Homework 1

1 Exercise 1

The so defined time inversion process

$$B(t;\alpha) = \begin{cases} t^\alpha W(1/t) & t > 0 \\ 0 & t = 0 \end{cases} \tag{1}$$

in order to be a Brownian Motion has to satisfy the following properties:

1. with probability 1, the mapping $t \mapsto W(t)$ is continuous and $W(0) = 0$;
2. if $0 = t_0 < t_1 < \cdots < t_N = T$, then the increments

$$W(t_N) - W(t_{N-1}), \ldots, W(t_1) - W(t_0)$$

   are *independent*; and
3. for all $t > s$ the increment $W(t) - W(s)$ has the *normal* distribution, with $E[W(t) - W(s)] = 0$ and $E[(W(t) - W(s))^2] = t - s$

We start by looking at the property (1). It's clear from the property of the Brownian Motion W that $B(t,\alpha)$ is continuous in $(0,\infty)$. The only point that we need to check the continuity is in fact at $t = 0$.
As the hint suggests, we start by computing the following for an arbitrary $h > 0, t > 0$.

$$\mathrm{Cov}\left[B(t+h,\alpha), B(t,\alpha)\right] = \mathbb{E}\left[(t+h)^\alpha t^\alpha W\left(\frac{1}{t}\right) W\left(\frac{1}{t+h}\right)\right] - \mathbb{E}\left[B(t+h,\alpha)\right]\mathbb{E}\left[B(t,\alpha)\right].$$

We note that $\mathbb{E}\left[B(t,\alpha)\right] = t^\alpha \mathbb{E}\left[W\left(\frac{1}{t}\right)\right] = 0$. Thus, the right hand side can be simplified as follows.

$$\begin{aligned} \mathrm{Cov}\left[B(t+h,\alpha), B(t,\alpha)\right] &= \mathbb{E}\left[(t+h)^\alpha t^\alpha W\left(\frac{1}{t}\right) W\left(\frac{1}{t+h}\right)\right] \\ &= (t+h)^\alpha t^\alpha \frac{1}{t+h} \\ &= (t+h)^{\alpha-1} t^\alpha \end{aligned}$$

Where in the one to last equality we used the fact that $\mathbb{E}\left[W_s W_t\right] = \min(s,t)$. This gives a necessary condition for $B(t,\alpha)$ to be a Brownian Motion, that is $\alpha = 1$. We will proceed in showing the other properties of $B(t,\alpha)$ assuming $\alpha = 1$.

For a mesh $\Pi = t_1 < t_2 < \cdots < t_n$ we write the process B_t as follows.

$$\begin{bmatrix} B(t_1) \\ \vdots \\ B(t_n) \end{bmatrix} = A \begin{bmatrix} W\left(\frac{1}{t_1}\right) \\ \vdots \\ W\left(\frac{1}{t_n}\right) \end{bmatrix} \tag{2}$$

Where just by the definition of the process B_t, we have $A = \mathrm{diag}(t_1, \ldots, t_n)$. Moreover, we can write the matrix on the right hand side as follows

$$\begin{bmatrix} W\left(\frac{1}{t_1}\right) \\ \vdots \\ W\left(\frac{1}{t_n}\right) \end{bmatrix} = O^n + \begin{bmatrix} \frac{1}{\sqrt{t_1}} \\ \vdots \\ \frac{1}{\sqrt{t_n}} \end{bmatrix} \begin{bmatrix} \mathcal{N}(0,1) \\ \vdots \\ \mathcal{N}(0,1) \end{bmatrix}. \tag{3}$$

Upon defining the matrix D^n as follows

$$D_n = \begin{bmatrix} 0 & \cdots & 0 \\ & I_{n-1} & \end{bmatrix}. \tag{4}$$

We can then write the matrix of the increments with the ingredients we have prepared so far.

$$B' = \begin{bmatrix} B_{t_1} \\ \vdots \\ B_{t_n} \end{bmatrix} - \begin{bmatrix} 0 \\ B_1 \\ \vdots \\ B_{t_{n-1}} \end{bmatrix} \tag{5}$$

It follows that $B' = A\mathcal{W} - D^n A\mathcal{W} = (I - D^n)A\mathcal{W}$, where $\mathcal{W} = \begin{bmatrix} W\left(\frac{1}{t_1}\right) \\ \vdots \\ W\left(\frac{1}{t_n}\right) \end{bmatrix}$. From this representation we get that B' is a multivariate Gaussian Since, implying that each marginal is also Gaussian.

2 Exercise 2

2.1

In this exercise we will prove the following equality:

$$\int_0^T t\, dW(t) = TW(T) - \int_0^T W(t)\, dt \tag{6}$$

Look at the left hand side, by taking it's forward Euler we obtain

$$I_1 = \int_0^T t dW(t) = \sum_{n=0}^{N-1} t_n (W(t_{n-1} - W(t_n))). \tag{7}$$

Applying the hint, i.e. using the Abel's summation by parts we get

$$\begin{aligned}\sum_{n=0}^{N-1} t_n (W(t_{n-1} - W(t_n))) &= t_N W(t_N) - t_0 W(t_0) - \sum_{k=1}^{N-1} W(t_k)(t_k - t_{k-1}) \\ &= TW(T) - \sum_{k=1}^{N-1} W(t_k)(t_k - t_{k-1}).\end{aligned}$$

What is left to prove is the convergence in L_2 of the right hand side, i.e. the following equation.

$$\lim_{n\to\infty} \mathbb{E}\left[\left(\int_0^T W(t)dt - \sum_{k=1}^{N-1} W(t_k)(t_k - t_{k-1})\right)^2\right] = 0. \tag{8}$$

Let's fix $t_n - t_{n-1} = \Delta t$ for every $n \geq 0$. Using the linearity of the integral, we can rewrite it as follows

$$\begin{aligned}\mathbb{E}\left[\left(\sum_{n=0}^{N-1} \int_{t_{n-1}}^{t_n} W(t)dt - \sum_{k=1}^{N-1} W(t_k)(t_k - t_{k-1})\right)^2\right] &= \mathbb{E}\left[\left(\sum_{n=0}^{N-1} \int_{t_{n-1}}^{t_n} W(t) - W(t_n)dt\right)^2\right] \\ &= \sum_{n=0}^{N-1} \mathbb{E}\left[\int_{t_{n-1}}^{t_n} W(t) - W(t_n)dt\right] \\ &= \sum_{n=0}^{N-1} \mathbb{E}\left[e_n^2\right].\end{aligned}$$

Where e_n is n-th error term. Moreover, in the last equation we dropped the cross terms since $\mathbb{E}[e_i e_j] = 0$ by the properties of the Brownian motion.
Next, we compute $\mathbb{E}[e_n^2]$ for every $n \geq 0$.

$$\begin{aligned}\mathbb{E}\left[e_n^2\right] &= \mathbb{E}\left[\left(\int_{t_n}^{t_{n-1}} W(t) - W(t_n)dt\right)^2\right] \\ &= \mathbb{E}\left[\int_{t_n}^{t_{n-1}} \int_{t_n}^{t_{n-1}} \left(W(t) - W(t_n)dt\right)\left(W(s) - W(t_n)ds\right)\right] \\ &= \mathbb{E}\left[\int_{t_n}^{t_{n-1}} \int_{t_n}^{t_{n-1}} \left(W(t) - W(t_n)\right)\left(W(s) - W(t_n)\right) dtds\right].\end{aligned}$$

Where we used twice Fubini's Theorem. We also use Fubini for the following equality.

$$\mathbb{E}\left[\int_{t_n}^{t_{n-1}}\int_{t_n}^{t_{n-1}}\left(W(t)-W(t_n)\right)\left(W(s)-W(t_n)\right)dtds\right] = \int_{t_n}^{t_{n-1}}\int_{t_n}^{t_{n-1}}\mathbb{E}\left[\left(W(t)-W(t_n)\right)\left(W(s)-W(t_n\right.\right.$$
$$= \int_{t_n}^{t_{n-1}}\int_{t_n}^{t_{n-1}}\min(t,s)-t_k dtds$$
$$= \frac{1}{3}(\Delta t)^3.$$

Finally, by summing all the values, we get

$$\sum_{n=0}^{T}\mathbb{E}\left[e_n\right] = \sum_{n=0}^{t} frac13(\Delta t)^3 = \frac{T^3}{n^2}. \tag{9}$$

It follows that by letting n go to infinity, the Forward Euler converges in L_2.

2.2

As in the hint, we now prove the following

$$\sum_{n=0}^{N-1} W(t_n)\big(W(t_{n+1})-W(t_n)\big) = \sum_{n=0}^{N-1}\frac{W(t_{n+1})^2-W(t_n)^2}{2} - \frac{\big(W(t_{n+1})-W(t_n)\big)^2}{2} \tag{10}$$

We start by looking at the right hand side.

$$\sum_{n=0}^{N-1}\frac{W(t_{n+1})^2-W(t_n)^2}{2} - \frac{\big(W(t_{n+1})-W(t_n)\big)^2}{2} =$$
$$= \sum_{n=0}^{N-1}\frac{W(t_{n+1})^2-W(t_n)^2}{2} - \frac{W(t_{n+1})^2-2W_{t_{n+1}}W(t_n)+W_{t_n}^2}{2}$$
$$= \sum_{n=0}^{N-1}\frac{2W_{t_n}W_{t_{n+1}}-2W(t_n)}{2} = \sum_{n=0}^{N-1}W(t_n)\big(W(t_{n+1})-W(t_n)\big)$$

Note that the

$$\sum_{n=0}^{N-1}\frac{W(t_{n+1})^2-W(t_n)^2}{2} = \frac{W(T)}{2} \tag{11}$$

as it is a telescopic sum. To finalize the proof, we need to show that the second part converges to $\frac{T}{2}$ or in other words that

$$\mathbb{E}\left[\left(\sum_{n=0}^{N-1}\left[W(t_{n+1})-W(t_n)\right]^2-T\right)^2\right]\to 0. \tag{12}$$

We define the summation $S_N \coloneqq \sum_{n=0}^{N-1} \left[W(t_{n+1}) - W(t_n)\right]^2$. It follows that

$$\mathbb{E}\left[(S_N - T)^2\right] = \mathbb{E}\left[S_N^2\right] - T^2 - 2T\mathbb{E}\left[S_N\right]. \tag{13}$$

We focus now on $\mathbb{E}\left[S_N\right]$ as follows.

$$\begin{aligned}
\mathbb{E}\left[S_N\right] &= \sum_{n=0}^{N-1} \mathbb{E}\left[W_{t_{n-1}}^2 + W_{t_n}^2 - 2W_{t_{n+1}}W_{t_n}\right] \\
&= \sum_{n=0}^{N-1} t_{n+1} + t_n - 2t_n = \sum_{n=0}^{N-1} t_{n+1} - t_n = T.
\end{aligned}$$

Where we used the fact that $\mathbb{E}\left[W_s W_t\right] = \min(s,t)$. It follows that $\mathbb{E}\left[(S_N - T)^2\right] = \mathbb{E}\left[S_N^2\right] - T^2 = \mathbb{E}\left[S_n^2\right] - \mathbb{E}\left[S_N\right]^2 = \mathrm{Var}\left[S_n\right]$.
The problem reduces then to compute $\mathbb{E}\left[S_N^2\right]$.

$$\mathbb{E}\left[S_N^2\right] = \sum_{n}^{N-1} \sum_{m}^{N_1} \mathbb{E}\left[\Delta W_n^2 \Delta W_m^2\right]$$

For $m = n$ we get $\mathbb{E}\left[\Delta W_m^4\right] = 3\Delta t^2$ by using the forth moment of the standard normal. On the other hand, for $n \neq m$ we get the following

$$\mathbb{E}\left[\Delta W_m^w \Delta W_n^2\right] = \mathbb{E}\left[\Delta W_m^2\right] \mathbb{E}\left[\Delta W_n^w\right] = (\Delta t)^2. \tag{14}$$

Putting everything together we obtain

$$\begin{aligned}
\mathbb{E}\left[S_N^2\right] &= 3N(\Delta t)^2 + N(N-1)(\Delta t)^2 \\
&= \frac{2T^2}{N} - T^2 N \\
\implies \mathbb{E}\left[(S_N - T)^2\right] &= \mathrm{Var}\left[S_N\right] = \frac{2T^2}{N}
\end{aligned}$$

Which clearly converges to 0 as $n \to \infty$.