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PETRI NETS, EVENT STRUCTURES AND DOMAINS, PART I

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Abstract. The general aim of this paper is to find a theory of concurrency combining the approaches of Petri and Scott (and others).

In part I we introduce our formalisms. To connect the abstract ideas of events and domains of information, we show how causal nets induce certain kinds of domains where the information points are certain sets of events. This allows translations between the languages of net theory and domain theory. Following the idea that events of causal nets are occurrences, we generalise causal nets to occurrence nets, by adding forwards conflict. Just as infinite flow charts unfold finite ones, so transition nets can be unfolded into occurrence nets. Next we extend the above connections between nets and domains to these new nets. Event structures which are intermediate between nets and domains play an important part in all our work. Finally, as an example of how concepts translate from one formalism to the other, we show how Petri's notion of confusion ties up with Kahn and Plotkin's concrete domains.

In part II we shall continue the job of connecting up notions within net theory and the theory of domains. In particular, we shall examine the idea of states of computations.

1. Introduction

The motivation of the present work is the search for a theory of concurrency which incorporates, on the one hand, the insights of Petri and his school [9, 10] on events, causality, etc. and on the other hand the insights of Scott [12, 14] and Stoy [15] on how to give denotational semantics using domains, which are partial orders of information. The work is rather abstract in that it attempts to connect up the ideas of events and (partial orders of) information. We hope that attending to the main intuitive ideas first will lead naturally to more practical applications later.

This paper consists of two parts. Part I which we present here has three sections. In Section 2, we consider *causal nets* [9] and show how they give domains whose (information) points are sets of events of the nets, that have occurred by some stage of the process (or computation) described by the net. This correspondence allows comparison of lattice-theoretic ideas and ideas expressed in the language of net theory; this is extended further in Section 3 where we introduce *occurrence nets* which are like causal nets but with forwards conflict added on. In both cases intermediate structures – called *event structures* – prove to be of use; they are like nets but with the conditions removed. We hope they are of some independent interest.

In both causal and occurrence nets the events are thought of as occurrences and the nets are acyclic and often infinite and describe, somehow, a process or computation. This contrasts with the systems approach of considering *transition nets* [10] as the main subject of study; indeed occurrence nets can be obtained (see Proposition 4) by unfolding transition nets forwards from an initial marking. It is in the same spirit as considering infinite acyclic flow diagrams, [13], or infinite terms [2].

As an end to part I we give an example of the translation of concepts from one formalism to the other: Petri's notion of confusion (in transition nets) is tied up with Kahn and Plotkin's concrete domains.

In part II we shall continue the job of connecting up notions within net theory and the theory of domains. In particular, we shall examine the idea of states of computations.

There are clearly many gaps in the present treatment. For one thing we would like a better understanding of what we mean when we say that a net describes a computation or process. Also the categorically minded will note that we have not discussed morphisms; this is particularly important for domains where the continuous functions play a major role in the denotations of programs. Finally we note a curious mismatch. We call our nets (descriptions of) processes or computations and each such net gives rise to a *whole* domain; on the other hand, in so far as processes are considered in the lattice-theoretic approach (as in [8]) they are only *elements* of domains. Resolution of this problem will no doubt involve separating and relating the different uses of the word 'process'.

2. Causal nets

We start off with an explanation of our computational interpretation of causal nets – the process level of net theory. To define these nets, we follow the axiomatic approach of Petri [9] and Best [1]:

Definition 1. A *Petri net* is a triple $N=(B, E, F)$, where
- B is a set of *conditions*,
- E is a set of *events*,
- $F \subseteq (B \times E) \cup (E \times B)$ is the *causal dependency relation*,

satisfying

A1 $\quad B \cap E = \emptyset$.

A2 $\quad F \neq \emptyset$.

A3 $\quad B \cup E = \mathrm{Field}(F)\ (=_{\mathrm{def}}\{x \in B \cup E \mid \exists y\colon (xFy) \vee (yFx)\})$.

For any $x \in B \cup E$, ${}^{\cdot}x$ ($x^{\cdot}$) denotes $\{y \mid yFx\}$ ($\{y \mid xFy\}$).

We call N a *causal net* iff further

A4 $\quad \forall b \in B: |b^{\cdot}| \leq 1$,

A5 $\quad \forall b \in B: |{}^{\cdot}b| \leq 1$,

A6 $\quad F^{+}$ is irreflexive,

A7 $\quad \forall b_1, b_2 \in B: ({}^{\cdot}b_1 = {}^{\cdot}b_2) \wedge (b_1^{\cdot} = b_2^{\cdot}) \Rightarrow b_1 = b_2$.

There is a well-known standard graphical representation of Petri nets, which we shall use throughout this paper. Conditions are represented by circles: ○, and events by boxes: □. The relation F is represented by oriented arcs between circles and boxes, so that there is an arc from x to y iff xFy.

Example 1. The graph in Fig. 1 represents the (causal) net $N = (B, E, F)$, where

$$B = \{b_1, b_2, b_3, b_4\},$$

$$E = \{e_1, e_2, e_3, e_4\},$$

$$F = \{(e_1, b_1), (e_1, b_2), (b_1, e_2), (b_2, e_3), (e_2, b_3), (e_3, b_4), (b_3, e_4), (b_4, e_4)\}.$$

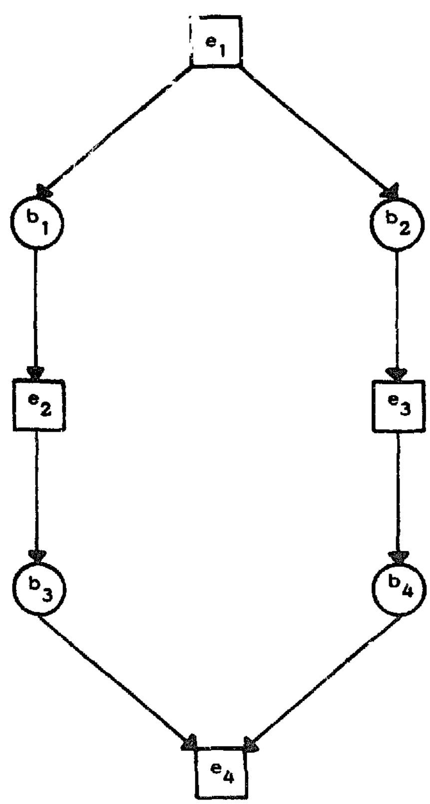

Fig. 1.

In [9] Petri gives a deeply considered discussion of causal nets and how they provide the foundation for general net theory. The notion of concurrency plays an important role in this analysis, and, as noted by Petri himself, is easily defined in the context of causal nets.

Definition 2. For a causal net $N=(B,E,F)$ the *concurrency relation* $\mathrm{co}_N \subseteq (B\cup E)\times(B\cup E)$ is defined by

$$\mathrm{co}_N = ((B\cup E)\times(B\cup E))\backslash(F^+\cup(F^+)^{-1}).$$

It follows that co_N is symmetrical and reflexive (from $A6$). It also follows that in causal nets any two elements of $B\cup E$ are either causally dependent or concurrent. We shall not go into Petri's careful arguments for the axioms for causal nets (based on the ideas behind general net theory), but only briefly outline our intuition behind causal nets as representing computations.

The events of causal nets represent *occurrences* of certain 'atomic events', and a state of a computation is represented by *holdings* of certain conditions. An occurrence of an event e is associated with a state in which all its preconditions (${}^{\cdot}e$) hold, and the effect of its occurrence is that all its preconditions cease to hold, and all its postconditions ($e^{\cdot}$) begin to hold. Furthermore, each event e is 'caused by' a unique subprocess ($\{x\in B\cup E\mid xF^+e\}$), and 'causes' a unique subprocess ($\{x\in B\cup E\mid eF^+x\}$). (Causality is probably not the right English word in this context – necessity may be better.) This is not necessarily true in higher level nets, in which events (and conditions) may be repeatable, and (forwards or backwards) conflict may be present.

As an illustration of the ideas behind these different levels, suppose we have a program which runs on various possible input data d, and which contains a definition of a procedure, P. Then we have the following possible events at process or system levels:

- *Process level*: The third call of P in the run with input data d.
- *System level*: Any call of P in the run with input data d.
  Any call of a procedure in the run with input data d.
  Any call of a procedure in any run of the program.

We now focus on the pattern of occurrences of events of causal nets. The relation F specifies a certain dependency, in the sense that if eF^+e', for $e,e'\in E$, then in the process described by the net, e' cannot occur without e having already occurred. This leads to the following definition of a 'causality' structure on events:

Definition 3. An *elementary event structure* is just a partial order $S=(E,\leqslant)$, where
- E is a set of *events*, and
- $\leqslant$ the partial order over E is called the *causality relation*.

The relationship between causal nets and elementary event structures is obvious. It is made precise by the next two theorems.

Theorem 1. *Let $N = (B, E, F)$ be a causal net. Then $\xi[N] =_{\text{def}} (E, F^* \restriction E^2)$ is an elementary event structure.*

Proof. Only asymmetry is non-trivial, and that follows from $A6$.

Theorem 2. *Let $S = (E, \leqslant)$ be an elementary event structure (with $E \neq \emptyset$). Then there is a causal net $\eta[S]$ such that $S = \xi[\eta[S]]$.*

Proof. We construct $\eta[S]$ as $N = (B, E, F)$, where

$$B = \{\langle e, e' \rangle \mid e, e' \in E, e \neq e', e \leqslant e'\} \cup \{\langle 0, e \rangle, \langle e, 1 \rangle \mid e \in E\},$$

$$F = \{(\langle e, e' \rangle, e'), (e, \langle e, e' \rangle) \mid e, e' \in E, \langle e, e' \rangle \in B\}$$

$$\cup \{(\langle 0, e \rangle, e), (e, \langle e, 1 \rangle) \mid e \in E\}.$$

Axioms A1, A3, A4, A5 and A7 are trivial; A2 follows from the assumption $E \neq \emptyset$, and A6 follows from the fact that $\leqslant$ is a partial order. It is also easy to see that $S = \xi[\eta[S]]$.

What these two trivial theorems say is that nets have "as much" structure as elementary event structures (ignoring the empty event structure); nothing is lost in the passage $S \to \eta[S]$. However, this does not work in the opposite direction, as in general N and $\eta[\xi[N]]$ are not isomorphic. Take the set N from Fig. 1. The elementary event structure $\xi[N]$ and the causal net $\eta[\xi[N]]$ are pictured in Fig. 2.

It should be clear from this example, that Theorem 2 holds for other definitions of η – the particular one we have chosen is somehow maximal – a point we shall return to later.

This raises the natural question, whether or not it is reasonable to identify causal nets using the equivalence relation:

$$N_1 \equiv N_2 \text{ iff } \xi[N_1] = \xi[N_2].$$

From our point of view, it seems that $\equiv$ is an acceptable equivalence relation, although from a net theory point of view, it might have undesirable properties.

However, we press on for the moment with the connection between elementary event structures and Scott [12, 14] domains of information. Given an elementary event structure, $S = (E, \leqslant)$, we want some idea of information about a certain set, x, of events having occurred (in the process $\eta[S]$). This information can be represented by the set itself, and the intuition behind the causality relation tells us that x must be left-closed, where

Definition 4. Let $S = (E, \leqslant)$ be an elementary event structure, and suppose $x \subseteq E$. Then x is *left-closed* iff

$$\forall e \in x\ \forall e' \in E: e' \leqslant e \Rightarrow e' \in x.$$

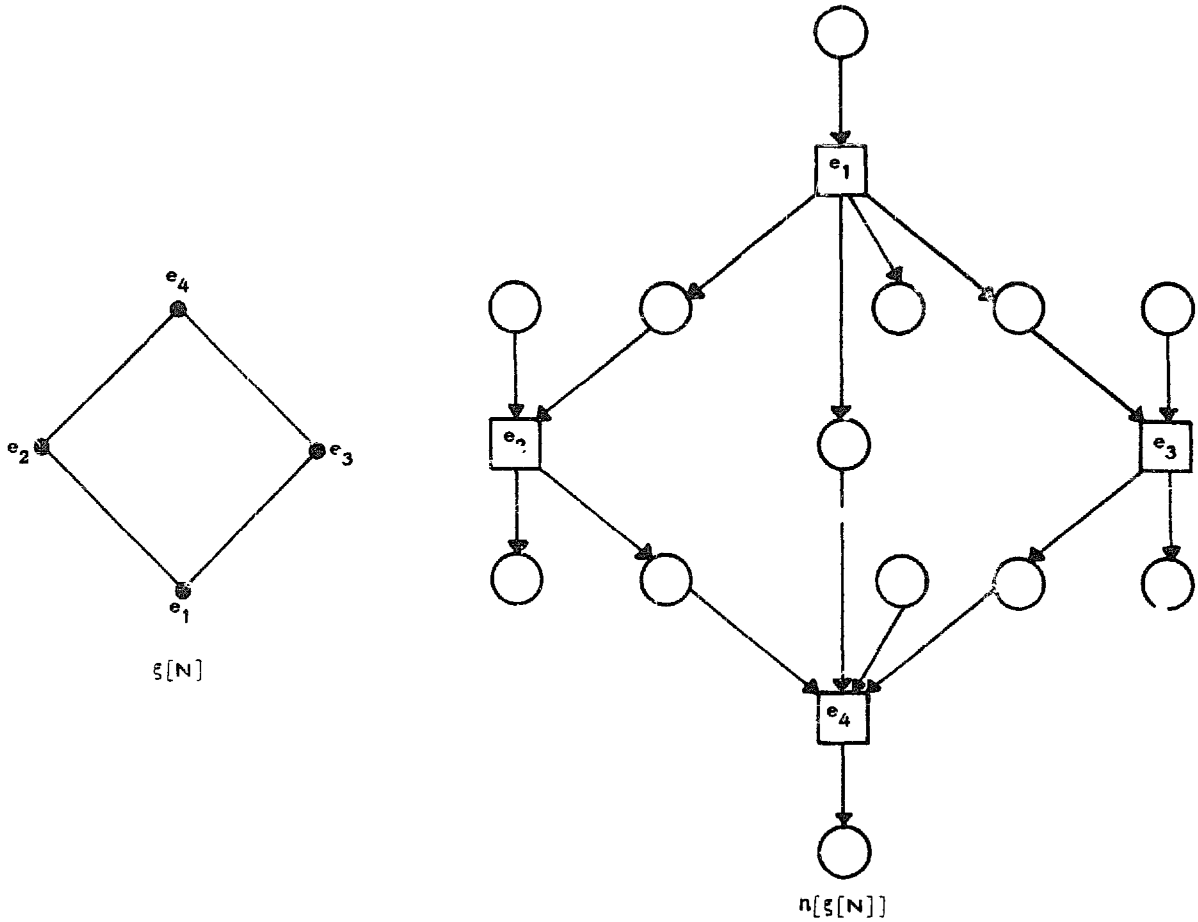

Fig. 2.

So, as information points we choose the left-closed subsets of E. What about the ordering? From the above it follows that x' contains more information than x precisely when x is a subset of x'.

Definition 5. Let $S=(E, \leqslant)$ be an elementary event structure. Then $\mathscr{L}[S]$ is the partial order of left-closed subsets of E ordered by inclusion.

It is quite easy to characterise the structures $\mathscr{L}[S]$. The only new concept we need is that of prime algebraicity.

Definition 6. Let $P=(D, \sqsubseteq)$ be a partial order. An element $p \in D$ is a *complete prime* (*prime*) iff for every $X \subseteq D$ (every finite $X \subseteq D$), if $\bigsqcup X$ exists and $p \sqsubseteq \bigsqcup X$, then there exists an $x \in X$ such that $p \sqsubseteq x$. The set of complete primes of P is denoted $\mathscr{C}_P$.

Definition 7. A partial order $P=(D, \sqsubseteq)$ is said to be *prime algebraic* iff for every element $d \in D$, $\bigsqcup P_d$ exists (where $P_d =_{\text{def}} \{p \sqsubseteq d \mid p$ is a complete prime$\}$), and $d=\bigsqcup P_d$.

In the graphical representation of partial orders in Fig. 3 the (complete) primes are circled, and it is easy to see that none of these partial orders are prime algebraic.

The next proposition relates the concept of prime algebraicity to more standard lattice-theoretic concepts.

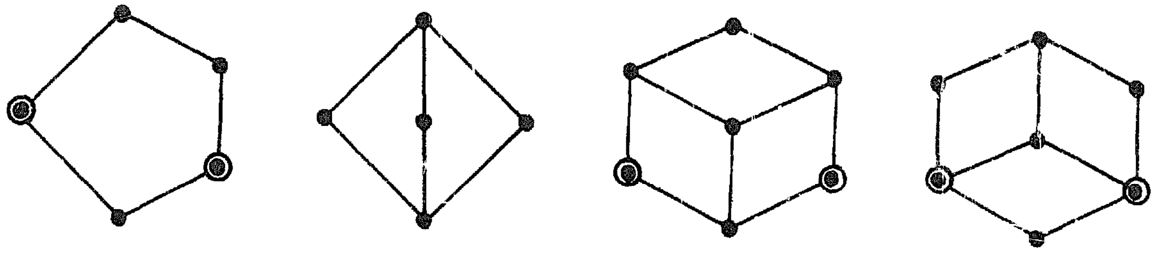

Fig. 3.

Proposition 1. *A complete lattice is prime algebraic iff it is algebraic and every finite element is a lub of complete primes. Further, such a lattice is completely distributive, every complete prime is finite, and an element is a complete prime iff it is completely irreducible.*

We now present results leading to the characterisation of the structures $\mathscr{L}[S]$.

Theorem 3. *Let $S = (E, \leqslant)$ be an elementary event structure. Then $\mathscr{L}[S]$ is a prime algebraic complete lattice. Its complete primes are those elements of the form $[e] =_{\text{def}} \{e' \in E \mid e' \leqslant e\}$ $(e \in E)$.*

Proof. The structure $\mathscr{L}[S]$ is a complete lattice with $\bigsqcup X = \bigcup X$ (and $\bigsqcap X = \bigcap X$).

Each $[e]$ is clearly left-closed, and is a complete prime as if $[e] \sqsubseteq \bigsqcup X = \bigcup X$, then $e \in [e] \subseteq \bigcup X$ and so for some x in X, $e \in x$, and so $[e] \sqsubseteq x$. As we have $x = \bigcup \{[e] \mid e \in x\}$ for any x in $\mathscr{L}[S]$, each element is a lub of the complete primes below it, and so $\mathscr{L}[S]$ is prime algebraic.

Finally, if x is a complete prime, then as we have $x = \bigcup \{[e] \mid e \in x\}$ we must have $x \subseteq [e]$ for some e in x. But then we must have $x = [e]$, which completes the proof.

This theorem indicates how to map our lattices to elementary event structures.

Definition 8. Let $P = (D, \sqsubseteq)$ be a prime algebraic complete lattice. The elementary event structure $\mathscr{P}[P]$ is defined as $(\mathscr{C}_P, \sqsubseteq \upharpoonright \mathscr{C}_P^2)$.

Before stating the characterisation of the structures $\mathscr{L}[S]$ we shall need the following general lemma.

Lemma 1. *Let $P = (D, \sqsubseteq)$ be a prime algebraic partial order. Then the map $\pi : P \to \mathscr{L}[(\mathscr{C}_P, \sqsubseteq \upharpoonright \mathscr{C}_P^2)]$ defined by*

$$\pi(d) =_{\text{def}} \{p \in \mathscr{C}_P \mid p \sqsubseteq d\}$$

is an order monic (i.e. $\pi(d) \sqsubseteq \pi(d')$ iff $d \sqsubseteq d'$), it preserves and reflects complete primes, and preserves those lubs that exist in P.

Proof. Clearly π is monotonic. If, on the other hand, $\pi(d) \subseteq \pi(d')$, then from prime algebraicity of P

$$d = \bigsqcup\{p \in \mathscr{C}_P \mid p \sqsubseteq d\} = \bigsqcup \pi(d) \subseteq \bigsqcup \pi(d') = d'.$$

Let p be a complete prime of P, then $\pi(p)$ is a complete prime in $\mathscr{L}[(\mathscr{C}_P, \sqsubseteq \upharpoonright \mathscr{C}_P^2)]$ from Theorem 3. On the other hand, it also follows from the theorem that if $\pi(d)$ is a complete prime, then d is a complete prime, too. So, π preserves and reflects complete primes. Finally, if $\bigsqcup_P X$ exists, then

$$\begin{aligned}\pi\left(\bigsqcup_P X\right) &= \left\{p \in \mathscr{C}_P \mid p \sqsubseteq \bigsqcup_P X\right\} \\ &= \bigcup_{x \in X} \{p \in \mathscr{C}_P \mid p \sqsubseteq x\} \quad \text{(by the definition of complete primeness)} \\ &= \bigcup_{x \in X} \pi(x).\end{aligned}$$

We shall often make use of the well-known fact that any mapping between partial orders which is onto and an order monic is an isomorphism. This happens in the proof of the next theorem, which states the very close relationship which exists between our lattices and event structures.

Theorem 4. *Let $S = (E, \leqslant)$ be an elementary event structure; then $S \cong \mathscr{P}[\mathscr{L}[S]]$. Similarly, let $P = (D, \sqsubseteq)$ be a prime algebraic complete lattice; then $P \cong \mathscr{L}[\mathscr{P}[P]]$.*

Proof. Define $\Psi : S \to \mathscr{P}[\mathscr{L}[S]]$ by $\Psi(e) = [e]$. Then Ψ is well-defined and onto from Theorem 3. Furthermore, Ψ is easily proved to be an order monic, and hence it is an isomorphism, which proves the first part of the theorem. As for the second part – π is known from Lemma 1 to be an order monic; π is also onto, since for any element X of $\mathscr{L}[\mathscr{P}[P]]$, $\bigsqcup_P X$ exists (P is a complete lattice) and

$$\begin{aligned}\pi\left(\bigsqcup_P X\right) &= \bigcup_{x \in X} \pi(x) \quad \text{(by Lemma 1)} \\ &= \bigcup\{[x] \mid x \in X\} \quad \text{(by the definition of } \pi) \\ &= X.\end{aligned}$$

So, π is indeed an isomorphism.

Take S to be the elementary event structure associated with the causal net from Fig. 1; S and $\mathscr{L}[S]$ are pictured in Fig. 4. The primes of $\mathscr{L}[S]$ are circled, and it is easy to see that $S \cong \mathscr{P}[\mathscr{L}[S]]$.

Theorem 4 shows that elementary event structures and prime algebraic complete lattices are equivalent structures, in the sense that one does not lose any structural

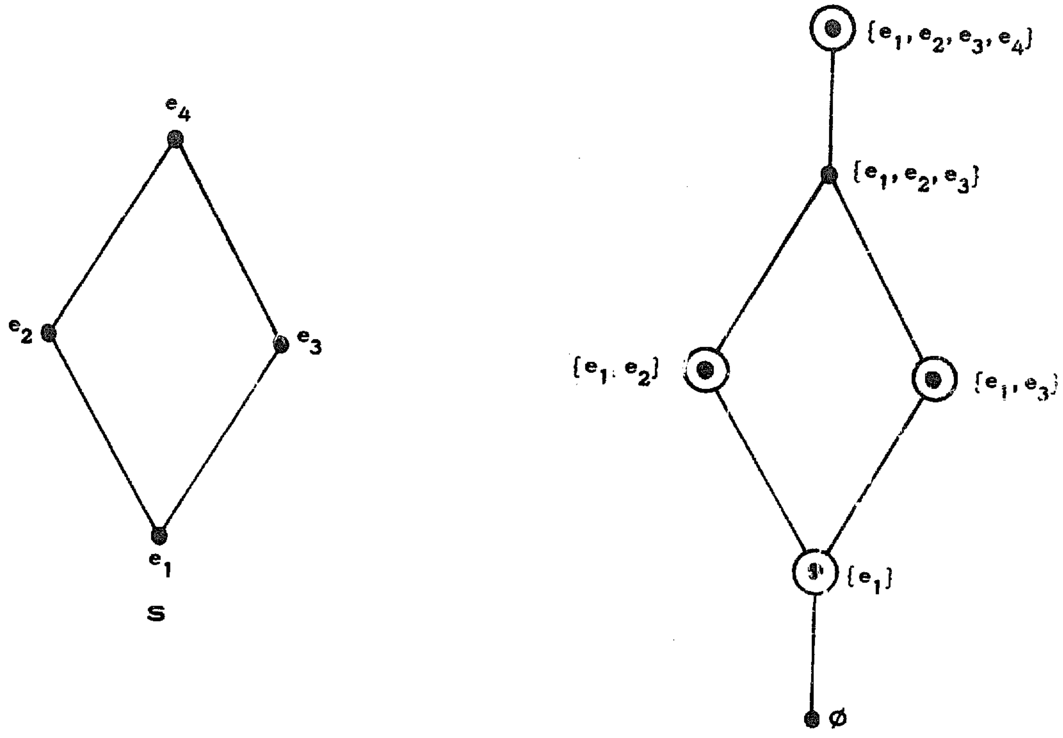

Fig. 4. $\mathscr{L}[S]$

information going from one to the other via the $\mathscr{L}$ and $\mathscr{P}$ mappings – in contrast to the earlier result about the relationship between causal nets and elementary event structures. A special case of Theorem 4 where S is finite is given in [3].

The framework we have set up so far, can be pictured as follows:

We are concerned with the translation of concepts and ideas from one side of this diagram to the other. From right to left we get an explanation in the framework of net theory of the Scott idea of information. From left to right we see how net concepts (like events and causality) translate into the idea of partial order of information. In the rest of this section we shall elaborate a little on this latter translation, but before that a few general remarks.

First, elementary event structures have been introduced here only as an intermediary technical device, but we do believe that they (and their generalisations introduced in the next section) are interesting in their own right, and as we shall see, they should be a more appropriate framework for a number of questions than their equivalent but more detailed lattice structures.

Second, we have deliberately established the links between net theory and the theory of partial orders at the process level. We strongly believe that an understanding of this low level is necessary for an understanding of similar links between higher level concepts within the two theories.

And now, let us see how some of the basic concepts of net theory translate using the mappings ξ and $\mathscr{L}$. Not surprisingly, since we specifically focussed our attention on event occurrences of nets, the concept of an event translates very well. From our construction of $\mathscr{P}[P]$, where P is a prime algebraic complete lattice, it seems that the notion of events translates into the complete primes of P. To see the intuition behind these primes, we shall need the notion of prime intervals of partial orders.

Definition 9. Let $(D, \sqsubseteq)$ be a partial order. Define the *interval* from d to d', $[d, d']$, for $d, d' \in D$ as

$$[d, d'] =_{\text{def}} \{d'' \in D \mid d \sqsubseteq d'' \sqsubseteq d'\}.$$

An interval $[d, d']$ is said to be *prime* iff $d \neq d'$ and $[d, d'] = \{d, d'\}$, in which case d' is said to *cover* d, which we write as $d \longrightarrow\!\!\!\!< d'$.

With our computational interpretation in mind, prime intervals correspond to steps of computations or more specifically, occurrences at particular states of the computations. To see how this works formally, define the relation $\leqslant$ between prime intervals by

$$[d_1, d_1'] \leqslant [d_2, d_2'] \text{ iff } d_2' = d_1' \sqcup d_2 \text{ and } d_1 = d_1' \sqcap d_2.$$

Next we define the equivalence relation $\approx$ between prime intervals as the equivalence generated by $\leqslant$. This relation represents the intuition behind 'occurrences of the same event'. How does this intuition tie up with the notion of complete primes? As a first step the following (easy to prove) proposition gives the relation between prime intervals and complete primes.

Proposition 2. *Let $P = (D, \sqsubseteq)$ be a prime algebraic complete lattice. Then for any prime interval $[d, d']$, $\pi(d') \backslash \pi(d)$ is a singleton. Hence if we put*

$$\text{pr}([d, d']) \in \pi(d') \backslash \pi(d),$$

then pr *is a well-defined mapping from the prime intervals of P to $\mathscr{C}_P$.*

And the following theorem states the relation between the equivalence and the mapping pr.

Theorem 5. *Let $P = (D, \sqsubseteq)$ be a prime algebraic complete lattice; then the following conditions are equivalent for prime intervals $[d_1, d_1']$ and $[d_2, d_2']$:*

(1) $[d_1, d_1'] \approx [d_2, d_2']$,

(2) $\text{pr}([d_1, d_1']) = \text{pr}([d_2, d_2'])$,

(3) *There exists a prime interval $[d_3, d_3']$ such that*

$$[d_1, d_1'] \geqslant [d_3, d_3'] \leqslant [d_2, d_2'].$$

Further, if p is a complete prime of P, then

$$p = \mathrm{pr}\left(\left[\bigsqcup\{p' \in \mathscr{C}_P \mid p \neq p' \wedge p' \sqsubseteq p\}, p\right]\right).$$

Proof. (1) ⇒ (2). It follows easily from the definition of $\leqslant$ that $[d_1, d_1'] \leqslant [d_2, d_2']$ implies $\mathrm{pr}([d_1, d_1']) = \mathrm{pr}([d_2, d_2'])$.

(2) ⇒ (3). Define $d_3 = d_1 \sqcap d_2$ and $d_3' = d_1' \sqcap d_2'$.

(3) ⇒ (1). Trivial.

The last part of the theorem is obvious.

Theorem 5 proves a one-to-one correspondence between the complete primes and the more intuitive equivalence classes of prime intervals. This justifies our translation of events into complete primes.

Now, it is easy to see that the events of a causal net N are in one-to-one correspondence with the events of $\xi[N]$, and the events of an elementary event structure S are in one-to-one correspondence with those of $\eta[S]$. On the other hand, the events of S are also in one-to-one correspondence with those of $\mathscr{L}[S]$, and the events of a prime algebraic complete lattice are in one-to-one correspondence with those of $\mathscr{P}[P]$.

The situation for translation of conditions is a good deal less pleasant. Our main tool for handling conditions is the extensionality axiom A7, which allows us to identify any condition b with its pre- and postevent (${}^{\cdot}b$ and $b^{\cdot}$). For simplicity, we shall only demonstrate how conditions translate into elementary event structures.

A condition of an elementary event structure S is taken to be any condition of $\eta[S]$. By definition this gives a nice one-to-one relationship between conditions of S and $\eta[S]$, but, obviously, it is more interesting to see how conditions of a causal net N correspond to certain conditions of $\xi[N]$. Define the map, bed, between these two sets of conditions as follows:

$$\forall b \in B\colon \mathrm{bed}(b) = \begin{cases} (0, e'), & \text{if } {}^{\cdot}b = \emptyset \text{ and } b^{\cdot} = \{e'\}, \\ (e, 1), & \text{if } {}^{\cdot}b = \{e\} \text{ and } b^{\cdot} = \emptyset, \\ (e, e'), & \text{if } {}^{\cdot}b = \{e\} \text{ and } b^{\cdot} = \{e'\}. \end{cases}$$

It follows from the axioms of causal nets that bed is well defined, and that it is one-to-one. However, in general bed will *not* be onto, obviously because of our construction of $\eta[S]$, which in general generates a lot of redundant conditions. One could try to remedy this by a characterisation of the 'essential' conditions of S. The following lemma is such an attempt.

Lemma 2. *Let $S = (E, \leqslant)$ be an elementary event structure, and b one of its conditions. Then the following two conditions are equivalent*:

(1) *for every causal net $N = (B, E, F)$ for which $S = \xi[N]$, $b \in \mathrm{bed}(B)$,*

(2) *$b = (e, e')$, where e' covers e (with respect to the relation $\leqslant$).*

Proof. Assume b of the required form, then clearly for every causal net $N = (B, E, F)$ for which $S = \xi[N]$, there must exist a condition $b' \in B$ such that $eFb'Fe'$, and hence $b = \text{bed}(b')$. On the other hand, if b is not of this form, construct a slightly modified form, N, of $\eta[S]$ leaving out the condition corresponding to b, such that $S = \xi[N]$ and $b \notin \text{bed}(B)$.

This lemma shows that the only essential conditions are the 'points of non-density'. However, the net consisting of the events of S and all essential conditions will not in general be mapped onto S by ξ. Indeed, considering, for instance, the elementary event structure associated with the rationals shows that it is even possible for *no* condition to be essential.

We leave it, for the moment, to the reader to see how the causal dependency and the concurrency relation of causal nets translate nicely into our event and lattice structures. We shall look closer into this in the next section.

3. Occurrence nets

From a computational point of view, all the structures introduced in Section 2 lack the important notion of conflict, branching or non-determinism. This is not inherent in either of our theories, and in this section we shall see a nice partial correspondence between their different ways of treating conflict.

Within net theory higher level nets may have (forward or backward) conflicts. Essentially this means that the subprocess 'caused by' or 'causing' an event or a condition is no longer unique. Net theory includes a thorough treatment of conflicts, mainly at the system level of transition nets. The process level semantics of a transition net is the *class* of causal nets it unfolds into, where all the choices associated with such an unfolding are 'made by the environment' [11]. However, from a computational point of view, we would prefer to deal with conflicts at the semantical level, and to express the meaning of a system with conflicts in *one* semantical object. We deliberately want to stay as close to causal nets as possible, looking for a class of nets with conflicts on a slightly higher level than causal nets. Graphically, this means that we may want to allow nets with the structures pictured in Fig. 5.

The intuition behind these structures is as follows:
- *forwards conflicts*: from the holding of b_1 either e_1 or e'_1 (but not both!) may occur, in either case with the same effect as in causal nets;
- *backwards conflicts*: the holding of b_2 may have begun from an occurrence of e_2 or e'_2 (but not from both!).

We reject backwards conflict as we wish to axiomatise a notion of condition occurrence which determines the event occurrence that caused it. So we keep A5 which rules out backwards conflict, and look for a replacement for A4 as that rules out forwards conflict.

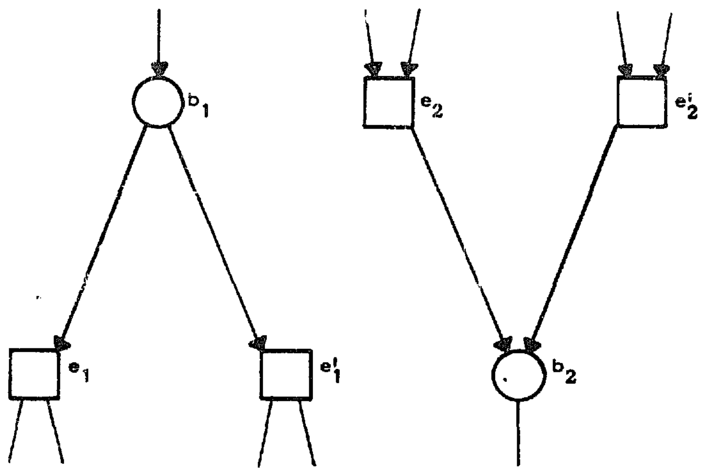

Fig. 5.

Definition 10. Let $N = (B, E, F)$ be a Petri net satisfying A5–A7. For any $a \in B \cup E$, let a^- denote the subset of E defined by

$$a^- =_{\text{def}} \{e \in E \mid eF^*a\}.$$

Two events e_1 and e_2 are said to be in *direct conflict*,

$$e_1 \#_{1N} e_2 \text{ iff } e_1 \neq e_2 \text{ and } {}^\bullet e_1 \cap {}^\bullet e_2 \neq \emptyset.$$

Two elements of $B \cup E$, a_1 and a_2, are said to be in *conflict*,

$$a_1 \#_N a_2 \text{ iff } \exists e_1, e_2 \in E: (e_1 \in a_1^-) \wedge (e_2 \in a_2^-) \wedge (e_1 \#_{1N} e_2).$$

Definition 11. A Petri net N is a (*forwards conflict*) *occurrence net* iff it satisfies A5–A7 and

A4′ $\quad \#_N$ is irreflexive.

Occurrence nets will be our new class of semantical nets. Elements of E and B still represent unique occurrences and holdings, respectively, and A4′ guarantees that no event (or condition) is in conflict with itself (can occur on two different branches of the computation, so to speak). More importantly, the concept of concurrency carries over nicely:

Definition 12. For an occurrence net $N = (B, E, F)$, the *concurrency relation* $\mathrm{co}_N \subseteq (B \cup E) \times (B \cup E)$ is defined by

$$\mathrm{co}_N =_{\text{def}} ((B \cup E) \times (B \cup E)) \setminus (F^+ \cup (F^+)^{-1} \cup \#_N).$$

The following proposition is an immediate consequence of our definitions.

Proposition 3. *Let $N = (B, E, F)$ be an occurrence net. Then co_N is symmetrical and reflexive. Furthermore, any two elements of $B \cup E$ are related in one of the three mutually exclusive ways: causally dependent, concurrent or in conflict.*

It should be noted that our occurrence nets have very little to do with the occurrence structures defined by Holt [4]. Any causal net is an occurrence net according to our definition.

Before introducing branching in our other theories, let us briefly illustrate how occurrence nets do describe the semantics of transition nets the way we wanted. We shall only give a brief introduction to transition nets – those readers not familiar with the theory are referred to [10].

A transition net N in the following is a finite Petri net with a dynamic behaviour. The conditions of N are called *places*, and the events *transitions*. The behaviour of N is defined in terms of its *markings* and the *firing rule*. A marking is a subset of places, usually represented by a token distribution (one token on every place in the marking). Markings of N may change dynamically via firings of transitions. A transition t may fire in a particular marking M iff $^{\cdot}t \subseteq M$ and $t^{\cdot} \cap M = \emptyset$. A firing of t will lead to a new marking $M' = (M \setminus {}^{\cdot}t) \cup t^{\cdot}$.

In the following we shall assume that N has associated with it an *initial marking* M_0 and the behaviour we are interested in is the "token game" you can play from M_0 as defined above. The set of markings you can get into playing this game from M_0 is called the set of *reachable markings*.

Furthermore, we shall assume that N is *contact-free* (1-safe), that is for any reachable marking M and transition t, $^{\cdot}t \subseteq M$ implies $t^{\cdot} \cap M = \emptyset$.

The idea behind our semantics of transition nets is that the behaviour of N will be described by an occurrence net with precisely one condition for each residence of a token on a place, and precisely one event for each firing possible for N. A particular *finite* (*sequential*) *behaviour* of N is given by a sequence

$$\sigma = M_0 t_0 M_1 t_1 \cdots t_n M_{n+1}, \tag{$*$}$$

where the M_i's are markings, M_0 is the initial marking, the t_i's transitions, and

$$\forall i \leqslant n: ({}^{\cdot}t_i \subseteq M_i) \wedge (M_{i+1} = (M_i \setminus {}^{\cdot}t_i) \cup t_i^{\cdot}).$$

Notice that we have assumed contact freeness. A particular firing of a transition, t_n, may now be identified with a certain equivalence class of sequences of this form. The equivalence will abstract away from the ordering of concurrent firings of transitions. Take a sequence of the form ($*$), and assume there exists an $i \leqslant n-1$ such that $^{\cdot}t_i \subseteq M_{i-1}$ (and hence $^{\cdot}t_i \cap {}^{\cdot}t_{i-1} = \emptyset$). Then '$\sigma$ represents the same firing as σ'' ($\sigma \equiv^{(1)} \sigma'$), where

$$\sigma' = M_0 t_0 \cdots M_{i-1} t_i M'_i t_{i-1} M_{i+1} \cdots t_n M_{n+1},$$

and M'_i is the unique marking guaranteeing that σ' is of the form ($*$). If $^{\cdot}t_n \subseteq M_{n-1}$ (and hence $^{\cdot}t_n \cap {}^{\cdot}t_{n-1} = \emptyset$), then also '$\sigma$ represents the same firing as σ''' ($\sigma \equiv^{(2)} \sigma''$), where

$$\sigma'' = M_0 t_0 \cdots M_{n-1} t_n M'_{n+1},$$

and M'_{n+1} is the unique marking guaranteeing that σ'' is of the form (*).

Now, let $\equiv$ denote the reflexive, symmetrical and transitive closure of $(\equiv^{(1)} \cup \equiv^{(2)})$, and let, for any σ of the form (*), $[\sigma]$ denote the equivalence class of σ with respect to $\equiv$. Basically, equivalence under $\equiv$ is the same kind of abstraction from orderings of concurrent firings as introduced in [7] for a different purpose.

It is easy to see that each element of an equivalence class has a unique final transition, and hence we may identify these equivalence classes with firings of the transition net. So, a firing is represented by 'the token game history that caused it', and the events of our semantical occurrence net, E, will be this set of equivalence classes. Residences of tokens on places are then represented by the set

$$B = \{\langle e, p\rangle \mid e \in E \text{ is a firing of transition } t, \text{ and the place } p \text{ belongs to } t^{\cdot}\}$$
$$\cup \{\langle 0, p\rangle \mid p \in M_0\}.$$

And finally, the F relation of our semantical net will be

- $(e, b) \in F$ iff there exists a place p of N such that $b = \langle e, p\rangle$;
- $(b, e) \in F$ iff either $b = \langle [M_0 t_0 \cdots t_{n-1} M_n], p\rangle$, $e = [M_0 t_0 \cdots t_{n-1} M_n t_n M_{n+1}]$ and $p \in {}^{\cdot}t_n$ or else $b = \langle 0, p\rangle$, $e = [M_0 t_0 M_1]$ and $p \in {}^{\cdot}t_0$.

Definition 13. Let N be a finite, contact-free transition net with initial marking M_0. Then $O[N, M_0]$ denotes the Petri net defined by the construction given above.

Proposition 4. *For any finite contact-free transition net N with initial marking M_0, $O[N, M_0]$ satisfies the axioms for occurrence nets. The map f, defined below, from $B \cup E$ to places and transitions of N is a folding* [9]:

$$f(\langle 0, p\rangle) = f(\langle e, p\rangle) = p, \qquad f([M_0 t_0 \cdots t_n M_{n+1}]) = t_n.$$

In Fig 6 a transition net N with initial marking is pictured with the occurrence net constructed from N.

Let us now see how branching is handled in our other theories. Since elementary event structures were our 'poorest' structures, it is not surprising that the only way of introducing branching is by adding structure.

Definition 14. An *event structure* is a triple $S = (E, \leqslant, \#)$, where

E1 $(E, \leqslant)$ is an elementary event structure,

E2 $\#$ is a symmetrical and irreflexive relation in E, called the conflict relation; it satisfies
$\forall e_1, e_2, e_3 \in E: e_1 \geqslant e_2 \# e_3 \Rightarrow e_1 \# e_3.$

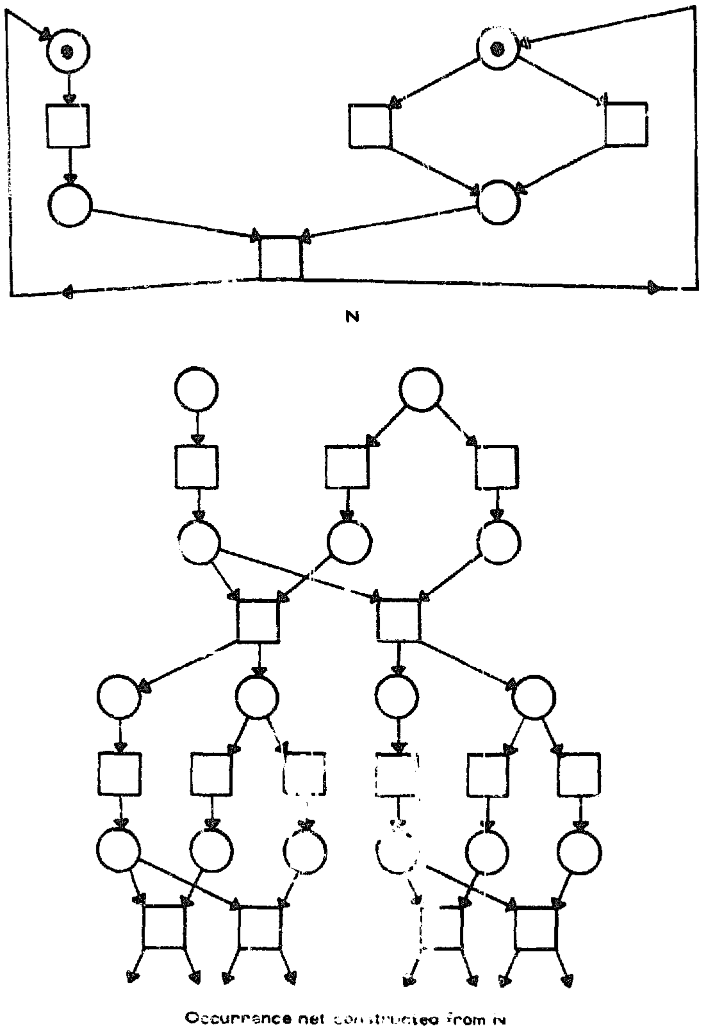

Fig. 6.

With these generalisations of causal nets and elementary event structures, the next two theorems provide straightforward generalisations of the mappings ξ and η and the results of Theorems 1 and 2.

Theorem 6. *Let $N = (B, E, F)$ be an occurrence net. Then*

$$\xi[N] =_{\mathrm{def}} (E, F^* \restriction E^2, \#_N \restriction E^2) \text{ is an event structure.}$$

Proof. The irreflexivity of $\#_N$ follows from A4′. Then E2 follows from the definition of $\#_N$.

Theorem 7. *Let $S = (E, \leqslant, \#)$ be an event structure (with $E \neq \emptyset$). Then there is an occurrence net $\eta[S]$ such that $S = \xi[\eta[S]]$.*

Proof. Define the set CE as follows:

$$\mathrm{CE} =_{\mathrm{def}} \{x \subseteq E \mid \forall e, e' \in x : e \neq e' \Rightarrow e \# e'\}.$$

The events of $\eta[S]$ are obviously those of E, and the set of conditions is defined by

$$B=\{\langle e, x\rangle \mid e\in E, x\in \mathrm{CE}, \text{and } \forall e'\in x: e\leq e'\}$$
$$\cup\{\langle 0, x\rangle \mid x\in \mathrm{CE}, x \text{ nonempty}\}.$$

Finally, the F relation is defined as

$$F=\{(\langle e, x\rangle, e') \mid \langle e, x\rangle\in B, e'\in x\}$$
$$\cup\{(\langle 0, x\rangle, e') \mid \langle 0, x\rangle\in B, e'\in x\}$$
$$\cup\{(e, \langle e, x\rangle) \mid \langle e, x\rangle\in B\}.$$

It follows that $\eta[S]$ is a well-defined occurrence net for which $\#_1 = \# = \#$, and hence $\xi[\eta[S]]=S$.

This construction of $\eta[S]$ may seem more unnecessarily complicated than the one from the proof of Theorem 2. Obviously, many simpler ones would do; however, we have again chosen a 'maximal' construction, in the sense that any condition in any occurrence net N for which $\xi[N]=S$ has a representative in $\eta[S]$ (which means that our treatment of conditions in elementary event structures discussed in Section 2 carries over to event structures).

Things get a bit more interesting when we move on to our lattice structures and generalisations of the mappings $\mathscr{L}$ and $\mathscr{P}$. Intuitively, an event structure represents a class of processes, where $e \# e'$ means that e and e' never occur in the same process. So, not all left-closed subsets of an event structure make sense as information points. Only the conflict free left-closed subsets can be the sets of occurrences at some stage of an associated process.

Definition 15. Let $S=(E, \leq, \#)$ be an event structure, and let x be a subset of E. Then x is *conflict free* iff

$$\forall e, e'\in x: \neg(e \# e').$$

Our idea about the ordering of information points is still the same, though.

Definition 16. Let $S=(E, \leq, \#)$ be an event structure. Then $\mathscr{L}[S]$ is the partial order of left-closed (w.r.t. $\leq$) and conflict free subsets of E, ordered by inclusion.

What about our charactrisation of the structures $\mathscr{L}[S]$? Obviously, we do not any longer get complete lattices. Two points will be inconsistent (have no upper bound) iff their union (as sets of events) contains conflict. But any consistent set of points will have a lub (their union), so the structures will be consistently complete. For a characterisation we need the even stronger condition of coherence (introduced in [6]).

Definition 17. Let $(D, \sqsubseteq)$ be a partial order. A subset x of D is *pairwise consistent* iff any two of its elements have an upper bound in D; $(D, \sqsubseteq)$ is said to be *coherent* iff every pairwise consistent subset of D has a lub. The consistency relation is denoted $\uparrow$; $\not\uparrow$ denotes inconsistency.

Theorem 8. *Let $S = (E, \leq, \#)$ be an event structure. Then $\mathscr{L}[S]$ is a prime algebraic coherent partial order. Its complete primes are those elements of the form $[e] =_{\text{def}} \{e' \in E \mid e' \leq e\}$.*

Proof. Let $X \subseteq \mathscr{L}[S]$ be pairwise consistent. Then $\bigcup X$ is conflict free, and so $\bigsqcup X = \bigcup X$, showing that $\mathscr{L}[S]$ is coherent.

The rest of the proof proceeds as in the proof of Theorem 3, noting that all elements of the form $[e]$ are conflict free from E2, and that for any x in $\mathscr{L}[S]$ the set $\{[e] \mid e \in x\}$ is pairwise consistent.

From this theorem we see how to generalise the mapping $\mathscr{P}$.

Definition 18. Let $P = (D, \sqsubseteq)$ be a prime algebraic coherent partial order. Then $\mathscr{P}[P]$ is defined as the event structure $(\mathscr{C}_P, \leq, \#)$, where $\leq$ is $\sqsubseteq$ restricted to $\mathscr{C}_P$, and for all $e, e' \in \mathscr{C}_P$: $e \# e'$ iff e and e' are inconsistent in P.

It is easy to see that $\mathscr{P}[P]$ is indeed an event structure, and we are now ready to prove the equivalence between event structures and prime algebraic coherent partial orders corresponding to Theorem 4. An isomorphism between two event structures is naturally any one to one and onto mapping, which respects and reflects both causality and conflict.

Theorem 9. *Let $S = (E, \leq, \#)$ be an event structure, then $S \cong \mathscr{P}[\mathscr{L}[S]]$. Similarly let $P = (D, \sqsubseteq)$ be any prime algebraic coherent partial order, then $P \cong \mathscr{L}[\mathscr{P}[P]]$.*

Proof. Define $\psi : S \to \mathscr{P}[\mathscr{L}[S]]$ by $\psi(e) = [e]$. It follows along the lines of the proof of Theorem 4 that ψ is an isomorphism with respect to $\leq$ and the corresponding relation in $\mathscr{P}[\mathscr{L}[S]]$. Furthermore, ψ is easily seen to respect and reflect the conflict relation.

The mapping π as defined in Lemma 1 is known to be an order monic from P to $\mathscr{L}[(\mathscr{C}_P, \sqsubseteq \restriction \mathscr{C}_P^2, \emptyset)]$ (from Lemma 1). By definition $\mathscr{L}[\mathscr{P}[P]]$ is a subordering of $\mathscr{L}[(\mathscr{C}_P, \sqsubseteq \restriction \mathscr{C}_P^2, \emptyset)]$, so all we have to prove is that the range of π is equal to the set of elements of $\mathscr{L}[\mathscr{P}[P]]$, i.e. for every left-closed set, X, of complete primes of P:

$$\exists d \in D: \pi(d) = X \text{ iff } \forall p, p' \in X: p \text{ and } p' \text{ are consistent.}$$

The 'only if' part is trivial. Assume X satisfies the right-hand side assumption. Coherence of P implies the existence of $\bigsqcup_P X$, and it follows that $\pi(\bigsqcup_P X) = X$ (just like in the proof of Theorem 4).

In Fig. 7 an occurrence net N is pictured with its associated event structure $\xi[N]$ and the coherent prime algebraic partial order $\mathscr{L}[\xi[N]]$.

So, we have now established a complete generalisation of the picture from the previous section:

$$\begin{tikzcd} \boxed{\text{Occurrence nets}} \arrow[r, shift left, "\xi"] & \boxed{\text{Event structures}} \arrow[l, shift left, "\eta"] \arrow[r, shift left, "\mathscr{L}"] & \boxed{\text{Prime algebraic coherent posets}} \arrow[l, shift left, "\mathscr{P}"] \end{tikzcd}$$

Our considerations about translation of events and conditions work just like in Section 2. Formally, Proposition 2 and Theorem 5 hold for prime algebraic coherent partial orders.

Restricting ourselves to these relations on events, the correspondences, as shown in Table 1, now be obvious to the reader.

Finally, let us see what these relations look like in terms of prime intervals of partial orders.

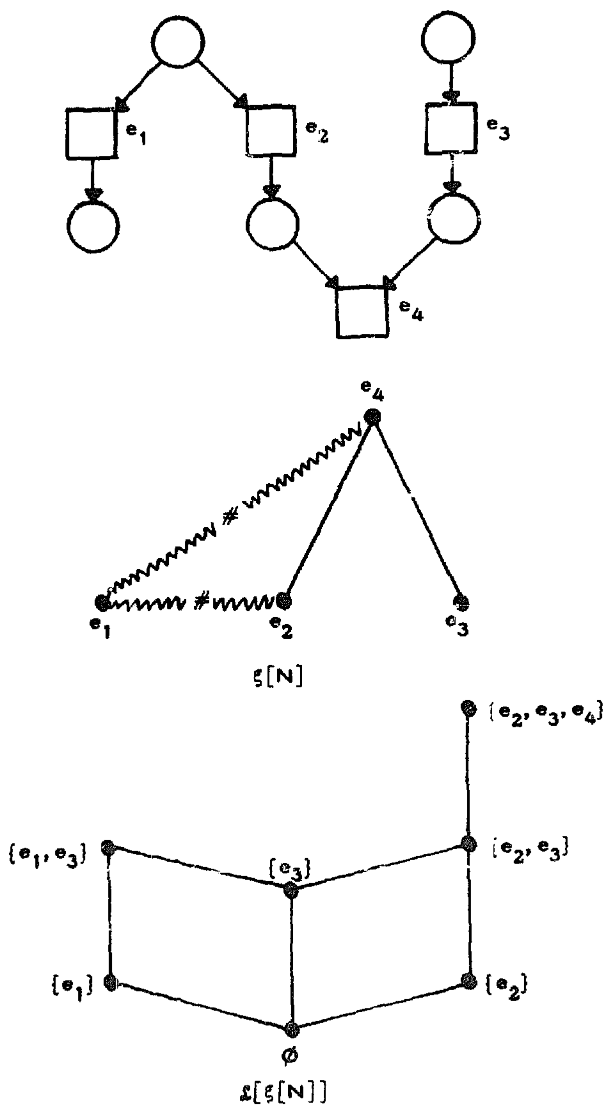

Fig. 7.

Table 1

	Occurrence nets $N=(B,E,F)$	Event structures $S=(E,\leqslant,\#)$	Prime algebraic Coherent posets $P=(D,\sqsubseteq)$
Causality	$F^{+}\restriction E^{2}$	$<$	$\sqsubsetneq\restriction\mathscr{C}_P^2$
Conflict	$\#_N\restriction E^2$	$\#$	$\not\uparrow\restriction\mathscr{C}_P^2$
Concurrency	$E^2\backslash(F^{+}\cup(F^{+})^{-1}\cup\#_N)$	$E^2\backslash((\cup)\cup\#)$	$\mathscr{C}_P^2\backslash(\sqsubsetneq\cup\sqsupsetneq\cup\not\uparrow)$

Definition 19. Let $P=(D,\sqsubseteq)$ be a prime algebraic coherent partial order. The relation $\rightarrowtail$ ('may occur before') on $\mathscr{C}_P$ is defined as follows: $p_1\rightarrowtail p_2$ iff there exist prime intervals of P, $[x_1,x_1']$, $[x_2,x_2']$, such that $\mathrm{pr}([x_1,x_1'])=p_1$, $\mathrm{pr}([x_2,x_2'])=p_2$ and $x_1\sqsubseteq x_2$. The complement of $\rightarrowtail$ is denoted $\not\rightarrowtail$.

Proposition 5. *Let $P=(D,\sqsubseteq)$ be a prime algebraic coherent partial order, and let p_1, $p_2\in\mathscr{C}_P$, $p_1\neq p_2$. Then*

$$p_1\sqsubsetneq p_2 \text{ iff } (p_1\rightarrowtail p_2)\wedge(p_2\not\rightarrowtail p_1),$$

$$p_1\not\uparrow p_2 \text{ iff } (p_1\not\rightarrowtail p_2)\wedge(p_2\not\rightarrowtail p_1).$$

and hence p_1 and p_2 are concurrent iff $(p_1\rightarrowtail p_2)\wedge(p_2\rightarrowtail p_1)$.

4. Confusion

So far we have provided techniques to show how, for instance, the notion of an event translates into domains. In part II we shall develop these further dealing with more sophisticated concepts such as states of a computation and confusion in detail. However, now we have enough machinery to connect confusion freeness in finite contact-free transition nets with concrete domains. Indeed an equivalent of confusion freeness was rediscovered in the work of Kahn and Plotkin on concrete domains [5] without their knowing it at the time. We set down some basic facts in

Proposition 6. *Let N be a finite contact-free transition net with initial marking, M_0. Each finite behaviour determines a finite element of $\mathscr{L}\circ\xi\circ O[N,M_0]$ under*

$$M_0t_0\cdots t_nM_{n+1}\overset{\mathrm{ev}}{\mapsto}\{[M_0t_0M_1],\ldots,[M_0t_0\cdots t_nM_{n+1}]\}.$$

Moreover there is a 1–1 *correspondence between finite sequential behaviours and finite chains of coverings from $\emptyset$ in $\mathscr{L}\circ\xi\circ O[N,M_0]$ namely $M_0t_0\cdots t_nM_{n+1}$ corresponding to*

$$\emptyset\prec\mathrm{ev}(M_0t_0M_1)\prec\cdots\prec\mathrm{ev}(M_0t_0\cdots t_nM_{n+1}).$$

(*Thus* ev *is onto finite elements and* $|[e]|<\infty$ *for all occurrences* e.)

We now give the definition of confusion for finite contact-free transition nets [10].

Definition 20. Let N be a finite contact-free transition net with initial marking, M_0. Then (N, M_0) is *symmetrically confused* iff there are a reachable marking M and transitions t, t', t'' such that $({}^{\cdot}t, {}^{\cdot}t', {}^{\cdot}t'' \subseteq M) \wedge ({}^{\cdot}t \cap {}^{\cdot}t' \neq \emptyset) \wedge ({}^{\cdot}t' \cap {}^{\cdot}t'' \neq \emptyset) \wedge ({}^{\cdot}t \cap {}^{\cdot}t'' = \emptyset)$.

Further, (N, M_0) is *asymmetrically confused* iff there are a reachable marking M and transitions t, t', t'' such that $({}^{\cdot}t, {}^{\cdot}t'' \subseteq M) \wedge ({}^{\cdot}t' \not\subseteq M) \wedge ({}^{\cdot}t' \subseteq (M \setminus {}^{\cdot}t) \cup t^{\cdot}) \wedge ({}^{\cdot}t \cap {}^{\cdot}t'' = \emptyset) \wedge ({}^{\cdot}t' \cap {}^{\cdot}t'' \neq \emptyset)$.

Finally, (N, M_0) is *confused* iff N is symmetrically or asymmetrically confused; otherwise (N, M_0) is said to be *confusion-free.*

In N_1 of Fig. 8 the conflict between t and t' may be resolved by the occurrence of t''. In N_2, t' and t'' may be brought into conflict by the occurrence of t.

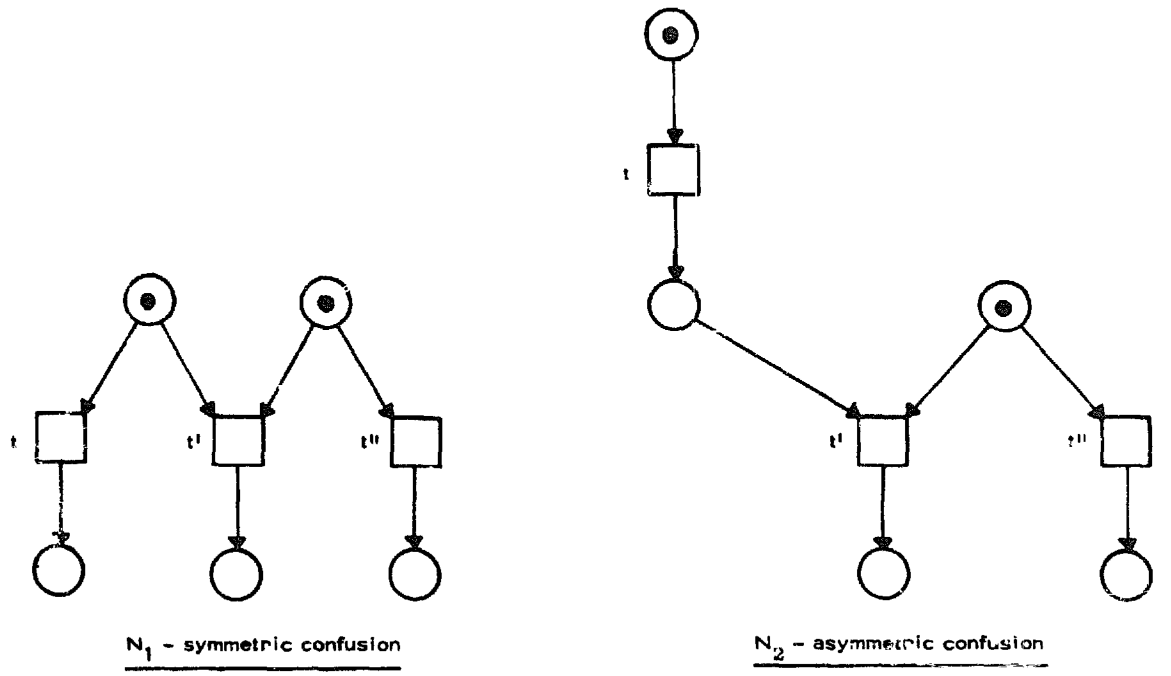

Fig. 8.

We remind the reader that axiom Q of concrete domains takes the following form (the elements are understood to be in a c.p.o.):

$$\text{Q} \qquad z \succ\!\!- x \sqsubseteq y \wedge z \not\uparrow y \Rightarrow \exists! t\colon x -\!\!\!< t \sqsubseteq y \wedge z \not\uparrow t.$$

Thus axiom Q has two parts, an existence part saying '$\exists t \cdots$', and a uniqueness part saying there is only one such t. The following proof demonstrates that these two parts correspond to banning asymmetric and symmetric confusion respectively.

Theorem 10. *Let N be a finite contact-free transition net with initial marking, M_0. Then*

$$(N, M_0) \textit{ is confusion free} \Leftrightarrow \mathscr{L} \circ \xi \circ O[N, M_0] \textit{ satisfies axiom } Q.$$

Proof. Suppose (N, M_0) is confused. Then it is either symmetrically or asymmetrically confused. In the first case

$$^{\cdot}t, ^{\cdot}t', ^{\cdot}t'' \subseteq M \wedge {}^{\cdot}t \cap {}^{\cdot}t' \neq \emptyset \wedge {}^{\cdot}t' \cap {}^{\cdot}t'' \neq \emptyset \wedge {}^{\cdot}t \cap {}^{\cdot}t'' = \emptyset$$

for some transitions t, t', t'' and some reachable marking M. We use the above proposition to translate this set-up into the domain. Take x to be the finite element of $\mathscr{L} \circ \xi \circ O[N, M_0]$ associated with the finite behaviour *up to M* and e, e', e'' the occurrences of the transitions t, t', t'' from it. Using Proposition 6 we get the picture of Fig. 9(a) in $\mathscr{L} \circ \xi \circ O[N, M_0]$, which contradicts the uniqueness part of axiom Q – take

$$y = x \cup \{e, e''\}, \qquad z = x \cup \{e'\}.$$

In a similar way the second case yields the picture of Fig. 9(b) in $\mathscr{L} \circ \xi \circ O[N, M_0]$, which contradicts the existence part of axiom Q – take

$$y = x \cup \{e, e'\}, \qquad z = x \cup \{e''\}.$$

Thus, (N, M_0) confused implies a violation of axiom Q.

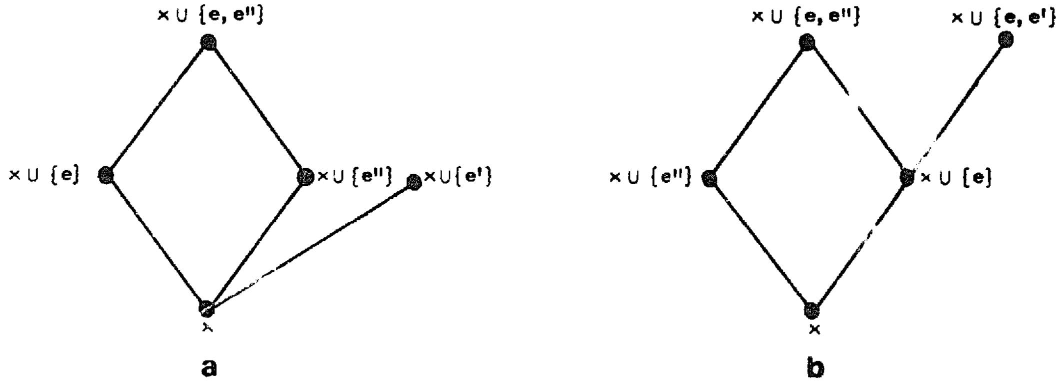

Fig. 9.

For the converse suppose axiom Q fails to hold. This can happen in two ways; either the uniqueness part fails, or the existence part fails. In the second case

$$\exists x, y, z \in \mathscr{L} \circ \xi \circ O[N, M_0]: z \succ\!\!- x \sqsubseteq y \wedge z \not\uparrow y \wedge \forall t: x -\!\!< t \sqsubseteq y \Rightarrow t \uparrow z.$$

We therefore have $z = x \cup \{e\}$ and $e \# e'$ for some occurrences $e' \in y$ and e. We may suppose e' is $\leqslant$-minimal in y such that $e \# e'$. Our hypothesis is maintained if we redefine

$$x = [e]\backslash\{e\}, \qquad z = [e], \qquad y = [e'] \cup [e]\backslash\{e\}.$$

Now we take a covering chain $x = x_0 -\!\!< x_1 -\!\!< \cdots -\!\!< x_n = y$. We must have $n \geqslant 2$ and $x_n \backslash x_{n-1} = \{e'\}$. Thus we get the picture Fig. 10 in $\mathscr{L} \circ \xi \circ O[N, M_0]$.

Using Proposition 6 we can translate this to asymmetric confusion in N. Similarly, but more directly, the first case yields a picture which translates to symmetric confusion in N.

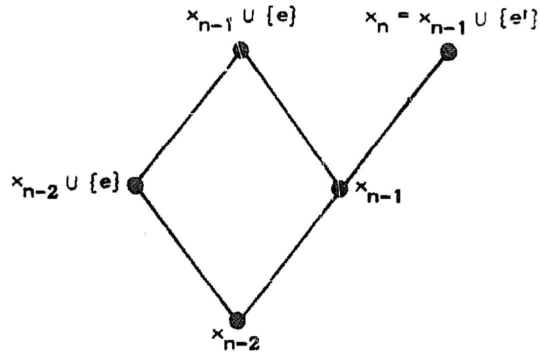

Fig. 10.

Corollary 1. *Let N be a finite contact-free transition net with initial marking M_0. Then (N, M_0) is confusion-free $\Leftrightarrow \mathscr{L} \circ \xi \circ O[N, M_0]$ is a distributive concrete domain.*

Proof. The theorem settles axiom Q. We know $\mathscr{L} \circ \xi \circ O[N, M_0]$ is distributive by the work of Section 3. Axioms C and R follow from distributivity and axiom F from $[e]$ being finite for all occurrences e. The fact that $\mathscr{L} \circ \xi \circ O[N, M_0]$ is ω-algebraic follows from N being finite.

Axiom Q evolved from the intuitions of Kahn and Plotkin in their work on concrete datatypes. There an event is imagined to occur at a fixed point in space and time; conflict between events is localised in that two conflicting events are enabled at the same time and are competing for the same point in space and time.

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