AIAA 80-0011R

Theory for Aerodynamic Force and Moment in Viscous Flows

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A general theory for aerodynamic forces and moments, developed through a rigorous analysis of the viscous flow equations, is presented. General formulas relating aerodynamic forces and moments to rates of change of vorticity moments in the fluid and the solid regions are presented. Analytical results for several problems are presented to demonstrate the application of the theory in interpreting complex aerodynamic phenomena and in computational fluid dynamics.

I. Introduction

THE problem of predicting aerodynamic force and moment acting on finite solid bodies immersed in and moving relative to a viscous fluid has occupied the center stage of aerodynamic research from the end of the nineteenth century onward. It is this focal problem that distinguished the science of aerodynamics from other branches of theoretical fluid mechanics. Studies of the motion of the fluid relative to solid bodies of course represent a fundamental aspect of fluid mechanics. In the majority of aerodynamic applications, however, such studies do not represent ends in themselves. Rather, they are undertaken in recognition of the fact that the motion of the fluid ultimately is responsible for the forces and moments exerted on the solid bodies by the fluid.

It is known that the problem of finding analytical solutions to equations of viscous fluid motion associated with high Reynolds numbers and with solid configurations of practical importance presents considerable, often insurmountable, mathematical difficulties. Historically, therefore, many important advances in aerodynamics were brought about by researchers who preceived approaches for the prediction of aerodynamic force and moment that avoid, as much as possible, entanglement with the details of the viscous fluid motion. In particular, the circulation theory is known to predict the lift force accurately for certain types of solid, e.g., thin airfoil, under certain flow environment, e.g., small angle of attack. The scope of applicability of the circulation theory and its extensions has not been established precisely. Considerable uncertainties exist regarding the application of the theory in cases where the solid does not possess a sharp trailing edge, where massive separation occurs, and where the solid is three dimensional and its motion is time dependent. These uncertainties arise mainly because of the perfect-fluid assumption used in the mathematical development of the theory. The viscous origin of circulation has been, of course, long recognized. Several well-known works, e.g., by von Kármán and Millikan,[1] Howarth,[2] and Sears,[3,4] have dealt in detail with certain aspects of viscous phenomena that produce circulation. Nevertheless, it is often difficult to interpret the application of the circulation theory as an approximation of the viscous flow phenomena.[4] Such interpretations are expected to contribute to the further advancement of aerodynamic theory. In addition, such interpretations may also suggest new and improved approaches for computational fluid dynamics in aerospace vehicle design process. The purpose of this paper is twofold: 1) to describe a general theory for aerodynamic forces and moments developed through a rigorous analysis of the viscous flow equations, and 2) to demonstrate that this general theory encompasses much of the existing aerodynamic theories. For this purpose, it is shown through analysis that many existing theories on aerodynamic force and moment are readily interpreted as various levels of approximation of the present theory.

A distinguishing feature of the present theory is that the concept of bound vortex, or that of singularity elements such as sources, sinks, and vortex filaments, is not embodied in the general formulas forming the theory. Rather, the actual vorticity distributions of the fluid and of the solid, the latter being twice the angular velocity of the solid body, enter these formulas. The starting point of the present theory is a rotational flow analysis. No simplifying assumptions are utilized in the derivation of the theory. The freedom from "bondage" is important in the interpretation of the various aerodynamic theories. For example, it permits a precise definition of the "circulation" about a two-dimensional solid not only for an unseparated flow but also for a flow containing an appreciable region of separation. While the Kutta-Joukowski theorem predicts zero drag, the general formula does relate the time variation of a vorticity-moment integral to a nonzero drag, including the profile drag. The formulas predict an unsteady drag without the customary energy or apparent mass consideration. These general formulas, in fact, clearly point out the basic principles for minimizing the drag and for maximizing the lift. Many of the measures proposed for drag reduction and for lift augmentation are readily interpretable on the basis of these principles.

In recent years, extensive efforts have been in progress at many research institutions to develop numerical methods for the solution of aerodynamic problems. These efforts are divisible into two major categories. In one category, numerical methods are being developed for the solution of inviscid flow equations. The free vortices are assumed to convect, but not diffuse, with the fluid. The solid body is represented by a singularity distribution.[5] Conceptually, these numerical methods utilize the basic assumptions of the circulation theory. These numerical methods are expected to be subject to the well-known limitations of the inviscid flow assumption. In this regard, the availability of the general formulas are expected to offer clearer interpretation of these numerical methods and better definition of their scope of application.

In the second category, numerical methods are being developed for the solution of differential equations governing viscous flows. Impressive progress has been made in recent years in the numerical solution of two-dimensional separated flow problems. For three-dimensional separated flows, the development of numerical methods is hindered by excessive computation requirements.[6,7] During the past few years, the present author and his co-workers have developed a new numerical approach that permits the confinement of the solution field to the vortical region of the flow.[8,9] This ap-

Presented as Paper 80-0011 at the AIAA 18th Aerospace Sciences Meeting, Pasadena, Calif., Jan. 14-16, 1980; submitted Feb. 8, 1980; revision received Aug. 27, 1980.

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proach has been shown to possess superior solution efficiency. In this approach, the vorticity field is computed. The pressure and the shear stress on the solid bodies that determine the aerodynamic force and moment are then computed from the vorticity field. The general formulas presented in this paper provide a convenient means of computing the aerodynamic forces and moments directly from vorticity distributions.

Several important theorems, utilized in the development of the present general theory for aerodynamic force and moment, are available in the literature in specialized forms, e.g., Refs. 10 and 11. These theorems are traditionally considered in the context of an infinite limitless fluid, i.e., an infinite fluid with no internal boundaries. In the present paper, these theorem are shown to be valid, and are utilized in studies of flows bounded internally by solid surfaces. The practical importance of this more general validity needs no emphasis since the interaction between the solid and the fluid is indeed what the subject of aerodynamics is about.

The subject of this work received some attention in the 1950's. Phillips[12] presented a formula relating the fluid momentum to an integral of vorticity moment for a two-dimensional flow around a nonrotating cylinder in translation. The formulas presented in this paper are more general and are valid for two- and three-dimensional flows associated with one or more finite solid bodies of any arbitrary shape executing any prescribed steady or time-dependent translation and/or rotation. Moreau[13] presented formulas for a limitless fluid and for a portion of fluid subject to certain "order" conditions at infinity. Truesdell[14] commented that "Moreau emphasized his application to a limitless fluid, all but a finite interior part of which is in irrotational or circulation preserving motion. In this connection we should beware of the extremely strong order conditions at infinity required in order to get simple results, order conditions, indeed, which possibly may never be satisfied." In this paper, formulas are rigorously derived for the viscous flow of fluids past finite bodies, using order conditions at infinity that are shown to be satisfied under quite general circumstances.

II. Vorticity Dynamics

The time-dependent motion of an infinite incompressible fluid with uniform viscosity relative to one or more immersed solid bodies is considered in the present study. The solid bodies are initially at rest in the fluid, also at rest, and are located within finite distances from one another. Subsequent prescribed motion of the solid bodies induces a corresponding motion of the fluid. At large time levels after the motion has initiated, if the solid bodies move uniformly at a constant translational velocity relative to the freestream, then the possibility of an asymptotic steady flow exists. Alternatively, the possibility of a time-dependent flow involving periodic vortex shedding, as evidenced by the well-known Kármán vortex street behind a circular cylinder, also exists. In the present work, for reasons to be explained later, a steady flow, when it exists, is considered to occur only near the solid bodies and is approached asymptotically at large time levels after the initiation of the solid motion. If the solid bodies do not move uniformly, or if the solid motion is time dependent, then the motion of the fluid is necessarily time dependent.

The differential equations describing the time-dependent incompressible fluid motion are the familiar continuity and Navier-Stokes equations.

For convenience, the region occupied by the fluid is designated R_f. A coordinate system with its origin located within finite distances from all solid surfaces, collectively designated by B_S, is used. Unless otherwise specified, this coordinate system is considered to be at rest relative to the freestream. The fluid region R_f is bounded internally by B_S and externally by a close boundary B_∞ at infinity. The region occupied by the jth solid body is designated R_j, which is bounded externally by B_j. The limitless region jointly occupied by all the solid bodies and the fluid is designated R_∞.

It is advantageous to introduce the vorticity vector ω, defined as the curl of the velocity vector v, and to consider the vorticity transport equation

$$\frac{\partial \omega}{\partial t} = \nabla \times (v \times \omega) + \nu \nabla^2 \omega \tag{1}$$

obtained by taking the curl of each term in the Navier-Stokes equation. Here t is the time and ν is the kinematic viscosity, considered to be uniform for simplicity.

There are several reasons for using ω in the formulation of the problem. In the first place, the circulation theory for the lift force suggests that the vorticity of the flow, that ultimately should be traced to the circulation, is responsible for the force and moment exerted by the fluid on the solid. Second, as shown in Ref. 8, the use of the vorticity vector, that is connected intimately with viscosity effects, permits the solution field for the incompressible flow problem to be confined to the viscous region only. Third, the set of equations in terms of ω decomposes conveniently into a kinematic aspect and a kinetic aspect, each aspect constituting an entity by itself.

The first feature stated provided the motivation for the present effort in developing general formulas relating aerodynamic forces and moments to the time variation of vorticity-moment integrals. The advantages offered by the second feature have been studied extensively by this author and his co-workers in a series of previous articles in the context of computation methods, e.g., Refs. 8 and 9.

The importance of the third feature in the present work is due to the fact that the physical processes of flow development are clearly delineated once the overall problem is decomposed into its kinematic and kinetic aspects. In particular, with vorticity as a field variable, considerable insight is gained by examining the differential equations describing each of the two aspects, without employing detailed mathematical or numerical analyses.

The kinematic aspect of the problem concerns the relationship between the vorticity distribution at any given instant of time and the velocity distribution at the same instant. The differential equations describing this aspect are the continuity and vorticity definition equations.[8] These equations are linear. Since the density of the solid bodies does not undergo appreciable change, the continuity equation obviously is valid within the solid regions R_j as well as in the fluid region R_f. With the vorticity field in R_j defined as the curl of the velocity field, the kinematics of the solid bodies and the fluid are described by the same differential equations. The stress-strain relations, which differentiate the fluid from the solid bodies kinetically, do not enter the kinematic relation between the velocity field and the vorticity field. As a consequence, the solid bodies and the fluid can be treated together as one kinematical system occupying a limitless space.

The kinetic aspect of the problem is concerned with the development of the vorticity field with time. This aspect is described by the vorticity transport Eq. (1). This equation is nonlinear in the sense that the first term on its right-hand side involve the product of v and ω, and v is kinematically a function of ω. This equation is valid only in the fluid domain R_f, which is limited, i.e., it is bounded internally by B_s.

Because of the nonlinearity of the differential equation and the necessity of treating a limited region, the analysis of the kinetic aspect of the problem presents greater mathematical difficulties than the analysis of the kinematic aspect, which, as stated earlier, is described by linear differential equations applicable to the entire limitless region R_∞.

For an inviscid fluid, the last term in Eq. (1) vanishes and the vorticity is convected with the fluid in the sense that the vorticity flux $\omega \cdot ds$ associated with each material element ds moving with the fluid remains a constant for all times. This well-known theorem of Helmholtz, a proof of which is available in many textbooks, e.g., Ref. 15, is modified in the case of a real fluid by the process of vorticity diffusion. According to Eq. (1), changes in the vorticity flux associated with each material element take place only by diffusion. Vorticity flux cannot be created or destroyed in the interior of a fluid.

For the problem under consideration, the vorticity is obviously everywhere zero prior to the impulsive start of the motion of the solid bodies. The interior of the fluid domain therefore can become vortical only if vorticity diffuses across the boundaries of the fluid region. Consequently, immediately after the onset of the motion, the vorticity is everywhere zero in the fluid except at the boundaries in contact with the solid bodies. That is, the fluid motion immediately after the onset of the solid motion has a nonzero tangential velocity relative to the solid bodies at the solid boundaries. The discontinuity in tangential velocity constitutes a sheet of concentrated vorticity (vortex sheet) at the boundaries. At subsequent time levels, this concentrated vorticity spreads into the interior of the fluid domain by diffusion and, once there, is transported away from the boundaries by both convection and diffusion. At the same time, the no-slip condition provides a mechanism for the continual generation of vorticity at the boundaries. The general flow pattern therefore contains vortical regions surrounding the solid bodies and vortical wakes trailing the solid bodies. Outside of these vortical regions and wakes, the flow is essentially free of vorticity and therefore irrotational. In particular, if the flow Reynolds number is not small, then the vorticity spreads by diffusion only a short distance from the boundaries before it is carried away with the fluid by convection. Therefore a large region of the fluid, ahead and to the side of the solid bodies, is essentially free of vorticity and is irrotational.

III. Selected Theorems

Asymptotic Behavior of Vorticity Field

The asymptotic behavior of the vorticity distribution at large distances from solid bodies is established in the following by considering the fundamental solution, i.e., the Green's function for an infinite unlimited region, of the diffusion equation[15-17]

$$F(r,t;r_0,t_0)=\frac{1}{[4\pi\nu(t-t_0)]^{d/2}}\exp\left\{-\frac{|r-r_0|^2}{4\nu(t-t_0)}\right\} \tag{2}$$

where r and r_0 are position vectors and d is the dimensionality of the problem, i.e., $d=1$, 2, and 3, respectively, for one-, two-, and three-dimensional problems. For the present problem, the fundamental solution F represents the vorticity distribution at the time level t resulting from the diffusion of a concentrated vorticity which is of "unit" strength and is located at the point r_0 at the time level t_0, with $t_0<t$, provided that the convection process is absent.

If, at the time level t_0, the vorticity distribution is known in the unlimited region R, then, in the absence of convection, the vorticity distribution at a subsequent time level t is expressible as an integral

$$\omega(r,t)=\int_R F\omega(r_0,t_0)\,\mathrm{d}R_0 \tag{3}$$

where the subscript for $\mathrm{d}R_0$ indicates that the integration is performed in the r_0 space.

The form of the fundamental solution F shows that in an unlimited region, if convection is absent and vorticity is nonzero at the time level t_0 only within finite distances from the origin, then the vorticity at any subsequent finite time level t approaches zero exponentially with increasing distance r from the origin, at large distances. Therefore, the vortical region is effectively confined to a finite region at any finite time level $t>t_0$.

For a fluid region bounded internally by solid surfaces and in which convective process is present, Eq. (3) needs to be generalized. It is obvious that the convective process, being one of finite rate, does not alter the preceding conclusions regarding the effective extent of vortical regions and the asymptotic behavior of vorticity at any finite time level. Similarly, the presence of solid surfaces in the fluid provides a mechanism for introducing vorticity at the boundaries of the fluid region and, as long as these boundaries are within finite distances from the origin, the introduction of vorticity does not alter the above conclusions. For two-dimensional flows, the generalized version of Eq. (3) is expressible as[17,18]

$$(\omega r,t)=\int_{R_f}(F\omega_0)_{t_0=0}\mathrm{d}R_0+\int_0^t \mathrm{d}t_0\int_{R_f}\omega_0 v_0\cdot\nabla_0 F\mathrm{d}R_0$$
$$+\nu\int_0^t \mathrm{d}t_0\oint_{B_S}(F\nabla_0\omega_0-\omega_0\nabla_0 F)\cdot n_0\mathrm{d}B_0 \tag{4}$$

where n is an outward directed unit normal vector, and the subscript 0 indicates that the variables, differentiations, and integrations are in the r_0, t_0 space, e.g., $\omega_0=\omega(r_0,t_0)$. The second and third integrals of Eq. (4) represent, respectively, the convective process and the effect of solid boundaries. Each of the integrands in the integrals of Eq. (4) is directly proportional to F or ∇F. At any given time level t, if the distance r is large, then F and ∇F decay exponentially with increasing r. It follows that, if the vorticity is initially nonzero only within finite distances from the origin, then the vorticity decays exponentially with increasing r, at large r, for all subsequent finite time levels. For the present problem, the vorticity is nonzero immediately after the onset of the motion only on the solid surfaces. Consequently, the vorticity field decays exponentially with distance from the solid body, at large distances, for all finite time levels. For three-dimensional flows, Eq. (4) is replaced by a vector equation. The statement just made regarding the exponential decay of vorticity is true for three-dimensional flows. The effective extent of the vorticity field is therefore finite for all finite time levels in both two- and three-dimensional flows.

Biot-Savart Law

The law of Biot-Savart is an expression of the kinematic relationship between the velocity and vorticity fields. Expressions of the law of Biot-Savart for the velocity field "induced" by a vortex filament is well known. For a distributed vorticity field $\boldsymbol{\omega}$ in an unlimited infinite region $\boldsymbol{R}$, an integral expression of the Biot-Savart law is[17]

$$v(r,t)=\nabla\times\int_{R_\infty}\omega_0 P\mathrm{d}R_0 \tag{5}$$

where P is the fundamental solution of the Poisson's equation, defined by

$$P=\begin{cases}\dfrac{1}{4\pi|r-r_0|} & \text{for three-dimensional problems}\\[2ex] \dfrac{1}{2\pi}\ln\dfrac{1}{|r-r_0|} & \text{for two-dimensional problems}\end{cases} \tag{6}$$

As discussed earlier, the solid bodies and the fluid can be treated together as one kinematical system occupying a limitless space. Equation (5) is therefore valid provided that R_∞ is the limitless region occupied jointly by the fluid and the solids. A derivation of Eq. (5) based on a separate treatment of the solid and the fluid regions is presented in Ref. 19. It has been shown[8] that Eq. (5) is completely equivalent to the differential continuity and vorticity definition equations.

Principle of Total Vorticity Conservation

In the present problem where the solid bodies are initially at rest in a fluid also at rest, it can be shown that the total vorticity in the infinite unlimited space occupied jointly by the fluid and the solid bodies is always zero. For three-dimensional flows, since the vorticity field is solenoidal and is effectively confined to a finite region, all vorticity lines in the combined solid-fluid system form closed curves. Consequently, one has

$$\int_{R_\infty} \omega \mathrm{d}R = 0 \tag{7}$$

Equation (7) is a consequence of the kinematics of the flow. It states that the total vorticity in an infinite limitless three-dimensional space is zero. Formal proofs of Eq. (7) are presented in Refs. 11 and 19.

For two-dimensional flows, the vorticity lines are directed perpendicular to the plane of the flow. The vorticity lines extend to infinity in the direction perpendicular to the flow, and they do not form closed curves in the plane of the flow. A proof of the principle of total vorticity conservation, utilizing the kinetics of the viscous flow problem, is possible.

For two-dimensional flows, the vorticity transport equation can be rewritten as

$$\frac{\mathrm{D}\omega}{\mathrm{D}t} = -\nu \nabla \times \nabla \times \omega \tag{8}$$

where D/Dt denotes a substantial derivative.

Integrating Eq. (8) over the fluid region R_f, which is a function of time, and using Stokes' theorem, one obtains,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_f(t)} \omega \mathrm{d}R = \nu \int_{B(t)} (\nabla \times \omega) \times \boldsymbol{n} \mathrm{d}B \tag{9}$$

In Eq. (9) the boundary B consists of the solid boundary B_S and the boundary at infinity B_∞. The contribution of B_∞ to Eq. (9) is zero provided that ω approaches zero as r^{-n} for large r, with $n>2$. Since ω approaches zero exponentially with increasing r for large r, the preceding order condition at large r is satisfied for the present problem. The boundary B in Eq. (9) can therefore be replaced by the solid boundary B_S.

The Navier-Stokes equation can be written as

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla p - \nu \nabla \times \omega \tag{10}$$

Thus one has

$$\int_{B_S} \frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} \times \boldsymbol{n} \mathrm{d}B = -\frac{1}{\rho}\oint_{B_S} \nabla p \times \boldsymbol{n} \mathrm{d}B$$
$$-\nu \oint_{B_S} (\nabla \times \omega) \times \boldsymbol{n} \mathrm{d}B \tag{11}$$

The first integral on the right-hand side of Eq. (11) is zero by virtue of the single valuedness of pressure on B_S. Combining Eqs. (11) and (9), one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_f(t)} \omega \mathrm{d}R = -\oint_{B_S} \frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} \times \boldsymbol{n} \mathrm{d}B \tag{12}$$

In Eq. (12), consider B_S to be the external boundary of the solid region R_S. With the no-slip condition, the substantial acceleration $\mathrm{D}\boldsymbol{v}/\mathrm{D}t$ on B_S is identical for the solid bodies and for the fluid. Using Stokes theorem, the right-hand side of Eq. (12) can be expressed as

$$\sum_j \int_{R_j} \nabla \times \left(\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t}\right) \mathrm{d}R = -\frac{\mathrm{d}}{\mathrm{d}t}\sum_j \int_{R_j} \nabla \times \boldsymbol{v} \mathrm{d}R \tag{13}$$

where the sum is over all solid bodies. One therefore obtains from Eqs. (12) and (13)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_f} \omega \mathrm{d}R + \frac{\mathrm{d}}{\mathrm{d}t}\sum_j \int_{R_j} \omega \mathrm{d}R = 0 \tag{14}$$

which states that the total vorticity is invariant for the combined fluid-solid system. For the present problem, the total vorticity is initially zero. Therefore Eq. (7) is satisfied for two-dimensional flows.

There are several conceptual differences between the principle of total vorticity conservation discussed here and the usual understanding of invariance of vorticity integral with respect to time.[14] In the present work, the solid regions are included in the evaluation of the total vorticity. The integrands of Eqs. (7) and (14) are piecewise continuous and the integrals converge. The meaning of the total vorticity in the unlimited infinite space jointly occupied by the fluid and the solid bodies is unambiguous. While previous discussions of the invariance of the total vorticity usually are made in the context of an inviscid fluid or of an unlimited infinite fluid region (in the absence of internal boundaries), the present study utilizes the no-slip condition at the solid boundaries and permits such boundaries to be present in the fluid.

It is of interest to note that Eq. (7) gives

$$\int_{R_f} \omega \mathrm{d}R = -2\sum_j \boldsymbol{\Omega}_j R_j \tag{15}$$

where $\boldsymbol{\Omega}_j$ is the angular velocity of the jth solid body. If the solid motion at any instant of time is known, then the total vorticity in the fluid at that instant is known. In Ref. 20, Sec. B.2, an equation similar to Eq. (15) is presented for a single solid in a three-dimensional flow. The equation of Ref. 20 includes an additional term representing a contribution at a surface enclosing the solid. In the present work, it is shown that this contribution is absent if the surface is sufficiently distant from the solid body. Equation (15) is valid for two-dimensional as well as three-dimensional flows.

Reference 20 also gives an equation for the rate of change of the total vorticity in the fluid. This equation contains a term representing conduction of vorticity through the solid boundary. It is pointed out in Ref. 20 that further dynamic (kinetic) equations are needed to evaluate this term. The present result, Eq. (14), shows that this term is given simply by the rate of change of the total vorticity of the solid bodies and is zero if the solid bodies do not undergo angular acceleration.

In Ref. 11, it is shown that in three dimensions the total vorticity is zero in a region which contains the fluid region and "a region extending beyond the actual boundaries." The present results show that the proper extension of the fluid region is simply the solid regions in which the correct vorticity values to assign are the actual vorticity of the solid bodies. For two-dimensional flows, the present study shows that the existence of a nonzero circulation around a closed path enclosing the solid body at a sufficiently large distance from the solid bodies is not possible for a real fluid for any finite time level after a motion has started from rest.

Asymptotic Behavior of Velocity Field

For two-dimensional flows, it can be shown[19] that the principle of conservation of total vorticity leads to the following asymptotic expression for the velocity field at large distances for the solid bodies:

$$v(r,t) = \frac{1}{2\pi} \nabla \left(\nabla \ln \frac{1}{r} \cdot \alpha \right) + \text{terms of order } r^{-n}, \quad n \geq 3 \tag{16}$$

where α is the first moment of the vorticity field, defined by

$$\alpha = \int_{R_\infty} r \times \omega \mathrm{d}R \tag{17}$$

For three-dimensional flows, it can be shown[11,19] that

$$v(r,t) = \frac{1}{8\pi} \nabla \left(\nabla \frac{1}{r} \cdot \alpha \right) + \text{terms of order } r^{-n}, \quad n \geq 4 \tag{18}$$

provided that the total vorticity is zero and the vorticity magnitude is of order r^{-n}, with $n > 4$, when r is large. These provisions are satisfied, as shown earlier in this paper. It should be pointed out that a derivation of Eq. (18), but not Eq. (16), is given in Ref. 11. The "strong restriction" that ω is of order r^{-n} $(n > 4)$ is used, without proof, in that derivation.

Equations (16) and (18) show that, if α, the first moment of vorticity integral, is not zero, then the velocity field is of order r^{-d} for large r, d being the dimensionality of the problem.

IV. Aerodynamic Force and Moment

Stress Outside the Vortical Regions

Outside the vortical regions, the vorticity is zero and the viscous stress is absent. The absence of vorticity implies the existence of a scalar potential Φ in the inviscid region such that the negative of the gradient of Φ gives v. The inviscid momentum equation then yields

$$p = \rho \frac{\partial \Phi}{\partial t} - \rho \frac{v^2}{2} + f(t) \tag{19}$$

for the inviscid region.

For three-dimensional flows past finite solid bodies, the region in which Φ exists is singly connected and therefore Φ is independent of path. For two-dimensional flows, the region in which Φ exists is multiply connected. However, the cyclic constant for Φ is zero since the effective vortical regions are of finite extent and the total vorticity is zero. In consequence, Φ is single valued in both two- and three-dimensional flows.

Velocity Integrals in Large Bounded Regions

From the continuity and vorticity definition equations, one obtains the following expression for the velocity vector in two-dimensional flows:

$$v = (r \times \omega) + \nabla \cdot (rv) - \nabla (r \cdot v) \tag{20}$$

Let R_L be a circular region $r \leq L$. Let L be sufficiently large so that R_L contains all the solid regions R_j's, and that the total vorticity and the total vorticity moment outside R_L are negligible. The velocity on the boundary B_L is accurately given by Eq. (16). Integrating Eq. (20) over R_L and using the divergence theorem and its corollaries, one obtains

$$\int_{R_L} v \mathrm{d}R = \int_{R_L} (r \times \omega) \mathrm{d}R - \oint_{B_L} r \times (n \times v) \mathrm{d}B \tag{21}$$

Equation (21) here is derived by treating the fluid and the solid bodies in R_L as a combined kinematic system. This treatment is permissible since Eq. (21) is a kinematic equation. Equation (21) can also be derived by treating the solid and fluid regions in R_L separately.[19]

Since the total moment of vorticity is negligible outside R_L, the first term on the right-hand side of Eq. (21) is recognized to be α, defined by Eq. (17). If L is sufficiently large, then only the first term on the right-hand side of Eq. (16) has a non-negligible contribution to the last integral in Eq. (21). This first term gives a velocity vector on B_L as follows:

$$v = -\frac{\alpha}{2\pi L^2} + \frac{(r \cdot \alpha) r}{L^4} \tag{22}$$

Noting that $r = Ln$ on B_L, one obtains from Eq. (22)

$$r \times (n \times v) = -\frac{n \times (r \times \alpha)}{2\pi L^2} \tag{23}$$

Using Stokes' theorem, one then obtains

$$\oint_{B_L} r \times (n \times v) \mathrm{d}B = -\frac{1}{2\pi L^2} \int_{R_L} \nabla \times (r \times \alpha) \mathrm{d}R \tag{24}$$

It is simple to show that the integrand on the right-hand side of Eq. (24) is equal to $-\alpha$. It follows that the contribution of the boundary integral to the velocity integral is simply $-\alpha/2$. Equation (21) therefore becomes

$$\int_{R_L} v \mathrm{d}R = \frac{1}{2} \alpha \tag{25}$$

It is important to note that Eq. (25) is valid for any sufficiently large but finite value of L. Although Eq. (25) is independent of the value of L, one may not consider L to be infinitely large. The integral $\int_{R_\infty} v \mathrm{d}R$ is known to be indeterminate if $\alpha \neq 0$ since, according to Eq. (16), v is of order r^{-2} for large r values.

Equation (20), and hence also Eq. (25), is valid only for two-dimensional problems. For three-dimensional problems, one has, in place of Eq. (20),

$$v = \tfrac{1}{2} [r \times \omega + \nabla \cdot (rv) - \nabla (r \cdot v)] \tag{26}$$

Using Eq. (26), it can be shown,[19] by using a procedure similar to that just described for two-dimensional problems, that the three-dimensional version of Eq. (25) is

$$\int_{R_L} v \mathrm{d}R = \frac{1}{3} \alpha \tag{27}$$

Equation (27) is valid for any sufficiently large but finite value of L. The integral $\int_R v \mathrm{d}R$ is indeterminate if $\alpha \neq 0$ (Ref. 19).

Reference 12 gives a formula similar to Eq. (25) but with the coefficient for α, ½ in Eq. (25), replaced by 1. That formula is derived by considering a finite cylindrical volume bounded by two parallel planes and using Eq. (20). The derivation of the formula neglected the contribution of the last integral in Eq. (21) (see the equation between Eqs. A-3 and A-4 of Ref. 12). The formula is in error for any finite cylinder. As it turns out, however, if the cylinder is infinite in length, then the formula in Ref. 12 yields a correct formula for the aerodynamic force given later in this paper.

Velocity-Moment Integrals in Large Bounded Regions

Using the identity

$$r\times v=-\frac{1}{2}r^2\omega+\frac{1}{2}\nabla\times(r^2v) \tag{28}$$

one obtains, with the help of Stokes' theorem

$$\int_R r\times v\mathrm{d}R=-\frac{1}{2}\int_R r^2\omega\mathrm{d}R-\frac{1}{2}\int_B r^2(v\times n)\mathrm{d}B \tag{29}$$

In Eq. (29), let R be R_L, a spherical region in three dimensions, or a circular region in two dimensions, centered on the origin with the radius L. Let L be sufficiently large so that the total vorticity and total vorticity moments outside R_L are negligible. One has then

$$\oint_{B_L} r^2(v\times n)\mathrm{d}B=L^2\oint_{B_L} v\times n\mathrm{d}B=-L^2\int_{R_L}\omega\mathrm{d}R \tag{30}$$

where Stokes' theorem is used to obtain the last integral. This last integral is zero because of the principle of total vorticity conservation discussed earlier. One obtains, therefore, upon noting $r^2\omega$ is negligible outside R_L

$$\int_{R_\infty} r\times v\mathrm{d}R=-\frac{1}{2}\beta \tag{31}$$

where β is the integral of the second moment of vorticity defined by

$$\beta=\int_{R_\infty} r^2\omega\mathrm{d}R \tag{32}$$

Equation (32) relates velocity-moment integral to an integral of the second moment of vorticity. This equation is valid for both two- and three-dimensional problems.

Force Acting on Large Boundary

The force F_L acting on the boundary B_L is expressible in terms of the pressure p. Since the vorticity decays exponentially with r, the shear stress on R_L is negligible for sufficiently large value of L. Using Eq. (19), one obtains

$$F_L=-\rho\oint_{B_L}\left[\frac{\partial\Phi}{\partial t}-\frac{v^2}{2}+\frac{f(t)}{\rho}\right]n\mathrm{d}B \tag{33}$$

The asymptotic behavior of v shows that the term $v^2/2$ does not contribute to Eq. (33). The function $f(t)$, being independent of position, also does not contribute to Eq. (33).

Using Eq. (16), one obtains for two-dimensional flows,

$$\oint_{B_L}\frac{\partial\Phi}{\partial t}n\mathrm{d}B=-\frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}t}\oint_{B_L}\left(\nabla\ln\frac{1}{r}\cdot\alpha\right)n\mathrm{d}B \tag{34}$$

The integrand on the right-hand side of Eq. (34) can be rewritten as $-r\cdot\alpha/L^2$. Using Stokes' theorem, one obtains then

$$\oint_{B_L}\frac{\partial\Phi}{\partial t}n\mathrm{d}B=\frac{\mathrm{d}}{2\pi L^2\mathrm{d}t}\int_{R_L}\nabla(r\cdot\alpha)\mathrm{d}R \tag{35}$$

It is simple to show that $\nabla(r\cdot\alpha)=\alpha$ and, therefore, the right-hand side of Eq. (35) gives $\frac{1}{2}\mathrm{d}\alpha/\mathrm{d}t$. Therefore, one obtains the following expression for F_L in the case of two-dimensional problems:

$$F_L=-\frac{\rho}{2}\frac{\mathrm{d}\alpha}{\mathrm{d}t} \tag{36}$$

For three-dimensional flows, it can be similarly shown[19] that

$$F_L=-\frac{\rho}{6}\frac{\mathrm{d}\alpha}{\mathrm{d}t} \tag{37}$$

Aerodynamic Force

Consider the control volume R_L bounded externally by B_L and containing the fluid occupying the region R_f' as well as the solid bodies occupying the regions R_js. The momentum theorem gives

$$F_t=\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_L}\bar\rho v\mathrm{d}R+\oint_{B_L}\rho v(v\cdot n)\mathrm{d}B \tag{38}$$

where F_t is the total force acting on the system within R_L, and $\bar\rho$ is either the fluid density ρ or the solid density ρ_j, depending on the specific region within R_L.

On account of the asymptotic behavior of v, the last integral in Eq. (38) is negligible for sufficiently large values of L. The integral over R_L can be written as a sum of integrals as follows:

$$F_t=\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_L}\rho v\mathrm{d}R-\sum_{j=1}^{N}\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_j}\rho v\mathrm{d}R+\sum_{j=1}^{N}\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_j}\rho_j v\mathrm{d}R \tag{39}$$

where N is the total number of solid bodies present.

The total force F_t acting on the system in R_L consists of the force F_L acting on the boundary B_L and the external force F_e which is exerted from outside the system and acts on the solid bodies. The total force acting on the solid bodies is F_e+F, where F is the aerodynamic force exerted by the fluid on the solid bodies. Newton's second law of motion states that the last term of Eq. (39) is equal to F_e+F. In consequence, one obtains from Eq. (39)

$$F=F_L-\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_L}\rho v\mathrm{d}R+\sum_{j=1}^{N}\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_j}\rho v\mathrm{d}R \tag{40}$$

For two-dimensional flows, placing Eqs. (25) and (36) into Eq. (40) yields

$$F=-\frac{\mathrm{d}\alpha}{\mathrm{d}t}+\rho\sum_{j=1}^{N}\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_j}v\mathrm{d}R \tag{41}$$

For three-dimensional flows, placing Eqs. (28) and (38) into Eq. (41) yields

$$F=-\frac{1}{2}\rho\frac{\mathrm{d}\alpha}{\mathrm{d}t}+\rho\sum_{j=1}^{N}\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_j}v_j\mathrm{d}R \tag{42}$$

Equations (41) and (42) show that the aerodynamic force exerted by a fluid on solid bodies immersed in and moving relative to the fluid is equal to the inertia force due to the mass of fluid displaced by the solid bodies plus a term proportional to the time rate of change of total first moment of the vorticity field in the solid bodies and the fluid.

Moment of Aerodynamic Force

The theorem of moment of momentum gives

$$l_t \times F_t = \frac{\mathrm{d}}{\mathrm{d}t}\int_{R_L} \bar{\rho}(r\times v)\mathrm{d}R + \oint_{B_L} (r\times v)(v\cdot n)\mathrm{d}B \quad (43)$$

where l_t is a position vector describing the line of action of the total force F_t.

The last integral in Eq. (43) is negligible for sufficiently large L. The total moment consists of the moment $l_L \times F_L$ acting on the boundary B_L and the externally applied moment $l_e \times F_e$. The shear stress is negligible on B_L. Consequently the moment $l_L \times F_L$ is negligible. One thus has

$$l_e \times F_e = \frac{\mathrm{d}}{\mathrm{d}t}\int_{R_L} \bar{\rho}(r\times v)\mathrm{d}R \quad (44)$$

or

$$l_e \times F_e = \rho\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_L} r\times v\mathrm{d}R - \rho\frac{\mathrm{d}}{\mathrm{d}t}\sum_{j=1}^{N}\int_{R_j} r\times v\mathrm{d}R + \frac{\mathrm{d}}{\mathrm{d}t}\sum_{j=1}^{N}\int_{R_j}\rho_j(r\times v)\mathrm{d}R \quad (45)$$

The total moment acting on the solid bodies consists of the externally applied moment and the moment exerted by the fluid on the solid bodies. This total moment is equal to the time rate of change of angular momentum of the solid bodies, given by the last term of Eq. (45). One therefore obtains the following expressions for the first moment of aerodynamic force in both two- and three-dimensional problems by using Eq. (31):

$$l\times F = \frac{1}{2}\rho\frac{\mathrm{d}\beta}{\mathrm{d}t} + \rho\sum_{j=1}^{N}\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_j} r\times v_j\mathrm{d}R \quad (46)$$

where l is a position vector describing the line of action of the aerodynamic force F. Equation (46) expresses the first moment of the aerodynamic force as the time rate of change of an integral of a second moment of the vorticity vector plus a moment of inertia term. Higher moments of the aerodynamic force can be related to integrals of higher moments of the vorticity vector.

Equations (7), (41), (42), and (46) form a general theory for aerodynamic force and moment in viscous flows. This general theory encompasses many earlier aerodynamics theories, as shall be shown in the next section of this paper. It is obvious that Eq. (7) is invariant under a Galilean transformation of coordinates. Using Eq. (7), it is simple to show that Eqs. (41) and (42) are also invariant under a Galilean transformation of coordinates.

V. Linkage to Existing Theories

As discussed earlier in this paper, the general pattern of a viscous flow at moderate or high Reynolds numbers includes vortical regions surrounding and trailing the solid bodies. For simplicity, consider an airfoil or a wing moving relative to an infinite fluid. The vortical regions are composed of a near vortical system and a trailing vortical system. The near vortical system contains attached boundary layers and, in general, also detached recirculating flows. The vorticity that exists in the trailing vortical system at any given instant of time represents the vorticity "shed" from the near vortical system at previous time levels. Shortly after the initiation of the motion of the solid body, the vorticity region is confined to a thin layer near the solid body. A concentrated dose of vorticity (starting vortex) may leave the vicinity of the solid body. This starting vortex moves in the general downstream direction and becomes diffused with increasing time. In the present study, a number of steady flow problems are considered. Steady state is reached near the solid body, say inside a contour C which encloses the solid body, at some large time level, say τ. The total flux of vorticity transported across C must be zero, because of steady state, for time levels greater than τ. For convenience, all the vorticity that has left the region enclosed by C prior to the time level τ is considered together as a starting vorticity complex. The remainder of the trailing vortical system is, for convenience, called the wake. This wake stretches between the near vortical system and the starting vortex complex. The line of demarcation between the near vortical system and the trailing system is not delineated clearly. For the present purposes, a clear delineation is not essential. The division of the overall vortical region into its several components, however, is useful since it permits the contributions of each of these components to the aerodynamic force to be considered individually.

Under some circumstances, the extent of the recirculating flow is small. The thin boundary layers surrounding the solid merge at the trailing edge of the solid body and feed into the wake. Since the boundary layer is thin, the vortical wake is also thin. The vorticity in the boundary layers and the wake can then be reasonably approximated by sheets of concentrated vorticity. If massive flow separation occurs, however, the vorticity in the vortical region cannot be represented accurately by vortex sheets. The present theory offers insight to the problem of aerodynamic forces and moments whether or not massive flow separation occurs.

Lift in Steady Viscous Flow

Consider an airfoil (or an infinitely long cylinder of any given cross-sectional profile) set into motion relative to a fluid. According to Eq. (41), the lift per unit length of the airfoil span is given by

$$L = \rho\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_f}\int x\omega\mathrm{d}x\mathrm{d}y \quad (47)$$

where x and y are Cartesian coordinates defining the plane of the flow, with x axis in the freestream direction. The reference frame is attached to the airfoil, which is moving relative to the freestream fluid at a constant velocity $-U$.

Consider two time levels, τ_1 and τ_2, with $\tau_1 < \tau_2$. The vorticity distributions at these two time levels are identical near the airfoil. During the time interval $\tau_1 < t < \tau_2$, however, the starting vorticity complex has moved further downstream and the wake has grown longer. The starting vorticity complex, being very far from the airfoil, is being convected at essentially the freestream velocity U. The diffusion process, according to Eq. (2), does not contribute to a change of the total x moment of vorticity. In consequence, if the total vorticity (circulation) of the starting vorticity complex is Γ_s, then the contribution of the starting vortex complex to the lift is $\rho\Gamma_s U$. The value of Γ_s is independent of time since it represents all the vorticity shed from the near vortical system before a steady state is reached. The growth of the wake causes the total x moment of vorticity to increase at the rate

$$K = \int_{-\infty}^{\infty} u\omega x_s\mathrm{d}y \quad (48)$$

where x_s is the x coordinate of the "downstream boundary" of the wake, and u is the x component of the velocity on this boundary. Since this downstream boundary is very far from the airfoil, u is approximately equal to the freestream velocity U. The quantity K is therefore approximately equal to $U\Gamma_a$, where Γ_a is the total vorticity in the portion of wake between the line $x=x_l$, a constant, and the boundary x_s. The contribution to lift due to the growth of the wake is ρK, or $\rho U\Gamma_a$. The total lift can now be written as

$$L=\rho U\Gamma \tag{49}$$

where

$$\Gamma=\Gamma_s+\Gamma_a \tag{50}$$

Equation (49) is formally identical to the well-known Kutta-Joukowski theory that states that the lift per unit span length acting on an airfoil is equal to the product of the fluid density and the freestream velocity and of the negative (clockwise) circulation about the airfoil. It is significant to note that the Kutta-Joukowski theory, based as it is on the concept of an ideal fluid, can be viewed as a specialization of the present general theory for a viscous fluid.

Circulation and Bound Vorticity

The circulation has an unambiguous meaning in the context of viscous flow. It is the total vorticity in the starting vortex complex plus the total vorticity in the wake downstream of an x station, say x_l, which is far from the airfoil so that the velocity there is approximately the freestream velocity.

In actual experimental or computational determination of Γ, it is more convenient to make use of the principle of total vorticity conservation, Eq. (7), and let Γ be the negative of the total vorticity upstream of the station x_l. Thus, Γ is the clockwise circulation around a large contour that encloses all the near vortical system and excludes all the starting vorticity complex and that cuts the wake in a straight line perpendicular to the freestream direction at a station where the flow velocity is approximately the freestream velocity.

After reaching the foregoing conclusion, the author's attention was drawn to the detailed experimental data of Bryant and Williams[21] who measured, some fifty years ago, the flow around a 19% thick airfoil placed in a large wind tunnel at a 10.1 deg angle of attack. Bryant and Williams observed that the circulation determined experimentally for contours of the type just described, when placed into the Kutta-Joukowski theory, i.e., Eq. (49), gives lift values in excellent agreement with that determined by integrating the measured pressure distribution on the airfoil.

It is important to note that, with Eq. (49), there is no need to know the actual distribution of vorticity within the contour just described. Neither is there a need to know the details of vorticity transport within the fluid. Equation (49) is therefore valid whether or not massive flow separation occurs. In this context, since Γ is independent of time, it is permissible to represent the total vorticity inside the contour by a "bound" vortex filament or a bound vortex sheet. The word bound, unfortunately, is not descriptive of the physical processes of vorticity transport. In reality, vorticity is continually being generated at the airfoil surface and transported in the fluid. In a steady flow, the vorticity does not stay stationary relative to the airfoil. However, the processes of convection and diffusion continually replenish the vorticity at all locations within the contour so that the distribution of vorticity within the contour is independent of time. The word bound describes this time-independency rather than an attachment of the vorticity to space. This interpretation of the word bound is consistent with physical reality and does not violate the well-known theorems of Helmholtz and of Stokes.

Profile Drag in Steady Flow

Equation (41) gives the following formula for drag per unit length of a solid body in time-independent two-dimensional translational motion

$$D=-\rho\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_f}\int y\omega\,\mathrm{d}x\mathrm{d}y \tag{51}$$

Neither the convection nor the diffusion of the starting vorticity system contribute to Eq. (51). The growth of the wake causes the total y moment of vorticity to increase at the rate

$$J=\int_{-\infty}^{\infty}u\omega y\,\mathrm{d}y \tag{52}$$

where the integration is performed at the "downstream boundary" of the wake. For large time levels, the x component of velocity at the downstream boundary may be written as $u=U-u'$, where U is the freestream velocity and u', the perturbation velocity, is much smaller than U. The vorticity can be approximated by

$$\omega=-\frac{\partial u}{\partial y}=\frac{\partial u'}{\partial y} \tag{53}$$

Using Eq. (53) in Eq. (52) and neglecting terms of order u'^2, one obtains

$$J=U\int_{-\infty}^{\infty}y\frac{\partial u'}{\partial y}\mathrm{d}y \tag{54}$$

Integrating by parts then gives, upon noting that $u'=0$ at $y=\pm\infty$,

$$J=-U\int_{-\infty}^{\infty}(U-u)\,\mathrm{d}y \tag{55}$$

The drag D is, therefore, according to Eq. (51), given by

$$D=\rho U\int_{-\infty}^{\infty}(U-u)\,\mathrm{d}y \tag{56}$$

where the integration is performed at an x station far from the solid body.

Equation (56) is a familiar formula easily derived by using the momentum theorem.

Induced Drag

Consider a finite wing set into motion in a fluid. The vorticity field, as shown earlier, is solenoidal and is effectively confined to a finite fluid region surrounding the wing. In the present three-dimensional flow, all vorticity lines in the fluid form closed loops. At sufficiently large time levels after the motion's onset, if the vorticity distribution near the wing is time independent in a reference frame attached to the wing, then the entire vorticity system can be represented by a "horseshoe" vorticity system. Each horseshoe vorticity line is composed of a bound vorticity line (the meaning of the word "bound" is as described earlier) and a pair of trailing vorticity lines that are connected far downstream by a "starting" vorticity line. These horseshoe loops grow with time.

According to Eq. (42), the drag acting on the wing is given by

$$D=\frac{\mathrm{d}}{\mathrm{d}t}\int\int_{R_f}\int(z\eta-y\zeta)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \tag{57}$$

where x, y, z form a right-handed Cartesian coordinate system, with the x axis in the freestream direction and the z axis in the span direction; η and ζ are, respectively, the y and z components of the vorticity vector.

The total drag is composed of the profile drag and the induced drag. The profile drag, given by Eq. (51), is related to the increase of the y moment of ζ due to the growth of the wake. The induced drag is related to the increase of the z moment of η due to the growth of the wake and the movement of the starting vortex complex (ζ component of vorticity) in the y direction. The vorticity component η can be related to the rate of change of circulation of the starting vorticity system in the z direction. In consequence, one obtains the following equation for the induced drag D_i:

$$D_i = \frac{1}{2}\rho\int_{-\infty}^{\infty} vz\left(\frac{d}{dz}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} dxdy\right)dz - \frac{1}{2}\rho\int_{-\infty}^{\infty} dz\int_{-\infty}^{\infty} dy\int_{x_1}^{\infty} \zeta v dx \tag{58}$$

where x_1 is a sufficiently large value.

Equation (58) is a generalized version of the familiar finite-wing formula derived in many treatises, e.g., Ref. 11, on the basis of downwash at the wing. To demonstrate the use of Eq. (58), let the starting vorticity system be represented by a single vortex filament and the trailing vorticity lines be represented by a sheet. The integration of ζ with respect to x and y in Eq. (58) gives simply $\Gamma(z)$, which is the circulation of the starting vorticity system. Equation (58) therefore reduces to

$$D_i = \frac{1}{2}\rho\int_{-\infty}^{\infty} vz\frac{d\Gamma}{dz}dz - \frac{1}{2}\rho\int_{-\infty}^{\infty} v\Gamma dz \tag{59}$$

Consider an elliptical variation of circulation of the starting vortex

$$\Gamma = \begin{cases} \Gamma_b 1-\left(\dfrac{z}{b/2}\right)^2, & -\dfrac{b}{2} < z < \dfrac{b}{2} \\ =0, & \text{elsewhere} \end{cases} \tag{60}$$

One obtains, from Eqs. (59) and (60), upn approximating the trailing vortex sheet by a semi-infinite flat stripe and noting that the trailing vortex sheet induces in this case a uniform y velocity component $-\Gamma b/2b$, the following formula:

$$D_i = \rho\Gamma_s^2/8 \tag{61}$$

Equation (61) is a well-known formula for the minimum induced drag associated with an "elliptical wing."

The analysis just presented shows that, with the present theory, the induced drag can be determined through a study of the vortical wake. The "downwash" is evaluated at a wake location rather than at the wing. The conceptual difficulties associated with "bound vorticity" is not present. The main utility of the present analysis is, of course, not in the rederivation of known formulas, such as Eq. (61), but in the alternative interpretation of aerodynamic phenomena it provides.

Apparent Mass Properties

The present theory is well suited for the study of time-dependent flows. As an example, the apparent mass properties of a solid body undergoing linear and angular accelerations in an incompressible fluid have been studied[22] using Eqs. (7), (40), and (46). Equations giving the apparent mass and the apparent moment of inertia have been obtained for arbitrarily prescribed motions of one or more solid bodies. For the special case where there is only one solid body, the equations can be simplified and shown[22] to be equivalent to the equations obtained using the concept of impulse and kinetic energy.[23]

VI. Concluding Remarks

A general theory for the aerodynamic force and the moment of the force has been developed through a rigorous analysis of the viscous flow equations. General formulas relating aerodynamic force and moment acting on one or more solid bodies to rates of change of vorticity moments in the fluid and the solid regions has been derived. Various elements of the aerodynamic force—the lift, the profile drag, and the induced drag—have been individually connected to the convection of vorticity in the far wake. An unambiguous definition of circulation for viscous flows has been obtained.

The starting point of the present theory is a rotational flow analysis. No simplifying assumptions, other than those contained in the familiar Navier-Stokes equations, are utilized in the derivation of the general formulas forming the theory. The concept of bound vortex, or that of other singularity elements, is not a part of the present analysis. This freedom from"bondage" has led to improved understanding of the viscous origin of the lift and the drag. Many existing theories on aerodynamic forces have been interpreted as various levels of approximations of the present viscous theory.

The concept of vorticity distribution in both the fluid region and the solid region has been utilized to decompose the overall incompressible viscous flow problem into its kinetic and kinematic aspects. Considerable insight has been gained by examining the mathematical descriptions of each of the two aspects without employing detailed mathematical or numerical analyses. This insight has enabled the rigorous development of the general theory for incompressible flows presented in this paper. In particular, the facts that the differential equations governing the kinematics of the problem are linear and that the solid bodies and the fluid can be treated together as one kinematical system occupying a limitless space have been recognized. This recognition facilitated the needed mathematical derivations of the present theory considerably. Extension of the present theory for compressible flow requires a study of the kinematics and the kinetics of a flow in which the dilatation field is not zero. This study is now in progress. It has been observed that, for the compressible flow, the differential equations for the kinematics are still linear and the solid bodies and the fluid can be still treated as one kinematical system occupying a limitless space.

The present paper emphasizes the analytical aspect of the present theory and represents the initial efforts in the utilization of the general theory. Additional results, including some results demonstrating the application of the theory to the computational branch of fluid dynamics, have been obtained.[22] It is planned to describe these results in a future paper with special emphasis given to the computational aspect of the theory.

Acknowledgment

The research is supported by the Office of Naval Research under Contract N00014-75-C-0249.

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