

Scale-free networks

Prof. Pascal Frossard
Signal Processing Laboratory (LTS4)



Network Machine Learning - EE452
Dr Dorina Thanou
Prof. Pascal Frossard



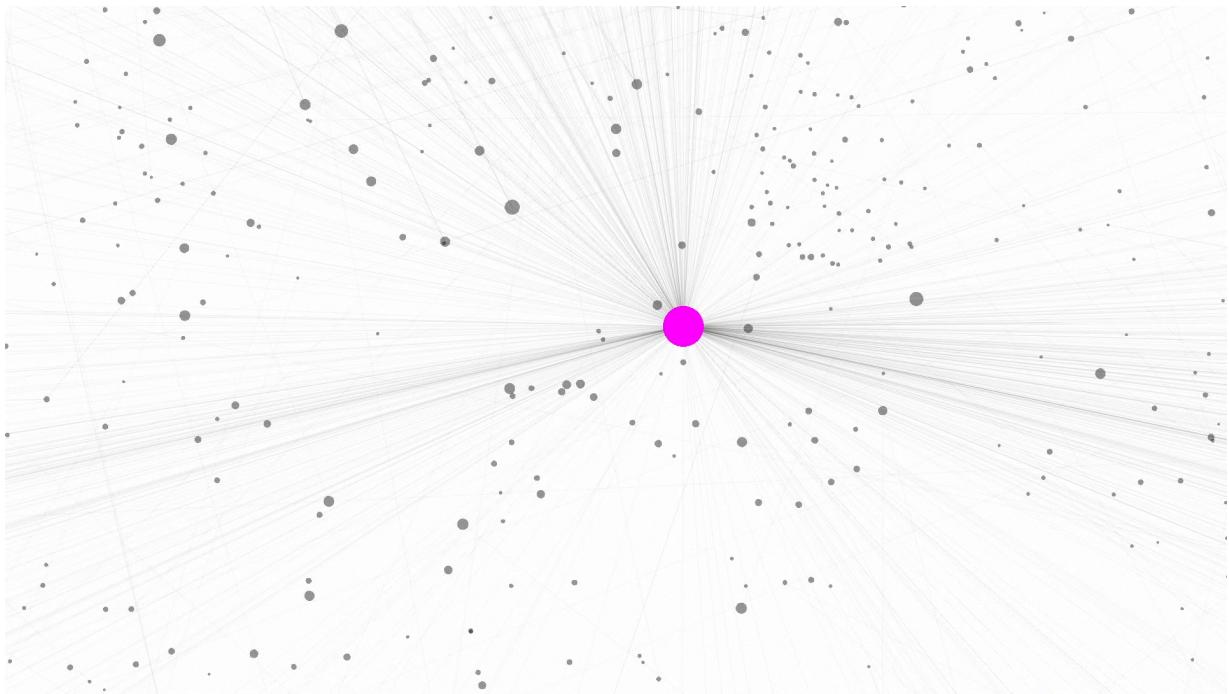
Outline

- Power Laws
- Hubs
- The Meaning of Scale-Free
- Ultra-Small Property
- Role of the degree exponent



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Zoom into the WWW



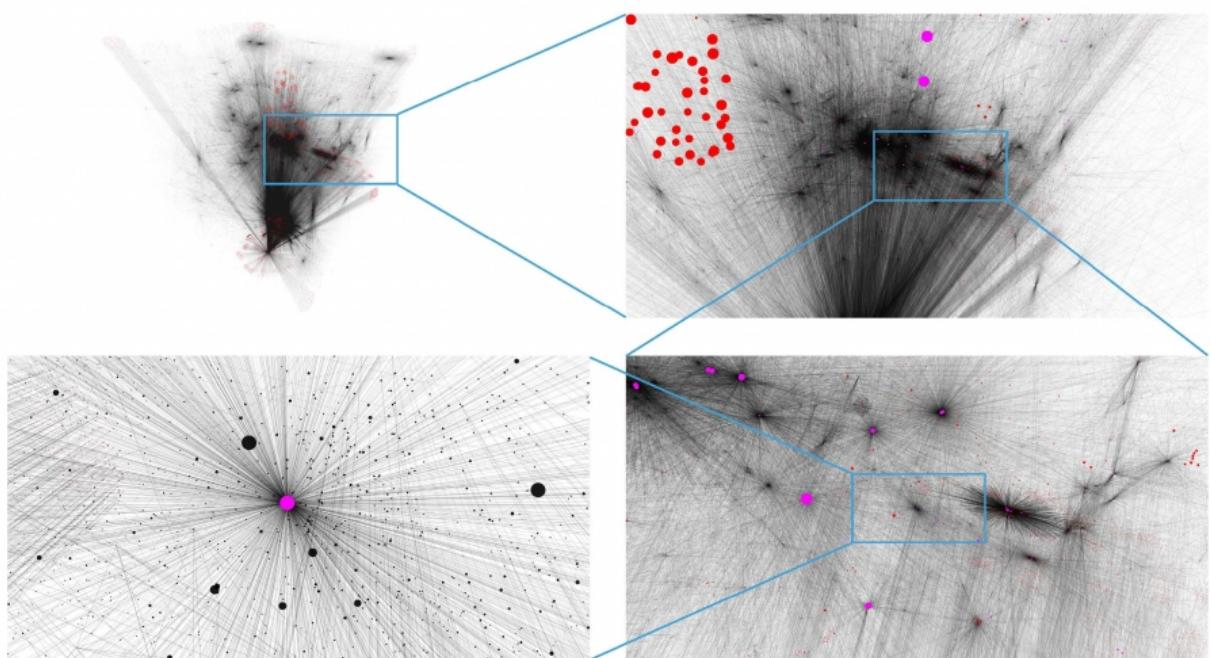
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The Web is not random



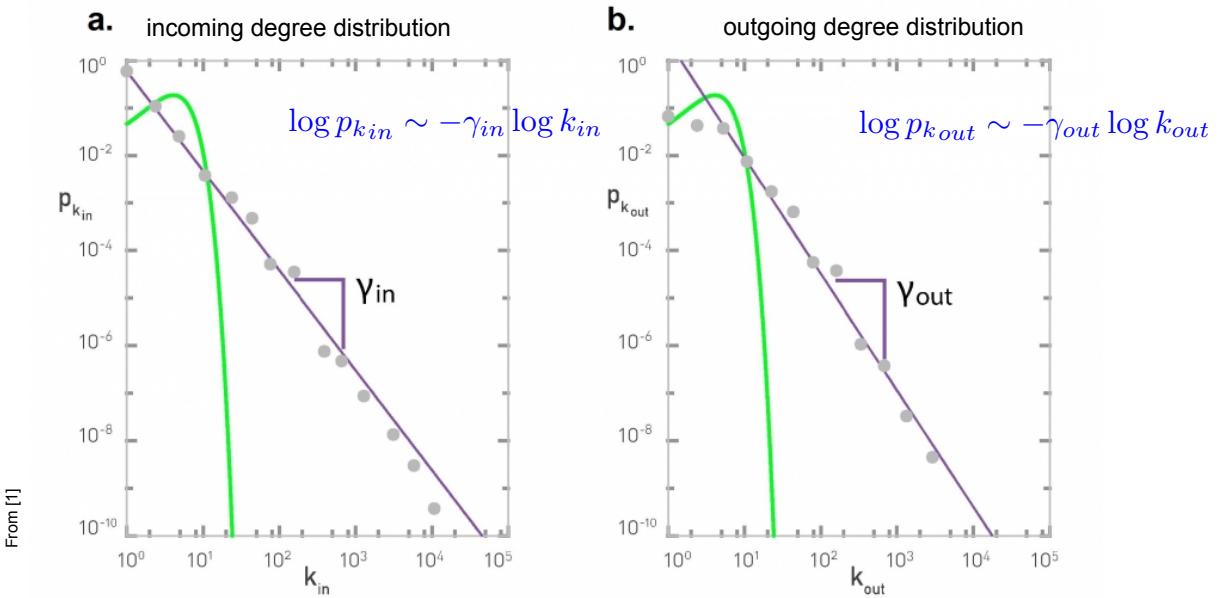
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WWW: degree distribution



- Degrees do not follow a Poisson distribution (like random networks) but rather a *power law distribution*, of the form

$$p_k \sim k^{-\gamma}$$

Scale-free network

A *scale-free network* is a network whose degree distribution follows a *power law*.

Discrete Formalism

Probability that a node has k links:

$$p_k = Ck^{-\gamma}$$

With normalisation constraints

$$\sum_{k=1}^{\infty} p_k = 1 \quad \text{or} \quad C \sum_{k=1}^{\infty} k^{-\gamma} = 1$$

The parameter C becomes

$$C = \frac{1}{\sum_{k=1}^{\infty} k^{-\gamma}} = \frac{1}{\zeta(\gamma)}$$

Riemann-zeta function

Finally: $p_k = \frac{k^{-\gamma}}{\zeta(\gamma)}$

p_0 can be specified separately

Continuum Formalism

Probability of node degree between k_1 and k_2

$$\int_{k_1}^{k_2} p(k) dk \quad \text{with} \quad p(k) = Ck^{-\gamma}$$

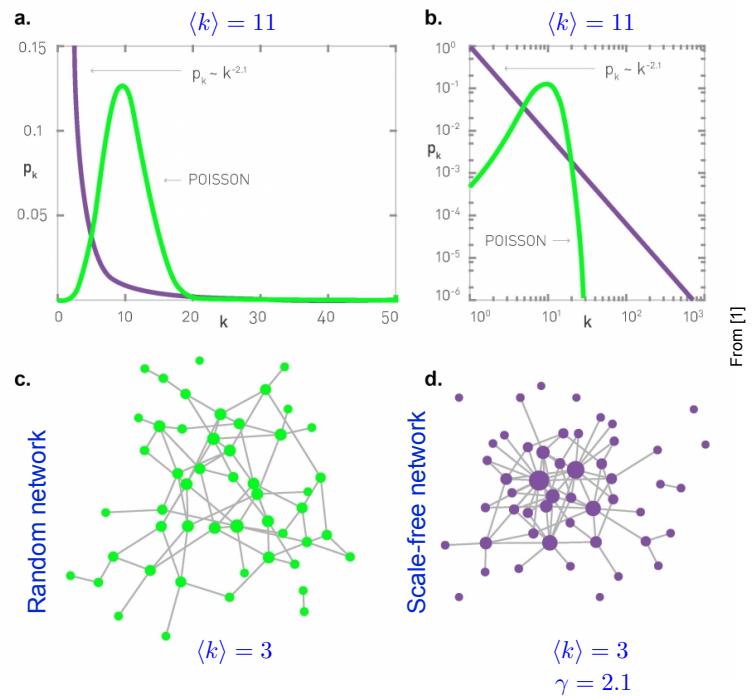
With normalisation constraints $\int_{k_{min}}^{\infty} p(k) dk = 1$

$$C = \frac{1}{\int_{k_{min}}^{\infty} k^{-\gamma} dk} = (\gamma - 1)k_{min}^{\gamma-1}$$

Finally:
$$p(k) = (\gamma - 1)k_{min}^{\gamma-1}k^{-\gamma}$$

Hubs

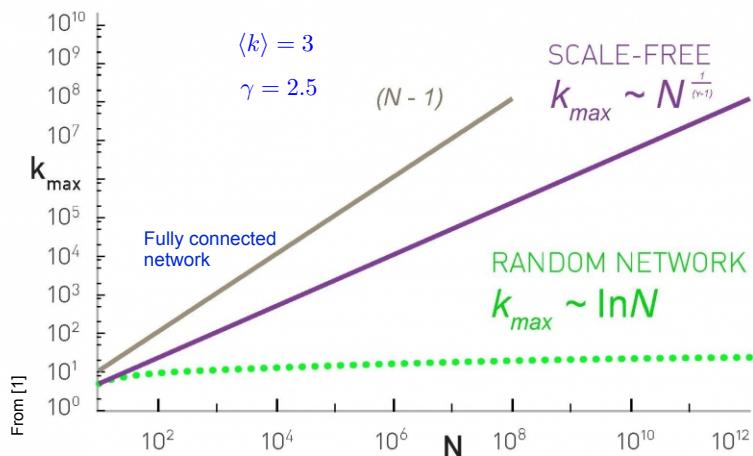
- Main difference between random and scale-free networks:
the tail of degree distribution!
- Scale-free networks have
 - many small degree nodes
 - not so many nodes around $\langle k \rangle$
 - high-degree nodes, aka *hubs*
- WWW example, $\langle k \rangle = 4.6$
 - with Poisson:
$$p_{100} = 10^{-94} \quad N_{k \geq 100} = \simeq 10^{-82}$$
 - with power-law, $\gamma_{in} = 2.1$
$$p_{100} = 4 \times 10^{-4} \quad N_{k \geq 100} = 4 \times 10^9$$



Largest hub

- The expected size of the largest hub is the maximum degree k_{max}
 - it is the *natural cutoff* of the degree distribution p_k
- For scale-free networks, the larger the network, the larger the biggest hub
 - we calculate the maximum degree such as there is at most one node in $[k_{max}, \infty[$
 - this means that
$$\int_{k_{max}}^{\infty} p(k)dk = \frac{1}{N}$$
 - with $p(k) = (\gamma - 1)k_{min}^{\gamma-1}k^{-\gamma}$, we can compute $k_{max} = k_{min}N^{\frac{1}{\gamma-1}}$
$$(\gamma - 1)k_{min}^{\gamma-1} \left\{ \frac{k^{-\gamma+1}}{-\gamma + 1} \right\}_{k_{max}}^{\infty} = k_{min}^{\gamma-1} \left\{ \frac{1}{k_{max}^{\gamma-1}} \right\} = \frac{1}{N}$$
- The polynomial dependence on N means that there could be orders of magnitude differences between k_{min} and k_{max}

Hubs are big in Scale-free networks



For random networks, we consider an approximation with $p(k) = Ce^{-\lambda k}$

Then, with the same definitions

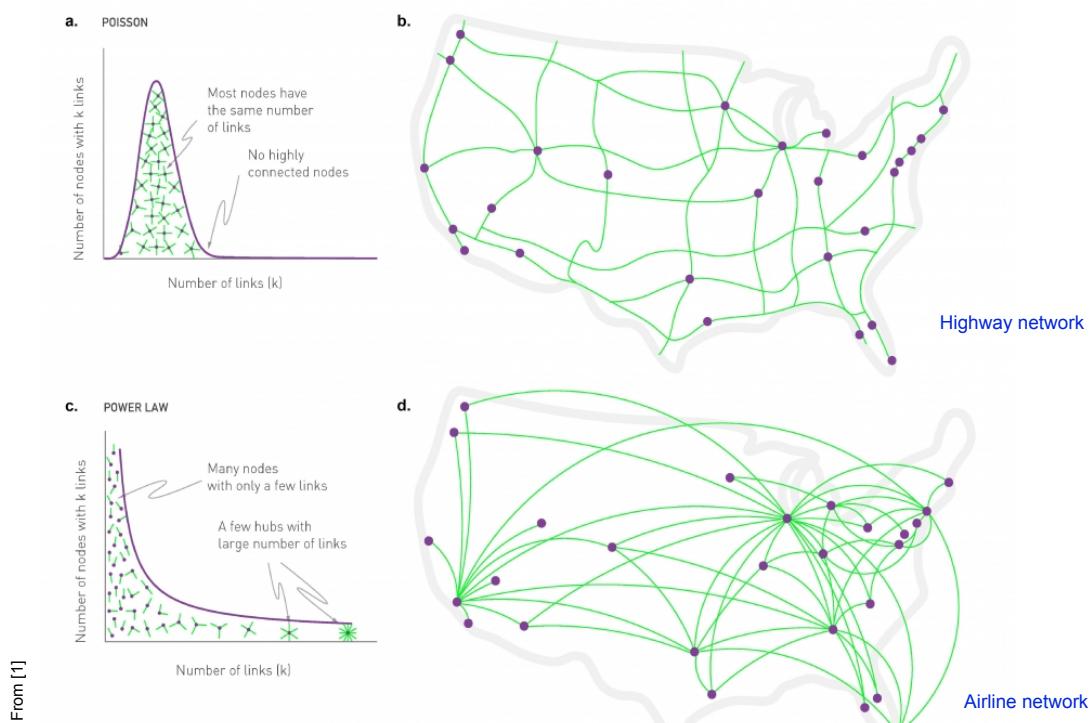
$$\int_{k_{\min}}^{\infty} p(k) dk = 1$$

$$\int_{k_{\max}}^{\infty} p(k) dk = \frac{1}{N}$$

$$\text{We get } k_{\max} = k_{\min} + \frac{\ln N}{\lambda}$$

- Hubs in a scale-free network are several orders of magnitude larger than the biggest node in a random network with the same N and $\langle k \rangle$!!
 - the dependence of k_{\max} on N is much slower for random networks

Random vs Scale-free networks



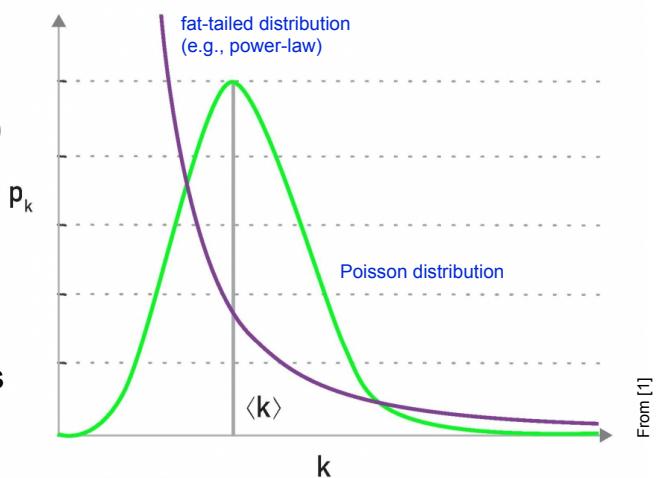
Why ‘Scale-free’?

- The n^{th} moment of the degree distribution: $\langle k^n \rangle = \sum_{k_{\min}}^{\infty} k^n p_k \approx \int_{k_{\min}}^{\infty} k^n p(k) dk$
 - for $n = 1$ we have the average degree $\langle k \rangle$
 - for $n = 2$ the second moment $\langle k^2 \rangle$ is related to the variance $\sigma^2 = \langle k^2 \rangle - \langle k \rangle^2$
 - for $n = 3$ the third moment $\langle k^3 \rangle$ determines the *skewness* σ : the standard deviation symmetry around $\langle k \rangle$
- For a scale-free network: $\langle k^n \rangle = \int_{k_{\min}}^{k_{\max}} k^n p(k) dk = C \frac{k_{\max}^{n-\gamma+1} - k_{\min}^{n-\gamma+1}}{n - \gamma + 1}$
 - k_{\min} is fixed, k_{\max} depends on N
 - limit when $k_{\max} \rightarrow \infty$
 - if $n - \gamma + 1 \leq 0$ then $k_{\max}^{n-\gamma+1} \rightarrow 0$: all moments with $n \leq \gamma - 1$ are finite
 - if $n - \gamma + 1 > 0$ then all moments diverge, i.e., $\langle k^n \rangle \rightarrow \infty$
- For many scale-free networks, $\gamma \in [2, 3]$
 - then $\langle k \rangle$ is finite, all other moments diverge
 - the degrees are typically in the range $k = \langle k \rangle \pm \sigma_k$, which gets ‘unbounded’

Networks with $\gamma < 3$ do not have a meaningful internal scale: they are scale-free!

Degree Variance

- For random networks:
 - we have $\sigma_k = \langle k \rangle^{\frac{1}{2}}$
 - hence, the internal scale is $\langle k \rangle$
- For regular lattices
 - all nodes have the same degree: $\sigma = 0$
- Scale-free networks
 - unbounded ‘variance’
 - nodes with widely different degrees coexist in the same network
 - divergence of second moment explains some of the most intriguing properties such as robustness to random failures and anomalous spread of viruses.



Lack of an Internal Scale

Degree Fluctuations in Real Networks

Network	N	L	$\langle k \rangle$	$\langle k_{in}^2 \rangle$	$\langle k_{out}^2 \rangle$	$\langle k^2 \rangle$	γ_{in}	γ_{out}	γ
Internet	192,244	609,066	6.34	-	-	240.1	-	-	3.42*
WWW	325,729	1,497,134	4.60	1546.0	482.4	-	2.00	2.31	-
Power Grid	4,941	6,594	2.67	-	-	10.3	-	-	Exp.
Mobile-Phone Calls	36,595	91,826	2.51	12.0	11.7	-	4.69*	5.01*	-
Email	57,194	103,731	1.81	94.7	1163.9	-	3.43*	2.03*	-
Science Collaboration	23,133	93,437	8.08	-	-	178.2	-	-	3.35*
Actor Network	702,388	29,397,908	83.71	-	-	47,353.7	-	-	2.12*
Citation Network	449,673	4,689,479	10.43	971.5	198.8	-	3.03*	4.00*	-
E. Coli Metabolism	1,039	5,802	5.58	535.7	396.7	-	2.43*	2.90*	-
Protein Interactions	2,018	2,930	2.90	-	-	32.3	-	-	2.89*-

From [1]



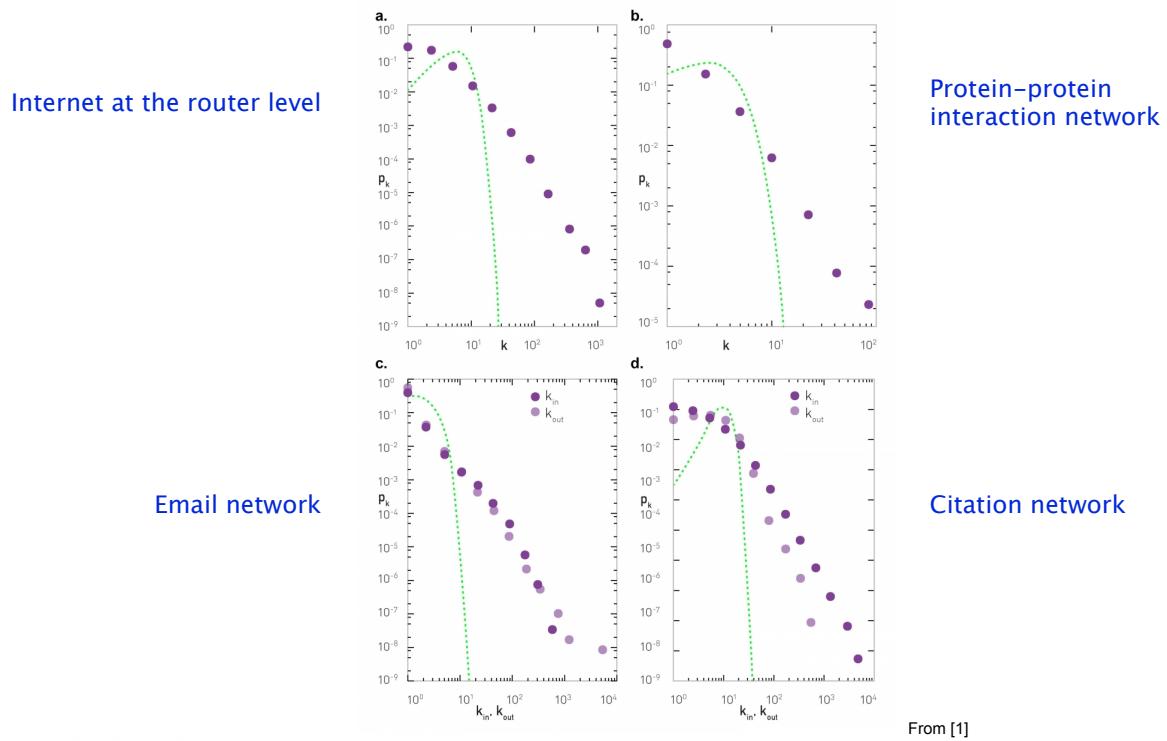
Standard Deviation is Large



From [1]



Universality of scale-free properties



Distributions in network science

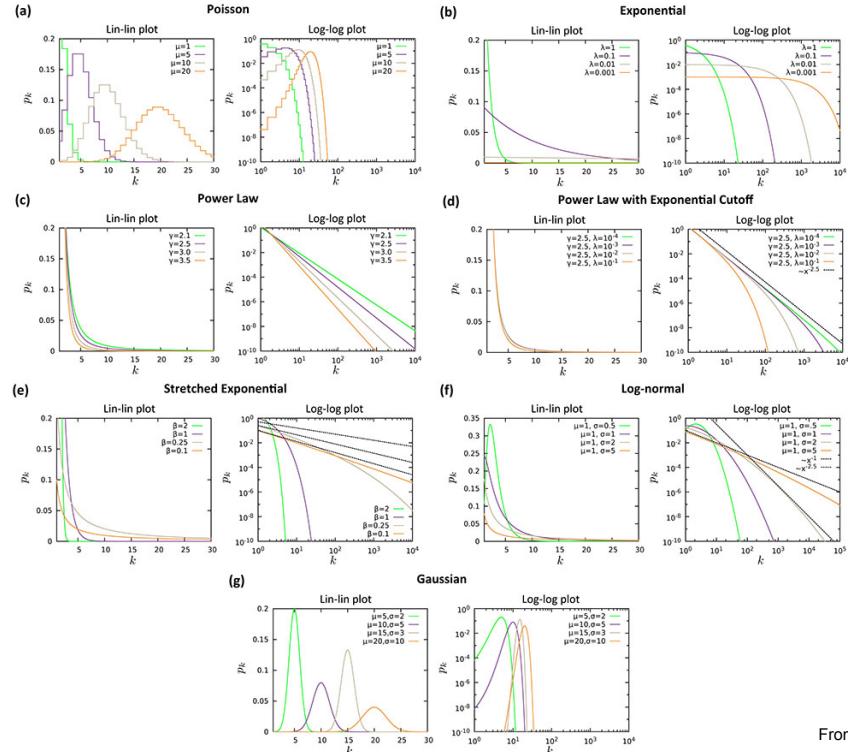
NAME	$p_x/p(x)$	$\langle x \rangle$	$\langle x^2 \rangle$
Poisson (discrete)	$e^{-\mu} \mu^x / x!$	μ	$\mu(1 + \mu)$
Exponential (discrete)	$(1 - e^{-\lambda}) e^{-\lambda x}$	$1/(e^\lambda - 1)$	$(e^\lambda + 1)/(e^\lambda - 1)^2$
Exponential (continuous)	$\lambda e^{-\lambda x}$	$1/\lambda$	$2/\lambda^2$
Power law (discrete)	$x^{-\alpha}/\zeta(\alpha)$	$\begin{cases} \zeta(\alpha - 2)/\zeta(\alpha), & \text{if } \alpha > 2 \\ \infty, & \text{if } \alpha \leq 1 \end{cases}$	$\begin{cases} \zeta(\alpha - 1)/\zeta(\alpha), & \text{if } \alpha > 1 \\ \infty, & \text{if } \alpha \leq 2 \end{cases}$
Power law (continuous)	$\alpha x^{-\alpha}$	$\begin{cases} \alpha/(\alpha - 1), & \text{if } \alpha > 2 \\ \infty, & \text{if } \alpha \leq 1 \end{cases}$	$\begin{cases} \alpha/(\alpha - 2), & \text{if } \alpha > 1 \\ \infty, & \text{if } \alpha \leq 2 \end{cases}$
Power law with cutoff (continuous)	$\frac{\lambda^{1-\alpha}}{\Gamma(1-\alpha)} x^{-\alpha} e^{-\lambda x}$	$\lambda^{-1} \frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha)}$	$\lambda^{-2} \frac{\Gamma(3-\alpha)}{\Gamma(1-\alpha)}$
Stretched exponential (continuous)	$\beta \lambda^\beta x^{\beta-1} e^{-(\lambda x)^\beta}$	$\lambda^{-1} \Gamma(1 + \beta^{-1})$	$\lambda^{-2} \Gamma(1 + 2\beta^{-1})$
Log-normal (continuous)	$\frac{1}{x \sqrt{2\pi\sigma^2}} e^{-(\ln x - \mu)^2/(2\sigma^2)}$	$e^{\mu + \sigma^2/2}$	$e^{2(\mu + \sigma^2)}$
Normal (continuous)	$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x - \mu)^2/(2\sigma^2)}$	μ	$\mu^2 + \sigma^2$

Exponentially Bounded
Distributions

Fat Tailed Distributions

From [1]

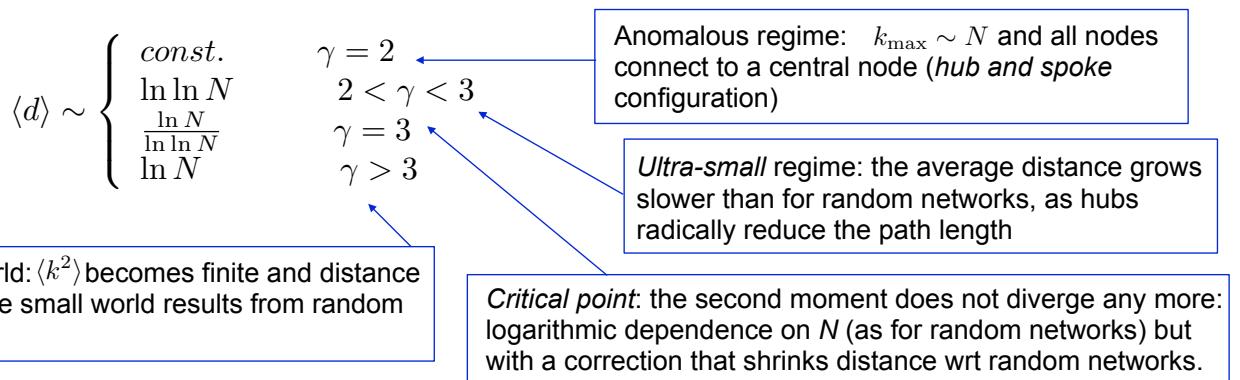
Distributions visualised



From [1]

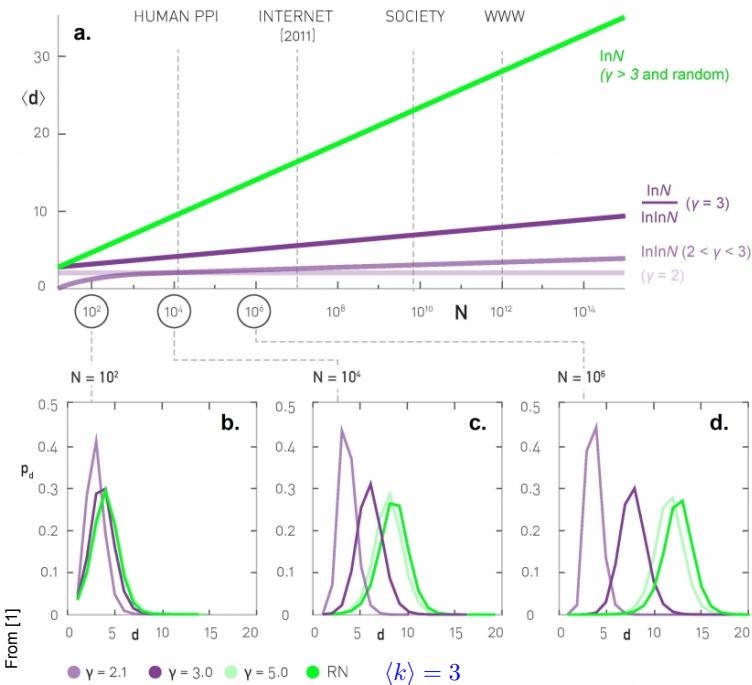
Ultra-Small Property

- Do hubs actually affect the small-world property?
 - Actually, distances in a scale-free network are smaller than the distances observed in an equivalent random network
- Dependence of the average distance on N and γ



R. Cohen and S. Havlin. Scale free networks are ultrasmall. Phys. Rev. Lett. 90, 058701, 2003.
B. Bollobás and O. Riordan. The Diameter of a Scale-Free Random Graph. Combinatorica, 24: 5-34, 2004.

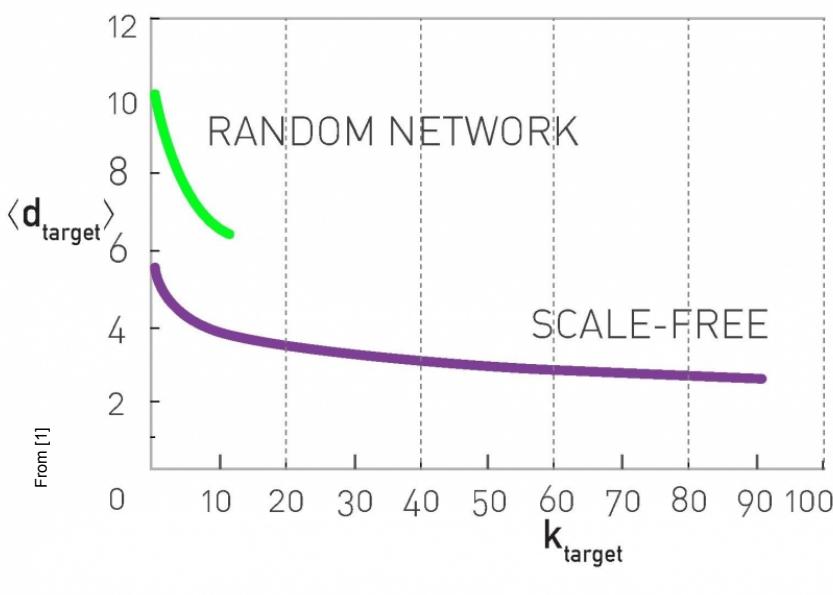
Distances in scale-free networks



- For small networks, all distances are comparable
- For large N , differences are important
- For small N , path length distribution overlap
- For large N , path length distributions are quite different for different γ

Scale-free property shrinks distances!

Always close to hubs



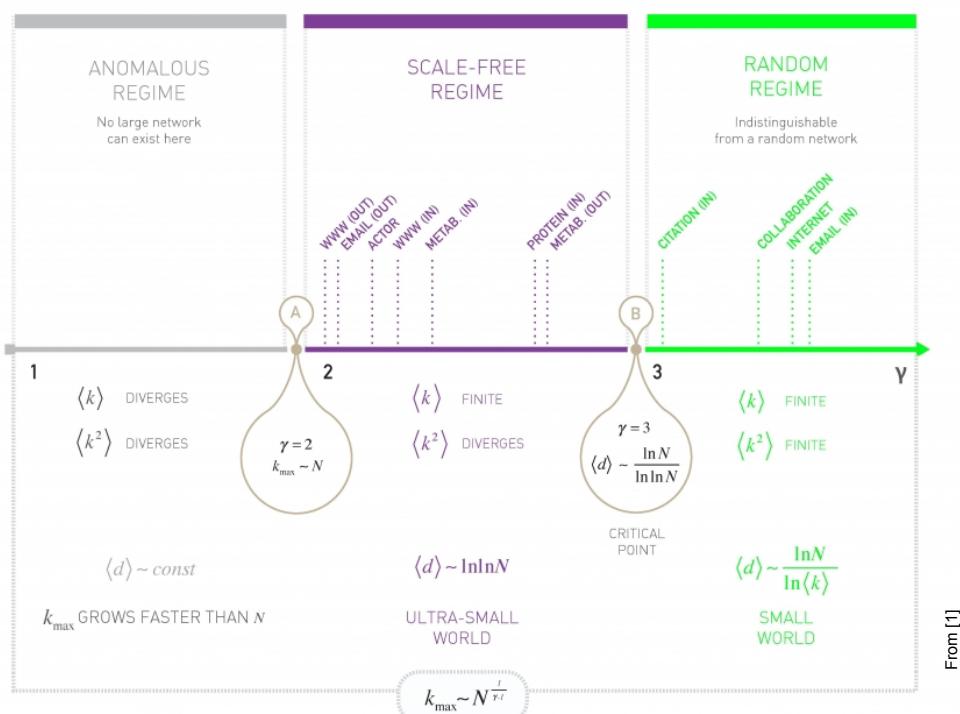
- In general, a node is typically closer to hubs than to less connected nodes.
- This effect is particularly pronounced in scale-free networks
 - There are always short paths linking hubs
 - Many of the shortest paths go through these hubs

The role of the degree exponent

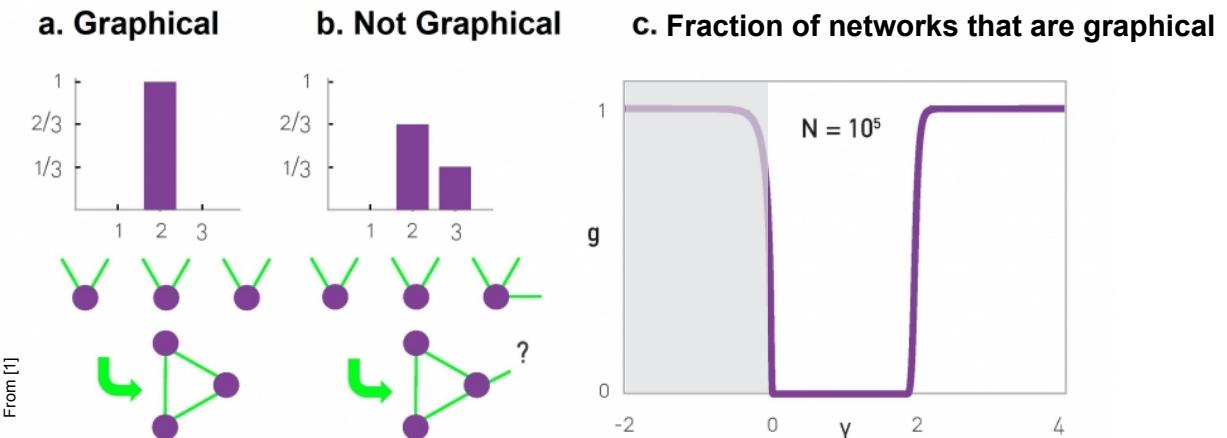
- Many properties of a scale-free network depend on γ
 - Anomalous Regime ($\gamma \leq 2$)
 - the number of links to the largest hub grows faster than N : not enough nodes!
 - the average degree $\langle k \rangle$ diverges for $N \rightarrow \infty$
 - Scale-Free Regime ($2 < \gamma < 3$)
 - first moment is the only finite moment as $N \rightarrow \infty$: ultra-small world regime
 - the market share of the largest hub decreases as $k_{\max}/N \sim N^{-\frac{\gamma-2}{\gamma-1}}$

$$k_{\max} = k_{\min} N^{\frac{1}{\gamma-1}}$$
 - Random Network Regime ($\gamma > 3$)
 - quite similar to random networks
 - first and second moments are finite: it has similar properties as a random network
 - for large γ , the degree distribution decays fast: hubs are small and less numerous
 - hard to observe a high degree power-law: we need large networks, at least
- $$N = \left(\frac{k_{\max}}{k_{\min}} \right)^{\gamma-1}$$
 with at least 2-3 orders of magnitude between k_{\min} and k_{\max}

γ - dependent properties



No scale-free networks for $\gamma < 2$?

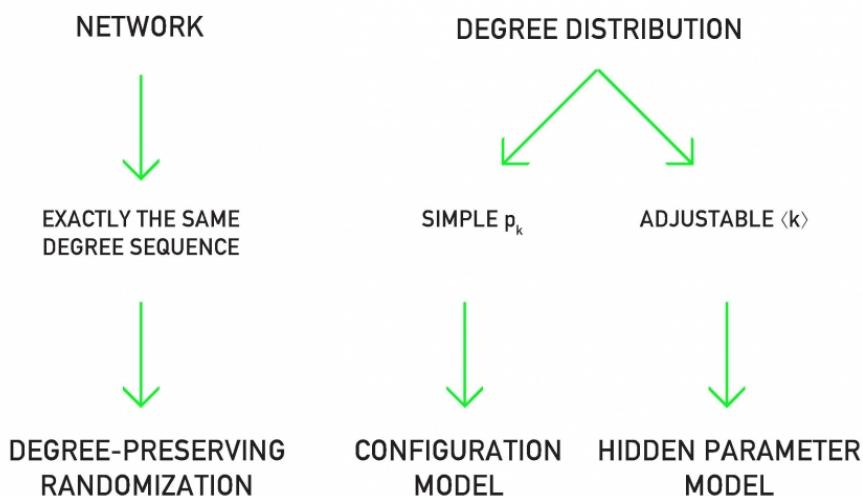


- A degree sequence that can be turned into simple graph (i.e., a graph lacking multi-links or self-loops) is called graphical - it can be verified by a specific algorithm
- If we apply the algorithm to scale-free networks we find that the number of graphical degree sequences drops to zero for $\gamma < 2$ (the largest hub grows faster than N !)

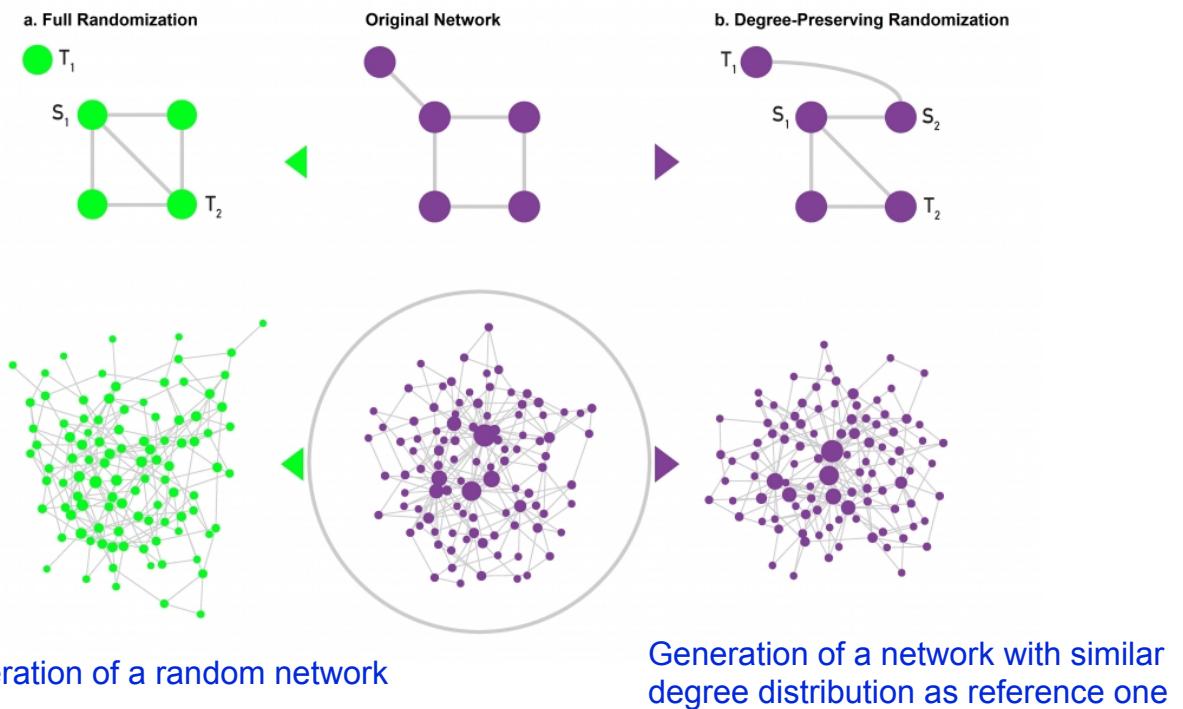
I. Charo Del Genio, G. Thilo, and K.E. Bassler. All scale-free networks are sparse. Phys. Rev. Lett. 107:178701, 10 2011.

Scale-free networks generation

- Analysis or comparisons might require to generate scale-free networks
- Different methods are available, depending on requirements and constraints



Degree-preserving randomisation



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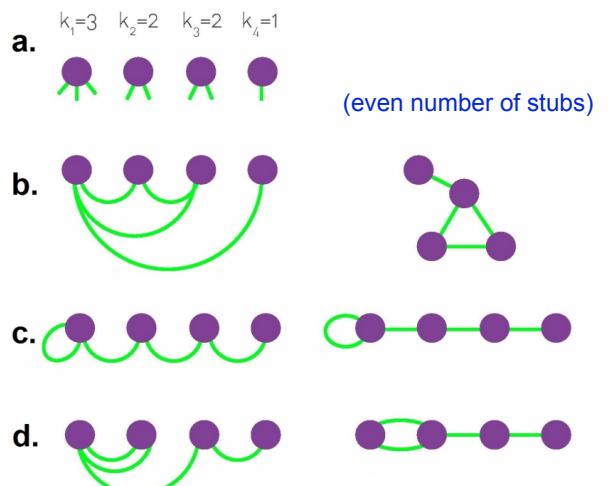
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Configuration model

- Generating a network with a pre-defined degree distribution
 - pick a random pair of stubs and connect them
 - repeat the process iteratively
- Probability of connecting i and j

$$p_{ij} = \frac{k_i k_j}{2L - 1}$$

- Self-loops and multi-links are possible
 - their probability goes to 0 for large N



From [1]

See [2] for details

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Hidden parameter model

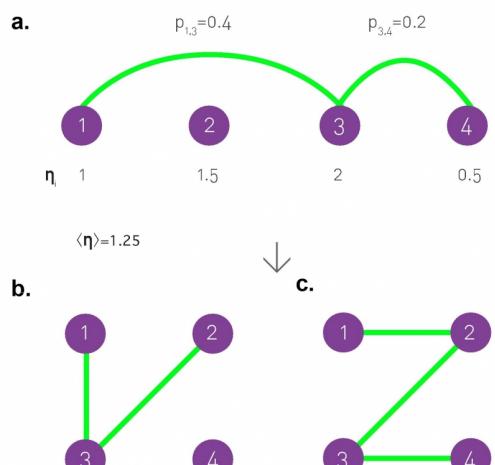
- Generation of scale-free networks with a predefined degree distribution and no self-loop or multi-links
 - start with isolated nodes, assign hidden parameter η_i chosen from $\rho(\eta)$ or given by $\{\eta_i\}$ leading resp. to
$$p_k = \int \frac{e^{-\eta} \eta^k}{k!} \rho(\eta) d\eta \quad \text{or} \quad p_k = \frac{1}{N} \sum_j \frac{e^{-\eta_j} \eta_j^k}{k!} \quad \langle k \rangle = \langle \eta \rangle$$
- connect each node with probability

$$p(\eta_i, \eta_j) = \frac{\eta_i \eta_j}{\langle \eta \rangle N}$$

- to have a scale-free network, use

$$\eta_i = \frac{c}{i^\alpha}, \quad i = 1, \dots, N \quad \longrightarrow \quad p_k \sim k^{-(1+\frac{1}{\alpha})}$$

tune γ through α



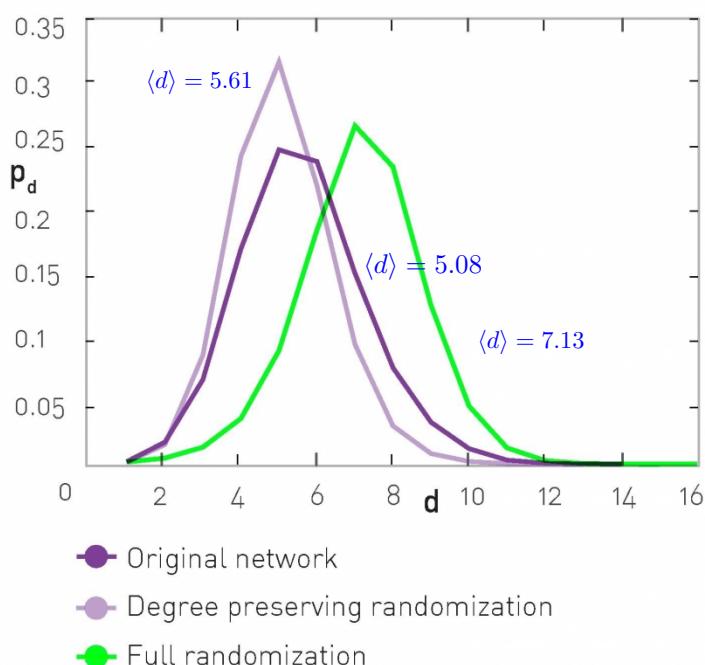
Sample network results

$$\langle L \rangle = \frac{1}{2} \sum_{i,j}^N \frac{\eta_i \eta_j}{\langle \eta \rangle N} = \frac{1}{2} \langle \eta \rangle N$$

B. Söderberg. General formalism for inhomogeneous random graphs. Phys. Rev. E 66: 066121, 2002.
 M. Boguñá and R. Pastor-Satorras. Class of correlated random networks with hidden variables. Phys. Rev. E 68: 036112, 2003.
 G. Caldarelli, I. A. Capocci, P. De Los Rios, and M.A. Muñoz. Scale-Free Networks from Varying Vertex Intrinsic Fitness. Phys. Rev. Lett. 89: 258702, 2002.

Testing the small-world property

Degree-preserving randomisation preserves the (small) expected distance value



Full randomisation increases the expected distance value: loss of scale-free property of the original network

Summary: scale-free networks

At a glance: Scale-free networks

- Degree distribution

$$p_k = \frac{k^{-\gamma}}{\zeta(\gamma)} \quad \text{or} \quad p(k) = (\gamma - 1)k_{\min}^{\gamma-1} k^{-\gamma}$$

- Size of the largest hub

$$k_{\max} = k_{\min} N^{\frac{1}{\gamma-1}}$$

- Moments of p_k for $N \rightarrow \infty$

- $2 < \gamma \leq 3$: $\langle k \rangle$ finite, $\langle k^2 \rangle$ diverges
- $\gamma > 3$: $\langle k \rangle$ and $\langle k^2 \rangle$ finite

- Distance

$$\langle d \rangle \sim \begin{cases} \text{const.} & \gamma = 2 \\ \ln \ln N & 2 < \gamma < 3 \\ \frac{\ln N}{\ln \ln N} & \gamma = 3 \\ \ln N & \gamma > 3 \end{cases}$$

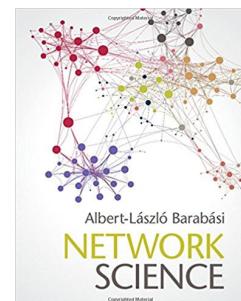
- Exponentially bounded networks (fast decreasing degree distribution for high k) lack outliers
 - for example, in highway networks or power grids, most nodes have comparable degrees
- Scale-free networks (fat tailed degree distributions) have hubs that change the system's behaviour
 - Many networks of scientific and practical interest, from the WWW to the subcellular networks or social ones
 - Outliers are expected in these networks
- Power-law rarely seen in 'pure' form:
 - If $\langle k^2 \rangle$ is large: scale-free behaviour; if $\langle k^2 \rangle$ is small and comparable to $\langle k \rangle$: random network approximation holds



References

[1] Network Science, by Albert-László Barabási, 2016 - Chapter 4

[2] Networks: An Introduction, by M. Newman, 2010



Some slides are inspired from Prof. Barabási's class on Network Science (www.BarabasiLab.com)

