Homework 1

1 Exercise 1

1.1

Let's define x_0 as the limit of the net

$$\sum_{a\in A} a \to x_0. \tag{1}$$

Since it converges, it means that there exist a set $F_0 \subset A$ such that for all $F > F_0$ the following holds

$$\|\sum_{a\in F} a - x_0\| < \epsilon \tag{2}$$

for a given $\epsilon > 0$. Since ϵ is arbitrary, we can choose $F_0' > F_0$ such that for all $F > F_0'$

$$\|\sum_{a\in F} a - x_0\| < \frac{\epsilon}{|\alpha|}. \tag{3}$$

Then, by properties of the norm we obtain

$$\begin{aligned}
&|\alpha|\|\sum_{a\in F} a - x_0\| < \frac{\epsilon}{|\alpha|}\alpha = \epsilon \\
&\|\alpha\sum_{a\in F} a - \alpha x_0\| < \epsilon \\
&\|\sum_{a\in F} \alpha a - \alpha x_0\| < \epsilon.
\end{aligned}$$

Where in the last step we used the fact that F is finite. This proves that $\alpha \sum_{a\in A} a$ converges to $\alpha x_0 = \alpha \sum_{a\in A} a$.

1.2

The hypothesis that $\sum_{a\in A} a$ and $\sum_{b\in B} b$ implies that there exists an F_0^a and F_0^b such that for every $F^a > F_0^a$ and $F^b > F_0^b$ the following holds

$$\|\sum_{a\in F^a} a - \sum_{a\in A} a\| < \frac{\epsilon}{2} \qquad \|\sum_{b\in F^b} b - \sum_{b\in B} b\| < \frac{\epsilon}{2}. \tag{4}$$

Denote $F_0 = F_0^a \cup F_0^b$, it follows that for every $F > F_0$

$$\begin{aligned}\|\sum_{x\in F} x - \sum_{a\in A} a - \sum_{b\in B} b\| &= \|\sum_{x\in F\cap A} + \sum_{x\in F\cap B} x - \sum_{a\in A} a - \sum_{b\in B} b\| \\ &\leq \|\sum_{x\in F\cap A} x - \sum_{a\in A} a\| + \|\sum_{x\in F\cap B} x - \sum_{b\in B} b\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Because ϵ was arbitrary, we conclude that $\sum_{x\in A\cup B} \to \sum_{a\in A} a + \sum_{b\in B} b$.

1.3

Let's star by proving that any converging net of positive numbers has at most a countable number of non zero elements.

Say that the net converges to M, i.e. $\sum_{a\in A} a = M < \infty$ where for every $a \in A, a > 0$. Consider now the sets $S_n = \{a \in A | a > \frac{1}{n}\}$, then

$$M \geq \sum_{a\in S_n} a \geq \sum_{a\in S_n} \frac{1}{n} = \frac{N}{n}.$$

As $M < \infty$ so is N which is the cardinality of the set S_n. It follows that

$$\#\{a \in A | a > 0\} = \#S = \# \bigcup_{n=\mathbb{N}}^{\infty} S_n \tag{5}$$

We conclude that A has at most countable number of non zero elements as countable union of finite sets.

Let's now prove the ($\implies$) direction. Given the previously proven statement, we can rewrite the net as countable sum and thus define a corresponding sequence $x_n = \sum_{i=0}^{n} a_i$ where w.log. we associated every non zero element a of A to an index i so that $a_i = a$. From standard analysis we obtain that every converging increasing sequence is bounded from above, i.e. there exists $N \in R$ so that $x_n < N$ for every n. It follows that for every finite $F \subset I$

$$\sum_{a\in F} a \leq \sum_{i\in\mathbb{N}} a_i \leq N. \tag{6}$$

We now prove the opposite implication ($\impliedby$). Assume that $\sup \left\{\sum_{a\in F} a : F \in \mathcal{F}\right\} = x_0$. We proceed now by contradiction, suppose that $\sum_{a\in A} a \to x_0 + t$ for an arbitrary $t > 0$. Let's define now what does it mean for a net to be increasing. Given a set F_0, it holds that for every $F > F_0$ we have that $\sum_{a\in F} a \geq \sum_{a\in F_0} a$.

By the definition of net convergence for standard set inclusion as order, we obtain that the net $\sum_{a\in A} a$ is increasing. It follows that for every F, $\sum_{a\in F} \leq x_0 + t$. Moreover, since

$\sum_{a\in A} a \to x_0+t$, there exists and F_t such that $\forall F > F_t$ $\|\sum_{a\in F} a - x_0 - t\| < \frac{t}{2}$. It follows

$$\begin{aligned}
&\|\sum_{a\in F} a - x_0 - t\| \leq 0\\
&\|\sum_{a\in F} a - x_0 - t\| = \sum_{a\in F} a - x_0 - t < \frac{t}{2}\\
&\sum_{a\in F} a > x_0 + \frac{t}{2}
\end{aligned}$$

Which is a contradiction. As this holds for any arbitrary t, $\sum_{a\in A} a \leq x_0$.
Let's conclude the proof by showing that

$$\sup\left\{\sum_{a\in F} a : F \in \mathcal{F}\right\} = \sum_{a\in A} a = x_0. \tag{7}$$

Take an arbitrary $x < x_0$. By contradiction, assume that $\sup\left\{\sum_{a\in F} a : F \in \mathcal{F}\right\} = x$. Again, by convergence of the net, there exists a F_0 such that $\forall F > F_0$ we have

$$\begin{aligned}
\|\sum_{a\in F} a - x_0\| &< |x - x_0|\\
x_0 - \sum_{a\in F} &x_0 - x\\
\sum_{a\in F} a &> x
\end{aligned}$$

Which is a contradiction. Combining it with the previous point we get the desired equality.
To double check that this is all we need.

2 Exercise 2

2.1

Let's define $H' = \bigvee \mathcal{F}$. By theorem 4.13 we have that $\forall h \in H'$, h can be written as $h = \sum_{e\in F}\langle h, e\rangle e$ as F is the basis for H' again by theorem 4.13.
Moreover, for every $x \in H$, we have that by definition $P_F x \in H'$. We define the operator Q as such

$$Qx := \sum_{e\in F}\langle x, e\rangle e \tag{8}$$

For Q to be equal to P_F, Qx has to be the unique elements in H' such that $x - Qx \perp H'$. We proceed by taking an orthogonal basis of H such that $\mathcal{E} \subset B$. By theorem 4.13 again, by trivially noting that $\bigvee H = H$, $x$can be represented as $x = \sum_{e\in B}\langle x, e\rangle e$. It follows that

$$\begin{aligned} x - Qx &= \sum_{e \in B} \langle x, e \rangle e - \sum_{e \in F} \langle x, e \rangle e \\ &= \sum_{e \in F} \langle x, e \rangle e + \sum_{e \in B \setminus F} \langle x, e \rangle e - \sum_{e \in F} \langle x, e \rangle e \\ &= \sum_{e \in B \setminus F} \langle x, e \rangle e \end{aligned}$$

We conclude that since B is orthogonal, it follows that $x - Qx \perp H' = \bigvee F$.

2.2

By the previous point, we can write $P_G x = \sum_{e \in G} \langle x, e \rangle e$ for every $G \subset \mathcal{G}$. It follows that

$$\begin{aligned} P_F P_G x &= P_F \left(\sum_{e \in G} \langle x, e \rangle e \right) \\ &= \sum_{e' \in F} \langle \sum_{e \in G} \langle x, e \rangle e, e' \rangle \end{aligned}$$

By using the orthogonality of the elements of $F, G \subset \mathcal{E}$ we rewrite the above as follows.

$$\begin{aligned} \sum_{e' \in F} \langle \sum_{e \in G} \langle x, e \rangle e, e' \rangle &= \sum_{e \in F \cap G} \langle \langle x, e \rangle e, e \rangle \\ &= \sum_{e \in F \cap G} \langle x, e \rangle e = P_{F \cap G} x \end{aligned}$$

As this holds for arbitrary F, G, proceeding the same fashion for $P_G P_F$ completes the proof.