AIAA Paper
No. 73-313

(3)

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FINITE THICKNESS WINGS

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AIAA PAPER 73-313

# AIAA Dynamics Specialists Conference

WILLIAMSBURG, VIRGINIA / MARCH 19-20, 1973



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# UNSTEADY SUBSONIC COMPRESSIBLE FLOW AROUND FINITE THICKNESS WINGS†

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## Abstract

A general formulation for the unsteady subsonic compressible potential flow around aircraft having arbitrary configurations is presented. An integral representation of the velocity potential is obtained. From this a linear integral equation relating the perturbation potential and its normal derivative (which is known from the boundary conditions) is derived. For the numerical solution of the integral equation, the surface of the aircraft is divided into small elements and the potential is assumed to be constant within each element. Numerical results are obtained for an oscillating finite-thickness wing and indicate good convergence and excellent agreement with existing lifting surface solutions.


## Nomenclature

| | |
|---|---|
| $a_\infty$ | free stream speed of sound |
| $b$ | span of wing |
| $b_k$ | Eq. (16) |
| $c$ | chord of wing |
| $C_p$ | pressure coefficient |
| $C_\ell$ | lift distribution coefficient |
| $C_{\ell\alpha}$ | $\partial C_\ell / \partial\alpha$ |
| $c_{ki}$ | Eq. (17) |
| $G$ | Green function |
| $h$ | thickness of wing |
| $M$ | Mach number, $U_\infty/a_\infty$ |
| $\vec{n}$ | normal to surface in x,y,z space |
| $\vec{N}$ | normal to surface in X,Y,Z space |
| NX,NY | number of elements in X,Y directions |
| $R$ | Eq. (8) |
| $S_B$ | equation describing the surface $\Sigma_B$ |
| $t$ | time |
| $T$ | $a_\infty \beta t$ |
| $x,y,z$ | space coordinate |
| $X,Y,Z$ | Prandtl-Glauert coordinate |
| $U_\infty$ | free-stream velocity |
| $\alpha$ | angle of attack |
| $\beta$ | $(1-M^2)^{1/2}$ |
| $\Sigma$ | surface surrounding body and wake |
| $\Sigma_B$ | surface surrounding body |
| $\tau$ | thickness ratio |
| $\Phi = U_\infty(x+\varphi)$ | velocity potential |
| $\varphi$ | perturbation velocity potential |
| $\hat{\varphi}$ | complex perturbation potential, Eq. (6) |
| $\varphi_n$ | normal derivative of $\varphi$ |
| $\hat{\varphi}_n$ | Eq. (9) |
| $\omega$ | frequency of oscillation |
| $\Omega$ | $\omega/a_\infty\beta$ |
| $A\!\!\!R$ | aspect ratio |
| $\xi$ | dummy variable of integration |
| $\Delta\hat{\varphi}$ | $\hat{\varphi}_u - \hat{\varphi}_\ell$ |

## I. Introduction

The evaluation of the pressure distribution over bodies in compressible unsteady potential flow is one of the fundamental problems in flight dynamics, aeroelasticity and related fields. An excellent review of the state of the art was given by Ashley and Rodden.[1] In case of interacting wings and tails the problem is solved by using well known lifting surface theories[2,3] which usually require complicated methods of solution. Considerable improvement was obtained with the method proposed by Albano and Rodden.[4] The problem of interaction of wings and bodies was analyzed by Giesieng, Kalman and Rodden[5] by lifting surface and slender-body theories combined with the method of images.

A general theory of the unsteady compressible potential flow around lifting bodies having arbitrary configurations and motions is presented in Refs. 6 and 7. The main results are summarized here. The basic tool employed in the theory is the Green function method. As well known, applying the Green function method for the wave equation yields the Huygens' Principle (or Kirchhoff's formula). The theory of Refs. 6 and 7 is a generalization of the Huygens' Principle to the equation of the aerodynamic potential in a frame of reference traveling at velocity $U_\infty$ with respect to the undisturbed air. In addition, there are cases (such as helicopter blades or spinning missiles, for instance) in which the surface of the aircraft cannot be assumed to be fixed with respect to the frame of reference, even when this is traveling at velocity $U_\infty$ with respect to the undisturbed air. Hence, the surface $\Sigma$ (surrounding body and wake) is assumed to be moving with respect to the frame of reference.

The general theory was applied to the problem of an oscillating aircraft in subsonic flow. The configuration is assumed to be completely arbitrary but the motion is restricted to harmonic oscillation of small amplitude. This implies[6] that the surface of the body can be assumed to be time-independent, while the boundary conditions take into account the motion of the body. The method is applied to wings in steady and oscillatory subsonic flow in Refs. 8 and 9, respectively. The results are presented here.

†This work was supported by NASA Langley Research Center under NASA Grant NGR 22-004-030
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73-313

## II. Formulation of the Problem

The linearized equation for the perturbation velocity potential, $\varphi$ , for a flow having free stream velocity $U_\infty$ in the direction of the positive x axis is given by

$$\nabla^2 \varphi = \frac{1}{a_\infty^2}\left(\frac{\partial}{\partial t} + U_\infty \frac{\partial}{\partial x}\right)^2 \varphi \tag{1}$$

Let the surface of the body be described in the general form*

$$S_B\,(x,y,z,t) = 0 \tag{2}$$

Then the boundary conditions on the body are given by $DS_B/Dt = 0$ or**

$$\varphi_n \equiv \frac{\partial \varphi}{\partial n} \equiv \frac{\nabla S_B}{|\nabla S_B|}\cdot\nabla\varphi = -\frac{1}{|\nabla S_B|}\left(\frac{1}{U_\infty}\frac{\partial S_B}{\partial t} + \frac{\partial S_B}{\partial x}\right) \tag{3}$$

Finally, the pressure coefficient is given by the linearized Bernoulli theorem

$$C_p = -2\left(\frac{1}{U_\infty}\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial x}\right) \tag{4}$$

In order to solve this problem, it is convenient to transform it into an integral equation. As mentioned above, the problem considered here is the one of small harmonic oscillations around a fixed configuration. In this case, introducing the generalized Prandtl-Glauert transformation,

$$X = x/\beta \qquad Y = y \qquad Z = z \qquad T = a_\infty \beta t \tag{5}$$

and the complex potential $\hat{\varphi}$ such that

$$\varphi(X,Y,Z,T) = \hat{\varphi}\,(X,Y,Z)\, e^{i\Omega(T+MX)} \tag{6}$$

one obtains (see Appendix A)

$$4\pi E\,\hat{\varphi}(X,Y,Z) = -\oint\!\!\oint_\Sigma \hat{\varphi}_n \frac{e^{-i\Omega R}}{R}\, d\Sigma + \oint\!\!\oint_\Sigma \hat{\varphi}\frac{\partial}{\partial N_1}\left(\frac{e^{-i\Omega R}}{R}\right) d\Sigma \tag{7}$$

where $\Omega = \omega/a_\infty\beta$, E is given by Eq. (A.7),

$$R = \left[(X_1-X)^2+(Y_1-Y)^2+(Z_1-Z)^2\right]^{\frac{1}{2}} \tag{8}$$

and

$$\hat{\varphi}_n = \varphi_n\, e^{-i\Omega(T+MX)} \tag{9}$$

with $\varphi_n$ prescribed by the boundary conditions, Eq. (3).

It should be noted that, outside $\Sigma$ , Eq. (7) is an integral representation of the potential $\hat{\varphi}$ in terms of the values of $\hat{\varphi}$ and $\hat{\varphi}_n$ on the surface $\Sigma$ . On the other hand, on $\Sigma$ , Eq. (7) is an integral equation relating the unknown values of $\hat{\varphi}$ on $\Sigma$ , to the known values of $\hat{\varphi}_n$ . Once this equation has been solved, Eq. (7) can be used for evaluating $\hat{\varphi}$ as well as its derivatives (perturbation velocity components) and also the pressure given by (see Eq. (4))

$$\hat{c}_p = c_p\, e^{-i\Omega(T+MX)} = -\frac{2}{\beta}\left(i\frac{\Omega}{M}\hat{\varphi} + \frac{\partial\hat{\varphi}}{\partial X}\right) = -\frac{2}{\beta}\, e^{-i\Omega X/M}\frac{\partial}{\partial X}\left(\hat{\varphi}\, e^{i\Omega X/M}\right) \tag{10}$$

The numerical procedure for the solution of the integral equation is given in the following.

## III. Numerical Procedure

If the point (x,y,z) is on $\Sigma$ , Eq. (7) reduces to (E = 1/2, Eq. (A.7))

$$\hat{\varphi}(X,Y,Z) = -\oint\!\!\oint_\Sigma \hat{\varphi}_n \frac{e^{-i\Omega R}}{2\pi R}\, d\Sigma + \oint\!\!\oint_\Sigma \hat{\varphi}\frac{\partial}{\partial N_1}\left(\frac{e^{-i\Omega R}}{2\pi R}\right) d\Sigma \tag{11}$$

It should be noted that, if $\hat{\varphi}_n$ is known everywhere on $\Sigma$ , Eq. (11) is a Fredholm integral equation of second kind, whose solution exists and is unique (solution of the external Von Neuman's problem for the Helmholtz equation). However, the branch of the surface $\Sigma$ , surrounding the wake, is not known and, for simplicity, is assumed to be composed of straight vortex lines emanating from the trailing edge and parallel to the direction of the undisturbed flow. Since the pressure is continuous across the wake,

$$\Delta\hat{\varphi}\, e^{i\Omega X/M} = \text{const} = \Delta\hat{\varphi}_{TE}\, e^{i\Omega X_{TE}/M} \tag{12}$$

Noting that the source contribution of the wake is equal to zero, Eq. (11) reduces to

$$\hat{\varphi} = -\oint\!\!\oint_{\Sigma_B} \hat{\varphi}_n \frac{e^{-i\Omega R}}{2\pi R}\, d\Sigma + \oint\!\!\oint_{\Sigma_B} \hat{\varphi}\frac{\partial}{\partial N_1}\left(\frac{e^{-i\Omega R}}{2\pi R}\right) d\Sigma + \frac{1}{2\pi}\int_{TE} \Delta\hat{\varphi}_{TE}\left[(Y-Y_1)\frac{\partial Z_{TE}}{\partial Y_1} - (Z-Z_1)\right] \hat{I}\; dY_1 \tag{13}$$

where $\Sigma_B$ is the surface of the body, $\int_{TE}$ indicates the integration along the trailing edge and

$$\hat{I} = \frac{1}{Z_1 - Z}\int_{X_{TE}}^{\infty} e^{i\frac{\Omega}{M}(X_{TE}-X_1)}\frac{\partial}{\partial Z_1}\left(\frac{e^{-i\Omega R}}{R}\right) dX_1 \tag{14}$$

The integral with respect to $X_1$ is evaluated analytically and is given in Appendix B. Except for a few special cases, Eq. (13) must be solved by using some approximate numerical method. The one used here was chosen mainly because of its flexibility (for wing-body-tail interaction, for instance). The surface $\Sigma_B$ is divided in N small finite surface elements, $\Sigma_i$ , see Fig. 1. The potential $\hat{\varphi}$ is assumed to be constant within each element and equal to the value $\hat{\varphi}_i$ , at the centroid of the element $\Sigma_i$ .

*For convenience, the arbitrary multiplicative sign in Eq. (2) is chosen so that $\nabla S$ has the direction of the outwardly directed normal $\vec{n}$.

**Note that: $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \nabla\Phi\cdot\nabla = \frac{\partial}{\partial t} + U_\infty \nabla(x+\varphi)\cdot\nabla$

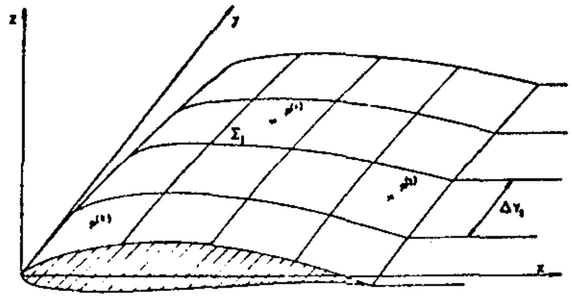

Fig. 1 Geometry of problem

Then, by satisfying Eq. (13), at the centroid of each element, one obtains a system of N linear equations and N unknowns

$$\hat{\varphi}_k = b_k + \sum_{i=1}^{N} c_{ki}\,\hat{\varphi}_i + \sum_{i=1}^{N} w_{ki}\,\hat{\varphi}_i \qquad (k=1,\dots,N) \tag{15}$$

where

$$b_k = -\oint_{\Sigma_B} \hat{\varphi}_n \, \frac{e^{-i\Omega R_k}}{2\pi R_k}\, d\Sigma \tag{16}$$

and

$$c_{ki} = \iint_{\Sigma_i} \frac{\partial}{\partial N_i}\left(\frac{e^{-i\Omega R_k}}{2\pi R_k}\right) d\Sigma_i \tag{17}$$

and $w_{ik}$, which represents the contribution of the wake, is given by ($\Delta Y_\ell$ is indicated in Fig. 1)

$$w_{ki} = \frac{s}{2\pi}\int_{\Delta Y_\ell}\left[(Y-Y_1)\frac{\partial Z_{TE}}{\partial Y_1} - (Z-Z_1)\right]\hat{I}\, dY_1 \tag{18}$$

($s = 1$ on the upper surface and $s = -1$ on the lower surface) for the elements, $\Sigma_\ell$, in contact with the trailing edge, and $w_{ki} = 0$ otherwise. In Eqs. (16) and (17), $R_k$ is the distance of the centroid of the element $\Sigma_k$ from the dummy point of integration. The coefficients $c_{ki}$ are evaluated analytically by assuming that the element $\Sigma_i$ is replaced by its tangent plane at the centroid and the exponential is replaced with its value at the centroid. A similar procedure is used for $b_k$. The corresponding analytical expressions for $c_{ki}$ and $b_k$ are given in Ref. 9.

The method presented above was applied to wings in steady and oscillating subsonic flow[8,9]. The wing geometry is given by (for zero angle of attack)

$$\begin{aligned} x &= x_{TE} + c\,\xi \\ y &= \frac{b}{2}\,\eta \\ z &= \pm\,\tau\, c_0\, \frac{3\sqrt{3}}{4}\sqrt{\xi}\,(1-\xi)\sqrt{1-\eta^2} \end{aligned} \tag{19}$$

where c is the chord and b is the span of the wing. In order to eliminate the square-root behavior of the potential at the leading edge and the tip of the wing, the transformation

$$\xi = \bar{X}^2 \qquad\qquad \eta = 1-(1-\bar{Y})^2 \tag{20}$$

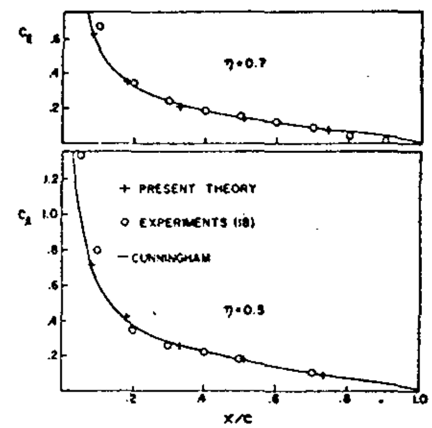


Fig. 2 Rectangular Wing ($A\!\!\!/ = 3$, M = .24 $\alpha = 5°$): $C_p$ at $\eta = .5$ and .7

was introduced. Surface elements with projections in the $\bar{X}$, $\bar{Y}$ plane of constant size

$$\Delta\bar{X} = 1/NX \qquad\qquad \Delta\bar{Y} = 1/NY \tag{21}$$

(NX and NY are the number of elements in $\bar{X}$ and $\bar{Y}$ direction, on half wing) were used. The centroid is the center of element in the $\bar{X}$, $\bar{Y}$ plane. Finally, the lift distribution coefficient

$$C_\ell = -\left(C_{p,u} - C_{p,\ell}\right)e^{-i\Omega T} \tag{22}$$

is evaluated as (see Eq. (10))

$$C_\ell = 2\, e^{-i\Omega\frac{\beta^2 X}{M}}\,\frac{\partial}{\partial\bar{X}}\left[(\hat{\varphi}_u - \hat{\varphi}_\ell)\, e^{i\frac{\Omega X}{M}}\right]\frac{d\bar{X}}{dx} \tag{23}$$

where $d\bar{X}/dx = 1/2c\bar{X}$ and the derivative with respect to $\bar{X}$ is evaluated by finite difference.

The numerical results for wings in steady state are presented in Figs. 2-6. The analysis of convergence and thickness effect are presented in Ref. 7 where it is shown that the convergence is very fast and that the effect of the thickness is small. In Fig. 2, the results for a rectangular wing at an angle of attack $\alpha = 5°$ (aspect ratio $A\!\!\!/ = 3$ and Mach number M = .24) are compared to experimental results of Lessing et al.[11], and theoretical results by Cunningham.[12] In Fig. 3, the results for a rectangular wing of aspect ratio $A\!\!\!/ = 1$ at M = .2 are compared to the ones of Hsu[13], Kulakowski and Haskell[14], and Cunningham[12], while in Fig. 4, the results for a delta wing with $A\!\!\!/ = 2.5$ and M = 0 are compared to the ones obtained by Widnall (see Ref. 15). Finally in Fig. 5, a tapered wing of $A\!\!\!/ = 3$, taper ratio T.R. = .5 and M = .8 are compared with the theoretical results of Cunningham[12] and the experimental ones of Kolbe and Boltz.[16]

In Fig. 6, the section lift coefficients $C_L$ (which are generally more significant than the pressure distribution since small error on $C_p$ near the leading edge, may yield large errors on $C_L$) are compared to the values obtained by Yates[17] for a rectangular wing with $A\!\!\!/ = 4$ and M = .507.

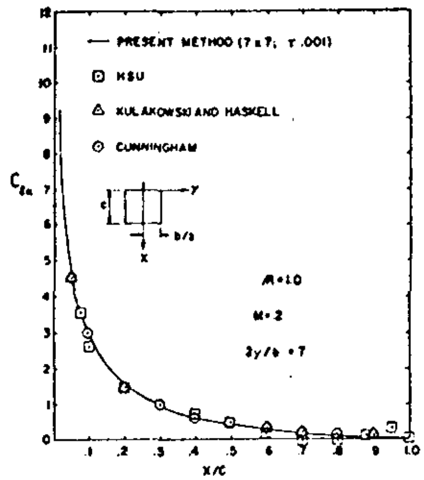


Fig. 3 Rectangular wing ( $AR$ = 1, M = .2): $C_{l\alpha}$ at $\eta$ = .7

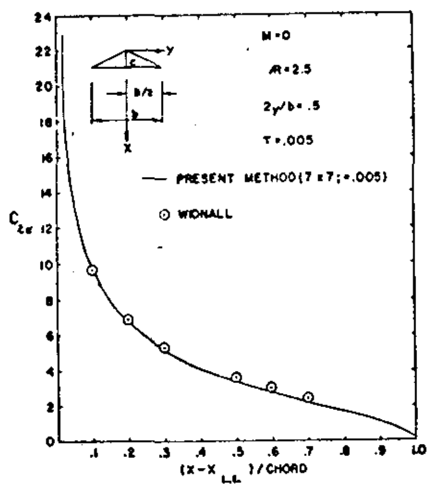


Fig. 4 Delta wing ( $AR$ = 2.5, M = 0): $C_{l\alpha}$ at $\eta$ = .5

Results for oscillating wings are presented in Figs. 7-11. Figs. 7-9 present the results for a rectangular wing with aspect ratio 2, M = 0 and reduced frequency $k = \omega c/U_\infty = 2$, rigidly oscillating around the axis x = c/2. Fig. 7 presents the analysis of convergence and Fig. 8 the thickness effect. These confirm the results obtained for the steady state problems that the convergence is very fast and that the thickness effect is very small. In Fig. 9, the results are compared to the ones obtained by Laschka.[18]

Figs. 10 and 11 present the results for a rectangular wing with aspect ratio $AR$ = 3, oscillating in a bending mode described by [4,11]

$$z = .18043\,|y/s| + 1.70255\,|y/s|^2 - 1.113688\,|y/s|^3 + .25387\,|y/s|^4 \qquad (24)$$

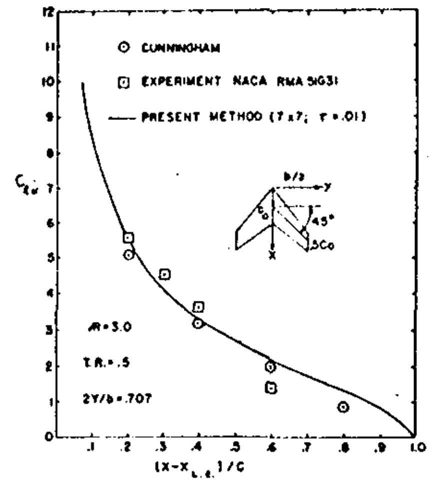


Fig. 5 Tapered Wing ( $AR$ = 3, T.R. = 5, M = .8): $C_{l\alpha}$ at $\eta$ = .707

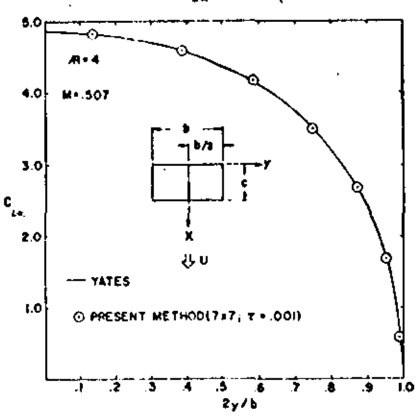


Fig. 6 Section lift coefficient $C_{L\alpha}$ for rectangular wing ( $AR$ = 4, M = .507)

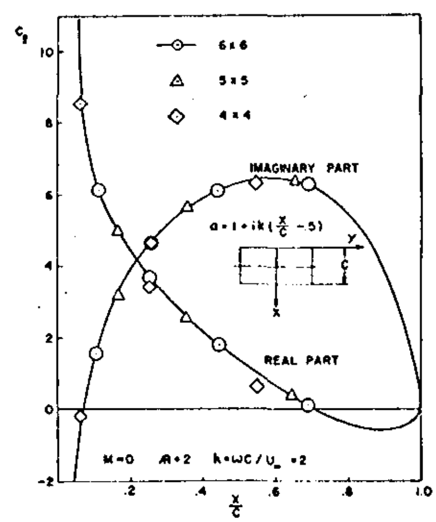


Fig. 7 Analysis of convergence: $C_l$ at root elements (NX = NY = 4,5,6; $\tau$ = .001) for rectangular wing oscillating around axis x = c/2 ( $AR$ = 2, M = 0, k = 2)

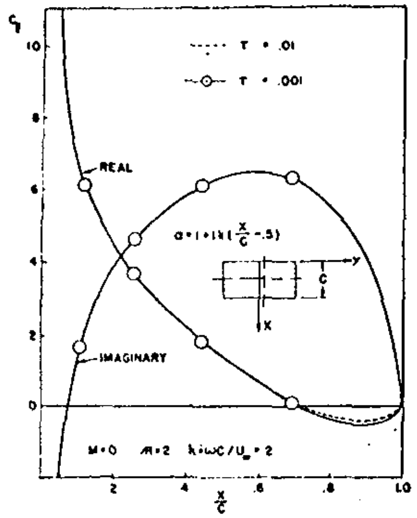


Fig. 8 Effect of thickness (NX = NY = 7) for rectangular wing oscillating around axis x = c/2 ( Æ = 2, M = 0, k = 2)

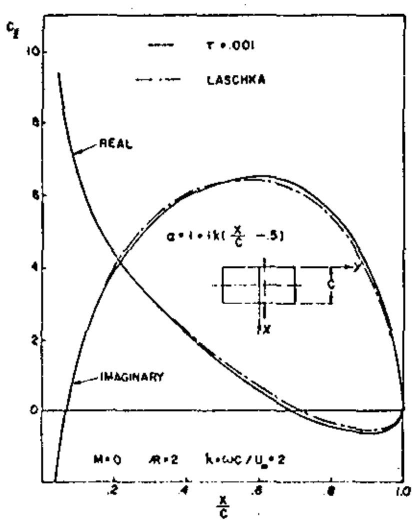


Fig. 9 Rectangular wing oscillating around axis x = c/2 ( Æ = 2, M = 0, k = 2): $C_l$ at $\eta$ = .15 (NX = NY = 7)

where s is the semispan of the wing. The results (for M = .24 and $k = \omega c/2U_\infty = .47$ and thickness ratio $\tau$ = .005) are normalized by dividing by the velocity of the wing at the tip. The results are compared to the ones obtained by Lessing et al.[11], and Albano and Rodden.[4]

In conclusion, if one takes into account the fact that the numerical formulation is still very crude, the results presented above indicate that the agreement of the present method with existing results is surprisingly good. More refined numerical formulations are now being explored.

*These results were obtained on the IBM 360/50 at Boston University. An advantage was taken of symmetry with respect to the y-axis and antisymmetry with respect to the z-axis.

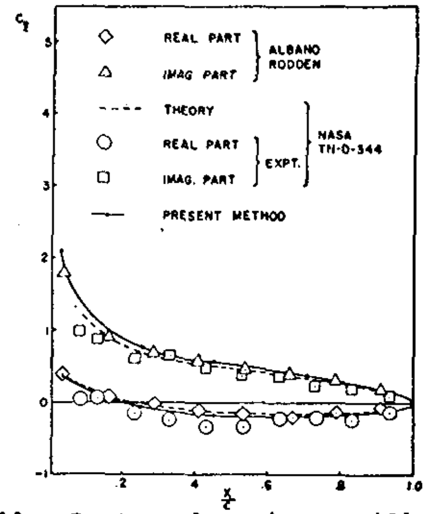


Fig. 10 Rectangular wing oscillating in bending mode ( Æ = 3 , M = .24, k = .47): $C_l$ at $\eta$ = 0

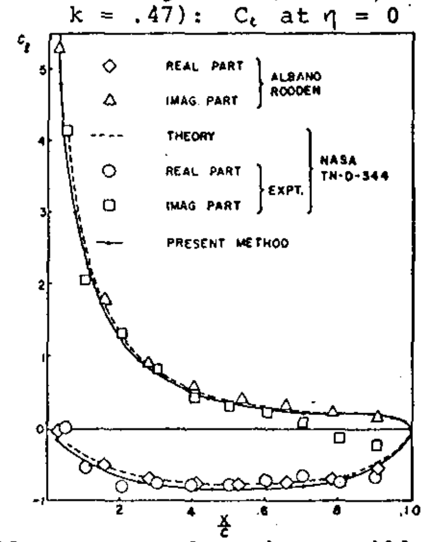


Fig. 11 Rectangular wing oscillating in bending mode ( Æ = 3, M = .24, k = .47): $C_l$ at $\eta$ = .9

## IV. Comments

The results obtained indicate that the method is accurate even for very small thickness ratios (good results were obtained even for thickness ratio $\tau$= 1/1000, although some elimination of significant figures was encountered). Furthermore, the computer time was surprisingly small (from 13 seconds for NX = NY = 4 to 128 seconds for NX = NY = 7 for the steady case and from 39 seconds for NX = NY = 4 to 159 seconds for NX = NY = 6 for the unsteady case).* In conclusion, the method has a very broad applicability since it is accurate, fast and can cover steady and unsteady flow around complex configurations. Application to complex configurations is now under way. Supersonic flow is also under consideration.

## V. Appendix A

In this Appendix, Eq. (7) is derived. Under the assumption that the surface is time independent, the steady and oscillatory components of the flow can be separated.[6] The unsteady flow is then given by

$$\varphi(x,y,z,t) = \tilde{\varphi}(x,y,z)\, e^{i\omega t} \tag{A.1}$$

and Eq. (1) reduces to

$$\nabla^2 \tilde{\varphi} = \frac{1}{a_\infty^2}\left(i\omega + U_\infty \frac{\partial}{\partial x}\right)^2 \tilde{\varphi} \tag{A.2}$$

Equation (A.2) simplifies considerable by introducing the Prandtl-Glauert transformation, Eq. (5) and the transformation (compare to Eq. (6))

$$\tilde{\varphi} = \hat{\varphi}\, e^{i\Omega M X} \tag{A.3}$$

which yields

$$\frac{\partial^2 \hat{\varphi}}{\partial X^2} + \frac{\partial^2 \hat{\varphi}}{\partial Y^2} + \frac{\partial^2 \hat{\varphi}}{\partial Z^2} + \Omega^2 \hat{\varphi} = 0 \tag{A.4}$$

The Green function for Eq. (A.4) is given by

$$G = -\frac{e^{-i\Omega R}}{4\pi R} \tag{A.5}$$

The corresponding Green formula is

$$4\pi E\, \hat{\varphi} = -\oint_{\Sigma} \left( \frac{\partial \hat{\varphi}}{\partial N_1} \frac{e^{-i\Omega R}}{R} - \hat{\varphi} \frac{\partial}{\partial N_1} \frac{e^{-i\Omega R}}{R} \right) d\Sigma \tag{A.6}$$

with E given by [6,9]

$$\begin{aligned} E &= 0 \quad && \text{inside } \Sigma \\ &= 1/2 && \text{on } \Sigma \\ &= 1 && \text{outside } \Sigma \end{aligned} \tag{A.7}$$

Neglecting terms of the same order of magnitude as the nonlinear terms (neglected in Eq. (1)), one obtains $\partial\hat{\varphi}/\partial N_1 \simeq \hat{\varphi}_n$ with $\hat{\varphi}_n$ given by Eq. (9) and thus, Eq. (A.6) reduces to Eq. (7).

## VI. Appendix B

Performing the integration in Eq. (14), one obtains [6,9]

$$\begin{aligned} \hat{I} = -e^{i\frac{\Omega}{M}(X_{TE}-X)} & \left[ \frac{\Omega\beta}{M R_0} \left\{ K_1(\kappa) + \frac{\pi i}{2} I_1(\kappa) \right\} \right. \\ & \left. + \frac{X - X_1}{R_0^2}\, e^{i\Omega\left(\frac{X - X_1}{M} - R\right)} + \frac{1}{R_0^2} F(u) \right]_{X_1 = X_{TE}} \end{aligned} \tag{B.1}$$

where $R_0^2 = (Y_1 - Y)^2 + (Z_1 - Z)^2$, $\kappa = \Omega\beta R_0/M$ and $I_1$ and $K_1$ are the modified Bessel functions of first order of first and second kind, respectively. Finally, the function F is given by

$$\begin{aligned} F(u) &= -i\kappa \int \frac{u\, e^{-i\kappa u}}{(1+u^2)^{1/2}}\, du \\ &= \sum_1^{\infty} F_n(u) \end{aligned} \tag{B.2}$$

where $F_n$ can be evaluated by using the recurrent formula

$$\begin{aligned} F_n(u) = \frac{1}{n!}(-i)^n \kappa^n \sqrt{1+u^2}\, u^{n-1} \\ + \frac{\kappa^2}{n(n-2)} F_{n-2}(u) \end{aligned} \tag{B.3}$$

with

$$\begin{aligned} F_1(u) &= -i\kappa\sqrt{1+u^2} \\ F_2(u) &= -\frac{\kappa^2}{2}\left[u\sqrt{1+u^2} - \ln\left(u + \sqrt{1+u^2}\right)\right] \end{aligned} \tag{B.4}$$

and $u = \left[M\sqrt{(x_{TE}-x)^2 + R_0^2} + (x_{TE}-x)\right]/\beta R_0$. Note that as shown in Ref. 6 for $Z = Z_1$, $\hat{I}$ is proportional to the Kernel function in Ref. 19.

## References


1. Ashley, H., and Rodden, W.P., "Wing-Body Aerodynamic Interaction", Annual Review of Fluid Mechanics, Vol. 4, 1972, pp. 431-472.

2. Ashley, H., Widnall, S., and Landahl, M.T., "New Directions in Lifting Surface Theory", AIAA J., Vol. 3, No. 1, Jan. 1965, pp. 3-16.

3. Landahl, M.T. and Stark, V.J.E., "Numerical Lifting-Surface Theory -- Problems and Progress", AIAA J., Vol. 6, No. 11, Nov. 1968, pp. 2049-2060.

4. Albano, E. and Rodden, W.P., "A Doublet-Lattice Method for Calculating Lift Distribution on Oscillating Surfaces in Subsonic Flows", AIAA J., Vol. 7, No. 2, Feb. 1969, pp. 279-285.

5. Giesieng, J.P., Kalman, T.P. and Rodden, W.P., "Subsonic Steady and Oscillatory Aerodynamics for Multiple Interfering Wings and Bodies", Journal of Aircraft, Vol. 9, No. 10, Oct. 1972, pp. 693-702.

6. Morino, L., "Unsteady Compressible Potential Flow Around Lifting Bodies Having Arbitrary Shapes and Motions", TR-72-01, *Department of Aerospace Engineering*, Boston University, June 1972.

7. Morino, L., "Unsteady Compressible Potential Flow Around Lifting Bodies: General Theory", AIAA Paper No. 73-196, Jan. 1973.

8. Kuo, C.C. and Morino, L., "Steady Subsonic Flow Around Finite-Thickness Wings", Boston University, Department of Aerospace Engineering, TR-73-02, Feb. 1973.

9. Morino, L. and Kuo, C.C., "Unsteady Subsonic Flow Around Oscillating Finite-Thickness Wings", Boston University, Department of Aerospace Engineering, TR-73-03, Feb. 1973.

10. Djojodihardjo, R.H. and Widnall, S.E., "A Numerical Method for the Calculation of Nonlinear Unsteady Lifting Potential Flow Problems", AIAA J., Vol. 7, No. 10, Oct. 1969, pp. 2001-2009.

11. Lessing, H.C., Troutman, J.C. and Menees, G.P., "Experimental Determination of the Pressure Distribution on a Rectangular Wing Oscillating in the First Bending Mode for Mach Numbers from 0.24 to 1.30", NASA TN D-344, 1960.

12. Cunningham, A.M. Jr., "An Efficient, Steady Subsonic Collocation Method for Solving Lifting-Surface Problems", J. Aircraft, Vol. 8, No. 3, March 1971, pp. 168-176.

13. Hsu, P.T. "Flutter of Low Aspect-Ratio Wings, Part I, Calculation of Pressure Distributions for Oscillating Wings of Arbitrary Platform in Subsonic Flow by the Kernel-Function Methods", TR-64-1, Oct. 1957, M.I.T.

14 Kulakowski, L.J. and Haskell, R.N., "Solution of Subsonic Nonplanar Lifting Surface Problems by Means of High-Speed Digital Computers", Journal of Aerospace Sciences, Vol. 28, Feb. 1961, pp. 103-113.

15. Ashley, H. and Landahl, M., "Aerodynamics of Wings and Bodies", Addison-Wesley Pub. Co., Inc., 1965, p. 159.

16. Kolbe, C.D. and Boltz, F.W., "The Forces and Pressure Distribution at Subsonic Speeds on a Planar Wing Having 45° of Sweep Back, an Aspect Ratio of 3, and a Taper Ratio of 0.5", RMA51G31, Oct. 1951, NACA.

17. Yates, E.C. Jr., Private Communication.

18. Laschka, B., "Die Instationaren Luftkräfte an Harmonisch Schwingenden Tragflügeln Endlicher Spannweite Bei Unter- Und Überschallgeschwindigkeit", Jahrbuch, 1961, der WGL, pp. 207-219.

19. Watkins, C.E., Runyan, H.S. and Woolston, D.S., "On the Kernel Function of the Integral Equation Relating the Lift and Downwash Distributions of Oscillating Finite Wings in Subsonic Flow", NACA Report 1234, 1955. (Supercedes NACA TN 3131).