Reprinted from INFORMATION AND CONTROL
Vol. 57, Nos. 2–3, May/June 1983
Printed in Belgium

The Non-sequential Behaviour of Petri Nets

U. GOLTZ AND W. REISIG

Rheinisch-Westfälische Technische Hochschule Aachen,
Lehrstuhl für Informatik II, Büchel 29–31, 5100 Aachen, Germany

The idea of representing non-sequential processes as partially ordered sets (occurrence nets) is applied to place/transition nets (Petri nets), based on the well known notion of process for condition/event-systems. For occurrence nets some theorems relating K-density, cut finiteness, and discreteness are proved. With these theorems the result that a place/transition net is bounded if and only if its processes are K-dense is obtained.

1. INTRODUCTION

C. A. Petri (1977) suggested the representation of non-sequential processes as *occurrence nets* (causal nets). The elements of such nets are event occurrences and condition holdings. Two elements a, b are ordered ($a < b$), if a is a prerequisite for b. Consequently, a and b are unordered, if they are causally independent (concurrent). Similar partially ordered structures are widely used for the description of processes, for instance, in Mazurkiewicz (1977), Nielsen *et al.* (1981), and Winkowski (1982).

Many properties of occurrence nets (cut-finiteness, density, continuity, coherence, etc.) have been studied and related to each other (Petri, 1980; Best, 1980a, b; Best and Merceron, 1983; Fernandez and Thiagarajan, 1982). It has been asked which of them are adequate for a characterization of "reasonable" processes. One of the most signficant properties in this respect is *K-density*. It was introduced by Petri (1977) in order to "ensure that for every new, real observation a place can be found in an ordering scheme according to its relation to precisely made observations." Best (1980a) motivates K-density by the intuitive idea that every sequential subprocess of a process should always be in a well defined state.

In this paper we will not only consider occurrence nets as abstract models of some kind of real processes. Rather we also want to talk about *processes* which run on special kinds of systems, represented as *place/transition nets* (usually called Petri nets). The relationship between processes and systems corresponds to the connection between finite automata and character strings in the sequential case. There have been some approaches (Starke, 1981; Grabowski, 1979; Winkowski, 1982; Rozenberg and Verreadt, 1983) which

0019-9958/83 $3.00

define processes of place/transition nets by adopting concepts of formal language theory. In contrast we define processes as mappings from occurrence nets into the underlying place/transition net. This is a proper generalization of the well known notion of processes of condition/event-systems (Petri, 1977; Genrich and Stankiewicz-Wiechno, 1980) and allows for precise reasoning about concurrency and causality. In fact, our definition relies on a certain unfolding of the underlying place/transition net into a condition/event-system. We shall not explain this here in detail. The examples of processes we present should give some idea of the appearance of this unfolding.

In order to study this notion of processes of place/transition nets in detail, we examine properties of the underlying occurrence nets. It turns out that there is a close relationship between the boundedness of a Petri net (i.e., the existence of an upper bound for the number of tokens for all markings) and the K-density of its processes.

To establish this relationship, we prove a theorem relating K-density and cut-finiteness of occurrence nets, using the results about K-density of Best (1980a, b). Then we characterize the boundedness of a Petri net by the cut-finiteness of its processes. We then get the result that a Petri net is bounded if and only if each of its processes is based on a K-dense occurrence net.

2. Occurrence Nets

In part (a) of this section we introduce nets, especially occurrence nets, and some related notions for such nets. Part (b) deals with some relationships between K-density, discreteness, degree-finiteness, and cut-finiteness of occurrence nets. Finally, in part (c) we consider foundedness and initial subnets of occurrence nets.

(a) *Basic Notions*

The basic notions introduced in this part are well known, e.g., from Genrich and Stankiewicz-Wiechno (1980) and Best and Merceron (1983).

2.1. Definition. (i) $N = (S, T; F)$ is called a *net* iff

(a) S and T are disjoint sets (S-*elements* and T-*elements*, resp.),

(b) $F \subseteq (S \times T) \cup (T \times S)$, F is called the *flow relation*,

(c) $\forall t \in T \; \exists s \in S \;\; tFs \vee sFt$.

(ii) For $x \in S \cup T$, ${}^{\bullet}x := \{y \mid yFx\}$ is called the *preset* of x, $x^{\bullet} := \{y \mid xFy\}$ is called the *postset* of x. For $X \subseteq S \cup T$, let ${}^{\bullet}X := \bigcup_{x \in X} {}^{\bullet}x$, $X^{\bullet} := \bigcup_{x \in X} x^{\bullet}$.

(iii) Let ${}^{\circ}N := \{x \in S \cup T \mid {}^{\bullet}x = \varnothing\}$ and $N^{\circ} := \{x \in S \cup T \mid x^{\bullet} = \varnothing\}$.

(iv) $x \in S \cup T$ is called *isolated* iff ${}^{\bullet}x \cup x^{\bullet} = \varnothing$.

Note that property 2.1(i)(c) excludes isolated T-elements, but in contrast to Genrich and Stankiewicz-Wiechno (1980), we do not exclude isolated S-elements. This has no influence on the validity of the theorems of Section 2. Graphically we represent S-elements and T-elements as circles and boxes, respectively. The flow relation is indicated by arcs between the corresponding circles and boxes.

Given a net $N = (S, T; F)$ we often write S_N, T_N, F_N instead of S, T, F. We denote $S \cup T$ by N if no confusion is possible.

2.2. DEFINITION. (i) A net K is an *occurrence net* iff

(a) $\forall x, y \in K \quad xF_K^+ y \Rightarrow \neg(yF_K^+ x)$ (F_K^+ denoting the transitive closure of F_K),

(b) $\forall s \in S_K \; |{}^{\bullet}s| \leqslant 1 \wedge |s^{\bullet}| \leqslant 1$.

(ii) Let K be an occurrence net.

(a) $<_K := F_K^+$ is the *order relation* of K. The index K is omitted if it is obvious from the context.

(b) Let $\mathbf{li} \subseteq K \times K$ and $\mathbf{co} \subseteq K \times K$ be given by

$$x\,\mathbf{li}\,y :\Leftrightarrow x < y \vee y < x \vee x = y,$$

$$x\,\mathbf{co}\,y :\Leftrightarrow \neg(x\,\mathbf{li}\,y) \vee x = y.$$

li and **co** denote the orderedness and unorderedness of elements, respectively. Maximal sets of pairwise ordered or unordered elements, resp., are called *lines* and *cuts*:

(iii) $M \subseteq K$ is a *line* iff $\forall x, y \in M \;\; x\,\mathbf{li}\,y \wedge \forall z \in K \backslash M \; \exists x \in M \;\; \neg(x\,\mathbf{li}\,z)$.
$M \subseteq K$ is a *cut* iff $\forall x, y \in M \;\; x\,\mathbf{co}\,y \wedge \forall z \in K \backslash M \; \exists x \in M \;\; \neg(x\,\mathbf{co}\,z)$.

(iv) A cut $M \subseteq K$ is a *slice* iff $M \subseteq S_K$.

As an example, Fig. 1 shows an occurrence net with 2 lines and 11 cuts.

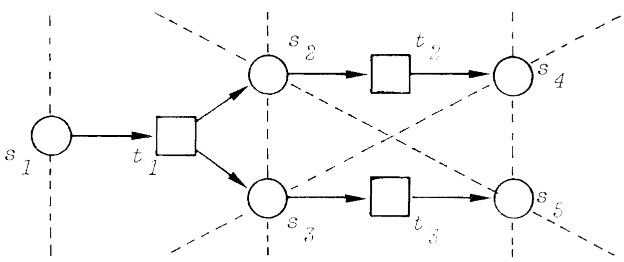

FIG. 1. This occurrence net has two lines, $\{s_1, t_1, s_2, t_2, s_4\}$ and $\{s_1, t_1, s_3, t_3, s_5\}$. Its slices are indicated by broken lines. For example $\{t_2, s_3\}$ is a cut but not a slice.

The following lemma follows from the axiom of choice (cf. Best, 1980a).

2.3. LEMMA. *Let K be an occurrence net.*

(i) *Let $L_0 \subseteq K$ such that $\forall x, y \in L_0 : x\,\mathbf{li}\,y$. Then there is a line L of K with $L_0 \subseteq L$.*

(ii) *Let $C_0 \subseteq K$ such that $\forall x, y \in C_0 : x\,\mathbf{co}\,y$. Then there is a cut C of K with $C_0 \subseteq C$.*

The following is obvious:

2.4. LEMMA. *Let K be a finite occurrence net. Then $°K$ and $K°$ are cuts.*

(b) *K-Dense Occurrence Nets*

K-dense occurrence nets (and, more general, K-dense partially ordered sets) have been studied in several papers (e.g., Best, 1980a, b; Best and Merceron, 1983; Fernandez and Thiagarajan, 1982; Nielsen *et al.*, 1981; Petri, 1977; Plünnecke, 1981). We relate K-density to other properties, i.e., to cut-finiteness, degree-finiteness, and discreteness.

Two key notions of this paper, K-density and cut-finiteness, are given as follows:

2.5. DEFINITION. Let K be an occurrence net.

(i) K is *K-dense* iff for every line L and every cut C of K: $L \cap C \neq \varnothing$.

(ii) K is *cut-finite* iff each cut of K is finite.

As $x\,\mathbf{li}\,y \wedge x\,\mathbf{co}\,y \Rightarrow x = y$, we get for lines L and cuts C of K-dense occurrence nets immediately: $|L \cap C| = 1$.

A first characterization of K-density requires the notion of causal components of occurrence nets. Roughly, a causal component of an occurrence net K is a net which consists of a subset of elements of K and of a flow relation which respects the $<$- (and hence the $\mathbf{co}$-) relation of K. This notion is incomparable to the notion of subnet which is defined as usual (cf. Fig. 2).

2.6. DEFINITION. Let K, K' be occurrence nets with $S_{K'} \subseteq S_K$ and $T_{K'} \subseteq T_K$.

(i) K' is called a *subnet* of K iff $F_{K'} = F_K \cap (S_{K'} \cup T_{K'})^2$.

(ii) K' is called a *causal component* of K iff $\forall x, y \in K'$ $x <_{K'} y \Leftrightarrow x <_K y$.

The main theorem proved in Best (1980a) shows that K-density can be characterized by means of the nets N_1 and N_2 shown in Fig. 3.

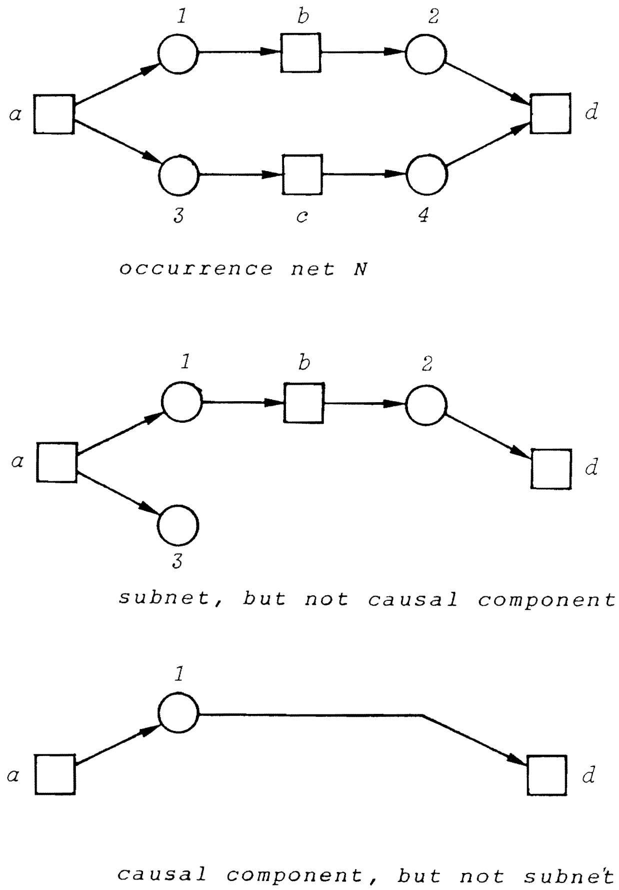

FIG. 2. Causal components and subnets.

2.7. THEOREM. *Let K be an occurrence net and let N_1 and N_2 be as shown in Fig.* 3. *K is K-dense iff there is no causal component of K shaped like N_1 or N_2.*

Using this theorem, we immediately see that cut-finiteness implies K-density. Furthermore we can show that an occurrence net is K-dense if all its slices are finite.

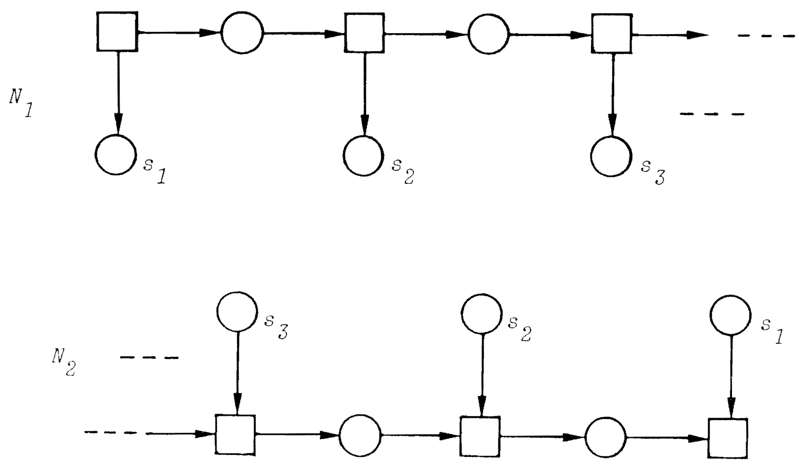

FIG. 3. Non-K-dense occurrence nets.

2.8. COROLLARY. *If all slices of an occurrence net K are finite then K is K-dense.*

Proof. Assume K is not K-dense. Then according to Theorem 2.7, N_1 or N_2 is a causal component of K. Let $S := \{s_1, s_2, ...\}$ as indicated in Fig. 3. According to Lemma 2.3(ii) there exists a cut C of K such that $S \subseteq C$. T-Elements in C can be replaced by their pre- or post-set, as in occurrence nets S-elements are not branched and T-elements are not isolated. This replacement yields an infinite slice of K. ∎

It is easy to find an example which shows that the converse of Corollary 2.8 is not true.

We now introduce two more properties of occurrence nets, discreteness, and degree-finiteness (Best and Merceron, 1983; Fernandez and Thiagarajan, 1982). Processes of place/transition nets, as introduced in Section 3, are based on degree-finite occurrence nets, which are not necessarily discrete.

2.9. DEFINITION. Let K be an occurrence net, let $x, y \in K$, and let L be a line of K.

(i) $[x, y] := \{z \in K \mid x \leqslant z \leqslant y\}$.

(ii) $[x, y; L] := [x, y] \cap L$.

(iii) K is *discrete* iff for all $x, y \in K$ and each line L $[x, y; L]$ is finite.

(iv) K is *degree-finite* iff $\forall t \in T_K$ ${}^{\bullet}t$ and $t^{\bullet}$ are finite.

As a direct consequence of a theorem stated in Best and Merceron (1983), we find that K-density implies discreteness.

2.10. THEOREM. *Each K-dense occurrence net is discrete.*

Now all preliminaries are given in order to relate K-density, degree-finiteness, and cut-finiteness for occurrence nets K starting with a finite cut ${}^{\circ}K$. (Remember that ${}^{\circ}K$ denotes the set of all "initial" elements of K as defined in 2.1(iii).)

2.11. THEOREM. *Let K be a degree-finite, K-dense occurrence net and let ${}^{\circ}K$ be a finite cut. Then all cuts of K are finite.*

Proof. Assume an infinite cut C of K and let $C' := C \setminus {}^{\circ}K$. We construct inductively elements z_i of K and infinite subsets $D_i \subseteq C'$ as follows: As ${}^{\circ}K$ is a cut, $\forall x \in C'$ $\exists y \in {}^{\circ}K$ with $y < x$. Since ${}^{\circ}K$ is finite and C' is infinite, there exists some $z_0 \in {}^{\circ}K$ such that the set $D_0 := \{x \in C' \mid z_0 < x\}$ is infinite (cf. Fig. 4).

Now assume z_i is given such that the set $D_i := \{x \in C' \mid z_i < x\}$ is infinite. Since K is degree-finite, $z_i^{\bullet}$ is finite and there exists an element $z_{i+1} \in z_i^{\bullet}$ such that $D_{i+1} := \{x \in C' \mid z_{i+1} < x\}$ is infinite (cf. Fig. 4).

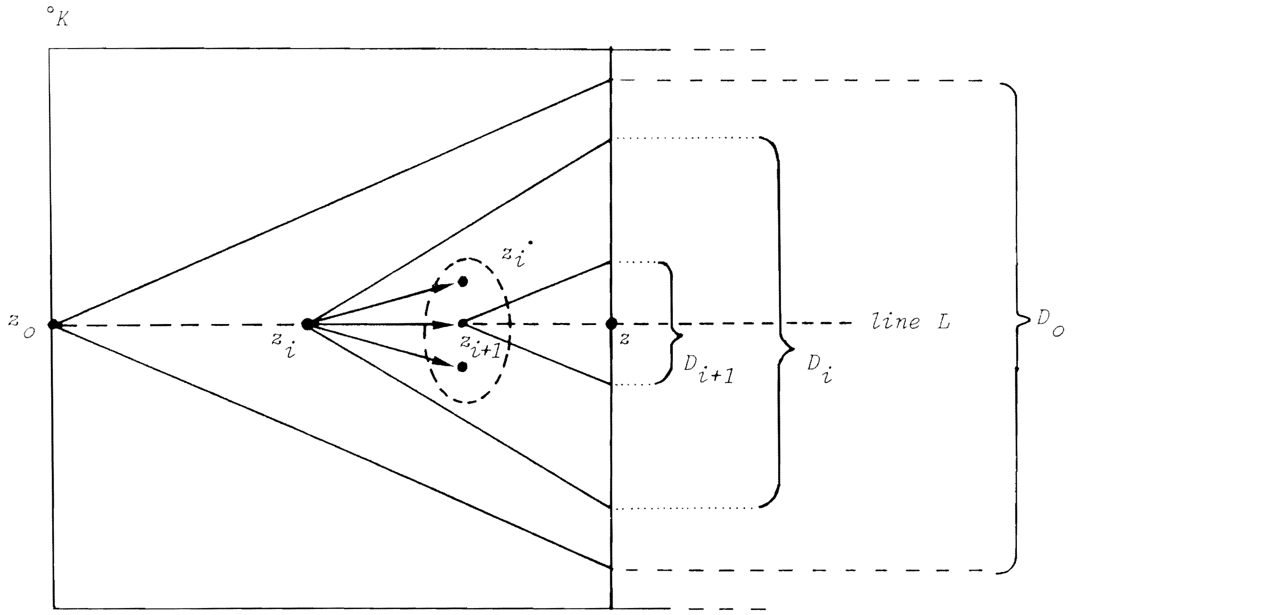

FIGURE 4

In this way we obtain an infinite totally ordered set $Z = \{z_0, z_1, z_2, \ldots\}$ and according to Lemma 2.3(i) there is a line L such that $Z \subseteq L$. As K is K-dense, there is an element $z \in L \cap C$. Assume that there is $i_0 \in \mathbb{N}$ such that $z < z_{i_0}$. There is $x \in C$ such that $z_{i_0} < x$, so $z < z_{i_0} < x$. This is impossible, since C is a cut. So $z_i < z$ for all $i \in \mathbb{N}$, and $[z_0, z; L]$ is infinite. Hence K is not discrete and therefore not K-dense (Theorem 2.10). ∎

This theorem can also be derived from some results on partial orders of Plünnecke (1981).

2.12. COROLLARY. *Let K be a degree-finite occurrence net such that $°K$ is a finite cut. K is K-dense iff all slices of K are finite.*

Proof. "$\Rightarrow$": Theorem 2.11,
"$\Leftarrow$": Corollary 2.8. ∎

(c) *Foundedness and Discrete Initial Subnets*

We now turn back to the property of discreteness defined in 2.9. We shall show that a degree-finite occurrence net starting with a finite cut is discrete if and only if all its elements have "a finite history."

2.13. DEFINITION. Let K be an occurrence net, let $x \in K$, and let $A \subseteq K$.

(i) $\downarrow x := \{y \in K \mid y \leqslant x\}$, $\downarrow A := \bigcup_{x \in A} \downarrow x$.

(ii) x is called *founded* iff $\downarrow x$ is finite.

2.14. THEOREM. *Let K be a degree-finite occurrence net such that $°K$ is a* finite *cut. K is discrete iff all $x \in K$ are founded.*

Proof. "$\Rightarrow$" Assume an element $y \in K$ which is not founded. Since $\downarrow y$ is infinite and K is degree-finite we find an infinite set $L_0 \subseteq \downarrow y$ such that $\forall x, z \in L_0$, $x \mathbf{li} z$ (analogously to the proof of König's lemma; see Knuth, 1973). Using Lemma 2.3(i), there exists a line L with $L_0 \subseteq L$. Since $°K$ is a finite cut, the set $L_x := \{z \in L_0 \mid x \leqslant z\}$ is infinite for some $x \in °K$. Then $x \leqslant z \leqslant y$ for all $z \in L_x \subseteq L$ and therefore $[x, y; L]$ is infinite. Hence K is not discrete.

"$\Leftarrow$" Assume that K is not discrete. Then there exist $x, y \in K$ and a line L such that $[x, y; L]$ is infinite. Since $\downarrow y \supseteq [x, y; L]$, y is not founded. ∎

Now we consider initial subnets of occurrence nets. These will be left-closed subsets of occurrence nets which satisfy two further requirements: They contain all initial elements of the net and, for every T-element contained in the subnet, all its postelements are also included. These requirements guarantee that any process restricted to some initial subnet will again yield a process.

2.15. DEFINITION. Let K be an occurrence net and let $M \subseteq K$. Let $A := °K \cup \downarrow M \cup ((\downarrow M)^{\cdot} \cap S_K)$. Then the subnet $K_M := (S_K \cap A, T_K \cap A; F_K \cap A^2)$ is called the *initial subnet* of K induced by M.

Clearly, an initial subnet of an occurrence net K is a causal component of K.

2.16. LEMMA. *Let K be a degree-finite occurrence net and let $M \subseteq K$. If K is discrete, $°K$ is a finite cut, and M is finite then K_M is finite.*

Proof. If K is discrete then all $x \in M$ are founded according to Theorem 2.14. Then $\downarrow M = \bigcup_{x \in M} \downarrow x$ is finite because M is finite. ∎

If an occurrence net contains an infinite slice, we can find an infinite slice contained in a discrete initial subnet.

2.17. THEOREM. *Let K be a degree-finite occurrence net with an infinite slice, $°K$ a finite cut. Then K has a discrete initial subnet with an infinite slice.*

Proof. If K is discrete, the proposition is trivially true, since K is an initial subset of itself.

Assuming that K is not discrete, we construct inductively for all $n \in \mathbb{N}$ non-founded elements x_n, non-founded T-elements t_n, and founded S-elements $s_n \in {}^{\cdot}t_n$ as shown in Fig. 5. According to Theorem 2.14, there exists a non-founded element x_0. Assuming a non-founded element x_n, there exists an element $y \in °K$ with $y < x_n$ (as $°K$ is a cut) and a line L such that $x_n, y \in L$ and $[y, x_n; L]$ is finite (due to Definition 2.2(ii)(a), there exists some $k \in \mathbb{N}$ such that $yF^k x_n$). Since $x_n \in [y, x_n; L]$, the set

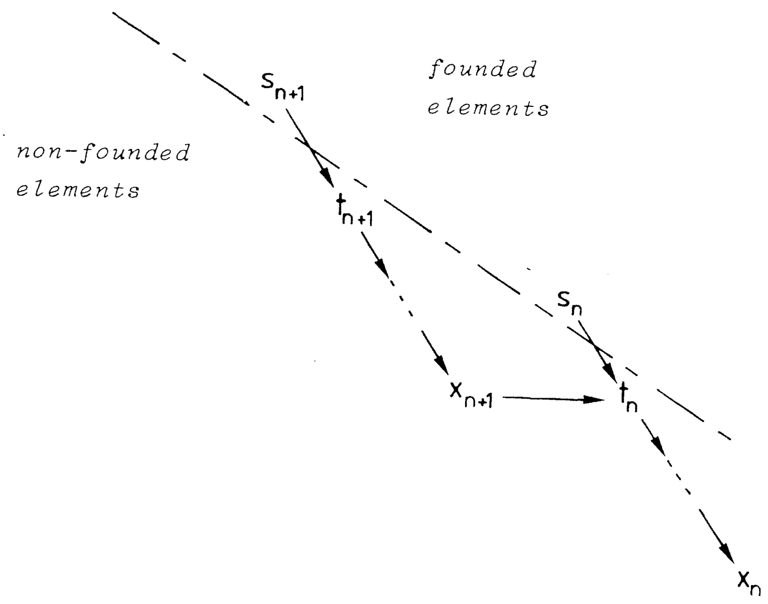

FIG. 5. The inductive construction of x_n, t_n and s_n.

$\{z \in [y, x_n; L] \mid z$ is not founded$\}$ has a minimal element t_n (obviously $t_n \leqslant x_n$). Due to this construction, there exists an element $s_n \in {}^{\bullet}t_n \cap L$ such that s_n is founded. As t_n is not founded, there exists an element $z \in {}^{\bullet}t_n$ which is not founded (as ${}^{\bullet}x_n$ is finite and $\downarrow x_n$ is infinite). As $s_n \neq z$, $|{}^{\bullet}t| > 1$, hence $t_n \in T_K$ and $s_n \in S_K$. With $x_{n+1} := z$, the induction step is completed.

Next we show $\forall i, j \in \mathbb{N}$: $i \neq j \Rightarrow s_i \neq s_j$: Assume $s_i = s_j$. Since s_i and s_j are S-elements and therefore non-branched we have $\{t_i\} = s_i^{\bullet} = s_j^{\bullet} = \{t_j\}$ hence $t_i = t_j$. But if (w.l.o.g.) $j < i$, we get from the above construction immediately $t_i < x_i < t_j$.

It is easy to show that $\forall i, j \in \mathbb{N}$ $s_i \mathbf{co} s_j$: Assume (w.l.o.g) $s_i < s_j$. Because $s_i^{\bullet} = \{t_i\}$, we obtain $s_i < t_i < s_j$ and $\downarrow t_i \subseteq \downarrow s_j$. But $\downarrow t_i$ is infinite (as t_i is not founded) whereas $\downarrow s_j$ is finite (as s_j is founded)!

Thus, $S = \{s_0, s_1, \ldots\}$ is an infinite set of founded, pairwise concurrent S-elements of K. According to Definition 2.15, all elements of the initial subnet K_S are founded. Hence, K_S is a discrete initial subnet of K (Theorem 2.14) and S is contained in some infinite slice of K_S. ∎

3. Processes of Place/Transition Nets

In part (a) of this section we introduce the well known model of place/transition nets, often called Petri nets. Furthermore, as the central concern of this part, we define processes for such nets as mappings from occurrence nets to place/transition nets and we discuss the intuitition of this notion. In part (b) we consider processes which are based on occurrence nets with special properties, as defined in Section 2. We show that a place/transition net is bounded if and only if all its processes are based on K-dense occurrence nets. Finally, we discuss discreteness of processes.

(a) *Place/Transition Nets and Their Processes*

Place/transition nets, also called marked nets or Petri nets, are the most widespread model of Net Theory. Such nets consist of S-elements (called *places*) which hold *tokens*, and of T-elements (called *transitions*) which can be *fired*. Upon firing a transition, the token count of all places in its preset is decreased, and the token count of all places in its postset is increased.

Continuing firing of transitions is usually represented as a firing sequence (Peterson, 1981). We suggest an alternative representation as a *process* in order to represent precisely causality and concurrency of transition firings.

According to Genrich and Stankiewicz-Wiechno (1980) we define:

3.1. DEFINITION. A 5-tuple $N = (S, T; F, W, M)$ is a *marked place/transition net* (a *marked net*, for short) iff

(i) $(S, T; F)$ is a net, $S \cup T$ is finite (the elements of S and T are called *places* and *transitions*, respectively),

(ii) $W: F \to \mathbb{N}$ assigns a positive *weight* to each arc,

(iii) $M: S \to \mathbb{N}$ is the *initial marking* of N.

We omit place capacities. (If wanted, they can be simulated by complementary places (cf. Genrich and Stankiewicz-Wiechno, 1980).) According to the notation of Section 2 we denote the components of N by S_N, T_N, F_N, W_N, M_N, respectively. Sometimes we assume $W_N(x, y) = 0$ for $(x, y) \notin F_N$.

In graphical representations of marked nets, arcs are inscribed by their weights and markings M are represented by $M(s)$ dots (called *tokens*) in each place s. Figure 6 shows an example of a marked net in its most general

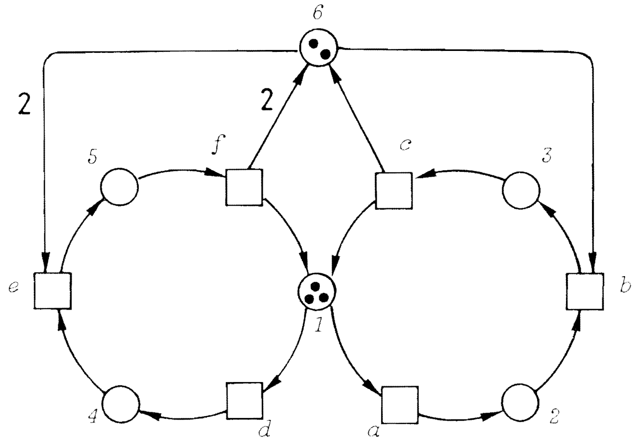

FIG. 6. A marked net.

form. The dynamic behaviour of marked nets is given by the usual firing rule:

3.2. DEFINITION. Let N be a marked net.

(i) A mapping $M: S_N \to \mathbb{N}$ is a *marking*.

(ii) Let M be a marking. A transition $t \in T_N$ is *M-activated iff* $\forall s \in {}^{\bullet}t \; M(s) \geqslant W(s, t)$.

(iii) Each M-activated transition t yields a *follower marking* M' by $\forall s \in S_N \; M'(s) = M(s) - W_N(s, t) + W_N(t, s)$.

In this case, *t fires from M to M'* and we write $M[t\rangle M'$.

(iv) Let M be a marking of N. The set $[M\rangle$ is the smallest set of markings such that (a) $M \in [M\rangle$ and (b) $M' \in [M\rangle, M'[t\rangle M'' \Rightarrow M'' \in [M\rangle$. The set $[M_N\rangle$ is the set of *reachable markings* of N.

As mentioned above, a common way to trace consecutive transition firings is the construction of firing sequences $M_0[t_1\rangle M_1 \cdots M_{n-1}[t_n\rangle M_n$, whereby t_i fires from M_{i-1} to M_i. As an example consider Fig. 7. In the net N of Fig. 7, a and b may fire concurrently, but c is delayed until both a and b have fired. The concurrency between a and b, and the causal dependency of a and c, and of b and c, cannot be derived from the firing sequence shown in Fig. 7. The net K of Fig. 8 suggests an alternative representation: The S- and T-elements of K are inscribed by places and transitions of N, respectively, indicating the places which change their token count and the transitions which fire. Obviously, K is a (labelled) occurrence net. Its initial slice represents the initial marking of N: one token on place 1, one token on place 2 and no token on all other places. In this way, all slices of K represent markings of N.

K represents a *process* of N. To be more precise, a process maps the elements of an occurrence net to the elements of a marked net. It is obvious that occurrence nets are adequate to represent such processes. If a place s is branched, e.g., the place 1 in the net N, one of the transitions in $s^{\bullet}$ is fired in

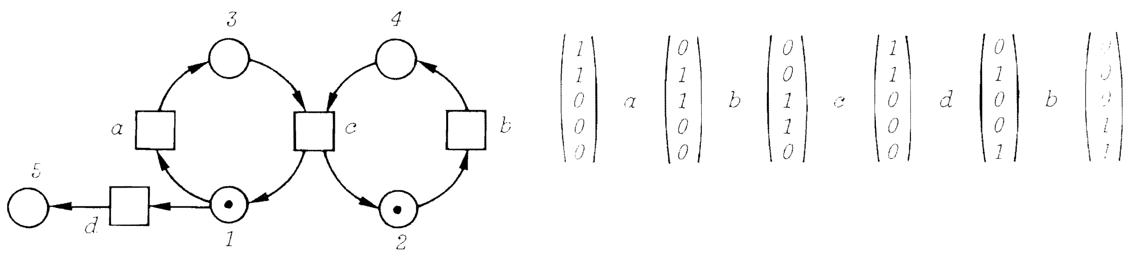

FIG. 7. A Petri net N and a firing sequence, in which markings M are represented by

$$\begin{pmatrix} M(1) \\ \vdots \\ M(5) \end{pmatrix}.$$

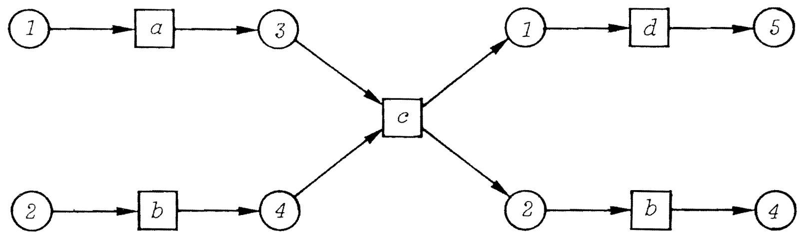

FIG. 8. A process of the net N shown in Fig. 7.

each actual situation. In the above example, first a and then d is chosen. Hence, S-elements of process representations are not branched. Furthermore, process representations are acyclic, because each instance of firing a transition is represented separately.

In the above example, the net N may be considered as a contact free condition/event-system. For such systems, the notion of process is, for instance, defined in Genrich and Stankiewicz-Wiechno (1980).

We shall define processes of place/transition nets as a proper generalization of this notion of process. As an example, Fig. 9 shows a marked net and Fig. 10 shows a process of this net. The key properties of processes are: (1) the initial S-elements represent the marking at which the process starts, and (2) the process respects the environment of transitions. For a process $p: K \to N$, (2) implies that ${}^\bullet p(t) = p({}^\bullet t)$ and $p(t^\bullet) = p(t)^\bullet$ for all $t \in T_K$.

3.3. DEFINITION. Let N be a marked net and let $M \in [M_N\rangle$. Let K be an occurrence net. A mapping $p: K \to N$ is called a *process* (of N starting at M) iff

(i) $p(S_K) \subseteq S_N \wedge p(T_K) \subseteq T_N$,

(ii) ${}^\circ K$ is a cut and $\forall s \in S_N\ M(s) = |p^{-1}(s) \cap {}^\circ K|$,

(iii) $\forall t \in T_K\ \forall s \in S_N$

$$W_N(s, p(t)) = |p^{-1}(s) \cap {}^\bullet t|,$$

$$W_N(p(t), s) = |p^{-1}(s) \cap t^\bullet|.$$

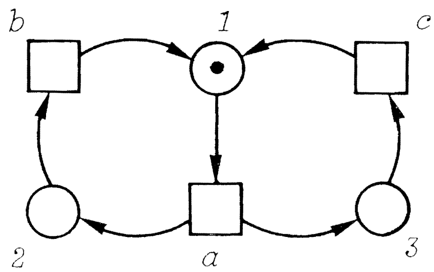

FIG. 9 A net which can get arbitrary many tokens on each place according to the firing rule of marked nets.

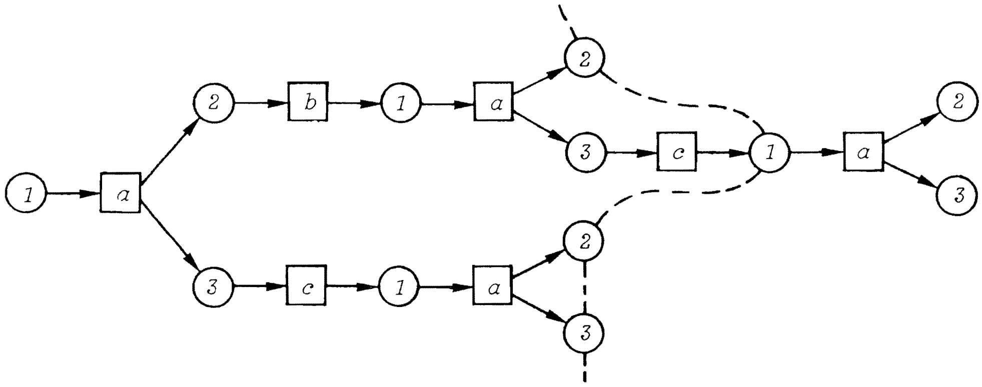

FIG. 10. A process of the marked net shown in Fig. 9. The dotted line reflects the marking $M(1) = M(3) = 1$, $M(2) = 2$.

As in the examples shown above, we represent processes graphically by labelling each element x of the occurrence net K with its image $p(x)$.

Since every marked net is finite it is obvious from Definition 3.3(iii) that the underlying occurrence nets of processes are always degree-finite.

In a process $p: K \to N$, each T-element t of K, together with its inscription, denotes a firing of the transition $p(t) \in T_N$. On the other hand, each S-element s of K, with its inscription, denotes a token in the place $p(s) \in S_N$. Furthermore, we shall show now that each slice of a finite process (K finite) corresponds to a reachable marking of N.

3.4. DEFINITION. Let $p: K \to N$ be a process, let S be a finite slice of K. We define the marking $m(p, S): S_N \to \mathbb{N}$ of N by $m(p, S)(s) = |p^{-1}(s) \cap S|$ for each $s \in S_N$.

As an example, consider Fig. 10. We obtain the marking corresponding to a slice by counting how often each place is represented in this slice.

3.5. THEOREM. *Let K be finite, let $p: K \to N$ be a process, and let S be a slice of K. Then $m(p, S) \in [M_N\rangle$.*

Proof. For any slice S_0 of K, let $T_{S_0} := {\downarrow} S_0 \cap T_K$. We prove the result by induction on $|T_S|$. $|T_S| = 0 \Rightarrow S = {}^\circ K$. $m(p, {}^\circ K) \in [M_N\rangle$ follows from Definition 3.3. Let $|T_S| = n + 1$.

We first show, by contradiction, that there exists $t \in {}^\cdot S$ such that $t^\cdot \subseteq S$. Assuming $\forall t \in {}^\cdot S$, $t^\cdot \nsubseteq S$, we construct inductively an infinite number of T-elements $t_0 < t_1 < \cdots$ (contradicting the finiteness of K) as follows: As ${}^\cdot S \neq \varnothing$, $\exists t_0 \in {}^\cdot S$. Assuming t_n, $\exists s \in t_n^\cdot$ such that $s \notin S$. Since S is a cut, $\exists s' \in S$ with $s \,\mathrm{li}\, s'$. Clearly $s < s'$ (since assuming $s' < s$ we get $s' < t_n \in {}^\cdot S$). Then there exists a T-element $t_{n+1} \in {}^\cdot s'$ such that $t_n < s < t_{n+1} < s'$. As $t_0, t_1, \ldots$ are ordered, they are mutually distinct.

We showed that there exists $t \in {}^\cdot S$ such that $t^\cdot \subseteq S$. Clearly, $S' := (S \backslash t^\cdot) \cup {}^\cdot t$ is a slice of K and $|T_{S'}| = n$. By the induction hypothesis,

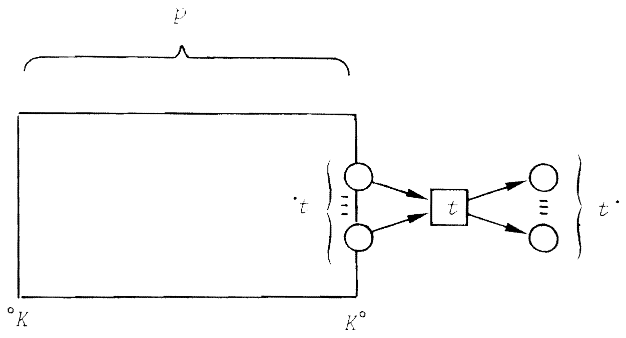

FIG. 11. The construction of p' from p.

$m(p, S') \in [M_N\rangle$. Since $m(p, S')[p(t)\rangle\, m(p, S)$ according to the firing rule and the definition of processes, we find $m(p, S) \in [M_N\rangle$. ∎

Conversely, for two reachable markings M, M' of a net N such that $M' \in [M\rangle$, there exists a process leading from M to M'.

3.6. THEOREM. *Let N be a marked net and let $M \in [M_N\rangle$. Then there exists for each $M' \in [M\rangle$ a finite process $p: K \to N$ with $m(p, {}^\circ K) = M$ and $m(p, K^\circ) = M'$.*

Proof. By induction on the structure of $[M\rangle$: (a) It is trivial to construct a process consisting only of a slice S such that $\forall s \in S_N\; M(s) = |p^{-1}(s) \cap S|$. (b) Assume $M'' \in [M\rangle$, $t \in T_N$, and $M''[t\rangle M'$. By the induction hypothesis, there exists a finite process $p: K \to N$ such that $m(p, {}^\circ K) = M$ and $m(p, K^\circ) = M''$. Now we construct a process p' in the following way. We add to K a new T-element which is mapped to t and new S-elements corresponding to $t^\bullet$. The flow relation is completed, respecting the environment of t. Note that in general this is not unique but always possible since $M''[t\rangle M'$ (cf. Fig. 11). p' satisfies the requirements. ∎

To conclude part (a) of this section, we shall discuss now some consequences of the notion of process as defined in 3.3.

If N is a marked net with an arc f weighted by $W(f) = n$, then in a process $p: K \to N$ f is "unfolded" in K into n arcs, as Fig. 12 shows. The representation of dynamic behaviour as processes shows immediately which

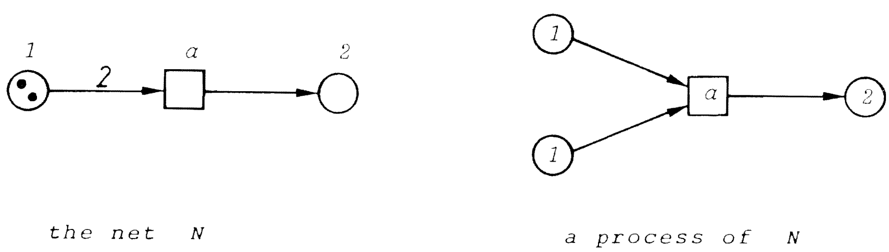

FIG. 12. A process of a net with arc weights.

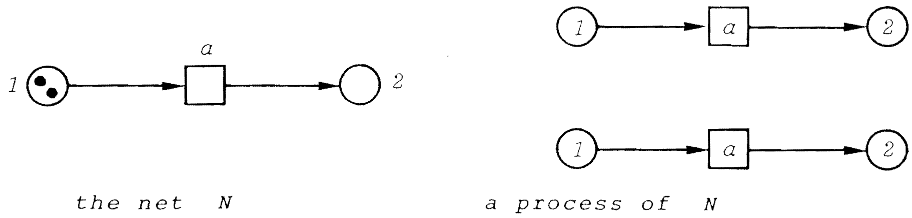

FIG. 13. A transition firing concurrently to itself.

transition firings are concurrent and which are ordered: by application of the relations **li** and **co** given in Definition 2.2.

In order to compare concurrent and ordered transition firings, consider the net N of Fig. 13 with the following interpretation: Each token in place 1 represents a file to be printed; each token in place 2 represents a file after being printed and each firing of transition a represents the action of printing a file. In the situation given in Fig. 13, two files are to be printed. There is no order specified for printing them, and assuming two printers are available, both files may be printed concurrently. This is represented by the process given in Fig. 13.

Now let us assume that only one printer is available. In the net of Fig. 14 this printer is represented as a token in place 3 and the two firings of a are serialised (the process shown in Fig. 14 has a sequential subprocess $\textcircled{3}\to\boxed{a}\to\textcircled{3}\to\boxed{a}\to\textcircled{3}$). Clearly, the two marked nets of Figs 13 and 14 represent two different real systems with different behaviours (concurrent or sequential firings of a, respectively). In contrast to firing sequences, processes reflect this difference. As a further example, skipping both arrows between place 3 and transition a in the net of Fig. 14 has much impact on the behaviour, but no impact on the firing sequences.

As a final example, assume a second token on place 3 in the net of Fig. 14, representing a second available printer. This net has two different processes, as Fig. 15 shows: p_1 reflects both printers acting concurrently, whereas p_2 reflects one lazy printer and one printer executing both tasks sequentially.

As motivated by these examples, we allow a transition to fire concurrently to itself (this is excluded in Genrich and Stankiewicz-Wiechno, 1980, and in the approach of Starke, 1981).

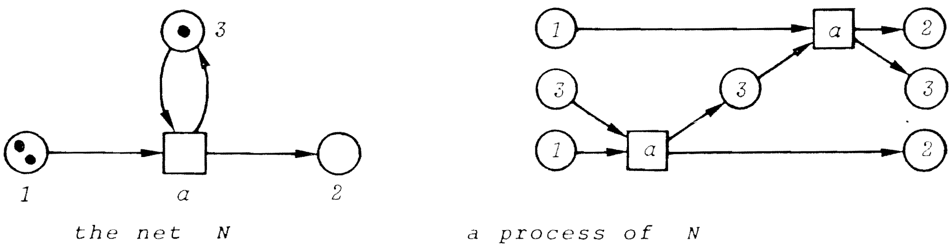

FIG. 14. Sequentialization by means of loops.

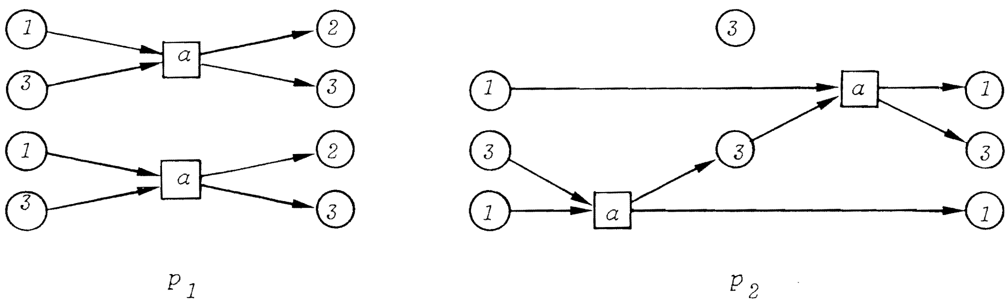

FIG. 15. Two processes of the net in Fig. 14, assuming two tokens on place 3.

(b) *Boundedness of Place/Transition Nets and K-density of Processes*

A marked net is called bounded if and only if there exists a natural number $n \in \mathbb{N}$ such that each place contains under each reachable marking not more that n tokens. This is equivalent to the requirement that the set of reachable markings be finite.

3.7. DEFINITION. A marked net N is called *bounded* iff $\exists n \in \mathbb{N}$ $\forall M \in [M_N\rangle$ $\forall s \in S_N$ $M(s) \leqslant n$.

In general, processes $p: K \to N$ may be infinite ($S_K \cup T_K$ may be infinite). Furthermore there may be infinite slices of K. These slices do not correspond to reachable markings, since all those markings are finite. (See Fig. 16.) We will show now that a marked net N is bounded if and only if all slices of all processes of N are finite. As a consequence we get, using the results of Section 2, that a marked net is bounded if and only if all its processes are based on K-dense occurrence nets.

3.8. LEMMA AND DEFINITION. *Let $p: K \to N$ be a process, let K' be an initial subnet of K. Then $p' := p \mid K'$ is a process of N. p' is called an* initial subprocess *of p.*

3.9. THEOREM. *Let $p: K \to N$ be a process of a marked net N such that K has an infinite slice. Then N is not bounded.*

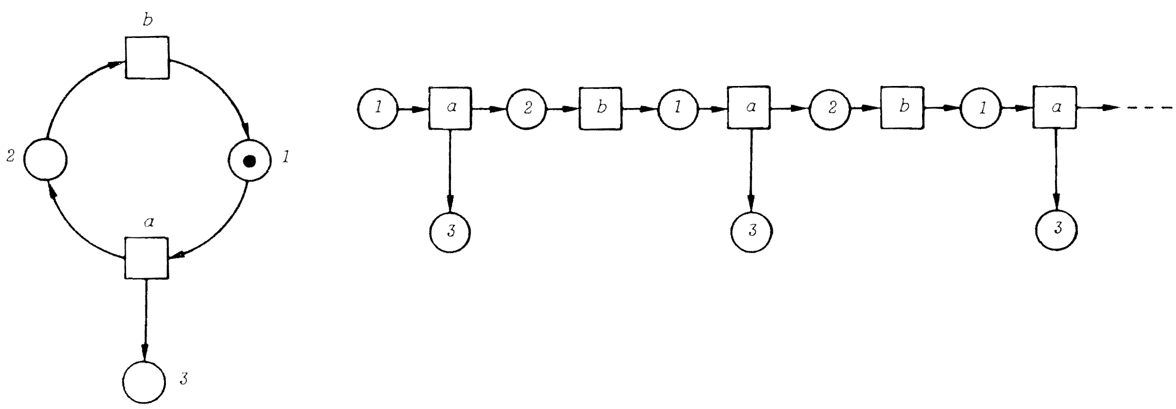

FIG. 16. An infinite process with an infinite slice.

Proof. If there exists $t \in T_N$ with ${}^{\bullet}t = \varnothing$ then we see immediately that N is not bounded (t may be fired arbitrarily often). So assume ${}^{\circ}N \subseteq S_N$. Then ${}^{\circ}K$ is a finite cut, since $M(s) \in \mathbb{N}$ for all $M \in [M_N\rangle$ and all $s \in S_N$. Using Theorem 2.17, there exists a discrete initial subnet K' of K with an infinite slice S. According to 3.8, $p' := p \mid K'$ is a process of N. There exists an S-element $s_0 \in S_N$ such that p' maps infinitely many elements of S onto s_0 (since N is finite). For $n \in \mathbb{N}$, let $A_n \subseteq S$ be a finite set such that p' maps n elements of A_n to s_0. $K_n := K'_{A_n}$ is a finite initial subnet of K' (Lemma 2.16). Since $x\,\mathbf{co}\,y$ in K_n for all $x, y \in A_n$, there exists a slice S_n of K_n with $A_n \subseteq S_n$. Using Theorem 3.5, $m(p, S_n) \in [M_n\rangle$. Clearly, $m(p, S_n)(s_0) \geqslant n$. Since this construction works for all $n \in \mathbb{N}$, the result follows. ∎

We shall show that the converse of this theorem is also true. But for this we first need some preparations.

3.10. Definition. Let M, M' be markings of some marked net N. We call M *smaller than* M' ($M < M'$) iff $M \neq M'$ and $\forall s \in S_N\ M(s) \leqslant M'(s)$.

3.11. Lemma. *Each infinite sequence $\sigma = M_1, M_2, \ldots$ of mutually distinct markings of some marked net N has a strongly increasing infinite subsequence $\sigma' = M_{i_1}, M_{i_2}, \ldots$.*

Proof. By induction on $|S_N|$.

If $|S_N| = 1$ then $M_i < M_j$ or $M_j < M_i$ for all $i, j \in \mathbb{N}$. In this case, let $M_{i_1} := M_1$ and, given M_{i_j}, there exist only finitely many markings M in σ such that $M < M_{i_j}$ (as descending sequences of naturals are finite), hence there exists some index $i_{j+1} > i_j$ such that $M_{i_{j+1}} > M_{i_j}$.

For $S_N = \{s_1, \ldots, s_{n+1}\}$, there exists by the induction hypothesis an infinite subsequence $\sigma'' = M_{l_1}, M_{l_2}, \ldots$ of σ such that

$$M_{l_j}(s_k) \leqslant M_{l_{j+1}}(s_k) \qquad \text{for} \quad 1 \leqslant k \leqslant n \quad \text{and all } j \in \mathbb{N}. \qquad (*)$$

With $M_{i_1} := M_{l_1}$ we construct $\sigma' = M_{i_1}, M_{i_2}, \ldots$ as a subsequence of σ'': Given M_{i_j}, there are only finitely many markings M in σ'' such that $M(s_{n+1}) \leqslant M_{i_j}(s_{n+1})$. Hence, there exists some index $i_{j+1} > i_j$ such that $M_{i_{j+1}}$ in σ'' and $M_{i_{j+1}}(s_{n+1}) > M_{i_j}(s_n)$. With $(*)$, we have $M_{i_{j+1}} > M_{i_j}$. ∎

3.12. Theorem. *If all slices of all processes of a marked net N are finite then N is bounded.*

Proof. Assume N is not bounded, hence $[M_N\rangle$ is infinite. We want to show first that there exist $M, M' \in [M_N\rangle$, $M < M'$, and $M' \in [M\rangle$. For this we construct inductively a tree T as follows. The root of T is M_N. For each node M_1 of T and each step $M_1 [t\rangle M_2$, M_2 becomes a son of M_1 if and only if M_2 is not already contained as a node in the path from the root M_N to the node M_1. Constructing T in this way, each reachable marking $M \in [M_N\rangle$ is a node of T because there exists a firing sequence $M_0[t_1\rangle \cdots [t_n\rangle M_n$ such that $M_0 = M_N$, $M_n = M$ and $M_i \neq M_j$ for $0 \leqslant i \neq j \leqslant n$. As $[M_N\rangle$ is assumed to be infinite, T has infinitely many nodes. Since T is finitely branched (N is finite), by König's lemma (Knuth, 1973) there exists an infinite path w in T. The markings on w are mutually distinct and if M_1 is nearer to the root M_N than M_2 then $M_2 \in [M_1\rangle$. Using Lemma 3.11, there exist two markings M, M' on this path such that $M < M'$ and $M' \in [M\rangle$.

As shown in Theorem 3.6, there is a process $p: K \to N$ with $m(p, {}^\circ K) = M$ and $m(p, K^\circ) = M'$. The idea is now to construct an infinite process p' by iterating the process p infinitely often. This is possible since $M < M'$ (cf. Fig. 17). Assume $K = (S, T; F)$ and ${}^\circ S := {}^\circ K \cap S$, $S^\circ := K^\circ \cap S$. Since $M < M'$ we have $|S^\circ| - |{}^\circ S| \geqslant 1$. Hence when iterating the process p, we shall get with each iteration at least one S-element more with an empty postset.

To make this precise, we construct inductively processes $p_n: K_n \to N$, $K_n = (S_n, T_n; F_n)$, such that p_n corresponds to iterating p n times. Let ${}^\circ S_n := {}^\circ K_n \cap S_n$, $S_n^\circ := K_n^\circ \cap S_n$. For each p_n, we shall show that $|S_n^\circ| \geqslant n$. Furthermore, we shall have $m(p_n, K_n^\circ) > m(p, {}^\circ K)$ such that the procedure may be continued.

Let $p_1 := p$. Clearly $|S_1^\circ| = |S^\circ| \geqslant 1$ and $m(p_1, K_1^\circ) = M' > M = m(p_1\ {}^\circ K_1)$. Now let p_n be constructed such that the requirements given above are satisfied.

Since $m(p_n, K_n^\circ) > m(p, {}^\circ K)$ we can assume that K_n and K have just the initial S-elements of K as common elements, i.e., $K_n \cap (K \setminus {}^\circ S) = \varnothing$ and ${}^\circ S \subseteq S_n^\circ$, in such a way that $\forall s \in {}^\circ S\ p_n(s) = p(s)$. This construction is always possible though in general not unique. Let the occurrence net K_{n+1} be given as the union of K_n and K: $K_{n+1} := (S_n \cup S, T_n \cup T; F_n \cup F)$. We then define $p_{n+1}: K_{n+1} \to N$ in the obvious way:

$$p_{n+1}(x) := \begin{cases} p_n(x) & \text{iff} \quad x \in K_n, \\ p(x) & \text{iff} \quad x \in K. \end{cases}$$

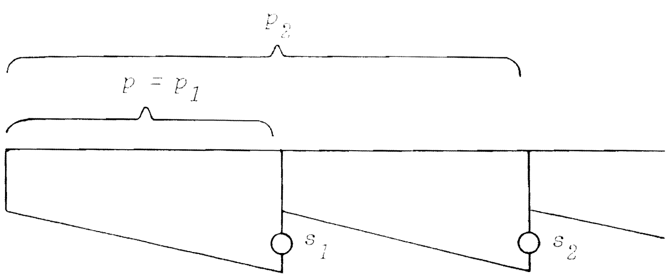

FIG. 17. The construction of p'.

Clearly p_{n+1} is a process of N and we have

$$|S^\circ_{n+1}| = |S^\circ_n| - |{}^\circ S| + |S^\circ| \geqslant n+1 \qquad \text{and}$$
$$m(p, K^\circ_{n+1}) \geqslant m(p, K^\circ) = M' > M = m(p, {}^\circ K).$$

Constructing the processes p_n in this way, we have $K_n \subseteq K_{n+1}$ and $|S^\circ_n| \geqslant n$ for all $n \in N$. With $K' := \bigcup_{n \in \mathbb{N}} K_n$, we construct the infinite process $p' : K' \to N$ in the obvious way such that, for each n, p_n is an initial subprocess of p'. Clearly K' contains infinitely many S-elements $s_1, s_2, \ldots$ with $s_i^{\bullet} = \varnothing$. They are pairwise concurrent and hence contained in some infinite slice of p'. ∎

From Theorems 3.9 and 3.12 we get:

3.13. COROLLARY. *A marked net N is bounded iff all slices of all processes of N are finite.*

We saw in Fig. 16 an unbounded net with a process containing an infinite slice. Theorem 2.7 shows that this process is not K-dense. We shall show now that the boundedness of a net can also be characterized by the K-density of its processes.

3.14. COROLLARY. *Let N be a marked net. If there is a non-K-dense process of N then N is not bounded.*

Proof. Corollaries 3.13 and 2.8. ∎

We can prove the converse of this corollary, under one restriction: We do not allow the preset of transitions to be empty. A transition with an empty preset is always enabled and produces "something out of nothing" (cf. Fig. 18).

3.15. THEOREM. *Let N be a marked net so that ${}^\circ N \subseteq S_N$. Then N is bounded $\Leftrightarrow$ all processes of N are K-dense.*

Proof. "$\Rightarrow$" Corollary 3.14.

"$\Leftarrow$" Let $p: K \to N$ be a process of N. ${}^\circ K$ is a finite cut of K, since $\forall t \in T_N\ {}^{\bullet}t \neq \varnothing$ and $M(s) \in \mathbb{N}$ for all $M \in [M_N\rangle$ and all $s \in S_N$. Using

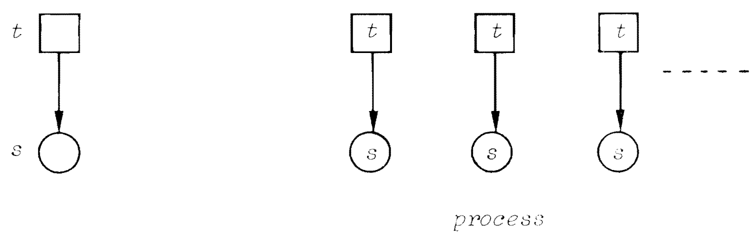

FIG. 18. All processes of the net are K-dense, but it is not bounded.

Corollary 2.12, all slices of K are finite if K is K-dense. The result follows from Corollary 3.13. ∎

(c) *Discreteness*

We remarked in Section 2 that K-density implies discreteness. However, we have seen processes which are not K-dense. Are there also processes which are not discrete? Figure 19 shows an example of such a process. By our intuitive understanding this process should not be considered as a representation of some "real process."

By Corollary 3.14 we know that non-discrete processes can only occur for unbounded nets, since K-density implies discreteness. We will show now for the class of marked nets as considered in Theorem 3.15 that a process is discrete if and only if it can be approximated by finite processes. To formalise what we mean by approximation, we use a partial order on the class of processes of a marked net, given by the notion of initial subprocess as defined in 3.8.

3.16. DEFINITION. *Let* p, p' be processes of some marked net N. $p \leqslant p' :\Leftrightarrow p$ is an initial subprocess of p'.

Next, we show that we have least upper bounds of ω-chains with respect to this partial order (similar to a construction of Winskel (unpublished manuscript)).

3.17. THEOREM. *Let N be a marked net. Let $p_0 \leqslant p_1 \leqslant \cdots \leqslant p_n \leqslant \cdots$ be an ω-chain of processes $p_n : K_n \to N$, $K_n = (S_n, T_n; F_n)$. Then the least upper*

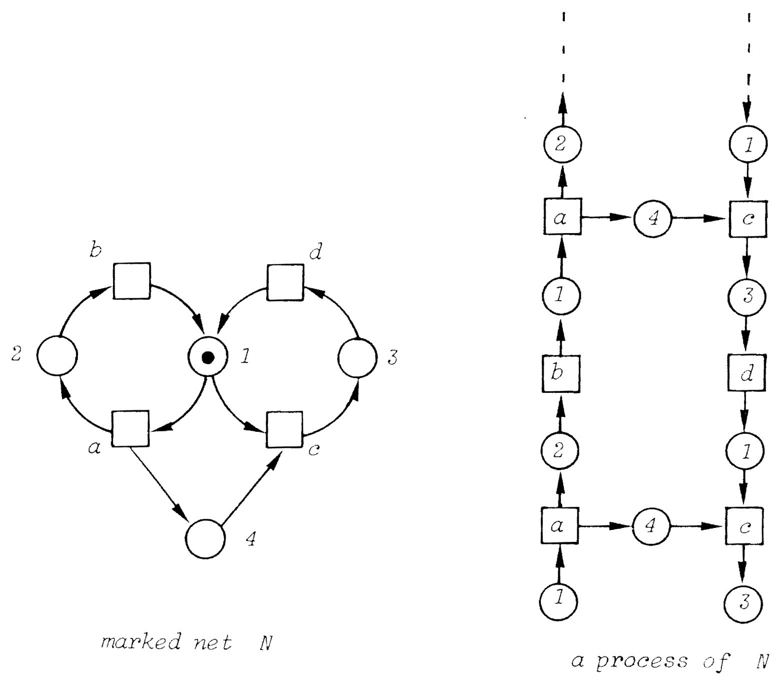

FIG. 19. A marked net with a process which is not discrete.

bound of this chain is the process $p: K \to N$, $K = (\bigcup_{n\in\omega} S_n, \bigcup_{n\in\omega} T_n; \bigcup_{n\in\omega} F_n)$, $p(x) := p_n(x)$ *for some* n *with* $x \in K_n$.

Proof. The definition of p is unique since $\forall i, j \in \mathbb{N}\ x \in K_i \wedge x \in K_j \Rightarrow p_i \leqslant p_j \vee p_j \leqslant p_i$; hence using Definition 3.8, $p_i(x) = p_j(x)$. Clearly, K is an occurrence net and p is a process and $\forall i\ p_i \leqslant p$. Assume $p': K' \to N$ such that $\forall i \in \mathbb{N}\ \ p_i \leqslant p'$. We intend to show: $p \leqslant p'$. Obviously, $K \subseteq K'$, as $x \in K \Rightarrow x \in K_i$ for some $i \Rightarrow x \in K'$ since K_i is initial subnet of K' $(K_i \leqslant K')$ for all i.

It remains to be shown that K is initial subnet of K', i.e., (according to Definition 2.15), (1) ${}^\circ K' \subseteq K$, (2) $\downarrow K \subseteq K$, ($\downarrow K$ to be understood in K'), (3) $\downarrow K^{\bullet} \cap S_{K'} \subseteq K$ ($K^{\bullet}$ to be understood in K').

(1) $x \in {}^\circ K' \Rightarrow x \in {}^\circ K_i$ for all i since $K_i \leqslant K' \Rightarrow x \in K$.

(2) If $x \in K$ and $y \in \downarrow K$, $x \in K_i$ for some i, hence $y \in K_i$ since $K_i \leqslant K'$. $y \in K_i$ follows.

(3) $s \in \downarrow K^{\bullet} \cap S_{K'} \Rightarrow \exists t \in \downarrow K \cap T_K$ with $s \in t^{\bullet} \Rightarrow t \in K_i$ for some $i \Rightarrow s \in K_i$ since $K_i \leqslant K' \Rightarrow s \in K$.

Obviously, $p'(x) = p_i(x) = p(x)$ for all $x \in K$ since $p_i \leqslant p'$ for all i. ∎

3.18. Theorem. *Let* N *be a marked net so that* ${}^\circ N \subseteq S_N$. *An infinite process* p *of* N *is discrete if and only if it is the least upper bound of some* ω-*chain of finite processes.*

Proof. "$\Rightarrow$" Let $p: K \to N$, K discrete. We construct inductively an ω-chain of processes $p_n: K_n \to N$ (using Definition 2.15)

$$M_0 := {}^\circ K,$$
$$M_{n+1} := M_n \cup M_n^{\bullet} \cup (M_n^{\bullet})^{\bullet},$$
$$K_n := K_{M_n},$$
$$p_n := p \mid K_n.$$

Clearly, all K_n are finite occurrence nets, $p_n \leqslant p_{n+1}$ for all n.

Using Theorem 2.14, all elements of K are founded and therefore $\forall x \in K$ $\exists n$ such that $x \in K_n$. Hence p is the least upper bound of this chain, according to the construction and Theorem 3.17.

"$\Leftarrow$" Let $p: K \to N$ be the least upper bound of the ω-chain $p_0 \leqslant p_1 \leqslant \cdots \leqslant p_n \leqslant \cdots$, $p_n: K_n \to N$ finite. For all $x \in K$ $\exists n$ such that $x \in K_n$. K_n is finite and therefore x is founded. Hence all $x \in K$ are founded and K is discrete (using Theorem 2.14). ∎

Based on this result, we suggest restricting the definition of process, allowing only finite processes and those infinite processes which may be approximated by finite ones as shown above.

4. Conclusion

One main concern of this paper was to introduce a notion of process for place/transition nets as a proper generalization of the definition of process for condition/event-systems. Then we showed, using this notion of process, that a place/transition net is bounded if and only if all its processes are K-dense. In particular for applications of place/transition nets as models of real systems, boundedness is an interesting notion. So our result is a further hint that K-density is a significant property of non-sequential processes. Finally we argued that non-discrete processes do not correspond to our intuition of reasonable "real" processes.

Acknowledgments

The first ideas for this paper appeared in a discussion with P. S. Thiagarajan. Eike Best contributed to this paper by many discussions and helpful suggestions. In particular, we owe to him an idea for shortening the proof of Theorem 2.11.

Received: March 3, 1983

References

Best, E. (1980a), A theorem on the characteristics of non-sequential processes, *Fund. Inform.* **3**, No. 1, 77–94.

Best, E. (1980b), The relative strength of K-density, *in* "Lecture Notes in Computer Science, Vol. 84," pp. 261–276, Springer-Verlag, Berlin/New York.

Best, E., and Merceron, A. (1983), Discreteness, K-density and D-continuity of occurrence nets, *in* "6th GI-Conference on Theoretical Computer Science, January 1983," Lecture Notes in Computer Science, Vol. 145, Springer-Verlag, Berlin/New York.

Fernandez, C., and Thiagarajan, P. S. (1982), Some properties of D-continuous causal nets, *in* "ICALP, 1982," Lecture Notes in Computer Science, Vol. 140, Springer-Verlag, Berlin/New York.

Genrich, H. J., and Stankiewicz-Wiechno, E. (1980), A dictionary of some basic notions of net theory, *in* "Advanced Course on General Net Theory of Processes and Systems, Hamburg," Lecture Notes in Computer Science, Vol. 84, Springer-Verlag, Berlin/New York.

Grabowski, J. (1979), On partial languages, Preprint No. 40/79, Sekt. Mathematik. Humboldt-Univ.

Knuth, D. E. (1973), "The Art of Computer Programming," Vol. 1, "Fundamental Algorithms," 2nd ed., Addison–Wesley, Reading, Mass.

MAZURKIEWICZ, A. (1977), "Concurrent Program Schemes and Their Interpretation," Report DAIMI PB-78, Computer Science Department, Aarhus University, Aarhus, Denmark.

NIELSEN, M., PLOTKIN, G., AND WINSKEL, G. (1981), Petri nets, event structures and domains, *Theoret. Comput. Sci.* **13**, 85–108.

PETERSON, J. L. (1981), "Petri Net Theory and the Modeling of Systems," Prentice–Hall, Englewood Cliffs, N.J.

PETRI, C. A. (1977), "Non-sequential Processes," GMD-ISF Report 77–05.

PETRI, C. A. (1980), Concurrency, *in* "Lecture Notes in Computer Science, Vol. 84," pp. 251–260, Springer-Verlag, Berlin/New York.

PLÜNNECKE, H. (1981), "Schnitte in Halbordnungen," GMD-ISF Report 81-09.

REISIG, W. (1980), Processes of marked, arc-weighted nets, *in* "Petri Nets and Related System Models," Newsletter No. 5, June.

ROZENBERG, G., AND VERREADT, R. (1983), Subset languages of Petri nets, *in* "Third European Workshop on Applications and Theory of Petri Nets," Informatik Fachberichte, Vol. 66, Springer-Verlag, Berlin/New York.

STARKE, P. (1981), Processes in Petri nets, *in* "Elektronische Informationsverarbeitung und Kybernetik," Vol. 17, Nos. 8–9.

WINKOWSKI, J. (1982), An algebraic description of system behaviours, *Theoret. Comput. Sci.* **21**, No. 3, 315–340.

WINSKEL, G. A new definition of morphism on Petri nets, unpublished manuscript.

Printed by the St. Catherine Press Ltd., Tempelhof 41, Bruges, Belgium