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The unfolding of general Petri nets*

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ABSTRACT. The unfolding of (1-)safe Petri nets to occurrence nets is well understood. There is a universal characterization of the unfolding of a safe net which is part and parcel of a coreflection from the category of occurrence nets to the category of safe nets. The unfolding of general Petri nets, nets with multiplicities on arcs whose markings are multisets of places, does not possess a directly analogous universal characterization, essentially because there is an implicit symmetry in the multiplicities of general nets, and that symmetry is not expressed in their traditional occurrence net unfoldings. In the present paper, we show how to recover a universal characterization by representing the symmetry in the behaviour of the occurrence net unfoldings of general Petri nets. We show that this is part of a coreflection between enriched categories of general Petri nets with symmetry and occurrence nets with symmetry.

1 Introduction

There is a wide array of models for concurrency. In [16], it is shown how category theory can be applied to describe the relationships between them by establishing adjunctions between their categories; the adjunctions often take the form of coreflections. This leads to uniform ways of defining constructions on models and provides links between concepts such as bisimulation in the models [5].

Only partial results have been achieved in relating Petri nets to other models for concurrency since, in general, there is no coreflection between occurrence nets and more general forms of net that allow transitions to deposit more than one token in any place or in which a place can initially hold more than one token. The reason for this, as we shall see, is that the operation of unfolding such a net to form its associated occurrence net does not account for the *symmetry* in the behaviour of the original net due to places being marked more than once. In this paper, we define the symmetry in the unfolding and use this to obtain a coreflection between general nets and occurrence nets *up to symmetry*.

Of course, there are undoubtedly several ways of adjoining symmetry to nets. The method we use was motivated by the need to extend the expressive power of event structures and the maps between them [14, 15]. Roughly, a symmetry on a Petri net is described as a relation between its runs as causal nets, the relation specifying when one run is similar to another up to symmetry; of course, if runs are to be similar then they should have similar futures as well as pasts. Technically and generally, a relation of symmetry is expressed as a span of open maps which form a pseudo equivalence.

This general algebraic method of adjoining symmetry is adopted to define symmetry in (the paths of) nets, which we use to relate the categories of general nets with symmetry and occurrence nets with symmetry. Another motivation for this work is that Petri nets provide a useful testing ground for the general method of adjoining symmetries. For example, the

*An extended version is available as a Computer Laboratory Technical Report.

present work has led us to drop the constraint in [14, 15] that the morphisms of the span should be jointly monic, in which case the span would be an equivalence rather than a pseudo equivalence. (A similar issue is encountered in the slightly simpler setting of nets without multiplicities [4].) Motivated by the categories of nets encountered, the method for adjoining symmetry is also extended to deal with more general forms of model such as those without all pullbacks.

2 Varieties of Petri nets

We begin by introducing Petri nets. It is unfortunately beyond the scope of the current paper to give anything but the essential definitions of the forms of net that we shall consider; we instead refer the reader to [9, 16] for a fuller introduction.

DEFINITION 1. *A general Petri net is a 5-tuple,*

$$G = (P, T, Pre, Post, \mathbb{M}),$$

comprising a set P of places *(or* conditions*); a set T of* transitions *(or* events*) disjoint from P; a* pre-place *multirelation, $Pre \subseteq_{\mu} T \times P$; a* post-place *$\infty$-multirelation, $Post \subseteq_{\mu_\infty} T \times P$; and a set $\mathbb{M}$ of ∞-multisets of P forming the set of* initial markings *of G. Every transition must consume at least one token:*

$$\forall t \in T\ \exists p \in P.\ Pre[t, p] > 0.$$

This is a mild generalization of the standard definition of Petri net in that we allow there to be a *set* of initial markings rather than just one initial marking, and will prove important later. In the case where a general net has precisely one initial marking, we say that the net is *singly-marked*.

A morphism of general nets embeds the structure of one net into that of another in way that preserves the token game for nets — see [13].

DEFINITION 2. *Let $G = (P, T, Pre, Post, \mathbb{M})$ and $G' = (P', T', Pre', Post', \mathbb{M}')$ be general Petri nets. A morphism $(\eta, \beta) : G \to G'$ is a pair consisting of a partial function $\eta : T \to_* T'$ and an ∞-multirelation $\beta \subseteq_{\mu_\infty} P \times P'$ which jointly satisfy:*

- *for all $M \in \mathbb{M}$: $\beta \cdot M \in \mathbb{M}'$*
- *for all $t \in T$: $\beta \cdot (Pre \cdot t) = Pre' \cdot \eta(t)$ and $\beta \cdot (Post \cdot t) = Post' \cdot \eta(t)$*

We write $\eta(t) = *$ if $\eta(t)$ is undefined and in the above requirement regard $*$ as the empty multiset, so that if $\eta(t) = *$ then $\beta \cdot (Pre \cdot t)$ and $\beta \cdot (Post \cdot t)$ are both empty.

The category of general Petri nets with multiple initial markings is denoted $\mathbf{Gen}^\sharp$, and we denote by **Gen** the category of singly-marked general nets (nets with one initial marking).

One simplification of general nets is to require that multirelations Pre and $Post$ are *relations* rather than (∞)-multirelations and that every initial marking must be a *set* of places rather than an ∞-multiset. We shall call such nets *P/T nets*. The relations Pre and $Post$ of a

P/T net may equivalently be seen as a *flow* relation $F \subseteq (P \times T) \cup (P \times T)$ describing how places and transitions are connected:

$$p\ F\ t \stackrel{\triangle}{\iff} Pre(p,t) \qquad t\ F\ p \stackrel{\triangle}{\iff} Post(t,p).$$

Any P/T net can therefore be defined as a 4-tuple $G = (P, T, F, \mathbb{M})$ by giving its flow relation. An important property that a P/T net can possess is (1-)*safety*, which means that any reachable marking is a set (*i.e.* there is no reachable marking that has more than one token in any place) — we say that a marking is *reachable* if it can be reached by any sequence of transitions from any initial marking according to the standard token game for nets.

Safe nets can be refined further to obtain *occurrence nets*.

DEFINITION 3. *An* occurrence net $O = (B, E, F, \mathbb{M})$ *is a safe net satisfying the following restrictions:*

1. $\forall M \in \mathbb{M} : \forall b \in M : (Pre \cdot b = \emptyset)$
2. $\forall b' \in B : \exists M \in \mathbb{M} : \exists b \in M : (b\ F^*\ b')$
3. $\forall b \in B : (|Pre \cdot b| \leq 1)$
4. F^+ *is irreflexive and, for all* $e \in E$, *the set* $\{e' \mid e'\ F^*\ e\}$ *is finite*
5. *# is irreflexive, where*

$$\begin{array}{rcl} e\#_m e' & \iff & e \in E \;\&\; e' \in E \;\&\; e \neq e' \;\&\; Pre \cdot e \cap Pre \cdot e' \neq \emptyset \\ b\#_m b' & \iff & \exists M, M' \in \mathbb{M} : (M \neq M' \;\&\; b \in M \;\&\; b' \in M') \\ x\#x' & \iff & \exists y, y' \in E \cup B : y\#_m y' \;\&\; y\ F^*\ x \;\&\; y'\ F^*\ x' \end{array}$$

Singly-marked occurrence nets can be seen to coincide with the original definition of occurrence net [8].

By ensuring that any condition occurs as the postcondition of at most one event, the constraints above allow the flow relation F to be seen to represent causal dependency. Since the flow relation is required to be irreflexive, as is the *conflict* relation #, every condition can occur in some reachable marking and every event can take place in some reachable marking. Two elements of the occurrence net are in conflict if the occurrence of one precludes the occurrence of the other at any later stage.

The *concurrency* relation $\mathrm{co}_O \subseteq (B \cup E) \times (B \cup E)$, indicating that two elements of the occurrence net are concurrent (may occur at the same time in some reachable marking) if they neither causally depend on nor conflict with each other, is defined as:

$$x\ \mathrm{co}_O\ y \stackrel{\triangle}{\iff} \neg(x\#y \text{ or } x\ F^+\ y \text{ or } y\ F^+\ x)$$

We often drop the subscript O and write co for the relation. The concurrency relation is extended to sets of conditions A in the following manner:

$$\mathrm{co}\ A \stackrel{\triangle}{\iff} (\forall b, b' \in A :\ b\ \mathrm{co}\ b') \text{ and } \{e \in E \mid \exists b \in A. e\ F^*\ b\} \text{ is finite}$$

The final class of net that we shall make use of is *causal* nets. These are well-known representations of paths of general nets, recording how a set of consistent events (events that do not conflict) causally depend on each other through the encountered markings of conditions.

DEFINITION 4. *A causal net* $C = (B, E, F, \mathbb{M})$ *is an occurrence net with at most one initial marking for which the conflict relation* $\#$ *is empty.*

2.1 Unfolding

Occurrence nets can be used to give the semantics of more general forms of net. The process of forming the occurrence net semantics of a net is called *unfolding*, first defined for safe nets in [8]. The result of unfolding a net G is an occurrence net $\mathcal{U}(G)$ accompanied by a morphism $\varepsilon_G : \mathcal{U}(G) \to G$ relating the unfolding back to the original net.

For a safe net N, we are able to say that the occurrence net $\mathcal{U}(N)$ and morphism $\varepsilon_N : \mathcal{U}(N) \to N$ are *cofree*. That is, for any occurrence net O and morphism $(\pi, \gamma) : O \to N$, there is a *unique* morphism $(\theta, \alpha) : O \to \mathcal{U}(N)$ such that the following triangle commutes:

$$\begin{tikzcd} \mathcal{U}(N) \arrow[r, "\varepsilon_N"] & N \\ O \arrow[u, "{(\theta,\alpha)}"] \arrow[ur, "{(\pi,\gamma)}"'] & \end{tikzcd}$$

This result, first shown in [12] (for singly-marked nets; the generalization to multiply-marked nets is straightforward), ensures that $\mathbf{Occ}^\sharp$ is a coreflective subcategory of the category of safe nets, the operation of unfolding giving rise to a functor that is right-adjoint to the obvious inclusion functor. In fact, the result also applies to give a coreflection between occurrence nets and P/T nets and, more generally still, to give a coreflection between occurrence nets and nets with single multiplicity in the post-places of each transition and that have at most one token in each place in their initial markings, as shown in [6].

A coreflection is not, however, obtained when we consider the unfoldings of arbitrary general nets (either singly- or multiply-marked). The problem does not lie in defining the unfolding of general nets, which is characterized as follows:

PROPOSITION 5. *The unfolding* $\mathcal{U}(G) = (B, E, F, \mathbb{M}_0)$ *of* $G = (P, T, Pre, Post, \mathbb{M})$ *is the unique occurrence net to satisfy*

$$\begin{aligned} B &= \{(M, p, i) \mid M \in \mathbb{M} \;\&\; p \in P \;\&\; 0 \le i < M[p]\} \\ &\quad \cup \;\; \{(\{e\}, p, i) \mid e \in E \;\&\; p \in P \;\&\; 0 \le i < (Post \cdot \eta(e))[p]\} \\ E &= \{(A, t) \mid A \subseteq B \;\&\; t \in T \;\&\; \mathrm{co}\, A \;\&\; \beta \cdot A = Pre \cdot t\} \\ & b\, F\, (A, t) \iff b \in A \\ & (A, t)\, F\, b \iff \exists p, i : (b = (\{(A, t)\}, p, i)) \\ \mathbb{M}_0 &= \{\{(M, p, i) \mid (M, p, i) \in B\} \mid M \in \mathbb{M}\}, \end{aligned}$$

where co *and* $\#$ *are the concurrency and conflict relations arising from* F *on* B *and* E. *Furthermore,* $\eta : E \to P$ *defined as* $\eta(A, t) = t$ *and* $\beta : B \to P$ *defined as* $\beta(X, p, i) = p$ *form a morphism* $\varepsilon_G = (\eta, \beta) : \mathcal{U}(G) \to G$ *in* $\mathbf{Gen}^\sharp$*, regarding the function* β *as a multirelation.*

The reason why we do not obtain a coreflection between the categories $\mathbf{Occ}^\sharp$ and $\mathbf{Gen}^\sharp$ (or $\mathbf{Occ}$ and $\mathbf{Gen}$) is that the uniqueness property required for cofreeness fails. That is,

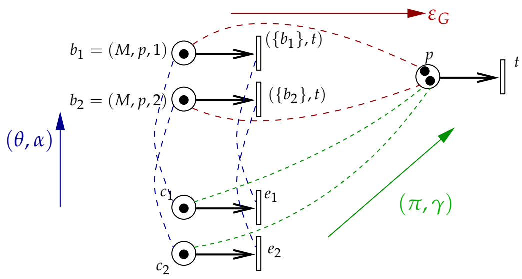

Figure 1: Non-uniqueness of mediating morphism (all multiplicities 1)

the morphism (θ, α) need not be the *unique* such morphism making the diagram above commute. In Figure 1, we present a general net G, its unfolding $\mathcal{U}(G)$ with morphism ε_G and an occurrence net O (which happens to be isomorphic to $\mathcal{U}(G)$) with morphism $(\pi, \gamma) : O \to G$ alongside two distinct morphisms $(\theta, \alpha), (\theta', \alpha') : O \to \mathcal{U}(G)$ making the diagram commute.

In the net $\mathcal{U}(G)$ in Figure 1, the two conditions b_1 and b_2 are symmetric: they arise from there being two indistinguishable tokens in the initial marking of G in the place p. The events $(\{b_1\}, t)$ and $(\{b_2\}, t)$ are also symmetric since they are only distinguished by their symmetric pre-conditions; they have common image under ε_G. Our goal shall be to show that there is a unique mediating morphism *up to symmetry*, *i.e.* any two morphisms from O to $\mathcal{U}(G)$ making the diagram commute are only distinguished through their choice of symmetric elements of the unfolding. We first summarize the part of the cofreeness property that does hold.

THEOREM 6. *Let G be a general Petri net, O be an occurrence net and $(\pi, \gamma) : O \to G$ be a morphism in* $\mathbf{Gen}^\sharp$*. There is a morphism $(\theta, \alpha) : O \to \mathcal{U}(G)$ in* $\mathbf{Gen}^\sharp$ *such that the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{U}(G) \arrow[r, "(\eta,\beta)=\varepsilon_G"] & G \\
O \arrow[u, "(\theta,\alpha)"] \arrow[ur, "(\pi,\gamma)"'] &
\end{tikzcd}$$

Furthermore, if the net G is a P/T net then (θ, α) is the unique such morphism.

It will be of use later to note that if the multirelation γ above is a function then so is α.

2.2 Pullbacks

The framework for defining symmetry in general nets, to be described in the next section, will require a subcategory which has pullbacks. Whereas it was shown in [3] that the category of singly-marked safe nets has pullbacks, the category of singly-marked general nets does not. Roughly, this is for two reasons: the category with multirelations as morphisms does not have pullbacks; and allowing only singly-marked nets obstructs the existence of pullbacks. It is the latter obstruction that led us to the earlier relaxation of the definition of nets, to permit them to have a set of initial markings rather than precisely one initial marking. To obtain a category of general nets with pullbacks, we restrict attention to *folding* morphisms between general nets (with multiple initial markings):

DEFINITION 7. *A morphism $(\eta, \beta) : G \to G'$ is a* folding *if both η and β are total functions.*

Denote the category of general nets with folding morphisms $\mathbf{Gen}^\sharp_{\mathrm{f}}$, its full subcategory of occurrence nets $\mathbf{Occ}^\sharp_{\mathrm{f}}$, and the full subcategory of causal nets $\mathbf{Caus}^\sharp_{\mathrm{f}}$.

PROPOSITION 8. *The category $\mathbf{Gen}^\sharp_{\mathrm{f}}$ has pullbacks.*

The category $\mathbf{Occ}^\sharp$ has pullbacks, though we will only need pullbacks of folding morphisms. Pullbacks in $\mathbf{Occ}^\sharp_{\mathrm{f}}$ are obtained by taking the corresponding pullbacks in $\mathbf{Gen}^\sharp_{\mathrm{f}}$. The following lemma expresses how pullbacks in subcategories with folding morphisms are not disturbed in moving to larger categories with all morphisms, though in the case of general nets we have to settle for them becoming weak pullbacks.†

LEMMA 9. *(i) The inclusion functor $\mathbf{Occ}^\sharp_{\mathrm{f}} \hookrightarrow \mathbf{Occ}^\sharp$ preserves pullbacks.*
(ii) The inclusion functor $\mathbf{Occ}^\sharp_{\mathrm{f}} \hookrightarrow \mathbf{Gen}^\sharp_{\mathrm{f}}$ preserves pullbacks.
(iii) The inclusion functor $\mathbf{Gen}^\sharp_{\mathrm{f}} \hookrightarrow \mathbf{Gen}^\sharp$ preserves weak pullbacks.

3 Categories with symmetry

It is shown in [14] how *symmetry* can be defined between the paths of event structures, and more generally on any category of models satisfying certain properties. The absence of pullbacks in the category $\mathbf{Gen}^\sharp$ obliges us to extend the method when introducing symmetry to general nets and their unfoldings.

The definition of symmetry makes use of *open* morphisms [5]. Let $\mathcal{C}_0$ be a category (typically a category of models such as Petri nets) with a distinguished subcategory $\mathcal{P}$ of path objects (such as causal nets), to describe the shape of computation paths, and morphisms specifying how a path extends to another. A morphism $f : X \to Y$ in $\mathcal{C}_0$ is *$\mathcal{P}$-open* if, for any morphism $s : P \to Q$ in $\mathcal{P}$ and morphisms $p : P \to X$ and $q : Q \to Y$, if the diagram on the left commutes, *i.e.* $f \circ p = q \circ s$, then then there is a morphism $h : Q \to X$ such that the diagram on the right commutes, *i.e.* $h \circ s = p$ and $f \circ h = q$:

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "s"'] & X \arrow[d, "f"] \\ Q \arrow[r, "q"'] & Y \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, "p"] \arrow[d, "s"'] & X \arrow[d, "f"] \\ Q \arrow[r, "q"'] \arrow[ur, "h"] & Y \end{tikzcd}$$

The path-lifting property expresses that via f any extension of a path in Y can be matched by an extension in X, and captures those morphisms f which are bisimulations, though understood generally with respect to a form of path specified by $\mathcal{P}$. It can be shown purely

†Recall a *weak* pullback is defined in a similar way to a pullback, but without insisting on uniqueness of the mediating morphism.

diagrammatically that open morphisms compose, and therefore form a subcategory, and are preserved under pullbacks in $\mathcal{C}_0$.

Assume categories

$$\mathcal{P} \subseteq \mathcal{C}_0 \subseteq \mathcal{C}$$

where $\mathcal{P}$ is a distinguished subcategory of path objects and path morphisms, $\mathcal{C}_0$ has pullbacks and shares the same objects as the (possibly larger) category $\mathcal{C}$, with the restriction that the inclusion functor $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ preserves weak pullbacks. Then, we will be able to add symmetry to $\mathcal{C}$, and at the same time maintain constructions dependent on pullbacks of open morphisms which will be central to constructing symmetries on unfoldings.‡ (The earlier method for introducing symmetry used in [14] corresponds to the situation where $\mathcal{C}_0$ and $\mathcal{C}$ coincide.)

The role of $\mathcal{P} \subseteq \mathcal{C}_0$ is to determine open morphisms; the role of the subcategory $\mathcal{P}$ is to specify the form of path objects and extension, while the, generally larger, category $\mathcal{C}_0$ fixes the form of paths $p : P \to C$ from a path object P in an object C of $\mathcal{C}_0$. Now, just as earlier, we can define open morphisms in $\mathcal{C}_0$, and so by definition those in $\mathcal{C}$.

Now we show how $\mathcal{C}$ can be extended with symmetry to yield a category $\mathcal{SC}$. The objects of $\mathcal{SC}$ are tuples (X, S, l, r) consisting of an object X of $\mathcal{C}$ and two $\mathcal{P}$-open morphisms $l, r : S \to X$ in $\mathcal{C}_0$ which make l, r a pseudo equivalence [1] in the category $\mathcal{C}$ (see Appendix A). The requirements on l and r are slightly weaker than those in [14] in that we do not require that the morphisms l and r are jointly monic.§

The morphisms of $\mathcal{SC}$ are morphisms of $\mathcal{C}$ that *preserve symmetry*. Let $f : X \to X'$ be a morphism in $\mathcal{C}$ and (X, S, l, r) and (X', S', l', r') be objects of $\mathcal{SC}$. The morphism $f : X \to X'$ preserves symmetry if there is a morphism $h : S \to S'$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[d, "f"'] & S \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, dotted, "h"] & X \arrow[d, "f"] \\ X' & S' \arrow[l, "l'"] \arrow[r, "r'"'] & X' \end{tikzcd}$$

With the definition of symmetry on objects, we can define the equivalence relation $\sim$ expressing when morphisms are *equal up to symmetry*:
Let $f, g : (X, S, l, r) \to (X', S', l', r')$ be morphisms in $\mathcal{SC}$. Define $f \sim g$ iff there is a morphism $h : X \to X'$ in $\mathcal{C}$ such that following diagram commutes in $\mathcal{C}$:

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[d, dotted, "h"] \arrow[dr, "g"] & \\ X' & S' \arrow[l, "l'"'] \arrow[r, "r'"] & X' \end{tikzcd}$$

Composition of morphisms in $\mathcal{SC}$ coincides with composition in $\mathcal{C}$ and the two categories share the same identity morphisms. The category $\mathcal{SC}$ is more fully described as a category enriched in equivalence relations.

‡We have chosen general conditions that work for our purposes here. It might become useful to replace the role of $\mathcal{P} \subseteq \mathcal{C}_0$ by an axiomatization of a subcategory of open morphisms in $\mathcal{C}$ and in this way broaden the class of situations in which we can adjoin symmetry.

§See [4] for an example of a symmetry on a safe net that cannot be expressed with the jointly-monic condition.

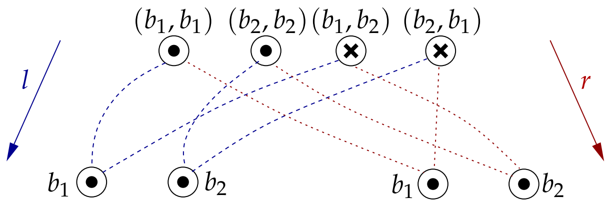

Figure 2: Symmetry in a net with two places

For nets, a reasonable choice for the paths $\mathcal{P}$ would be $\mathbf{Caus}^\sharp_\mathrm{f}$, taking path objects to be causal nets and expressing path extensions by foldings between them. (There are other possibilities, say restricting to finite causal nets, or the causal nets associated with finite elementary event structures, which would lead to less refined equivalences up to symmetry.) The categories $\mathbf{Caus}^\sharp_\mathrm{f} \subseteq \mathbf{Gen}^\sharp_\mathrm{f} \subseteq \mathbf{Gen}^\sharp$ meet the requirements needed to construct $\mathcal{S}\mathbf{Gen}^\sharp$ — in particular by Lemma 9 (iii), so adjoining symmetry to general nets. The requirements are also met by $\mathbf{Caus}^\sharp_\mathrm{f} \subseteq \mathbf{Occ}^\sharp_\mathrm{f} \subseteq \mathbf{Occ}^\sharp$ yielding $\mathcal{S}\mathbf{Occ}^\sharp$ (this time using Lemma 9 (ii)).

We remark that a folding morphism between general nets is $\mathbf{Caus}^\sharp_\mathrm{f}$-open in $\mathbf{Gen}^\sharp_\mathrm{f}$ iff it is $\mathbf{Caus}^\sharp$-open in $\mathbf{Gen}^\sharp$, and a folding morphism between occurrence nets is $\mathbf{Caus}^\sharp_\mathrm{f}$-open in $\mathbf{Occ}^\sharp_\mathrm{f}$ iff it is $\mathbf{Caus}^\sharp$-open in $\mathbf{Occ}^\sharp$.

4 Symmetry in unfolding

In Section 2.1, we showed how a general Petri net may be unfolded to form an occurrence net. This was shown not to yield a coreflection due to the mediating morphism not necessarily being unique. The key observation was that uniqueness might be obtained by regarding the net up to the evident symmetry between paths in the unfolding. This led us to define a category of general nets with symmetry. To give an example of the forms of symmetry that can be expressed, consider the simple net with two places, b_1 and b_2, both initially marked once. Suppose that we wish to express that the two places are symmetric; for instance, the net might be thought of as the unfolding of the general net with a single place initially marked twice. The span to express that symmetry is presented in Figure 2. Without our extension of the definition of net to allow multiple initial markings, this simple symmetry would be inexpressible. This accompanies the fact that the category of singly-marked general nets (even when restricted to folding morphisms) does not have pullbacks.

In general, the symmetry in an unfolding is obtained by unfolding the *kernel* of the morphism $\varepsilon_G : \mathcal{U}(G) \to G$, which is the pullback of ε_G against itself in $\mathbf{Gen}^\sharp_\mathrm{f}$:

$$\begin{tikzcd} S \arrow[r, "r"] \arrow[d, "l"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{U}(G) \arrow[d, "\varepsilon_G"] \\ \mathcal{U}(G) \arrow[r, "\varepsilon_G"'] & G \end{tikzcd}$$

To see that $(\mathcal{U}(G), \mathcal{U}(S), l \circ \varepsilon_S, r \circ \varepsilon_S)$ is a symmetry, we must show that the morphisms $l \circ \varepsilon_S$ and $r \circ \varepsilon_S$ are $\mathbf{Caus}^\sharp_\mathrm{f}$-open and form a pseudo equivalence. The latter point follows a purely diagrammatic argument. Open morphisms from occurrence nets into general nets can be characterized in the following way:

PROPOSITION 10. *Let O be an occurrence net and G be a general net. A morphism $f : O \to G$ is $\mathbf{Caus}^\sharp_\mathrm{f}$-open in $\mathbf{Gen}^\sharp_\mathrm{f}$ if, and only if, it reflects any initial marking of G to an initial marking of O and satisfies the following property:*

> *for any subset A of conditions of O such that* co A *for which there exists a transition t of G such that $f \cdot A = Pre_G \cdot t$, there exists an event e of O such that $A = Pre_O \cdot e$ and $f(e) = t$.*

The morphism $\varepsilon_G : \mathcal{U}(G) \to G$ of Proposition 5 is readily seen to satisfy this property for any G, and is therefore $\mathbf{Caus}^\sharp_\mathrm{f}$-open. The pullback of open morphisms is open [5] so the morphisms l and r are $\mathbf{Caus}^\sharp_\mathrm{f}$-open, and therefore $l \circ \varepsilon_S$ and $r \circ \varepsilon_S$ are both open since open morphisms compose to form open morphisms [5]. Note that a morphism between occurrence nets is $\mathbf{Caus}^\sharp_\mathrm{f}$-open in $\mathbf{Occ}^\sharp_\mathrm{f}$ iff it is $\mathbf{Caus}^\sharp_\mathrm{f}$-open in $\mathbf{Gen}^\sharp_\mathrm{f}$.

PROPOSITION 11. *The tuple $(\mathcal{U}(G), \mathcal{U}(S), l \circ \varepsilon_S, r \circ \varepsilon_S)$ is an occurrence net with symmetry.*

With the symmetry on $\mathcal{U}(G)$ at our disposal, we obtain the equivalence relation $\sim$ on morphisms from any occurrence net to $\mathcal{U}(G)$. This is used to extend Theorem 6 to obtain cofreeness 'up to symmetry'.

THEOREM 12. *Let G be a general Petri net and O be an occurrence net. For any morphism $(\pi, \gamma) : O \to G$ in $\mathbf{Gen}^\sharp$, there is a morphism $(\theta, \alpha) : O \to \mathcal{U}(G)$ in $\mathbf{Gen}^\sharp$ such that*

$$\begin{tikzcd} \mathcal{U}(G) \arrow[r, "\varepsilon_G"] & G \\ O \arrow[u, "(\theta,\alpha)"] \arrow[ur, "(\pi,\gamma)"'] & \end{tikzcd}$$

commutes, i.e. $\varepsilon_G \circ (\theta, \alpha) = (\pi, \gamma)$. Furthermore, any morphism $(\theta', \alpha') : O \to \mathcal{U}(G)$ in $\mathbf{Gen}^\sharp$ such that $\varepsilon_G \circ (\theta', \alpha') = (\pi, \gamma)$ satisfies $(\theta, \alpha) \sim (\theta', \alpha')$ with respect to the symmetry (S, l, r) on $\mathcal{U}(G)$ defined above (and the identity symmetry on O).

5 A coreflection up to symmetry

We show how the results of the last section are part of a more general coreflection from occurrence nets *with symmetry* to general nets *with symmetry*. In the last section, we showed how to unfold a general net to an occurrence net with symmetry. For the coreflection, we need to extend this construction to unfold general nets themselves with symmetry.

To show that the 'inclusion' $I : \mathcal{S}\mathbf{Occ}^\sharp \to \mathcal{S}\mathbf{Gen}^\sharp$ taking an occurrence net with symmetry (O, S, l, r) to a general net with symmetry is a functor, it is necessary to show that the transitivity property holds of the symmetry in $\mathcal{S}\mathbf{Gen}^\sharp$. For this it is important that pullbacks are not disturbed in moving from $\mathbf{Occ}^\sharp_\mathrm{f}$ to the larger category $\mathbf{Gen}^\sharp_\mathrm{f}$, as is assured by Lemma 9.

We now have a functor $I : \mathcal{S}\mathbf{Occ}^\sharp \to \mathcal{S}\mathbf{Gen}^\sharp$, respecting $\sim$, regarding an occurrence net with symmetry (O, S, l, r) itself directly as a general net with symmetry.

It remains for us to define the unfolding operation on objects of the category of general nets with symmetry. Its extension to a *pseudo* functor will follow from the biadjunction. Let (G, S_G, l, r) be a general net with symmetry. Let $\varepsilon_G : \mathcal{U}(G) \to G$ be the folding morphism given earlier in Proposition 5. It is open by Proposition 10. The general net (G, S_G, l, r) is 'unfolded' to the occurrence net with symmetry $\mathcal{U}(G, S_G, l, r) = (\mathcal{U}(G), S_0, l_0, r_0)$; its symmetry, $S_0 \triangleq \mathcal{U}(S')$, $l_0 \triangleq l' \circ \varepsilon_{S'}$ and $r_0 \triangleq r' \circ \varepsilon_{S'}$, is given by unfolding the *inverse image* S', l', r' of the symmetry in G along the open morphism $\varepsilon_G : \mathcal{U}(G) \to G$:

$$\begin{tikzcd}
 & & \mathcal{U}(S') \arrow[d, "\varepsilon_{S'}"] & & \\
 & & S' \arrow[dl] \arrow[dr] \arrow[ddll, bend right, "l'"'] \arrow[ddrr, bend left, "r'"] \arrow[dd, phantom, "\vee"] & & \\
 & \cdot \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee"] & & \cdot \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee"] & \\
\mathcal{U}(G) \arrow[dr, "\varepsilon_G"'] & & S_G \arrow[dl, "l"] \arrow[dr, "r"'] & & \mathcal{U}(G) \arrow[dl, "\varepsilon_G"] \\
 & G & & G &
\end{tikzcd}$$

The pullbacks are in $\mathbf{Gen}_{\mathrm{f}}^{\sharp}$. The diagram makes clear that ε_G is a morphism preserving symmetry.

The construction of the symmetry above depends crucially on the existence of pullbacks in $\mathcal{C}_0$ and the property that pullbacks of open morphisms are open (here weak pullbacks do not suffice) — without this we would not know that l' and r' were open.

Now that we have the inclusion $I : \mathcal{S}\mathbf{Gen}^{\sharp} \to \mathcal{S}\mathbf{Occ}^{\sharp}$ and the operation of unfolding a general net with symmetry, we are able to generalize Theorem 6 to give a cofreeness result:

THEOREM 13. *Let $\widehat{G} = (G, S_G, l_G, r_G)$ be a general net with symmetry and $\widehat{O} = (O, S_O, l_O, r_O)$ be an occurrence net with symmetry. For any $(\pi, \gamma) : \widehat{O} \to \widehat{G}$ in $\mathcal{S}\mathbf{Gen}^{\sharp}$, there is a morphism $(\theta, \alpha) : \widehat{O} \to \mathcal{U}(\widehat{G})$ in $\mathcal{S}\mathbf{Gen}^{\sharp}$ such that the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{U}(\widehat{G}) \arrow[r, "\varepsilon_G"] & \widehat{G} \\
\widehat{O} \arrow[u, "(\theta,\alpha)"] \arrow[ur, "(\pi,\gamma)"'] &
\end{tikzcd}$$

Furthermore, (θ, α) is unique up to symmetry: any $(\theta', \alpha') : \widehat{O} \to \mathcal{U}(\widehat{G})$ such that $\varepsilon_{\widehat{G}} \circ (\theta', \alpha') \sim (\pi, \gamma)$ satisfies $(\theta, \alpha) \sim (\theta', \alpha')$.

Technically, we have a biadjunction from $\mathcal{S}\mathbf{Occ}^{\sharp}$ to $\mathcal{S}\mathbf{Gen}^{\sharp}$ with I left biadjoint to $\mathcal{U}$ (which extends to a pseudo functor). Its counit is ε and its unit is a natural isomorphism $\widehat{O} \cong \mathcal{U}(\widehat{O})$. In this sense, we have established a coreflection from $\mathcal{S}\mathbf{Occ}^{\sharp}$ to $\mathcal{S}\mathbf{Gen}^{\sharp}$ up to symmetry.

6 Conclusion

Occurrence nets were first introduced in [8] together with the operation of unfolding singly-marked safe nets. The coreflection between occurrence nets and safe nets was first shown in [11]. A number of attempts have been made since then to characterize the unfoldings of more general forms of net.

Engelfriet defines the unfolding of (singly-marked) P/T nets in [2]. Rather than giving a coreflection between the categories, the unfolding is characterized as the greatest element of a complete lattice of occurrence nets embedding into the P/T net.

A coreflection between a subcategory of (singly-marked) general nets and a category of embellished forms of transition system is given in [7]. There, the restriction to particular kinds of net morphism is of critical importance; taking the more general morphisms of general Petri nets presented here would have resulted in the cofreeness property failing for an analogous reason to the failure of cofreeness of the unfolding of general nets to occurrence nets without symmetry.

An adjunction between a subcategory of singly-marked general nets and the category of occurrence nets is given in [6]. The restriction imposed on the morphisms of general nets there, however, precludes in general there being a morphism from $\mathcal{U}(G)$ to G in their category of general nets if $\mathcal{U}(G)$, the occurrence net unfolding of G, is regarded directly as a general net. To obtain an adjunction, the functor from the category of occurrence nets into the category of general nets is not regarded as the direct inclusion, but instead occurs through a rather detailed construction and does not yield a coreflection apart from when restricted to the subcategory of semi-weighted nets.

In this paper, we have shown that there is an implicit symmetry between paths in the unfolding of a general net arising from multiplicities in its initial marking and multiplicities on arcs from its transitions. By placing this symmetry on the unfolding, extending the scheme in [14], we are able to obtain its cofreeness up to symmetry, thus characterizing the unfolding up to the symmetry. We then adjoin symmetry to the categories of general nets and occurrence nets (using the standard definition of net morphism) to obtain a coreflection up to symmetry.

It is becoming clear from this and other work [10] that sometimes, in adjoining symmetry, models do not fit the simple scheme outlined in [14] appropriate to event structures and stable families. For example, the category of general nets with *all* morphisms does not have pullbacks as is required for the scheme in [14]. Alongside [10], the consideration of how symmetry may be placed on nets here and in [4] has suggested that we allow more liberal axioms on categories of models which enable their extension with symmetry.

The generalization of nets presented here to allow them to have more than one initial marking is also necessary for equipping other, less general, forms of net, such as safe nets or occurrence nets, with symmetry. In the companion paper [4], we extend the existing coreflection between singly-marked occurrence nets and P/T nets to this setting and show that this yields a coreflection between occurrence nets with symmetry and P/T nets with symmetry. In [4], we exhibit coreflections between event structures and multiply-marked occurrence nets.

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A Pseudo equivalences

Assume a category $\mathcal{C}$. Let $l, r : S \to G$ be a pair of morphisms in $\mathcal{C}$. They form a *pseudo equivalence* (and if jointly monic, an *equivalence*) iff there exist morphisms ρ, σ and τ such that the following diagrams commute, for some weak pullback Q, f, g of l against r:

Reflexivity

$$\begin{tikzcd} & G \arrow[dl, "\mathrm{id}_G"'] \arrow[d, dashed, "\rho"] \arrow[dr, "\mathrm{id}_G"] & \\ G & S \arrow[l, "l"] \arrow[r, "r"'] & G \end{tikzcd}$$

Symmetry

$$\begin{tikzcd} & S \arrow[dl, "r"'] \arrow[d, dashed, "\sigma"] \arrow[dr, "l"] & \\ G & S \arrow[l, "l"] \arrow[r, "r"'] & G \end{tikzcd}$$

Transitivity

$$\begin{tikzcd} & & Q \arrow[dl, "f"'] \arrow[d, dashed, "\tau"] \arrow[dr, "g"] & & \\ & S \arrow[dl, "l"'] \arrow[dr, "r"] & S \arrow[dll, "l"] \arrow[drr, "r"] & S \arrow[dl, "l"] \arrow[dr, "r"] & \\ G & & G & & G \end{tikzcd}$$