# Homework 1

## 1 Exercise 1

### 1.1

Let's define $x_0$ as the limit of the net

$$\sum_{a\in A} a \to x_0. \tag{1}$$

Since it converges, it means that there exist a set $F_0 \subset A$ such that for all $F > F_0$ the following holds

$$\|\sum_{a\in F} a - x_0\| < \epsilon \tag{2}$$

for a given $\epsilon > 0$. Since $\epsilon$ is arbitrary, we can choose $F_0' > F_0$ such that for all $F > F_0'$

$$\|\sum_{a\in F} a - x_0\| < \frac{\epsilon}{|\alpha|}. \tag{3}$$

Then, by properties of the norm we obtain

$$\begin{aligned}
|\alpha|\|\sum_{a\in F} a - x_0\| &< \frac{\epsilon}{|\alpha|}\alpha = \epsilon \\
\|\alpha\sum_{a\in F} a - \alpha x_0\| &< \epsilon \\
\|\sum_{a\in F} \alpha a - \alpha x_0\| &< \epsilon.
\end{aligned}$$

Where in the last step we used the fact that $F$ is finite. This proves that $\alpha\sum_{a\in A} a$ converges to $\alpha x_0 = \alpha\sum_{a\in A} a$.

### 1.2

The hypothesis that $\sum_{a\in A} a$ and $\sum_{b\in B} b$ implies that there exists an $F_0^a$ and $F_0^b$ such that for every $F^a > F_0^a$ and $F^b > F_0^b$ the following holds

$$\|\sum_{a\in F^a} a - \sum_{a\in A} a\| < \frac{\epsilon}{2} \qquad \|\sum_{b\in F^b} b - \sum_{b\in B} b\| < \frac{\epsilon}{2}. \tag{4}$$

Denote $F_0 = F_0^a \cup F_0^b$, it follows that for every $F > F_0$

$$\begin{aligned}\|\sum_{x\in F} x - \sum_{a\in A} a - \sum_{b\in B} b\| &= \| \sum_{x\in F\cap A} + \sum_{x\in F\cap B} x - \sum_{a\in A} a - \sum_{b\in B} b\| \\ &\leq \| \sum_{x\in F\cap A} x - \sum_{a\in A} a\| + \| \sum_{x\in F\cap B} x - \sum_{b\in B} b\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Because $\epsilon$ was arbitrary, we conclude that $\sum_{x\in A\cup B} \to \sum_{a\in A} a + \sum_{b\in B} b$.

## 1.3

Let's start by proving the following proposition:

**Proposition 1.** *Any converging $\sum_{a\in A} a \to x$ of positive numbers has at most a countable number of non zero elements. That is, $\sum_{a\in A} x = \sum_{i\in\mathbb{N}} a_i$, meaning that they converge to the same value $x$.*

*Proof.* Say that the net converges to $M$, i.e. $\sum_{a\in A} a = M < \infty$ where for every $a \in A, a > 0$. Consider now the sets $S_n = \{a \in A | a > \frac{1}{n}\}$, then

$$M \geq \sum_{a\in S_n} a \geq \sum_{a\in S_n} \frac{1}{n} = \frac{N}{n}.$$

As $M < \infty$ so is $N$ which is the cardinality of the set $S_n$. It follows that

$$\#\{a \in A | a > 0\} = \#S = \# \bigcup_{n=\mathbb{N}}^{\infty} S_n \tag{5}$$

We conclude that $A$ has at most countable number of non zero elements as countable union of finite sets. □

Let's now prove the ( $\implies$ ) direction. Given the previously proven statement, we can rewrite the net as countable sum and thus define a corresponding sequence $x_n = \sum_{i=0}^{n} a_i$ where w.log. we associated every non zero element $a$ of $A$ to an index $i$ so that $a_i = a$. From standard analysis we obtain that every converging increasing sequence is bounded from above, i.e. there exists $N \in R$ so that $x_n < N$ for every $n$. It follows that for every finite $F \subset I$

$$\sum_{a\in F} a \leq \sum_{i\in\mathbb{N}} a_i \leq N. \tag{6}$$

We now prove the opposite implication ( $\impliedby$ ). Assume that $\sup \left\{\sum_{a\in F} a : F \in \mathcal{F}\right\} = x_0$. We proceed now by contradiction, suppose that $\sum_{a\in A} a \to x_0 + t$ for an arbitrary $t > 0$. Let's define now what does it mean for a net to be increasing. Given a set $F_0$, it holds that for every $F > F_0$ we have that $\sum_{a\in F} a \geq \sum_{a\in F_0} a$.

By the definition of net convergence for standard set inclusion as order, we obtain that the net $\sum_{a\in A} a$ is increasing. It follows that for every $F$, $\sum_{a\in F} \leq x_0 + t$. Moreover, since $\sum_{a\in A} a \to x_0 + t$, there exists and $F_t$ such that $\forall F > F_t$ we have $\| \sum_{a\in F} a - x_0 - t\| < \frac{t}{2}$. It follows

$$\begin{aligned}
&\| \sum_{a\in F} a - x_0 - t\| \leq 0 \\
&\| \sum_{a\in F} a - x_0 - t\| = \sum_{a\in F} a - x_0 - t < \frac{t}{2} \\
&\sum_{a\in F} a > x_0 + \frac{t}{2}
\end{aligned}$$

Which is a contradiction. As this holds for any arbitrary $t$, $\sum_{a\in A} a \leq x_0$.
Let's conclude the proof by showing that

$$\sup \left\{ \sum_{a\in F} a : F \in \mathcal{F} \right\} = \sum_{a\in A} a = x_0. \tag{7}$$

Take an arbitrary $x < x_0$. By contradiction, assume that $\sup\left\{\sum_{a\in F} a : F \in \mathcal{F}\right\} = x$. Again, by convergence of the net, there exists a $F_0$ such that $\forall F > F_0$ we have $\| \sum_{a\in F} a - x_0\| < |x - x_0|$. As both quantities are negative, it follows that $\sum_{a\in F} a - x_0 > x - x_0$ which leads to the contradiction $\sum_{a\in F} a > x$. Combining it with the previous point we get the desired equality.

# 2 Exercise 2

## 2.1

Let's define $H' = \bigvee \mathcal{F}$. By theorem 4.13 we have that $\forall h \in H'$, $h$ can be written as $h = \sum_{e\in F} \langle h, e\rangle e$ as $F$ is the basis for $H'$ again by theorem 4.13.
Moreover, for every $x \in H$, we have that by definition $P_F x \in H'$. We define the operator $Q$ acting on $x$ as such

$$Qx \coloneqq \sum_{e\in F} \langle x, e\rangle e. \tag{8}$$

For $Q$ to be equal to $P_F$, $Qx$ has to be the unique elements in $H'$ such that $x - Qx \perp H'$. We proceed by taking an orthogonal basis of $H$ such that $\mathcal{E} \subset B$, this basis is guaranteed to exist given the proposition 4.2. By theorem 4.13 again, by trivially noting that $\bigvee H = H$,

$x$ can be represented as $x = \sum_{e \in B} \langle x, e \rangle e$. It follows that

$$\begin{aligned} x - Qx &= \sum_{e \in B} \langle x, e \rangle e - \sum_{e \in F} \langle x, e \rangle e \\ &= \sum_{e \in F} \langle x, e \rangle e + \sum_{e \in B \setminus F} \langle x, e \rangle e - \sum_{e \in F} \langle x, e \rangle e \\ &= \sum_{e \in B \setminus F} \langle x, e \rangle e \end{aligned}$$

We conclude that since B is orthogonal, it follows that $x - Qx \perp H' = \bigvee F$.

## 2.2

By the previous point, we can write $P_G x = \sum_{e \in G} \langle x, e \rangle e$ for every $G \subset \mathcal{G}$. It follows that

$$\begin{aligned} P_F P_G x &= P_F \left( \sum_{e \in G} \langle x, e \rangle e \right) \\ &= \sum_{e \in G} P_F \langle x, e \rangle e. \\ &= \sum_{e \in G} \sum_{e' \in F} \langle \langle e, x \rangle e, e' \rangle e' \\ &= \sum_{e \in G} \sum_{e' \in F} \langle e, x \rangle \langle e, e' \rangle e' \\ &= \sum_{e \in G \cap F} \langle e, x \rangle e. \end{aligned}$$

Where we used the orthogonality of $P_F$ and the fact that the set $F, G$ are orthonormal.

## 2.3

Given an arbitrary finite family of disjoin orthonormal basis $(F_n)_{n \in \mathbb{N}}$.

$$P_{\bigcup_{i=1}^{n} F_i} = P_{\bigcup_{i=1}^{n-1} \cup F_n} = P_{F_j \cup F_n}$$

Where $F_j := \bigcup_{i=1}^{n-1} F_i$. Thus, we just have to prove the above statement for two sets. This follows directly from the linearity of the sum, i.e.

$$P_{F_i} x + P_{F_j} x = \sum_{e \in F_i} \langle x, e \rangle e + \sum_{e \in F_j} \langle x, e \rangle e = \sum_{e \in F_i \cup F_j} \langle x, e \rangle e = P_{F_i \cup F_j}.$$

For the convergence, we have that for a fixed $\epsilon > 0$, for all $n > n_0$ the following holds true:

$$\|P_{\bigcup_{i=1}^{\infty} F_i} x - \sum_{e \in \bigcup_{i=1}^{n} F_i} \langle x, e \rangle e\| = \|P_{\bigcup_{i=1}^{\infty} F_i} x - \sum_{i=1}^{n} \sum_{e \in F_i} \langle x, e \rangle e\| < \epsilon. \tag{9}$$

Equivalently, for the same $\epsilon$, there exists a $n_0'$ such that for all $n > n_0$

$$\|\sum_{i}^{\infty} P_{F_i} x - \sum_{i}^{n} P_{F_i} x\| = \|\sum_{i}^{\infty} P_{F_i} x - \sum_{i}^{n} \sum_{e \in F_i} \langle e, x \rangle x\| < \epsilon \tag{10}$$

By taking $\overline{n} = \max(n_0, n_0')$ we get that both limits converge to the same value. By uniqueness of the limit we have that

$$P_{\bigcup_{i=1}^{\infty} F_i} x = \sum_{i=1}^{\infty} P_{F_i} x. \tag{11}$$

### 2.4

We prove by giving a counterexample that $\sum_i^{\infty} P_{F_i} x$ does not converge in the operator norm. Let's define $F_i = e_i$ where $e_i \cap e_j = \varnothing$ and $\|e_i\| = 1$ for every $i$. Take now $\epsilon = \frac{1}{2}$, then, for for every $n > N$

$$\|\sum_{i=0}^{\infty} P_{F_i} x - \sum_{i=0}^{n} P_{F_i} x\| = \|\sum_{i=n}^{\infty} P_{F_i} x\| = 1 > \frac{1}{2}$$

Therefore, for $\epsilon = \frac{1}{2}$ there exists no $n > 0$ such that the the difference of the norm converges.

## 3 Exercise 3

## 4 Exercise 4

### 4.1

Let's start noting how the operator $S$ affects the inner product.

$$\langle Sx, y \rangle = \langle (0, x_1, \dots), y \rangle = \sum_{i=1}^{\infty} x_i y_{i+1}$$

We can see that the inner product with $x$ and the adjoint on $y$ must be equal $\langle x, S^* y \rangle = \sum_{i=1}^{\infty} x_i y_{i+1}$ which is the left shift operator

$$S^*(x_1, x_2, \dots) = (x_2, x_3, \dots)$$

## 4.2

Let's compute now te concatenation of $S$ and its adjoint.

$$SS^*(x_1, x_2, \dots) = S(x_2, x_3, \dots) = (0, x_2, x_3, \dots)$$

Conversely

$$S^*S(x_1, x_2, \dots) = S(0, x_1, x_2, x_3, \dots) = (x_1, x_2, \dots) = x$$

## 4.3

And we can extend this to $S^n$ and $(S^*)^n$

$$S^n(S^*)^n x = (0^n, x_{n+1}, n_{n+2}, \dots)$$

and similarly

$$(S^*)^n S^n = x$$