Portfolio Theory
Homework 1

Sembenico, Alvise
12380288

1 1.1

As in this exercise we are looking at several stochastic processes and have to classify as predictable or just adapted, we will first show that the indicator function is adapted/predictable if the argument is adapted/predictable. This is true almost just by definition as the preimage is the following.

$$\mathbb{1}_A^{-1}(x) = \begin{cases} A \text{ if } x = 1 \\ \Omega \setminus A \text{ if } x = 0 \end{cases} \tag{1}$$

Where $A \in \mathcal{F}$ so does $\Omega \setminus A = A^C$. With this in mind, we can proceed on considering the following processes.

- $\varphi_t = \mathbb{1}_{\{S_t^{(1)} > S_{t-1}^{(1)}\}}$;

  ϕ_t is merely adapted as S_t is just adapted.

- $\varphi_1 = 1$ and $\varphi_t = \mathbb{1}_{\{S_{t-1}^{(1)} > S_{t-2}^{(1)}\}}$ for $t \geq 2$;

  ϕ_t is predictable as both process are $\mathcal{F}_t$ measurable, thus, ϕ_{t-1} is $\mathcal{F}_t$ measurable.

- $\varphi_t = \mathbb{1}_A \cdot \mathbb{1}_{\{t > t_0\}}$, where $t_0 \in \{0, \ldots, T\}$ and $A \in \mathcal{F}_{t_0}$;

  We can see that $\mathbb{1}_A$ and $\mathbb{1}_{t > t_0}$ are both deterministic functions, moreover, given that $A \in \mathcal{F}_0$, we have that for every $t \geq 1$, ϕ_t is F_{t+1} measurable, therefore the process is predictable.

- $\varphi_t = \mathbb{1}_{\{S_t^{(1)} > S_0^{(1)}\}}$;

  Again by looking at the argument of the argument of the indicator function, we see that S_t is merely adapted. It follows that ϕ_t is also just adapted...

- $\varphi_1 = 1$ and $\varphi_t = 2\varphi_{t-1}\mathbb{1}_{\{S_{t-1}^{(1)} < S_0^{(1)}\}}$ for $t \geq 2$.

  We can see that the argument of the indicator function is again predictable. We have to be careful about the ϕ_{t-1} component. However, using an induction argument, we can see that each ϕ_{t-1} is $\mathcal{F}_t$ measurable, making it predictable. It follows that ϕ_t is predictable as well.

2 1.2

Proof. We will prove the statement with a series of double direction implications.

A strategy is self financing if and only if

$$W_t(\phi) = W_0(\phi) + G_t(\phi) = W_0(\phi) + (\phi \cdot X)_t$$

For every t. It follows that

$$\begin{aligned} \phi_t^T S_t &= \phi_0^T S_0 + \sum_i^t \phi_i^T (S_i - S_{i-1}) \\ \sum_i^t \phi_i^T S_i - \phi_{i-1}^T S_{i-1} &= \sum_i^t \phi_i^T (S_i - S_{i-1}) \\ \sum_i^t (\phi_i^T + \phi_{i-1}^T) S_{i-1} &= 0 \end{aligned}$$

For every $t = 0, \dots, T$. As it has to be true for all the t, by induction, we deduce that

$$(\phi_t^T - \phi_{t-t}^T) S_{t-1} = 0.$$

for all $t = 1, \dots T$. In other words, given that it's true for $t = 1$ and by the inductive step

$$\begin{aligned} \sum_i^2 (\phi_i^T + \phi_{i-1}^T) S_{i-1} &= (\phi_1^T + \phi_0^T) S_0 + (\phi_2^T + \phi_1^T) S_1 \\ &= (\phi_2^T + \phi_1^T) S_1 = 0. \end{aligned}$$

By diving both sides of the equation, we get that

$$(\phi_t^T - \phi_{t-t}^T) \widetilde{S}_{t-1} = 0. \tag{2}$$

Given that this is true for every t, again by induction argument, we conclude that

$$\widetilde{W}_t(\phi) = \widetilde{W}_0(\phi) + (\phi \cdot \widetilde{X})_t. \tag{3}$$

For every $t = 0, \dots, 1$. □

3 1.3

Proof.

$$\begin{aligned} \Delta W_t(\phi) &= \phi_t^T \Delta X_t \\ \iff \phi_t^T S_t - \phi_{t-1}^T S_{t-1} &= \phi_t S_t - \phi S_{t-1} \\ \iff \phi_{t-1}^T S_{t-1} &= \phi_t^T S_{t-1} \\ \iff \phi_t^T S_t &= \phi_{t+1}^T S_t \end{aligned}$$

where the last statement holds true as the previous ones are true for every $t = 0, \ldots, T$. ☐

4 1.4

Let's start by noting that the reciprocal absolute continuity of $\mathcal{P}, \mathbb{Q}$ implies that the Radon Nikodym is well defined. Moreover, we can define

$$Z_\infty := \left.\frac{dQ}{dP}\right|_{\mathcal{F}_\infty}. \tag{4}$$

Furthermore, we define

$$Z_t = \mathbb{E}\left[Z_\infty | \mathcal{F}_t\right]. \tag{5}$$

We now show that Z_t is indeed a martingale. We know that it's squared integrable, thus $\mathbb{E}\left[|Z_t|\right] < \infty \forall t > 0$. So we focus on the martingality property. For $s \leq t$;

$$\begin{aligned} \mathbb{E}\left[Z_t | \mathcal{F}_s\right] &= \mathbb{E}\left[\mathbb{E}\left[Z_\infty | F_t\right] | \mathcal{F}_s\right] \\ &= \mathbb{E}\left[Z_\infty | \mathcal{F}_{s \wedge t}\right] \\ &= Z_s \end{aligned}$$

Let's now show that

$$Z_t = \left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} \tag{6}$$

Take $A \in F_t$, then

$$Q(A) = \mathbb{E}_P\left[\mathbb{1}_A \left.\frac{dQ}{dP}\right|_{\mathcal{F}_t}\right] \tag{7}$$

But also, for $A \in \mathcal{F}_t \subset \mathcal{F}_\infty$

$$\begin{aligned} Q(A) &= \mathbb{E}_P\left[\mathbb{1}_A Z_\infty\right] && (8) \\ &= \mathbb{E}_P\left[\mathbb{E}_P\left[\mathbb{1}_A Z_\infty | \mathcal{F}_t\right]\right] && (9) \\ &= \mathbb{E}_P\left[\mathbb{1}_A Z_t\right] && (10) \end{aligned}$$

From both equalities we conclude that

$$Z_t = \left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} \tag{11}$$

5 1.5

5.1 a

Let's prove that

$$S_t = S_0 \prod_{k=1}^{t} (1 + R_k) \tag{12}$$

is a Martingale when $R_1, \ldots, R_T$ are independent and $\mathbb{E}[R_t] = 0$.

Let's start by proving the integrability, for any positive t we have

$$\begin{aligned}
\mathbb{E}[|S_t|] &= \mathbb{E}[S_t] && \text{(stricly positive)} \\
&= \mathbb{E}\left[S_0 \prod_{k=1}^{t} (1 + R_k)\right] \\
&= S_0 \mathbb{E}\left[\prod_{k=1}^{t} (1 + R_k)\right] && \text{(independence)} \\
&= S_0 < \infty.
\end{aligned}$$

We now prove the martingale property of the process S.

$$\begin{aligned}
\mathbb{E}[S_{t+1}|\mathcal{F}_t] &= \mathbb{E}\left[S_0 \prod_{k=1}^{t+1} (1 + R_k)|\mathcal{F}_t\right] \\
&= S_0 \mathbb{E}\left[\prod_{k=1}^{t} (1 + R_k) \cdot (1 + R_{t+1})|\mathcal{F}_t\right] \\
&= S_0 \prod_{k=1}^{t} (1 + R_k) \mathbb{E}[1 + R_{t+1}|\mathcal{F}_t] \\
&= S_0 \prod_{k=1}^{t} (1 + R_k) (1 + \mathbb{E}[R_{t+1}|\mathcal{F}_t]) \\
&= S_0 \prod_{k=1}^{t} (1 + R_k) (1 + \mathbb{E}[R_{t+1}]) \\
&= S_0 \prod_{k=1}^{t} (1 + R_k) \\
&= S_t.
\end{aligned}$$

In the previous series of equivalences we used the measurability of S_{t-1} and independence of the R_t and the fact that $\mathbb{E}[R_t] = 0$.

5.2 b

Let's now derive the necessary and sufficient conditions for S_t to be a Martingale. Let's start by looking at the martingale property.

$$\begin{aligned}
\mathbb{E}\left[S_{t+1}|\mathcal{F}_t\right] &= \mathbb{E}\left[S_0 \prod_{k=1}^{t+1}(1+R_k)|\mathcal{F}_t\right] \\
&= S_0\mathbb{E}\left[\prod_{k=1}^{t}(1+R_k)(1+R_{t+1})|\mathcal{F}_t\right] \\
&= S_0 \prod_{k=1}^{t}(1+R_k)\mathbb{E}\left[1+R_{t+1}|\mathcal{F}_t\right].
\end{aligned}$$

The above is equal to S_t if and only if $\mathbb{E}\left[1+R_{t+1}|\mathcal{F}_t\right]=1$ which is true if and only if $\mathbb{E}\left[R_{t+1}|\mathcal{F}_t\right]=0$. Therefore, the necessary and sufficient condition is that

$$\mathbb{E}\left[R_{t+1}|\mathcal{F}_t\right]=0 \tag{13}$$

So we conclude that R_t must be a Martingale with $R_0=0$.

5.3 c

To define R_t such that S_t is a martingale but the returns are not independent we can define a process R_t such that is's a martingale with expectation 0 and dependent returns. Let's define $R_0 \sim N(0,1)$, next we define

$$R_t = \begin{cases} R_{t-1}, & \text{with } p=\frac{1}{2}; \\ -R_{t-1}, & \text{with } p=\frac{1}{2}; \end{cases} \tag{14}$$

It's obvious that the returns R_t are not independent, as well as that $\mathbb{E}\left[R_{t+1}|\mathcal{F}_t\right]=0$ as requested by the previous point for S_t to be a Martingale.

6 1.6

Given the process

$$X_t^{(1)} := X_0^{(1)} \prod_{i=1}^{t} e^{\sigma_i Z_i + m_i}, \quad t=0,\dots,T. \tag{15}$$

We want to construct a measure $\mathbb{Q}$ such that X_t is still log-normally distributed under $\mathbb{Q}$. Define

$$Z_t := \left.\frac{d\mathbb{Q}}{d\mathcal{P}}\right|_{\mathcal{F}_t} \tag{16}$$

Using equation 9.2 from Stochastic integration, we know that

$$\mathbb{E}_{\mathcal{Q}}\left[X_t|\mathcal{F}_{t-1}\right] = \frac{\mathbb{E}\left[X_t Z|\mathcal{F}_{t-1}\right]}{\mathbb{E}\left[Z|\mathcal{F}_{t-1}\right]} \tag{17}$$

For $t = 1$, we get

$$\mathbb{E}_{\mathcal{Q}}\left[X_1|\mathcal{F}_0\right] = \mathbb{E}\left[X_1\right] = X_0 \frac{\mathbb{E}\left[e^{\sigma_1 \widetilde{Z}_1 + m_1} Z_1\right]}{\mathbb{E}\left[Z_1\right]} \tag{18}$$

Since we want X_1 to be a Martingale, it must hold that

$$\frac{\mathbb{E}\left[e^{\sigma_1 \widetilde{Z}_1 + m_1} Z_1\right]}{\mathbb{E}\left[Z_1\right]} = 1 \tag{19}$$

Assume now that $Z_1 = e^{\overline{\sigma}_1 \widetilde{Z}_1 + \overline{m}_1}$. Then, by plugging it in, we get the following equations

$$\exp\left(\overline{m_1} + \frac{\overline{\sigma}_1^2}{2}\right) = 1 \tag{20}$$

and

$$\exp\left(\overline{m_1} + m_1 + \frac{(\overline{\sigma}_1 + \sigma_1)^2}{2}\right) = 1 \tag{21}$$

Which are solved by

$$\overline{\sigma_1} = -\frac{2m_1 + \sigma_1^2}{\sigma_1} \qquad \overline{m}_1 = \left(\frac{2m_1 + \sigma_1^2}{\sigma_1}\right)\frac{1}{2}. \tag{22}$$

Which characterizes the Z_1 distribution.

For $t = 2$ we get

$$\mathbb{E}\left[X_2|\mathcal{F}_1\right] = \frac{\mathbb{E}\left[X_2 Z_2|\mathcal{F}_1\right]}{\mathbb{E}\left[Z_2|\mathcal{F}_1\right]} = \frac{X_1 \mathbb{E}\left[e^{\sigma_2 \widetilde{Z}_2 + m_2} Z_2|\mathcal{F}_1\right]}{\mathbb{E}\left[Z_2|\mathcal{F}_1\right]}. \tag{23}$$

Which can be further simplified by

$$X_1 \frac{\mathbb{E}\left[e^{\sigma_2 \widetilde{Z}_2 + m_2} Z_2|\mathcal{F}_1\right]}{\mathbb{E}\left[Z_2|\mathcal{F}_1\right]} = X_1 \frac{\mathbb{E}\left[e^{\sigma_2 \widetilde{Z}_2 + m_2} Z_2|\mathcal{F}_1\right]}{Z_1} \tag{24}$$

So it must hold that

$$\mathbb{E}\left[e^{\sigma_2 \widetilde{Z}_2 + m_2} Z_2|\mathcal{F}_1\right] = Z_1 \tag{25}$$

As well as $\mathbb{E}\left[e^{\sigma_2 \widetilde{Z}_2 + m_2} Z_2|\mathcal{F}_1\right]$ be log-normally distributed. A sensitive guess it that

$$Z_2 = Z_1 Y_2. \tag{26}$$

Plugging this in we obtain the following.

$$\mathbb{E}\left[e^{\sigma_2 \widetilde{Z}_2 + m_2} Z_2|\mathcal{F}_1\right] = \mathbb{E}\left[e^{\sigma_2 \widetilde{Z}_2 + m_2} Z_1 Y_2|\mathcal{F}_1\right] = Z_1 \mathbb{E}\left[e^{\sigma_2 \widetilde{Z}_2 + m_2} Y_2|\mathcal{F}_1\right]. \tag{27}$$

Again, we obtain the condition that $\mathbb{E}\left[e^{\sigma_2 \widetilde{Z}_2 + m_2} Y_2 | \mathcal{F}_1\right] = 1$. This is the same as before, just with different indices. We thus obtain

$$\overline{\sigma_2} = -\frac{2m_2 + \sigma_2^2}{\sigma_2} \qquad \overline{m}_2 = \left(\frac{2m_2 + \sigma_2^2}{\sigma_2}\right)\frac{1}{2}. \tag{28}$$

By induction, we obtain that

$$Z_t = \prod_i^t Y_t. \tag{29}$$

Where Y_t is log normal distributed with the coefficients described above.