-a(s+t)} \mathbb{E}\left[\int_0^s e^{au}\, dW_u \left(\int_0^s e^{au}\, dW_u + \int_s^t e^{au}\, dW_u\right)\right] = \\
&\qquad = b^2 e^{-a(s+t)} \mathbb{E}\left[\left(\int_0^s e^{au}\, dW_u\right)^2 + \int_s^t e^{au}\, dW_u \int_0^s e^{au}\, dW_u\right] \\
&\qquad = b^2 e^{-a(s+t)} \left(\mathbb{E}\left[\left(\int_0^s e^{au}\, dW_u\right)^2\right] + \mathbb{E}\left[\int_s^t e^{au}\, dW_u \int_0^s e^{au}\, dW_u\right]\right) \\
&\qquad = b^2 e^{-a(s+t)} \mathbb{E}\left[\left(\int_0^s e^{au}\, dW_u\right)^2\right]
\end{aligned}
$$

By Ito isometry again

$$
\begin{aligned}
b^2 e^{-a(s+t)} \mathbb{E}\left[\left(\int_0^s e^{au}\, dW_u\right)^2\right] &= b^2 e^{-a(s+t)} \mathbb{E}\left[\int_0^s e^{2au}\, du\right] \\
&= b^2 e^{-a(s+t)} \frac{1}{2a} \left(e^{2at} - 1\right) \\
&= \frac{b^2}{2a} \left(e^{-a(t-s)} - e^{-a(s+t)}\right) \\
&= \frac{b^2}{2a} e^{-a(t-s)} \left(1 - e^{-2as}\right) \\
&= \frac{b^2}{2a} e^{-a\tau} \left(1 - e^{-2as}\right)
\end{aligned}
$$

Where we replaced $\tau = t - s$. It follows

$$\text{Cov}\left[X_s, X_t\right] \to 0 \text{ as } t \to \infty. \tag{21}$$

3.3 interpretation

When we look at the expectation we see that it starts at x_0 and will decay or expand until it comes to x_∞ in the infinite time limit.

We note that we start with a variance of zero, that will reach $\frac{b}{2a}$ asymptotically.
The correlation between $X(t)$ and $X(t+\tau)$ converges over time to the variance that is $\frac{b^2}{2a}e^{-a\tau}$. This limit will exponentially decrease for larger τ. This shows that the closer $X(t)$ and $X(t+\tau)$ are to each other, the bigger the correlation between the two. For larger separation time τ, the correlation will diminish.

3.4 Numerics

3.4.1 Setup

For the numeric simulation, we set the following parameters:

- $x_0 = 0$
- $x_{\text{inf}} = 1$
- $a = b = \frac{1}{2}$
- $T = 10$
- $dt = 0.1$

We generated 1000 sample paths, using the following formula obtained just by differentiation of the stochastic process.

$$dX_t = -a(X_t - x_\infty)dt + bdW_t. \tag{22}$$

In order to use Euler-Maruyama, we adjusted the above with the following approximation

$$X_{t+1} = X_t - a(X_{t-1} - x_\infty)\Delta t + bW_{\Delta t}. \tag{23}$$

3.4.2 Results

The first thing to notice is the limit of the expectation. This is x_∞, therefore, the process will converge in mean to such value.

To corroborate this, by running numerical experiments, we can see that indeed it converges in mean to x_∞. In the plot 1 we plotted only 10 sample paths, however, for the mean we use a sample size of 1000 to make it converge.

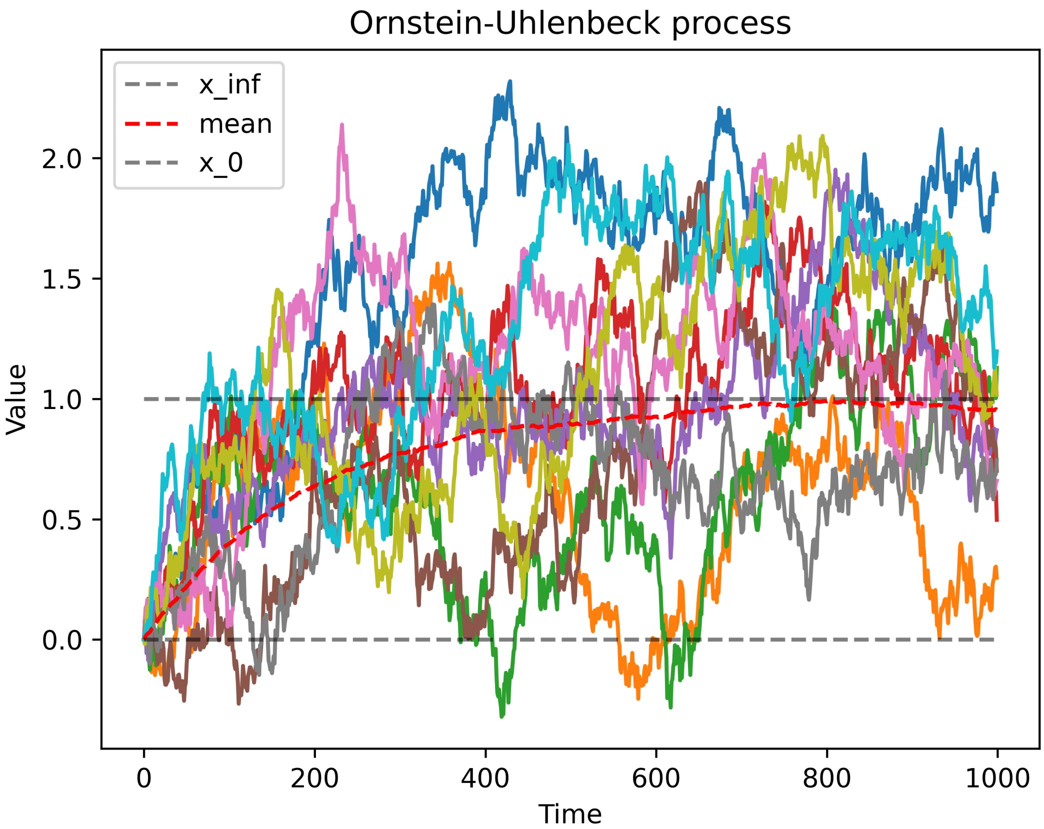

Figure 1: Sample paths of the Ornstein-Uhlenbeck (OU) process over time, highlighting key statistical features. The dashed gray line represents the long-term equilibrium x_∞, while the red dashed line shows the mean of the process, which converges to x_∞ as time increases. The initial condition x_0 is indicated by the dashed black line. Each colored line corresponds to a different realization of the OU process.

The variance of the process is plotted against its asymptotic limit in figure 2

Finally, regarding the covariance, we plot in 3 the theoretical variance as well as the numeric one.

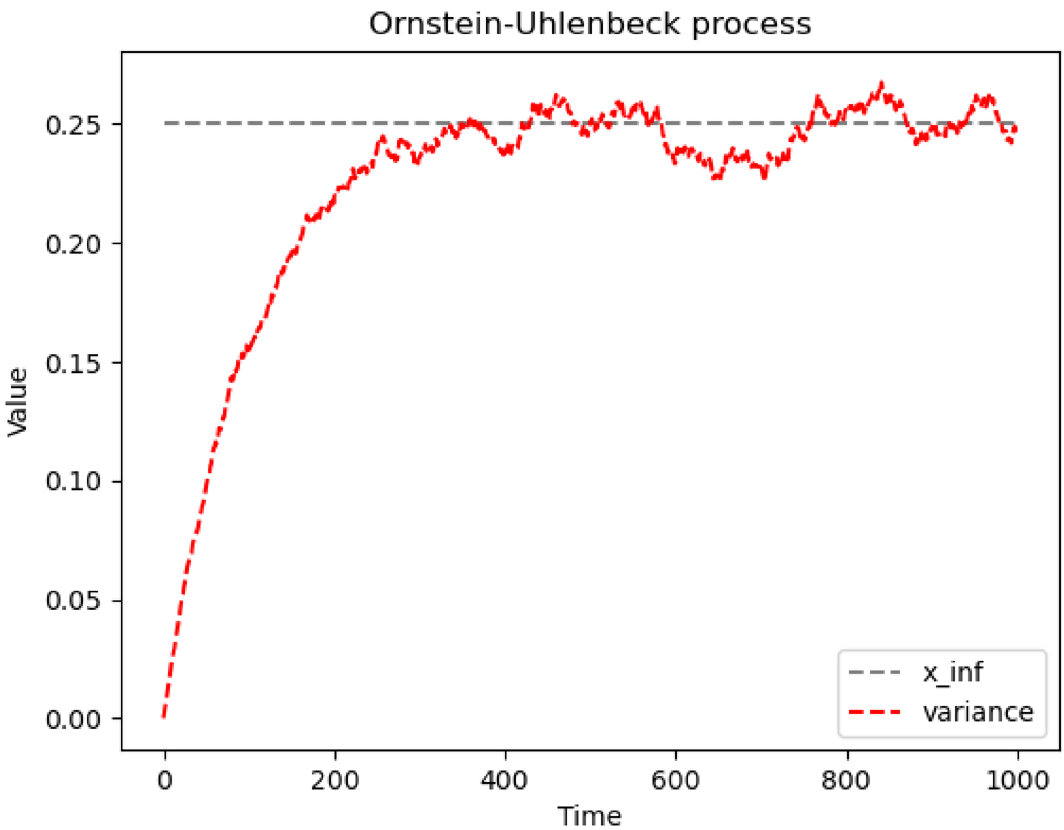

Figure 2: Variance of the Ornstein-Uhlenbeck (OU) process X_t over time (red dashed line) compared to the theoretical long-term variance $\frac{\sigma^2}{2\alpha}$ (dashed gray line). As expected, the variance converges to the theoretical limit $x_\infty = \frac{\sigma^2}{2\alpha}$ as time increases.

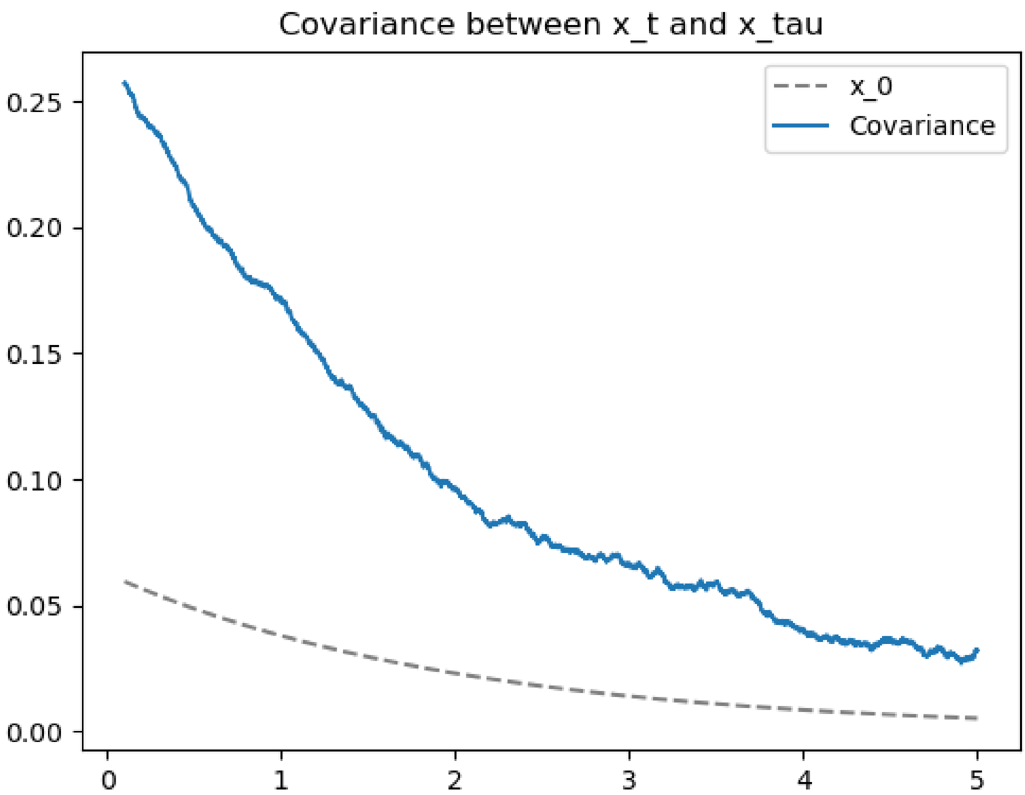

Figure 3: Plot showing the covariance between X_t and $X_{t+\tau}$ for $\tau > 0$. The solid blue line represents the empirical covariance of the process over time, while the dashed gray line shows the decay of the covariance starting from the initial condition x_0. The covariance decreases and asymptotically approaches zero as τ increases, indicating that the process becomes less correlated as the time difference grows.