

Homework 1

1 Exercise 1

1.1

Let's define x_0 as the limit of the net

$$\sum_{a \in A} a \rightarrow x_0. \quad (1)$$

Since it converges, it means that there exist a set $F_0 \subset A$ such that for all $F > F_0$ the following holds

$$\left\| \sum_{a \in F} a - x_0 \right\| < \epsilon \quad (2)$$

for a given $\epsilon > 0$. Since ϵ is arbitrary, we can choose $F'_0 > F_0$ such that for all $F > F'_0$

$$\left\| \sum_{a \in F} a - x_0 \right\| < \frac{\epsilon}{|\alpha|}. \quad (3)$$

Then, by properties of the norm we obtain

$$\begin{aligned} |\alpha| \left\| \sum_{a \in F} a - x_0 \right\| &< \frac{\epsilon}{|\alpha|} \alpha = \epsilon \\ \left\| \alpha \sum_{a \in F} a - \alpha x_0 \right\| &< \epsilon \\ \left\| \sum_{a \in F} \alpha a - \alpha x_0 \right\| &< \epsilon. \end{aligned}$$

Where in the last step we used the fact that F is finite. This proves that $\alpha \sum_{a \in A} a$ converges to $\alpha x_0 = \alpha \sum_{a \in A} a$.

1.2

The hypothesis that $\sum_{a \in A} a$ and $\sum_{b \in B} b$ implies that there exists an F_0^a and F_0^b such that for every $F^a > F_0^a$ and $F^b > F_0^b$ the following holds

$$\left\| \sum_{a \in F^a} a - \sum_{a \in A} a \right\| < \frac{\epsilon}{2} \quad \left\| \sum_{b \in F^b} b - \sum_{b \in B} b \right\| < \frac{\epsilon}{2}. \quad (4)$$

Denote $F_0 = F_0^a \cup F_0^b$, it follows that for every $F > F_0$

$$\begin{aligned} \left\| \sum_{x \in F} x - \sum_{a \in A} a - \sum_{b \in B} b \right\| &= \left\| \sum_{x \in F \cap A} x + \sum_{x \in F \cap B} x - \sum_{a \in A} a - \sum_{b \in B} b \right\| \\ &\leq \left\| \sum_{x \in F \cap A} x - \sum_{a \in A} a \right\| + \left\| \sum_{x \in F \cap B} x - \sum_{b \in B} b \right\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Because ϵ was arbitrary, we conclude that $\sum_{x \in A \cup B} x \rightarrow \sum_{a \in A} a + \sum_{b \in B} b$.

1.3

Let's start by proving that any converging net of positive numbers has at most a countable number of non zero elements.

Say that the net converges to M , i.e. $\sum_{a \in A} a = M < \infty$ where for every $a \in A, a > 0$. Consider now the sets $S_n = \{a \in A | a > \frac{1}{n}\}$, then

$$M \geq \sum_{a \in S_n} a \geq \sum_{a \in S_n} \frac{1}{n} = \frac{N}{n}.$$

As $M < \infty$ so is N which is the cardinality of the set S_n . It follows that

$$\#\{a \in A | a > 0\} = \#S = \# \bigcup_{n \in \mathbb{N}} S_n \quad (5)$$

We conclude that A has at most countable number of non zero elements as countable union of finite sets.

Let's now prove the (\implies) direction. Given the previously proven statement, we can rewrite the net as countable sum and thus define a corresponding sequence $x_n = \sum_{i=0}^n a_i$ where w.l.o.g. we associated every non zero element a of A to an index i so that $a_i = a$. From standard analysis we obtain that every converging increasing sequence is bounded from above, i.e. there exists $N \in \mathbb{R}$ so that $x_n < N$ for every n . It follows that for every finite $F \subset I$

$$\sum_{a \in F} a \leq \sum_{i \in \mathbb{N}} a_i \leq N. \quad (6)$$

We now prove the opposite implication (\impliedby). Assume that $\sup \{\sum_{a \in F} a : F \in \mathcal{F}\} = x_0$. We proceed now by contradiction, suppose that $\sum_{a \in A} a \rightarrow x_0 + t$ for an arbitrary $t > 0$. Let's define now what does it mean for a net to be increasing. Given a set F_0 , it holds that for every $F > F_0$ we have that $\sum_{a \in F} a \geq \sum_{a \in F_0} a$.

By the definition of net convergence for standard set inclusion as order, we obtain that the net $\sum_{a \in A} a$ is increasing. It follows that for every F , $\sum_{a \in F} a \leq x_0 + t$. Moreover, since

$\sum_{a \in A} a \rightarrow x_0 + t$, there exists and F_t such that $\forall F > F_t \parallel \sum_{a \in F} a - x_0 - t \parallel < \frac{t}{2}$. It follows

$$\begin{aligned} \parallel \sum_{a \in F} a - x_0 - t \parallel &\leq 0 \\ \parallel \sum_{a \in F} a - x_0 - t \parallel &= \sum_{a \in F} a - x_0 - t < \frac{t}{2} \\ \sum_{a \in F} a &> x_0 + \frac{t}{2} \end{aligned}$$

Which is a contradiction. As this holds for any arbitrary t , $\sum_{a \in A} a \leq x_0$.

Let's conclude the proof by showing that

$$\sup \left\{ \sum_{a \in F} a : F \in \mathcal{F} \right\} = \sum_{a \in A} a = x_0. \quad (7)$$

Take an arbitrary $x < x_0$. By contradiction, assume that $\sup \{ \sum_{a \in F} a : F \in \mathcal{F} \} = x$. Again, by convergence of the net, there exists a F_0 such that $\forall F > F_0$ we have

$$\begin{aligned} \parallel \sum_{a \in F} a - x_0 \parallel &< |x - x_0| \\ x_0 - \sum_{a \in F} a &< x_0 - x \\ \sum_{a \in F} a &> x \end{aligned}$$

Which is a contradiction. Combining it with the previous point we get the desired equality.

To double check that this is all we need.