Homework 1

1 Exercise 1

2 Exercise 2

2.1

In this exercise we will prove the following equality:

$$\int_0^T t\,dW(t) = TW(T) - \int_0^T W(t)\,dt \tag{1}$$

Look at the left hand side, by taking it's forward Euler we obtain

$$I_1 = \int_0^T tdW(t) = \sum_{n=0}^{N-1} t_n(W(t_{n-1} - W(t_n))). \tag{2}$$

Applying the hint, i.e. using the Abel's summation by parts we get

$$\begin{aligned}\sum_{n=0}^{N-1} t_n(W(t_{n-1} - W(t_n))) &= t_N W(t_N) - t_0 W(t_0) - \sum_{k=1}^{N-1} W(t_k)(t_k - t_{k-1}) \\ &= TW(T) - \sum_{k=1}^{N-1} W(t_k)(t_k - t_{k-1}).\end{aligned}$$

What is left to prove is the convergence in L_2 of the right hand side, i.e. the following equation.

$$\lim_{n\to\infty} \mathbb{E}\left[\left(\int_0^T W(t)dt - \sum_{k=1}^{N-1} W(t_k)(t_k - t_{k-1})\right)^2\right] = 0. \tag{3}$$

Let's fix $t_n - t_{n-1} = \Delta t$ for every $n \geq 0$. Using the linearity of the integral, we can rewrite it as follows

$$\begin{aligned}\mathbb{E}\left[\left(\sum_{n=0}^{N-1}\int_{t_{n-1}}^{t_n} W(t)dt - \sum_{k=1}^{N-1} W(t_k)(t_k - t_{k-1})\right)^2\right] &= \mathbb{E}\left[\left(\sum_{n=0}^{N-1}\int_{t_{n-1}}^{t_n} W(t) - W(t_n)dt\right)^2\right] \\ &= \sum_{n=0}^{N-1} \mathbb{E}\left[\int_{t_{n-1}}^{t_n} W(t) - W(t_n)dt\right] \\ &= \sum_{n=0}^{N-1} \mathbb{E}\left[e_n^2\right].\end{aligned}$$

Where e_n is n-th error term. Moreover, in the last equation we dropped the cross terms since $\mathbb{E}\left[e_i e_j\right] = 0$ by the properties of the Brownian motion.
Next, we compute $\mathbb{E}\left[e_n^2\right]$ for every $n \geq 0$.

$$\begin{aligned}
\mathbb{E}\left[e_n^2\right] &= \mathbb{E}\left[\left(\int_{t_n}^{t_{n-1}} W(t) - W(t_n)dt\right)^2\right] \\
&= \mathbb{E}\left[\int_{t_n}^{t_{n-1}} \int_{t_n}^{t_{n-1}} \left(W(t) - W(t_n)dt\right)\left(W(s) - W(t_n)ds\right)\right] \\
&= \mathbb{E}\left[\int_{t_n}^{t_{n-1}} \int_{t_n}^{t_{n-1}} \left(W(t) - W(t_n)\right)\left(W(s) - W(t_n)\right) dtds\right].
\end{aligned}$$

Where we used twice Fubini's Theorem. We also use Fubini for the following equality.

$$\begin{aligned}
\mathbb{E}\left[\int_{t_n}^{t_{n-1}} \int_{t_n}^{t_{n-1}} \left(W(t) - W(t_n)\right)\left(W(s) - W(t_n)\right) dtds\right] &= \int_{t_n}^{t_{n-1}} \int_{t_n}^{t_{n-1}} \mathbb{E}\left[\left(W(t) - W(t_n)\right)\left(W(s) - W(t_n\right.\right. \\
&= \int_{t_n}^{t_{n-1}} \int_{t_n}^{t_{n-1}} \min(t,s) - t_k dtds \\
&= \frac{1}{3}(\Delta t)^3.
\end{aligned}$$

Finally, by summing all the values, we get

$$\sum_{n=0}^{T} \mathbb{E}\left[e_n\right] = \sum_{n=0}^{t} frac13(\Delta t)^3 = \frac{T^3}{n^2}. \tag{4}$$

It follows that by letting n go to infinity, the Forward Euler converges in L_2.

2.2

As in the hint, we now prove the following

$$\sum_{n=0}^{N-1} W(t_n)\big(W(t_{n+1}) - W(t_n)\big) = \sum_{n=0}^{N-1} \frac{W(t_{n+1})^2 - W(t_n)^2}{2} - \frac{\big(W(t_{n+1}) - W(t_n)\big)^2}{2} \tag{5}$$

We start by looking at the right hand side.

$$\sum_{n=0}^{N-1} \frac{W(t_{n+1})^2 - W(t_n)^2}{2} - \frac{\big(W(t_{n+1}) - W(t_n)\big)^2}{2} =$$

$$= \sum_{n=0}^{N-1} \frac{W(t_{n+1})^2 - W(t_n)^2}{2} - \frac{W(t_{n+1})^2 - 2W_{t_{n+1}}W(t_n) + W_{t_n}^2}{2}$$

$$= \sum_{n=0}^{N-1} \frac{2W_{t_n}W_{t_{n+1}} - 2W(t_n)}{2} = \sum_{n=0}^{N-1} W(t_n)\big(W(t_{n+1}) - W(t_n)\big)$$

Note that the

$$\sum_{n=0}^{N-1} \frac{W(t_{n+1})^2 - W(t_n)^2}{2} = \frac{W(T)}{2} \tag{6}$$

as it is a telescopic sum. To finalize the proof, we need to show that the second part converges to $\frac{T}{2}$ or in other words that

$$\mathbb{E}\left[\left(\sum_{n=0}^{N-1} [W(t_{n+1}) - W(t_n)]^2 - T\right)^2\right] \to 0. \tag{7}$$

We define the summation $S_N \coloneqq \sum_{n=0}^{N-1} [W(t_{n+1}) - W(t_n)]^2$. It follows that

$$\mathbb{E}\left[(S_N - T)^2\right] = \mathbb{E}\left[S_N^2\right] - T^2 - 2T\mathbb{E}\left[S_N\right]. \tag{8}$$

We focus now on $\mathbb{E}[S_N]$ as follows.

$$\mathbb{E}[S_N] = \sum_{n=0}^{N-1} \mathbb{E}\left[W_{t_{n-1}}^2 + W_{t_n}^2 - 2W_{t_{n+1}}W_{t_n}\right]$$

$$= \sum_{n=0}^{N-1} t_{n+1} + t_n - 2t_n = \sum_{n=0}^{N-1} t_{n+1} - t_n = T.$$

Where we used the fact that $\mathbb{E}[W_sW_t] = \min(s,t)$. It follows that $\mathbb{E}[(S_N - T)^2] = \mathbb{E}[S_N^2] - T^2 = \mathbb{E}[S_n^2] - \mathbb{E}[S_N]^2 = \mathrm{Var}[S_n]$.
The problem reduces then to compute $\mathbb{E}[S_N^2]$.

$$\mathbb{E}\left[S_N^2\right] = \sum_{n}^{N-1} \sum_{m}^{N_1} \mathbb{E}\left[\Delta W_n^2 \Delta W_m^2\right]$$

For $m = n$ we get $\mathbb{E}[\Delta W_m^4] = 3\Delta t^2$ by using the forth moment of the standard normal.

On the other hand, for $n \neq m$ we get the following

$$\mathbb{E}\left[\Delta W_m^w \Delta W_n^2\right] = \mathbb{E}\left[\Delta W_m^2\right]\mathbb{E}\left[\Delta W_n^w\right] = (\Delta t)^2. \tag{9}$$

Putting everything together we obtain

$$\begin{aligned} \mathbb{E}\left[S_N^2\right] &= 3N(\Delta t)^2 + N(N-1)(\Delta t)^2 \\ &= \frac{2T^2}{N} - T^2 N \\ \implies \mathbb{E}\left[(S_N - T)^2\right] &= \mathrm{Var}\left[S_N\right] = \frac{2T^2}{N} \end{aligned}$$

Which clearly converges to 0 as $n \to \infty$.