

Homework 1

1 Exercise 1

1.1

Let's define x_0 as the limit of the net

$$\sum_{a \in A} a \rightarrow x_0. \quad (1)$$

Since it converges, it means that there exist a set $F_0 \subset A$ such that for all $F > F_0$ the following holds

$$\left\| \sum_{a \in F} a - x_0 \right\| < \epsilon \quad (2)$$

for a given $\epsilon > 0$. Since ϵ is arbitrary, we can choose $F'_0 > F_0$ such that for all $F > F'_0$

$$\left\| \sum_{a \in F} a - x_0 \right\| < \frac{\epsilon}{|\alpha|}. \quad (3)$$

Then, by properties of the norm we obtain

$$\begin{aligned} |\alpha| \left\| \sum_{a \in F} a - x_0 \right\| &< \frac{\epsilon}{|\alpha|} \alpha = \epsilon \\ \left\| \alpha \sum_{a \in F} a - \alpha x_0 \right\| &< \epsilon \\ \left\| \sum_{a \in F} \alpha a - \alpha x_0 \right\| &< \epsilon. \end{aligned}$$

Where in the last step we used the fact that F is finite. This proves that $\alpha \sum_{a \in A} a$ converges to $\alpha x_0 = \alpha \sum_{a \in A} a$.

1.2

The hypothesis that $\sum_{a \in A} a$ and $\sum_{b \in B} b$ implies that there exists an F_0^a and F_0^b such that for every $F^a > F_0^a$ and $F^b > F_0^b$ the following holds

$$\left\| \sum_{a \in F^a} a - \sum_{a \in A} a \right\| < \frac{\epsilon}{2} \quad \left\| \sum_{b \in F^b} b - \sum_{b \in B} b \right\| < \frac{\epsilon}{2}. \quad (4)$$

Denote $F_0 = F_0^a \cup F_0^b$, it follows that for every $F > F_0$

$$\begin{aligned} \left\| \sum_{x \in F} x - \sum_{a \in A} a - \sum_{b \in B} b \right\| &= \left\| \sum_{x \in F \cap A} x + \sum_{x \in F \cap B} x - \sum_{a \in A} a - \sum_{b \in B} b \right\| \\ &\leq \left\| \sum_{x \in F \cap A} x - \sum_{a \in A} a \right\| + \left\| \sum_{x \in F \cap B} x - \sum_{b \in B} b \right\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Because ϵ was arbitrary, we conclude that $\sum_{x \in A \cup B} x \rightarrow \sum_{a \in A} a + \sum_{b \in B} b$.

1.3

Let's start by proving that any converging net of positive numbers has at most a countable number of non zero elements.

Say that the net converges to M , i.e. $\sum_{a \in A} a = M < \infty$ where for every $a \in A, a > 0$. Consider now the sets $S_n = \{a \in A | a > \frac{1}{n}\}$, then

$$M \geq \sum_{a \in S_n} a \geq \sum_{a \in S_n} \frac{1}{n} = \frac{N}{n}.$$

As $M < \infty$ so is N which is the cardinality of the set S_n . It follows that

$$\#\{a \in A | a > 0\} = \#S = \# \bigcup_{n \in \mathbb{N}} S_n \quad (5)$$

We conclude that A has at most countable number of non zero elements as countable union of finite sets.

Let's now prove the (\implies) direction. Given the previously proven statement, we can rewrite the net as countable sum and thus define a corresponding sequence $x_n = \sum_{i=0}^n a_i$ where w.l.o.g. we associated every non zero element a of A to an index i so that $a_i = a$. From standard analysis we obtain that every converging increasing sequence is bounded from above, i.e. there exists $N \in \mathbb{R}$ so that $x_n < N$ for every n . It follows that for every finite $F \subset I$

$$\sum_{a \in F} a \leq \sum_{i \in \mathbb{N}} a_i \leq N. \quad (6)$$

We now prove the opposite implication (\impliedby). Assume that $\sup \{\sum_{a \in F} a : F \in \mathcal{F}\} = x_0$. We proceed now by contradiction, suppose that $\sum_{a \in A} a \rightarrow x_0 + t$ for an arbitrary $t > 0$. Let's define now what does it mean for a net to be increasing. Given a set F_0 , it holds that for every $F > F_0$ we have that $\sum_{a \in F} a \geq \sum_{a \in F_0} a$.

By the definition of net convergence for standard set inclusion as order, we obtain that the net $\sum_{a \in A} a$ is increasing. It follows that for every F , $\sum_{a \in F} a \leq x_0 + t$. Moreover, since

$\sum_{a \in A} a \rightarrow x_0 + t$, there exists and F_t such that $\forall F > F_t \parallel \sum_{a \in F} a - x_0 - t \parallel < \frac{t}{2}$. It follows

$$\begin{aligned} \parallel \sum_{a \in F} a - x_0 - t \parallel &\leq 0 \\ \parallel \sum_{a \in F} a - x_0 - t \parallel &= \sum_{a \in F} a - x_0 - t < \frac{t}{2} \\ \sum_{a \in F} a &> x_0 + \frac{t}{2} \end{aligned}$$

Which is a contradiction. As this holds for any arbitrary t , $\sum_{a \in A} a \leq x_0$.

Let's conclude the proof by showing that

$$\sup \left\{ \sum_{a \in F} a : F \in \mathcal{F} \right\} = \sum_{a \in A} a = x_0. \quad (7)$$

Take an arbitrary $x < x_0$. By contradiction, assume that $\sup \{ \sum_{a \in F} a : F \in \mathcal{F} \} = x$. Again, by convergence of the net, there exists a F_0 such that $\forall F > F_0$ we have

$$\begin{aligned} \parallel \sum_{a \in F} a - x_0 \parallel &< |x - x_0| \\ x_0 - \sum_{a \in F} x_0 - x \\ \sum_{a \in F} a &> x \end{aligned}$$

Which is a contradiction. Combining it with the previous point we get the desired equality.

To double check that this is all we need.

2 Exercise 2

2.1

Let's define $H' = \bigvee \mathcal{F}$. By theorem 4.13 we have that $\forall h \in H'$, h can be written as $h = \sum_{e \in F} \langle h, e \rangle e$ as F is the basis for H' again by theorem 4.13.

Moreover, for every $x \in H$, we have that by definition $P_F x \in H'$. We define the operator Q as such

$$Qx := \sum_{e \in F} \langle x, e \rangle e \quad (8)$$

For Q to be equal to P_F , Qx has to be the unique elements in H' such that $x - Qx \perp H'$. We proceed by taking an orthogonal basis of H such that $\mathcal{E} \subset B$. By theorem 4.13 again, by trivially noting that $\bigvee H = H$, x can be represented as $x = \sum_{e \in B} \langle x, e \rangle e$. It follows

that

$$\begin{aligned}
 x - Qx &= \sum_{e \in B} \langle x, e \rangle e - \sum_{e \in F} \langle x, e \rangle e \\
 &= \sum_{e \in F} \langle x, e \rangle e + \sum_{e \in B \setminus F} \langle x, e \rangle e - \sum_{e \in F} \langle x, e \rangle e \\
 &= \sum_{e \in B \setminus F} \langle x, e \rangle e
 \end{aligned}$$

We conclude that since B is orthogonal, it follows that $x - Qx \perp H' = \bigvee F$.

2.2

By the previous point, we can write $P_G x = \sum_{e \in G} \langle x, e \rangle e$ for every $G \subset \mathcal{G}$. It follows that

$$\begin{aligned}
 P_F P_G x &= P_F \left(\sum_{e \in G} \langle x, e \rangle e \right) \\
 &= \sum_{e' \in F} \left\langle \sum_{e \in G} \langle x, e \rangle e, e' \right\rangle e'
 \end{aligned}$$

By using the orthogonality of the elements of $F, G \subset \mathcal{E}$ we rewrite the above as follows.

$$\begin{aligned}
 \sum_{e' \in F} \left\langle \sum_{e \in G} \langle x, e \rangle e, e' \right\rangle e' &= \sum_{e \in F \cap G} \langle \langle x, e \rangle e, e \rangle \\
 &= \sum_{e \in F \cap G} \langle x, e \rangle e = P_{F \cap G} x
 \end{aligned}$$

As this holds for arbitrary F, G , proceeding the same fashion for $P_G P_F$ completes the proof.

2.3

$$P_{\bigcup_{i=1}^n F_i} = P_{\bigcup_{i=1}^{n-1} F_i \cup F_n} = P_{F_j \cup F_n} = P_{F_j} + P_{F_n}$$

Where we used the exercise above for the last equation. By induction, we show that

$$P_{\bigcup_{i=1}^n F_i} = \sum_{i=1}^n P_{F_i} \quad (9)$$

We are now interest at the limit of those objects. Assume $\lim_{n \rightarrow \infty} \sum_{i=1}^n P_{F_i} x \rightarrow x_0$ in the norm sense. Thus, there exists an $n_0 > 0$ such that for every $n > n_0$

$$\left\| \sum_{i=1}^n P_{F_i} x - x_0 \right\| < \epsilon \quad (10)$$

It follows that

$$\|P - x_0\| < \epsilon$$

2.4

We prove by giving a counterexample that $\sum_i^\infty P_{F_i} x$ does not converge in the operator norm. Let's define $F_i = e_i$ where $e_i \cap e_j = \emptyset$ and $\|e_i\| = 1$ for every i . Take now $\epsilon = \frac{1}{2}$, then, for every $n > N$

$$\left\| \sum_{i=0}^\infty P_{F_i} x - \sum_{i=0}^n P_{F_i} x \right\| = \left\| \sum_{i=n}^\infty P_{F_i} x \right\| = 1 > \frac{1}{2}$$

Therefore, for $\epsilon = \frac{1}{2}$ there exists no $n > 0$ such that the the difference of the norm converges.

3 Exercise 3

4 Exercise 4

4.1

Let's start noting how the operator S affects the inner product.

$$\langle Sx, y \rangle = \langle (0, x_1, \dots), y \rangle = \sum_{i=1}^\infty x_i y_{i+1}$$

We can see that the inner product with x and the adjoint on y must be equal $\langle x, S^* y \rangle = \sum_{i=1}^\infty x_i y_{i+1}$ which is the left shift operator

$$S^*(x_1, x_2, \dots) = (x_2, x_3, \dots)$$

4.2

Let's compute now the concatenation of S and its adjoint.

$$SS^*(x_1, x_2, \dots) = S(x_2, x_3, \dots) = (0, x_2, x_3, \dots)$$

Conversely

$$S^*S(x_1, x_2, \dots) = S(0, x_1, x_2, x_3, \dots) = (x_1, x_2, \dots) = x$$

4.3

And we can extend this to S^n and $(S^*)^n$

$$S^n(S^*)^n x = (0^n, x_{n+1}, x_{n+2}, \dots)$$

and similarly

$$(S^*)^n S^n = x$$