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The Generation and Decay of Vorticity

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Vorticity, although not the primary variable of fluid dynamics, is an important derived variable playing both mathematical and physical roles in the solution and understanding of problems. The following treatment discusses the generation of vorticity at rigid boundaries and its subsequent decay. It is intended to provide a consistent and very broadly applicable framework within which a wide range of questions can be answered explicitly. The rate of generation of vorticity is shown to be the relative tangential acceleration of fluid and boundary without taking viscosity into account and the generating mechanism therefore involves the tangential pressure gradient within the fluid and the external acceleration of the boundary only. The mechanism is inviscid in nature and independent of the no-slip condition at the boundary, although viscous diffusion acts immediately after generation to spread vorticity outward from boundaries. Vorticity diffuses neither out of boundaries nor into them, and the only means of decay is by cross-diffusive annihilation within the fluid.

1. INTRODUCTION

The Helmholtz vorticity equation for an incompressible homogeneous fluid,

$$\partial\boldsymbol{\omega}/\partial t+(\mathbf{v}\cdot\nabla)\boldsymbol{\omega}=(\boldsymbol{\omega}\cdot\nabla)\mathbf{v}+\nu\nabla^2\boldsymbol{\omega},$$

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includes the processing term $(\boldsymbol{\omega}\cdot\nabla)\mathbf{v}$ which describes the effects of local amplification (or concentration) of vorticity by vortex filament stretching and local turning of filaments, and the term $\nu\nabla^2\boldsymbol{\omega}$ representing the spread of vorticity due to viscosity, where $\boldsymbol{\omega}=(\xi,\eta,\zeta)$ is the vorticity. It contains no true generation term that would correspond with the creation of fresh vorticity where none existed before, and it has long been recognized that all sources of vorticity in homogeneous fluids must lie at the boundaries of fluid regions. Vorticity may be generated at interior points of inhomogeneous fluids, but is also generated at their boundaries, presumably by the same mechanism as operates in homogeneous fluids. We may therefore restrict the following discussion of the generation of vorticity at boundaries to the case of homogeneous fluids, although we shall later show that the same generation mechanism operates universally.

The complete determination of a vorticity field requires boundary conditions as well as a differential equation: for example, the Helmholtz equation for the Rayleigh problem (Section 4.1) of a semi-infinite region of fluid bounded by a plane boundary set impulsively in motion with steady velocity in its own plane is represented by the reduced equation

$$\partial\boldsymbol{\omega}/\partial t=\nu\nabla^2\boldsymbol{\omega},$$

and to obtain a unique solution we require a condition on the creation of vorticity at the boundary. We find a curious situation here, for although most of the many texts on fluid mechanics at least introduce the Helmholtz vorticity equation, very few so much as mention boundary conditions. One of the very few exceptions is Batchelor (1967, p. 280) who notes that the boundary condition on vorticity is provided in effect by the no-slip condition, though this seems scarcely a satisfactory condition on vorticity which is a physically distinct quantity with different dimensions from velocity. Moreover, boundaries in homogeneous fluids are the source of all vorticity, and we shall clearly need to consider what are the appropriate boundary conditions.

Some authors have suggested that the generation of vorticity in a region of homogeneous fluid is related to its diffusion out of solid boundaries, and its decay to diffusion into other boundaries. If, however, vorticity is a physical entity relating to fluid rotation, we may reasonably ask what is the physical effect on a boundary suffering continuous loss of vorticity, and what on a boundary steadily gaining vorticity? Plane Poiseuille flow has been interpreted as a steady motion resulting from the generation of vorticity at one boundary and its equal loss at the other; and as vorticity provides a measure of rotation in a fluid we might look for an effect of torque on an experimental channel, although none has been reported! Moreover, a mere reversal of the coordinate frame would interchange the boundary of generation and that of loss of vorticity. There can be no doubt that vorticity is a genuine physical entity, corresponding with the particular constituent of fluid motion in the neighborhood of a point associated instantaneously with rotation about an axis through that point. We shall, therefore, need to consider more critically the notion that vorticity may diffuse out of or into boundaries. One certain physical effect at boundaries is the wall stress exerted by moving fluid and the equal and opposite tangential stress exerted by the wall on nearby moving particles of fluid. In rigid body dynamics tangential forces exert torques which generate angular acceleration, and we must determine whether wall stress generates vorticity and indeed what roles are played by torque and angular momentum in fluid dynamics.

We recall that fluid dynamics is a branch of mechanics and that fluid motion is fully represented by the Navier–Stokes equation (a form of Newton's equation of motion), together with a continuity equation to keep track of mass and an energy equation if heat is important in addition to mechanical energy. All flow problems can, in principle if seldom in practice, be solved from these equations without introducing vorticity; but although we do not need vorticity for the description of fluid motions it satisfies approximately a number of simple and far-reaching conservation relationships and provides a powerful alternative physical basis for the discussion of complex three-dimensional flows. One further advantage of the use of vorticity which is often emphasized is the absence of pressure from the Helmholtz equation, a fact that has led many to assert that pressure plays no role in vorticity dynamics, an assertion that we shall question.

We may reasonably argue that vorticity is a well-defined variable having a clear relationship with the physical concept of rotation in fluid motion, satisfying a known differential equation and of widely accepted value in the description of fluid flows. It is therefore quite unsatisfactory that so many who work with fluids remain uneasy over its use, and in particular that there should persist such widespread uncertainty as to the behavior of vorticity near boundaries, exemplified by responses to the following questions:

i) what are the boundary conditions on vorticity and why are they so generally overlooked;

ii) does pressure play any role in vorticity dynamics;

iii) is vorticity generated by wall stress;

iv) what, if any, is the role of torque in fluid dynamics;

v) what is the physical mechanism or mechanisms for the generation of vorticity at boundaries; and

vi) what are the mechanisms for loss of vorticity, and in particular can vorticity be lost by diffusion to boundaries?

2. PREVIOUS TREATMENTS OF THE GENERATION AND DECAY OF VORTICITY

Few authors have seriously discussed either the generation of vorticity at boundaries or its subsequent decay, with two notable exceptions: Lighthill and Batchelor, who have resolved parts of the problem, without resolving it as a whole.

Lighthill (1963), in an elegant and wide-ranging introduction to boundary layer theory, espoused the use of vorticity as an effective means of solution for aerodynamic problems. He noted that at almost all points of the boundary there is a non-zero gradient of vorticity along the normal, with flux density of "total vorticity" having x-component $-\nu(\partial\xi/\partial z)_{z=0}$ "out of the solid surface", where z is normal distance from the boundary, assumed locally plane. By applying the Navier–Stokes equation at a stationary plane boundary $z=0$,

$$-\nu(\partial\xi/\partial z, \partial\eta/\partial z, 0)_{z=0}=\rho^{-1}(\partial p/\partial y, -\partial p/\partial x, 0)_{z=0},$$

which he took as the local strength of a distribution of vorticity sources spread over the solid boundary. It follows that tangential vorticity must be created at the boundary in the direction of the surface isobars at a rate proportional to the tangential pressure gradient. What does not immediately follow is why this should be so.

Although Lighthill described the boundary as a distributed source of vorticity, similar to a distributed source of heat, it is clear from his wider discussion of flow development that he envisaged the generation process as taking place at the boundary surface. In discussing two-dimensional boundary layers he then invoked vorticity sources in a region of falling pressure along the boundary and vorticity sinks (at which vorticity is abstracted at the surface) in a following region of rising pressure. At this stage the situation does not appear to be fully resolved; in the absence of a physical mechanism for the generation of vorticity, the relation between vorticity flux density and tangential pressure gradient does not distinguish between the outward diffusion of positive vorticity and the inward diffusion of negative vorticity. Nor is it clear whether vorticity can be lost by diffusion to boundaries. We cannot, therefore, say whether vorticity of a particular sense is being first generated at and subsequently lost to the boundary as the pressure falls and rises again, or whether there is continuous generation of vorticity first of one sense and then of the other as fluid moves along the boundary. Indeed, it is not clear whether there is any meaningful distinction between these two.

The difficulty with the vorticity flux density relationship as a boundary condition for the Helmholtz equation in calculating flow fields past aerofoils is that it involves the pressure field. Lighthill sidestepped this implicitly by considering an adjustment process in which an initial inviscid flow field is used to determine the free-slip velocity at the boundary and the production of vorticity is inferred from changes in slip velocity; from this point our interests diverge.

Lighthill made two cautionary statements to which we shall need to return. That although vorticity relates instantaneously and locally to the angular momentum of infinitesimal spherical fluid particles, the flow of vorticity cannot be viewed as diffusion of angular momentum as it is continuously transported to fresh fluid elements which suffer rearrangement in the flow. Hence there is no continuing axis about which it is meaningful to calculate vorticity related angular momenta, and it is angular velocity and not angular momentum to which vorticity should be related. Secondly, the diffusion of vorticity relates to the diffusion of (linear) momentum, to which we choose the interpretation that vorticity diffuses in effect because linear momentum diffuses.

Batchelor gives a careful discussion of vorticity in his *Introduction to Fluid Dynamics* (1967), but although he devotes a substantial section (Section 5.4) to the source of vorticity in motions generated from rest, his conclusions remain general and do not result in an expression for generation rates. Again he identifies the free-slip velocity in the inviscid solution for flow past a body set in motion as the effective source of vorticity in real flows, and hence solid boundaries as vorticity sources. He takes the further step, however, of explicitly identifying the no-slip condition at the boundary in real fluids as a mechanism for the production of vorticity, though he uses this only in a broadly descriptive manner. He argues also (Section 5.2) that vorticity cannot be destroyed in the interior of a homogeneous fluid and this appears to lead to the concept of loss of vorticity by diffusion to boundaries. It is perhaps unfortunate that the expressions "diffusion of vorticity out of", "from", "to" and "across" solid boundaries have become established in the literature without any very clear differentiation. There is little doubt that the generation process takes place at the boundary and neither within the wall nor within the fluid, but it is less clear what is intended when vorticity is regarded as lost by diffusion to a wall, as for instance when Batchelor describes steady motion as due to the steady flux of vorticity out of one solid boundary being balanced by an equal steady flux into another boundary (1967, p. 281). These are matters that will be discussed more precisely below.

In these two treatments Lighthill has concentrated on the role of pressure gradients over stationary surfaces and Batchelor on surfaces accelerated or set impulsively in motion. Many relevant matters have been raised but they do not seem to have come into focus sufficiently for us to answer the questions posed in the foregoing section. The following analysis is intended to clarify these aspects of the behavior of vorticity; much of it is interpretive, but the questions are so fundamental to our effective use of the concept of vorticity and the confusion they engender is so widespread among those using the methods of fluid dynamics that no further excuse seems to be needed.

3. VORTICITY AT AND NEAR BOUNDARIES

For motion near a plane boundary take origin O in the boundary with Oz normal and $\mathbf{n}$ the unit normal vector (Figure 1). The following discussion may be restricted to two-dimensional flow in (x, z) planes without loss of generality. Our state of understanding makes it difficult to specify boundary conditions on the vorticity $\boldsymbol{\omega}$ directly, but these can be derived from the condition on velocity,

$$\mathbf{v}=(u, w)=0 \quad \text{on} \quad z=0, \qquad \text{for all } x \text{ and } t.$$

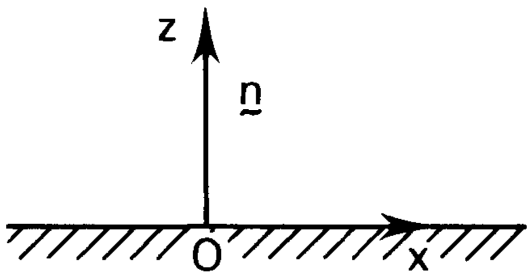

FIGURE 1 The coordinate system for the neighborhood of a boundary.

It follows that on $z=0$,

$$\partial(u, w)/\partial x=0, \quad \partial^2(u, w)/\partial x^2=0, \qquad \text{for all } x, t;$$

and from the continuity equation for an incompressible fluid

$$\partial u/\partial x+\partial w/\partial z=0,$$

$$\partial w/\partial z=0 \quad \text{on } z=0, \qquad \text{for all } x, t.$$

The remaining component of $\partial\mathbf{v}/\partial z$ relates to the tangential boundary stress $\boldsymbol{\tau}=(\tau_x, 0)$ through the relation

$$\tau_x=\mu\, \partial u/\partial z,$$

and is non-zero except at a separation point. Hence,

$$\boldsymbol{\omega}=(\xi, \eta, \zeta)=\{0, (\partial u/\partial z)-(\partial w/\partial x), 0\}$$

takes the value on $z=0$,

$$\mathbf{w}_0=(0, \mu^{-1}\tau_x, 0),$$

so that $\boldsymbol{\omega}_0 \cdot \boldsymbol{\tau}_0=0$. Thus vortex filaments are tangential to a stationary boundary and inclined at angle $+\pi/2$ to the wall stress or skin friction $\boldsymbol{\tau}$, a result which holds also in three dimensions. This boundary condition on $\boldsymbol{\omega}$ appears again to be of limited value because $\boldsymbol{\tau}$ is unknown at this stage. It is, however, a result of some importance, and the first indication that wall stress cannot generate vorticity; indeed, wall stress and vorticity are closely related velocity gradients.

3.1 The flux density of vorticity at a stationary plane boundary

At the boundary $z=0$ the Navier–Stokes equation reduces to

$$0=-\rho^{-1}(\nabla p)_0+\nu(\nabla^2\mathbf{v})_0,$$

and hence on $z=0$,

$$\frac{\partial^2}{\partial z^2}(u, w)=\frac{1}{\mu}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right)p.$$

Thus,

$$(\partial\boldsymbol{\omega}/\partial z)_0=[\partial(0, \eta, 0)/\partial z]_0=\{0, \partial^2 u/\partial z^2, 0\}_0$$

$$=\{0, \mu^{-1}\,\partial p/\partial x, 0\}_0=\mu^{-1}\mathbf{n}\times(\nabla p)_0,$$

with a corresponding result for $\partial\boldsymbol{\omega}/\partial x$ on $z=0$.

We recall that Lighthill (1963) has already identified $-\nu(\partial\boldsymbol{\omega}/\partial z)_0$ as the diffusive flux density (or flow per unit area per unit time) of (positive) vorticity outwards from the wall and interpreted

$$-\nu(\partial\boldsymbol{\omega}/\partial z)_0=-\rho^{-1}(\mathbf{n}\times\nabla)p$$

as a local boundary source of vorticity. This is obviously an important result, but in the absence of any physical mechanism we are unable to determine the balance of positive and negative vorticity generated at the wall, lost by diffusion from or gained by diffusion to the wall, or indeed whether the vorticity field might be produced entirely by a pattern of generation and diffusive loss of a single sense of vorticity to the boundaries.

3.2 Conditions at a wall moving in its own plane

We have seen already from Batchelor (1967) that boundary motion must be included in our discussion, and we shall therefore generalize the foregoing results to the case of a rigid boundary $z=0$ moving in its own plane with velocity $\mathbf{V}=\{U(t),0\}$. We consider tangential motion only here, but will return to the differences between tangential and normal motion of boundaries.

At the boundary $z=0$,

$$\mathbf{v}=(U,0), \quad \text{for all} \quad x,t,$$

and as there is no spatial variation of boundary motion,

$$[(\partial/\partial x, \partial^2/\partial x^2)(u,w)]_0=0.$$

Continuity in an incompressible fluid involves only spatial variation of velocity, and as before

$$(\partial w/\partial z)_0=0, \quad \text{and} \quad \boldsymbol{\omega}_0=\{0,\mu^{-1}\tau_x,0\}.$$

Substitution in the Navier–Stokes equation at the boundary yields

$$[\partial^2(u,w)/\partial z^2]_0=\{\mu^{-1}(\partial p/\partial x)+\nu^{-1}(dU/dt),\mu^{-1}\,\partial p/\partial z\}_0,$$

and

$$-\nu(\partial\boldsymbol{\omega}/\partial z)_0=\{0,-[\rho^{-1}(\partial p/\partial x)+dU/dt],0\}_0$$

$$=-\rho^{-1}[(\mathbf{n}\times\nabla)p]_0-\mathbf{n}\times(d\mathbf{V}/dt).$$

We note that in two-dimensional motion the flux of vorticity from and to boundaries depends on tangential pressure gradients and boundary acceleration; and particularly the symmetry involved in the acceleration of fluid by the tangential pressure gradient and the tangential acceleration of the boundary.

3.3 The relationship of vortex lines and streamlines at the boundary

We can expand the velocity in a Taylor series for the neighborhood of the origin O on the boundary,

$$\mathbf{v}(x,z)=\mathbf{v}_0+x\left(\frac{\partial \mathbf{v}}{\partial x}\right)_0+z\left(\frac{\partial \mathbf{v}}{\partial z}\right)_0+\tfrac{1}{2}x^2\left(\frac{\partial^2 \mathbf{v}}{\partial x^2}\right)_0+zx\left(\frac{\partial^2 \mathbf{v}}{\partial z\,\partial x}\right)_0$$
$$+\tfrac{1}{2}z^2\left(\frac{\partial^2 \mathbf{v}}{\partial z^2}\right)_0+O(r^3),$$

where the suffix 0 refers to conditions at the origin and $r^2=x^2+z^2$. Using the results from the previous sections,

$$\mathbf{v}(x,z)=\mu^{-1}\{z\boldsymbol{\tau}+zx(\partial\boldsymbol{\tau}/\partial x)+\tfrac{1}{2}z^2(\nabla p)_0\}+O(r^3).$$

This may be expressed wholly in terms of $\boldsymbol{\tau}=(\tau_x,0)$ using

$$(\nabla p)_0=\{\partial\tau_x/\partial z,0,-\partial\tau_x/\partial x\};$$

or in terms of vorticity $\boldsymbol{\omega}=(0,\eta,0)$ using $\mathbf{n}\times\boldsymbol{\tau}=\mu\boldsymbol{\omega}_0$.

The streamlines are given by

$$dx/u=dz/w,$$

and for small values of z

$$dx/\tau_x\approx dz/0;$$

thus in the limit $z\to 0$ the limiting streamlines touch the boundary and are in the direction of the wall stress, and hence orthogonal to the limiting vortex lines which also touch the boundary.

A stream function is available for two-dimensional flows, and has expansion valid in a neighborhood of the boundary origin O,

$$\psi(x,z)=\psi_0+\tfrac{1}{2}z^2(\partial u/\partial z)_0+O(r^3)\approx\psi_0+\tfrac{1}{2}\eta_0 z^2.$$

4. CASE STUDIES OF EXACT SOLUTIONS

Further insight can be gained into the significance of these results by reinterpretation of the small number of exact solutions of the Navier–Stokes or Helmholtz equations. These solutions are simple in form and so familiar that we take them for granted, but like all exact solutions they provide a source of relevant information whether or not we appreciate it. We shall be concerned especially with the behavior of vorticity at and near boundaries and we seek a consistent approach to matters concerning its generation and decay.

4.1 A plate started impulsively into motion in its own plane

Motion in the semi-infinite region of fluid $z>0$ initiated from rest when the plane boundary $z=0$ is set impulsively into tangential motion with speed $u=UH(t)$ at time $t=0$ is represented by the equation

$$\partial u/\partial t=\nu\,\partial^2 u/\partial z^2,$$

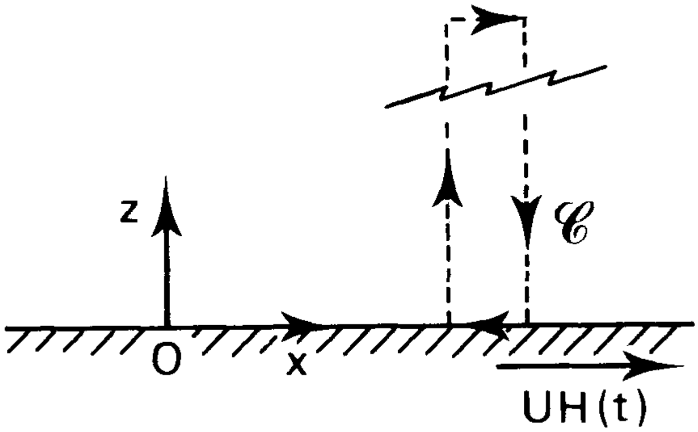

FIGURE 2 The integration circuit $\mathscr{C}$ for the impulsively started plate.

for which there is the similarity solution

$$u = U\{1 - \pi^{-1/2} \int_0^s e^{-\phi^2}\, d\phi\},$$

where $s = z/(2\nu t)^{1/2}$ is the similarity variable and the initial and boundary conditions are:

$$t < 0, \qquad u = 0 \quad 0 \leqq z;$$

$$t \geqq 0, \qquad u = U \text{ on } z = 0,\ u \to 0 \text{ as } z \to \infty.$$

The corresponding vorticity is

$$\boldsymbol{\omega} = (0, \eta, 0) = \{0, -(2\pi\nu t)^{-1/2} U e^{-s^2}, 0\}$$

and has maximum magnitude $U/(2\pi\nu t)^{1/2}$ at the plate for all $t \geqq 0$.

The wall stress at the boundary,

$$\boldsymbol{\tau} = \{\mu(\partial u/\partial z)_0, 0, 0\} = \{-\mu U(2\pi\nu t)^{-1/2}, 0, 0\},$$

is unbounded at the initial instant and thereafter decreases in proportion to $t^{-1/2}$; and the circulation around circuit $\mathscr{C}$ of unit x-width and unbounded z-height (Figure 2),

$$\oint_{\mathscr{C}} \mathbf{v} \cdot d\mathbf{r} = -U,$$

increases impulsively from zero to $-U$ at $t = 0$ and is thereafter constant.

In this case all the vorticity is generated at the initial instant as the plate is impulsively accelerated, and it is generated entirely at the boundary surface s (or z) $= 0$. Thereafter vorticity diffuses away from the boundary but its gross amount, which is the circulation in the contour $\mathscr{C}$ per unit width, remains constant, and in particular it is neither lost to nor gained from the boundary despite the continuing wall stress and the fact that the boundary remains the point of greatest vorticity concentration. Note that,

$$\text{flux density of } \eta = -\nu\, \partial\eta/\partial z = -Uz(2\pi\nu)^{-1/2} t^{-3/2} e^{-z^2/2\nu t},$$

and that the flux density at the boundary ($z=0$) is zero except at $t=0$; and that at $s=0$, $t=0$ it is infinite.

4.2 A plate accelerated uniformly into motion in its own plane

If the boundary is accelerated from rest at $t=0$ with uniform acceleration $(A, 0, 0)$ there is again a similarity solution with velocity

$$u(s,t)=At\left\{(1+s^2)\left[1-(2/\pi)^{1/2}\int_0^s e^{-1/2\phi^2}\,d\phi\right]-(2/\pi)^{1/2}s\,e^{-1/2s^2}\right\},$$

and vorticity

$$\eta(s,t)=A(t/2\nu)^{1/2}\left\{2s\left[1-(2/\pi)^{1/2}\int_0^s e^{-1/2\phi^2}\,d\phi\right]-2(2/\pi)^{1/2}\,e^{-1/2s^2}\right\}.$$

The vorticity at the boundary, $\eta(0,t)=-2A(t/\pi\nu)^{1/2}$, is always of largest magnitude, and vorticity (uniformly of negative sign in this frame) diffuses progressively outwards. Circulation in the contour $\mathscr{C}$ of unit width is

$$k(t)=\oint_{\mathscr{C}}\mathbf{v}\cdot d\mathbf{r}=(2\nu t)^{1/2}\int_0^\infty \eta\,ds=-At,$$

and the rate of increase in circulation is equal to the rate at which vorticity diffuses out from the boundary,

$$dk/dt=-A=-\nu(\partial\eta/\partial z)_0.$$

Here the boundary is continuously accelerated, vorticity is generated at uniform rate $-A$ per unit area of boundary, and this vorticity diffuses out at precisely its rate of generation. It is again consistent that there is no loss of vorticity by diffusion to the boundary; the total circulation per unit width increases at the uniform rate of generation and, once generated, vorticity is never lost. Note that the rate of generation is associated with tangential acceleration, $-A$, of the boundary and not the wall stress $\tau_y=\mu\eta_0=-2\rho A(\nu t/\pi)^{1/2}$.

4.3 A plate oscillating in its own plane

When the boundary is oscillated with velocity $(U\cos\omega t, 0, 0)$ the solution is

$$u(z,t) = U\,e^{-kz}\cos(\omega t - kz)$$

$$\eta(z,t) = -2^{1/2}kU\,e^{-kz}\cos(\omega t - kz + \tfrac{1}{4}\pi),$$

where $k = (\omega/2\nu)^{1/2}$.

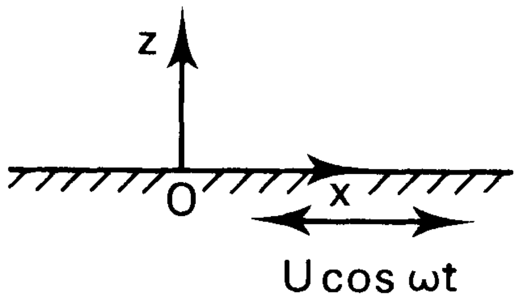

FIGURE 3 The oscillating plate.

In this case

$$-dU/dt = \omega U\sin\omega t = -\nu(\partial\eta/\partial z)_0,$$

and the rate of generation of vorticity at the plate is equal to its rate of diffusion outwards. Circulation (and hence gross vorticity) in the contour $\mathscr{C}$ of unit width is $-U\cos\omega t$. Thus the vorticity generated by acceleration of the boundary is alternatively positive and negative in each half cycle; it diffuses out, but is rapidly annihilated by cross-diffusion, so that the mean amplitude of vorticity decreases exponentially with distance and the mean circulation of the field is zero.

4.4 Plane Couette flow

When a plane boundary $z=h$ is moved in its own plane with uniform velocity $(U, 0, 0)$ over a stationary boundary $z=0$ the intervening layer of fluid at uniform pressure has velocity $u = Uz/h$ and vorticity $\eta = U/h$. The steady wall stresses on the upper and lower plates are $\tau(h) = -\mu U/h$ and $\tau(0) = \mu U/h$, and the gross vorticity per unit length of channel is $(U/h)h = U$. This flow might be set up from rest by impulsively starting the upper plate at speed $UH(t)$, where $H(t)$ is the Heavyside step function. An amount of vorticity U per unit length of plate would be generated at the upper plate at the initial instant $t=0$, and this would diffuse out into the layer to produce in time the uniform distribution $\eta = U/h$. Thereafter the flow is steady and vorticity is neither generated at the plates nor lost to them.

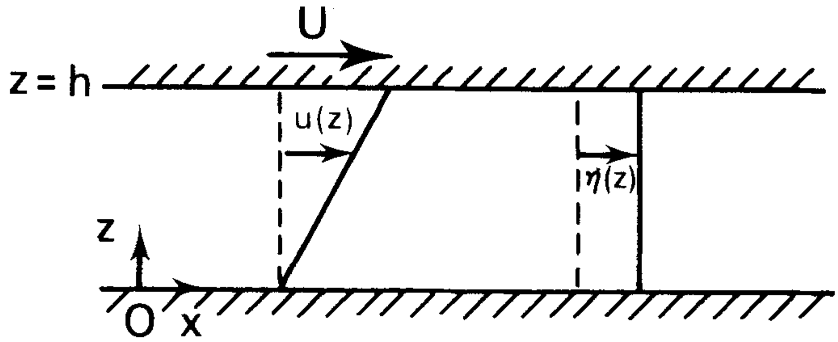

FIGURE 4 Plane Couette flow.

4.5 Plane Poiseuille flow

Steady flow between fixed parallel planes, $z = \pm h$, under a uniform pressure gradient $dp/dx = -\gamma$ has velocity $u = (\gamma h^2/2\mu)(1 - z^2/h^2)$ and vorticity $\eta = -\gamma z/\mu$. Vorticity is generated continuously by the tangential pressure gradient at the lower boundary at rate $+\gamma/\rho$, and at the upper boundary at rate $-\gamma/\rho$ (sense of normal reversed), and each diffuses towards the centre plane where the positive and negative fluxes suffer annihilation. The circulation per unit length of channel is zero.

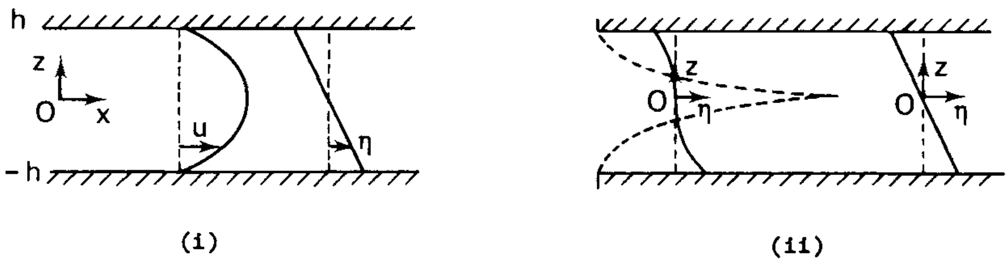

FIGURE 5 Plane Poiseuille flow: (i) profiles for developed flow; (ii) profiles for entry length.

We may ask whether Poiseuille flow could be represented equally well by the generation of vorticity of a single sense, say at the rate γ/ρ at the lower boundary, diffusion across the layer with flux density $-\nu\,\partial\eta/\partial z=\gamma/\rho$, and absorption at the upper boundary. Although this appears to be a viable alternative in the region of developed flow, it cannot explain the entry length with upper/lower boundary layers wholly of negative/positive vorticity, generated along the boundaries and diffusing out to fill a progressively larger proportion of the channel until the two boundary layers meet at the midplane and the upward/downward fluxes of positive/negative vorticity are mutually annihilated by cross diffusion. We note that there is nothing special about the positiveness or negativeness of vorticity as such, because the sign depends on our choice of axes; however, it is essential that there exist vorticity of opposite senses or sign.

4.6 Blasius boundary layer

For two-dimensional flow over a semi-infinite flat plate in the absence of pressure gradients, we introduce a stream function ψ with $\mathbf{v}=\{\partial\psi/\partial z, 0, -\partial\psi/\partial x\}$ and find the solution

$$\psi=(2\nu Ux)^{1/2}f(s)$$

$$\eta=U^{3/2}(2\nu x)^{-1/2}\{f''-(2Re)^{-1}(f-sf'-s^2f'')\},$$

where $s=(U/2\nu x)^{1/2}z$ is a similarity variable, $f(s)$ is a universal profile function and $Re=xU/\nu$ is a local Reynolds number. The circulation around the contour $\mathscr{C}$ of unit width is U, and is independent of distance along the plane. Thus all vorticity is generated at the leading edge, and is thereafter merely advected

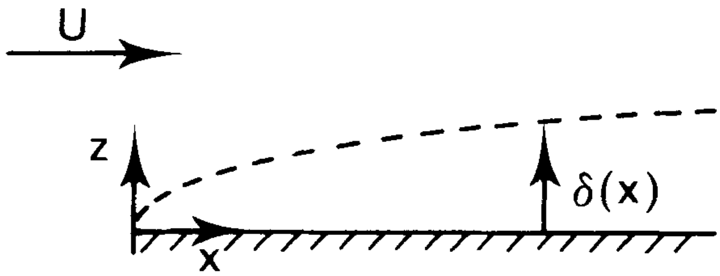

FIGURE 6 Blasius boundary layer.

downstream and diffuses into a thickening boundary layer. None is lost by diffusion to the boundary.

In this formulation the leading edge, 0, is a singular point, but in reality there must be a leading edge of finite radius of curvature around which a pressure gradient will exist and vorticity will be generated and advected away downstream at the rate $\frac{1}{2}U^2$.

4.7 Summary of case studies

We may divide the foregoing cases into two groups:

i) fast generation, including the impulsively started plate, Couette flow and the Blasius boundary layer; and
ii) slow generation, including the accelerated plate, the oscillating plate and Poiseuille flow.

In the former group the generation of vorticity occurs (or has occurred) instantaneously and we are concerned with the developing (or developed) distribution of a fixed amount of vorticity by diffusion or a mixture of diffusion and advection. These cases, and notably that of the impulsively started plate, are specially important because the generation process is separated momentarily from the subsequent viscous redistribution. The latter group comprises cases in which there is continuing generation of vorticity, so that generation and viscous redistribution are inextricably associated.

We can also identify a number of properties that must be taken into account in developing any comprehensive treatment of the generation and decay of vorticity:

i) that generation results from tangential acceleration of a boundary, from tangential initiation of boundary motion and from tangential pressure gradients acting along a boundary;
ii) that generation is instantaneous;
iii) that vorticity once generated cannot subsequently be lost by diffusion to boundaries;
iv) that reversal of the sense of acceleration or of the sense of pressure gradient results in reversal of the sense of vorticity generated;
v) that wall stress relates to the presence of vorticity but is not a cause of its generation;

vi) that the generation process is independent of the prior presence of vorticity;

vii) that both senses of vorticity are needed to explain observations;

viii) that walls play no direct role in the decay or loss of vorticity;

ix) that vorticity decay results from cross-diffusion of two fluxes of opposite sense and takes place in the fluid interior.

5. THE GENERATION OF VORTICITY AT BOUNDARIES

The foregoing case studies have drawn attention to rapid, and hence probably inertial, consequences of changing transverse motion, and because of the physical interpretation of vorticity we anticipate that rotation will play a role. We shall first consider the effect of an instantaneous impulse on a rigid body, partly because our grasp of rigid body dynamics is more complete than that of fluid dynamics, and partly to contrast the two.

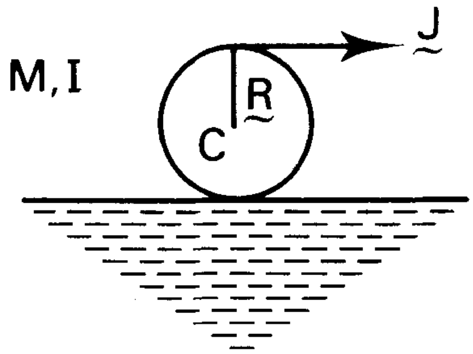

FIGURE 7 The effect of an off-centre impulse on a rigid body.

Suppose that a sphere of radius R, mass M and moment of inertia I about horizontal axes through its centre of mass C, rests on a smooth horizontal ice sheet (to avoid bother with a reactionary static friction impulse). If an instantaneous horizontal impulse $\mathbf{J}$ is now applied to the highest point, the sphere acquires at that instant:

i) linear momentum $\mathbf{J}$, translational velocity $\mathbf{v}=\mathbf{J}/M$, and kinetic energy of translation $\mathbf{J}^2/2M$; and

ii) angular momentum $\mathbf{R}\times\mathbf{J}$, angular velocity $\mathbf{\Omega}=\mathbf{R}\times\mathbf{J}/I$, and kinetic energy of rotation $(\mathbf{R}\times\mathbf{J})^2/2I$.

The impulse has done total work $\mathbf{J}^2/2M+(\mathbf{R}\times\mathbf{J})^2/2I$ in the instant of its application and the sphere has acquired both translational and rotational motion. Every part of the sphere partakes of this response because force is transmitted at infinite (wave) speed in a rigid body. Thus at the initial instant each particle of the sphere acquires a velocity but no particle has yet moved; and we have chosen to consider an instantaneous impulse because in doing so we separate inertial and frictional effects.

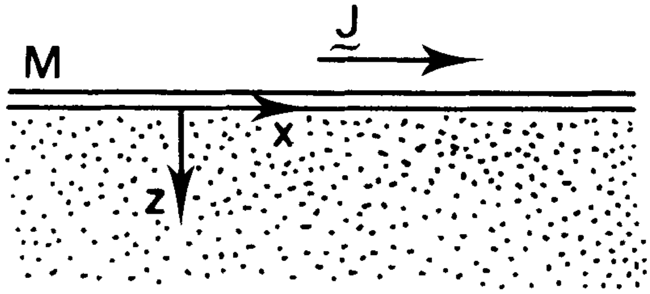

FIGURE 8 The effect of an impulse on a fluid boundary.

To find the effect of an instantaneous tangential impulse $\mathbf{J}$ applied to the free surface of a large volume of fluid we provide a rigid lid of mass M which will distribute the impulse uniformly over the surface. The impulse will set the plate instantaneously into tangential motion with initial speed U, say, and at this instant we have the Rayleigh problem of stationary fluid lying over a plate started impulsively into uniform motion in its own plane, considered in Section 4.1 and identified as specially important in Section 4.7; the solution has initial behaviour for small time t,

vorticity	$\eta \sim (\nu t)^{-1/2} U$
velocity	$u \sim U$
linear momentum	$\sim \rho(\nu t)^{1/2} U$ per unit volume

moment of momentum relative to $z=0 \sim \rho \nu t U$ per unit volume,

and is restricted to a fluid layer of thickness $\delta \sim (\nu t)^{1/2}$. Thus at the initial instant there is no communication of momentum to the fluid, the impulse goes wholly into generating lid momentum, and

$$\mathbf{U}=\mathbf{J}/M.$$

In the initial instant the instantaneous impulse does work $\mathbf{J}^2/2M$ on the lid but no work whatsoever on the fluid. Neither linear momentum nor moment of momentum are communicated to the fluid at the instant of impulse, but an exceedingly thin layer $(\delta \sim \nu^{1/2}t^{1/2})$ of exceedingly strong vorticity $(\eta \sim -\nu^{-1/2}t^{-1/2}U)$ is generated at the common surface of fluid and lid. It is only after the lid has started to move that fluid is drawn into motion with it by the tangential stress exerted by the moving lid.

There are two ways in which a moving boundary generates motion in fluid:

i) normal motion of the boundary generates normal stress or pressure which is communicated at sound speed to all parts of the fluid;

ii) tangential motion of the boundary generates shearing stress which is communicated through the fluid only by the diffusion of momentum, relatively a very slow process except at very small time when the gradients are very large.

Thus, when we accelerate a plate, vorticity is generated instantaneously, but the fluid is drawn into motion only by the continuing wall stress as motion of the plate is maintained by a continuing force which does work on the fluid. We must now return to the lid which we set into transverse motion impulsively. It will continue to move at uniform speed U only under the action of a maintained tangential force

$$-\tau_x = -\mu(\partial u/\partial z)_0 = \mu U(2\pi\nu t)^{-1/2};$$

and in the absence of this force it will suffer progressive deceleration with corresponding generation of positive vorticity.

5.1 The generation mechanism

The vorticity distribution above a plate started impulsively into motion at time $t=0$ with speed $(U,0)$ in its plane is (Section 4.1)

$$\omega = \{0, -U(2\pi\nu t)^{-1/2} e^{-z^2/2\nu t}, 0\}.$$

All the vorticity ($-U$ per unit x-width) is generated at the instant

$t=0$ of the impulse, and at that instant the vorticity is infinite at the boundary but zero everywhere within the fluid; and its gradient is infinite at the boundary and zero elsewhere. At any later time, *no matter how soon after the impulsive start*, the vorticity is finite and its gradient zero at the wall $z=0$. Thus diffusion of vorticity does not begin until after its generation, but the effect of diffusion is then exceedingly rapid very close to the wall; at later times near the wall and at all times in the body of the fluid viscous diffusion acts slowly as we have come to expect. When vorticity is generated continuously, diffusion has taken control of all except the elementary addition $\delta\eta$ in the last element of time δt in the limit $\delta t \to 0$. In such cases (cf. Section 4.2) the vorticity is always bounded at the wall and the effects of generation and diffusion appear to be interrelated. In these cases and where vorticity is generated by the action of pressure gradients we may need to separate the generation and diffusion processes artifically in order to understand what is taking place.

The generation of vorticity is generally instantaneous, and the vorticity generated is instantaneously unbounded in magnitude but in an infinitely thin sheet at the boundary. It is, therefore, more appropriate to work in terms of the circulation in a small circuit $\mathscr{C}$ with two arms of length δx parallel to the boundary and just in fluid or solid, respectively, and two short closing arms δz normal to the boundary. The circulation is then

$$\delta k = \oint_{\mathscr{C}} \mathbf{v} \cdot d\mathbf{r} \approx (u-U)\delta x$$

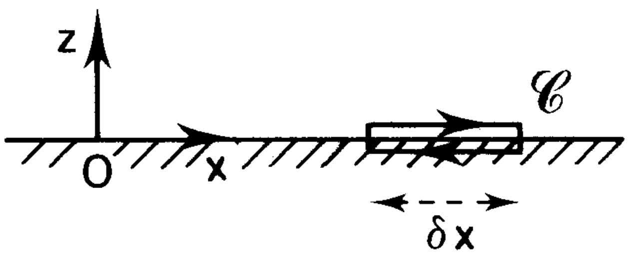

FIGURE 9 The circulation circuit $\mathscr{C}$ for generation at a boundary.

since the contribution from the closing arms normal to the boundary is $O[(\delta z)^3]$ from Section 3.3; and in the limit for small δx the circulation per unit length of boundary in a contour $\mathscr{C}$ normal to Oy

is

$$k_\eta = u - U.$$

It follows that the rate of generation of y-vorticity is

$$\frac{dk_\eta}{dt} = \frac{du}{dt} - \frac{dU}{dt} = \frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} - \frac{dU}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{dU}{dt} + \nu\nabla^2 u,$$

where we have substituted from the Navier–Stokes equation for x-component motion within the fluid. If we restrict attention for the moment to the case of the impulsively started plate, the pressure gradient term should be retained because of the possibility of large impulsive pressures, but the viscous term $\nu\nabla^2 u$ can have no effect in the initial instant of the impulse and may be neglected for that instant (although it will become very large immediately after). Hence,

$$dk_\eta/dt = -\rho^{-1}\,\partial p/\partial x - dU/dt,$$

and for an impulsive change

$$[k_\eta] = -\rho^{-1}(\partial/\partial x)\int p\,dt - [U].$$

This component generation rate corresponds precisely with the flux density obtained in Section 3.2.

The gross vorticity per unit length of a thin layer is the difference in tangential velocity across the layer, and the rate of generation of vorticity in the layer is the relative tangential acceleration across the layer. In the absence of viscosity, only pressure gradients can produce acceleration within the fluid and in a homogeneous fluid they produce homogeneous acceleration. Thus all relative tangential acceleration produced by the pressure field in a homogeneous fluid must be at boundary surfaces; relative tangential acceleration due to motion of the boundaries can only be at boundary surfaces. Viscosity plays no role in the generation of vorticity precisely because the generation is an instantaneous response of fluid and boundaries to inertial forces, and it follows that the generation of vorticity at the boundaries of a homogeneous fluid is an inviscid process, as it must surely be since Lighthill and Batchelor were able to use the free-slip layer of potential theory solutions to infer the rates of generation of vorticity. The no-slip boundary condition can in no way provide a mechanism for the generation of vorticity, and indeed plays no role in its generation.

Viscosity does, however, play a major role in the redistribution of vorticity, and begins to do so immediately after generation. It follows that in the latter group of flows considered above we shall have artificially to separate the generation and viscous redistribution processes to incorporate the individual cases into our scheme of vorticity generation. The six cases are summarized below.

1) The impusively started plate is a case of "fast generation" in which the rate of generation is infinite and the gross impulsive generation per unit area is $[u-U]=-U$. Viscous redistribution begins immediately after generation.
2) The uniformly accelerated plate is a case of "slow generation" in which viscous redistribution co-exists with generation, and to identify the rate of generation of vorticity we must isolate it by taking $\nu=0$. The relative acceleration $a-A=-A$ then gives the rate of vorticity generation.†
3) The oscillating plate is another case of "slow generation" and we obtain the rate of generation of vorticity as

$$a-A=-A=-dU/dt=\omega U \sin \omega t.$$

by neglecting viscous redistribution.
4) Couette flow is in vortical terms a steady state in which vorticity generated previously when the upper plate was set moving has been viscously redistributed uniformly between the plates.
5) Poiseuille flow is a case of "slow generation" in which the generation rate at the upper and lower boundaries can be found by taking $\nu=0$ as $A-a=-du/dt=\rho^{-1}\,dp/dx=-\gamma/\rho$ at the upper boundary and $a-A=du/dt=\gamma/\rho$ at the lower boundary.

†This is analogous to the conduction of heat in a semi-infinite solid bounded by a plane face which receives radiation at uniform rate, another case in which the supply and diffusion of, in this case, heat are governed by independent physical processes.

6) The Blasius boundary layer is a case of "fast generation" at the leading edge followed by advection along the boundary together with viscous redistribution outwards from it.

The six case studies have now been incorporated into our scheme for the generation of vorticity at boundaries, and the way is clear for further applications. For example, in mixed Couette–Poiseuille flow along a parallel channel with upper plate speed U and pressure gradient $\rho^{-1}\, dp/dx = -\gamma/\rho$, if the lower plate is oscillated in its own plane with velocity $V \cos \omega t$ the instantaneous rate of generation per unit area of lower plate is $\gamma/\rho + \omega\, V \sin \omega t$, and negative vorticity will be generated during part of the cycle if $\rho\omega V/\gamma > 1$.

The same mechanism for vorticity generation acts in the interior of inhomogeneous fluids where relative tangential acceleration is produced whenever there is a component of $\nabla\rho$ normal to ∇p, leading to the customary generation rate $-\nabla(\rho^{-1}) \times \nabla p$ per unit volume.

5.2 Vorticity and rate-of-strain

Limiting streamlines must always touch the boundary and those nearby are approximately parallel to it except near points and lines of separation, but in general both exhibit curved patterns in surfaces (approximately) parallel to the boundary. These streamline patterns simplify considerably in two-dimensional flows, since each streamline must then lie in one of a family of parallel planes normal to the boundary. We may use these two-dimensional flows to gain further insight into the nature of vorticity generation at plane boundaries.

We can identify three limiting types of flow which exhibit specially simple structure:

i) pure rotation, consisting of "rigid body rotation" of fluid with curved streamlines and uniform vorticity but without rate-of-strain;

ii) pure rate-of-strain, comprising all potential flows and in particular two-dimensional potential flows where the vorticity is necessarily zero and the curvature of streamlines results from rate-of-strain;

iii) simple shear, consisting of flow with parallel streamlines but different velocity on different streamlines, where there must be a specific relationship between vorticity and rate-of-strain.

Sufficiently close to the boundary in two-dimensional (x, z) flow we have from Section 3.3 for a neighborhood of an origin O on a plane boundary,

$$u = z\eta_0 + zx(\partial\eta/\partial x)_0 + \tfrac{1}{2}z^2(\partial\eta/\partial z)_0 + O(r^3)$$

$$w = -\tfrac{1}{2}z^2(\partial\eta/\partial x)_0 + O(r^3),$$

and

$$\psi = \psi_0 + \tfrac{1}{2}\eta_0 z^2 + O(r^3).$$

Thus the vorticity is approximately uniform in small neighborhoods of the boundary, which are therefore regions of approximately uniform shear where the streamlines, $z \approx$ constant, are approximately straight lines parallel to the boundary.

Motion generated near a plane boundary has, inevitably, to be parallel to the boundary, and it follows that boundaries can seldom act as pure sources of vorticity. In general they serve as joint sources of vorticity and rate-of-strain, and in the case of two-dimensional flows over a plane boundary $z=0$ the vorticity $\eta = \partial u/\partial z - \partial w/\partial x$ and the related component of rate-of-strain $\varepsilon = \partial u/\partial z + \partial w/\partial x$ are equal at the boundary,

$$\varepsilon_0 = \eta_0,$$

and are generated with equal boundary flux densities. All but one of the cases considered in Section 4 consist of parallel flow with $\varepsilon = \eta = \partial u/\partial z$ throughout. In these cases the rate-of-strain is generated with vorticity at a boundary, is spread through the layer by the diffusion of momentum, and is then permanent (and in particular cannot be lost by diffusion to boundaries). Naturally, as in the case of Poiseuille flow, opposing rates of strain will cancel, in this case at the mid-level.

We have considered cases with either uniform pressure or very simple pressure fields. In more complicated two-dimensional flows, especially those with inertial pressure gradients, the vorticity equation reduces to

$$D\eta/Dt = \nu\nabla^2\eta;$$

and the rate-of-strain ε satisfies the differential equation

$$D\varepsilon/Dt = -2\rho^{-1}(\partial^2 p/\partial z\,\partial x) + \nu\nabla^2\varepsilon,$$

and there is generation of rate-of-strain in the interior of the flow. In the cases considered previously $\partial^2 p/\partial z\,\partial x = 0$ and rate-of-strain is generated only at boundaries, but in flow towards a step or towards a cylinder there is interior generation of rate-of-strain corresponding with the deflection of streamlines past the body.

5.3 The decay of vorticity

We have seen from our case studies that vorticity is not lost by diffusion to boundaries other than in circumstances in which vorticity of counter sign is being generated and suffering immediate cross-diffusive annihilation with pre-existing vorticity. On the other hand, vorticity of alternating sense generated at an oscillating plate diffuses outwards, the positive and negative bands interdiffusing and suffering cross-diffusive annihilation so that mean amplitudes decrease exponentially with increasing distance from the wall. Again, in two-dimensional Poiseuille flow vorticity is generated at equal and opposite rates at the two boundaries producing equal and opposite flux densities diffusing towards mid-channel where they annihilate by cross diffusion.

It has sometimes been implied that because vorticity is solenoidal $\nabla \cdot \boldsymbol{\omega} = 0$, it cannot be lost in the interior of a fluid and must therefore be permanent or lost only at boundaries. However, if we integrate over an arbitrary volume V with closed surface S,

$$0 = \int_V \nabla \cdot \boldsymbol{\omega}\, dV = \int_S \boldsymbol{\omega} \cdot \mathbf{n}\, dS,$$

implying that each vortex line or vortex tube must cut the surface S an even number of times, usually twice (entry and exit).

In Figure 10 sketches (i) and (ii) both satisfy this requirement, but will be affected quite differently by diffusion: (i) can suffer cross-diffusion with annihilation of the vorticity in its interior and reconnection near the poles, as in (iii); whereas (ii) can suffer little immediate change by diffusion except to lose or gain some part of a tube across its surface. The diffusive process (i)→(iii) corresponds

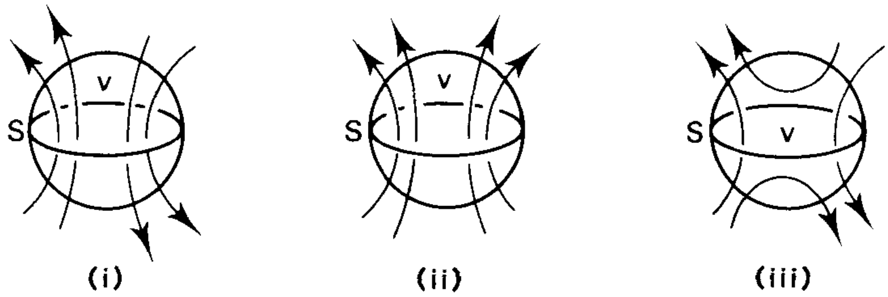

FIGURE 10 The effects of diffusion on vortex filaments passing through a volume V enclosed by a surface S.

with cross-diffusive annihilation of vorticity in the fluid interior and is the sole cause of the decay of vorticity fields.

The importance of cross-diffusion as a mechanism for decay of vorticity fields is emphasized by the global conditions on circulation. In simply connected regions of fluid the circulation in an enveloping contour which lies everywhere on the boundary of the region is a measure of the gross vorticity in the region. The gross circulation in an enveloping contour of a bounded region with outer boundary at rest must always be zero. Thus, no matter how we stir a cup of tea, the circulation in a path around the inner surface of the cup remains zero, and at any instant there must be equal strength of upwardly oriented as of downwardly oriented vortex tubes. Diffusion will, therefore, always lead in time to the total annihilation of all vorticity, and simultaneously of all motion, in the cup. It is not, of course, the diffusion of vorticity which brings the tea to rest, but the diffusion of momentum and the corresponding wall stress of cup on fluid.

Gross circulation in enveloping contours of infinite fluid regions containing bounded interior subregions of disturbance is again zero; and, unless we rotate boundaries or consider regions bounded partly by moving surfaces, all vorticity-producing disturbances in the interior of a fluid region must produce equal amounts of vorticity of opposite sense. Stirring our cup of tea with a circular motion may produce clockwise rotation with an inner core of negative vorticity surrounded by an annulus of positive vorticity (for an appropriate frame); but the gross circulation is zero and the vorticity will decay totally by cross-diffusion. A two-dimensional jet issues from a slit source from which equal amounts of vorticity of opposite sense are discharged from the two sides of the orifice. There is zero gross circulation around the jet as a whole, and in time the jet vorticity will decay everywhere to zero by cross diffusion. If, perchance, you cool your tea by blowing over its surface, you will direct your mouth-jet tangentially over the liquid surface, bringing against the surface tangential vorticity of sense appropriate to that side of the jet. Does this vorticity diffuse across the surface into the tea? In case of doubt, always return to the fundamental flow variables, in this case stress and momentum. The jet will produce surface stress on the tea and surface layers of tea will gain momentum, thereby creating a normal velocity gradient below the surface and hence vorticity in the tea. In effect, vorticity has diffused across the surface, but the circulation in an appropriate circuit (now following the inner cup wall from the right-hand rim to bottom and up to left-hand rim, and then continuing up and over the top of the jet to close at the right-hand rim) is and remains zero as long as the circuit embraces the whole jet cross-section. We are normally concerned with flows in which gross circulation is zero, and in these flows cross-diffusion will ultimately annihilate all vorticity.

5.4 The diffusion of vorticity

We have maintained a certain ambiguity on the diffusion of vorticity, but we now return to Lighthill's statement that this relates to the diffusion of momentum. One possible view is to note that the Helmholtz equation contains the term $\nu\nabla^2\omega$, analogous to the term $\nu\nabla^2\mathbf{v}$ representing diffusion of momentum in the Navier–Stokes equation, and therefore vorticity, like momentum, suffers viscous diffusion.

The further case study of steady two-dimensional Couette flow in a channel of depth $2h$ occupied to depth h from the bottom with

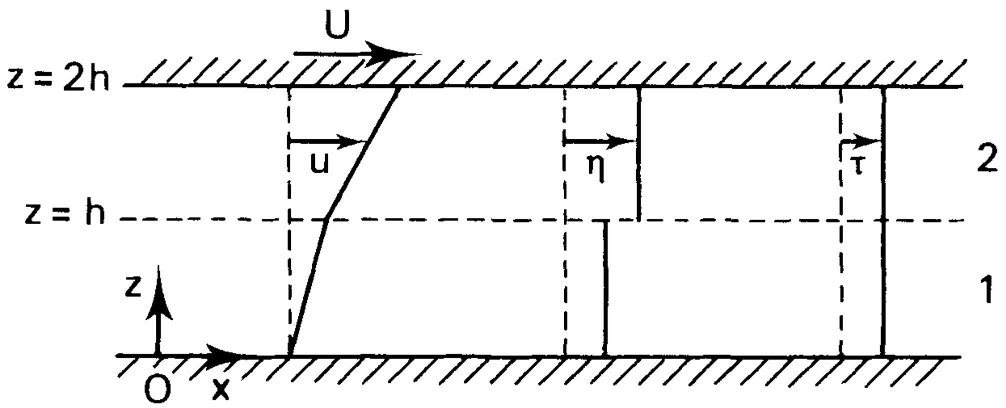

FIGURE 11 Couette flow in a two-layer fluid.

fluid of density ρ_1, and coefficient of viscosity μ_1, and from h to $2h$ with a fluid ρ_2, μ_2, and with $\rho_2 < \rho_1$ and $\mu_2 < \mu_1$, provides additional insight. The solution is

$$u_1 = \frac{\mu_2}{\mu_1 + \mu_2} \frac{U}{h} z, \qquad u_2 = \frac{\mu_1}{\mu_1 + \mu_2} \frac{U}{h} z - \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2},$$

or in terms of vorticity

$$\eta_1 = \frac{\mu_2}{\mu_1 + \mu_2} \frac{U}{h}, \qquad \eta_2 = \frac{\mu_1}{\mu_1 + \mu_2} \frac{U}{h},$$

where suffix 1 relates to the lower layer and 2 to the upper. Thus there is a discontinuity in gradient of the velocity at the interface, and an *absolute discontinuity in vorticity*. The solution therefore, predicts an infinite vorticity gradient at the interface, although it should be noted that the shear stress,

$$\tau_1 = [\mu_1 \mu_2 / (\mu_1 + \mu_2](U/h) = \tau_2,$$

is continuous through the interface and, indeed, uniform across the entire double layer; note also that the flux of vorticity is zero throughout the layer in this steady flow. We cannot accommodate a singular level within the flow region, but it is reasonable to note that the viscosity changes discontinuously at $z = h$ and therefore we should exclude this level from the solution, solve separately in the upper and lower layers and match the solutions across the mid-level, as of course we have done above. The boundary conditions for the Navier–Stokes equation are:

$$u_2(h^+) = u_1(h^-), \qquad \mu_2(du_2/dz)_{h^+} = \mu_1(du_1/dz)_{h^-};$$

and the rate of generation of vorticity at the interface is

$$(\partial u_2/\partial t)_{h^+} - (\partial u_1/\partial t)_{h^-} \equiv 0.$$

There is no real dilemma in this result, although we may too easily have accepted that diffusion of momentum provides an acceptable model for diffusion of vorticity.

The case of heat conduction through a two-layer parallel slab with different thermal conductivities in the two layers and outer faces maintained at different temperatures is in some ways analogous. There is again a discontinuity of gradient at the interface, in this case of temperature, but no discontinuity in heat flux, which is uniform throughout the slab. The discontinuity in temperature gradient is perhaps of less concern, as we are aware that it is heat that is conducted down the temperature gradient, although in material of uniform density and thermal conductivity the equation can be written as well in terms of temperature as heat, and usually is so.

Returning to vorticity, the primary mechanical variables are force and momentum. We may draw on the kinetic theory of gases to illustrate our argument. Gradient diffusion is the result of transport involving molecular collisions, and in each collision there is an exchange of properties. Two molecules in collision interact solely through a joint impulse which changes the linear momentum of each and does work on each. This work goes partly into the kinetic energy of mean motion and partly into random motion including translation, rotation, and vibration. The former relates to the macroscopic motion of the gas, the latter to its random microscopic motion and therefore to its heat content. Thus in collisions there is interchange of linear momentum and thermal energy, and it is linear momentum and heat that are directly diffused in a gas. Vorticity is not related to molecular spin but to mean velocity gradients averaged over a number of mean free paths; it is therefore a genuine continuum variable, and is transported not by direct molecule–molecule interactions but as a consequence of the diffusion of linear momentum which follows from the existence of velocity gradients and in turn modifies them and extends them through the fluid. It is only in special circumstances, however, that we may have to be circumspect, and we can usually treat vorticity as a diffusing quantity with coefficient of diffusivity ν.

6. SUMMARY

The motion of a rigid body is determined by its linear momentum and its angular momentum, the former generated by force and the latter by torque or the turning effect of a force. A force with line of action passing through the centre of mass of a body generates linear momentum only, in every part of the body. A force that does not act through the centre of mass generates both linear momentum and angular momentum simultaneously throughout the body. Angular momentum is important in the response of rigid bodies to forces because every part of the body moves in concert.

There is no limit to the degree to which a fluid may be deformed, but the only available forces that can produce this deformation are pressure gradients acting throughout the body of the fluid and normal and tangential stresses applied across the boundaries. In a homogeneous fluid vorticity is generated only at boundaries and its generation is instantaneous and the result of inertial forces; rate of strain is generated both internally and at boundaries. Effects of tangential forces are transmitted only by diffusion, and there is no turning effect of force-at-a-distance in fluids. It follows that torque and angular momentum have only the most restricted significance in fluid dynamics, relating to regions of rotating fluid in which the residence time of fluid elements is long relative to the motion of the whole, as in tropical cyclones.

Vorticity is generated at boundaries by the relative acceleration of fluid and wall produced instantaneously:

i) from the fluid side by tangential pressure gradients, although the generation is masked by viscous redistribution of the vorticity immediately upon generation;

ii) from the wall side by acceleration of the boundary, where generation is again partially masked by viscous diffusion when there is continuing generation.

Wall stress is a force and produces changes in fluid momentum; it does not produce relative acceleration of fluid and wall in the instantaneous and inviscid sense required for the generation of vorticity, and indeed tangential stress acts throughout all fluid regions where there is vorticity (or transverse velocity gradient). Wall stress does not produce vorticity.

Momentum can and does diffuse to boundaries producing wall stress, but there is no mechanism by which relative tangential velocity can diffuse to and be lost at a boundary. Relative acceleration of the opposite sign can, however, be produced at boundaries, and as the corresponding momentum diffuses out the two vorticity distributions of opposite sign will in time cancel. The only means of decay or loss of vorticity is by cross-diffusion and annihilation of vorticity of opposite signs.

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