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James C. Wu

Elements of Vorticity Aerodynamics

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To my wife
Mei Ying Wu
whose love and support made this work possible

Preface

Helmholtz' ground breaking vortex theorems in the mid-nineteenth century provided the tools for the momentous discoveries in theoretical aerodynamics by Prandtl and others nearly a century ago. Since then, studies of vorticity dynamics have received continual impetus from diverse applications in engineering, physics, and mathematics. A number of books on vortex theory and vortex dynamics addressing a broad range of topics of theoretical and practical interests are presently available. In comparison with these books, the present monograph has a narrower focus. It is aimed at sharing my understanding of theoretical aerodynamics with the reader interested in the classical circulation theory and in the role of modern vorticity dynamics in theoretical aerodynamics involving unsteady and non-streamlined flows.

My lectures in a short course at the Tsinghua University and at the Second Biennial Retreat on Vorticity Aerodynamics, both scheduled for September in Beijing, presented an ideal occasion for preparing my notes: an audience and a firm target date. As it turned out, however, my initial estimate of required time was overly optimistic. In the end, to meet target dates, certain compromises had to be made.

One major compromise is the omission of a chapter discussing vorticity-based flow computations. While computational aerodynamics is one of my favorite subjects, time limitations prevented the inclusion of this subject as a component of this monograph. Topics discussed in the present work, however, form the core of vorticity-based computation methods. Detailed discussions of these methods are available in some of the references quoted in Chaps. 1 and 3.

Other compromises involve editorial issues such as curtailing redundancies and adding figures, exercises, and more sample problems. Chapters of my notes were prepared more or less as independent articles, each with its own themes, references, and introductory discussions, intermittently over an extended period of time. Efforts to tie the chapters together and to implement the obviously desirable improvements were ultimately limited by available time.

The present monograph is based essentially on my lecture notes completed in August of 2004. It is my intent to prepare a "**Version 2.0**" of the monograph in the reasonable future. It is my hope that my colleagues would kindly provide commentaries and critiques about this initial version. One issue of special concern during my preparation of the present version is the proper discussion of certain viewpoints and strategies about vorticity aerodynamics, acquired and used over the years in my research and teaching. These viewpoints and strategies are obviously not the only ones that work; they are by no means a panacea for all applications of vorticity dynamics. I am, however, convinced that they are consistent, rational, and very effective within the perimeters defined in this monograph. Advocating these viewpoints and strategies is not meant to underplay the merits of alternative viewpoints and strategies, especially the classical ones. In this regard, I wish to acknowledge the special help of Prof. J.Z. Wu, who kindly reviewed my draft manuscripts on very short notices.

During my teaching and research career, I had the good fortune of associations with many brilliant and marvelous individuals—teachers, colleagues, and former students—who provided indispensable inspiration for my work. I wish to take this opportunity to express my gratitude for their contribution to my understanding of vorticity aerodynamics.

I would be remiss not to mention again the love and support of my wife Mei-Ying Wu, especially during the past year, as the preparation of my notes took up more and more, eventually virtually all, of the time at my disposal.

Shanghai, China 吴镇远
September 2004 James C. Wu

Contents

Chapter 1
Introduction

1.1 Preliminaries

The science of *aerodynamics* is often defined *as the branch of dynamics*, *which treats of the motion of air and other gases and of the forces acting on bodies in motion through air or on fixed bodies in a current of air (or other gases) (Webster* 1953*)*. The aerodynamicist, however, views his discipline in a more focused context as *an applied science that deals with forces produced by air or other fluids on bodies moving through it*. He recognizes that, as a branch of fluid dynamics, aerodynamics deals with the motion of fluids. Yet, in aerodynamics, studies of fluid motions do not represent ends by themselves. Rather, such studies are a means for evaluating the aerodynamic force. Historically, successes in theoretical aerodynamics were linked to discoveries that permit the prediction of the aerodynamic force without requiring the knowledge of complete flow details. The focal subject—the force—distinguishes the discipline of aerodynamics from all other branches of fluid dynamics.

The task of finding theoretical solutions to practical aerodynamic flow problems was and still is formidably difficult. Bypassing flow details as much as possible was indeed the only strategy open to the pioneering aerodynamicist. Together with simplifications of the inviscid-fluid assumption, this strategy served as the springboard for the dazzling developments of classical aerodynamics a century ago by leading European scholars: Kutta, Joukowski, Lanchester, Prandtl, among others. The power of the resulting theory of lift and induced drag, called the circulation theory today, is well documented.

Also well recognized are the vortex theorems of Helmholtz and others that paved the way for the development of the circulation theory. Less well acknowledged, especially outside the community of aerodynamicists, is the fact that the efficacy of the circulation theory is limited for the most part to steady and quasi-steady aerodynamics involving attached flows over streamlined airfoils and wings. For convenience, such attached flows are referred to as streamlined flows in the present

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study. Efforts to extend the circulation theory to general non-streamlined and fully unsteady flows are by no means straightforward. Attempts to estimate unsteady forces using the circulation theory at times lead to musings such as: *It is impossible for the bumblebee to fly*. The fact of course is bumblebees do fly, but they do not rely on streamlined steady forces that the aerodynamicist understands so well. There exists a vast unexplored and potentially fertile territory—in non-streamlined and in fully unsteady realms—where the aerodynamicist's understanding of the principles of flight is woefully lacking.

With the circulation theory, the steady aerodynamic force is predictable with amazingly little information about flow details. For the two-dimensional flow, there is the Kutta–Joukowski theorem stating that the lift on an airfoil is equal to the product of the fluid density, the flight speed, and the circulation. This theorem can be proved in several different ways. For the three-dimensional flow, the legendary Prandtl (1921) used a simple model—the horseshoe vortex system—to represent the vorticity field in the flow and to create the lifting-line theory. The fact that this simple model produces the powerful lifting-line theory for predicting both the lift and the induced drag never ceased to marvel the aerodynamicist, students and leading scholars alike. In a recent article, for example, Kroo (2001) discussed a *force-free wake* interpretation of why *Prandtl's mold works well with a very poor representation of the wake shape*. Rather unexpected is a remark by Max Munk (1981), who made important contributions to the lifting-line theory, first as Prandtl's student in Germany and subsequently as a distinguished scientist at the National Advisory Committee for Aeronautics (NACA) in the U.S.: *My principal paper on induced drag was still under the spell of Prandtl's vortex theory. Everything that Prandtl said was correct, but it was not the right approach*.

Munk was over 90 when he made the above remark. Munk neither elaborated nor proffered an alternative, perhaps more correct, approach to the lifting-line theory. Advances during the past century, however, made it timely to revisit classical theories of aerodynamics from the modern viewpoint of the dynamics of the vorticity field in the viscous fluid rather than from the viewpoint of the vortex in the inviscid fluid. In this Chapter, the general parameters of a revisit, undertaken by this writer intermittently over the past three decades, are described. In subsequent chapters, certain outcomes of the revisit are reviewed, attributes of vorticity dynamics as applied to aerodynamics are summarized, new interpretations of the classical circulation theory are provided and a framework for additional studies of non-streamlined unsteady aerodynamics is advocated.

The foundation of the present study, as in all theoretical studies of fluid dynamics, is the first principles of fluid dynamics: the laws of mass and energy conservation and Newton's second law of motion. Many familiar assumptions are employed in the past to simplify the description of the flow. Assumptions are indispensable since the first principles are intractable without them. An assumption, of course, should not be admitted merely because it simplifies, but should be justifiable as an approximation of the physical reality being described. On this basis, the inviscid-fluid assumption is not adopted for the present study from the outset. This assumption is abandoned not because inviscid results are thought to be less powerful than commonly believed. It is simply that this assumption can only be viewed as an idealization, not as an approximation.

The exclusion of the inviscid-fluid assumption, as it turns out, does not invalidate many useful inviscid theories. Instead, by excluding this assumption from the outset, advantages of treating the kinematic aspect of vorticity dynamics separately from the kinetic aspect are brought into focus. It then becomes evident that vorticity kinematics, unlike vorticity kinetics, is not linked to the viscosity of the fluid. A defining strategy of the present study is to identify, differentiate, and analyze these two aspects individually as much as possible before merging the results. This strategy offers remarkable advantages in the aerodynamic analysis because it helps to clarify certain conceptual difficulties and paradoxes associated with several familiar assumptions of classical aerodynamics, including the inviscid-fluid assumption.

1.2 Differential Equations, Initial and Boundary Conditions

This book is concerned with external aerodynamics. The time-dependent incompressible flow of an infinite viscous fluid, primarily air and water, in three-dimensional space relative to a finite and rigid solid body immersed and moving in the fluid is selected as the *reference flow problem*. To underscore external aerodynamic applications, the solid body is at times referred to as a wing. The flow can be steady, quasi-steady, or fully unsteady. The wing is not constrained to operate at a small angle of attack in a steady streamlined environment. It may flap either while flying forward or while hovering. It experiences a lift, a drag, and possibly also a side force.

To formulate the reference flow problem mathematically, the region occupied by the fluid is denoted R_f and that occupied by the wing R_s. The solid–fluid interface is denoted S. The combined region of the fluid and the wing, denoted R_∞, is infinite and unlimited. Both the wing and the fluid are at rest at the initial time level $t = 0$. Subsequent prescribed motion of the wing during the time period $0 < t < t_1$ induces a corresponding motion of the fluid. The viscosity of the fluids assumed uniform for simplicity.

The fluid region R_f is assumed to be bounded internally by the closed surface S and externally by S_∞, a closed surface on which every points infinitely far from S. The flow in R_f is then described by the following set of two equations:

$$\nabla \cdot v = 0 \tag{1.1}$$

$$\frac{\partial v}{\partial t} = -(v \cdot \nabla)v - \frac{1}{\rho}\nabla p + \nu\nabla^2 v + \nabla q \tag{1.2}$$

where v is the flow velocity, p, ρ, and ν are respectively the pressure, the density, and the kinematic viscosity of the fluid, and ∇q represents a conservative external body force.

Equation (1.1) is a statement of the law of conservation of mass for an incompressible flow. This equation is called the continuity equation because it is based on the assumption that the fluid is a continuous medium, i.e., a continuum (Shapiro 1953). The continuum assumption is also a part of (1.2), the Navier–Stokes momentum equation, which is a mathematical statement of Newton's second law of motion.

Equations (1.1) and (1.2) comprise a set of four scalar equations containing four scalar unknown field variables: the pressure and the three components of the velocity vector. In fluid dynamics, the fact that the number of equations and the number of unknown field variables are equal is often considered important. For example, with the addition of the energy equation, the energy density of the fluid is an additional field variable. For the compressible flow, the mass density also becomes an unknown field variable. Some authors, therefore conclude that an additional equation, the equation of state, is required. This conclusion is correct. The number of scalar differential equations describing a physical problem, however, needs not be equal to the number of scalar field variables. In fact, in the vorticity-dynamic formulation of the flow problem, these numbers are unequal.

To complete the mathematical formulation of the flow problem, values of v in R_f at the initial time level, t = 0 and on the boundaries S and S_∞ for the time period $0 < t < t_1$ need to be specified. With the flow initially at rest, the initial condition is

$$v(r, t = 0) = 0 \text{ in } R_f \tag{1.3}$$

where r is the position vector.

With the wing motion prescribed for the time period $0 < t < t_1$, one writes

$$v_s = f(r_s, t) \tag{1.4}$$

where r_s is the position vector of a point on the wing surface, v_s is the velocity of the wing surface at a time level t, with $0 < t < t_1$, and f is a known function of r_s and t. With the familiar no-penetration and no-slip conditions, the velocity is continuous across the fluid-solid interface S. Therefore (1.4) gives, with r_s considered a position vector on the fluid boundary, the velocity boundary condition on S for the problem during $0 < t < t_1$. It is important to note that (1.1) and (1.2) describe the incompressible flow in R_f. Prescribed initial and boundary conditions, (1.3) and (1.4), allow the solution of the two equations in R_f.

For the incompressible flow of either air or water, an astonishingly rich variety of fluid motions are observed in laboratories (van Dyke 1982). These motions are described by the same differential Equations, (1.1) and (1.2). They differ not only quantitatively, but also in character, solely because of differences in initial and boundary conditions. Thus the proper prescription of boundary and initial conditions is critically important in fluid dynamics. This issue is mathematically complex. It is generally assumed that, if the motion of the wing is prescribed for the time period $0 < t < t_1$, then the fluid motion in R_f surrounding the wing is determinate in this period of time.

In external aerodynamics, the fluid on S_∞ is typically assumed to be undisturbed by the wing motion. The flow velocity on S_∞ is therefore called the *freestream velocity*. Observed in a stationary reference frame, the freestream velocity is zero. For a non-rotating wing moving at a constant velocity, it is convenient to observe the flow in a reference frame attached to and moving with the wing. In this *attached reference frame, the freestream* velocity is uniform and constant, being the negative of the velocity of the wing in the *stationary reference frame*. The formulation (1.1)–(1.4), together with the *freestream condition*, is assumed to describe the flow in R_f completely. In other words, (1.3) and (1.4), plus the implied freestream condition, are assumed to be the appropriate conditions for the unique solution of (1.1) and (1.2) in R_f for the time period $0 < t < t_1$. The correctness of this conclusion is confirmed in Sects. 3.8 and 4.7.

1.3 Major Assumptions

Many simplifying assumptions, not all stated explicitly, are adopted in the formulation of the reference flow problem given in Sect. 1.2. Each assumption should be reasonably justifiable as an approximation. Each assumption implies certain restrictions on the range of applicability of the formulation. Detailed discussions of these issues are found in well-known fluid dynamic treatises, e.g. Batchelor (1967). In the following paragraphs, major assumptions involved in the formulation of the reference flow problem and the resulting restrictions on the applications of the present study are briefly described.

1. **Boundary and Initial Conditions**

The no-slip and no-penetration conditions, which yield (1.4), assume the continuity of the velocity vector across the fluid-solid interface S. In other words, the relative velocity of the two material media, the solid and the fluid, on the two sides of the interface S is zero. Batchelor (1967) observed: *All the available evidence does show that, under common conditions of moving fluids, temperature and velocity* (*both tangential and normal components*) *are continuous across a material boundary between a fluid and another medium.* The present study is concerned for the most part with flights of aerial animals and manmade flying machines at moderate speeds and altitudes and of locomotion of aquatic animals. The assumption of zero relative velocity between the solid and the fluid at their interface is an excellent approximation. Extremely small animals (microorganisms) for which this assumption is not accurate are excluded from the present work. Readers interested in motions of microorganisms and ideas of drag reduction based on slipping flows may wish to study the molecular behavior of fluids on solid surfaces and may find the introductory discussions of Lighthill (1963) on this topic revealing.

The assumption of a *rigid solid body* is an excellent approximation in many aeronautical applications. This assumption places a restriction on the prescription of velocity boundary conditions. Specifically, with a rigid wing, the integral of the velocity component normal to S over S must be zero. That is, the function f must satisfy

$$\oiint_S \mathrm{f} \cdot \mathrm{n} \, d\mathrm{S} = 0 \tag{1.5}$$

where n is the unit vector normal to S.

The requirement (1.5) modifies neither the initial condition (1.3) nor the differential equations (1.1) and (1.2). For a deformable or flexible body changing its shape, but not its volume, (1.5) is satisfied. Therefore the formulation of the reference flow problem is applicable without revision to motions induced by such bodies. For expanding and contracting bodies, the right-hand side of (1.5) is non-zero. To generalize the present study to such bodies, consequences of this fact need to be included in analyses.

The reference flow problem defined in Sect. 1.2 involves a single solid body. The formulation (1.1) through (1.4) remain valid under the more general circumstance where several finite solid bodies are immersed in the fluid and moving relative to one another. The surface S is then comprised of several closed surfaces S_i each bounding a specific solid body. If each solid body is assumed to be rigid, then the restriction (1.5) applies to each surface S_i individually. For several moving bodies that remain at finite distances from one another, the generalization does not require additional analysis.

The assumption that the fluid and the solid object are *at rest initially* is readily generalized by writing, in place of (1.3),

$$\mathrm{v}(\mathrm{r}, t_0) = \mathrm{g}(\mathrm{r}) \tag{1.6}$$

where g is a known function of r.

This generalization does not alter the analyses since the differential Eqs. (1.1) and (1.2) and the boundary condition (1.4) are not affected. The function g is required to be solenoidal, i.e., its divergence mutt be zero, otherwise (1.1) is not satisfied initially. This requirement is obviously satisfied in the special case g = 0, i.e., (1.3).

In summary, selecting a single rigid body initially at the rest for the reference flow problem implies a number of assumptions. These assumptions simplify the discussion of the physical and mathematical aspects of the flow problem. For more general flows that are not initially at rest and that contain multiple non-rigid bodies, the differential equations (1.1) and (1.2), remain unaltered. Extensions of the present analyses for such more general flows do not involve conceptual difficulties.

Three-dimensional regions in external flows are typically simply connected. Two-dimensional external flow regions, in contrast, are generally multiply connected. The discussions of the present book are primarily concerned with simply connected regions. It is, however, not difficult to generalize the conclusions to multiply connected regions.

2. **Fluids**

The assumption that the fluid is a continuum appears to contradict the knowledge that matter is made up, at the microscopic level, of molecules. It is known that the average molecular diameter of air is about 3.7×10^{-8} cm. The average spacing between molecules of atmospheric air at standard sea level conditions is about ten times greater. Much of the space occupied by a gas is therefore devoid of a material medium. Hence the idea of a continuum is sometimes said to be an abstraction or a convenient fiction, e.g., Shapiro (1953). In reality, in aerodynamic applications where the fluid is not a highly rarefied gas, the smallest length and time scales of practical interest are typically vastly greater than the molecular scales. Therefore the continuum description of the fluid is an excellent approximation, at the practical or macroscopic level, of the aggregated and statistically averaged behavior of a very large number of molecules. The present study uses the continuum assumption and excludes applications involving highly rarefied gases and extremely small bodies (e.g., microorganisms.) for which the macroscopic and molecular or microscopic scales are comparable. Effects of vibratory and rotational excitations of polyatomic molecules, dissociations and other chemical reactions, ionizations, etc. are also excluded. Readers interested in these effects are referred to books on high temperature and rarefied gas dynamics, e.g., Vincenti and Kruger (1967). The present study is restricted to liquid flows under normal conditions; free surfaces, cavitations, and other phenomena peculiar to liquid flows are excluded. The study of airflows is restricted to those in the troposphere at moderate speeds. Studies of continuum behaviors of the gas as a consequence of molecular behaviors are in the province of kinetic theory of gases. Reader interested in the kinetic theory will find the book of Jeans (1940) informative.

Batchelor (1967) prefaced his enlightening book on fluid dynamics with the statement: *I regard flow of a viscous incompressible fluid as being at the center of fluid dynamics by virtue of its fundamental nature and its practical importance*. Liquids are typically considered incompressible fluids because the spatial and temporal variations of their density are in general quite small.

The expression *flow of an incompressible fluid* is often used in the literature to describe the low-speed flow of a gas. Gases are obviously compressible. Thus this expression is intended not to indicate the gas is incompressible, but to say the density of the flowing gas is approximately constant. This expression is an acceptable description of the gas flowing at subsonic speeds. The alternative expression, incompressible flow of a gas, however, is a more apt description of the flow of a gaseous medium at low speeds.

In most aerodynamic applications, external body forces, including the buoyancy force, are conservative and therefore expressible as the gradients of single-valued scalar potential functions, as in (1.2). Conservative forces, like the pressure force, do not enter into the vorticity-dynamic formulation of the flow problem, as shown in Sect. 1.4.

To obtain (1.2), the assumption of a Newtonian fluid is used. The stress experienced by a Newtonian fluid depends linearly on its rate of deformation. Extensive experiments on flows of air and water show that the Newtonian-fluid assumption holds over a remarkably broad range of conditions. More detailed discussions of this topic are available in many books, e.g. Batchelor (1967). In low-speed external aerodynamics, the viscosity of the fluid typically does not change appreciably and the assumption of uniform viscosity is an excellent approximation.

The fluid medium of primary concern in this study, air and water, have small viscosities. The assumption of an inviscid fluid is therefore often adopted with the expectation that inviscid results provide approximate answers to aerodynamic problems. It is known, however, while the inviscid fluid assumption simplifies the mathematical formulation of flow problems, it also creates conceptual difficulties and paradoxes. As stated earlier, in the present book, the assumption of an inviscid fluid not adopted. Hence the idea of *the flow of an inviscid fluid* is not admitted. The idea of the inviscid flow, however, is very useful in the study of the external flow of a viscous fluid. It is known that viscous effects are significant only in relatively small regions of flows, e.g., boundary layers and wakes. Outside these regions, the viscous effects are insignificant and the flow, not the fluid, is approximately inviscid. Scholars in fluid dynamics generally recognize the distinction between *the flow of an inviscid fud and the inviscid flow of a viscous fluid*. Yet the two expressions are often used interchangeably for convenience, thus obscuring the importance of their conceptual distinction.

With the inviscid fluid assumption, the last term in the Navier–Stokes momentum equation (1.2) vanishes. One then has the Euler's momentum equation in R_f.

$$\frac{\partial \mathbf{v}}{\partial t} = -(\mathbf{v} \cdot \nabla)\mathbf{v} - \frac{1}{\rho}\nabla p + \nabla q \tag{1.7}$$

While the Navier–Stokes momentum equation is a second order partial differential equation, the Euler's momentum equation is first order. The inviscid fluid assumption thus alters the mathematical characters of the flow problem. To solve (1.1) and (1.7), the boundary condition (1.4) is typically replaced by

$$v_s \cdot n = f(r_s, t) \tag{1.8}$$

where f is a known scalar function.

The use of (1.8) in place of (1.4) implies that the tangential component of the velocity field is discontinuous on S. The inviscid-fluid assumption therefore raises the question as to what, if any, physical role s played by the tangential velocity of the solid surface.

It is recognized that an inviscid fluid does not exist in the real world. Suppose a super fluid, a fluid that is truly inviscid, is created, one would then expect the behavior of this super fluid to be quite different from that of the real viscous fluid. Surely, general agreements cannot be expected to prevail between the flow of the super fluid and the observed flow of the real fluid in nature and in laboratories. Agreements, however, are known to exist between some aspects of inviscid flow solutions, which describe the flow of super fluids, and real flows under special circumstances, for example, streamlined flows. It is fortunate that these special circumstances are exceedingly important in aeronautics. The question as to why such agreements exist, however, is as important as the question why they do not exist under more general circumstances. Definitive answers to both questions are crucial; they cannot be simply bypassed in one's search for the understanding the principles of aerodynamics involving non-streamlined unsteady flows.

3. **Flows**

In low subsonic flows of air or other gases, changes of the density of gaseous media in motion are known to be small. For example, at a Mach number of 0.14 (corresponding to an air speed at sea level of about 50 m/s or 110 m/h), changes of density in the isentropic flow of air are less than 1% (Shapiro 1953). Therefore, as an approximation, the *incompressible flow* assumption is sufficiently accurate in very many important applications in aeronautics. This assumption decouples the law of energy conservation and the equation of state of the gas from the law of mass conservation and Newton's second law, thereby substantively simplifies the mathematical description of the flow.

It is well known that the sound speed demarcates gaseous flows into subsonic and supersonic regimes. In the subsonic regime, the assumption of an incompressible flow is interpretable as an approximation. The approximation is accurate at low subsonic speeds and less accurate at higher subsonic speeds. In the supersonic regime, the incompressible flow assumption is not merely inaccurate. It is inadmissible in certain regions of the flow because physical features such as shock waves, excluded by this assumption, may be present. It is of interest to note, however, analogous to the presence of inviscid regions in the flow of a viscous fluid, the flow of air surrounding a wing in supersonic flight contains regions where these supersonic flow features are absent.

Earlier efforts to extend the vorticity-dynamic approach to studies of compressible flows were reported by, for example, EI Raface, Lekoudis and Wu (1981). These efforts demonstrated the feasibility of using vorticity dynamics in studies of flows at high subsonic speeds, but did not progress much beyond the proof of concept stage. Issues of solution accuracy and efficiency remain as major undertakings deserving future attention.

As mentioned earlier, the idea of the inviscid flow is useful in analyses. The effects of viscosity, represented by the term $\nu\nabla^2 \mathrm{v}$, in (1.2), are small in regions where the Laplacian of the velocity field is small. In such regions, the flow is reasonably approximated by an inviscid flow. It is important to recognize that viscous effects can be, and generally is, large in certain flow regions near the solid boundary S. Therefore, not the entire flow is approximately inviscid. The inviscid fluid assumption requires the viscous effects to be small in the entire flow, which does not conform to the real world.

It is important to note that the viscosity of the fluid is not a part of the continuity equation (1.1). The issue as to whether the fluid is viscous or inviscid is thus irrelevant if one were to study (1.1) alone. In the present study, (1.1) is identified with the kinematics of the flow. More detailed discussions of this issue are presented in Chaps. 2 and 3.

Lighthill (1963) remarked: … *It must be admitted that this and other simplifying features have tempted theoretical hydrodynamicists into an unwarranted concentration on two-dimensional flows, leading experimentalists to similar restrictions for the sake of comparison with theory*. All aircraft, all aerial, and aquatic animals are of course three-dimensional. Two-dimensional flows do provide valuable approximations to real flows under special circumstances. Consider, for example, a wing with a symmetric planform and a very large aspect ratio moving perpendicular to its span. Near the mid-span of the wing, gradients of flow variables in the span direction are expected to be small. In other words, the spatial coordinate in the span direction is not explicitly involved in the formulation of the flow problem and the flow is approximately two-dimensional. There are nevertheless important regions of the aerodynamic flow, for example, regions near the tips of a lifting wing, where the two-dimensional approximation is clearly unacceptable. Lighthill (1963) observed that *in flows which do not contain rotating bodies*, all vorticity appears in closed tubes. In the two-dimensional flow, the vorticity s directed perpendicular to the plane of the flow. To visualize and make use of the fact that vorticity tubes are closed, one needs to return to a three-dimensional picture and envision two-dimensional planes of flow continuing unchanged in the span direction (see Chap. 6). Vorticity tubes can then be considered closed in planes very far from the plane of the flow. To avoid such interpretations, the reference flow in the present study is assumed three-dimensional. Discussions of two-dimensional flows are made from time to time primarily for the purpose of elucidation.

All lift-producing flows are considered unsteady in the present study. Obviously, for the reference flow, if the wing motion is time-dependent, then the flow about the wing is unsteady. Steady state is possible only if the wing is undergoing steady rectilinear motion. At large time levels after the onset of the wing's motion, for example, the flow near the wing observed in the wing-attached reference frame may asymptotically approach steady state. However, for the lifting wing, the starting vortex shed after the motion's onset continues to move away from the wing. Therefore, the flow is unsteady in regions near the starting vortex. At large time levels, the starting vortex is far from the wing. The existence of the starting vortex, however, cannot be simply ignored; otherwise the law of total vorticity conservation (Wu 1981 and Sect. 5.2) is violated. On other hand, recognizing the presence of the starting vortex at infinity means the assumption of a uniform freestream velocity on S_∞ is not satisfied in the limit as $t \to \infty$. Clearly, the flow cannot be uniform near the starting vortex. This dilemma is discussed in more details in Sect. 4.8. It is noted that in the stationary reference frame the flow is always unsteady, especially near the wing, simply because the wing is in motion. If the wing translates at a constant velocity, but does not have a streamlined shape, or if the angle of attack is large, then the boundary layers generally separate from the wing surface. At Reynolds numbers of interest in aeronautics, flow separation typically leads to periodic shedding of vortices exemplified by the well-known Karman vortex street behind a circular cylinder. Thus, in addition to steady-state solutions, time-independent boundary values on S admit unsteady (periodic) solutions near S. In short, steady flows are rare exceptions while unsteady flows are the norm in aerodynamic applications.

As noted earlier, principles of steady-state streamlined aerodynamics are today reasonably well understood. The aeronautical engineer utilizes these principles very effectively in his design of modern airplanes. The flight speed, range and efficiency of conventional fixed-wing transport airplanes have reached remarkably impressive levels. In comparison, the designs of helicopters, vertical and short takeoff and landing (V/STOL) aircraft, and unmanned air vehicles (UAVs) are at the present handicapped by the lack of understanding of principles of non-streamlined unsteady aerodynamics. It is noted that, in the animal kingdom, flapping wings, tails, and fins are used pervasively to produce unsteady forces for locomotion. Manmade unsteady aerodynamic devices need not mimic aerial or aquatic animals: these devices need not be limited to flapping surfaces. With the pervasive use of unsteady aerodynamic forces in nature, it appears that, once the principles of fully unsteady aerodynamics are adequately known, useful devices can be build for unsteady applications in aeronautics and in other areas of fluid engineering.

1.4 Vorticity-Dynamic Formulation

The continuity Equation (1.1) is repeated below for convenience:

$$\nabla \cdot \mathrm{v} = 0 \tag{1.9}$$

The vorticity field ω is defined as the curl of the velocity field:

$$\nabla \times \mathrm{v} = \omega \tag{1.10}$$

Taking the curl of each term in Equation (1.2) and rearranging the result, one obtains the following differential equation, known as the vorticity transport equation:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\mathbf{v} \times \boldsymbol{\omega}) + \nu \nabla^2 \boldsymbol{\omega} \tag{1.11}$$

Equations (1.9), (1.10), and (1.11) together describe the dynamics of the vorticity fields in the fluid region R_f.

There are at least three major reasons for formulating the flow problem using the idea of the vorticity. First of all, the viscous term in (1.2), $\nu\nabla^2 v$, is expressible as $-\nu\nabla\times\omega$. Therefore, viscous effects are important only in flow regions where the vorticity field is significant or, more specifically, where the curl of the vorticity field is significant. Second, the evolution of the vorticity field is directly related to the aerodynamic force, as shown by the circulation and vorticity-moment theorems (Wu 1981, Chaps. 5 and 6). In consequence, regions where the vorticity field is effectively zero need not enter into aerodynamic analyses. Thirdly with the vorticity formulation, the flow problem is clearly decomposed into two distinct entities—*vorticity kinematics and vorticity kinetics*—each amendable to individual analyses. These analyses in turn contribute to physical insights about non-streamlined and fully unsteady aerodynamics (Chap. 6).

These attributes of the vorticity formulation offer remarkable advantages in computational fluid dynamics. Many of these advantages are demonstrated conclusively and reported previously (e.g., Wu and Gulcat 1981; Tuncer et al. 1990; Kim et al. 1996). This book is about the theoretical side of vorticity dynamics. The vorticity-based computational approach is outlined in this Chapter to facilitate discussions of vorticity dynamics in subsequent Chapters.

The term dynamic is defined (Webster 1953) *as the branch of mechanic treating of the motion of bodies (kinematics) and the forces in producing or changing their motion (kinetics)*. Equations (1.9) and (1.10) are equations of vorticity kinematics, since they deal with motions of the fluid apart from their causes. Specifically, (1.9) imposes the solenoidal (divergence free) condition on the velocity field v and (1.10) relates v at any specific instant of time to ω at the same instant. Equation (1.11) is concerned with the evolution of the vorticity field w in the flow and causes of this evolution. Equation (1.11) does not contain a force term explicitly. It is a kinetic equation because it is a direct consequence of (1.2) and because t deals with causes and effects.

In computing the flow, consider marching the numerical solution from an old time level, say t_0, to a new time level, say $t_0 + \delta t$. The fluid region R_f is composed of the boundary S and the open domain R_f'. The computation grid contains boundary points on S and interior points in R_f'. The values of ω and v are known at all points in R_f at the old time level $t = t_0$. The values of v at all boundary points on S at the new time level $t = t_0 + \delta t$ are prescribed and known. To compute new values of ω at all grid points in R_f' and on S and new value of v at all interior points in R_f', a computation loop composed of the following three major steps s available (Wu and Gulcat 1981):

(i) Place the known values of ω and v into the right-hand side of (1.11) to determine the time rate of change of ω at the old time level t_o. Compute new values of ω at the interior points using values of the time rate of change of ω just determined.

(ii) Place new values of ω and v obtained in Steps (i) and prescribed boundary values of v into (1.9) and (1.10) to compute new boundary values of ω.

(iii) Place new values of ω obtained in Steps (i) and (ii) and new prescribed boundary values of v into (1.9) and (1.10) and solve the equations to obtain new interior values of v at the new time level $t_0 + \delta t$.

This three-step loop involves four sets of data, namely, velocity values on the boundary S, velocity values in the interior of the domain R_f', boundary vorticity values, and interior vorticity values. These four sets of data are known at the old time level t_0. Boundary velocity values are given at the new time level $t_0 + \delta t$; and each step of the loop computes one of the three remaining data sets at the new time level. At the completion of the loop, all four sets of data are known at the new time level. The loop can be repeated to advance the solution progressively with time. In each loop, new velocity boundary values are used. The loop can be repeated and the solution can progress as long as new boundary velocity values are prescribed.

The first step in the loop is a vorticity-kinetic step. The third step is a vorticity-kinematic step. The two fields ω and v appear in both steps but have separate and distinct roles. Specifically, in Step (i), ω is the focal field variable whole evolution with time is evaluated. The field v is involved only because it is a part of the physical processes that cause ω to change with time. In Step (iii), the roles of ω and v are reversed. Here, the central task is the determination of v. In this step, (1.9) and (1.10) are solved to determine a new velocity field that corresponds to the set of new vorticity values. Thus, in Step (iii), v is the focal field to be evaluated and ω is an inhomogeneous term playing a passive role.

Step (ii) is needed to provide the missing new values of ω on the boundary S. To recognize this need, consider first the scalar diffusion equation $\partial f/\partial t - \upsilon\nabla^2 f = g$ in R_f. This equation expresses the relation between the spatial and the time variation of f. The inhomogeneous term g in this equation influences this relationship. Under general circumstances, g can be afunction of the spatial coordinates, the function f, and first order spatial derivatives of f. Theorems of partial differential equations, as discussed in many textbooks in applied mathematics (e.g., Morse and Feshbach 1953), states that the solution of this equation for the time period $0 < t < t_1$ exists and is unique if initial ($t = 0$) values of f at all points in R_f and boundary values of f at all points on S for the period $0 < t < t_1$ are known. The diffusion equation is applicable in the open domain R_f'. The function f is required to be a very smooth function in R_f. This means that, in the limit as an interior point approaches a boundary point, the value of f at the interior point approaches the known values of f at the boundary point. Since the diffusion equation is satisfied by f at all interior points, it is also satisfied at all boundary points. Boundary values of f, however, are not a part of the solution of the diffusion equation. Rather, they are specified at all boundary points for a unique solution of the equation to exist in the interior domain R_f'.

The vorticity transport Equation (1.11) is a vector diffusion equation in which ω is the unknown field and the inhomogeneous terms contain v and ω. Based on the above discussions, the solution of (1.11) requires the initial and boundary conditions:

$$\omega_0 = p(r) \tag{1.12}$$

$$\omega_s = q(r_s, t) \tag{1.13}$$

where ω_0 is the vorticity field in R_f at the time level t = 0 and ω_s the vorticity at the boundary point r_s on S during the time interval $0 < t < t_1$.

With (1.11), the specification of the functions p and q determines ω uniquely in R_f' for the time period $0 < t < t_1$. Using (1.10), the initial condition for (1.11) is determinate once the initial velocity field is given. For the reference flow, g(r) = 0 in (1.6) and (1.12) simplifies to ω_0.

With the no-slip condition, once the solid motion is prescribed, values of the tangential components of the velocity vector are determined at all points on S. Values of the normal component of the vorticity vector are then determined by (1.10). Furthermore, if velocity values are known at all points on S, and vorticity values are known at all points in R_f', then values of tangential components of vorticity at all boundary point are uniquely determined by (1.9) and (1.10). In other words, Step (ii) is a kinematic step. A proof of this facts presented in Sect. 3.8. Alternative types of vorticity boundary condition (e.g., values of the normal gradient of ω on S) that also determine ω uniquely in R_f' are permissible, but are not considered in the present study.

The kinematic Eqs. (1.9) and (1.10) are linear. Classical mathematical methods, including the principle of superposition, are useful in vorticity kinematics. Theoretical conclusions about vorticity kinematics are obtainable mathematically rigorously. In contrast, the equation of vorticity kinetics (1.11) is nonlinear since the inhomogeneous terms in (1.11) involve $v \times \omega$, and v is a function of ω. This means that vorticity kinetics is mathematically difficult. It is possible, however, to obtain useful conclusions about vorticity kinetics through a study of (1.11) without actually solving t (Chap. 4).

For the kinematic equations (1.9) and (1.10), no initial condition is required because these equations are concerned only with the instantaneous relation between v and ω. Since only instantaneous relations are involved, the time coordinate t is omitted in the formulation of the vorticity-kinematic problem; and initial conditions are not involved.

Consider the boundary condition for the equation-set (1.9) and (1.10), in particular, the following two types of boundary conditions on S:

$$v_s \cdot n = q(r_s) \tag{1.14}$$

$$v_s \times n = h(r_s) \tag{1.15}$$

For convenience, the specification of the function q is said to be the prescription of Neumann's condition (normal velocity component). The specification of h is said to be the prescription of Dirichlet's condition (tangential velocity components), and of both q and h Cauchy's condition (all components of v).

Suppose, for the moment, that values of ω are given at all boundary and interior points. It is known from theories of partial differential equations that (1.9) and (1.10), with the prescription of either Neumann's condition (Lamb 1932) or Dirichlet's condition (see Sect. 3.8), determines v uniquely in R_f'. If v is a smooth function in R_f', then the solution obtained with Neumann's condition gives values of v × n on S. Therefore, with the prescription of Neumann's condition, Dirichlet's condition cannot also be prescribed because the prescribed values maybe in conflict with the values of the solution obtained using Neumann's condition. Similarly, once Dirichlet's condition is prescribed, Neumann's condition cannot be also prescribed. One therefore concludes that Cauchy's condition over-specifies the vorticity kinematics.

The condition (1.4) is Cauchy's condition. Therefore it appears that (1.4) over-specifies vorticity kinematics. In reality, however, the supposition that ω values are given at all points, including all boundary point, requires revision. As noted earlier, values of the normal component (and not the tangential components) of ω on S are determined directly from the prescribed boundary velocity values using (1.10). Values of tangential components of ω on S are not a part of the solution of (1.11) and need to be computed. In other words, on S, the following information is missing:

$$\omega_s \times n = m(r_s) \tag{1.16}$$

In the present study, m is viewed as a third type of prescriptible boundary condition, along with q and h in (1.14) and (1.15). In other words, with ω values known at all interior points, there are three types of prescriptible conditions on S. If $m(r_s)$ is prescribed, then the prescription of either $q(r_s)$ (Neumann's condition) or $h(r_s)$ (Dirichlet's condition) determines v in R_f' uniquely. In Chap. 3, it is shown that, if m is not prescribed (w is known in R_s', but ω × n is not known on S), then the specification of q and h also determines v in R_f' uniquely. In short, the prescription of any two of the three conditions—the Neumann condition (1.14), the Dirichlet condition (1.15), and the third type, the tangential boundary vorticity (1.16)—renders the solution of (1.9) and (1.10) for v in R_f unique. The three combinations, m and q, m and h, and q and h are in reality equivalent. The combination q and h (Cauchy's condition), however, is preferred, because the specified solid motion gives q and h forthwith.

In summary, with the kinematic-kinetic decomposition, the overall flow problem is formulated as two interlaced component problems: vorticity kinematics and vorticity kinetics. Vorticity kinematics is described by the differential equations (1.9) and (1.10), which applies in R_f', and the Cauchy boundary condition both (1.14) and (1.15) on S. In this formulation, values of the inhomogeneous term w are known at all points in R_f' Cauchy's boundary condition gives values of ω_s × n forthwith through (1.10) alone; and t gives value of ω_s × n indirectly through (1.9) and (1.10).

The solution of (1.9) and (1.10) with Cauchy's velocity boundary condition provides, along with values of v in R_f', values of $\omega_s \times n$ on S. The problem of vorticity kinetics is described by (1.11) in R_f' and the vorticity boundary condition (1.16) on S. The normal part of the vorticity boundary condition is obtained forthright from Cauchy's condition. The tangential part of the vorticity boundary condition is determined by vorticity kinematics, i.e., (1.9) and (1.10). Vorticity kinematics and vorticity kinetics are interlaced because the field variables v and ω appear in both component problems. They are also interlaced because they share the same boundary condition: Cauchy's velocity boundary condition.

The issue of boundary conditions is obviously crucially important in all problems of physical sciences. This issue is discussed in details in Sects. 3.8 and 4.7. It is noted in passing that the primitive field variable p that appears in the Navier–Stokes momentum equation is replaced by the derived variable w in the vorticity transport equation. The issue of boundary condition for the variable p is not without ambiguity in aerodynamic analyses. In contrast, (1.13) is known to be correct boundary condition for vorticity kinetics, in which w is the focal variable.

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Chapter 2
Theorems of Helmholtz and Kelvin

2.1 Introduction

Helmholtz' (1858) vortex theorems paved the way for the legendary Ludwig Prandtl (1921) to invent the lifting line theory, a crowning advancement in theoretical aerodynamics. Sommerfeld (1950) remarked that *the main contents of Helmholtz's theory* are *conservation laws: It is impossible to produce or destroy vortices, or, in more general terms, the vortex strength is constant in time*. Sommerfeld noted that Helmholtz' theorems are correct under the conditions *the fluid is inviscid and incompressible; the external forces possess a single-valued potential within the space filled by the fluid*. Sommerfeld also pointed out that in Helmholtz' theorems, *apart from the conservation of the vortex strength in time*, *there is also a spatial conservation: the vortex strength is constant along each vortex line or vortex tube, which must either be closed or end at the boundary of the fluid*. In the current literature, Helmholtz' theorem on the spatial conservation of the vorticity strength is called Helmholtz' first vortex theorem, and the theorem on the time conservation is called his second vortex theorem.

Lighthill (1986) discussed Kelvin's (1869) theorem on the persistence of circulation and pointed out that *Kelvin's theorem is exact for the Euler model*, which contains the key assumptions that the fluid is *inviscid and incompressible*. Lighthill noted: *One especially valuable deduction from Kelvin's theorem is concerned with "the movement of vortex lines". This is Helmholtz's theorem (also exact on the Euler model), which states that vortex lines move with the fluid.*

In vorticity dynamics, the strategy of partitioning the overall flow problem into its kinematic and kinetic aspects offers advantages in aerodynamic analyses (Wu 1981; Tuncer et al. 1990). In the present study, this strategy is used in a revisit of the vortex theorems of Helmholtz and Kelvin and in generalizing these theorems to flows of viscous fluids. It is learned from this revisit that Helmholtz' first vortex theorem on spatial conservation is a theorem of vorticity kinematics and, as such, is valid not only in the fluid region but also in the solid region. This observation leads

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to the portrayal of the vorticity distribution as a system of closed vorticity tubes. With this portrayal, classical aerodynamic theories, including the lifting line theory, are interpretable on the basis of the vorticity-moment theorem (Wu 1981, Chap. 6). This portrayal also brings forth opportunities for establishing new approaches for viscous and unsteady aerodynamic analysis (Wu et al. 2002, Chap. 7).

Helmholtz' second vortex theorem, or its equivalence Kelvin's theorem, is a vorticity-dynamic theorem based on both kinetics and kinematics. The generalized second vortex theorem states that the vorticity strength in the viscous fluid is not conserved in time; it diffuses at a predictable rate.

2.2 Kinematic-Kinetic Partition of Flow Problems

As discussed in Chap. 1, the present study deals with the incompressible flow of a viscous fluid as described by the following set of two differential equations:

$$\nabla \cdot \mathbf{v} = 0 \tag{2.1}$$

$$\frac{\partial \mathbf{v}}{\partial t} = -(\mathbf{v} \cdot \nabla)\mathbf{v} - \frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v} + \nabla q \tag{2.2}$$

where $\mathbf{v}$ is the flow velocity, p, ρ, and ν are respectively the pressure, the density, and the kinematic viscosity of the fluid, and ∇q represents a conservative body force.

Equation (2.1), the continuity equation, is a mathematical statement of the law of conservation of mass for the incompressible flow. Equation (2.2), Navier–Stokes' momentum equation, expresses Newton's second law of motion for the viscous fluid. For the idealized inviscid fluid, the last term in (2.2) is zero and one has Euler's momentum equation for the incompressible flow

$$\frac{\partial \mathbf{v}}{\partial t} = -(\mathbf{v} \cdot \nabla)\mathbf{v} - \frac{1}{\rho}\nabla p + \nabla q \tag{2.3}$$

The overall problem of the incompressible flow of a viscous fluid is partitioned into two aspects by ascribing (2.1) to kinematics and (2.2), or (2.3), to kinetics. Kinematics and kinetics, the two branches of dynamics, are typically defined as follows: *Kinematics* is *the study of motions of themselves, apart from their causes*. *Kinetics is the study of changes of motions produced by forces*. In ascribing (2.1) to kinematics, the view is taken that the law of mass conservation imposes a condition on the incompressible motion of the fluid. Namely, the incompressible velocity field must be divergence-free. This condition is imposed on the flow at each instant of time, and is not regarded as a cause of change of the flow with time. This view is generally accepted in classical studies of the incompressible flow.

In an elegant treatise, Truesdell (1954) stated: *In general the flow of a fluid, whether perfect or viscous, can be defined by kinematical conditions.* In Truesdell's view, *circulation preservation* in time is one of the kinematical conditions on the flow. He then presented extensive results based on this view. In the present study, circulation preservation is viewed not as a kinematic condition, but as a kinetic consequence. Based on this perspective, it is shown that, in the case of the viscous flow, the kinetics as described by (2.2) causes the circulation to diffuse. However, in regions of the flow where viscous effects are insignificant, (2.3) brings about circulation preservation with time.

Scholars in fluid dynamics with different research focuses often interpret the terms kinematics and kinetics differently. Differences in interpretations are, by themselves, not important to the present study. The recognition that the overall flow problem contains two aspects, each with its own physical and mathematical characteristics is, however, crucially important. Viewing (2.1) as a constraint and (2.2) as a cause–effect relationship makes it feasible to treat these two interlaced aspects individually before uniting the individual results. The advantages of this strategy are not obvious if (2.1) and (2.2) are both viewed as kinematic constraints. These advantages become pronounced in studies of the flow of the viscous fluid using concepts of vorticity dynamics.

The familiar definition of the vorticity field, as stated in Chap. 1, is

$$\nabla \times \mathbf{v} = \boldsymbol{\omega} \tag{2.4}$$

Equations (2.1), (2.2), and (2.4) constitute a set of vorticity-dynamic equations describing the flow of the viscous fluid. Of these equations, (2.1) and (2.4) are kinematic since they deal with motions without reference to forces that cause motions. The kinetic Eq. (2.2) can be restated in terms of $\boldsymbol{\omega}$ as

$$\frac{\partial \mathbf{v}}{\partial t} = -\nabla(h - q) - \omega \times \mathbf{v} - \nu \nabla \times \omega \tag{2.5}$$

where $h = p/\rho + (\mathbf{v} \cdot \mathbf{v})/2$ is the total head.

Taking the curl of each term in (2.5) yields the vorticity-transport equation:

$$\frac{\partial \omega}{\partial t} = -\nabla \times (\omega \times \mathbf{v}) - \nu \nabla \times (\nabla \times \omega) \tag{2.6}$$

For the viscous fluid, the flow kinetics is represented by either (2.5) or (2.6). Although (2.6) does not contain a force term explicitly, it is considered a kinetic equation because it is a forthwith consequence of (2.2). This equation describes the change of the vorticity field with time. The terms on the right-hand side of (2.6) represent physical processes causing the change. For the inviscid fluid, the flow kinetics is described by (2.7) or (2.8)

$$\frac{\partial \mathbf{v}}{\partial t} = -\nabla h - \omega \times \mathbf{v} \tag{2.7}$$

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = -\nabla \times (\omega \times \mathbf{v}) \tag{2.8}$$

Several useful mathematical identities are derived in §2.3. These identities are not new. They are included here to facilitate the discussions of the theorems of Helmholtz and Kelvin. The reader wishing to verify the derivations may wish to refer to vector identities listed in §3.2.

2.3 Kinematic Preliminaries

In the following discussions, sufficient smoothness of lines (curves) and surfaces in space and sufficient differentiability of field variables are assumed so that the divergence theorem, Stokes' theorem, and other mathematical relations are meaningful.

A curve in space whose tangent at each point on it is in the direction of a vector field **f** is called a vector line of **f**. A surface comprising all the vector lines of **f** passing through a circuit (a closed curve) in space is called a vector tube of **f**. Vector lines and vector tubes of the velocity field **v** are called *streamlines* and *stream tubes*. Vectors lines and vector tubes of the vorticity field **ω** are called vorticity lines and vorticity tubes.

A vector field **f** is said to be solenoidal if its divergence, $\nabla \cdot \mathbf{f}$, is zero and lamellar if its curl, $\nabla \times \mathbf{f}$, is zero. A solenoidal velocity field **v** is said to be incompressible and a lamellar velocity field irrotational. In the following discussions, the terms irrotational and lamellar are used interchangeably: **f** is said to be irrotational or lamellar if $\nabla \times \mathbf{f} = 0$.

With Helmholtz' decomposition, a general vector field **f** is expressible in the form

$$\mathbf{f} = \nabla g + \nabla \times \mathbf{h} \tag{2.9}$$

where g is a scalar potential function and **h** is a vector potential function. By virtual of the vector identities $\nabla \times (\nabla g) = 0$ and $\nabla \cdot (\nabla \times \mathbf{h}) = 0$, ∇g is lamellar and $\nabla \times \mathbf{h}$ is solenoidal. Thus (2.9) decomposes the general vector field **f** into a solenoidal field, denoted $\mathbf{f}^*$ in this study, and a lamellar field denoted $\mathbf{f}'$. A lamellar field $\mathbf{f}'$ is expressible as ∇g alone and a solenoidal field $\mathbf{f}^*$ as $\nabla \times \mathbf{h}$ alone.

Consider two circuits C_1 and C_2 that lie on the same vector tube of **f**, each circuit encircles the tube once. Let S_1 be a cap of C_1 and S_2 a cap of C_2. Denote the section of the vector tube between C_1 and c by S_e. The three surfaces S_1, S_2, and S_e together form a closed surface S_o bounding a simply connected region R_o. The divergence theorem then gives

$$\iint_{S_1} \mathbf{f}\cdot\mathbf{n}\,dS + \iint_{S_2} \mathbf{f}\cdot\mathbf{n}\,dS + \iint_{S_e} \mathbf{f}\cdot\mathbf{n}\,dS = \iint_{R_0} \nabla\cdot\mathbf{f}\,dR \tag{2.10}$$

where $\mathbf{n}$ is the unit normal vector on the surfaces S_o directed outward from R_o.

The field $\mathbf{f}\cdot\mathbf{n}$ is a measure of the strength of the normal component of $\mathbf{f}$ on S and is called the *flux of* $\mathbf{f}$ on S. Since $\mathbf{f}$ is tangent to the vector tube, $\mathbf{f}\cdot\mathbf{n} = 0$ on S_e. For the solenoidal field $\mathbf{f}^*$, the right-hand side of (2.10) is zero. Let $\mathbf{n}_1$ and $\mathbf{n}_2$ be two unit normal vectors on S_1 and S_2 respectively, with their positive directions given by the axial direction of the vector tube. With this orientation, if $\mathbf{n}_1 = \mathbf{n}$ on S_1, then $\mathbf{n}_2 = -\mathbf{n}$ on S_2. On the other hand, if $\mathbf{n}_1 = -\mathbf{n}$, then $\mathbf{n}_2 = \mathbf{n}$ on S_2. In either case, one obtains from (2.10)

$$\iint_{S_1} \mathbf{f}^*\cdot\mathbf{n}_1\,dS + \iint_{S_2} \mathbf{f}^*\cdot\mathbf{n}_2\,dS \tag{2.11}$$

The two integrals in (2.11) are called respectively the total flux of the field $\mathbf{f}^*$ across S_1 and S_2 or the strength of $\mathbf{f}^*$ of the vector tube at the S_1 and S_2. This equation states that the strength of the vector tube of $\mathbf{f}^*$ is constant along the path of the tube. Equation (2.11) expresses a spatial, or kinematic, conservation relationship. If the solenoidal vector field $\mathbf{f}^*$ is time-dependent, then the spatial conservation is valid at each instant of time. Representing $\mathbf{f}^*$ by $\nabla\times\mathbf{h}$, one obtains, using Stokes' theorem

$$\oint_C \mathbf{h}\cdot\boldsymbol{\tau}\,ds = \iint_S \mathbf{f}^*\cdot\mathbf{n}\,dS \tag{2.12}$$

In (2.12), $\boldsymbol{\tau}$ is a unit tangent vector on the circuit C. The positive senses of $\mathbf{n}$ and $\boldsymbol{\tau}$ are related by the right-handed screw convention. Namely, as a right-handed screw turns in the positive $\boldsymbol{\tau}$ direction, it advances in the positive $\mathbf{n}$ direction. The left-hand side of (2.12) is called the circulation of $\mathbf{h}$ around C. Equation (2.12) states the total flux of a solenoidal field $\mathbf{f}^*$ across S is equal to the circulation of the vector potential $\mathbf{h}$ of $\mathbf{f}^*$ around C, of which S is a cap. The three terms *total flux across a surface*, *strength of vector tube*, and *circulation* are equivalent. These terms are used interchangeably in the literature, each describing the surface integral, or its equivalent line integral, in (2.12).

Let C be a time-dependent circuit. Denote the positions of C at the time levels $t_1 = t$ and $t_2 = t + \delta t$ respectively by C_1 and C_2. Consider the time derivative of the integration of $\mathbf{f}\cdot\boldsymbol{\tau}$ over C, where $\mathbf{f}$ is time-dependent. One writes

$$\frac{d}{dt}\oint_{C}\mathbf{f}(\mathbf{r},t)\cdot\boldsymbol{\tau}\,ds \equiv \lim_{\delta t\to 0}\left\{\left[\oint_{C_2}\mathbf{f}(\mathbf{r},t+\delta t)\cdot\boldsymbol{\tau}\,ds-\oint_{C_1}\mathbf{f}(\mathbf{r},t)\cdot\boldsymbol{\tau}\,ds\right]/\delta t\right\}$$
$$=\lim_{\delta t\to 0}\oint_{C_2}\left[\frac{\mathbf{f}(\mathbf{r},t+\delta t)-\mathbf{f}(\mathbf{r},t)}{\delta t}\right]\cdot\boldsymbol{\tau}\,ds+\lim_{\delta t\to 0}\left\{\left[\oint_{C_2}\mathbf{f}(\mathbf{r},t)\cdot\boldsymbol{\tau}\,ds-\oint_{C_1}\mathbf{f}(\mathbf{r},t)\cdot\boldsymbol{\tau}\,ds\right]/\delta t\right\} \tag{2.13}$$

Let S_1 be a cap of C_1 and S_2 a cap of C_2. Using Stokes' theorem, the last two integrals in (2.13) can be restated as integrations of $(\nabla \times \mathbf{f}) \cdot \mathbf{n}_i$, i = 1 or 2, over the surfaces S_i. Let S_e be a surface joining C_1 and C_2 at the time level t. The three surfaces S_1, S_2, and S_e together form a closed surface S_o bounding the region R_o. If $\boldsymbol{\tau}$ is in a direction such that $\mathbf{n}_1 = \mathbf{n}$, the outward unit normal vector on S_1, then, with the right-handed screw convention, $\mathbf{n}_2 = -\mathbf{n}$ on S_2. Thus

$$\oint_{C_2}\mathbf{f}(\mathbf{r},t)\cdot\boldsymbol{\tau}\,ds-\oint_{C_1}\mathbf{f}(\mathbf{r},t)\cdot\boldsymbol{\tau}\,ds=-\oiint_{S_0}(\nabla\times\mathbf{f})\cdot\mathbf{n}\,dS+\iint_{S_e}(\nabla\times\mathbf{f})\cdot\mathbf{n}\,dS \tag{2.14}$$

The first integral on the right-hand side of (2.14) is an integral over the closed surface S_o. This integral can be restated, using the divergence theorem, as an integration of $\nabla \cdot (\nabla \times \mathbf{f})$ over R_o and is therefore zero. On S_e, as $\delta t \to 0$, one has $\mathbf{n}dS \to (\mathbf{v}_c\delta t) \times (\boldsymbol{\tau}ds)$, where $\mathbf{v}_c$ is the velocity of C_1. One therefore has

$$\iint_{S_e}(\nabla\times\mathbf{f})\cdot\mathbf{n}\,dS=\delta t\oint_{C_1}(\nabla\times\mathbf{f})\cdot(\mathbf{v}_c\times\boldsymbol{\tau})dS=\delta t\oint_{C_1}[(\nabla\times\mathbf{f})\times\mathbf{v}_c]\cdot\boldsymbol{\tau}\,dS \tag{2.15}$$

Placing (2.15) into (2.14), and the result into (2.13), one obtains, upon noting that, as $\delta t \to 0$, the integrand in the first term on right-hand side of (2.13) gives $(\partial\mathbf{f}/\partial t) \cdot \boldsymbol{\tau}$,

$$\frac{d}{dt}\oint_{C(t)}\mathbf{f}(\mathbf{r},t)\cdot\boldsymbol{\tau}\,ds=\oint_{C}\left(\frac{\partial\mathbf{f}}{\partial t}\right)\cdot\boldsymbol{\tau}\,ds+\oint_{C}[(\nabla\times\mathbf{f})\times\mathbf{v}_c]\cdot\boldsymbol{\tau}\,ds \tag{2.16}$$

where the superfluous subscripts for the circuit C are dropped. The symbol C(t) is used, rather than C, in the left-hand side of (2.16) to emphasize the time dependency of the path of integration. The two integrals on the right-hand side of (2.16) are not differentiated with respect to time. Therefore the time dependency of the integration path C is not an issue of concern. The line integral of $\mathbf{f} \cdot \boldsymbol{\tau}$ over a moving (and/or deforming) loop C(t) changes with time because of two contributing factors:

the local change of **f** on C and the motion of the loop C. The right-hand side integrals in (2.16) express separately the contributions.

For the lamellar vector field $\mathbf{f}' = \nabla g$, the last term in (2.16) vanishes since $\nabla \times (\nabla g) = 0$. The integral on the left-hand side of (2.16) vanishes since it can be restated as an integral of $(\nabla \times \mathbf{f}') \cdot \mathbf{n}$ over a cap S of C. The first term on the right-hand side of (2.16) is similarly zero because $\partial \mathbf{f}'/\partial t = \nabla(\partial g/\partial t)$. Therefore (2.16) is trivial for the lamellar field. Only the solenoidal part of the general vector field **f**, as represented by (2.9), plays a role in (2.16).

Consider the time derivative of the total flux of $\mathbf{f}(\mathbf{r},t)$ across S(t)

$$\begin{aligned}\frac{d}{dt}\iint_S \mathbf{f}(\mathbf{r},t)\cdot\mathbf{n}_1\,dS &\equiv \lim_{\delta t\to 0}\left\{\left[\iint_{S_2}\mathbf{f}(\mathbf{r},t+\boldsymbol{\delta}t)\cdot\mathbf{n}_2\,dS - \iint_{S_1}\mathbf{f}(\mathbf{r},t)\cdot\mathbf{n}_1\,dS\right]/\delta t\right\}\\ &= \lim_{\delta t\to 0}\iint_{S_2}\left[\frac{\mathbf{f}(\mathbf{r},t+\delta t)-\mathbf{f}(\mathbf{r},t)}{\delta t}\right]\cdot\mathbf{n}_2\,dS\\ &+ \lim_{\delta t\to 0}\left\{\left[\iint_{S_2}\mathbf{f}(\mathbf{r},t)\cdot\mathbf{n}_2\,dS - \iint_{S_1}\mathbf{f}(\mathbf{r},t)\cdot\mathbf{n}_1\,dS\right]/\delta t\right\}\end{aligned} \tag{2.17}$$

With the surface S_e defined earlier, one obtains, using the divergence theorem

$$\iint_{S_2}\mathbf{f}\cdot\mathbf{n}_2\,dS - \iint_{S_1}\mathbf{f}\cdot\mathbf{n}_1\,dS = -\iiint_{R_0}\nabla.\mathbf{f}\,dR + \iint_{S_e}\mathbf{f}\cdot\mathbf{n}_e\,dS \tag{2.18}$$

As $\delta t \to 0$, $dR \to -(\mathbf{v}_{s1}\delta t)\cdot\mathbf{n}_1 dS$ and $\mathbf{n}_e dS \to (\mathbf{v}_c \times \boldsymbol{\tau})ds\delta t$, where $\mathbf{v}_{s1}$ is the velocity of the cap S_1 and $\mathbf{v}_c$ is the velocity of the circuit C_1. One thus obtains

$$\iint_{S_2}\mathbf{f}\cdot\mathbf{n}_2\,dS - \iint_{S_1}\mathbf{f}\cdot\mathbf{n}_1\,dS = \delta t\iint_{S_1}(\nabla\cdot\mathbf{f})(\mathbf{v}_s.\mathbf{n})dS + \delta t\oint_{C_1}\mathbf{f}.\left(\mathbf{v}_C\times\boldsymbol{\tau}\right)dS \tag{2.19}$$

The integrand in the last integral of (2.19) can be restated as $(\mathbf{f} \times \mathbf{v}_c)\cdot\boldsymbol{\tau}$. Thus Stokes' theorem yields

$$\oint_{C_1}\mathbf{f}\cdot(\mathbf{v}_c\times\boldsymbol{\tau})ds = \iint_{S_1}\nabla\times(\mathbf{f}\times\mathbf{v}_{s1})\cdot\mathbf{n}\,dS \tag{2.20}$$

Placing (2.20) into (2.19) and the results into (2.17), one obtains, upon noting the integrand in the first integral in the right-hand side of (2.17) is $\partial\mathbf{f}/\partial t$ and dropping the subscripts for S:

$$\frac{d}{dt}\iint\limits_{S(t)} \mathbf{f}(\mathbf{r},t)\cdot\mathbf{n}\,dS = \iint\limits_{S} \frac{\partial \mathbf{f}}{\partial t}\cdot\mathbf{n}\,dS + \iint\limits_{S}[(\nabla\cdot\mathbf{f})\mathbf{v}_s]\cdot\mathbf{n}\,dS + \iint\limits_{S}[\nabla\times(\mathbf{f}\times\mathbf{v}_s)]\cdot\mathbf{n}\,dS \tag{2.21}$$

The total flux of **f** over S(t) changes with time because the field **f** is time-dependent and also because the surface S is time-dependent. With (2.21), the time rate of change of the flux of **f** over S(t) is given by the sum of the three integrals on the right-hand side. The first integral represents the local rate of change of **f** on S. The second and third integrals together represent the motion and deformation of S.

For the solenoidal field $\mathbf{f}^* = \nabla \times \mathbf{h}$, (2.21) reduces to

$$\frac{d}{dt}\iint\limits_{S(t)} \mathbf{f}^*\cdot\mathbf{n}dS = \iint\limits_{S} \frac{\partial \mathbf{f}^*}{\partial t}\cdot\mathbf{n}dS + \iint\limits_{S}\nabla\times(\mathbf{f}^*\times\mathbf{v_s})\cdot\mathbf{n}dS \tag{2.22}$$

Consider the time derivative of the integral of $\mathbf{f}(\mathbf{r}, t)$ over S(t), which is a cap of C(t)

$$\begin{aligned}\frac{d}{dt}\iint\limits_{S}\mathbf{f}(\mathbf{r},t)dS &\equiv \lim_{\delta t\to 0}\left\{\left[\iint\limits_{S_2}\mathbf{f}(\mathbf{r},t+\delta t)dS - \iint\limits_{S_1}\mathbf{f}(\mathbf{r},t)dS\right]/\delta t\right\}\\ &= \lim_{\delta t\to 0}\iint\limits_{S_2}\left[\frac{\mathbf{f}(\mathbf{r},t+\delta t)-\mathbf{f}(\mathbf{r},t)}{\delta t}\right]dS + \lim_{\delta t\to 0}\left\{\left[\iint\limits_{S_2}\mathbf{f}(\mathbf{r},t)dS - \iint\limits_{S_1}\mathbf{f}(\mathbf{r},t)dS\right]/\delta t\right\}\end{aligned} \tag{2.23}$$

Through analyses similar to those leading to (2.19) and (2.21), one obtains

$$\iint\limits_{S_2}\mathbf{f}(\mathbf{r},t)dS - \iint\limits_{S_1}\mathbf{f}(\mathbf{r},t)dS = \delta t\oint\limits_{C_1}\mathbf{f}(\mathbf{v}_c\cdot\mathbf{n})ds \tag{2.24}$$

$$\frac{d}{dt}\iint\limits_{S(t)}\mathbf{f}(\mathbf{r},t)dS = \iint\limits_{S}\frac{\partial \mathbf{f}}{\partial t}dS + \oint\limits_{C_1}\mathbf{f}(\mathbf{v}_c\cdot n)ds \tag{2.25a}$$

$$= \iint\limits_{S}\left[\frac{\partial \mathbf{f}}{\partial t} + (\nabla\cdot\mathbf{v}_s)\mathbf{f} + (\mathbf{v}_s\cdot\nabla)\mathbf{f}\right]dS \tag{2.25b}$$

A similar identity for the time derivative of the integral of $\mathbf{f}(\mathbf{r},t)$ over the three-dimensional region R(t) bounded by S(t) is

$$\frac{d}{dt}\iiint\limits_{R(t)}\mathbf{f}(\mathbf{r},t)dR = \iiint\limits_{R}\frac{\partial \mathbf{f}}{\partial t}dR + \oiint\limits_{S}\mathbf{f}(\mathbf{v}_s\cdot\mathbf{n})dS \tag{2.26a}$$

$$= \iiint_R \left[\frac{\partial \mathbf{f}}{\partial t} + (\nabla \cdot \mathbf{v}_v)\mathbf{f} + (\mathbf{v}_v \cdot \nabla)\mathbf{f} \right] dR \tag{2.26b}$$

where $\mathbf{v}_v$ is the velocity of points in R and $\mathbf{v}_s$ is the velocity of points on the boundary S.

Equations (2.25a) and (2.26a) remain valid with the vector field $\mathbf{f}(\mathbf{r}, t)$ replaced by the scalar field $f(\mathbf{r}, t)$. The mathematical identities presented in this §2.3 are valid independently of the material medium occupying the space R. Whether a solid body, a viscous fluid, an inviscid fluid, or a combination of these media is present in R does not alter these identities. They are kinematic identities in which C needs not be a material circuit and S needs not be a material surface. $\mathbf{v}_c$, $\mathbf{v}_s$, and $\mathbf{v}_v$ represent the motions of the circuit, the surface, and points in R and need not be the flow velocity $\mathbf{v}$.

2.4 Helmholtz' First Vortex Theorem

In (2.11), let $\mathbf{f}^*$ be the incompressible flow velocity $\mathbf{v}$. One obtains

$$\iint_{S_1} \mathbf{v} \cdot \mathbf{n}_1 dS = \iint_{S_2} \mathbf{v} \cdot \mathbf{n}_2 \, dS \tag{2.27}$$

Equation (2.27) states a well-known fact: The flow rate (total velocity flux) through a stream tube in an incompressible flow is constant.

Let $\mathbf{f}^*$ be $\mathbf{v}$ and $\mathbf{h}$ be the vector potential $\boldsymbol{\psi}$ of $\mathbf{v}$ in (2.12). One has

$$\iint_S \mathbf{v} \cdot \mathbf{n} \, dS = \oint_C \boldsymbol{\psi} \cdot \boldsymbol{\tau} \, ds \tag{2.28}$$

Equation (2.28) states that the total rate of an incompressible flow across the area S is identical to the circulation of the vector potential $\boldsymbol{\psi}$ around the circuit C, of which S is a cap. Equation (2.27) therefore states that the circulation of the vector potential $\boldsymbol{\psi}$ around a stream tube in the incompressible flow is constant along the path of the stream tube.

Let $\mathbf{f}^*$ be $\boldsymbol{\omega}$ in (2.11). One obtains

$$\iint_{S_1} \boldsymbol{\omega} \cdot \mathbf{n}_1 dS = \iint_{S_2} \boldsymbol{\omega} \cdot \mathbf{n}_2 \, dS \tag{2.29}$$

Let $\mathbf{f}^*$ be $\boldsymbol{\omega}$ and $\mathbf{h}$ be $\mathbf{v}$. Equation (2.12) yields

$$\iint_S \boldsymbol{\omega} \cdot \mathbf{n}\, dS = \oint_C \mathbf{v} \cdot \tau\, ds \tag{2.30}$$

Equation (2.30) states that the total vorticity flux across S is identical to the circulation (of velocity) around C, of which S is a cap. Therefore (2.29) states: *The circulation of a vorticity tube is constant along each vorticity tube*. This statement paraphrases the first part of Sommerfeld's (1950) statement about spatial conservation of vorticity: *The vortex strength* (circulation) *is constant along each vortex line or vortex tube, which must either be closed or end at the boundary of the fluid*.

In classical fluid dynamics, the term vortex is typically used in place of the word vorticity, often in the context of the inviscid fluid idealization. In the present study, the idea of a vortex always means an approximation of a part of a vorticity field in a real flow, not a singular element in an inviscid fluid. The terms *vorticity tube*, *vorticity line*, and *vorticity flux* are therefore preferred in the present study. Sommerfeld used the terms vortex line and vortex tube interchangeably. In the present study, except in direct quotation of classical literature, a *vortex filament* (not a line) gives the approximate position of a thin vorticity tube in space. The filament has an infinitesimal (and nonzero) cross-sectional area and is not a line. It represents (approximates) the vorticity tube and has the circulation of the tube.

If a vortex filament of finite strength Γ ends in the fluid, then the strength of the vorticity tube represented by the filament changes abruptly at the ending point from Γ to zero. The strength along the vorticity tube is then not a constant. The second part of Sommerfeld's statement, that a vortex tube must either be closed or end at the boundary of the fluid, is therefore a consequence, a corollary, of the first part of his statement.

Consider a solid body immersed and moving in an infinite fluid. Denote the solid region by R_s, the fluid region by R_f, the fluid–solid interface by S, and the infinite unlimited space occupied jointly by the fluid and solids by R_∞. The velocity $\mathbf{v}_r$ of a rigid body rotating at the angular velocity $\boldsymbol{\Omega}$ is defined by $\mathbf{v}_r = \mathbf{v}_a + \boldsymbol{\Omega} \times (\mathbf{r} - \mathbf{r}_a)$, where $\mathbf{v}_a$ is the translation velocity of the point $\mathbf{r} = \mathbf{r}_a$. It is simple to show that $\nabla \cdot \mathbf{v}_r = 0$ and $\nabla \times \mathbf{v}_r = 2\boldsymbol{\Omega}$. In other words, (2.1) is satisfied in R_s and, as defined by (2.4), $\boldsymbol{\omega} = 2\boldsymbol{\Omega}$ in R_s. Equations (2.1) and (2.4) therefore describe the kinematics of both the fluid and the solid; they are valid equations in the infinite unlimited region R_∞. It is not difficult to generalize the above discussions to flows containing multiple rigid bodies.

With the *no-slip condition*, the normal vorticity component is necessarily zero at the surface of a non-rotating rigid solid body. Lighthill (1963), on the basis of this observation, concluded: *In flows which do not contain rotating bodies, all vorticity appears in closed loops*. Lighthill's conclusion can also be obtained by observing that with non-rotating solids, the solid region R_s contains no vorticity. With the vorticity field present only in R_f, S can only be a part of the surfaces of vorticity tubes. In consequence, vorticity tubes cannot continue into R_s and must be closed in R_f.

With the no-slip condition, the tangential components of the velocity vector are continuous on S. Hence the normal component of the vorticity vector is continuous on S whether or not the solids are rotating. Approaching S from the side of the solid or the side of the fluid gives the same value of the normal vorticity on S. The kinetics of the motion in the solid region differs from that in the fluid. The gradients of the velocity components, hence also the tangential components of vorticity, are therefore discontinuous on S. The discontinuity occurs whether or not the solid is rotating. In either case, because the normal component of vorticity is continuous on S, vorticity flux is continuous on S and Lighthill's observation can be generalized to the statement: *All vorticity appears in closed loop in the infinite unlimited region* R_∞.

Vorticity lines need not be smooth for the idea of vorticity tubes and vorticity loops to be meaningful. Vorticity tubes may bend at finite angles; they only need to continue in space. A vorticity line, being a curve in space, has a zero cross-sectional area and is not associated with a total flux or a circulation. Along the path of a vorticity line, the value of the vorticity can change. The line may pass through points of zero vorticity, giving the impression that it ends in space. This occurrence does not mean that a vorticity tube ends in space.

Vorticity tubes may overlap, or merge. The merged tube appears as a single vorticity tube with a circulation that is equal to the sum of the circulations of the component tubes. For the flow containing rotating solids, some of the vorticity tubes pass through S. On S, a vorticity tube in R_s merges with a vorticity tube in R_f. In most aerodynamic applications, the circulation of the vorticity tube in R_s (the cross-sectional area of the tube times $2\boldsymbol{\Omega}$) is weak compared to the circulation of the tube in R_f. The merging of the two tubes near S gives the misleading appearance that the vorticity tube in R_s does not continue into R_f and vice versa. This appearance obscures the truth that, because the vorticity flux is continuous on S, individual vorticity tubes are continued on S; they do not end on S either from the fluid side or from the solid side. Helmholtz' first vortex theorem therefore generalizes to the statement: *The vorticity strength* (circulation) *is constant along each vorticity tube, which must be closed in space*. In other words, a vorticity field in R_∞ is portrayable in general as a set of closed vorticity tubes (vorticity loops), each with a constant circulation along its path. This generalized Helmholtz' first vortex theorem is a kinematic theorem stating the spatial conservation of circulation.

Equations (2.1) and (2.4) contain first order spatial derivatives of the velocity field $\mathbf{v}$. For these equations to be meaningful, $\mathbf{v}$ must be not only continuous but piecewise smooth (once differentiable). The generalized Helmholtz' first vortex theorem and many vorticity-kinematic issues involve only integrations of $\boldsymbol{\omega}$. Therefore the vorticity field $\boldsymbol{\omega}$ needs only be piecewise continuous. Smoothness requirements for the vorticity in kinetics are more stringent since the vorticity transport equation (2.6), contains derivatives of $\boldsymbol{\omega}$. The kinematic-kinetic partition of the overall flow problem is in this context important in flow analyses. Without this partition, it is difficult to resolve many well-known conceptual difficulties and paradoxes in classical fluid dynamics.

2.5 Vorticity Loops

As a consequence of the generalized Helmholtz' first vortex theorem, every vorticity field in the infinite unlimited space R_∞ occupied jointly by the fluid and the solid is portrayable by a set of vorticity loops (closed tubes). The cross-sectional area S of each loop varies along the path of the loop, but the total flux of vorticity (circulation) of the loop is constant along the path. Each vorticity loop can be divided into n thinner vorticity loops simply by partitioning S into smaller areas S_i, i = 1, 2, … , n, with vorticity strengths Γ_i. These thinner loops are contiguous and together they occupy the space of the original undivided vorticity loop. The sum of the vorticity strengths of the thinner loops is the total strength of the original vorticity loop. A very thin vorticity loop is accurately approximated by a closed vortex filament (vortex loop), as discussed in §2.4. Thus any vorticity field can be approximated by a large number of vortex loops in space.

Let the closed path of the vortex loop representing a thin vorticity tube be C. If C is divided into two open paths C_1 and C_2 and the two dividing points are connected by the path C_3, then there are two closed paths: a closed path $C_1{}^*$ formed by combing C_3 and C_1 and a second closed path $C_2{}^*$ formed by combining C_3 and C_2. Consider two vortex loops: $\Gamma_1{}^*$ on $C_1{}^*$ and $\Gamma_2{}^*$ on $C_2{}^*$. Let $\Gamma_1{}^* = \Gamma_2{}^* = \Gamma$. On the open path C_3, the directions of the loops $C_1{}^*$ and $C_2{}^*$ are opposite. Thus the combined circulation of the loops $\Gamma_1{}^*$ and $\Gamma_2{}^*$ is zero on C_3. The two smaller loops $\Gamma_1{}^*$ and $\Gamma_2{}^*$ together are therefore equivalent, kinematically, to the original loop Γ. Thus any vortex loop Γ is divisible into two smaller vortex loops with the same vortex strength. Successive divisions give a set of small loops, each with the circulation Γ. In aggregate, the small vortex loops are equivalent to the large loop Γ. Since the paths of division can be arbitrarily chosen, the set of small loops can be configured with great flexibility.

Under general circumstances, the vorticity field spreads over a relatively large flow region. Therefore a large number of vortex loops, each representing a thin vorticity loop, are needed to produce a reasonably accurate approximation of the vorticity field. In the limit as the cross-sectional area of each vorticity loop approaches zero, there is an infinite number of vortex loops and the approximation becomes precise.

Dividing thin vorticity loops into sets of small vorticity loops produces a large number of thin and small vorticity loops well suited for flow computations. Dealing with vorticity loops rather than the vorticity field ensures the satisfaction of the principle of total vorticity conservation (see Chap. 5), which is necessary for solution stability in three-dimensional flow computations. The use of a small number of vorticity loops is convenient, but significant computational inaccuracy is expected. For streamlined flows, however, the vorticity-loop concept is powerful in interpreting the lifting-line theory of aerodynamics, as discussed in Chap. 6.

This suggests the possibility of simplified solution procedures for computing certain types of flows, including unsteady flows. Additional discussions of vorticity-loop portrayals are given by Wu et al. (2002) in a study of the lift, the profile drag, and the induced drag on hovering rotors.

2.6 Kelvin's Theorem

In (2.16), let C(t) be a material circuit C_m moving with the fluid and $\mathbf{f} = \boldsymbol{\psi}$, the vector potential of an incompressible velocity field $\mathbf{v}$. The integrand of the last integral in (2.16) vanishes since $(\nabla \times \boldsymbol{\Psi}) \times \mathbf{v} = \mathbf{v} \times \mathbf{v} = 0$. Thus one has

$$\frac{d}{dt}\oint_{C_m} \boldsymbol{\Psi}(\mathbf{r}, t) \cdot \boldsymbol{\tau}\, ds = \oint_{C_m} \left(\frac{\partial \boldsymbol{\Psi}}{\partial t}\right) \cdot \boldsymbol{\tau}\, ds \tag{2.31}$$

Equation (2.31) states that the rate of change of the material circulation of the vector potential $\boldsymbol{\Psi}$ is caused by only the local change of $\boldsymbol{\Psi}$. The motion of the material circuit does not contribute to the change.

Let $\mathbf{f} = \mathbf{v}$, the velocity field of an incompressible flow. Equation (2.16) gives

$$\frac{d}{dt}\oint_{C_m} \mathbf{v} \cdot \boldsymbol{\tau}\, ds = \oint_{C_m} \left(\frac{\partial \mathbf{v}}{\partial t}\right) \cdot \boldsymbol{\tau}\, ds + \oint_{C_m} (\boldsymbol{\omega} \times \mathbf{v}) \cdot \boldsymbol{\tau}\, ds \tag{2.32}$$

where the left-hand side integral gives the *material circulation* Γ_m of the velocity field.

The first integral on the right-hand side of (2.32) represents the contribution of the local change of $\mathbf{v}$ to the change of the material circulation Γ_m. The last integral is the circulation of Lamb's vector $\boldsymbol{\omega} \times \mathbf{v}$ around the material circuit C_m and represents the contribution to the rate of change of Γ_m by the motion of the material circuit C_m.

Equation (2.32) is a kinematic equation obtained using only (2.1) and (2.4). Placing the kinetic equation (2.5) into (2.32) yields

$$\frac{d\Gamma_m}{dt} = -\oint_{C_m} [\nabla(h - q)] \cdot \tau\, ds - \nu \oint_{C_m} (\nabla \times \boldsymbol{\omega}) \cdot \tau\, ds \tag{2.33}$$

The first integral in (2.33) can be restated using Stokes' theorem as an integral over S_m, a cap of C_m, of $\nabla \times [\nabla(h - q)]$, which is identically zero. One thus has

$$\frac{d\Gamma_m}{dt} = -\nu \oint_{C_m} (\nabla \times \boldsymbol{\omega}) \cdot \tau\, ds \tag{2.34}$$

Equation (2.34) is a vorticity-dynamic relation based on the kinematic equations (2.1) and (2.4) and the kinetic equation (2.5). Placing (2.7) in (2.32) or letting $\nu = 0$ in (2.34) yields

$$\frac{d\Gamma_m}{dt} = 0 \tag{2.35}$$

Equation (2.35) is a statement of Kelvin's theorem on the persistence of circulation: *The circulation around a material circuit moving with an inviscid fluid remains constant.*

For the viscous fluid, (2.34) states: *The circulation around a material loop moving with a viscous fluid changes with time as the result of viscous diffusion.* It is known that the viscosities of air and water, the primary flow media of interest in aerodynamics, are very small. More precisely, the dimensionless Reynolds number is very large. Therefore, except in flow regions close to solid surfaces, where the gradient of the vorticity field can be very steep, the right-hand side of (2.34) is small and the circulation around the material circuit changes very slowly. In many regions far from solid surfaces, the vorticity gradients are so small that circulation changes are negligible slow. Such flow regions are inviscid flow regions even though the fluid is not inviscid.

2.7 Helmholtz' Second Vortex Theorem

Let $\mathbf{f}^* = \boldsymbol{\omega}$ and $\mathbf{v}_s = \mathbf{v}$. One obtains using (2.22) and (2.6) the following equations:

$$\frac{d}{dt}\iint\limits_{S_m} \boldsymbol{\omega}\cdot\mathbf{n}\,dS = \iint\limits_{S_m} \frac{\partial\boldsymbol{\omega}}{\partial t}\cdot\mathbf{n}\,dS + \iint\limits_{S_m} \nabla\times(\boldsymbol{\omega}\times\mathbf{v})\cdot\mathbf{n}\,dS \tag{2.36}$$

$$\frac{d}{dt}\iint\limits_{S_m} \boldsymbol{\omega}\cdot\mathbf{n}\,dS = -\nu\iint\limits_{S_m} [\nabla\times(\nabla\times\boldsymbol{\omega})]\cdot\mathbf{n}\,dS \tag{2.37a}$$

$$= -\nu\oint\limits_{C_m} (\nabla\times\omega)\cdot\tau\,dS \tag{2.37b}$$

Letting $\nu = 0$ in (2.37), or placing (2.7) into (2.36), one obtains

$$\frac{d}{dt}\iint\limits_{S_m} \boldsymbol{\omega}\cdot\mathbf{n}\,dS = 0 \tag{2.38}$$

Sommerfeld's statement (1950) of Helmholtz' second vortex theorem is: *The vortex strength* (total flux of vorticity across a material surface) *is constant in time*. In other words, the circulation around a material vorticity tube is independent of time in an inviscid fluid. Thus Helmholtz' second vortex theorem and Kelvin's theorem on the persistence of circulation are equivalent. In fact, by virtue of (2.12), (2.38) is equivalent to (2.35) and (2.37) is equivalent to (2.34). The proof provided for (2.37) in this Section is therefore redundant.

The surface S_m is a cap of the material loop C_m. There exist an infinite number of caps for each material loop in space. Thus (2.34) is more general than (2.37a). In other words, using Stokes' theorem, (2.34) can be restated as (2.37) with S_m replaced by S. S can be an arbitrary cross-sectional surface of a material vorticity tube, a cap of a material circuit, but not necessarily a material cap moving with the fluid.

Equation (2.38) states that the total vorticity flux across the cross-sectional area of a material vorticity tube in an inviscid flow is independent of time. Letting the cross-sectional area of a material tube approach zero, one arrives at Lighthill's (1986) interpretation of Helmholtz' second theorem: *Vortex lines move with the* (inviscid) *fluid.* Saffman (1992) reviewed the works of Lamb (1932) and others that culminated in Lighthill's statement. Alternative proofs of this statement are presented in the works of Lamb, Lighthill, Lugt (1996), Saffman, Whitham (1963), and others.

As discussed in §2.6, the circulation of a material vorticity tube in a viscous fluid changes with time because of viscous diffusion, a process that spreads the vorticity in the flow. Consider, for simplicity, a Cartesian coordinate system (x,y,z) and a planar flow with the velocity field $\mathbf{v}(x,y)$ and $\boldsymbol{\omega} = \omega\,\mathbf{k}$, where $\mathbf{k}$ is the unit vector in the z-direction. For this planar flow, the vorticity flux $\boldsymbol{\omega} \cdot \mathbf{n}$ is identical to ω since $\mathbf{k} = \mathbf{n}$. Thus (2.37) yields

$$\frac{d}{dt}\iint_{S_m} \omega \cdot dS = \nu \iint_{S_m} \left(\nabla^2\omega\right) dS \tag{2.39}$$

Using (2.25), (2.39) can be restated in the form

$$\iint_{S} \left(\frac{D\omega}{Dt} - \nu\nabla^2\omega\right) dS = 0 \tag{2.40}$$

where $D\omega/Dt = \partial\omega/\partial t + (\mathbf{v} \cdot \nabla)\omega$ is the material (substantial) derivative of ω, a time derivative following the motion of the fluid. Equation (2.39) is given by Lamb (1932) and a more general equation for a non-planer surface S_m by Wu and Wu (1998).

Equation (2.40) is obtainable directly by integrating the two-dimensional version of the vorticity transport equation (2.6), which is expressible in the form $D\omega/Dt = \nu\nabla^2\omega$. In this form, the vorticity is a scalar field and the vorticity transport equation is analogous to the familiar diffusion equation. Consider, for example, the heat conduction (diffusion) equation $\partial T/\partial t = \kappa\nabla^2 T$, where T is the temperature field and κ is the heat conductivity (diffusivity). In heat conduction, heat energy, measured in terms of temperature, is transported in a medium at rest. This process is irreversible. It equalizes the heat energy, spreading it from higher temperature regions to lower temperature regions. Observed in a reference frame moving with the fluid, the viscous diffusion of vorticity is analogous to heat conduction in a stationary medium. Without viscous diffusion, the total vorticity over any material surface in the plane of the flow is independent of time. This is analogous to the temperature associated with each material element of a stationary non-conducting solid remaining unchanged relative to time. In the flow of the viscous fluid, the total vorticity in a material surface changes with time as a result of viscous diffusion.

In three-dimensional flows, the vorticity is a vector field. Equation (2.37) is valid for an arbitrarily chosen material surface, including an elemental material surface δS_m. Therefore the total vorticity flux over δS_m is spread by viscous diffusion when viewed in a *material reference frame* moving with the fluid. It is worth underscoring that the total flux $\boldsymbol{\omega} \cdot \mathbf{n}\ \delta S_m$, not the vector $\boldsymbol{\omega}$, is conserved in the inviscid flow. This total flux is diffused in the viscous flow. In three-dimensions, the surface element δS_m translates and rotates. The total vorticity flux translates and rotates with δS_m.

2.8 Concluding Remarks

The theorems of Helmholtz and Kelvin are most conveniently interpreted using the idea of the vorticity flux $\boldsymbol{\omega} \cdot \mathbf{n}$. Helmholtz' first vortex theorem is recognized as a theorem of vorticity kinematics stating that *the total vorticity flux in a vorticity tube is constant along the path of the tube*. This theorem is valid in the infinite unlimited region occupied jointly by the solid and the fluid. In consequence, *in external aerodynamics, all vorticity fields are portrayable as sets of vorticity loops, each with a constant circulation along the path of the loop*.

Helmholtz' second vortex theorem, or its equivalence Kelvin's theorem, is a theorem of vorticity dynamics. A generalized statement of this theorem is: *The total vorticity flux in each material vorticity tube changes with time only as a result of vorticity diffusion across the boundary surface of the tube*. It is important to note that, because this generalized theorem is a vorticity-dynamic theorem, it is applicable in the interior of the fluid region, and not in the solid region. In an inviscid flow, the total vorticity flux in each vorticity tube remains unchanged with time. In many flow regions far from solid surfaces, the vorticity gradients are so small that circulation changes are negligible slow. Such flow regions are effectively inviscid even though the fluid is viscous. In such flow regions, vorticity lines are effectively material lines. It is convenient to think of vorticity loops portraying the vorticity field as moving with the fluid while retaining their strengths (circulation) in such regions.

In flow regions where viscous effects are important, vorticity lines are not material lines. They do not move with the fluid. Since the circulation of each material circuit changes with time, the strength of each vorticity loop is not conserved in time. It is not sufficient to merely keep track of the movements of the vorticity loops; the strength and the shape of each loop must be undated continually. At the present stage of development, potential applications of the generalized theorems of Helmholtz and Kelvin in aerodynamics are evolving. Based on the present understanding, new interpretations of classical aerodynamic theories and of the connection between two- and three-dimensional aerodynamics are possible. Discussions of these topics are presented by Wu (1981) and in Chap. 6.

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Chapter 3
Vorticity Kinematics

3.1 Differential Equations of Vorticity Kinematics

Kinematics is a branch of dynamics defined as *the science concerned with motions in themselves, apart from their causes* (Webster 1953). *Vorticity kinematics* is a branch of *vorticity dynamics* concerned with the instantaneous relationship between the velocity field $\mathbf{v}$ and the vorticity field $\boldsymbol{\omega}$. The fields $\mathbf{v}$ and $\boldsymbol{\omega}$ in unsteady flows are time-dependent. In studies of vorticity kinematics, however, only the instantaneous relationship between $\mathbf{v}$ and $\boldsymbol{\omega}$, and not their change with time, is of concern. It is thus not necessary to view $\mathbf{v}$ and $\boldsymbol{\omega}$ as time-dependent fields. For the following discussions, $\mathbf{v}$ and $\boldsymbol{\omega}$ are treated as functions only of $\mathbf{r}$, the position vector; their time-dependencies are not stated explicitly.

As discussed in Chap. 1, the differential equations of vorticity kinematics for the incompressible flow are

$$\nabla \cdot \mathbf{v} = 0 \tag{3.1}$$

$$\nabla \times \mathbf{v} = \boldsymbol{\omega} \tag{3.2}$$

Equation (3.1), known as the continuity equation, is a mathematical statement of the law of mass conservation for the incompressible flow. This equation states that the divergence, or dilatation, of the velocity field is zero and contains the assumption that the density of the flow medium does not change significantly in the flow. This assumption is acceptable as an approximation for most flows of gases at low subsonic speeds and of liquids. Equation (3.2) defines the vorticity field $\boldsymbol{\omega}(\mathbf{r})$ as the curl of the velocity field $\mathbf{v}(\mathbf{r})$. Equations (3.1) and (3.2) are viewed as restraints on the velocity field $\mathbf{v}$: The divergence, or dilatation, of $\mathbf{v}$ must be zero and its curl must be equal to $\boldsymbol{\omega}$.

As mentioned earlier, the central task of vorticity kinematics is the evaluation of the incompressible velocity field $\mathbf{v}(\mathbf{r})$ that corresponds to a known vorticity field $\boldsymbol{\omega}(\mathbf{r})$ in a region R where (3.1) and (3.2) are valid. For this evaluation, (3.2) is

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integrated, and the resulting field $\mathbf{v}$ must be solenoidal to satisfy (3.1). The field $\mathbf{v}$ must be unique. In other words, if a field $\mathbf{v} = \mathbf{v}_1(\mathbf{r})$ that satisfies (3.1) and (3.2) is found, no other field that differs from $\mathbf{v}_1(\mathbf{r})$ can be found that also satisfies (3.1) and (3.2). Two attributes of vorticity kinematics render the task of finding $\mathbf{v}$ amenable to classical mathematical analyses. The first attribute is obvious: The differential equations of vorticity kinematics, (3.1) and (3.2), are linear. Thus many classical methods, including the principle of superposition, are useful in the study of vorticity kinematics.

The second, less obvious, attribute is that the viscosity of the fluid is not a part of (3.1) and (3.2). In fact, the stress–strain relationship, which distinguishes a solid from a fluid and a viscous fluid from an inviscid fluid, is not a part of (3.1) and (3.2). In other words, whether the fluid is viscous or inviscid and whether a solid or a fluid is present in the region of interest are not relevant in the study of vorticity kinematics.

The vorticity $\boldsymbol{\omega}$ describes the rotation of the fluid. In a Cartesian coordinate system (x, y, z) with unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, let the x-, y-, and z- components of $\mathbf{v}$ be u, v, and w and those of $\boldsymbol{\omega}$ be ξ, η, ζ respectively. One then has $\xi = \partial w/\partial y - \partial v/\partial z$. It is simple to show that at any point $\mathbf{r}$ in a fluid, the term $\partial w/\partial y$ represents the angular velocity of the differential line elements dy rotating about the x-axis and the term $-\partial v/\partial z$ represents the angular velocity of the line element dz about the x- axis. Thus ξ is the sum of the angular velocities of the line elements dy and dz about the x-axis. This conclusion is independent of the choice of the directions of the coordinate axes. Thus ξ represents the sum of the angular velocities of any two mutually perpendicular line elements in the y-z plane at the point $\mathbf{r}$. More generally, $\boldsymbol{\omega}(\mathbf{r})$ represents the sum of angular velocities of any two mutually perpendicular line elements in the plane normal to $\boldsymbol{\omega}$ at the point $\mathbf{r}$.

In a rigid solid body, the angular velocities of all line elements are identical, each being equal to the angular velocity $\boldsymbol{\Omega}$ of the solid. The field $\mathbf{v}$ in the solid is given by

$$\mathbf{v}(\mathbf{r}) = \mathbf{v}_a + \boldsymbol{\Omega} \times (\mathbf{r} - \mathbf{r}_a), \tag{3.3}$$

where $\mathbf{v}_a$ is the rectilinear velocity of the solid body at the point $\mathbf{r}_a$ in the solid region R_s. It is simple to show, using vector identity (3.8), that the divergence of the field $\mathbf{v}(\mathbf{r})$ given by (3.3) is zero. Thus (3.1) is satisfied in the solid region R_s. This is a foregone conclusion since a rigid solid body is obviously incompressible. If $\boldsymbol{\omega}$ in the solid is defined by (3.2), then, according to (3.9), $\mathbf{v}(\mathbf{r})$ as given by (3.3) corresponds to $\boldsymbol{\omega} = 2\boldsymbol{\Omega}$. In the solid rotating at the angular velocity $\boldsymbol{\Omega}$, the angular velocity of each of the two mutually perpendicular line elements in the plane normal to the axis of rotation is $\boldsymbol{\Omega}$. Thus the vorticity in a solid, like that in the fluid, represents the sum of angular velocities of two line elements.

Consider the reference flow problem defined in Chap. 1: a solid immersed in and moving relative to an infinite viscous fluid. The solid region is denoted R_s and the fluid region R_f. The infinite unlimited region jointly occupied by the fluid and the solid is denoted R_∞. With (3.1) and (3.2) satisfied in both R_s and R_f, the fluid and the solid can be treated together in R_∞ as a single combined kinematic system, provided certain conditions of continuity and smoothness of the fields $\mathbf{v}$ and $\boldsymbol{\omega}$ are met. The task of evaluating the field $\mathbf{v}$ that corresponds to a known field $\boldsymbol{\omega}$ involves the integration, and not the differentiation, of the field $\boldsymbol{\omega}$. In consequence, provided $\boldsymbol{\omega}$ is piecewise continuous in R_∞, it is permissible to treat the fluid and the solid together as a single kinematic system jointly occupying R_∞. The velocity field $\mathbf{v}$ is continuous and smooth separately in R_s and in R_f. With the no-penetration and no-slip conditions, $\mathbf{v}$ (all three components) is continuous across S. Therefore $\mathbf{v}$ is continuous and piecewise smooth in R_∞.

As discussed in Chap. 2, with the no-slip condition, the tangential velocity components on S are the same whether one approaches S from the fluid side or the solid side. The normal component of the vorticity field, defined by (3.2), is determined forthwith by the tangential derivatives of the tangential velocity components on S. In consequence, the normal component of vorticity is continuous across S. The tangential components of vorticity, however, are in general discontinuous across S because the normal gradients of velocity components are discontinuous across S. Since $\boldsymbol{\omega}$ is piecewise continuous in R_∞, the fluid and the solid can be treated together as one kinematic system in R_∞.

In classical aerodynamics, the no-penetration condition is generally accepted as the proper boundary condition for the flow of the inviscid fluid. Without the no-slip condition, there is a discontinuity in the tangential components of the velocity across S. This discontinuity is typically viewed as a vortex sheet on S. In the present study, the vortex sheet represents (approximates) a thin layer of vorticity surrounding S. The strength of the vortex sheet is the integrated value of vorticity across the layer that the sheet represents. That is, one envisions the thickness of the layer approaches zero while the strength of the layer remains finite. There is thus an infinitesimally thin layer of vorticity with a fixed finite strength. It is important to recognize that the vortex sheet has two sides and, as an approximation of a vorticity layer, has an infinitesimal but non-zero thickness. The fluid side of the sheet is distinct from the solid side and the vorticity in the layer that the sheet represents is between the two sides. The fact that the normal velocity component is continuous across the vortex sheet does not mean that the two sides collapse onto a surface in space. The boundary of the fluid is on the solid side of the sheet and the tangential velocity components there must agree with those of the solid. This is in fact the no-slip condition that the fluid, whether viscous or inviscid, satisfies.

In external aerodynamics, the shape of S is typically complex. If the kinematics of the fluid motion is treated separately from that of the solid, then the complexity of the shape of S makes analyses difficult (Wu 1976). For two-dimensional flow problems, S can be mapped onto a simple shape, for example a circle, using conformal transformation. This mapping facilitates analyses not only in vorticity kinematics, but also in vorticity kinetics (Wang and Wu 1986; Peterson et al. 1987). For three-dimensional flows, no method comparable to conformal mapping is available and complex boundary shape is a major issue of concern.

Many classical vorticity-kinematic theorems were derived assuming an inviscid fluid occupies the unlimited region R_∞. The recognition that there is no need to differentiate viscous fluids, inviscid fluids, and solids means that these theorems are directly applicable to motions of viscous fluids sharing the unlimited region R_∞ with solid bodies. There are many applications for which special theorems for the fluid region R_f are needed. The strategy used in the present study is to first bypass the need to deal with complex boundary shapes and work with the unlimited region R_∞. The results obtained are then used as benchmarks guiding the derivation of special theorems applicable only in R_f.

3.2 Vector Identities

Tensor analyses and notations offer many advantages in theoretical fluid dynamics. Vector analyses are, however, adequate for the purposes of the present study. The following list contains useful vector identities. In the list, lower case letters denote scalars, boldface capital letters denote vectors, a, b, c, **A, B**, and **C** are constant scalars and vectors, and f, g, h, **F, G**, and **H** are sufficiently smooth fields.

Algebraic Identities

$$(\mathbf{F}\times\mathbf{G})\cdot\mathbf{H} = (\mathbf{G}\times\mathbf{H})\cdot\mathbf{F} \tag{3.4}$$

$$\mathbf{F}\times(\mathbf{G}\times\mathbf{H}) = -(\mathbf{F}\cdot\mathbf{G})\mathbf{H} + (\mathbf{F}\cdot\mathbf{H})\mathbf{G} \tag{3.5}$$

Differential Identities

$$\nabla\cdot(\mathrm{f}\,\mathbf{F}) = (\nabla\,\mathrm{f})\cdot\mathrm{F} + \mathrm{f}\,\nabla\cdot\mathrm{F} \tag{3.6}$$

$$\nabla\times(\mathrm{f}\,\mathbf{F}) = (\nabla\,\mathrm{f})\times\mathbf{F} + \mathrm{f}(\nabla\times\mathbf{F}) \tag{3.7}$$

$$\nabla\cdot(\mathbf{F}\times\mathbf{G}) = -\mathbf{F}\cdot(\nabla\times\mathbf{G}) + \mathbf{G}\cdot(\nabla\times\mathbf{F}) \tag{3.8}$$

$$\nabla\times(\mathbf{F}\times\mathbf{G}) = -(\mathbf{F}\cdot\nabla)\mathbf{G} + \mathbf{F}(\nabla\cdot\mathbf{G}) + (\mathbf{G}\cdot\nabla)\mathbf{F} - \mathbf{G}(\nabla\cdot\mathbf{F}) \tag{3.9}$$

$$\nabla\times(\mathbf{F}\cdot\mathbf{G}) = \mathbf{F}\times(\nabla\times\mathbf{G}) + (\mathbf{F}\cdot\nabla)\mathbf{G} + \mathbf{G}\times(\nabla\times\mathbf{F}) + (\mathbf{G}\cdot\nabla)\mathbf{F} \tag{3.10}$$

$$\nabla\times(\nabla\,\mathrm{f}) = 0 \tag{3.11}$$

$$\nabla\cdot(\nabla\times\mathbf{F}) = 0 \tag{3.12}$$

$$\nabla\times(\nabla\times\mathbf{F}) = -\nabla^2\,\mathbf{F} + \nabla(\nabla\cdot\mathbf{F}) \tag{3.13}$$

A vector field $\mathbf{F}'$ is lamellar (irrotational) if $\nabla\times\mathbf{F}' = 0$. The lamellar field $\mathbf{F}'$ is expressible as the gradient of a scalar potential field f:

$$\mathbf{F}' = \nabla f \tag{3.14}$$

A vector field $\mathbf{F}^*$ is solenoidal if $\nabla \cdot \mathbf{F}^* = 0$. The solenoidal field $\mathbf{F}^*$ is expressible as the curl of a vector potential field $\mathbf{G}$

$$\mathbf{F}^* = \nabla \times \mathbf{G} \tag{3.15}$$

The vector $\mathbf{r}'$ and its magnitude are defined by

$$\mathbf{r}' = \mathbf{r} - \mathbf{r}_o \tag{3.16}$$

$$\mathrm{r}' = |\mathbf{r} - \mathbf{r}_o| \tag{3.17}$$

One has then

$$(\mathbf{F} \cdot \nabla)\mathbf{r}' = \mathbf{F} \tag{3.18}$$

and the following identities for $\mathbf{r}' \neq 0$:

$$\nabla(1/\mathrm{r}') = -\mathbf{r}'/\mathrm{r}'^3 \tag{3.19}$$

$$\nabla \cdot [\nabla(1/\mathrm{r}')] = \nabla^2(1/\mathrm{r}') = 0 \tag{3.20}$$

$$\nabla \times [\nabla(1/\mathrm{r}')] = 0 \tag{3.21}$$

Integral Identities

For single-valued smooth functions $\mathbf{F}$ and f in a simply connected region R bounded by a closed surface S, there are the following relationships between surface and volume integrals, in which $\mathbf{n}$ is the unit vector normal to S directed outward from R:

$$\iiint_R \nabla \cdot \mathbf{F} dR = \oiint_S \mathbf{F} \cdot \mathbf{n}\, dS \tag{3.22}$$

$$\iiint_R \nabla \times \mathbf{F} dR = \oiint_S \mathbf{F} \times \mathbf{n}\, dS \tag{3.23}$$

$$\iiint_R \nabla f dR = \oiint_S f\mathbf{n}\, dS \tag{3.24}$$

There are also relationships between an integral over a closed circuit C and that over its cap S in space, in which $\mathbf{n}$ is the unit vector normal to S and $\boldsymbol{\tau}$ is the unit tangent vector on C, the positive senses of $\mathbf{n}$ and $\boldsymbol{\tau}$ are related by the right-handed screw convention defined in Sect. 2.3. Only one such relation is given here:

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \oint_C \mathbf{F} \cdot \boldsymbol{\tau}\, ds \tag{3.25}$$

Equation (3.22) is called the divergence theorem, Gauss' theorem, or Green's theorem. Equations (3.23) and (3.24) are corollaries of the divergence theorem. Equation (3.25) is called Stokes' theorem. The surface S in the identities needs not be a solid surface.

3.3 Poisson's Equation and Integral Representations

Since $\mathbf{v}$ is solenoidal in the incompressible flow, a vector potential $\boldsymbol{\psi}$ exists such that

$$\mathbf{v} = \nabla \times \boldsymbol{\psi} \tag{3.26}$$

Equation (3.2) then gives

$$\nabla \times (\nabla \times \psi) = \boldsymbol{\omega} \tag{3.27}$$

Let $\nabla \cdot \psi = 0$, one obtains using (3.13) a Poisson's equation for the vector field $\boldsymbol{\psi}$:

$$\nabla^2 \psi = -\omega \tag{3.28}$$

It is informative, before treating (3.27), to review the properties of the scalar Poisson's equation describing a scalar field φ in a simply connected finite region R bounded by S:

$$\nabla^2 \varphi \;=\; f(\mathbf{r}) \tag{3.29}$$

For the special case where the inhomogeneous term $f(\mathbf{r})$ is zero in R, one has the Laplace's equation $\nabla^2 \varphi = 0$. Fields satisfying the Laplace's equation are called harmonic fields. Suppose the Laplacian of a field $\psi(\mathbf{r})$ is $f(\mathbf{r})$, then $\varphi = \psi$ is a solution of (3.29). This solution is not unique since the Laplacian of the field $\psi + \psi'$, where ψ' is an arbitrary harmonic field, is also $f(\mathbf{r})$.

For the general case where $f(\mathbf{r}) \neq 0$ in R, since $\nabla^2 \varphi = \nabla \cdot (\nabla \varphi)$, (3.22) gives

$$\iiint\limits_R \nabla^2 \varphi \, dR = \oiint\limits_S \nabla\varphi \cdot \mathbf{n} \, dS \tag{3.30}$$

which can be restated, using (3.29), as

$$\iiint\limits_R f(\mathbf{r}) dR = \oiint\limits_S \frac{\partial \varphi}{\partial n} dS \tag{3.31}$$

Equations (3.30) and (3.31) are mathematical identities that require no special physical content. For convenience, the term f($\mathbf{r}$) is called the *source strength* (or simply the *source*) in accordance with accepted conventions. The integral on the right-hand side of (3.31) is the total flux of φ across dS and the one on the left-hand side is the total source strength in R. According to (3.31), the total source in R is equal to the total flux of φ over S.

Consider a spherically symmetric field $\varphi_a = g(\rho)$ in a spherical coordinate system (ρ, θ, ϕ), where ρ is the radial coordinate and θ and ϕ are angular coordinates. The gradient of φ_a is $(\partial g/\partial \rho)\mathbf{e}_\rho$, $\mathbf{e}_\rho$ being the unit vector in the ρ-direction. For the case $g(\rho) = -1/\rho$, (3.19) gives $\nabla\varphi_a = (1/\rho^2)\mathbf{e}_\rho$, $\rho \neq 0$. The total flux of φ_a over the spherical surface S of radius ρ_1 centered at the point ρ = 0 is thus 4π. As $\rho_1 \to 0$, the left-hand side of (3.31) remains finite. According to (3.20), $\nabla^2\varphi_a = 0$ at all points $\rho \neq 0$. Thus the source in R is present only at the point ρ = 0 and the total source strength there is 4π. This means that the field −1/ρ corresponds to a total source strength 4π at the point ρ = 0. Thus the field $\varphi_u = -1/(4\pi\rho)$ corresponds to a unit of total source strength at ρ = 0.

Consider a small region δR with a characteristic length λ. The point $\mathbf{r} = 0$ is inside this small region. In the limit as $\lambda \to 0$, δR approaches the differential volume element dR and the total source strength in δR approaches f(0)dR. Based on the conclusion of the last paragraph, the spherical symmetric field –f(0)dR/(4πρ) corresponds to the total source strength f(0)dR at $\mathbf{r} = 0$. Suppose a differential volume element dR at the point $\mathbf{r} = \mathbf{r}_o \neq 0$ contains a total source strength $f(\mathbf{r}_o)$dR. The corresponding field is then spherically symmetric about the point $\mathbf{r} = \mathbf{r}_o$ and is $-f(\mathbf{r}_o)dR/(4\pi|\mathbf{r} - \mathbf{r}_o|)$. Integrating over R in which the source distribution is f($\mathbf{r}$), one obtains an integral representing for (3.29)

$$\varphi(\mathbf{r}) = \iiint\limits_R F(\mathbf{r}, \mathbf{r}_o) f(\mathbf{r}_o) \, dR_o \tag{3.32}$$

where F, call the *fundamental* or *principal solution* of Poisson's equation (3.29), is given by

$$F(\mathbf{r}, \mathbf{r}_o) = -1/(4\pi \, r') \tag{3.33}$$

This field $F(\mathbf{r}, \mathbf{r}_o)$ is the field corresponding to a unit source at $\mathbf{r}_o$ and is a function of $\mathbf{r}$ and $\mathbf{r}_o$. To distinguish the source distribution $f(\mathbf{r}_o)$ from the field $\varphi(\mathbf{r})$, the expression '$\mathbf{r}_o$-space' is used. The subscript 'o' is used to indicate field variables and operations in the $\mathbf{r}_o$-space. Fields such as $f(\mathbf{r}_o)$ are denoted f_o at times. In (3.32), $\mathbf{r}_o$ is the dummy variable of integration. No subscripts are used for field variables and operations performed in the $\mathbf{r}$-space. The fundamental solution links the two spaces $\mathbf{r}$ and $\mathbf{r}_o$. Using (3.19), (3.20) and (3.21) and defining the vector field $\mathbf{Q}$ as the gradient of F, one obtains at all points $\mathbf{r} \neq \mathbf{r}_o$:

$$\mathbf{Q}(\mathbf{r}, \mathbf{r}_o) = \nabla F = -\nabla_o F = \mathbf{r}'/(4\pi r'^3) \tag{3.34}$$

$$\nabla \cdot \mathbf{Q} = \nabla^2 F = \nabla_o^2 F = 0 \tag{3.35}$$

$$\nabla \times \mathbf{Q} = \nabla_o \times \mathbf{Q} = 0, \tag{3.36}$$

where ∇_o is the nabla differential operator in the $\mathbf{r}_o$-space.

Provided f is finite, the singularity of F at the point $\mathbf{r} = \mathbf{r}_o$ in (3.32) does not contribute to the integral on the right-hand side of (3.32) since, in the limit as $\mathbf{r}_o \to \mathbf{r}$, the integrand in (3.32) increases as r'^{-1} and the elemental volume δR_o, as it approaches dR_o, decreases as r'^3.

Take the Laplacian of the two sides of (3.32) in the $\mathbf{r}$-space. For the right-hand side of (3.32), the integration is in the $\mathbf{r}_o$-space. Therefore the Laplacian differential operator in the $\mathbf{r}$-space can be moved inside the integral. Furthermore, since $f(\mathbf{r}_o)$ is not a function of $\mathbf{r}$, the Laplacian operator operates only on $F(\mathbf{r}, \mathbf{r}_o)$, yielding

$$\nabla^2 \varphi = \iiint_{R_0} \nabla^2 F(\mathbf{r}, \mathbf{r}_o) f(\mathbf{r}_o) dR_o \tag{3.37}$$

According to (3.35), the integrand in (3.37) is zero at all points $\mathbf{r} \neq \mathbf{r}_o$. The singularity of F is such that, in the limit as $\mathbf{r}_o \to \mathbf{r}$, $\nabla^2 F dR_o \to 1$. Thus the right-hand side of (3.37) gives $f(\mathbf{r})$. The integral representation (3.32) is therefore equivalent to (3.29). The function $\varphi(\mathbf{r})$ given by (3.32) is a solution of (3.29). It is, however, not a unique solution because the sum of the integral in (3.32) and any arbitrary harmonic field ψ' also satisfies (3.29).

Using (3.6), one obtains $\nabla \cdot (\varphi \nabla F - F \nabla \varphi) = \varphi \nabla^2 F - F \nabla^2 \varphi$. Therefore (3.22) gives

$$\iiint_{R'} (\varphi \nabla^2 F - F \nabla^2 \varphi) dR = \oiint_{S'} (\varphi \nabla F) \cdot \mathbf{n}\, dS - \oiint_{S'} (F \nabla \varphi) \cdot \mathbf{n}\, dS \tag{3.38}$$

where R' is a simply connected region bounded by S' and S' is composed of S, which bounds R' externally, and S_ε, a spherical surface of radius ε centered at the point $\mathbf{r} = \mathbf{r}_o$ and bounds R' internally. The singular point $\mathbf{r} = \mathbf{r}_o$ is excluded from R'.

Using (3.29) and (3.35), one obtains from (3.38)

$$-\iiint_{R'} \mathrm{FfdR} = \oiint_{S} (\varphi\nabla \mathrm{F})\cdot\mathbf{n}\mathrm{dS} - \oiint_{S}(\mathrm{F}\nabla\varphi)\cdot\mathbf{n}\,\mathrm{dS} + \oiint_{S_\varepsilon}(\varphi\nabla\mathrm{F})\cdot\mathbf{n}\,\mathrm{dS} - \oiint_{S_\varepsilon}(\mathrm{F}\nabla\varphi)\cdot\mathbf{n}\,\mathrm{dS} \tag{3.39}$$

In the limit as $\varepsilon \to 0$, F increases as ε^{-1} and ∇F increases as ε^{-2}. Since the surface area of S_ε approaches zero as ε^{-2}, the last integral in (3.39) vanishes. The outward (from the region R′) unit normal vector $\mathbf{n}$ is equal to $-\mathbf{r}'/r'$ on S_ε. Therefore, according to (3.34), $\nabla \mathrm{F} \to -\mathbf{n}/(4\pi\varepsilon^2)$ on S_ε. As $\varepsilon \to 0$, $\varphi(\mathbf{r}) \to \varphi(\mathbf{r}_o)$. Therefore the third integral in the right-hand side of (3.39) gives $-\varphi(\mathbf{r}_o)$. The volume integral of Ff over the interior of S_ε goes to zero as $\varepsilon \to 0$. Equation (3.39) therefore yields, upon interchanging the **r**- and $\mathbf{r}_o$-spaces,

$$\varphi(\mathbf{r}) = \iiint_{R_o} \mathrm{F}(\mathbf{r},\mathbf{r}_o)\mathrm{f}(\mathbf{r}_o)\,\mathrm{dR}_o + \oiint_{S_o} \varphi_o\nabla_o\mathrm{F})\cdot\mathbf{n}_o\,\mathrm{dS}_o - \oiint_{S_o}(\mathrm{F}\nabla_o\varphi_o)\cdot\mathbf{n}_o\,\mathrm{dS}_o \tag{3.40}$$

Equation (3.40) is an integral representation of (3.29). This integral representation, unlike (3.32), contains two boundary integrals involving φ and $\partial\varphi/\partial n$. If values of f(**r**) at all points in R and values of φ and $\partial\varphi/\partial n$ at all points on S are known, then φ can be evaluated explicitly, point by point, in R using (3.40). There is the concern that, if φ and $\partial\varphi/\partial n$ are both prescribed on S, then the problem described by (3.29) in R is over-specified. This concern is studied in Sect. 3.4.

3.4 Poisson's Equation and Boundary Conditions

Suppose values of the inhomogeneous term f(**r**) in (3.29) are known at all points **r** in R. Consider three fields κ_1, κ_2, and κ_3, with κ_3 defined by $\kappa_3 = \kappa_1 - \kappa_2$. Let $\nabla^2\kappa_1 = \nabla^2\kappa_2 = \mathrm{f}$ in R. It follows that $\nabla^2\kappa_3 = 0$ and the field κ_3 is harmonic in R. If the imposition of an additional condition on κ_1 and κ_2 makes $\kappa_3 = 0$ at all points **r** in R, then $\kappa_1 = \kappa_2$ and the solution $\varphi = \kappa_1$, or equivalently $\varphi = \kappa_2$, is unique in R. Consider the values of φ on S

$$\varphi(\mathbf{r}_S) = \beta(\mathbf{r}_S) \tag{3.41}$$

where $\mathbf{r}_s$ is the position vector on S and $\beta(\mathbf{r}_s)$ is a function to be prescribed.

If the condition $\kappa_1 = \kappa_2$ on S is imposed, then $\kappa_3 = 0$ on S. According to the principle of extrema—also called the principle of maximum and minimum, see, e.g. (Morse and Feshbach 1953)—a harmonic function cannot have a maximum or a minimum value in R. In other words, a point where the value of a harmonic function is either a maximum or a minimum must be a boundary point. Thus, since κ_3 is harmonic and is zero on S, it is zero at all points in R, for otherwise there is a minimum or a maximum value of κ_3 in R. In consequence, the solution of (3.29) subject to the prescription of $\beta(\mathbf{r}_s)$ is unique. The prescription of the function $\beta(\mathbf{r}_s)$ is commonly called Dirichlet's condition.

Alternatively, consider the values of $\partial\varphi/\partial n$ on S and let

$$\partial\varphi(\mathbf{r}_S/\partial\mathbf{n} = \chi(\mathbf{r}_S) \tag{3.42}$$

If the condition $\partial\kappa_1/\partial n = \partial\kappa_2/\partial n$ on S is imposed, then $\partial\kappa_3/\partial n = 0$ on S. It follows that, since κ_3 cannot have a maximum or a minimum value in R, $\nabla\kappa_3 = 0$ and κ_3 is a constant in R. Therefore, the prescription of $\chi(\mathbf{r}_s)$ makes the solution of (3.29) unique to within an arbitrary constant. This prescription is called Neumann's condition. It is noted that, on account of (3.31), the function $\chi(\mathbf{r}_s)$ needs to satisfy the auxiliary condition

$$\oiint_S \chi(\mathbf{r})dS = \iiint_R f(\mathbf{r})dR \tag{3.43}$$

Dirichlet's and Neumann's conditions are discussed in many applied mathematics textbooks. The well-known conclusion is that the prescription of one of these two conditions renders the solution unique. The prescription of both $\lambda(\mathbf{r}_s)$ and $\chi(\mathbf{r}_s)$, called Cauchy's condition, over-specifies the problem. Obviously, the values of $\partial\varphi/\partial n$ on S are determined as a part of the unique solution obtained by prescribing Dirichlet's condition on S. Therefore Neumann's condition on S cannot be also prescribed, because values of $\partial\varphi/\partial n$ prescribed cannot be expected to agree with values determined from Dirichlet's solution. Similarly, values of φ on S are determined by the solution obtained using Neumann's condition. Therefore Dirichlet's condition cannot be specified in addition to Neumann's condition. There are, however, circumstances where Cauchy's condition is permissible.

Suppose φ is specified on S and used to obtain a solution of (3.29) in R. The solution gives values of $\partial\varphi/\partial n$ on S. The values of $\partial\varphi/\partial n$ on S thus obtained and the prescribed values of φ on S are *compatible*. It is obviously acceptable to prescribe values of $\partial\varphi/\partial n$ on S that are compatible with prescribed values of φ. This conclusion appears trivial. Yet it is relevant if (3.40) is to be used to evaluate φ in R.

The three integrals in (3.40) represent the contributions to the field $\varphi(\mathbf{r})$ respectively by the source in the region R, values of φ on S (Dirichlet's condition), and values of $\partial\varphi/\partial n$ on S (Neumann's condition). Each integral represents a field. For convenience, the three fields are designated respectively $\varphi_s(\mathbf{r})$, $\varphi_d(\mathbf{r})$, and $\varphi_n(\mathbf{r})$.

Suppose $f(\mathbf{r})$ is known at all points in R and φ is known at all points on S (Dirichlet's condition). The fields $\varphi_s(\mathbf{r})$ and $\varphi_d(\mathbf{r})$ are then determinate and (3.40) gives

$$\varphi(\mathbf{r}) = \varphi_S(\mathbf{r}) + \varphi d(\mathbf{r}) - \oiint_{S_o} F(\partial\varphi/\partial n)_o \, dS_o \tag{3.44}$$

where $\varphi_s(\mathbf{r})$ and $\varphi_d(\mathbf{r})$ are known.

On each boundary point $\mathbf{r}_s$, the value of $\varphi(\mathbf{r}_s)$ is known (Dirichlet's condition). Thus (3.44) yields, on the boundary S,

$$\oiint_{S_o} F(\mathbf{r}_s, \mathbf{r}_{so})(\partial\varphi/\partial n)_o \, dS_o = \varphi_s(\mathbf{r}_s) + \varphi_d(\mathbf{r}_s) - \varphi(\mathbf{r}_s) \tag{3.45}$$

Similarly, suppose $f(\mathbf{r})$ is known at all points in R and $\partial\varphi/\partial n$ is known at all points on S (Neumann's condition). In this case, $\varphi_s(\mathbf{r})$ and $\varphi_n(\mathbf{r})$ are known and (3.40) yields

$$\oiint_{S_o} \varphi(\mathbf{r}_{so})\partial F/\partial n_o \, dS_o - \varphi(\mathbf{r}_s) = -\varphi_s(\mathbf{r}_s) + \varphi_n(\mathbf{r}_s) \tag{3.46}$$

The right-hand sides of (3.45) and (3.46) are known functions of $\mathbf{r}_s$. Equation (3.45) is a Fredholm integral equation of the first kind in which the unknown function is $\partial\varphi(\mathbf{r}_s)/\partial n$. Equation (3.46) is a Fredholm integral equation of the second kinds in which $\varphi(\mathbf{r}_s)$ is the unknown function. Theorems concerning solutions of Fredholm equations are discussed extensively in mathematical treatises, e.g., Kellogg (1953) and Garabedian (1967). Conditions for the existence of a solution are met for either (3.45) or (3.46). The two equations are conceptually equivalent. Consider (3.46), which is preferred for numerical solution methods because it leads to diagonally dominant matrices. Obviously, this equation determines $\varphi(\mathbf{r}_s)$ values that are compatible with prescribed $\partial\varphi/\partial n$ values on S.

The above discussions confirm the earlier conclusion that the imposition of Dirichlet's condition makes Neumann's condition determinate and vice versa. This outcome is equivalent to the conclusion that Cauchy's condition over-specifies the problem described by the scalar Poisson equation. Both conclusions are based on the assumption that the source $f(\mathbf{r})$ is known at all points in R. Consider now a closed surface S* dividing R into two parts, R* and R^-. R^- is the closed region bounded externally by S* and R* is a narrow layer bounded internally by S* and externally by S. Denote the thickness of the layer by δn, and the average source strength across the layer by f*. Suppose both φ and $\partial\varphi/\partial n$ are known at all points on S (Cauchy's condition) and $f(\mathbf{r})$ is known at all points in R^-, but not known in the layer R*. Here it is convenient to write $\varphi_s(\mathbf{r}) = \varphi_s^-(\mathbf{r}) + \varphi_s^*(\mathbf{r})$, where $\varphi_s^-(\mathbf{r})$ is the integral of Ff_o over R^- and $\varphi_s{}^*(\mathbf{r})$ is the integral of Ff_o over R^*. Since $\varphi_s^-(\mathbf{r})$, $\varphi_d(\mathbf{r})$, and $\varphi_n(\mathbf{r})$ are known, (3.40) yields

$$\varphi(\mathbf{r}) = \iiint_{R_o^*} F(\mathbf{r}, \mathbf{r}_o) f(\mathbf{r}_o) dR_o + \varphi_s^-(\mathbf{r}) + \varphi_d(\mathbf{r}) - \varphi_n(\mathbf{r}) \tag{3.47}$$

For very narrow layers (very small values of δn), the integral in (3.47) is approximately a surface integral. Since $\varphi(\mathbf{r}_s)$ is known (Dirichlet's condition) on S, (3.47) gives

$$\oiint_{S_o} F(\mathbf{r}_s, \mathbf{r}_o)\sigma(\mathbf{r}_o) dS_o \cong h(\mathbf{r}_s) \tag{3.48}$$

where $\sigma = f^*\delta n$ and $h(\mathbf{r}_s) = \varphi(\mathbf{r}_s) - \varphi_s^-(\mathbf{r}_s) - \varphi_d(\mathbf{r}_s) + \varphi_n(\mathbf{r}_s)$ is a known function of $\mathbf{r}_s$.

In the limit as $\delta n \to 0$, (3.48) becomes precise. In this limit, if $\sigma(\mathbf{r}_o)$ retains a finite value, then $f \to \infty$ and the source in R* approaches a sheet of source of finite strength σ. This sheet is between the surfaces S* and S. In the limit as $\delta n \to 0$, the surfaces are spaced apart at infinitesimally small (but not zero) distances. In other words, the sheet has two sides, S* and S, that do not coincide. The values of $\partial\varphi/\partial n$ on the two sides of the source sheet differ by the amount σ. Equation (3.48) is a Fredholm's integral equation of the first kind. If the strength of the source layer σ is not known, then the specification of Cauchy's condition on S determines $\sigma(\mathbf{r}_s)$. Values of σ, as determined by (3.48), is approximately $f^*\delta n$ for small values of δn, and precisely fdn in the limit as $\delta n \to dn$. The form of (3.48) is identical to that of (3.45). Thus one obtains from (3.40),

$$\oiint_{S_o} F(\mathbf{r}_s, \mathbf{r}_o)[\sigma(\mathbf{r}_o) - (\partial\varphi/\partial n)_o] dS_o \approx \varphi(\mathbf{r}_s)\varphi_s^-(\mathbf{r}_s) - \varphi_d(\mathbf{r}_s) \tag{3.49}$$

Equation (3.49) shows an equivalence of the *boundary source strength* σ and $-\partial\varphi/\partial n$.

In summary, with the source strength f known in the interior region R^-, there are three types of available boundary conditions for the solution of (3.29). These are: (i) Dirichlet's condition φ, (ii) Neumann's condition $-\partial\varphi/\partial n$, and (iii) the boundary source condition σ. With any two of the three conditions specified on S, the solution of (3.29) in R is unique. The third condition is determinate as a part of this unique solution.

3.5 Law of Biot–Savart

Sections 3.3 and 3.4 provide a roadmap for studies of the kinematics of the vorticity field. Consider the vector Poisson's equation (3.28), which can be restated, using (3.32), as

$$\psi(\mathbf{r}) = -\iiint\limits_{R_o} F(\mathbf{r}, \mathbf{r}_o)\boldsymbol{\omega}(\mathbf{r}_o)\, dR_o \tag{3.50}$$

Taking the curl of both sides of (3.50) and using (3.26), one obtains the following integral representation for the velocity field $\mathbf{v}$:

$$\mathbf{v}(\mathbf{r}) = \iiint\limits_{R_o} \boldsymbol{\omega}(\mathbf{r}_o) \times \mathbf{Q}(\mathbf{r}, \mathbf{r}_o)\, dR_o \tag{3.51}$$

In deriving (3.51), differentiations are performed in the $\mathbf{r}$-space. The order of differentiation and integration for the right-hand side of (3.50) is interchanged. Since $\boldsymbol{\omega}(\mathbf{r}_o)$ is not a function of $\mathbf{r}$, one obtains (3.51) using (3.7) and (3.34). For a flow with vorticity non-zero only in a finite region, the integrals in (3.50) and (3.51) are over this finite region only. The field variables $\boldsymbol{\psi}$ and $\mathbf{v}$ can be evaluated at points inside or outside this finite region.

Consider an infinitely long cylinder centered on the z-axis in the cylindrical coordinate system (ρ, θ, z), where the coordinates are related to the Cartesian coordinates (x, y, z) by $x = \rho\cos\theta$, $y = \rho\sin\theta$, and $z = z$. This cylinder contains vorticity in the z-direction. This vorticity field is not a function of z and the total vorticity strength, or circulation, of the cylinder is Γ. Denote the characteristic length of the tube's cross-sectional area by ρ_c. In the limit as $\rho_c \to 0$, it can be shown that (3.51) yields

$$\mathbf{v} = \mathbf{e}_\theta \Gamma/(2\pi\rho) \tag{3.52}$$

where $\mathbf{e}_\theta$ is the unit vector in the θ-direction of the cylindrical coordinate system.

Equation (3.52) gives the velocity field corresponding to an infinitely long and straight vortex filament containing the z-axis and a circulation Γ. As discussed in Sect. 2.4, this vortex filament is not a line with a zero cross-sectional area. In the limit as $\rho_c \to 0$, the cross-sectional area of the filament becomes infinitesimal, but not zero. For a circular cylinder centered on the z-axis with an axially symmetric distribution of vorticity in it, the limiting process is not necessary. In other words, (3.52) applies to all points outside this infinitely long circular cylinder with a finite radius and a circulation Γ.

The relationship expressed by (3.52), or more generally (3.51), is called the law of Biot–Savart. This relationship parallels the relationship between the magnetic field and the electric current in electromagnetism. It is generally understood that a magnetic field is *induced*, or caused, by a current. Therefore it is said that a vortex induces a velocity field. It is worth underscoring that the law of Biot–Savart is kinematic. The relationship between $\mathbf{v}$ and $\boldsymbol{\omega}$ is not causal, but reciprocal. In other words, corresponding to each field $\mathbf{v}$ there is a field $\boldsymbol{\omega}$ described by (3.2) and corresponding to each field $\boldsymbol{\omega}$ there are fields $\mathbf{v}$ described by (3.1) and (3.2). The issue of uniqueness of solution is not relevant in evaluating the field $\boldsymbol{\omega}$ that corresponds to a known field $\mathbf{v}$, since this evaluation involves only a differentiation based on (3.2). The central task of vorticity kinematics, as discussed in Sect. 3.1, is, however, the evaluation of the solenoidal field $\mathbf{v}$ that corresponds to a known field $\boldsymbol{\omega}$. For this task, the issue of uniqueness is critically important since there are an infinite numbers of solenoidal fields $\mathbf{v}$ that give the same known vorticity field $\boldsymbol{\omega}$.

If the curl of a field $\mathbf{u}(\mathbf{r})$ is $\boldsymbol{\omega}$, then $\mathbf{v}(\mathbf{r}) = \mathbf{u}(\mathbf{r})$ is a solution of (3.2). According to (3.11), $\mathbf{v}(\mathbf{r}) = \mathbf{u}(\mathbf{r}) + \nabla\phi$, where ϕ is an arbitrary scalar field, is also a solution of (3.2). If $\mathbf{u}$ is solenoidal so that (3.1) is also satisfied, then $\mathbf{v}(\mathbf{r}) = \mathbf{u}(\mathbf{r})$ is a solution of (3.1) and (3.2), but so is $\mathbf{v}(\mathbf{r}) = \mathbf{u}(\mathbf{r}) + \nabla\phi$, provided ϕ is harmonic ($\nabla^2\phi = 0$). Therefore a field $\mathbf{u}(\mathbf{r})$ that satisfies (3.1) and (3.2) in R is not necessarily a unique solution of these two equations.

To understand the meanings and implications of the Biot–Savart law and the role of $\nabla\phi$ in the solution of (3.1) and (3.2), it is informative to examine first certain aspects of the solution of ordinary differential equations. Consider, for example, the equation $d^2f/dx^2 = 2$ in the interval $0 < x < a$. This equation has a general solution in the form $f(x) = x^2 + c_1x + c_2$, c_1 and c_2 being arbitrary constants. The solution is particularized by fixing values to c_1 and c_2. For example, suppose $a = 1$. Letting $c_1 = -6$ and $c_2 = 5$ yields the particular solution $f(x) = x^2 - 6x_1 + 5$. This solution gives the values of $f = 5$ at $x = 0$ and $f = 0$ at $x = 1$, but is incompatible with all other possible values of f at these points. For the sample equation, c_1 and c_2 are expressible in term of values of f at the points $x = 0$ and $x = a$. Specifically, one has $f(x) = x^2 + [(f_a - f_o)/a - a]x + f_o$, where $f_a = f(x = 0)$ and $f_o = f(x = 0)$. This very useful solution is unique in that, for any pair of given values f_a and f_o, it is the only solution in the interval $0 < x < a$. Furthermore, it agrees with all possible pair of values f_a and f_o. The general solution can be viewed as composing of two parts: x^2 and $c_1x + c_2$. The part x^2 is akin to an indefinite integral of a first order ordinary differential equation and is a particular solution obtained by letting $c_1 = c_2 = 0$ in the general solution. This particular solution, $f = x^2$, is compatible with only one pair of values of f at two selected points. The part $c_1x + c_2$ contains uncertainties in the arbitrariness of c_1 and c_2. In the unique solution just mentioned, $c_1x + c_2$ is explicitly expressed in terms of boundary values of f, thereby removing its uncertainties.

The solution of (3.1) and (3.2) is obviously more complex than the solution of the sample ordinary differential equation. Nevertheless, there exist useful parallels. For example, the Biot–Savart law (3.50), like the solution x^2 of the sample equation, is interpretable as a particular solution of the equation set (3.1) and (3.2). Specifically, the general solution of (3.1) and (3.2) contains, in addition to the integral in (3.51), the uncertain part $\nabla\phi$, where ϕ is harmonic. The law of Biot–Savart (3.51) is obtained from the general solution by letting $\nabla\phi = 0$. In Sect. 3.6, the uncertain part $\nabla\phi$ is expressed explicitly in terms of boundary values of $\mathbf{v}$. The result is the very useful identity called the generalized law of Biot–Savart.

3.6 Generalized Law of Biot–Savart

As discussed in Sect. 3.5, a general solution of (3.1) and (3.2) is

$$\mathbf{v}(\mathbf{r}) = \iiint_{R_o} \boldsymbol{\omega}(\mathbf{r}_o) \times \mathbf{Q}(\mathbf{r}, \mathbf{r}_o)\, dS_o + \nabla\varphi \tag{3.53}$$

where φ is an arbitrary harmonic function.

The Biot–Savart law (3.51) is obtained from (3.53) by assigning the zero to the uncertain field $\nabla\varphi$. The result is a solution of (3.1) and (3.2) compatible with the boundary condition $\mathbf{v} = 0$ on S if the vorticity is non-zero only in a finite region and all points on S are infinitely far from this region. Under general circumstances, (3.51) is not expected to be compatible with values of $\mathbf{v}$, zero or non-zero, on S positioned at finite or infinite distances from the region of non-zero vorticity. To derive a solution of (3.1) and (3.2) that satisfies all boundary values on S, the function $\mathbf{P}(\mathbf{r}, \mathbf{r}_o)$ defined below is used

$$\mathbf{P} = \mathbf{A} \times \mathbf{Q} \tag{3.54a}$$

$$= -\nabla \times (F\mathbf{A}) \tag{3.54b}$$

where $\mathbf{A}$ is an arbitrary constant vector and $\mathbf{Q} = \nabla F$, and F is defined by (3.33).

The equivalence of (3.54a) and (3.54b) is easily shown using (3.7). Using (3.36), (3.37), (3.9), (3.10), (3.11), and (3.12), one obtains at all points $\mathbf{r} \neq \mathbf{r}_o$

$$\nabla \cdot \mathbf{P} = 0 \tag{3.55}$$

$$\nabla \times \mathbf{P} = -\nabla(\mathbf{A} \cdot \mathbf{Q}) \tag{3.56}$$

$$\nabla \times (\nabla \times \mathbf{P}) = 0 \tag{3.57}$$

Equations (3.8) and (3.57) give,

$$\nabla \cdot [\boldsymbol{\psi} \times (\nabla \times \mathbf{P}) - \mathbf{P} \times (\nabla \times \boldsymbol{\psi})] = -\mathbf{P} \cdot [\nabla \times (\nabla \times \boldsymbol{\psi})] \tag{3.58}$$

Using (3.56) and (3.7), one obtains

$$\boldsymbol{\psi} \times (\nabla \times \mathbf{P}) = \nabla \times [(\mathbf{A} \cdot \mathbf{Q})\boldsymbol{\psi}] - (\mathbf{A} \cdot \mathbf{Q})(\nabla \times \boldsymbol{\psi}) \tag{3.59}$$

Placing (3.59) into (3.58) and using (3.54a), (3.12), (3.26) and (3.27), one has

$$-\nabla \cdot [(\mathbf{A} \cdot \mathbf{Q})\mathbf{v} + (\mathbf{A} \times \mathbf{Q}) \times \mathbf{v}] = (\mathbf{A} \times \mathbf{Q}) \cdot \boldsymbol{\omega} \tag{3.60}$$

The right-hand side term in (3.60) is re-expressible using (3.4) as $-\mathbf{A} \cdot (\boldsymbol{\omega} \times \mathbf{Q})$. One then obtains, using the divergence theorem (3.22),

$$\oiint_{S} [(\mathbf{A}\cdot\mathbf{Q})\mathbf{v} + (\mathbf{A}\times\mathbf{Q})\times\mathbf{v}]\cdot\mathbf{n}\,dS + \oiint_{S_\varepsilon} [(\mathbf{A}\cdot\mathbf{Q})\mathbf{v} + (\mathbf{A}\times\mathbf{Q})\times\mathbf{v}]\cdot\mathbf{n}\,dS$$
$$= \mathbf{A}\cdot\iiint_{R'} \boldsymbol{\omega}\times\mathbf{Q}\,dR \tag{3.61}$$

where S, S_ε, and R′ are defined in Sect. 3.3 in connection with (3.38).

Using (3.4) and (3.5), one obtains

$$[(\mathbf{A}\times\mathbf{Q})\times\mathbf{v}]\cdot\mathbf{n} = -\mathbf{A}\cdot[(\mathbf{v}\times\mathbf{n})\times\mathbf{Q}] \tag{3.62a}$$

$$= \mathbf{A}\cdot[-(\mathbf{Q}\cdot\mathbf{v})\mathbf{n} + (\mathbf{Q}\cdot\mathbf{n})\mathbf{v}] \tag{3.62b}$$

As shown in Sect. 3.3, $\mathbf{Q}\to -\mathbf{n}/(4\pi\varepsilon^2)$ on S_ε in the limit as $\varepsilon\to 0$. Using (3.62b), the integrand of the second integral in (3.61) goes to $-(\mathbf{A}\cdot\mathbf{v})(\mathbf{n}\cdot\mathbf{n})/(4\pi\varepsilon^2) = -(\mathbf{A}\cdot\mathbf{v})/(4\pi\varepsilon^2)$ in this limit. Since, as $\mathbf{r}\to\mathbf{r}_o, \mathbf{v}(\mathbf{r})\to\mathbf{v}(\mathbf{r}_o)$, the integral gives $-\mathbf{A}\cdot\mathbf{v}(\mathbf{r}_o)$. Using (3.62a) to restate the first integral in (3.61) yields

$$\mathbf{A}\cdot\oiint_{S} [(\mathbf{v}\cdot\mathbf{n})\mathbf{Q} - (\mathbf{v}\times\mathbf{n})\times\mathbf{Q}]\,dS - \mathbf{A}\cdot\mathbf{v}(\mathbf{r}_o) = \mathbf{A}\cdot\iiint_{R'} \boldsymbol{\omega}\times\mathbf{Q}\,dR \tag{3.63}$$

Noting that **A** is an arbitrary constant vector and that the volume integral over the interior of S_ε goes to zero as $\varepsilon\to 0$, one obtains, upon interchanging the **r**- and $\mathbf{r}_o$- spaces and noting $\mathbf{Q}_o(\mathbf{r},\mathbf{r}_o) = -\mathbf{Q}(\mathbf{r},\mathbf{r}_o) = -\mathbf{r}'/(4\pi r'^3)$, the following solution of (3.1) and (3.2):

$$\mathbf{v}(\mathbf{r}) = \iiint_{R_o} \boldsymbol{\omega}_o\times\mathbf{Q}\,dR_o - \oiint_{S_o} (\mathbf{v}_o\cdot\mathbf{n}_o)\mathbf{Q}\,dS_o + \oiint_{S_o} (\mathbf{v}_o\times\mathbf{n}_o)\times\mathbf{Q}\,dS_o \tag{3.64}$$

In (3.64), $\omega_o = \omega(\mathbf{r}_o)$ and $\mathbf{v}_o = \mathbf{v}(\mathbf{r}_o)$. The field $\nabla\varphi$ in (3.53) is expressed in (3.64) by

$$\nabla\varphi^* = -\oiint_{S_o} (\mathbf{v}_o\cdot\mathbf{n}_o)\mathbf{Q}\,dS_o + \oiint_{S_o} (\mathbf{v}_o\times\mathbf{n}_o)\times\mathbf{Q}\,dS_o \tag{3.65}$$

Equation (3.64) is a generalized Biot–Savart law. While (3.51) is obtained from (3.28) with the assumption $\nabla\cdot\boldsymbol{\psi} = 0$, (3.64) is based on (3.1) and (3.2) and does not rely on this assumption. The divergence of each of the integrals in (3.64) is zero at all points in the interior of the region R. This fact can be shown by moving the divergence operator inside each integral, noting that $\boldsymbol{\omega}_o$, $\mathbf{v}_o$, and $\mathbf{n}_o$ are independent of **r**, and using (3.6), (3.8), (3.35) and (3.36). It is not difficult to show that the curl of the first integral in (3.64) is $\boldsymbol{\omega}(\mathbf{r})$ and the curls of the second and third integrals are zero. Thus φ is harmonic. In Sects. 3.7 and 3.8, it is shown that (3.64) is a unique solution of (3.1) and (3.2) compatible with all boundary conditions on S. With (3.64), the velocity field R is explicitly determinate by the vorticity field in R and Cauchy's velocity boundary condition.

3.7 Uniqueness of Solution

Suppose values of $\boldsymbol{\omega}$ are known at all points in R. Consider three fields $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ in R, with $\mathbf{u}_3$ defined by $\mathbf{u}_3 = \mathbf{u}_1 - \mathbf{u}_2$. Let $\nabla \cdot \mathbf{u}_1 = \nabla \cdot \mathbf{u}_2 = 0$ and $\nabla \times \mathbf{u}_1 = \nabla \times \mathbf{u}_2 = \omega$ in R. It follows that $\nabla \times \mathbf{u}_3$ and $\nabla \times \mathbf{u}_3 = 0$. Since $\mathbf{u}_3$ is irrotational, there is a scalar potential function $\beta(\mathbf{r})$ such that $\mathbf{u}_3 = \nabla\beta$. Since $\mathbf{u}_3$ is also solenoidal, one has $\nabla^2\beta = 0$. Therefore β is harmonic in R. $\mathbf{v} = \mathbf{u}_1$ and $\mathbf{v} = \mathbf{u}_2$ are conceivably different solutions of (3.1) and (3.2). If certain additional common conditions are imposed on the fields $\mathbf{u}_1$ and $\mathbf{u}_2$ so that $\mathbf{u}_3 = 0$ at all points in R, then the solutions $\mathbf{v} = \mathbf{u}_1$ and $\mathbf{v} = \mathbf{u}_2$ are identical. That is, the imposed conditions make the solution $\mathbf{v} = \mathbf{u}_1$ (or equivalently $\mathbf{v} = \mathbf{u}_2$) unique.

On S, consider a set of right-handed mutually perpendicular unit vectors, $\mathbf{n}$, $\boldsymbol{\tau}$ and $\mathbf{b}$, $\boldsymbol{\tau}$ and $\mathbf{b}$ being tangential to S. Denote the velocity components in the $\mathbf{n}$-, $\boldsymbol{\tau}$-, and $\mathbf{b}$- directions respectively by v_n, v_τ, and v_b. One has then $\mathbf{v} \cdot \mathbf{n} = v_n$ and $\mathbf{v} \times \mathbf{n} = v_b\tau - v_\tau\mathbf{b}$ on S. Consider values of v_n on S:

$$v_n(\mathbf{r}_s) = h(\mathbf{r}_s) \tag{3.66}$$

where $h(\mathbf{r}_s)$ is a function to be prescribed.

If $\mathbf{u}_1 \cdot \mathbf{n} = \mathbf{u}_2 \cdot \mathbf{n}$ on S, then $\mathbf{u}_3 \cdot \mathbf{n} = 0$, i.e., $\partial\beta/\partial n = 0$, on S. Since β is a harmonic function, according to the principle of extrema, it cannot have a maximum or a minimum value in R. The field β therefore is uniform in R because its normal derivative must be zero at every point on S. Hence $\nabla\beta = \mathbf{u}_3 = 0$ at all points in R. The solution of (3.1) and (3.2), subject to the prescription of the function $h(\mathbf{r}_s)$ in (3.66), is thus unique. Familiar expressions associated with the scalar Poisson's equation are adopted in the present study of vorticity kinematics. The prescription of $h(\mathbf{r}_s)$ is thus referred to as Neumann's condition. This condition is discussed in many well-known fluid dynamics treatises. For example, Lamb (1932) concluded: *The motion of a fluid occupying any limited simply-connected region is determinate when we know the values of the expansion, and of the component of vorticities, at every point of the region and of the value of the normal velocity at every point of the boundary*. For the present study, the values of the expansion, i.e., the divergence of $\mathbf{v}$, are known: These values are zero at all points in R. It is noted in passing that Lamb also concluded: *The motion of a fluid which fills infinite space, and is at rest at infinity, is determinate when we know the values of the expansion (*θ*, say) and the component of vorticities* ξ, η, ζ, *at all points of the region*. The Biot–Savart law (3.51) is a confirmation of this conclusion. From the vorticity-kinematic viewpoint, since there is no need to differentiate fluids and solids, the Biot–Savart law is applicable to external flows where the fluid does not fill the infinite space, but shares this space with solids.

As an alternative to (3.66), consider values of $\mathbf{v} \times \mathbf{n} = v_b \boldsymbol{\tau} - v_\tau \mathbf{b}$ on S

$$\mathbf{v} \times \mathbf{n} = \mathbf{g}(\mathbf{r}_s) \tag{3.67}$$

where $\mathbf{g}(\mathbf{r}_s)$ is a function with components in the directions of $\boldsymbol{\tau}$ and $\mathbf{b}$.

If $\mathbf{u}_1 \times \mathbf{n} = \mathbf{u}_2 \times \mathbf{n}$ on S, then $\mathbf{u}_3 \times \mathbf{n} = 0$ on S. Thus $(\nabla\beta) \times \mathbf{n} = 0$. In other words, β is constant on S. In consequence, according to the principle of extrema, β is constant and $\mathbf{u}_3 = 0$ in R. Therefore the solution of (3.1) and (3.2) subject to the boundary condition (3.67) is unique. The prescription of $\mathbf{g}(\mathbf{r}_s)$ is referred to in this study as Dirichlet's condition. While Neumann's condition (3.66) is often used in studies of flows of inviscid fluids, Dirichlet's condition (3.67) is not always emphasized in classical vortex dynamic studies.

The imposition of either Neumann's condition or Dirichlet's condition renders the solution of (3.1) and (3.2) unique in R. If $h(\mathbf{r}_s)$ is prescribed, then a unique solution of (3.1) and (3.2) is determined. In other words, Neumann's condition determines the function $\mathbf{g}(\mathbf{r}_s)$. Similarly, once $\mathbf{g}(\mathbf{r}_s)$ is prescribed, $h(\mathbf{r}_s)$ is determinate. The prescription of both $h(\mathbf{r}_s)$ and $\mathbf{g}(\mathbf{r}_s)$, called Cauchy's condition here, over-specifies the problem. This conclusion assumes that values of $\boldsymbol{\omega}$ are known at all points in R.

3.8 Boundary Conditions in Vorticity Kinematics

As shown in Sect. 3.4, with the source strength $f(\mathbf{r})$ known at all points in R, the imposition of either the values of $\partial\varphi/\partial n$ (Neumann's condition) or the values of φ (Dirichlet's condition) at all points on S renders the solution of the scalar Poisson's equation $\nabla^2\varphi = f(\mathbf{r})$ unique in R. In other words, the prescription of boundary values of φ determines values of φ on S uniquely. Conversely, the prescription of boundary values of φ determines boundary values of $\partial\varphi/\partial n$ uniquely. In Sect. 3.7, it is shown that, with vorticity values known at all points in R, the imposition of either the values of the normal velocity component v_n (Neumann's condition) or the values of the tangential velocity components v_b and v_τ (Dirichlet's condition) at all points on S renders the solution of the vorticity-kinematic Eqs. (3.1) and (3.2) unique in R. There is interchangeability between Neumann's condition and Dirichlet's condition. For the scalar Poisson's equation, each of the two conditions involves a scalar function of $\mathbf{r}_s$. In vorticity kinematics, Neumann's condition involves a scalar function $v_n(\mathbf{r}_s)$ and Dirichlet's condition involves two scalar functions v_b and v_τ.

Each of the three integrals in (3.64) represents a vector field that contributes to the field $\mathbf{v}(\mathbf{r})$ in R. The integrals are labeled respectively $\mathbf{v}_\omega$, $-\mathbf{v}_n$, and $\mathbf{v}_d$ in the following discussions. The field $\mathbf{v}_\omega$ represents the contribution of the vorticity in R to $\mathbf{v}(\mathbf{r})$ in R. This field is solenoidal and its curl is $\boldsymbol{\omega}$. The fields $\mathbf{v}_n$, and $\mathbf{v}_d$ are both solenoidal and irrotational and they represent respectively the contributions of Neumann's boundary condition, and Dirichlet's boundary condition.

Suppose $\boldsymbol{\omega}(\mathbf{r})$ is known at all points in R. With Dirichlet's condition, (3.64) gives

$$\oiint_{S_o} (\mathbf{v}_o \cdot \mathbf{r}_o)\mathbf{Q}\, dS_o + \mathbf{v}(\mathbf{r}) = \mathbf{v}_d(\mathbf{r}) + \mathbf{v}_\omega(\mathbf{r}) \tag{3.68}$$

Apply (3.68) to the boundary point $\mathbf{r}_s$ and take the dot product of $\mathbf{n}$ and each term in the resulting equation, one obtains, upon denoting $\mathbf{Q}(\mathbf{r}_s, \mathbf{r}_o)$ by $\mathbf{Q}_s$ and noting $\mathbf{v} \cdot \mathbf{n} = v_n(\mathbf{r}_s)$:

$$\oiint_{S_o} \mathbf{v}_n(\mathbf{r}_o)(\mathbf{Q}_s \cdot \mathbf{n})\, dS_o + \mathbf{v}_n(\mathbf{r}_s) = \mathbf{n} \cdot \mathbf{v}_d(\mathbf{r}_s) + \mathbf{n} \cdot \mathbf{v}_\omega(\mathbf{r}_s) \tag{3.69}$$

The right-hand side of (3.69) is a known scalar function of $\mathbf{r}_s$. (3.69) is a scalar Fredholm's integral equation of the second kind in which $v_n(\mathbf{r}_s)$ is the unknown function.

Suppose $\boldsymbol{\omega}(\mathbf{r})$ is known at all points in R. With Neumann's condition, (3.64) yields

$$\oiint_{S_o} (\mathbf{v}_o \times \mathbf{n}_o) \times \mathbf{Q}\, dS_o - \mathbf{v}(\mathbf{r}) = -\mathbf{v}_n(\mathbf{r}) - \mathbf{v}_\omega(\mathbf{r}) \tag{3.70}$$

Apply (3.70) to the boundary point $\mathbf{r}_s$, one obtains, upon taking the cross product of $\mathbf{n}$ of each term in the resulting equation

$$\mathbf{n} \times \oiint_{S_o} (\mathbf{v}_o \times \mathbf{n}_o) \times \mathbf{Q}_S\, dS_o + \mathbf{v}(\mathbf{r}_S) \times \mathbf{n} = -\mathbf{n} \times \mathbf{v}_n(\mathbf{r}_S) - \mathbf{n} \times \mathbf{v}_\omega(\mathbf{r}_S) \tag{3.71}$$

Since $\mathbf{v}_n(\mathbf{r}_s)$ is determinate from Neumann's condition and $\mathbf{v}_\omega(\mathbf{r}_s)$ is determinate from known $\boldsymbol{\omega}$ in R, the right-hand side of (3.71) is known. Thus (3.71) is an inhomogeneous Fredholm's integral equation of the second kind. This equation is a vector equation with components in the directions of $\boldsymbol{\tau}$ and $\mathbf{b}$. v_τ and v_b are the unknown functions and they appear in both components of (3.71). With (3.69), values of $v_n(\mathbf{r}_s)$ can be determined using known values of $\boldsymbol{\omega}(\mathbf{r})$, $v_b(\mathbf{r}_s)$, and $v_\tau(\mathbf{r}_s)$. With (3.71), values of $v_b(\mathbf{r}_s)$ and $v_\tau(\mathbf{r}_s)$ can be determined using known values of $\boldsymbol{\omega}(\mathbf{r}_s)$ and $v_n(\mathbf{r}_s)$. This observation confirms the earlier conclusion that, if values of $\boldsymbol{\omega}(\mathbf{r})$ are known at all points in R, then the imposition of Dirichlet's condition (v_i and v_b values on S) determines Neumann's condition (values of v_n on S) and vice versa.

Equation (3.71) is a vector equation (with two components) in three-dimensional flows. For the two-dimensional flow in the x-y plane with $\mathbf{b} = \mathbf{k}$, one has $v_b = 0$

and $\mathbf{n} \times \mathbf{v} = -\mathbf{v}_\tau \mathbf{k}$; and (3.71) reduces to a scalar Fredholm's equation with the unknown scalar function v_τ .

Paralleling the discussions in Sect. 3.4, consider a closed surface S* dividing R into two parts, R* and R^-. Suppose values of $\boldsymbol{\omega}(\mathbf{r})$ are known at all points in R^- but not in the narrow layer R*. Suppose also Cauchy's condition is imposed and hence values of $\boldsymbol{\omega} \cdot \mathbf{n}$ are known on S (see Sect. 3.1). Denote the average strength of the vorticity in the narrow region R* by $\boldsymbol{\omega}^*$ and re-write (3.64) as

$$\mathbf{v}(\mathbf{r}) = \iiint\limits_{R_o^-} \boldsymbol{\omega}_o \times \mathbf{Q}\, dR_o + \iiint\limits_{R_o^*} \boldsymbol{\omega}_o \times \mathbf{Q}\, dR_o + \oiint\limits_{S_o} (\mathbf{v}_o \times \mathbf{n}_o) \times \mathbf{Q}\, dS_o - \oiint\limits_{S_o} (\mathbf{v}_o \cdot \mathbf{n}_o)\mathbf{Q}\, dS_o \tag{3.72}$$

Denoting the first integral in (3.72) by $\mathbf{v}^-$, one obtains

$$\iiint\limits_{R_o^*} \boldsymbol{\omega}(\mathbf{r}_o) \times \mathbf{Q}(\mathbf{r}, \mathbf{r}_o)\, dR_o = \mathbf{v}(\mathbf{r}) - \mathbf{v}^-(\mathbf{r}) - \mathbf{v}_d(\mathbf{r}) - \mathbf{v}_n(\mathbf{r}) \tag{3.73}$$

On S, (3.73) yields

$$\iiint\limits_{R_o^*} \boldsymbol{\omega}(\mathbf{r}_o) \times \mathbf{Q}_s\, dR_o = \mathbf{v}(\mathbf{r}_s) - \mathbf{v}^-(\mathbf{r}_s) - \mathbf{v}_d(\mathbf{r}_s) - \mathbf{v}_n(\mathbf{r}_s) \tag{3.74}$$

For very small values of δn, the approximation of the integral in (3.74) by a surface integral leads to the following integral expression for the unknown function $\gamma(\mathbf{r}_S) = \omega^* \delta n$:

$$\oiint\limits_{S_o} \boldsymbol{\gamma}(\mathbf{r}_o) \times \mathbf{Q}(\mathbf{r}_s, \mathbf{r}_o)\, dR_o \approx \mathbf{v}(\mathbf{r}_s) - \mathbf{v}^-(\mathbf{r}_s) - \mathbf{v}_d(\mathbf{r}_s) - \mathbf{v}_n(\mathbf{r}_s) \tag{3.75}$$

In the limit as $\delta n \to 0$, (3.75) becomes precise. In this limit, the surfaces S* and S are apart at infinitesimal distances. If $\boldsymbol{\gamma}(\mathbf{r}_s)$ retains a finite value, then $\boldsymbol{\omega} \to \infty$ and the layer of vorticity in R* becomes a *vorticity sheet* of strength $\boldsymbol{\gamma}$ between S* and S. These two do not coincide. The thickness of the vorticity sheet becomes infinitesimal, but not zero. The values of $\mathbf{v} \times \mathbf{n}$ on S* side of the layer is greater than that on S by the amount $\boldsymbol{\gamma}$. Equation (3.75) is a Fredholm's integral equation of the first kind. If the strength $\boldsymbol{\gamma}$ of the vorticity layer is not known, then the specification of the Cauchy's condition on S determines $\boldsymbol{\gamma}(\mathbf{r}_s)$. Values of $\boldsymbol{\gamma}$ determined by (3.75) are approximate for small values of δn and precise in the limit as $\delta n \to 0$. The form of (3.75) is identical to that of (3.70). Thus one obtains

$$\oiint_{S_o} [(\mathbf{v}_o \times \mathbf{n}_o) + \boldsymbol{\gamma}_o] \times \mathbf{Q}(\mathbf{r}_s, \mathbf{r}_o)\, dS_o = \mathbf{v}(\mathbf{r}_s) - \mathbf{v}^-(\mathbf{r}_s) - \mathbf{v}_n(\mathbf{r}_s) \tag{3.76}$$

Equation (3.76) shows an equivalence of the components of $\boldsymbol{\omega}^*\delta n$ (= $\boldsymbol{\gamma}$) and the tangential velocity components $\mathbf{v} \times \mathbf{n}$ on S. This equivalence is approximate and it becomes precise in the limit as $\delta n \rightarrow 0$. In Sect. 1.4, it is asserted that the prescription any two of the three conditions—(i) Neumann's condition, (ii) Dirichlet's condition, and (iii) the condition $\boldsymbol{\omega} \times \mathbf{n}$ on S—renders the solution of (3.1) and (3.2) unique. The above discussions confirm the correctness of this assertion.

Consider the computation loop outlined in Sect. 1.4. For Step (i) of the loop, obviously velocity values are needed only in flow zones where the vorticity values are non-zero. For Step (iii), the discretization of (3.1) and (3.2) invariably yields a set of implicit algebraic equations containing values of $\mathbf{v}$ as unknowns. With these implicit equations, step (iii) must be performed over a large flow region that includes not only flow zones where the vorticity is non-zero, but also irrotational flow zones where the vorticity is zero. With (3.64), velocity values can be computed point by point. Therefore, the computations in Step (iii), like those in Step (i), can be confined to the relatively small zones where the vorticity is non-zero. The use of (3.64) therefore offers major advantages in solution efficiency. Also, (3.76) permits highly accurate evaluations of boundary vorticity values and offers major advantages in solution accuracy (Wu 1976).

The attribute of (3.64) for computing $\mathbf{v}$ values explicitly is unique and is of great value. This equation can be used, for example, to compute velocity values at all boundary points of any selected flow zone. Once this is carried out, the flow in the zone can be treated separately from all other zones of the flow. Special procedures well suited for the selected zone can then be used. For example, in the boundary layer zone, boundary-layer momentum equations can be solved in place of Navier–Stokes' momentum equations. Zonal procedures based on (3.64) are well established for routine, efficient, and accurate computations of viscous flows in two dimensions (Patterson et al. 1987). For computing three-dimensional viscous flows, (3.64) also offers major advantages (Kim et al. 1996). Vorticity-based methods for computing three-dimensional viscous flows, however, are not yet in a fully mature stage of development.

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Chapter 4
Vorticity Kinetics

4.1 Navier–Stokes Momentum Equation

Newton's second law of motion, as applied to the fluid medium, yields the well-known momentum theorem that states: *The time rate of change of the total momentum of a fluid system of fixed identity is equal to the force acting on the system*. A mathematical statement of this theorem for the incompressible flow of a viscous fluid is the familiar Navier–Stokes momentum equation

$$\frac{\partial(\rho\mathbf{v})}{\partial t} = -(\mathbf{v}\cdot\nabla)(\rho\mathbf{v}) - \nabla p + \nu\nabla^2(\rho\mathbf{v}) + \nabla q \tag{4.1}$$

where ∇q is a conservative external force, ρ, $\mathbf{v}$, p, and ν are respectively the density, the velocity, the pressure, and the kinematic viscosity of the fluid. Under general circumstances, these properties are functions of the position $\mathbf{r}$ and the time t. For the present study ρ and ν are assumed uniform and constant, independent of $\mathbf{r}$ and t.

Equation (4.1) contains, in addition to the assumption that ρ and ν are uniform and constant, a number of additional simplifying assumptions reviewed in Chap. 1. Derivations of (4.1) and extensive discussions of the assumptions involved are available in many fluid dynamic textbooks, e.g., Batchelor (1967). This equation is a *kinetic* equation: it deals with motions and forces that cause changes of motion.

The field variable of primary concern in the momentum equation is $\rho\mathbf{v}$, the momentum of the fluid per unit volume. For the incompressible flow, it is convenient to consider $\mathbf{v}$ as the primary variable and restate (4.1) as

$$\frac{\partial\mathbf{v}}{\partial t} = -(\mathbf{v}\cdot\nabla)\mathbf{v} - \frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v} + \frac{1}{\rho}\nabla q \tag{4.2}$$

Since (4.1) contains the term $(\mathbf{v}\cdot\nabla)\mathbf{v}$, it is a non-linear equation. Because of the non-linearity and the necessity to treat flow regions with complex boundary shapes, the kinetics of the flow problem is mathematically difficult to solve. It is, however,

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possible to obtain useful information about the kinetics of the flow without solving (4.1). An effective approach to derive information from (4.1) is through the idea of the vorticity field $\boldsymbol{\omega}$, defined as the curl the velocity field $\mathbf{v}$

$$\nabla \times \mathbf{v} = \boldsymbol{\omega} \tag{4.3}$$

The definition (4.3) is kinematics since it is concerned only with the instantaneous relation between two field variables, $\boldsymbol{\omega}$ and $\mathbf{v}$. Both variables describe the motion of the fluid. Equation (4.3) expresses a relationship between these two fields, apart from the forces that cause either $\mathbf{v}$ or $\boldsymbol{\omega}$ to change. The fields $\mathbf{v}$ and $\boldsymbol{\omega}$, however, have kinetic contents and are a part of both vorticity kinetics and vorticity kinematics. In this chapter, several topics of vorticity kinetics are reviewed. These topics are discussed in many textbooks in fluid dynamics, e.g., Batchelor (1967), in great details. They are revisited here to bring into focus the framework for the present study of external aerodynamics.

4.2 Convection and Material Derivative

The left-hand side of (4.1) represents the time rate of change of momentum per unit volume. A stationary observer in space sees this rate of change of momentum. Each of the terms on the right-hand side of (4.1) represents a physical process that causes the momentum of the fluid to change. The contributions of the processes to the momentum-change rate can be individually evaluated and their sum gives the momentum change rate seen by the stationary observer.

The first term on the right-hand side of (4.1) represents the process of *convection*. In this process, the fluid, while in motion, carried certain physical properties with it. Consider, as an example, elements of the fluid containing certain amounts of heat energy. As elements of the fluid move about, they carry heat energies with it. In this process, heat energy is transferred, or transported, from certain points in space to other points. The momentum of the fluid is carried by fluid elements and transported by convection in the same way.

A *material element of the fluid*, also called a *fluid element or simply* an *element*, is a small quantity of fluid with a fixed identity occupying a small closed region. A fluid element is identified by its mass content. A *material fluid region* is a collection of a large number of fluid elements, each with its individual mass content. The fluid elements, and hence also the material fluid region, move about and deform with time, but each element retains the same mass. In this study, the shapes of the fluid elements are chosen to be not uncommon, so that a single length, say λ, characterizes each element and the volume of the element δR is expressible as λ^3. An infinitesimal fluid element occupies an infinitesimal volume and is often called a *fluid particle* in the literature. To avoid the molecular connotation of the word particle, the expression *a fluid element* is used in the present study. A point $\mathbf{r}$ in space has a zero characteristic length. A fluid element has an infinitesimal, but nonzero, characteristic length λ. An infinitesimal fluid element is not a point, for otherwise the element cannot have a mass content. When needed to emphasize this distinction, the expression *a fluid element containing the point* **r**, or *a fluid element at* **r**, is used in this study in place of the simpler term *the point* **r**. Consider a time-dependent field f(**r**, t) that represents a property of the fluid such as its density or momentum. Taylor's formula with remainder gives

$$f(\mathbf{r}+\delta\,\mathbf{r},\,t+\delta\,t) = f(\mathbf{r},t) + \delta\,\mathbf{r} \cdot \nabla f\big|_{\mathbf{r},t} + +\delta\,t\,(\partial\,f/\partial\,t)\big|_{\mathbf{r},t} + O\Big(|\delta\,\mathbf{r}|^2\big|, \delta\,t^2, |\delta\,\mathbf{r}|\delta\,t\Big) \tag{4.4}$$

Suppose, at the time level t_1, an observer is at the point $\mathbf{r}_1$ in space and is moving at the velocity $\mathbf{v}_a$. This observer sees the value of the field changing. During a small time interval δt, the observer travels, to the order δt, from the point $\mathbf{r}_1$ to the point $\mathbf{r}_1 + \mathbf{v}_a\delta t$. The value of f the observer sees at the time level t_1 is $f(\mathbf{r}_1, t_1)$ and at the time level $t_1 + \delta t$ is $f(\mathbf{r}_1 + \mathbf{v}_a\delta t, t_1 + \delta t)$. Placing $\delta\mathbf{r} = \mathbf{v}_a\delta t$ into (4.4) and denoting the values of f as seen by the moving observer by $f_a(t)$, one obtains, upon omitting the superfluous subscript '1',

$$[f_a(t+\delta\,t) - f_a(t)]/\delta\,t = \mathbf{v}_a \cdot \nabla f\big|_{\mathbf{r},t} + +(\partial\,f/\partial\,t)\big|_{\mathbf{r},t} + O(\delta\,t) \tag{4.5}$$

The left-hand side of (4.5) is, in the limit as $\delta t \to 0$, the time-rate of change of f seen by the moving observer. Denoting this rate by D_af/D_at, one obtains in this limit from (4.5)

$$D_a f(\mathbf{r},t)/D_a t = \partial\,f/\partial\,t + \mathbf{v}_a \cdot \nabla f \tag{4.6}$$

Two factors contribute to the change of f as seen by the observer. First, because the field f is time-dependent, the observer sees a local change of f if he is stationary. This contribution is represented by the first term on the right-hand side of (4.6): the partial derivative of f with respect to time. The last term in (4.6) represents the second factor that contributes to the change of f as seen by the observer: the motion of the observer.

In (4.6), $\mathbf{v}_a$, the velocity of the observer, need not be uniform or independent of time. In fluid dynamics, it is convenient to let $\mathbf{v}_a = \mathbf{v}$, the fluid velocity, to denote the rate of change of f seen by the observer moving with the fluid by Df/Dt, and to write

$$\frac{Df}{Dt} = \frac{\partial\,f}{\partial\,t} + \mathbf{v} \cdot \nabla f \tag{4.7}$$

Df/Dt is called either the material derivative or the substantial derivative of f. The words material and substantial indicate that this derivative gives the rate of change of f observed while moving with the *material* or *substance* of the fluid. Since f is a field variable, one envisions an infinite number of observers covering the field, each observer moving at the local velocity of a fluid element. With (4.7), the material derivative of f is expressed in two parts: the local part $\partial f/\partial t$ representing the rate of change of f seen in the stationary reference frame and the second part $\mathbf{v} \cdot \nabla f$ representing the effect of the observer's motion with the fluid. This second part is called the convection term.

The idea of the material derivative is easily illustrated by the continuity equation for the flow of a compressible fluid, which is a mathematical statement of the law of mass conservation. This equation is obtainable from (2.26b) by letting $f = \rho$ and $\mathbf{v}_v = \mathbf{v}$ in that equation. For the fluid occupying the material fluid region R(t), one has

$$\frac{d}{dt}\iiint_{R(t)} \rho(\mathbf{r}, t)\, dR = \iiint_{R} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\, \mathbf{v})\right] dR \tag{4.8}$$

Since the same fluid elements, identified by their masses, occupy R(t) at different time levels, the integral of the left-hand side of (4.8) is time invariant by virtue of the law of mass conservation. Therefore the left-hand side of (4.8) vanishes. Because it vanishes for all choices of R(t), the integrand of the right-side of (4.8) is zero at every point in R. One thus obtains the following continuity equation for the flow of a compressible fluid:

$$\frac{\partial \rho}{\partial t} = -(\mathbf{v} \cdot \nabla)\, \rho - \rho\, \nabla \cdot \mathbf{v} \tag{4.9}$$

In (4.9), the left-hand side term is the rate of change of ρ observed in a stationary reference frame and the first term on the right-hand side is the convection term. Equation (4.9) can be restated as

$$\frac{1}{\rho}\frac{D\rho}{Dt} = -\nabla \cdot \mathbf{v} \tag{4.10}$$

According to (4.10), the observer moving with the fluid element in the compressible flow sees the density of the fluid element changing with time. The negative of the rate of this change divided by the density ρ is the divergence of the velocity field.

For the incompressible flow, the continuity equation simplifies to (4.11) or (4.12) below

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + (\mathbf{v} \cdot \nabla)\, \rho = 0 \tag{4.11}$$

$$\nabla \cdot \mathbf{v} = 0 \tag{4.12}$$

Equations (4.11) and (4.12) are equivalent since, according to (4.10), if $D\rho/Dt = 0$, then $\nabla \cdot \mathbf{v} = 0$ and, conversely, if $\nabla \cdot \mathbf{v} = 0$, then $D\rho/Dt = 0$. The use of (4.12), which states that the incompressible velocity filed must be solenoidal, is in general preferred over that of (4.11). This information is used in the study of vorticity kinematics in Chap. 3.

By definition, an incompressible medium is one that occupies a volume that does not change with time. Since the fluid element is identified by its mass content that, according to the law of mass conservation, does not change with time, the density of each fluid element, as seen by the observer moving with the element, is independent of time. This observation leads immediately to (4.11), and to (4.12) via (4.10). Alternatively, (4.12) is derivable using, again, (2.26b). Let $f = 1$ and $\mathbf{v}_v = \mathbf{v}$ in this equation, one obtains

$$\frac{d}{dt}\iiint_{R(t)} dR = \iiint_{R} \nabla \cdot \mathbf{v} dR \tag{4.13}$$

The integral on the left-hand side of (4.13) is simply R(t), the material fluid region comprising of individual fluid elements. Since the fluid is incompressible, the volume of each fluid element, hence also R(t), is independent of time. Thus the left-hand side of (4.13) is zero. Since it is zero for all choices of R(t), the integrand of the right-hand side integral in (4.13) is zero at every point in R(t). Thus (4.12) is derived without invoking explicitly the law of mass conservation. This derivation is based on identifying fluid elements by the region they occupy, and is kinematic.

If ρ is independent of $\mathbf{r}$ and t, then $D\rho/Dt = 0$ (and $\nabla \cdot \mathbf{v} = 0$). There is, however, the more general circumstance where the fluid is incompressible but has a non-uniform density. In the flow of this fluid, the observer moving with a fluid element sees no change of density. A stationary observer, however, sees the density changing because, as time progresses, different fluid elements with different densities are passing through the fixed point in space where the observer is stationed.

In the flow of a compressible fluid, the stationary observer sees the density changing not only because different fluid elements with different densities are passing by him, but also because another physical process, a process represented by the term $-\rho\nabla \cdot \mathbf{v}$, is influencing what he observes. Suppose R(t) in (4.13) is an infinitesimal volume V(t) occupied by a fluid element at $\mathbf{r}$. An observer moving with the fluid element sees the density changing at the rate $D\rho/Dt$ and the volume changing at the rate DV/Dt. Since the mass content ρV of the fluid element is a constant, one has $D(\rho V)/Dt = 0$ and $(1/\rho)D\rho/Dt = -(1/V)DV/Dt = -D(\ln V)/Dt$. Thus (4.10) states $\nabla \cdot \mathbf{v}$ at the point $\mathbf{r}$ is the rate of change of volume of the fluid element at $\mathbf{r}$ divided by that volume. This rate of change of logarithmic volume is called either the rate of expansion or the rate of dilatation. The term $\nabla \cdot \mathbf{v}$ therefore represents the rate of expansion (or contraction) of the volume occupied by the fluid element as it moves about, with corresponding dilution or strengthening of the density, which is simply the mass per unit volume of the fluid.

In summary, the convection of a fluid property, such as the density, means that the property is transported, or carried, by fluid elements as they move about. Convection by itself does not cause the property to change. That is, to an observer moving with the fluid element, the fluid property is not altered by convection: The material derivative of the property is zero if convection is the only process involved. There are, however, other processes that cause the property to change while being transported by convection. The expansion of the fluid element's volume in a compressible flow is an example. The rate of this expansion, represented by the term $\nabla \cdot \mathbf{v}$ in (4.9), causes the density to change.

The momentum Eq. (4.1) can be restated in terms of the material derivative as

$$\frac{D(\rho\,\mathbf{v})}{Dt} = -\nabla p + \nu\nabla^2(\rho\,\mathbf{v}) + \nabla q \tag{4.14}$$

This form of the momentum equation shows that the observer moving with a fluid element experiences three types of forces: a pressure force, a viscous force and an external force. The field $\rho\mathbf{v}$ describes the momentum of the fluid per unit volume. The momentum of the moving fluid element is not changed by convection, but is changed by the pressure, the viscous, and the body forces.

4.3 Viscous Force and Diffusion of Momentum

The term $\nu\nabla^2(\rho\mathbf{v})$ in (4.1) represents the contribution of the viscous force to the change of momentum per unit volume of the fluid. This term is also interpretable as the viscous diffusion of momentum from one fluid element to another. Consider an insulated and stationary solid in which the heat energy, characterized by the temperature, is initially distributed unevenly. The diffusion process, called heat conduction in this case, transports the heat energy from higher temperature regions to lower temperature regions within the solid. As a consequence, the energy differences within the solid diminish as time progresses and they approach zero asymptotically as time extends. In this way the diffusion process is an *equalizer*: It evens out the heat energy (temperature) in the solid. In fluid flows, the momentum of the fluid is equalized by viscous diffusion. The fluid elements are in motion, and viscous diffusion slows the motion of faster fluid elements, transferring their momentum to slower fluid elements. Unlike heat conduction, the diffusive transport of momentum takes place only when parts of the fluid are in motion, for otherwise the momentum is zero (and therefore already equal) for all fluid elements.

Denote a property of a material medium per unit volume by f. Fourier's law of diffusion states that the time rate of diffusive transport of f across a unit area in a material medium is proportional to $-\nabla f$, where the negative sign accounts for the fact that f is transported in the direction of decreasing f. Denote the constant of proportionality, called the diffusion coefficient, by κ. The rate of diffusive transport of f across a unit area is then $-\kappa\nabla f$. Consider a finite simply connected region R fixed in space and bounded externally by the closed surface S. The total amount of f in the region is the integration of f over R. The rate of f being transported by diffusion out of R across a differential surface element dS is $-\kappa(\nabla f \cdot \mathbf{n})\,dS$, $\mathbf{n}$ being the unit normal vector directed outward from R. If f is not created or destroyed in R, and no other physical process transporting f into or out of R is present, then the integration of $-\kappa(\nabla f \cdot \mathbf{n})$ dS over S is equal to the rate of decrease of the total f in R due to diffusion. That is,

$$\frac{d}{dt}\iiint\limits_R f\, dR \oiint\limits_S \kappa\,(\nabla f)\cdot \mathbf{n}\, dS \tag{4.15}$$

Since R is a fixed volume in space, the order of temporal differentiation and spatial integration on the left-hand side of (4.15) can be interchanged. Suppose κ is uniform. Using the divergence theorem to restate the surface integral as a volume integral, one obtains

$$\iiint\limits_R \left(\frac{\partial f}{\partial t} - \kappa\,\nabla^2 f)\, dR = 0\right) \tag{4.16}$$

Since (4.16) is valid for all choices of R in the material medium, the integrand in (4.16) must be zero at all points in the medium. Thus one has the following diffusion equation for all points $\mathbf{r}$ in R:

$$\frac{\partial f}{\partial t} - \kappa\,\nabla^2 f = 0 \tag{4.17}$$

If the only physical process causing changes in f in a stationary fluid is diffusion, then $\partial f/\partial t = \kappa\nabla^2 f$. The term $\kappa\nabla^2 f$ thus represents the diffusion effects on the time rate of change of the field f in R. For the fluid in motion, a stationary observer sees changes caused by convection and by diffusion. An observer moving with a fluid element, however, sees only the change caused by diffusion. Therefore $Df/Dt = \kappa\nabla^2 f$ or, equivalently, $\partial f/\partial t = -\mathbf{v}\cdot\nabla f + \kappa\nabla^2 f$.

The spatial differential operator in (4.17), the Laplacian operator, is second order and elliptic. The time–space relation of (4.17) is parabolic. Equation (4.17) is linear and homogeneous. The mathematical properties of this equation are discussed extensively in the literature, e.g., Morse and Feshbach [1953]. The problem described by (4.17) is known as an initial-boundary value problem. If values of $f(\mathbf{r}, t_0)$ at an initial time level $t = t_0$ are known at all points in R (initial condition) and values of f are known at all points on the boundary S (Dirichlet's condition) for the time period $t_0 < t < t_1$, then (4.17) determines values of $f(\mathbf{r}, t)$ at every point in R uniquely for this period. Alternatively, if initial values of f are known at all points in R and values of $\partial f/\partial n$ (Neumann's condition), are known at every point on S for the time period $t_0 < t < t_1$, then (4.17) also determines values of $f(\mathbf{r}, t)$ at every point in R uniquely for this period. It is permissible to specify a linear combination of the two conditions on S. The specification both of f and $\partial f/\partial n$ on S (Cauchy's conditions), however, over-specifies the problem.

As mentioned, the diffusion process is an equalizer and is, for this reason, irreversible. The solution can only march forward in time. It is possible to predict future ($t > t_0$) values of f in R based on (4.17). Marching backward in time leads to unstable solutions. That is, in all solutions marching backwards in time, errors grow and eventually overwhelm the true solutions, making the results unstable and worthless. In the heat conduction problem, if there is no input of heat to the solid either internally or though the solid boundaries, the temperature of the solid ultimately becomes uniform in the solid. The value of this uniform temperature depends only on the total heat energy, and not on the initial distribution of energy in R. In other words, there are an infinite number of initial temperature distributions with the same total heat energy that give the same final uniform temperature solution. Knowing the final uniform temperature, to go back in time and sort out which one of these possible distributions existed initially is not feasible.

In a moving fluid, the convection term $-\mathbf{v} \cdot \nabla f$ involves first order derivatives in space and therefore can be treated as an inhomogeneous term in the diffusion equation. The mathematical properties of the diffusion equation described above are not changed by the inhomogeneous term.

Equation (4.1) is interpretable as a diffusion equation for the momentum field $\rho\mathbf{v}$ with inhomogeneous terms $-\rho\mathbf{v} \cdot \nabla\mathbf{v} - \nabla p$. Since $\rho\mathbf{v}$ is the momentum of the fluid per unit volume, $\mathbf{v}$ is the momentum of the fluid per unit mass. Thus (4.2) is interpretable either as a diffusion equation for the velocity field or a diffusion equation for the momentum per unit mass of the incompressible fluid. The viscous term, $\nu\nabla^2\mathbf{v}$, in (4.2) can be viewed either as a viscous force (acting on a unit mass of the fluid) or as viscous diffusion term. As a diffusion term, it represents a process that transfers, or transports, the momentum (per unit mass) of the fluid by viscous action in accordance with the diffusion equation. The inhomogeneous terms in (4.2) involve first order differentiations in space. These terms are analogous to sources and sinks terms in the heat conduction problem. If these inhomogeneous terms are known, or can be evaluated, and initial and boundary conditions of $\mathbf{v}$ are specified, then the velocity field in R_f is determined uniquely by (4.2).

4.4 Vorticity and Flow Regions

Vorticity kinetic is concerned with the change of the vorticity field with time. A familiar interpretation of the vorticity vector is that it describes the rotation of fluid elements. This interpretation is kinematic since it is based on (4.3), a kinematic equation that does not deal with the causes of fluid motion. Specifically, as discussed in Chap. 3, the vorticity vector $\boldsymbol{\omega}(\mathbf{r}, t)$ is the sum of the angular velocities of two perpendicular line elements in the plane normal to $\boldsymbol{\omega}$ at the point $\mathbf{r}$ and the time level t. The vorticity field, however, also has a central role in the kinetics of the flow problem.

As discussed, a vector field $\mathbf{f}(\mathbf{r}, t)$ is *lamellar* if $\nabla \times \mathbf{f} = 0$. A lamellar velocity field is said to be *irrotational* because, in the absence of vorticity, fluid elements are not rotating. More specifically, if the sum of the angular velocities of any two perpendicular lines in a fluid element (which, as discussed gives the vorticity $\boldsymbol{\omega}$) is zero, then the fluid element is considered to be not rotating. The lines may rotate with opposite velocities so that the fluid element deforms while its angular momentum is conserved.

The viscous term in (4.2) can be restated as $-\nu\nabla \times \boldsymbol{\omega}$. Hence the viscosity of the fluid has no effects on the flow in a region where the vorticity field is lamellar. Obviously, a uniform (including zero) vorticity region is a special region of lamellar vorticity. More generally, according to (3.14), in a lamellar vorticity region, there exists a scalar potential field χ such that $\boldsymbol{\omega} = \nabla\chi$. The consequences of this observation deserve further investigations.

Water and air have very small viscosities. Hence the viscous term in the flow of water and air is small wherever $\nabla \times \boldsymbol{\omega}$ is not large. In external flows, the field $\boldsymbol{\omega}$ in general is small in regions far from the solid boundary S. In these regions, the viscous term in (4.2) is very small and the flow is approximately inviscid. The inviscid flow approximation, however, is not admissible in flow regions near S, such as the boundary-layer region, where $\boldsymbol{\omega}$ and $\nabla \times \boldsymbol{\omega}$ are large.

The term $(\mathbf{v} \cdot \nabla)\mathbf{v}$ can be restated as $\boldsymbol{\omega} \times \mathbf{v} + \nabla(\mathbf{v} \cdot \mathbf{v})/2$. Equation (4.2) can thus be recast in the form

$$\frac{\partial \mathbf{v}}{\partial t} = -\boldsymbol{\omega} \times \mathbf{v} - \nabla(h - q) + \nu\nabla \times \boldsymbol{\omega} \tag{4.18}$$

where $h = p/\rho + (\mathbf{v} \cdot \mathbf{v})/2$ is the total head, and $\boldsymbol{\omega} \times \mathbf{v}$ is called the Lamb vector.

Equation (4.18) reveals two levels of flows in which viscous effects are absent: the inviscid level and the irrotational level. At the inviscid level, the viscous term $-\nu\nabla \times \boldsymbol{\omega}$ vanishes, but the vorticity field $\boldsymbol{\omega}$ and the Lamb vector $\boldsymbol{\omega} \times \mathbf{v}$ need not be zero. With the inviscid-fluid assumption, obviously the viscous term vanishes and viscous effects are absent in all flow regions. In vorticity dynamics, however, the fluid is not assumed to be inviscid. Viscous effects are absence in lamellar vorticity flow regions. They are negligible in regions where the vorticity is negligibly small or, more generally, where the vorticity field is nearly lamellar. Flows in such regions are at the inviscid level although the fluid is viscous. Physically, only parts of the flow, not the entirety of the flow, can be at the inviscid flow level.

At the irrotational level, $\mathbf{v}$ is lamellar, $\boldsymbol{\omega} = 0$, and both the viscous term and the Lamb vector are zero. Since $\mathbf{v}$ is expressible as $\nabla\phi$, where ϕ is a scalar velocity potential, (4.18) yields the following two equivalent forms of Bernoulli's equation

$$\nabla\left(\frac{\partial \phi}{\partial t} + \frac{p - q}{\rho} + \frac{\mathbf{v} \cdot \mathbf{v}}{2}\right) = 0 \tag{4.19}$$

$$\frac{\partial \phi}{\partial t} + \frac{p - q}{\rho} + \frac{\mathbf{v} \cdot \mathbf{v}}{2} = g\,(t) \tag{4.20}$$

For the steady flow, one has the familiar version of the Bernoulli's equation stating $(p-q)/\rho + \mathbf{v} \cdot \mathbf{v}/2$ is a constant. In external flows of viscous fluids, the flow is approximately irrotational in some regions of the flow. In such regions, (4.20) provides an approximate description of the flow. It is worth noting that (4.20) contains the non-linear term $\mathbf{v} \cdot \mathbf{v}$ (equivalently $\nabla\phi \cdot \nabla\phi$). Neither the inviscid fluid assumption nor the irrotational assumption renders the differential equation describing the flow linear. Equations (4.19) and (4.20) are valid at the irrotational level and not necessarily valid at the inviscid flow level.

In summary, there are three types of flow regions coexisting in the overall incompressible flowfield of the viscous fluid: regions where viscous effects are important, regions where vorticity is present but viscous effects are negligibly small, and regions where vorticity is absent or negligibly small. From the vorticity-dynamic viewpoint, distinctions between various flow regions are attributable to the relative magnitudes of the Lamb vector $\boldsymbol{\omega} \times \mathbf{v}$ and the viscous term $-\nu\nabla \times \boldsymbol{\omega}$ in (4.18), as compared to the term $\partial\mathbf{v}/\partial t$. Interactions between these regions produce a rich array of flow patterns (see van Dyke 1982) that served as a happy stamping ground for the fluid dynamicist in centuries past.

4.5 Appearance and Transport of Vorticity

Taking the curl of each term in (4.18), one obtains the vorticity transport equation:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = -\nabla \times (\boldsymbol{\omega} \times \mathbf{v} + \nu\nabla \times \boldsymbol{\omega}) \tag{4.21a}$$

$$= -(\mathbf{v} \cdot \nabla)\boldsymbol{\omega} + (\boldsymbol{\omega} \cdot \nabla)\mathbf{v} + \nu\nabla^2\boldsymbol{\omega} \tag{4.21b}$$

Since the curl of the gradient of a scalar function is always zero, the total head h in (4.18), which contains the pressure p, is absent in (4.21a). In fact, all terms expressible as the gradient of a single valued scalar function, including the term $\nabla(\mathbf{v} \cdot \mathbf{v})$ in (4.19) and conservative force term ∇q, do not enter (4.21a). These terms are not a part of vorticity dynamics. As it turns out, however, the scalar potential function $\phi(t)$ and (4.20) plays a critical role in the vorticity-moment theorem of aerodynamics derived in this Chapter.

Equation (4.21a) does not contain a force term. This equation is a kinetic equation because it is a direct consequence of the Navier–Stokes momentum equation, which is kinetic. This equation deals with physical processes that cause the vorticity field to change with time. As discussed, the velocity is interpretable as the momentum of the fluid per unit mass. Similarly, the vorticity is a description of the rotation (equivalently the angular momentum per unit mass) of fluid elements. In this context, (4.21) is a differential equation describing the rate of change of rotation of the fluid with time. In (4.21b), the first term on the right-hand side represents the transport of vorticity (angular momentum) by convection. The last term represents the transport of vorticity by viscous diffusion. These two transport processes are discussed in Sects. 4.2 and 4.3. The term $(\boldsymbol{\omega} \cdot \nabla)\mathbf{v}$ has no counterpart in (4.1). This term is associated with the stretching and turning of vorticity tubes with fluid motion discussed in Chap. 2. In short, (4.21) states that the vorticity is transported in the fluid by the physical processes of convection and diffusion. A fluid element representing a small segment of a vorticity tube turns and stretches as it moves about, and an observer moving, turning, and stretching with the fluid element sees only the change of vorticity content in the fluid element caused by diffusion. In the absence of viscosity, the observer sees no change of the vorticity content in the element.

In inviscid flow regions containing vorticity, the last term in (4.21b) vanishes and the vorticity is transported only by convection in the sense that the vorticity flux $\boldsymbol{\omega} \cdot d\mathbf{S}$ associated with each material element $d\mathbf{S} = \mathbf{n}dS$ moving with the fluid remains constant for all times. This is the second vortex theorem of Helmholtz, proofs of which are given in many fluid dynamic textbooks (e.g., Sommerfeld 1950) on the basis of the inviscid fluid assumption. From the vorticity-dynamic viewpoint, Helmholtz' second vortex theorem is valid in the flow of the viscous fluid in inviscid rotational regions where the viscous term $\nu\nabla \times \boldsymbol{\omega}$ is negligibly. In viscous flow regions, Helmholtz' vortex theorem is modified by the process of viscous diffusion (see Chap. 2).

Convection and diffusion are transport processes that re-distribute the vorticity in space. These processes do not create or destroy the vorticity in the interior of the fluid domain. Turning and stretching of vorticity tubes also do not create or destroy the vorticity. Hence the vorticity can neither be created nor be destroyed in the interior of the fluid domain. This is a principle of vorticity conservation in the fluid domain.

Consider the reference flow problem defined in Chap. 1: the incompressible flow of a fluid external to a finite solid body immersed and moving in a viscous fluid. The fluid region R_f is bounded internally by the fluid/solid interface S and externally unlimited. The solid and the fluid are at rest initially. For this reference flow, the vorticity is obviously zero everywhere in R_f prior to the motion's onset. Since vorticity can neither be created nor be destroyed in the interior of R_f, all the vorticity in the interior of the fluid domain must be introduced to the fluid at the boundary S. From S, the vorticity is transported into the interior of the fluid domain. In a viscous fluid, with the no-slip and no-penetration conditions, the relative velocity between the fluid and the solid at the boundary S is zero. Therefore the vorticity introduced on S cannot be transported away from the boundary by convection. This vorticity is transported away from S only through diffusion. Once in the interior of the fluid domain, the vorticity is transported by both convection and diffusion. As time progresses, new vorticity continues to be introduced on S and is first transported away from S by diffusion and then transported within the fluid by both diffusion and convection. The central task of vorticity kinetics is the study of the evolution of the vorticity field with time.

4.6 Speeds of Vorticity Transport

The velocity of convection of the vorticity in the fluid is obviously the fluid velocity **v**. Since **v** is finite, convective transport is a finite-rate process. In aeronautics, **v** is characterized by the flight speed. Thus 10 m/s (36 km/h) is a suitable scale for measuring the speed of convective transport of vorticity in low speed aerodynamics.

Let $\kappa = \nu$ in the diffusion Eq. (4.17). The fundamental solution F_d, i.e. Green's function for an infinite unlimited region, of the diffusion equation is

$$F_d(\mathbf{r}, t; \mathbf{r}_0, t_0) = [4\pi\nu(t-t_0)]^{-\frac{3}{2}} \exp\left[-\frac{|\mathbf{r}-\mathbf{r}_0|^2}{4\nu(t-t_0)}\right] \tag{4.22}$$

For vorticity transport by diffusion, F_d represents the vorticity distribution in the unlimited space **r** at the time level t resulting from a concentrated vorticity of unit strength located at the point $\mathbf{r}_0$ at the time level t_0. If, at the time level t_0, the vorticity distribution is $\boldsymbol{\omega}(\mathbf{r}_0, t_0)$, then the vorticity distribution at the subsequent time level t in the unlimited **r**-space is given by the following integral:

$$\boldsymbol{\omega}(\mathbf{r}, t) = \iiint_{R_0} F_d \boldsymbol{\omega}(\mathbf{r}_0, t_0) dR_0 \tag{4.23}$$

A generalized form of (4.23) for a fluid region bounded internally by S and in which convective transport of vorticity is in progress is given by Wu (1981).

The form of (4.22), the fundamental solution F_d of the diffusion equation shows an infinite signal speed of vorticity transport by viscous diffusion. It also shows that the vorticity decays exponentially as $\exp(-\alpha)$, where $\alpha = |\mathbf{r} - \mathbf{r}_0|^2 \big/ [4\nu\,(t - t_0)]$, as a result of viscous diffusion. Using the time scale of 1 s and the velocity scale of 10 m/s, one has the corresponding length scale of 10 m for convection. For diffusive transport of vorticity, using the same time and length scales, i.e. $(t - t_0) = 1$ s and $|\mathbf{r} - \mathbf{r}_0| = 10$ m, one obtains $\alpha \sim 2 \times 10^6$ based on the kinematic viscosity of sea level air at 0 °C, which is 1.3×10^{-5} m^2/s. Therefore $\exp(-\alpha)$ is exceedingly small. This means that the effective speed of vorticity transport by diffusion is extremely small compared to the speed of convection, except in regions where the velocity is extremely small. This speed is in any event effectively finite.

For the reference flow problem, vorticity is introduced to the fluid at the solid boundary S. Because the transport of vorticity by convection is finite rate and that by diffusion is effectively finite-rate, the vorticity can travel only a finite distance away from the boundary S within a finite period of time after the motion's onset. Suppose a steep vorticity gradient exists in the interior of a fluid region. According to (4.22), the diffusion process smoothes out this steep gradient rapidly in a very short period of time. Thus, very steep vorticity gradients can persist only near the boundary S, where the vorticity is introduced. Since viscous effects are important only in flow regions where the vorticity gradient is very steep, most regions of the flow, excepting those very close to the solid boundary S, are effectively inviscid flow regions. This well-known conclusion is a cornerstone of the boundary-layer theory. From the viewpoint of vorticity kinetics, the boundary layer is a thin vorticity region in which the viscous term and the Lamb vector are comparable in magnitude (See Sect. 4.4). Certain terms in the Navier–Stokes momentum equations are negligible in thin regions near S where the vorticity and the velocity vectors are both essentially tangential to S. In three-dimensional flow, all vorticity tubes present in boundary layers must be closed in space. They generally continue beyond the layers into regions where viscous effects are unimportant. This means there exist regions outside boundary layers where the flow is at the inviscid level, but not the irrotational level. These and many other observations are useful in interpreting various conceptual issues related to the inviscid fluid assumption and the three-dimensional boundary layer flow. They are not discussed further in the present study, except the general conclusions that, for the reference flow, the vorticity field is confined effectively to a relatively small finite region surrounding and trailing the solid at all finite time levels after the motion's onset and that, at large distances from the solid, the vorticity field decays exponentially with increasing distance from the solid.

4.7 Boundary Conditions in Vorticity Kinetics

Consider the reference flow defined in Chap. 1. The vorticity transport equation, (4.21), can be stated in the form of an inhomogeneous diffusion equation in the fluid domain R_f

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \nu \nabla^2 \boldsymbol{\omega} = \nabla \times (\mathbf{v} \times \boldsymbol{\omega}) \tag{4.24}$$

As discussed in Sect. 4.3, the diffusion problem is an initial-boundary value problem. The initial values of vorticity can be stated generally as

$$\boldsymbol{\omega}(\mathbf{r}, t_o) = \mathbf{f}(\mathbf{r}) \text{ in } R_f, \tag{4.25}$$

where $\mathbf{f}(\mathbf{r})$ is a specified function of the position vector $\mathbf{r}$.

For the reference flow, if t_0 is a time level prior to the onset of the motion, then obviously $\mathbf{f}(\mathbf{r}) = 0$.

The fluid region R_f is bounded internally by the surface S and is externally unlimited. The idea of a flow region containing vorticity and extending to infinity is, however, not a clear-cut issue in external aerodynamics. This issue is discussed in Chap. 1. For the present discussion, it is appropriate to limit the time of interest to a finite period, say $t_0 < t < t_1$, and to consider a large but finite region R_f' bounded internally by S and externally by a large finite surface S_e that encloses S.

The following discussions are applicable to three-dimensional flows where $R_f^{'}$ is a simply connected region. It is simple to generalize the discussions for applications involving multiply connected flow regions.

Appropriate vorticity boundary conditions for (4.24) are

$$\boldsymbol{\omega}\,(\mathbf{r}_s, t_0 < t < t_t) = \mathbf{g}(\mathbf{r}_s, t)\text{ on } S \tag{4.26}$$

$$\boldsymbol{\omega}\,(\mathbf{r}_e,\ t_0 < t < t_1) = \mathbf{h}(\mathbf{r}_e, t)\text{ on } S_e \tag{4.27}$$

where $\mathbf{r}_s$ and $\mathbf{r}_a$ are position vectors on S and S_e respectively and **g** and **h** are prescribed functions of $\mathbf{r}_s$, $\mathbf{r}_{a,}$ and t.

For the reference flow, the function $\mathbf{h}(\mathbf{r}_e, t)$ is typically assumed to be zero. The vorticity in the fluid domain is initially zero everywhere. As discussed in Sects. 4.5 and 4.6, the vorticity cannot be created in the interior of the fluid domain. It is introduced at S and is transported in the fluid only by the finite-rate convection and diffusion processes. In consequence, the extent of the nonzero vorticity field is finite at all finite time levels subsequent to the motion's onset. According to (4.22) and (4.23), in an infinite unlimited space occupies by the fluid, if the vorticity is non-zero only within finite distances from the origin at a given time level and only the diffusion process transports vorticity in the fluid, then at any subsequent finite time level the vorticity at large distances from the origin decays exponentially with increasing distance from the origin. Equation (4.23) is generalized for the reference flow in which convective transport of vorticity takes place, in addition to diffusive transport, and the solid surface S bounds the fluid region internally (Wu 1981). The generalized equation leads to the same conclusion just stated for the special case of an infinite fluid in which no convective transport of vorticity takes place, namely, the vorticity decays exponentially far from the solid. This is an expected outcome. The convective transport of vorticity, being a finite rate process, obviously cannot alter the earlier conclusion. The presence of the surface S, which is finite and provides a mechanism for introducing the vorticity into the fluid region, also cannot alter the earlier conclusion. Thus the assumption that $\mathbf{h}(\mathbf{r}_{e,}\ t) = 0$ in (4.27) is an excellent approximation, provided all points on S_e are sufficiently far from S. In fact, the approximation becomes precise in the limit as these distances approaches infinity. It needs to be emphasized, however, that $\mathbf{h}(\mathbf{r}_{e,}\ t) = 0$ is an acceptable approximation only if the time interval $t_0 < t < t_1$ is finite. If t_1 is infinitely large, then the presence of the starting vortex on S_e means this approximation needs to be reexamined. This issue, as it turns out, is not an inconsequential matter that can be simply disregarded.

For the flow of the viscous fluid, with the no-slip condition, the tangential component of the velocity field **v** is continuous across S. Therefore tangential velocity values at all points on S, the internal boundary of the fluid region, are determined by the prescribed solid motion during the time interval $t_0 < t < t_1$. These known values in turn determine the values of the normal component of the vorticity vector at all points on S. The tangential components of the vorticity vector on S are determined by vorticity kinematics, as discussed in Chap. 3.

4.8 Approaching Steady Flow

According to the well-known Kutta–Joukowski theorem, the lift on a two-dimensional solid body in steady motion is proportional to the *circulation* around the body. Therefore a solid body experiencing a steady lift must have a non-zero circulation around it. Consider an airfoil initial at rest in a fluid also initially at rest. This airfoil is set into motion suddenly and thereafter kept moving rectilinearly at a constant velocity and a small angle of attack. The motion of the airfoil creates a corresponding flow of the fluid. Near the airfoil, the flow approaches steady state asymptotically at large time levels after the onset of the motion. The lift on the airfoil also asymptotically approaches a steady value and hence, at large time levels, there is a non-zero constant circulation around the airfoil. According to the principle of conservation of total vorticity derived in Chap. 5, the total vorticity in the fluid must be zero. In consequence, the fluid away from the airfoil must have a circulation that balances the circulation around the airfoil. This circulation is shed from the vicinity of the airfoil in the form of a starting vortex shortly after the motion's onset [see, e.g. Batchelor (1967), Von Karman and Burgers (1934)]. The circulation around the starting vortex, i.e. the total vorticity shed from the vicinity of the airfoil, is equal and opposite the circulation around the airfoil, so that the principle of total vorticity conservation is satisfied.

For a wing experiencing a steady lift, a starting vortex is also shed shortly after the onset of the wing motion. According to Helmholtz' first vortex theorem, a vortex tubes must not end in the fluid. In fact, in Chap. 2, it is shown that a vortex tube cannot end in space, and must form a loop in the infinite unlimited region R_∞ jointly occupied by the wing and the fluid. Vortex tubes near the wing therefore must continue into the fluid. Prandtl (1921) observed that vorticity tubes near the wing are directed mostly along the wingspan. On the basis of Helmholtz' first vortex theorem, Prandtl concluded that vortex tubes in the flow near the wing must continue in the fluid with the same strength and leave the vicinity of the wing in the form of vortex tubes trailing the wing. Lanchester in England independently theorized, near the end of nineteenth century [see Anderson (1997)], the existence of the trailing vortices near the wing tips, which he called vortex trunks, and about the role of this trailing vortex in aerodynamics. Prandtl envisioned a system of U-shaped vortex tubes approximating the vorticity surrounding and trailing a lifting wing. The two parallel sectors of the tubes are in the flow direction and they approximate the trailing vorticity. The middle sections of the tubes are in the wingspan direction and they approximate the vorticity near the wing. The U-shaped vortex tube is today called a horseshoe vortex. Traveling along the U-path, one observes constant vortex strength. That is to say, the circulation around the vorticity tube does not change, as required by Helmholtz' first vortex theorem.

The middle section of the horseshoe vortex is today called the *bound vortex*. The position of this section of the horseshoe vortex is called the lifting line. Scholars in classical aerodynamics, including Prandtl, used the word 'bound' for convenience. In some recent textbooks on aerodynamics, however, the term *bound vortex* is said to be a vortex somehow fastened to a wing position and is distinct from 'free vortices' that move with the fluid. As discussed in Sects. 4.3 and 4.4, the vorticity is transported in the fluid by convection and diffusion. Helmholtz' second vortex theorem is interpretable as a specialization of this observation to an inviscid fluid in which diffusion is absent. The idea that the vorticity near the wing belongs to a privileged class of vorticity that needs not obey Helmholtz' theorem was clearly not intended by scholars in classical aerodynamics. In the present discussions, the expression *ambient vorticity* (*vortex*) is used in place of the term bound vortex to avoid the unintended connotation of the word *bound*. The word *ambient* indicates this vorticity surrounds and is near the wing. Ambient vorticity is not a part of the wake vorticity, which includes the vorticity in the trailing and starting vortices. In a steady flow, the ambient vorticity is independent of time when viewed in a reference frame attached to the wing. This time independency does not mean the ambient vorticity is fixed in space in the wing-attached reference frame. Rather, in the vicinity of the wing, as in all other regions of the flow, the vorticity is continually transported by diffusion and convection. The time-independency in the ambient region means the rate of transport by diffusion is equal and opposite to the rate of convection. As a result, the sum of these rates is zero and a stationary observer at each point in the vicinity of the wing sees no change of vorticity with time.

The horseshoe vortex and the starting vortex are in reality parts of a closed vortex loop. The two parallel sectors of the U are closed downstream of the wing by the starting vortex. In classical aerodynamics, the starting vortex is typically envisioned as infinitely far downstream of the wing and has no significant influence on the flow near the wing. This observation is particularly useful in steady two-dimensional streamlined flows for which the well-known Kutta conditions can be used to establish the circulation value around the airfoil independently of the strength of the starting vortex. For applications involving unsteady or non-streamlined flows and/or three-dimensional flows, it is in general necessary to recognize the principle of total vorticity conservation in aerodynamic analysis. This means that, although the effects of the starting vortex on the velocity field near the wing is insignificant, the vorticity content of the starting vortex is important and must be properly recognized in aerodynamic analysis. The vorticity-moment theorem derived in Chap. 5, in particular, requires the recognition of the existence of the starting vortex.

As time progresses, the starting vortex continually moves away from the solid and thus, at an infinitely large time level after the motion's onset, the starting vortex is infinitely far from the solid. This observation has two consequences in aerodynamic analysis. First, with the starting vortex moving away from the solid, the flow associated with a lifting body cannot be truly steady. Second, with the presence of the starting vortex, the flow far from the solid is not truly a freestream: The velocity field near the starting vortex is not negligibly small. These consequences are not trivial and needs attention in aerodynamic analysis.

Two scenarios, both for large but finite time levels after the motion's onset, can be considered. In the first scenario, the flow is in a sufficiently large region that does not enclose the starting vortex. The starting vortex is envisioned to be sufficiently far outside this region and have negligible effects on the flow inside the region, which can be steady or unsteady. In either case, in three-dimensional flows, the trailing vorticity connecting the starting vorticity to the ambient vorticity must cut through the external boundary S_e of the region. The velocity values at points on S_e that are near the trailing vorticity differ from the freestream value significantly. The assumption that, since S_e is sufficiently far from the wing, the flow on this external boundary is undisturbed by the wing's motion (freestream) is not applicable. In a limiting process where the distances of all points on S_e from the wing approach infinity, the undisturbed freestream assumption is inapplicable as long as the starting vortex is considered to be outside S_e.

In the second scenario, the flow is in an exceedingly large region that encloses all regions of non-zero vorticity in the fluid, including the trailing vorticity and the starting vorticity. The external boundary S_e of this region in this scenario is in a potential flow region. No trailing vorticity crosses S_e. The flow in the region enclosed by S_e and internally bounded by S, because of the presence of the starting vortex, is unsteady. The unsteady Bernoulli's Eq. (4.20), and not its more familiar steady version, is applicable on S_e. The time level is allowed to be as large as imaginable, but remains finite. All points on S_e, however, are permitted to approach infinity so that the starting vortex remains enclosed by S_e. This second scenario is used to derive the vorticity-moment theory of aerodynamics in Chap. 5.

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Chapter 5
Vorticity-Moment Theorem

5.1 Preliminary Remarks

The vorticity-moment theorem of aerodynamics discussed in this chapter is valid for viscous and unsteady flows. Earlier derivations of the theorem (Wu 1978, 1981) are updated and important features of the theorem are revisited in the present study. The theorem is shown to encompass much of the classical inviscid steady theories of aerodynamics, including the lifting-line theory, and is a rigorous mathematical consequence of the incompressible continuity and Navier–Stokes momentum equations:

$$\nabla \cdot \mathbf{v} = 0 \tag{5.1}$$

$$\rho \frac{\partial \mathbf{V}}{\partial \mathbf{t}} = -\rho(\mathbf{v} \cdot \nabla)\mathbf{v} - \nabla p + \rho \nu \nabla^2 \mathbf{v} + \nabla q \tag{5.2}$$

where $\mathbf{v}$ is the flow velocity, ρ, p, and ν are respectively the density, the pressure and the kinematic viscosity of the fluid, and ∇q represents a conservative external body force.

The theorem begins with the idea of the vorticity field $\boldsymbol{\omega}$, defined as the curl of $\mathbf{v}$:

$$\nabla \times \mathbf{v} = \boldsymbol{\omega} \tag{5.3}$$

Taking the curl of each term in (5.2) and re-arranging the result, one obtains

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\mathbf{v} \times \boldsymbol{\omega}) - \nu \nabla \times (\nabla \times \boldsymbol{\omega}) \tag{5.4}$$

The set of equations (5.1), (5.3), and (5.4) describe the dynamics of the vorticity field in a viscous fluid, with (5.1) and (5.3) describing the kinematic aspect and (5.4) the kinetic aspect. Vorticity kinematics is concerned with the instantaneous

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relationship between $\boldsymbol{\omega}$ and $\mathbf{v}$. More specifically, it deals with the evaluation, on the basis of (5.1) and (5.3), of the velocity field $\mathbf{v}(\mathbf{r}, t)$ at a specific instant of time that corresponds to a known vorticity field $\boldsymbol{\omega}(\mathbf{r}, t)$ at the same instant. In unsteady flows, $\boldsymbol{\omega}$ and $\mathbf{v}$ are time-dependent fields. However, since only the instantaneous relationship between the two fields is of concern, explicit references to their time dependencies are omitted in the present discussions of vorticity-kinematics. As discussed in Chaps. 2 and 3, classical methods of mathematical analyses are useful in vorticity kinematics because (5.1) and (5.3) are linear.

Unlike vorticity kinematics, which deals with instantaneous relationships, *vorticity kinetics* is concerned with changes of the vorticity field with time and the causes of these changes. Vorticity kinetics, described by the vorticity transport equation (5.4), is difficult to analyze mathematically because (5.4) is non-linear. General understandings of physical processes of vorticity transport are described in Chap. 4. These understandings are sufficient for the present derivation of the vorticity-moment theorem of aerodynamics.

A reference flow problem for the present study of vorticity aerodynamics is defined in Chap. 1. This reference flow involves a finite rigid solid body, called a wing to emphasize the aerodynamic applications here, immersed and moving in an infinite fluid. The wing is initially at rest in a fluid also initially at rest. At a certain instant of time, the body is set into motion. Thereafter, the motion of the wing is prescribed and is not restricted to rectilinear and/or steady motions. The regions occupied by the body and the fluid are denoted as R_s and R_f, respectively, and the interface of the two regions is denoted S. For the reference flow, the assumption of a single rigid body initially at rest is made for convenience. The vorticity-moment theorem presented in this Chapter is applicable directly to flows involving multiple rigid bodies. The bodies may deform, although their volumes are not supposed to change with time, and they need not be at rest initially.

5.2 Total Vorticity Conservation

An important component of the vorticity-moment theorem is the principle of conservation of the total vorticity. This principle is expressible mathematically as

$$\frac{d}{dt}\iiint_{R_\infty} \boldsymbol{\omega} dR = 0 \tag{5.5}$$

where R_∞ is the infinite unlimited space jointly occupied by the fluid and the solid.

In Chap. 3, it is shown that the solid and the fluid can be treated together as a single kinematic system jointly occupying R_∞. In Chap. 4, it is shown that in external flows, the vorticity field in the fluid is confined to finite regions surrounding and trailing the solid at all finite time level. Consider a finite region R_ω that contains R_s, all parts of R_f where the vorticity is non-zero, and possibly also parts of R_f where the vorticity is zero. By properly selecting the zero vorticity parts, it is always possible to construct the region R_ω so that it is simply connected. The region R_p surrounding R_ω is then an irrotational flow region and is simply connected in three-dimensional flows.

Consider a closed circuit C in R_p. Let S_c be a cap of C. Using (5.3), Stokes' theorem gives:

$$\iint_{S_c} \boldsymbol{\omega} \cdot \mathbf{n} dS = \oint_C \mathbf{v} \cdot \tau \, ds = \Gamma \tag{5.6}$$

where $\mathbf{n}$ is the unit vector normal to S_c, the integral on the left-hand side is the total flux of the vorticity over S_c, $\boldsymbol{\tau}$ is the unit vector tangential to C, and Γ is the circulation around C. The directions of $\mathbf{n}$ and $\boldsymbol{\tau}$ are related by the rule of the right-handed screw.

Suppose C and all points on a cap S_c of C are in R_p. Since $\boldsymbol{\omega} = 0$ on this cap, (5.6) gives $\Gamma = 0$. The circulation around C is independent of the choice of the cap. It follows that, since $\Gamma = 0$ on the cap S_c, the total flux of vorticity overall caps of C are zero. Suppose C_c is a closed circuit lying in the x-y plane in a Cartesian coordinate system (x, y, z) and all points on C_c are in the region R_p. Since R_ω is finite and simply connected, it is obviously possible to construct a cap of C_c in such a way that all points on this cap are in R_p. This means that the total flux over this cap, and hence all caps, of C_c is zero. One therefore obtains the following equation for ζ, the z-component of $\boldsymbol{\omega}$:

$$\iint_{S_C} \zeta(x, y, z) dxdy = 0 \tag{5.7}$$

where S_c is the cap of C in the x-y plane. For the case S_c lies outside the region R_ω, (5.7) is obviously true. For the case S_c cuts through R_ω, the correctness of (5.7) can be confirmed by representing the vorticity field in R_ω by closed vorticity tubes. As shown in Chap. 2, all vorticity fields in external flows can be represented, without exception, by sets of closed tubes (loops) of vorticity in R_∞. With the vorticity non-zero only in R_ω, all the closed vorticity loops are contained within R_ω. Obviously, if a closed vorticity loop crosses S_c, going from one side of this cap to the other side, it must return to the initial side, crossing S_c in the opposite direction. Since the strength of the vorticity tube is constant along its path, the contribution to the integration of $\boldsymbol{\omega} \cdot \mathbf{n}$ over S_c of one of the two crossings must be equal and opposite to that of the other. More generally, if the tube crosses from one side of S_c to the other side n times, it must return n times, n being an integer including zero. The net contribution of each pair of crossings of each vorticity tube to the integration of $\boldsymbol{\omega} \cdot \mathbf{n}$ over S_c is zero. Hence, the total contribution of all the closed vorticity tubes that make up the vorticity field in R_ω is zero.

In (5.7), the cap S_c can be replaced by S_{xy}, the entire infinite x-y plane. Upon integrating the result with respect to z over the range $-\infty < z < \infty$, one obtains

$$\iiint_{R_\infty} \zeta dR = \iiint_{R_\omega} \zeta dR = 0 \tag{5.8}$$

Equation (5.8) states that the total z-component of vorticity, ζ, in R_∞ is zero. The total x- and y-components of $\boldsymbol{\omega}$ in R_∞ are also zero, as can be similarly proved. Thus one has

$$\iiint_{R_\infty} \boldsymbol{\omega} dR = 0 \tag{5.9}$$

The arguments leading to (5.7), and hence (5.8) and (5.9), assume that the potential flow region R_p, in which C is drawn, is simply connected. In two-dimensional flows, however, R_p is in general doubly connected. Therefore an alternative derivation of the principle of total vorticity conservation is needed for the two-dimensional flow. Discussions of the vorticity-moment theorem for the two-dimensional flow, including the principle of total vorticity conservation, are postponed to Sect. 5.7. The present Section and Sects. 5.3 through 6 deal with three-dimensional flows only.

Equation (5.9) is more exacting than (5.5). It states that the total vorticity in the infinite unlimited region jointly occupied by the fluid and the solid is not only time-invariant: It is zero at all instants of time. The study of the fluid and the solid together as a single kinematic system is not a familiar and conventional approach. An alternative proof of (5.9), treating the kinematics of the fluid motion in R_f alone is therefore desirable.

One possible approach to prove (5.9) is to use the divergence theorem and write

$$\iiint_{R_f'} \boldsymbol{\omega} dR = -\oiint_{S} \mathbf{v} \times \mathbf{n} dS \oiint_{S_e} \mathbf{v} \times \mathbf{n} dS \tag{5.10}$$

where R_f' is the fluid region bounded internally by S and externally by S_e, a finite closed surface enclosing S, and $\mathbf{n}$ is the normal unit vector directed outward from R_f'.

With the no-slip condition, one obtains for the solid

$$\iiint_{R_S} \boldsymbol{\omega} dR = \oiint_{S} \mathbf{v} \times \mathbf{n} dS \tag{5.11}$$

where $\mathbf{n}$ is the normal unit vector directed outward from R_f' and inward into R_s.

Let S_e be a spherical surface of radius r. In the limit as $r \longrightarrow \infty$, the region R_∞ represents R_f' and R_s combined. If in this limit the last integral in (5.10) approaches zero, then one obtains (5.9) by adding (5.10) and (5.11). The law of Biot–Savart (see Chap. 3) indicates that, unless (5.9) is valid, the velocity field approaches zero asymptotically for large values of r as r^{-2}. Since S_e approaches infinity also as r^{-2}, it is not assured that the last integral in (5.10) goes to zero in the limit as $r \longrightarrow \infty$. Therefore, (5.10) and (5.11) do not provide a rigorous proof of (5.9).

Equation (5.9), however, can be proved rigorously using the vector identity $\mathbf{A} \cdot \boldsymbol{\omega} = \nabla \cdot [(\mathbf{A} \cdot \mathbf{r})\boldsymbol{\omega}]$, where $\mathbf{A}$ is an arbitrary constant vector. Upon integrating the above identity over R_f' and using the divergence theorem, one obtains:

$$\mathbf{A} \cdot \iiint_{R_f'} \boldsymbol{\omega} dR = \mathbf{A} \cdot \oiint_{S} \mathbf{r}(\boldsymbol{\omega} \cdot \mathbf{n}) dS + \mathbf{A} \cdot \oiint_{S_e} \mathbf{r}(\boldsymbol{\omega} \cdot \mathbf{n}) dS \tag{5.12}$$

Since $\mathbf{A}$ is an arbitrary constant vector, one obtains immediately from (5.12)

$$\iiint_{R_f'} \boldsymbol{\omega} dR = \oiint_{S} \mathbf{r}(\boldsymbol{\omega} \cdot \mathbf{n}) dS + \oiint_{S_e} \mathbf{r}(\boldsymbol{\omega} \cdot \mathbf{n}) dS \tag{5.13}$$

With the no-slip condition, the tangential components of $\mathbf{v}$ are continuous across S. In consequence, $\boldsymbol{\omega} \cdot \mathbf{n}$ is continuous across S and one obtains,

$$\iiint_{R_S} \boldsymbol{\omega} dR = - \oiint_{S} \mathbf{r}(\boldsymbol{\omega} \cdot \mathbf{n}) dS \tag{5.14}$$

It is shown in Chap. 4 that $\boldsymbol{\omega}$ decays exponentially with increasing r. Thus, in the limit as $r \longrightarrow \infty$, the last integral in (5.13) approaches zero and adding (5.13) and (5.14) yields (5.9).

The principle of total vorticity conservation discussed here differs conceptually from the time-invariance of vorticity integrals discussed in earlier literature, e.g., (Lagerstrom 1964; Batchelor 1967). The present principle goes beyond stating that the total vorticity of an inviscid fluid filling the infinite space is conserved; it states that the total vorticity is zero in the unlimited space occupied jointly by the fluid, viscous or inviscid, and one or more rigid or deforming solid body. For a solid rotating at the angular velocity Ω, the total vorticity in the fluid region R_f is simply $-2V\Omega$, where V is the volume of the solid.

The fact that R_ω is a finite region is a consequence of vorticity kinetics. The proof of (5.9) is otherwise based on vorticity kinematics only. The continuity equation for the incompressible flow, (5.5), is not used in the derivation of (5.9).

5.3 Asymptotic Behavior of Velocity Field

The total momentum of the fluid in the incompressible flow is equal to the density of the fluid times the total velocity of the fluid. In Sect. 5.5, it is shown that the total velocity of the material media in R_∞ is related kinematically to the total $\mathbf{r} \times \boldsymbol{\omega}$, the first moment of vorticity, in R_∞. In preparation for the derivation of this relationship, the asymptotic behavior of the velocity field far from the solid is examined in the present Section. Consider the law of Biot–Savart, (3.51) of Chap. 3:

$$\mathbf{v}(\mathbf{r}) = \iiint_{R_\omega} \boldsymbol{\omega}(\mathbf{r}_o) \times \mathbf{Q}(\mathbf{r}, \mathbf{r}_o) dR_o \tag{5.15}$$

where $\mathbf{Q} = \nabla F$ and F is the fundamental solution of the elliptic equation defined by:

$$F = -1/(4\pi r') \tag{5.16}$$

$$r' = |\mathbf{r} - \mathbf{r}_o| \tag{5.17}$$

$$\mathbf{r}' = \mathbf{r} - \mathbf{r}_o \tag{5.18}$$

Various aspects of the fundamental solution are discussed in Chap. 3. The function $\mathbf{Q}$, defined as the gradient of F, is

$$\mathbf{Q} = \mathbf{r}'/(4\pi r'^3) \tag{5.19}$$

The fundamental solution F and its gradient $\mathbf{Q}$ are functions of both $\mathbf{r}$ and $\mathbf{r}_o$. In (5.15), $\mathbf{r}_o$ is a dummy variable of integration and the integral in (5.15) gives the velocity field in the $\mathbf{r}$-space. The integration is over the finite region R_ω, which includes the parts of R_f in which $\boldsymbol{\omega} \neq 0$ and R_s if the solid is rotating. Replacing R_ω by R_∞ in (5.15) obviously does not change $\mathbf{v}(\mathbf{r})$. The form of $\mathbf{Q}$ suggests that in the limit as $r \longrightarrow \infty$, $\mathbf{v}$ approaches zero as r^{-2}. In reality, $\mathbf{v}$ approaches zero as r^{-3} because of the principle of total vorticity conservation (5.9), as shown below.

Taylor's series expansion of the $1/r'$ in the $\mathbf{r}_o$-space about the point $\mathbf{r}_o = 0$ gives:

$$\frac{1}{r'} = \left.\frac{1}{r'}\right|_{r_o=0} + r_o \cdot \nabla_o \left(\frac{1}{r'}\right)_{r_o=0} + \text{higher order terms} \tag{5.20}$$

The first term on the right-hand side of (5.20) is equal to $1/r$. Since $\nabla_o(1/r') = -\nabla(1/r')$, the second term in (5.20) is equal to $-\mathbf{r}_o \cdot \nabla(1/r) = \mathbf{r}_o \cdot \mathbf{r}/r^3$. The higher order terms in (5.20) involve higher order derivatives of $1/r'$ evaluated at the point $\mathbf{r}_o = 0$. These terms are in the form r^{-m}, $m \geq 3$. Thus, $\mathbf{Q} = \nabla F$ is expressible as

$$\mathbf{Q} = -\frac{1}{4\pi}\left\{\nabla\left(\frac{1}{r}\right) - \nabla\left[\mathbf{r}_o \cdot \nabla\left(\frac{1}{r}\right)\right]\right\} + \text{terms of order } r^{-n},\ n \geq 4 \tag{5.21}$$

Denoting $\nabla(1/r)/4\pi$ by $\mathbf{g}$ and placing (5.21) into (5.15), one obtains

$$\mathbf{v}(\mathbf{r}) = -\iiint\limits_{R_\omega} \boldsymbol{\omega}_o \times \mathbf{g} dR_o + \iiint\limits_{R_\omega} \boldsymbol{\omega}_o \times \nabla(\mathbf{r}_o \cdot \mathbf{g}) dR_o + \text{terms of order } r^{-n},\ n \geq 4 \tag{5.22}$$

In (5.22), the integrations are performed in the $\mathbf{r}_o$-space. Since, $\mathbf{g}$ is a function of $\mathbf{r}$ only, it can be moved outside the first integral. Because of the principle of total vorticity conservation (5.9) and because $\boldsymbol{\omega} = 0$ outside R_ω, the first integral in (5.22) is zero. The second integral can be expressed in terms of $\mathbf{r} \times \boldsymbol{\omega}$, the first moment of vorticity.

Consider the vector fields $\mathbf{F}$ and $\mathbf{G}$ and the identity $\nabla \cdot (F_x\mathbf{G}) = F_x\nabla \cdot \mathbf{G} + (\nabla F_x) \cdot \mathbf{G}$ in the Cartesian coordinate system (x, y, z), where F_x is the x-component of F. Using the divergence theorem, one obtains for the closed region R bounded by S_e:

$$\iiint\limits_{R} (F_X \nabla \cdot \mathbf{G} + \mathbf{G} \cdot \nabla F_X) dR = \oiint\limits_{S_e} F_X \mathbf{G} \cdot \mathbf{n} dS \tag{5.23}$$

The vector identity that corresponds to (5.23) is

$$\iiint\limits_{R} [\mathbf{F}(\nabla \cdot \mathbf{G}) + (\mathbf{G} \cdot \nabla)\mathbf{F}] dR = \oiint\limits_{S_e} \mathrm{F}(\mathbf{G} \cdot \mathbf{n}) dS \tag{5.24}$$

Let $\mathbf{F} = \mathbf{r}$ and $\mathbf{G} = (\mathbf{r} \cdot \mathbf{g}_o)\, \boldsymbol{\omega}$. One obtains $\mathbf{F}\,(\nabla \cdot \mathbf{G}) = (\mathbf{g}_o \cdot \boldsymbol{\omega})\, \mathbf{r}$ and $(\mathbf{G} \cdot \nabla)\, \mathbf{F} = (\mathbf{r} \cdot \mathbf{g}_o)\, \boldsymbol{\omega}$. Upon interchanging $\mathbf{r}$ and $\mathbf{r}_o$ and letting $R = R_\omega$, one obtains from (5.24):

$$\iiint\limits_{R_\omega} [(\mathbf{g} \cdot \boldsymbol{\omega}_o)\mathbf{r}_o + (\mathbf{g} \cdot \mathbf{r}_o)\boldsymbol{\omega}_o] dR_o = \oiint\limits_{S_e} \mathbf{r}_o(\mathbf{g} \cdot \mathbf{r}_o)(\boldsymbol{\omega}_o \cdot \mathbf{n}_o) dS_o \tag{5.25}$$

It can be shown that $\boldsymbol{\omega}_o \times \nabla(\mathbf{r}_o \cdot \mathbf{g}) = -\nabla \times [(\mathbf{r}_o \cdot \mathbf{g})\boldsymbol{\omega}_o]$ and $(\mathbf{r}_o \cdot \mathbf{g})\boldsymbol{\omega}_o = -½[\mathbf{g} \times (\mathbf{r}_o \times \boldsymbol{\omega}_o)] + ½[(\mathbf{g} \cdot \boldsymbol{\omega}_o)\, \mathbf{r}_o + (\mathbf{g} \cdot \mathbf{r}_o)\boldsymbol{\omega}_o]$. Thus one obtains from (5.22) using (5.25):

$$\mathbf{v}(\mathbf{r}) = 1/2\, \nabla \times \left[\iiint\limits_{R_\omega} \mathbf{g} \times (\mathbf{r}_o \times \boldsymbol{\omega}_o) dR_o - \oiint\limits_{S_e} \mathbf{r}_o(\mathbf{r}_o \cdot \mathbf{g})(\boldsymbol{\omega}_o \cdot \mathbf{n}_o) dS_o \right] + \text{terms of order } r^{-n},\ n \geq 4 \tag{5.26}$$

In the limit as $r \longrightarrow \infty$, the vorticity on S_e decays exponentially with increasing r and the last integral in (5.26) vanishes. For the first integral, since the integration is in the $\mathbf{r}_o$-space, the function $\mathbf{g} = \nabla(1/r)/4\pi$, being dependent only on $\mathbf{r}$, can be moved outside the integral sign. Therefore, (5.26) yields:

$$\mathbf{v}(\mathbf{r}) = \nabla \times [\nabla(1/r) \times \alpha]/(8\pi) + \text{terms of order } r^{-n},\ n \geq 4 \tag{5.27a}$$

$$= \nabla[\nabla(1/r) \cdot \alpha]/(8\pi) + \text{terms of order } r^{-n},\ n \geq 4 \tag{5.27b}$$

where $\boldsymbol{\alpha}$ is defined by:

$$\alpha = \iiint_{R_\infty} \mathbf{r} \times \boldsymbol{\omega}\, dR = \iiint_{R_\omega} \mathbf{r} \times \boldsymbol{\omega}\, dR \tag{5.28}$$

The vector $\boldsymbol{\alpha}$ is the total first moment of the vorticity in R_∞, which is the infinite unlimited region jointly occupied by the fluid and the solid. The vector $\boldsymbol{\alpha}$ is time-dependent, but is a constant vector at each specific instant of time. The integration of the vorticity moment $\mathbf{r} \times \boldsymbol{\omega}$ can be performed in either the $\mathbf{r}$-space or the $\mathbf{r}_o$-space over either R_∞ or the finite region R_ω that contains vorticity. Thus, according to (5.27), $\mathbf{v}$ decays as r^{-3} for large r, provided $\boldsymbol{\alpha}$ is not zero. If $\boldsymbol{\alpha} = 0$, then $\mathbf{v}$ decays as r^{-4}.

5.4 Total Velocity

Letting $\mathbf{F} = \mathbf{v}$ and $\mathbf{G} = \mathbf{r}$ in (5.24), one obtains

$$\iiint_R [\mathbf{v}(\nabla \cdot \mathbf{r}) + (\mathbf{r} \cdot \nabla)\mathbf{r}]dR = \oiint_{S_e} \mathbf{v}(\mathbf{r} \cdot \mathbf{n})dS \tag{5.29}$$

It can be shown that $\mathbf{v}(\nabla \cdot \mathbf{r}) + (\mathbf{r} \cdot \nabla)\mathbf{v} - \nabla(\mathbf{r} \cdot \mathbf{v}) = 2\,\mathbf{v} - \mathbf{r} \times \boldsymbol{\omega}$. Integrating this identity over R, using the divergence theorem to restate the resulting left-hand side terms, one obtains, upon noting $\mathbf{v}(\mathbf{r} \cdot \mathbf{n}) - (\mathbf{r} \cdot \mathbf{v})\mathbf{n} = \mathbf{r} \times (\mathbf{v} \times \mathbf{n})$,

$$\iiint_R \mathbf{v}dR = \frac{1}{2}\iiint_R (\mathbf{r} \times \boldsymbol{\omega})dR + \frac{1}{2}\oiint_{S_e} [\mathbf{v}(\mathbf{r} \cdot \mathbf{n}) - (\mathbf{r} \cdot \mathbf{v})\mathbf{n}]dS \tag{5.30a}$$

$$= \frac{1}{2}\iiint_R (\mathbf{v} \times \boldsymbol{\omega})\, dR + \frac{1}{2}\oiint_{S_e} \mathbf{r} \times \mathbf{n}(\mathbf{v} \times \mathbf{n})dS \tag{5.30b}$$

Consider a spherical coordinate system (r, θ, ψ) with unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_\psi$. The coordinates r, θ, and ψ are related to the Cartesian coordinates (x, y, z) by $x = r \sin\theta \cos\psi$, $y = r \sin\theta \sin\psi$, and $z = r \cos\theta$. Suppose S_e is a spherical surface with a radius r centered at $\mathbf{r} = 0$. Let $\boldsymbol{\alpha} = \alpha\mathbf{k}$, $\mathbf{k}$ being the unit vector in the z-direction. One then has $\nabla(1/r) \cdot \boldsymbol{\alpha} = -\alpha\, r^{-2} \cos\theta$. Equation (5.27b) gives

$$\mathbf{v}(\mathbf{r}) = \alpha(2\cos\theta\, \mathbf{e}_r + \sin\theta\, \mathbf{e}_\theta)/\left(8\pi r^3\right) + \text{terms of order}\, r^{-n},\ n \geq 4 \tag{5.31}$$

On S_e, $\mathbf{n} = \mathbf{e}_r$, $\mathbf{r} = r\mathbf{e}_r$, and $\mathbf{e}_\theta = \cos\theta\cos\psi\, \mathbf{i} + \cos\theta\sin\psi\, \mathbf{j} - \sin\theta\, \mathbf{k}$. Thus, one has

$$\begin{aligned}\mathbf{r} \times (\mathbf{v} \times \mathbf{n}) = \alpha(&\sin\theta\cos\theta\cos\psi\, \mathbf{i} + \sin\theta\cos\theta\sin\psi\, \mathbf{j} \\ &- \sin^2\theta\, \mathbf{k})/(8\pi r^2) + \text{terms of order}\ r^{-n},\ n \geq 3\end{aligned} \tag{5.32}$$

The last integral in (5.30b) is expressible as

$$\oiint_{S_e} \mathbf{r} \times (\mathbf{v} \times \mathbf{n})dS = r^2 \int_0^\pi \left[\sin\theta \int_0^{2\pi} \mathbf{r} \times (\mathbf{v} \times \mathbf{n})d\psi\right] d\theta + \text{terms of order}\ r^{-n},\ n \geq 1 \tag{5.33}$$

Using (5.32), one obtains from (5.33):

$$\oiint_{S_e} \mathbf{r} \times (\mathbf{v} \times \mathbf{n})dS = -\alpha\mathbf{k}/3 + \text{terms of order}\ r^{-n},\ n \geq 1 \tag{5.34}$$

Since the direction of $\mathbf{k}$ can be chosen arbitrarily, the first term on right-hand side of (5.34) can be stated simply as $-\boldsymbol{\alpha}/3$. Placing (5.34) into (5.30b) and noting that the first integral in (5.30) is $\boldsymbol{\alpha}/2$, one obtains the following expression as an approximate expression for the total velocity in the spherical region R bounded by S_e.

$$\iiint_{R_L} \mathbf{v}dR = \alpha/3 \tag{5.35}$$

In (5.35), the region of integration is designated R_L instead of R_∞ to emphasize the fact that (5.35) is obtained through the limiting process $r \longrightarrow \infty$, as discussed in more details in Sect. 5.8. In this limit, (5.35) is precise.

5.5 Aerodynamic Force

The material system in R_L defined at the end of Sect. 5.4 is composed of the solid body that occupies R_s and the fluid in the region R_f' surrounding the solid. The spherical surface S_e with radius r centered at the point r = 0 is the control surface enclosing the control volume R_L. In the following analyses, the body force on the fluid in R_f' is omitted. The total force $\mathbf{F}_t$ experienced by the total system in R_L is then the sum of the surface forces on S_e and the external force $\mathbf{F}_e$ acting on the solid. In wind-tunnel experiments, the external force $\mathbf{F}_e$ on the solid, the test model, is exerted through the model mount. For an aircraft, this force is a thrust—a propulsive force—produced by the propulsion system. In aeronautics, the propulsive force is typically analyzed separately from the aerodynamic force $\mathbf{F}$ exerted on the fluid on the solid body. In bio-fluid dynamic studies, however, this force is often analyzed as a part of $\mathbf{F}$.

The surface forces on S_e consist of the pressure force and the viscous force. The viscous force is represented by the last term $\rho\nu\nabla^2\mathbf{v} = -\rho\nu\nabla \times \boldsymbol{\omega}$ in (5.2). This force per unit area is given by $\rho\nu\boldsymbol{\omega} \times \mathbf{n}$ on S_e. Let r be sufficiently large so that all points on S_e are in the potential flow region where $\boldsymbol{\omega}$ is negligibly small. Then only the pressure force contributes to the force on S_e and one has

$$\mathbf{F}_t = \mathbf{F}_e - \oiint_{S_e} p\mathbf{n}dS \tag{5.36}$$

According to the momentum theorem, the total force is equal to the time rate of change of the total momentum of the system in the control volume R_L plus the rate of momentum leaving R_L through S_e. One thus obtains, denoting the solid density by ρ_s:

$$\mathbf{F}_t = \rho\frac{d}{dt}\iiint_{R'} \mathbf{v}dR + \frac{d}{dt}\iiint_{R_S} \rho_S\mathbf{v}dR + \rho\oiint_{S_e} \mathbf{v}(\mathbf{v}\cdot\mathbf{n})dS \tag{5.37}$$

As $r \longrightarrow \infty$, $\mathbf{v}$ goes to zero as r^{-3}. Therefore, the last integral vanishes in the limit as $r \longrightarrow \infty$. The second integral of (5.37) is the change of momentum of the solid. This change is equal to $\mathbf{F} + \mathbf{F}_e$, the total force acting on the solid. Thus, (5.36) and (5.37) yield

$$\mathbf{F} = -\rho\frac{d}{dt}\iiint_{R'} \mathbf{v}dR - \oiint_{S_e} \rho\mathbf{n}dS \tag{5.38}$$

In the limit as $r \longrightarrow \infty$, the first integral in (5.38) is the total velocity in R_∞ subtracted by the total velocity in R_s. Therefore, (5.38) and (5.35) give

$$\mathbf{F} = -\frac{1}{3}\frac{d\boldsymbol{\alpha}}{dt} + \rho\frac{d}{dt}\iiint_{R_S} \mathbf{v}dR - \oiint_{S_e} \rho\mathbf{n}dS \tag{5.39}$$

Since S_e is outside R_ω, the unsteady Bernoulli's equation (4.20) gives:

$$\oiint_{S_e} p\mathbf{n}dS = -\rho \oiint_{S_e} \left[\frac{\partial \phi}{\partial t} + \frac{\mathbf{v}\cdot\mathbf{v}}{2} - g(t)\right]\mathbf{n}dS \tag{5.40}$$

The asymptotic behavior of $\mathbf{v}$, given in (5.27), shows that the term $\mathbf{v} \cdot \mathbf{v}/2$ does not contribute to (5.40). The term g(t), being independent of position, also does not contribute. Letting $\boldsymbol{\alpha} = \alpha\mathbf{k}$, one obtains from (5.27a, b) $\phi \approx \nabla(1/r) \cdot \boldsymbol{\alpha}/(8\pi) = -\alpha \cos\theta/(8\pi r^2)$. Since $\mathbf{n} = \mathbf{e}_r = \sin\theta\cos\psi\,\mathbf{i} + \sin\theta\sin\psi\,\mathbf{j} + \cos\theta\,\mathbf{k}$, one has

$$\oiint_{S_e} \phi\mathbf{n}dS = -\frac{\alpha}{8\pi}\int_0^{\pi}\left\{\sin\theta\cos\theta\int_0^{2\pi}[\sin\theta(\cos\psi\,\mathbf{i} + \sin\psi\,\mathbf{j}) + \cos\theta\,\mathbf{k}]d\psi\right\}d\theta \tag{5.41}$$

The x-and y-components of (5.41) are zero because the integrations of cos ψ and sin ψ over 2π are zero. The z-component of (5.41) gives $-\alpha\mathbf{k}/6$. Since the direction of $\mathbf{k}$ is chosen arbitrarily, the right-hand side of (5.41) can be stated simply as $-\boldsymbol{\alpha}/6$. In the limit as $r \longrightarrow \infty$, (5.41) is precise. Placing this result into (5.40) and then into (5.38), one obtains,

$$\mathbf{F} = -\frac{\rho}{2}\frac{d}{dt}\iiint_{R_f} \mathbf{r}\times\boldsymbol{\omega}dR + \rho\frac{d}{dt}\iiint_{R_S} \mathbf{v}dR \tag{5.42}$$

Equation (5.42) relates the aerodynamic force on a solid body immersed and moving in a fluid to the time rate of change of the total first moment of the vorticity field in R_∞. If the solid is rotating at an angular velocity Ω, then $\boldsymbol{\omega} = 2\Omega$ in R_s and one has

$$\mathbf{F} = -\frac{\rho}{2}\frac{d}{dt}\iiint_{R_\infty} \mathbf{r}\times\boldsymbol{\omega}dR + \rho\frac{d}{dt}\iiint_{R_S} (\mathbf{v} + \boldsymbol{\Omega}\times\mathbf{r})dR \tag{5.43}$$

For a non-rotating solid moving at a constant speed, the last term in (5.43) is zero.

5.6 Moment of Aerodynamic Force

It is relatively simple to show that the moment of the aerodynamic force is related to the time rate of change of the total second moment of the vorticity field. Using the vector identity $\nabla \times (r^2\mathbf{v}) = 2\mathbf{r} \times \mathbf{v} + r^2\boldsymbol{\omega}$ and the divergence theorem, one obtains

$$\iiint_{R_L} \mathbf{r} \times \mathbf{v} dR = -\frac{1}{2} \iiint_{R_L} r^2 \boldsymbol{\omega} dR - \frac{1}{2} \oiint_{S_e} r^2 (\mathbf{v} \times \mathbf{n}) dS \tag{5.44}$$

Since S_e is a spherical surface centered at the point $\mathbf{r} = 0$, r is a constant on S_e. Therefore, the factor r^2 can be moved outside the last integral in (5.44). The integration of $\mathbf{v} \times \mathbf{n}$ over S_e can be restated as the total vorticity in R_L using the divergence theorem. According to (5.9), this total vorticity is identically zero for sufficiently large values of r. Thus the last term in (5.44) vanishes.

The total moment of force $\mathbf{M}_t$ acting on the material system in the control volume R_L is the sum of the external applied moment $\mathbf{M}_e$ on the solid and the moment of the surface forces on S_e. The moment of the surface forces on S_e is composed of the moment of the pressure force and that of the viscous force. Since the vorticity decays exponentially with distance from S, the viscous force does not contribute to the moment of force on S_e. The moment of the pressure force on an element dS of the control surface S_e is $p\mathbf{n} \times \mathbf{r}dS$. This moment of force is zero since $\mathbf{r} = r\mathbf{n}$ and $\mathbf{n} \times \mathbf{n} = 0$. Therefore the total moment of force on the system in R_L is equal to $\mathbf{M}_e$. This moment of force causes the moment of momentum of the system, inclusive of the fluid and the solid, in R_L to change. Thus

$$\mathbf{M}_e = \rho \frac{d}{dt} \iiint_{R_\infty} \mathbf{r} \times \mathbf{v} dR - \rho \frac{d}{dt} \iiint_{R_S} \mathbf{r} \times \mathbf{v} dR + \frac{d}{dt} \iiint_{R_S} \rho_S \mathbf{r} \times \mathbf{v} dR \tag{5.45}$$

The last term in (5.45) is the rate of change of the moment of momentum of the solid and is equal to the total moment of force on the solid, which is the sum of $\mathbf{M}_e$ and the moment of the aerodynamic force $\mathbf{M}$. Equations (5.44) and (5.45) therefore give

$$\mathbf{M} = \frac{\rho}{2} \frac{d}{dt} \iiint_{R_\infty} r^2 \boldsymbol{\omega} dR + \rho \frac{d}{dt} \iiint_{R_S} \mathbf{r} \times \mathbf{v} dR \tag{5.46}$$

5.7 Two-Dimensional Flows

The derivation of the principle of total vorticity conservation for two-dimensional flows uses the following identity obtainable from (2.25):

$$\frac{d}{dt} \iint_{R(t)} f(\mathbf{r}, t) dR = \iint_{R(t)} \left[\frac{\partial f}{\partial t} + (\mathbf{v} \cdot \nabla) f \right] dR \tag{5.47}$$

where R(t) is a two-dimensional material region moving with the fluid.

For two-dimensional flows, one has $\nabla \times (\mathbf{v} \times \boldsymbol{\omega}) = -(\mathbf{v} \cdot \nabla)\boldsymbol{\omega}$. Placing this relation into (5.4), integrating the result over the fluid region R_f', and using (5.47), one obtains:

$$\frac{d}{dt}\iint\limits_{R_f'(t)} \boldsymbol{\omega} dR = -\nu \iint\limits_{R_f'(t)} \nabla \times (\nabla \times \boldsymbol{\omega}) dR \tag{5.48}$$

where $R_f'(t)$ is bounded internally by the solid–fluid interface S and externally by S_e, a boundary outside the region R_ω that contains vorticity. The divergence theorem therefore gives

$$\frac{d}{dt}\iint\limits_{R_f(t)} \boldsymbol{\omega} dR = \nu \oint\limits_S (\nabla \times \boldsymbol{\omega}) \times \mathbf{n} dS \tag{5.49}$$

Integrating the cross product of (5.2) and $\mathbf{n}$ over the solid surface S and noting that $\nu\nabla^2\mathbf{v} = -\nu\nabla \times \boldsymbol{\omega}$ gives:

$$\oint\limits_S \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}\right] \times \mathbf{n} dS = -\frac{1}{\rho}\oint\limits_S \nabla p \times \mathbf{n} dS - \nu \oint\limits_S \nabla \times \boldsymbol{\omega}) \times \mathbf{n} dS \tag{5.50}$$

The first integral on the right-hand side of (5.49) is zero by virtue of the Stoke's theorem and the fact that the curl of the gradient of any function is zero. Thus, (5.48) and (5.50) yield

$$\frac{d}{dt}\iint\limits_{R_f(t)} \boldsymbol{\omega} dR = -\oint\limits_S \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}\right] \times \mathbf{n} dS \tag{5.51}$$

In the rigid solid region R_s, the velocity field is expressible as

$$\mathbf{v}(\mathbf{r}) = \mathbf{v}_a + \boldsymbol{\Omega} \times (\mathbf{r} - \mathbf{r}_a) \tag{5.52}$$

where Ω is the solid's angular velocity and $\mathbf{v}_a$ is its rectilinear velocity at the point $\mathbf{r} = \mathbf{r}_a$.

As discussed earlier, corresponding to (5.52), $\boldsymbol{\omega}(t) = 2\Omega(t)$ in R_s. Since $\boldsymbol{\omega}$ is uniform, $(\mathbf{v} \cdot \nabla)\boldsymbol{\omega} = 0$ in R_s. Furthermore, $(\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{2}\nabla(\mathbf{v} \cdot \mathbf{v}) - \mathbf{v} \times \boldsymbol{\omega}$. For two-dimensional flows, corresponding to (5.52), one has $\nabla \times [(\mathbf{v} \cdot \nabla)\mathbf{v}] = -\nabla (\mathbf{v} \times \boldsymbol{\omega}) = 0$. In consequence, (5.47) and the divergence theorem give, in $R_s(t)$:

$$\frac{d}{dt}\iint\limits_{R_S(t)} \boldsymbol{\omega} dR = -\iint\limits_{R_S(t)} \nabla \times \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}\right] dR \tag{5.53a}$$

$$= -\oint_S \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right] \times \mathbf{n}_S dS \tag{5.53b}$$

In (5.53b), $\mathbf{n}_s$ is the unit normal vector pointing outward from R_s and into R_f. Therefore $\mathbf{n}_s = -\mathbf{n}$, the unit normal vector in (5.51). With the no-penetration and no-slip conditions, $\mathbf{v}$ is continuous across S. Therefore the right-hand side of (5.53b) is equal to the negative of that of (5.51). Adding (5.51) and (5.53b) then gives

$$\frac{d}{dt}\iint_{R_\infty} \boldsymbol{\omega} dR = 0 \tag{5.54}$$

Equation (5.54) states that the total vorticity in R_∞, the infinite unlimited two-dimensional region jointly occupied by the fluid and the solid, is conserved in time.

The two-dimensional flow is often considered a planar flow. Suppose the plane of the flow is the x-y plane in a Cartesian coordinate system (x, y, z) and the z-component of $\mathbf{v}$ is zero (or at most a constant). The vorticity vector is then directed perpendicular to the flow of the plane ($\boldsymbol{\omega} = \omega\mathbf{k}$) and thus $\boldsymbol{\omega}\cdot\mathbf{n} = 0$. Therefore (5.12) and (5.13) are not meaningful. The proof that leads to (5.9), which states that the total vorticity in the infinite unlimited three-dimensional space is not merely conserved, but is zero at all times, is not applicable to the two-dimensional space. For the reference flow, since the solid and the fluid are initially at rest, the total vorticity in R_∞ in the two-dimensional space is initially zero. Because this total vorticity is conserved, it is zero at all subsequent time levels. This conclusion can also be obtained by envisioning vorticity tubes extending perpendicularly from the plane of the flow and these tubes close infinitely far from the plane of the flow. This issue is discussed further in Chap. 6. For the following discussions, the total vorticity in R_∞ is supposed to be zero in the two-dimensional space.

The law of Biot–Savart is expressible for the two-dimensional flow in the form:

$$\mathbf{v}(\mathbf{r})\iint_{R_\infty} \boldsymbol{\omega}_0 \times \nabla f dR_0 \tag{5.55}$$

where f is the fundamental solution of two-dimensional elliptic equation defined by:

$$f = -\ln(1/r')/(2\pi) \tag{5.56}$$

Consider the Taylor's series expansion of f in the $\mathbf{r}_o$-space about the point $\mathbf{r}_o = 0$:

$$f = -\{\ln(1/r) - \mathbf{r}_o\cdot\nabla[\ln(1/r)]\}/(2\pi) + \text{terms of order } r^{-m}, m \geq 2 \tag{5.57}$$

Thus $\mathbf{q} = \nabla f$ can be expressed as

$$\mathbf{q} = \{(\mathbf{r}/r^2) - \nabla[(\mathbf{r}_o \cdot \mathbf{r})/r^2]\}/(2\pi) + \text{terms of order } r^{-n},\ n \geq 3 \tag{5.58}$$

Hence, (5.55) yields,

$$\mathbf{v}(\mathbf{r}) = \frac{1}{2\pi}\iint_{R_\infty} \boldsymbol{\omega}_0 \times \left(\frac{\mathbf{r}}{r^2}\right) dR_o - \frac{1}{2\pi}\iint_{R_\infty} \boldsymbol{\omega}_0 \times \nabla\left(\frac{\mathbf{r}_o \cdot \mathbf{r}}{r^2}\right) dR_o + \text{terms of order } r^{-n},\ n \geq 3 \tag{5.59}$$

In (5.59), the integrations are performed in the $\mathbf{r}_o$-space. Since $\mathbf{r}/r^2$ is a function of $\mathbf{r}$ only, it can be moved outside the first integral. The first term is zero since the total vorticity in R_∞ is zero. The gradient in the second integral is performed in the $\mathbf{r}$-space. The integrand of this integral can be therefore rewritten as $-\nabla \times \{[\mathbf{r}_o \cdot (\mathbf{r}/r^2)]\boldsymbol{\omega}_o\}$. Noting that $-[\mathbf{r}_o \cdot (\mathbf{r}/r^2)]\boldsymbol{\omega}_o = (\mathbf{r}/r^2) \times (\mathbf{r}_o \times \boldsymbol{\omega}_o) - \mathbf{r}_o[(\mathbf{r}/r^2) \cdot \boldsymbol{\omega}_o]$, $\mathbf{r} \cdot \boldsymbol{\omega}_o = 0$ in two-dimensional flows, and $\nabla \times [(\mathbf{r}/r^2) \times (\mathbf{r}_o \times \boldsymbol{\omega}_o)] = \nabla\ [(\mathbf{r}/r^2) \cdot (\mathbf{r}_o \times \boldsymbol{\omega}_o)]$, one obtains

$$\mathbf{v} = -\nabla[\alpha \cdot (\mathbf{r}/r^2)]/(2\pi) + \text{terms of order } r^{-n},\ n \geq 3 \tag{5.60}$$

where $\boldsymbol{\alpha}$ is the total first vorticity moment in the two-dimensional region R_∞ defined by:

$$\alpha = \iint_{R_\infty} \mathbf{r} \times \boldsymbol{\omega} dR \tag{5.61}$$

Noting that $(\nabla \cdot \mathbf{r}) = 2$ in two-dimensional space, not 3, one obtains $\mathbf{v}(\nabla \cdot \mathbf{r}) + (\mathbf{r} \cdot \nabla)\mathbf{v} - \nabla(\mathbf{r} \cdot \mathbf{v}) = \mathbf{v} - \mathbf{r} \times \boldsymbol{\omega}$, and not $\mathbf{v}(\nabla \cdot \mathbf{r}) + (\mathbf{r} \cdot \nabla)\mathbf{v} - \nabla(\mathbf{r} \cdot \mathbf{v}) = 2\mathbf{v} - \mathbf{r}\ \boldsymbol{\omega}$. Thus, one obtains, paralleling the derivations of (5.29) and (5.30), the following equation in place of (5.30),

$$\iint_R \mathbf{v} dR = \iint_R \mathbf{r} \times \boldsymbol{\omega} dR + \oint_{S_e} \mathbf{r} \times (\mathbf{v} \times \mathbf{n}) dS \tag{5.62}$$

Let S_e be a circle of radius r. In a polar coordinate system (r, θ) with $x = r\cos\theta$, $y = r\sin\theta$, and unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$, there is $\boldsymbol{\alpha} \cdot (\mathbf{r}/r^2) = \alpha_r/r = (\alpha_x \cos\theta + \alpha_y \sin\theta)/r$. Thus the first term in (5.60) is $[(\alpha_x \cos\theta + \alpha_y \sin\theta)\mathbf{e}_r + (\alpha_x \sin\theta - \alpha_y \cos\theta)\mathbf{e}_\theta]/(2\pi r^2)$. One obtains, since $\mathbf{r} = r\,\mathbf{e}_r$, $\mathbf{n} = \mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$, and $\mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$:

$$\begin{aligned}\mathbf{r}\times(\mathbf{v}\times\mathbf{n}) &= \mathbf{e}_\theta(\alpha_X\sin\theta-\alpha_Y\cos\theta)/(2\pi r)+\text{terms of order } r^{-n},\ n\geq 2\\ &= (-\sin\theta\,\mathbf{i}+\cos\theta\,\mathbf{j})(\alpha_X\sin\theta-\alpha_Y\cos\theta)/(2\pi r)+\text{terms of order } r^{-n},\ n\geq 2\end{aligned} \tag{5.63}$$

In the limit as $r\longrightarrow\infty$, terms of order r^{-n}, $n\geq 2$ do not contribute to the last integral in (5.62) and one obtains, upon integrating (5.63) over S_e,

$$\oint_{S_e}\mathbf{r}\times(\mathbf{v}\times\mathbf{n})dS=-\alpha/2+\text{terms of order } r^{-n},\ n\geq 1 \tag{5.64}$$

Placing (5.64) into (5.62) yields the relatively simple expression for the total velocity in R_L, the circular region of radius r in the limit $r\longrightarrow\infty$:

$$\iint_{R_L}\mathbf{v}dR=\alpha/2 \tag{5.65}$$

Paralleling the derivation of (5.39), one obtains, in place of (5.39),

$$\mathbf{f}=-\frac{\rho}{2}\frac{d\alpha}{dt}-\oint_{S_e}\rho\mathbf{n}dS+\rho\frac{d}{dt}\iint_{R_S}\mathbf{v}dR \tag{5.66}$$

In (5.66), $\mathbf{f}$ is the aerodynamic force on a two-dimensional solid body. As discussed earlier, the two-dimensional flow is viewed as a planar flow around a long cylinder. In this flow, $\mathbf{f}$ has the dimension of force per unit length (of the cylinder).

The unsteady Bernoulli's equation, (4.20), yields

$$\oint_{S_e}\rho\mathbf{n}dS=-\rho\frac{d}{dt}\oint_{S_e}\left[\frac{\partial\phi}{\partial t}+\frac{\mathbf{v}\cdot\mathbf{v}}{2}-g(t)\right]\mathbf{n}dS \tag{5.67}$$

Bernoulli's equation is valid in the irrotational flow region surrounding R_ω. This region is in general doubly connected in the two-dimensional flow. The total vorticity in R_∞, and hence also in R_ω, however, is zero. Therefore, the cyclic constant for ϕ is zero in the region surrounding R_ω. Therefore, ϕ is single valued on S_e and, according to (5.60), $\phi=-(\alpha_x\cos\theta+\alpha_y\sin\theta)/(2\pi r)$.

Since $\mathbf{n}=\mathbf{e}_r=\cos\theta\,\mathbf{i}+\sin\theta\,\mathbf{j}$, $\phi\mathbf{n}=(\alpha_x\cos\theta+\alpha_y\sin\theta)(\cos\theta\,\mathbf{i}+\sin\theta\,\mathbf{j})/(2\pi r)$ and

$$\oint_{S_e}\phi\mathbf{n}dS=-\alpha/2 \tag{5.68}$$

Placing (5.68) into (5.67) and the result into (5.66) yields, upon noting that, in the limit as $r\longrightarrow\infty$, $\mathbf{v}$ goes to zero as r^{-2} and the only element in (5.67) contributing to the pressure force on S_e is the unsteady potential term $\partial\phi/\partial t$, one obtains

$$\mathbf{f} = -\rho \frac{d}{dt} \iint_{R_\infty} \mathbf{r} \times \boldsymbol{\omega} dR + \rho \frac{d}{dt} \iint_{R_S} \mathbf{v} dR \tag{5.69}$$

The derivation of (5.46) is applicable to both three-dimensional and two-dimensional flows. That is, for two-dimensional applications, one has

$$\mathbf{m} = \frac{\rho}{2} \frac{d}{dt} \iint_{R_\infty} \mathbf{r}^2 \boldsymbol{\omega} dR + \rho \frac{d}{dt} \iint_{R_S} \mathbf{r} \times \mathbf{v} dR \tag{5.70}$$

where $\mathbf{m}$ is the 'two-dimensional' moment of the force $\mathbf{f}$.

5.8 Observations

The vorticity-moment theorem for incompressible viscous aerodynamics is composed of the following three mathematical statements:

$$\iiint_{R_\infty} \boldsymbol{\omega} dR = 0 \tag{5.71}$$

$$\mathbf{F} = -\frac{\rho}{2} \frac{d}{dt} \iiint_{R_\infty} \mathbf{r} \times \boldsymbol{\omega} dR + \rho \frac{d}{dt} \iiint_{R_S} \mathbf{v} dR \tag{5.72}$$

$$\mathbf{M} = \frac{\rho}{2} \frac{d}{dt} \iiint_{R_\infty} \mathbf{r}^2 \boldsymbol{\omega} dR + \rho \frac{d}{dt} \iiint_{R_S} \mathbf{r} \times \mathbf{v} dR \tag{5.73}$$

Each of the three statements involves a total moment of the vorticity field in the infinite unlimited region R_∞ jointly occupied by the fluid and the solid. The first statement (5.71) is that the total zero moment of vorticity, i.e., the total vorticity, in R_∞ is zero. There is no conceivable method to introduce vorticity into any part of R_∞ without simultaneous taking away an equal and opposite amount from other parts of R_∞. For example, if the angular velocity of a solid body with the volume V changes from zero to Ω, then a total vorticity in the amount $\Omega(V/2)$ is introduced into solid region R_s. There must therefore be the same amount of total vorticity removed from the fluid region R_f. Whether an external applied moment of force or an aerodynamics moment causes the solid rotation is immaterial. As long as the angular velocity of the solid boy changes, the total vorticity in the fluid must change by a corresponding amount to maintain the total vorticity in R_∞ zero. The principle of total vorticity conservation, (5.71), is kinematic, but is an indispensable part of the derivations of (5.72) and (5.73). This theorem is derived without using the incompressible continuity equation, (5.1).

The second statement, (5.72), relates the aerodynamic force to the rate of change of the total first moment of vorticity in R_∞. Unlike the total zero moment, the total first moment of vorticity in R_∞ can change with time. Consider the case of a non-rotating solid body moving at a constant velocity. For this case, the last term in (5.72) is zero. Since $\boldsymbol{\omega}$ is zero in R_s, one has

$$\mathbf{F} = -\frac{\rho}{2}\frac{d}{dt}\iiint_{R_f} \mathbf{r} \times \boldsymbol{\omega} dR \tag{5.74}$$

Equation (5.74) states that the aerodynamic force on the solid body in steady motion is directly proportional to the time-rate of change of total first moment of vorticity in the fluid. Without a change of vorticity moment in the fluid, an aerodynamic force cannot exist. Therefore, for the solid body in steady motion, whether or not the aerodynamic force on the body is steady or time-dependent, the total first moment of vorticity must change with time. A steady aerodynamic force does not mean that the total vorticity moment is time-independent in R_∞. It merely means that the total vorticity moment is changing at a constant rate. Steady streamlined flows over wings in forward flight at small angles of attack are known to have large lift-to-drag ratios. This phenomenon is possible only if the rate of change of the component of the total vorticity moment in the flight direction is small compared to that in the lift direction. To reduce drag, ways need to be found to minimize the rate of change of vorticity momentum in the flight direction. Similarly, maximizing the rate of change of vorticity moment in the lift direction leads to increased lift. As it turns out, streamline-shaped wings at small angles of attack provides the proper steady-state parameters for minimizing drag and maximizing lift. These issues are discussed in Chap. 6.

As discussed in Sects. 1.3 and 4.8, external aerodynamic flows are not true steady flows because of the presence of the starting vortex and the trailing vortices. With the starting vortex moving away from the wing, the vorticity moment in the fluid always changes with time. It is important to note that (5.72) and (5.74) are valid only in three-dimensional space. For the two-dimensional flow, there is (5.69), in which the factor of $\frac{1}{2}$ in front of the vorticity-moment integral is absent. This topic is also discussed in Chap. 6.

Equation (5.30) states:

$$\iiint_R \mathbf{v} dR = \frac{1}{2}\iiint_R r \times \boldsymbol{\omega} dR + \frac{1}{2}\oiint_{S_e} r \times (\mathbf{v} \times \mathbf{n}) dS \tag{5.75}$$

Equation (5.75) is a kinematic relationship stating that the total velocity in an arbitrary simply connected finite region R is equal to one half of the total first vorticity-moment in R plus a surface integral over S_e, which bounds R externally. This equation is applicable to the solid region R_s bounded externally by S. In consequence, the inertia force term in (5.72) cannot be simply replaced by the density of the fluid times the rate of change of the total first moment of vorticity in the solid region R_s. The surface integral in (5.75) must also be taken into account. It is simple to see that, letting $R = R_s$ and $S_e = S$ in (5.75) and placing the result into (5.72) gives:

$$\mathbf{F} = -\frac{\rho}{2}\frac{d}{dt}\iiint_{R_f} \mathbf{r} \times \boldsymbol{\omega} dR + \rho\frac{d}{dt}\oiint_{S} \mathbf{r} \times (\mathbf{v} \times \mathbf{n}) dS \tag{5.76}$$

It is noted that for the non-rotating solid body, the total vorticity in R_f is zero. In consequence, the total first moment of vorticity in R_f is independent of the reference point about which the moment is evaluated (see Sect. 7.5). For the rotating solid, however, the total vorticity in R_f is not zero and the first moment of vorticity is dependent on the reference point.

The third statement of the vorticity-moment theorem relates the first moment of the aerodynamics force to the rate of change of the total second moment of vorticity in R_∞. The structure of (5.73) is similar to that of (5.72). The possibility exists that higher moments of the aerodynamic force can be similarly related to total moments of vorticity.

In the two-dimensional region R_∞, suppose there is a small circular region δR of radius δr centered at $\mathbf{r} = 0$ with the vorticity strength ω. The total vorticity in δR is then $\Gamma = \pi\ \omega(\delta r)^2$. According to the law of Biot–Savart, the velocity field in R_∞ corresponding to Γ is $\mathbf{e}_\theta\Gamma/(2\pi r)$. This observation has two consequences. First, the kinetic energy of the fluid per unit volume is $\rho\Gamma^2/(8\pi^2 r^2)$. Since this quantity approaches zero as r^{-2}, the total kinetic energy in R_∞ is indefinite. Suppose in addition to the vorticity in δR at $r = 0$ there is at another location, say $r = a$, $\theta = 0$, an additional total vorticity in the amount $-\Gamma$. For this case, the total vorticity in R_∞ is zero and, according to (5.60), the velocity corresponding to Γ and $-\Gamma$ combined decreases as r^{-2} in the limit as $r \longrightarrow \infty$. In consequence, the total kinetic energy in the circular region is finite in the limit as the radius of the circle approaches infinity. Therefore, for the total energy in two-dimensional flows to be finite in R_∞, the total vorticity in R_∞ must be zero.

Second, corresponding to the elemental vorticity $d\Gamma = \omega(\mathbf{r} = 0)dR$, there is $\mathbf{v}(r, \theta + \pi) = -\mathbf{v}(r, \theta)$ and hence the total velocity in R_∞ is zero. Since the choice of the point of origin in R_∞ is arbitrary, the total velocity in R_∞ corresponding to every elemental vorticity in R_ω is zero. In consequence, the total velocity in R_∞ corresponding to all distributions of vorticity is zero. As shown by (5.65), the total velocity in R_L, the circle of radius r, is not zero for all finite values of r. This holds true as long as r remains finite. In other words, in the evaluation of the total velocity, the idea that the finite value of r is allowed to approach infinity ($r \longrightarrow \infty$) in the limit cannot be simply replaced by the idea that r is infinite ($r = \infty$). The limiting process is used here as a strategy to connect the kinematics results, specifically (5.65), to the kinetics of the external flow problem. This process differs from more familiar situations where the limiting process is used in analyses, for example, to define a derived variable or a ratio of two variables. This conclusion is applicable to three-dimensional flows where r is the radius of a sphere. It demonstrates the central role of the boundary integrals in the vorticity-moment theorem of aerodynamics.

The study of the asymptotic behavior of the velocity field and its potential function at larges distances from the solid, described in Sect. 5.3, enables the evaluation of the two boundary integrals, one of $\mathbf{r} \times (\mathbf{v} \times \mathbf{n})$ and the other of $(p/\rho)\,\mathbf{n}$, over S_e, which is outside the vertical region R_ω. It is worth noting that the contributions of the two integrals to the aerodynamic force cancel each other because, as it turns out,

$$\frac{\rho}{2}\frac{d}{dt}\oiint_{S_e} \mathbf{r} \times (\mathbf{v} \times \mathbf{n})dS = -\oiint_{S_e} \rho\mathbf{n}dS \tag{5.77}$$

It is worth noting that (5.77) is obtained indirectly by separate studies of vorticity kinematics and vorticity kinetics and combining the results. At the present, a more direct proof of this relationship is not established. Also, an equation similar to (5.72), but with the integral covering only a part (not the entirety) of the region R_ω, where vorticity is non-zero, is presently not available. The use of the vorticity-moment theorem is currently in an opening stage of research and development. This theorem at its present stage nevertheless offers opportunities for the development of an understanding of the principles of aerodynamics beyond the current confines of the steady streamlined flow.

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Chapter 6
Classical Aerodynamics

6.1 Vorticity-Moment Theorems

The vorticity-moment theorem contains the following mathematical statements:

$$\iiint_{R_\infty} \omega dR = 0 \tag{6.1}$$

$$\mathbf{F} = -\frac{\rho}{2}\frac{d}{dt}\iiint_{R_\infty} \mathbf{r}\mathbf{x}\omega dR + \rho\frac{d}{dt}\iiint_{R_S} \mathbf{v}dR, \tag{6.2}$$

where $\mathbf{F}$ is the aerodynamic force exerted by the fluid on the solid immersed in and moving relative to the fluid, R_s is the region occupied by the solid, R_∞ is the infinite unlimited space occupied jointly by the fluid and the solid, $\mathbf{v}$ is the velocity field, $\boldsymbol{\omega}$ is the vorticity field, ρ is the density of the fluid, and $\mathbf{r}$ is the position vector.

Equation (6.1) states that the total vorticity in R_∞ is always zero. Equation (6.2) relates the aerodynamic force exerted by the fluid on the solid to the rate of change of the total first moment of vorticity in R_∞. A third statement that relates the moment of the aerodynamic force to the rate of change of the second moment of vorticity is presented in Chap. 5. The present and the next chapters discuss applications of (6.1) and (6.2) only.

Equations (6.1) and (6.2) can be restated in forms that involve integrations over only the fluid region R_f and its boundary S, and not over R_s. Specifically, one has

$$\iiint_{R_f} \omega dR = -2\boldsymbol{\Omega}V \tag{6.3}$$

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$$\mathbf{F} = -\frac{\rho}{2}\frac{d}{dt}\iiint_{R_f} \mathbf{r} \times \omega dR + \frac{1}{2}\rho\frac{d}{dt}\oiint_{S} \mathbf{r} \times (\mathbf{v} \times n)dS, \tag{6.4}$$

where $\mathbf{\Omega}$ is the angular velocity and V is the volume of the solid. In the special case of a non-rotating solid, the right-hand side of (6.3) is zero. The total vorticity in the fluid is always zero in this case. The form of (6.4) appears convenient in applications involving deformable or flexible solid bodies. For the nonrotating solid, one has

$$\oiint_{S} \mathbf{r} \times (\mathbf{v} \times n)dS = -2\iiint_{R_S} \mathbf{v} dR \tag{6.5}$$

and therefore the last term in (6.4) gives the inertia force of the fluid replaced by the solid. For the rotating solid, because the total vorticity in R_f is not zero, the total first moment of vorticity in R_f is dependent on the reference point about which the moment is evaluated. As a result, the integrals in (6.4) are not invariant to a Galilean transformation. That is, in evaluating the two integrals in (6.4), the point of origin of the coordinate system must be traced continually and must be incorporated in aerodynamic analyses.

With (6.2), the aerodynamic force is determinate once the motion of the solid is prescribed and the rate of change of $\mathbf{r} \times \boldsymbol{\omega}$ in R_∞ is known. The quantitative evaluation of the rate of change of $\mathbf{r} \times \boldsymbol{\omega}$ requires the solution of the non-linear vorticity transport equation. Classical theories of aerodynamics avoid this exceedingly difficult task by using simple flow models. The vorticity-momentum theorem offers unconventional interpretations of such models, as described in the remainder of this Chapter.

6.2 Kutta–Joukowski's Theorem and Vortical Flow Zones

For two-dimensional flows, there are the following vorticity-moment equations:

$$\iint_{R_\infty} \boldsymbol{\omega} dR = 0 \tag{6.6}$$

$$\mathbf{f} = -\rho\frac{d}{dt}\iint_{R_\infty} \mathbf{r} \times \omega dR + \rho\frac{d}{dt}\iint_{R_S} \mathbf{v} dR \tag{6.7}$$

As discussed earlier, the two-dimensional flow is viewed as an approximation of the flow near the mid-span of a long right cylindrical solid body moving perpendicular to its axis. In this flow, the velocity vector lies in planes perpendicular to the cylinder's axis and has a zero gradient in the axial direction. The term $\mathbf{f}$ in (6.7) is the force per unit span.

In the present section, a steady two-dimensional flow is examined. A Cartesian coordinate system (x, y, z) with the unit vectors **i**, **j**, and **k** in the x-, y-, and z-directions, respectively, and a reference frame attached to and moving with the solid is used. The plane of the flow is the x-y plane. The freestream velocity, i.e., the velocity of the fluid undisturbed by the cylinder's motion, is designated $U_\infty \mathbf{i}$ and the lift is in the y-direction. The z-direction is then the axial (span) direction of the cylinder. Thus, the velocity field **v** has a zero z-component and the vorticity field **ω** points in the z-direction.

The Kutta–Joukowski theoremfor the steady two-dimensional flow states (Kuethe and Schetzer 1959): *The force per unit length acting on a right cylinder of any cross section whatever is equal to* ρVΓ *and acts perpendicular to* **V**. In this statement, Γ is circulation, often call the bound vortex strength, around the cylinder and **V** is the freestream velocity with magnitude V. The theorem is expressible as simple formulas:

$$\mathbf{f} = \rho \boldsymbol{\Gamma} \times \mathbf{V} \tag{6.8a}$$

$$= \rho U_\infty \Gamma \mathbf{j} \tag{6.8b}$$

The circulation Γ of a vortex filament (see Chap. 2) is the total flux of vorticity in the vorticity tube that the filament represents. The circulation, therefore, has a direction associated with the vorticity in the tube. With the vortex filament representing the tube, the direction of the circulation is simply the direction of the filament. In two-dimensional flows, the vorticity **ω** is in the z-direction. In (6.8a), $\boldsymbol{\Gamma} = \Gamma \mathbf{k}$ is the circulation vector around the airfoil and is directed along the infinite span. Placing $\mathbf{V} = U_\infty \mathbf{i}$ in (6.8a) yields (6.8b) immediately.

A century ago, the Kutta–Joukowski theorem was a revolutionary development that allowed the quantitative evaluation of the lift on the airfoil. This theorem served, since its discovery, as the foundation and the core of classical aerodynamics. There are several ways to derive this theorem. For example, consider a solid circular cylinder of radius a centered at the point $\mathbf{r} = 0$ in the x-y plane. Consider a polar coordinate system (r, θ) with r and θ related to the Cartesian coordinates x and y by $x = r\cos\theta$ and $y = r\sin\theta$. It is simple to show that the velocity field **v** defined by (6.9) below, where $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are unit vectors in the r- and θ-directions, satisfies $\nabla \cdot \mathbf{v} = 0$ and $\nabla \times \mathbf{v} = 0$ in the region $a < r < \infty$:

$$\mathbf{v} = U_\infty (1 - a^2/r^2) \cos\theta\, \mathbf{e}_r - [U_\infty \left(1 + a^2/r^2\right) \sin\theta + \Gamma/(2\pi r)]\mathbf{e}_\theta \tag{6.9}$$

In the limit as $r \to \infty$, this field gives the freestream velocity $U_\infty \mathbf{i}$. At $r = a$, one has

$$\mathbf{v} = -[2U_\infty \sin\theta + \Gamma/(2\pi a)]\mathbf{e}_\theta \tag{6.10}$$

Accordingly, (6.9) describes a steady incompressible flow around the circular cylinder that satisfied the no-penetration condition $\mathbf{v} \cdot \mathbf{n} = 0$ on the surface r = a. This flow is irrotational and potential. Thus, Bernoulli's equation (4.20), is applicable.

In the flow region $a < r < \infty$, the scalar potential function, ϕ, in the Bernoulli equation is multi-valued. For the steady flow, one assumes $\partial\phi/\partial t = 0$ and writes,

$$p = -\rho(\mathbf{v} \cdot \mathbf{v})/2 + C \tag{6.11}$$

where C is a constant.

Consider a circle in the plane of the flow with the radius $b > a$ centered at the point $\mathbf{r} = 0$. Since viscous force is absent, the force $\mathbf{f}_b$ acting on the circle is equal to the integration of $-p\mathbf{n}$, $\mathbf{n}$ being a unit normal vector in the increasing r direction, over the circle. One therefore obtains, noting that $\mathbf{n} = \mathbf{e}_r = \cos\theta\, \mathbf{i} + \sin\theta\, \mathbf{j}$:

$$\mathbf{f}_b = b \int_0^{2\pi} \left(\frac{\rho \mathbf{v} \cdot \mathbf{v}}{2} - C \right) \mathbf{n} d\theta \tag{6.12a}$$

$$= (1/2)\rho U_\infty \Gamma (1 + a^2/b^2)\mathbf{j} \tag{6.12b}$$

In the limit as $b \to a$, this force becomes $\rho\, U_\infty \Gamma\, \mathbf{j}$, which agrees with (6.8b). With (6.12b), the implicit assumption is that there is zero viscous force on the surface r = a and in the region $a < r < \infty$. There is also the assumption that the flow is streamlined: The flow on all points of the circle is tangent to the cylinder, as shown by (6.10).

Kuethe and Schetzer (1959) considered source–sink pairs to be determining the shape of the cylinder and pointed out: *viewed from infinity, any source-sink pair appears to be a doublet* and *the doublet portion of the flow pattern makes no contribution to either the momentum or the force when* r *becomes large*. They concluded on this basis that "*the shape of the cylinder cannot influence the resultant force*". This conclusion can also be reached using conformal transformation, again assuming a streamlined flow with no viscous force. It is of interest to note that for all values of b, including the limit case $b \to \infty$, (6.12b) gives a non-zero pressure force on the surface r = b. It can be shown that the total flux of momentum of the fluid leaving the boundary r = b is $\frac{1}{2}\rho U_\infty \Gamma(1 - a^2/b^2)\mathbf{j}$. This momentum flux is also non-zero for all values of b, including the limit case $b \to \infty$.

The radius a of the cylinder does not appear in (6.8) and need not be specific. With the Kutta–Joukowski theorem, only three numbers are needed to determine the lift: the density ρ, the freestream velocity U_∞, and the circulation Γ. In the following paragraphs, this remarkable theorem of aerodynamics is interpreted using the vorticity-moment theory without assuming the flow is streamlined or inviscid.

For the nonrotating cylinder in steady rectilinear motion, the last term in (6.7) vanishes and the region of the first integral reduces to the fluid region R_f.

To facilitate discussions, the region of non-zero vorticity surrounding and trailing the cylinder is considered to be composed of three zones: the ambient zone, the near wake zone, and the far (or ultimate) wake zone. These three zones are denoted R_b, R_n, and R_u, respectively. R_b is a fixed zone attached to and enclosing the cylinder. R_n expands as time progresses and R_u moves continually downstream, away from the cylinder. The precise lines of demarcation separating the zones are unimportant. The zones are designated merely for convenience in the present discussions.

Since $\boldsymbol{\omega} = \omega\mathbf{k}$, one has $-\mathbf{r} \times \boldsymbol{\omega} = -y\omega\mathbf{i} + x\omega\mathbf{j}$. According to (6.7), the lift, being in the *y*-direction, is related to the change of $x\omega$ in the flow. The drag is related to the change of $-y\omega$. Suppose for the moment the cylinder is an airfoil and the angle of attack is sufficiently small so that the boundary layers surrounding the airfoil remain attached all the way to the trailing edge of the airfoil. In other word, the flow around the airfoil is a streamlined viscous flow. This is, of course, an approximation since in real flows boundary layers are not fully attached. In this streamlined flow, there exist two boundary layers, one on the upper surface of the airfoil and the other on the lower surface. In the reference frame attached to the airfoil, the vorticity in the boundary layers is time-independent. The vorticity in the upper boundary layer has a sense opposite to that in the lower layer. The total, integrated, vorticity value in the two layers is $\Gamma\mathbf{k}$ and there is a total vorticity moment in R_b. In an airfoil-attached reference frame, this vorticity moment does not change with time and thus does not contribute to (6.7).

The time independence of the vorticity in the boundary layers does not mean that this vorticity is “bound” to the airfoil. In reality, the vorticity in the boundary layers is continually introduced at the airfoil surface. Once introduced, the vorticity enters the fluid domain by diffusive transport and, once there, is transported downstream by convection while continually diffuse. Because these processes balance each other, an observer attached to the airfoil sees no change of vorticity in the boundary layers with time. The vorticity in both boundary layers is, in reality, transported downstream along the airfoil's surface. The two boundary layers merge near the airfoil's trailing edge to form a thin wake trailing the airfoil. The boundary-layer flow, carrying vorticity with it, leaves R_b and forms the near-wake flow in R_n. If the combined rate of vorticity fed into the R_n is not zero, then the total vorticity in R_b must changes with time and the flow in R_b is unsteady. In other words, the total rate at which the vorticity is fed into R_n by the two steady boundary layers in R_b must be zero. Since vorticity can neither be created nor destroyed in the interior of the fluid domain, the total vorticity in R_n is zero.

The near wake is composed of two sub-layers of vorticity. These sub-layers are continuations of the two boundary layers in R_b. For reasons stated in the preceding paragraph, the vorticity strength of the upper sub-layer, i.e., the integrated vorticity value across this sub-layer, is equal and opposite to that of the lower sub-layer. If the wake is thin, then the total integrated vorticity moment across the wake is small. The lift is therefore due primarily to the change of the vorticity moment associated with the far wake in R_u.

Consider the airfoil to be initially at rest in the fluid also initially at rest. The airfoil is set into motion and thereafter kept moving at a constant velocity. The vorticity in the far wake is the vorticity that left the region R_b shortly after the motion's onset. This vorticity is called the starting vortex. At the time level when the flow in R_b reaches steady state, the starting vortex is in R_u, far from the airfoil. At this *steady-state time level*, the starting vortex is moving away from the airfoil at approximately the freestream velocity $U_\infty\mathbf{i}$. Because $\nabla \times \boldsymbol{\omega}$ is small in this region (see Chap. 4), diffusive transport of vorticity in R_u is negligible. The predominant process of transport of vorticity in R_u is convection. As discussed, the total vorticity in R_n is zero. The total vorticity in R_b is the circulation around the region R_b. If this circulation is $\Gamma\mathbf{k}$, then, because of (6.6), the total vorticity in R_u, i.e., the circulation of the starting vorticity, is $-\Gamma\mathbf{k}$.

Assuming the velocity in R_u is the freestream velocity $U_\infty\mathbf{i}$, the rate of change of the total x-component of vorticity moment in R_u, as observed in the reference frame moving with the airfoil, is $-\Gamma U_\infty\mathbf{j}$. In consequence, since the vorticity in R_b and R_n does not contribute, (6.7) yields the lift $\rho U_\infty\Gamma\mathbf{j}$, as predicted by the Kutta–Joukowski theorem (6.8b).

In the above discussions, a reference frame attached to and moving with the airfoil is used. In a stationary reference frame, R_u is essentially stationary and the region R_b is moving at the velocity $-U_\infty\mathbf{i}$. With the above-described simplifications, (6.7) again gives $\rho U_\infty\Gamma\mathbf{j}$ as the lift. Thus, the Kutta–Joukowski theorem is interpretable as a special case, that of a two-dimensional steady flow producing a lift, of the vorticity-moment theorem.

As shown in Chap. 5, the vorticity-moment theorem is a rigorous mathematical consequence of the incompressible continuity and Navier–Stokes equations. This theorem is not restricted to streamlined inviscid flows. The only assumptions contained in (6.6) and (6.7) are those associated with the continuity and Navier–Stokes equations. The re-derivation of (6.8) based the vorticity-moment theorem involves only the additional assumption that the starting vortex is moving at the freestream velocity. In two-dimensional flows, with the starting vortex located in R_u, which is far from R_b, the freestream velocity is obviously a reasonable approximation of the actual velocity of the starting vortex. The power of Kutta–Joukowski's theorem to predict lift accurately is well-known for streamlined flows. Earlier experiments, however, did not provide clear evidences that this theorem is also applicable to the non-streamlined flows. The present re-derivation of (6.8) shows conclusively that (6.8) is applicable to non-streamlined flows.

6.3 Non-Streamlined Flow and Karman Vortex Street

Suppose the angle of attack of the airfoil is large and the upper boundary layer separates. In this situation, a recirculating flow region is present and a part of the vorticity in this region is transported downstream along with the vorticity from the boundary layers. For convenience, the re-circulating flow is considered to be in the ambient region R_b. Suppose the flow in R_b and R_n remains steady, except that R_n is growing far downstream. Much of the discussions of Sect. 6.2 related to the vorticity moment then remain applicable. However, with the re-circulating flow feeding vorticity into the near wake, the near wake is relatively thick. The vorticity moment in the near wake, like that in the streamlined flow, is primarily in the x-direction ($y\omega\mathbf{i}$). The change of this vorticity moment is, according to (6.7), related to a drag. Since the positive vorticity and the negative vorticity are not spaced closely together in the y-direction in the non-streamlined flow, the change of the total x-component of the vorticity moment, and hence the drag, is in general much larger in the non-streamlined flow than that in the streamlined flow. This is of course a well-recognized reality with decisive implications in aeronautics. If the flow in R_b is steady, then the net rate of vorticity transported into R_n, as discussed in Sect. 6.2, is zero. Since the vorticity in R_n moves with approximately the freestream velocity, the vorticity in R_n does not contribute to lift substantially. The growth of R_n, however, contributes substantively to the drag. In short, in the non-streamlined flows, one expects the Kutta–Joukowski theorem to predict the lift with reasonable accuracy, but the zero drag prediction of (6.8b) is expected to be off the mark.

For the streamlined flow, the circulation Γ represents the total vorticity in the boundary layers. For the non-streamlined flow, Γ represents the total vorticity in the boundary layers and the re-circulating region. As long as the total circulation in the region R_b is known with reasonable precision, the circulation in R_u is $-\Gamma$ and the lift on the body is reasonably accurately predicted by $\rho U_\infty \Gamma \mathbf{j}$. The circulation for the non-streamlined flow is in general substantially different from that for an assumed streamlined flow. Furthermore, for the streamlined flow, the Kutta condition provides a convenient way to determine Γ accurately. For the non-streamline flow, there is no theoretical (non-computational) method available for the evaluation of Γ.

For the streamlined flow, viscous diffusion spreads the vorticity in the boundary layers in the y-direction. This vorticity is continually fed into R_n. There is, therefore, a non-zero x-component of vorticity moment (ywi) continually entering R_n and a related drag. This drag, known as the profile drag, is small because the wake width is small. In the absence of viscous diffusion, the vorticity in the two boundary layers appears as vortex sheets, i.e., vorticity layers of finite strength and infinitesimal thickness attached to the airfoil surface. If the two boundary vortex sheets merge at the trailing edge to form a wake vortex sheet, then the wake sheet has zero strength because the strengths of the two boundary sheets are equal and opposite. Therefore, ywi = 0 and the profile drag is zero. The origin of the profile drag, as is well known, is the viscosity of the fluid. Without viscous diffusion, the idealized streamlined flow has no profile drag. For the non-streamlined flow, the profile drag is more than the integrated skin-friction force on the airfoil. Flow separation alters the pressure distribution on the airfoil, giving rise to a pressure drag which is often called the form drag. As discussed earlier, according to (6.7), the profile drag is related to the change of the x-component of vorticity moment in the near wake. Obviously, to minimize the profile drag, the first and most crucial step is to ensure that the flow is streamlined. This step curtails the form drag. Truly streamlined flows are not achievable in the real flow of a fluid. Minimizing the wake width diminishing the x-component of the vorticity moment in the wake. This reduces the friction drag and is important in airplane design.

In the above discussions, the flow in R_b is assumed to be steady. In the boundary layers, the vorticity is continually replenishes from the solid surface and carried downstream by convection. Viscous diffusion serves as a damping mechanism so that large amplitude perturbations from time-averaged vorticity field (and the corresponding velocity field) do not persist. The solid surface and the damping provide a viable steady flow environment in the streamlined flow. In the re-circulating flow region and the wake regions, the stabilizing influence of the solid boundary is absent. For the vorticity field to be steady, the diffusion process and the convection process must be in balance. If a net amount of vorticity is transported away from a fixed spatial element by convection, the same net amount must be brought into this element by diffusion. This requirement is difficult to meet and deviations from time-averaged vorticity typically amplify. In very low Reynolds number flows, viscous diffusion dominates and these deviations are damped out under some, though not all, circumstances. Reynolds numbers of flows of general interest in aeronautics, however, are typically so high that the flow in the re-circulation region is convection-dominated. Viscous damping is ineffective and deviations from time-averaged vorticity values often grow into oscillatory events such as Karman vortex streets downstream of circular cylinders (Karman and Burgers 1934). Vortex streets are observed not only in viscous flows past circular cylinder. They are typically present in flows past bluff (non-streamlined) bodies and airfoils at high angles of attack. With the shedding of the vortex street, the flow in R_b undergoes large amplitude oscillations. Vorticity is transported, or shed, into R_n in distinctive clusters. The sense of rotation of the vortices alternates. The shedding of a positive vortex is followed by the shedding of a negative vortex, which is in turn followed by the shedding of a third positive vortex, etc. After the shedding of a pair of vortices, the flow in R_b typically returns to nearly its initial state. The shedding of the next pairs of vorticity clusters then commences. The time-averaged load on the solid body, both the lift and the drag, is then periodical and interpretable using (6.7).

The total vorticity in each pair of vortices must be zero, or nearly zero, for otherwise, to satisfy (6.6), the total vorticity in R_b changes from one cycle of shedding to another; and this change means the flow in R_b is fully unsteady, not periodic. Denote the period of oscillation, i.e., a cycle off shedding of a pair of vortices, by τ, the total vorticity (the circulation) in one of the vortices by Γ_k, and the distance in the y-direction of the second vortex from the first by λ. The total x-component of vorticity moment shed from R_b during one period of oscillation is then $\lambda\Gamma_k$. Thus, $\lambda\Gamma_k/\tau$ gives the rate at which the x-component of the vorticity moment is added to the near wake. This means $\rho\lambda\Gamma_k/\tau$ is an indication of the average profile drag. Since the total circulation of the pair of vortices is zero, the near wake contributes only to unsteady lift. The time-averaged lift, if it exists, is associated with the circulation and movement of the starting vortex. Karman and Burgers (1934) studied vortex streets using the classical approach and presented a drag formula that they indicated *checks fairly well with measurements*.

Lighthill (1975) made the following remark about fish locomotion: *The famous Karman vortex street can work in reverse: it can, with the sense of rotation of all the vortices altered, be responsible for thrust instead of drag*. From the vorticity-moment viewpoint, the sign of the total x-component of vorticity moment shed from R_b during one period of oscillation is reversed. A fish moving at a more or less constant speed casts off pairs of vortices and $-\rho\lambda\Gamma_k/\tau$ is an indication of the time-averaged thrust. It should be noted that a fish, in its forward motion, must encounter a drag. The time-averaged thrust must be equal and opposite the time-average drag so as to maintain the constant forward speed of the fish. There is little doubt that the vortex street left behind by the fish is related to the thrust. The fish, however, must leave behind also a vortical wake related to the drag, a wake that is less prominently displayed compared with the vortex street, but is no less important in studies of the fish motion. It is worth underscoring that animal locomotion generally involves time-dependent flows. The vorticity-moment theorem, with its time-dependent format, offers a viable alternative to classical flow analyses such as the one employed by Karman and Burgers (1934).

It is of interest to note that, in the absence of stabilizing influences of the solid boundary and viscous damping, tubes and layers of vorticity directed perpendicular to the flow, in which the Lamb vector $\mathbf{v} \times \boldsymbol{\omega}$ is not small, typically do not persist long. For example, wake layers in R_n in two-dimensional streamlined flow contain vorticity in the z-direction and the velocity is essential in the x-direction. These layers typically break into small segments that roll up to form a sequence of counter-rotating pairs of vortices. These vortices bear similarity to the Karman vortex street. The total vorticity of each vortex is quite small. The vortices are nevertheless related to the drag on the solid.

6.4 Vorticity Loop and Vorticity Moment

As shown in Chap. 2, the vorticity field in the infinite unlimited region R_∞ jointly occupied by the fluid and the solids can be portrayed by a set of closed tubes (loops) of vorticity. For the case the solids are not rotating, the vorticity field is zero in the solid and all vorticity loops are in R_f. The walls of the vorticity tubes are composed of vorticity lines, i.e., lines whose tangent at each point is in the direction of the vorticity vector at that point. The strength of each vorticity tube (the integrated value of the normal component of the vorticity vector over a cross-sectional area of the tube) is the circulation Γ around the tube. This circulation is the same at all sections of the tube.

Consider a vorticty field in R_∞ (or R_f in the case of nonrotating solids) represented by a set of vorticity loops, each with a very small cross-sectional area. Approximate the path of a specific loop in the set by a circuit (closed curve) C in space and describe the circulation of the tube by the vector $\boldsymbol{\Gamma} = \Gamma\boldsymbol{\tau}$, where Γ is a constant and $\boldsymbol{\tau}$ is the unit tangent vector of the path C in the direction of $\boldsymbol{\omega}$ in the tube. The total vorticity in a segment of a vorticity loop of length ds is $\boldsymbol{\omega}dR = \Gamma\boldsymbol{\tau}ds$

and the first moment of vorticity of the segment is $\Gamma\mathbf{r} \times \boldsymbol{\tau}ds$. If the vorticity loop lies in the x-y plane where $z = z_1$, a constant, then $\boldsymbol{\tau}ds = dx\mathbf{i} + dy\mathbf{j}$ and $\mathbf{r} \times \boldsymbol{\tau}ds = -z_1 dy\mathbf{i} + z_1 dx\mathbf{j} + (xdy - ydx)\mathbf{k}$. The integration of $\mathbf{r} \times \boldsymbol{\omega}$ over the region R_l occupied by this loop of vorticity is approximately:

$$\iint_{R_l} \mathbf{r} \times \omega dR \approx \Gamma\left[-\mathbf{i}z_1 \oint_C dy + \mathbf{j}z_1 \oint_C dx + \mathbf{k}\oint_C (xdy - ydx)\right] \tag{6.13}$$

The first two integrals on the right-hand side of (6.13) are zero. The last integral can be shown, using Green's theorem, to be 2A, where A is the area enclosed by C. Hence, the moment of the vorticity in the vorticity loop is approximately $2A\Gamma\mathbf{k}$. More generally, the vorticity moment $\boldsymbol{\Lambda} = \Lambda_x\mathbf{i} + \Lambda_y\mathbf{j} + \Lambda_z\mathbf{k}$ of a planar loop of vorticity with the circulation Γ is approximately $2A\Gamma\mathbf{n}$, where $\mathbf{n}$ is the unit vector normal to the plane in which vorticity loop lies. The direction of $\mathbf{n}$ is determined by $\boldsymbol{\tau}$ (which gives a direction to the path of C) following the rule of the right-handed screw. The approximation becomes precise in the limit as the cross-sectional area of the closed tube approaches zero.

It is convenient to designate a plane in space by the direction of its normal. For example, an x-y ($z = z_1$) plane is designated as the z_1-plane. More generally, a planar area with the unit normal vector $\mathbf{n}$ is described by the vector $\mathbf{A} = A\mathbf{n} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$, where $A_x = An_x$, $A_y = An_y$, and $A_z = An_z$, n_x, n_y, and n_z being, respectively, the direction cosines of $\mathbf{n}$ in the x-, y- and z-directions. The vorticity moment $\boldsymbol{\Lambda}$ is therefore given approximately by $2\mathbf{A}\Gamma = 2\Gamma A_x\mathbf{i} + 2\Gamma A_y\mathbf{j} + 2\Gamma A_z\mathbf{k}$. In other words, in evaluating $\boldsymbol{\Lambda}$, it is permissible to decompose the vorticity loop into component-loops, each with the circulation Γ. The vorticity moment Λ_x of the component-loop enclosing the area $A_x\mathbf{i}$ is approximately $2\Gamma A_x$. Similarly, $\Lambda_y \approx 2\Gamma A_y$ and $\Lambda_z \approx 2\Gamma A_z$. Since a nonplanar circuit C is divisible into a set of very small (Chap. 2) and approximately planar circuits and the sum of the vorticity moments of these loops is approximately $\boldsymbol{\Lambda}$, one has for all loops:

$$\iiint_{R_l} \mathbf{r} \times \omega dR \approx 2\Gamma \iint_A \mathbf{n}dA \tag{6.14}$$

Batchelor (1967) presented a concise derivation of (6.14), which is approximate because the region R_l is represented by a closed tube with an infinitesimal cross-sectional area, i.e., the closed vorticity tube is represented by a vortex-filament loop. This relationship between the first vorticity moment of a vorticity tube and its circulation is kinematic.

There are an infinite number of caps for each specific C in space. For convenience, the integral on the right-hand side of (6.14) is called the vector area (of the cap) of the circuit C and denoted $\mathbf{A}$. According to the divergence theorem, the integration of $\mathbf{n}$ is zero over all closed and simply connection surfaces in space. Consider two caps A_1

and A_2 of C. A_1 and A_2 together form a closed surface that encloses a region R. The normal vector **n** on each of the two surfaces is determined by the right-handed screw convention in accordance with the direction of **τ** on the circuit C. For (6.14), **τ** is in the direction of **ω** in the vorticity tube (vortex filament). If **n** on A_1 points outward from R, then **n** on A_2 points inward into R. The converse is true. In consequence, while the scalar area of a cap depends on the selection of a specific cap, **A** is independent of the cap selection. Wu (2001) provided an alternative proof of this conclusion by considering a conical shaped cap with its apex at the point $\mathbf{r} = 0$. For this cap, the magnitude of the vector $\mathbf{r} \times \boldsymbol{\tau}ds$ is the area of the elemental parallelogram with sides **r** and $\boldsymbol{\tau}ds$. This area is twice the area of the triangle on the cap with its vertex at $\mathbf{r} = 0$ opposite the side ds. The direction of $\mathbf{r} \times \boldsymbol{\tau}ds$ is normal to the cap. In consequence, if the vectors **r**, **τ**, and **n** form a right-handed set, then the integration of $\mathbf{r} \times \boldsymbol{\tau}$ over C gives 2**A** and one obtains (6.14).

As shown in Sect. 2.5, a vorticity loop is divisible into a set of thinner loops by partitioning the cross-sectional area of the loop into smaller areas. The vector area **A** of the circuit C (not the cross-sectional area of the tube) is divisible into a set of smaller areas each enclosed by a smaller circuit. In the limit as the enclosed areas approach zero, the loop is represented precisely by a set of infinitesimal planar loops. Thus, for the purpose of evaluating the force **F** using (6.2), the vorticity field can be approximated by a large set of tiny planar vorticity loops. This offers a circulation-based method for discretization of the vorticity field. With each loop decomposable into its directional components, this method offers remarkable flexibility in aerodynamic computations.

6.5 Lifting-Line Theory: Uniform Circulation Lift

In the following discussions, the most popular conventions describing the flow past a finite wing are adopted. In a reference frame attached to and moving with the wing, the freestream (fluid motion undisturbed by the motion of the wing) velocity is in the x-direction and is designated $\mathbf{V} = U_\infty \mathbf{i}$. Thus in a stationary reference frame (a frame at rest relative to the freestream), the wing is moving at the velocity $-U_\infty \mathbf{i}$. The lift is in the z-direction. The y-axis is in the span direction of the wing. In the description of the two-dimensional flow in Sect. 6.2, the lift is in the y-direction and the span in the z-direction. Equation (6.8a) becomes $\mathbf{f} = -\rho \boldsymbol{\Gamma} \times \mathbf{V}$ with $\boldsymbol{\Gamma} = \Gamma \mathbf{j}$. The negative sign is introduced in the expression in order to retain the right-handedness of the coordinate system x-y-z.

Prandtl (1921) modeled the vorticity field in the steady incompressible flow around the wing by a system of horseshoe-shaped vortices. This system is composed of a lifting line representing the circulation $\Gamma(y)\mathbf{j}$ around the wing and a vortex sheet trailing the lifting line. Prandtl observed that the trailing vortex sheet induces a downwash, i.e., a velocity component w that is in the negative z-direction, at the lifting-line location. Prandtl then replaced the freestream velocity $\mathbf{V} = U_\infty \mathbf{i}$ by $\mathbf{V} = U_\infty \mathbf{i} + w\mathbf{k}$ and obtained, in place of the Kutta–Joukowski theorem (6.8b), the following relation:

$$\mathbf{f}(y) = -\rho w(y)\Gamma(y)\mathbf{i} + \rho U_\infty \Gamma(y)\mathbf{k} \tag{6.15}$$

The first term of the right-hand side of (6.15) is a force per unit span in x-direction, a drag. Note that w is a downwash and therefore has a negative value. The last term in (6.15) is the lift per unit span on the wing. In obtaining (6.15) from (6.8a), two assumptions are used. First, the Kutta–Joukowski theorem, which is proved for the two-dimensional flow only, is assumed applicable to all sections of the wing, each with its own circulation $\Gamma(y)$, in the evaluation of the aerodynamic force in the three-dimensional flow. Second, the downwash corresponding to the trailing vortex sheet is assumed to modify the freestream velocity as seen by the observer riding on the wing. In this and the next two Sections, (6.15) is re-derived on the basis of the vorticity-moment theorem using neither the two-dimensional theorem of Kutta–Joukowski nor the idea that the downwash alters the freestream.

Let the lifting line lie on the y-axis in the range $-b/2 \leq y \leq b/2$. Equation (6.15) yields the following expressions for the lift L on the wing:

$$L = \rho U_\infty \int_{-b/2}^{b/2} \Gamma(y)dy \tag{6.16}$$

Prandtl (1921) noted: *The density of the lift* (*lift per unit length*) *is not constant over the whole span, but in general falls off gradually from a maximum at the middle nearly to zero at the end. In accordance with what has been proved* (Kutta–Joukowski's theorem), *there corresponds to this a circulation decreasing from within outward. Therefore according to the theorem* (Helmholtz's first vortex theorem) *that by the displacement of the closed curve the circulation* Γ *can change only if a corresponding quantity of vortex filament are cut, we must assume that vortex filaments proceed off from the trailing edge wherever* Γ *changes. For a portion of this edge of length dx the vortex strength is therefore to be written* $(d\Gamma/dx)dx$, *and hence per unit length of the edge is* $d\Gamma/dx$.

The observation of Prandtl is revisited here using the idea of vorticity loops. The flow around and trailing the wing contains vorticity. Suppose the wing is symmetric and the flow is streamlined. As pointed out by Prandtl, the lift, and hence the circulation, is not constant over the span. For the moment, however, suppose all the vorticity tubes in the boundary layers continue unchanged in the span direction until they reach the wing tips. Equation (6.16) then gives the lift per unit span, L/b, predicted by the two-dimensional Kutta–Joukowski's theorem, namely, $L/b = \rho U_\infty \Gamma$.

Parallel to the discussions of Sects. 6.2 and 6.3, the region of non-zero vorticity is envisioned to be composed of three zones: the ambient zone R_b, the near wake zone R_n, and the far (or ultimate) wake zone R_u. In R_b, there are boundary layers containing vorticity that points predominately in the span direction. Therefore, vorticity tubes in the boundary layers are mainly in the span direction. These tubes, according to Helmholtz' theorem, cannot end at the wing tips. The flow outboard the wing tips are mainly in the freestream direction. Beyond the wing tips, the vorticity tubes do not continue in the span direction, but are bent and transported downstream by the flow. These vorticity tubes enter R_n as more or less concentrated stream-wise vortices trailing the wing tips. In an x-plane downstream and near the wing's trailing edge, the trailing vortices typically leave two distinctive traces near the two wing tips. The trailing vortices are often called tip vortices. They are continuations of the vorticity tubes in the boundary layers. Because the two tip vortices are continuations at the opposite ends of the vorticity tubes in R_b, their senses of rotation as seen in the x-plane are opposite. According to (6.15), with the lift in the positive z-direction, the circulation (the total vorticity strength of the two boundary layers on the upper and lower surfaces of the wing) around the wing is $\Gamma\mathbf{j}$. With Prandtl's horseshoe vortex model, a single vortex filament on the y-axis, called the lifting line, represents all the vorticity tubes in the boundary layers. The vorticity tubes in the tip vortices are represented by semi-infinite vortex filaments in the x-direction. The two tip vortices join the lifting line at the tip locations (0, −b/2, 0) and (0, b/2, 0). The circulations of the tip vortex at y = b/2 is $\Gamma\mathbf{i}$ and that at y = −b/2 is $-\Gamma\mathbf{i}$.

In external aerodynamics, the fluid surrounding the wing is not bounded externally. According to Helmholtz' first vortex theorem, a vortex filament must not end in the fluid. Therefore, the horseshoe vortex filament described in the preceding paragraph must close far downstream of the wing. The idea that the closing of the horseshoe vortex occurs infinitely far downstream is useful in theoretical aerodynamics. The second scenario discussed in Sect. 4.8, however, is more in keeping with the vorticity-moment theorem. Specifically, consider the reference flow problem defined in Chap. 1. The wing and the fluid are initially at rest. At a certain instant of time, the wing is set into motion and is thereafter kept moving at the constant velocity ($-U_\infty\mathbf{i}$) in a reference frame at rest initially and remains stationary after the onset of the wing motion. Within a short time period after the motion's onset, boundary layers appear on the upper and lower surfaces of the wing. The circulation, or total vorticity, of the boundary layers, according to (6.1), must be balanced by the starting vorticity shed into the wake. As discussed, in three-dimensional flows, the vorticity tubes in the boundary layers continue beyond the wing tips as vorticity tubes in the tip vortices. Thus, as the starting vortex leaves the vicinity of the wing, a closed tube (loop) of vorticity appears in the flow. This loop contains four segments: the boundary layers vorticity tube, or the lifting line, in R_b, the two tip vortices in R_n, and the starting vortex in R_u. As time progresses, the starting vortex is transported further and further downstream. The tip vortices lengthen with time. In other words, the near wake zone R_n continually grows in length. At all finite time levels, the vorticity tubes in R_b (boundary layers), R_n (tip vortices), and R_u (starting vortex) are connected to form closed loops. The width of the loop is approximately b, the span of the wing. The trailing vortices are approximately perpendicular to the wingspan and in the flow direction, as Prandtl assumed. At large time levels after the motion's onset, the flow near the wing is essentially steady. However, at any finite time level after the motion's onset, no matter how large is the time level, there is a closed and nearly rectangular vorticity loop of constant circulation.

The fact that the horseshoe vortex must close far downstream of the wing is no doubt recognized by Prandtl and by other leading scholars of classical aerodynamics. Typically, the starting vortex was considered to be infinitely downstream of the wing and therefore does not induce a downwash at the lifting-line position. The effects of the starting vortex on the aerodynamic force were traditionally ignored. The role of the starting vortex, however, is important in the vorticity-moment theorem.

As discussed in Sect. 6.4, if a closed tube of vorticity is approximated by a vortex-filament loop, then the total vorticity moment $\mathbf{r} \times \boldsymbol{\omega}$ of the vorticity tube is approximated by $2\Gamma\mathbf{A}$, where $\mathbf{A} = A\mathbf{n}$ is the vector area of the vortex loop. Therefore, (6.2) yields:

$$\mathbf{F} = -\rho\Gamma(d\mathbf{A}/dt) \tag{6.17}$$

where $\mathbf{F}$ is the force associated with a deforming vortex loop enclosing the vector area $\mathbf{A}$ and possessing the circulation Γ.

Equation (6.17) was given by Wu and Wu (1996) using (6.4). At very large (but finite) time levels after the motion's onset, the circulation Γ around the wing is steady. Using Prandtl's simplifications, but including the starting vortex in the model of the vorticity field, there is a closed rectangular vortex filament of strength Γ that lies in the $z = 0$ plane. The width of the rectangle is b, the wingspan, and the length λ of the rectangle increases with time. The direction $\mathbf{n}$ of the plane is $-\mathbf{k}$. Therefore, (6.17) yields:

$$\mathbf{F} = \rho\Gamma b(d\lambda/dt)\mathbf{k} \tag{6.18}$$

Since the starting vortex is moving at the velocity $U_\infty\mathbf{i}$, the length of the rectangle is increasing at a rate equal to U_∞. The lift predicted by (6.8a, b) is then $\rho\Gamma b U_\infty\mathbf{k}$, which is the lift given by (6.8b).

Provided the starting vortex is moving at the velocity $U_\infty\mathbf{i}$, the above conclusion remains correct under the more general circumstances where the vortex loop does not lie in a plane and/or the loop is not a rectangle. Designate the circuits representing the vortex loop at the time levels t_1 and $t_2 = t_1 + \delta t$, respectively, by C_1 and C_2. Suppose the position of the starting vortex at the time levels t_1 is given by $x = x_1$, $-b/2 < y < b/2$, and $z = z_1$. The position of the starting vortex at the time level t_2 is then given by $x = x_1 + U_\infty\delta t$, $-b/2 < y < b/2$, and $z = z_1$. Following the discussions of Sect. 2.5, C_2 is divisible into two circuits: the old circuit C_1 and the new addition which is rectangular and with the width b and the length $U_\infty\delta t$. The assumption that C_1 remains unchanged over the time period δt implies that the position of the trailing vortices in R_n relative to the wing (the lifting line) remains fixed. Thus, the length of the new rectangular circuit represents the growth of the trailing vortices and the width of the rectangle represents the distance between the new and the old positions of the starting vortex. The vector area of this new circuit is $bU_\infty\delta t\mathbf{k}$. Therefore during the time interval δt, the total vorticity moment in the fluid increases by the amount $-2b\Gamma U_\infty \delta t\mathbf{k}$ (note the circulation of the starting vortex is $-\Gamma\mathbf{j}$). The rate of increase of the total vorticity moment is $-2b\Gamma U_\infty\mathbf{k}$ and, according to (6.2), there is again $L/b = \rho\Gamma U_\infty$.

It is not necessary for the starting vortex to maintain the circulation $-\Gamma\mathbf{j}$ and the width b. Since the total vorticity in R_b is $\Gamma b\mathbf{j}$ and that in R_n is zero, the total vorticity in R_u, according to (6.1), must be equal to $-\Gamma b\mathbf{j}$. Suppose the circulation of the starting vortex is $-\Gamma_1\mathbf{j}$ at the time level t_1, then the length b_1 of the starting vortex is equal to $b\Gamma/\Gamma_1$. Assuming that this vortex line is moving at the velocity $U_\infty\mathbf{i}$, the new circuit described earlier has the area $b_1 U_\infty \delta t\mathbf{k} = b(\Gamma/\Gamma_1) U_\infty \delta t\mathbf{k}$. Therefore, during the time interval δt, the total vorticity moment in the fluid increases by the amount $-2b\Gamma U_\infty \delta t\mathbf{k}$. The rate of increase of the total vorticity moment is $-2b\Gamma U_\infty\mathbf{k}$ and one again has $L/b = \rho U_\infty \Gamma$.

Under the heading of *Motion of a Perfect Fluid Produced by External Forces*, von Karman and Burgers (1934) discussed the generation of a vortex ring by an impulsive pressure acting over a circular area and showed that the impulse I of a planar *vortex ring* of strength Γ is expressible as

$$I = \rho\Gamma\mathbf{A} \tag{6.19}$$

Letting $\mathbf{F} = -dI/dt$, one obtains (6.17). Thus, (6.2) is interpretable through the idea of the impulse. It is worth underscoring, however, that the idea of the vorticity moment is not an explicit part of (6.17) or (6.19). From the viewpoint of the vorticity-moment theorem, (6.17) is an outcome, and not a substitute, of (6.2). Furthermore, (6.2) is a consequence of the Navier–Stokes equations and is applicable to the flow (see Chap. 4) of the viscous fluid. This equation relates the aerodynamic force on the wing to the evolution of the vorticity field in the fluid. This relationship opens up a new avenue for understanding aerodynamic principles and for developing new methods of aerodynamic analysis.

Designate the x-, y-, and z-components of $\boldsymbol{\omega}$ by ξ, η, and ζ, respectively. Since the lift is in the *z*-direction, in a study of the lift based on (6.2), changes of only the z-component of $\mathbf{r} \times \boldsymbol{\omega}$, i.e., $x\eta - y\xi$, requires attention. With the rectangular vortex loop model, the vorticity in the starting vortex is in the y-direction (η). Suppose the starting vortex is moving with the velocity $\mathbf{v} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$. Clearly, v and w, the x- and y-component of $\mathbf{v}$, do not contribute to the lift since they do not cause the z-component of the vorticity moment to change. The movement of the starting vortex in the x-direction causes the z-component of the vorticity moment to change at the rate $-u\Gamma b$. Therefore, according to (6.2), the movement of the starting vortex corresponds to a lift per unit span of $\frac{1}{2}\rho u\Gamma$. If one assumes $u = U_\infty$, then the movement of the starting vortex gives one-half of the lift predicted by the Kutta–Joukowski theorem. Since the total vorticity $-\Gamma b\mathbf{j}$ of the starting vortex is determined by the total vorticity in R_b, the precise location of the vorticity in R_u is irrelevant. The prediction that the lift per unit length corresponding to the movement of the starting vortex is $\frac{1}{2}\rho U_\infty \Gamma$ is as accurate as the assumption $u = U_\infty$. The assumption that $u = U_\infty$ is an acceptable, in fact very accurate, approximation since the starting vortex is in a convection-dominated flow region far from the wing.

The remaining half of the lift predicted by the Kutta–Joukowski theorem corresponds to the lengthening of the tip vortices. With the rectangular vortex loop model, the tip vortices are in the x-direction. The senses of rotation of the two tip vortices are opposite to one another. Therefore, the z-component of the vorticity moment ($-y\xi$) that corresponds to the tip vortices per unit length is $-b\Gamma$. The tip vortices are connected to the starting vortex and grow at a rate that is equal to the velocity of the starting vortex. Therefore, the z-component of the vorticity moment of the tip vortices grows at the rate $-bu\Gamma$. In consequence, the lift per unit span corresponding to the lengthening of the tip vortices is $\frac{1}{2}\rho u\Gamma$, or $\frac{1}{2}\rho U_\infty\Gamma$ with the assumption $u = U_\infty$.

In a stationary reference frame, the freestream velocity is zero. The x-component of the velocity of the starting vortex is zero. Therefore, the lift corresponding to the starting vortex is zero. However, the wing (the lifting line) is moving at the velocity $-U_\infty\mathbf{i}$. The rate of change of the x-component of the vorticty moment associated with the wing's motion is thus $-bU_\infty\Gamma$. The lift corresponding to this change is $\frac{1}{2}\rho U_\infty\Gamma$ per unit span. The lift per unit span corresponding to the growth of tip vortices next to the lifting line is also $\frac{1}{2}\rho U_\infty\Gamma$. Therefore, the total lift per unit span of the wing is $\rho U_\infty\Gamma$.

6.6 Lift on Finite Wing with Varying Circulation

A finite wing with a constant circulation in a three-dimensional flow is an idealized situation. For this idealized situation, one half of the lift $\rho U_\infty\Gamma$ predicted by the two-dimensional Kutta–Joukowski theorem is related to the movement of the starting vortex and the other half to the growth of the tip vortices. Tip vortices are not present in two-dimensional flows. The movement of the starting vortex in the two-dimensional flow alone gives the entire lift ($\rho U_\infty\Gamma$) and not one-half of it. The connection between the two-dimensional and the three-dimensional flows is interpreted as follows.

As discussed, a two-dimensional flow is considered to be a description of the flow in the mid-span zone of a long right cylindrical solid body (wing with a long span) moving perpendicular to its axis (span). In this mid-span zone, the velocity vector lies essentially in the y-plane and has a zero y-gradient. Let the circulation around the wing be $\Gamma\mathbf{j}$, a constant along the span. Envision several adjoining segments of the wing in the inboard zone, far from the wing tips. These wing segments are called winglets for convenience. Let the span of each winglet be δb. The vorticity system associated with each winglet, viewed individually, is modeled by a rectangular vortex loop composed of a lifting line of length δb trailed by two tip vortices and closed by a starting vortex segment of length δb. The time rate of change of the z-component of the vorticity moment associated with each winglet is then $-2U_\infty\Gamma\delta b$ and the corresponding lift is $\rho U_\infty\Gamma\delta b$. Consider a winglet between two neighboring winglets. The two tip vortices of the middle winglet coincide with tip vortices of the neighboring winglets. Since the senses of rotation of the coinciding tip vortices are opposite, they cancel one another and no tip vortices are visible in the inboard zone. Each winglet nevertheless experiences a force that corresponds to the lengthening rectangular vortex loop. One half of this force is related to the movement of the starting vortex and the other half to the lengthening of the invisible tip vortices. In consequence, the lift per unit span of the wing in the mid-span zone is $\rho U_\infty \Gamma$. With the view that a two-dimensional flow describes the flow around the mid-span of a long finite wing, all vortcity tubes in the fluid are closed far from the mid-span zone. In the limit as the span of the wing approaches infinity, these vorticity tubes remain closed. In this limit, the flow in the mid-span zone is two-dimensional, but the vorticity tubes (in the y-direction) are parts of closed vorticity tubes, even though the closing of tubes are not directly visible.

As Prandtl noted, the lift per unit span *is not constant over the whole span, but in general falls off gradually from a maximum at the middle nearly to zero at the end.* Consider a finite wing with a nonuniform circulation $\Gamma(y)\mathbf{j}$ lying on the y-axis between $y = -b/2$ and $y = b/2$. Divide the wing into a set of J winglets each of the span $\delta b = b/J$. Label the winglets sequentially from 1 to J. Let $y_j = -b/2 + j\delta b$ and the span of the jth winglet be in the range $y_{j-1} < y < y_j$. Approximate the circulation around the jth winglet by the average circulation over its span and designate this circulation by $\Gamma_j\mathbf{j}$. Represent the vorticity field surrounding and trailing each winglet by a rectangular vortex loop. Under this circumstance, the tip vortices of two neighboring winglets coincide. Because the two adjoining winglets have different circulations, the combined circulation of the coinciding tip vortices is not zero. Specifically, at $y = y_j$, $j = 1, 2, 3, \ldots, J - 1$, the two coinciding tip vortices have, individually, circulations $\Gamma_j\mathbf{i}$ and $-\Gamma_{j+1}\mathbf{i}$. Thus, one sees a vortex filament with the circulation $(\Gamma_j - \Gamma_{j+1})\mathbf{i}$ trailing the wing (the lifting line) at the point $(0, y_j, 0)$. In the limit as $\delta b \to 0$, the trailing vortices becomes a vortex sheet with the strength $\boldsymbol{\gamma} = \gamma\mathbf{i}$ given by

$$\gamma = -d\Gamma/dy \tag{6.20}$$

Prandtl (1921) called the trailing vortex sheet a vortex ribbon. He observed, based on Helmholtz' first vortex theorem, that: *The circulation* Γ *can change only if a corresponding quantity of vortex filament is cut.* He then concluded that the strength (circulation) of the vortex ribbon per unit length of the edge is $d\Gamma/dx$. Prandtl's (1921) conclusion is identical to (6.20) if one adds a negative sign and replaces x by y to account for the fact that Prandtl used a left-handed Cartesian coordinate system and assigned x in as the span direction. In the limit as $\delta b \to dy$, the lift on the winglet approaches $\rho\Gamma(y)U_\infty(y)dy$. Upon integrating this expression over the span of the wing, one obtains the lifting-line expression (6.16). In summary, the vorticity-moment theorem provides an alternative derivation of the lifting-line theory for the lift on a finite wing, a derivation that does not rely on the use of the two-dimensional theorem of Kutta–Joukowski.

6.7 Induced Drag

Prandtl observed that the tip vortices induce a downwash $-w(y)$ at the lifting-line position. He treated this downwash as a modifier of the freestream velocity $\mathbf{V}$ in the Kutta–Joukowski theorem (6.8a). By letting $\mathbf{V} = U_\infty\mathbf{i} + w\mathbf{k}$ in (6.8a), one obtains, in addition to the lift given by (6.8b), or more generally by (6.16), a drag D_i given by

$$D_i = -\rho \int_{-b/2}^{b/2} \Gamma(y)w(y)dy \tag{6.21}$$

The drag D_i is called the induced drag because of Prandtl's assumption that the downwash "induced" by the trailing vortex sheet modifies the relative velocity between the wing and the freestream fluid and thereby causes this drag. To understand the downwash at the lifting line, consider first the finite wing with a constant circulation along its span. With Prandtl's model, the tip vortices are two semi-infinite vortex filaments lying in the z-plane and extending from the wing tips to infinity in the x-direction. The downwash corresponding to the two semi-infinite straight vortex filaments can be evaluated using the Biot–Savart law. This downwash on the y-axis is

$$w(0,0,y) = -[1/(y-b/2) + 1/(y+b/2)]\Gamma/4\pi \tag{6.22}$$

Placing (6.22) into (6.21) gives an expression for the induced drag on the finite wing that is determinate once the circulation Γ is known. Consider now the closed rectangular vortex filament defined in Sect. 6.5. In the wing-attached reference system, assume the tip vortices to be finite, but very long. Corresponds to the tip vortices, the downwash at the starting vortex position is obviously identical to the downwash at the lifting-line position. Because of the downwash, the velocity of the starting vortex, in addition to the x-component $U_\infty\mathbf{i}$, which is related to the lift, has a z-component $w\mathbf{k}$. According to (6.2), the downward movement of the starting vortex contributes to a drag.

Since the drag is in the x-direction, in a study of the drag based on (6.2), the change of only the x-component of $\mathbf{r} \times \boldsymbol{\omega}$, i.e., $y\zeta - z\eta$, requires attention. The circulation of the starting vortex representing the vorticity in R_u is $-\Gamma\mathbf{j}$. Thus, the downward movement of the starting vortex, which contains the vorticity component η, causes the x-component of the vorticity moment to change at the rate $w\Gamma b$. Therefore, according to (6.2), the downward movement of the starting vortex corresponds to a drag per unit span of $-\frac{1}{2}\rho w(y)\Gamma(y)$. Integrating this expression over the span of the finite wing gives one-half of the induced drag on the wing expressed by (6.21).

Consider the winglets defined in Sect. 6.6. The *j*th winglet has a span δb, a circulation $\Gamma_j\mathbf{j}$, and is trailed by two tip vortices at y_{j-1} and y_j. The two tip vortices are connected far downstream of the wing by the starting vortex segment of length δb with the circulation $-\Gamma_j\mathbf{j}$. As the starting vortex segment of a winglet moves, the corresponding tip vortices grow so that they remain connected to the starting vortex segment. Since the starting vortex of this winglet is moving with the velocity $U_\infty \mathbf{i} + w_j \mathbf{k}$, the tip vortices are growing at this rate. Since the downwash w is in general much smaller than U_∞ ($w_j << U_\infty$), the circulation of the tip vortex at $y = y_{j-1}$ is approximately $-\Gamma_{j-1}\mathbf{i} - \Gamma_{j-1}(w_j/U_\infty)\mathbf{k}$ and that at $y = y_j$ is approximately $\Gamma_j \mathbf{i} + \Gamma_j(w_j/U_\infty)\mathbf{k}$. Therefore the vorticity moment of the two tip vortices is $-\delta b \Gamma_j \mathbf{k} + \delta b \Gamma_j\ (w_j/U_\infty)\mathbf{i}$ per unit length. The motion of the starting vortex thus causes the x-component of the vorticity moment to change at the rate $\Gamma_j w_j \delta b$. The growth of the tip vortices causes the x-component of the vorticity moment to change by the same amount. Thus, the total rate of change of the x-component of the vorticity moment associated with the winglet is $2\Gamma_j w_j \delta b$. The corresponding drag on the winglet is $-\rho \Gamma_j w_j \delta b$. In the limit as $\delta b \to dy$, the drag on the winglet approaches $-\rho \Gamma(y) w(y) dy$. Upon integrating this expression over the span of the wing, one obtains the induced drag expression (6.21). In other words, (6.21) is derivable for the viscous fluid using neither the two-dimensional Kutta–Joukowski theorem nor the idea that the downwash modifies **V**.

For the wing with a varying circulation over its span, (6.22) is readily generalized to:

$$w(y) = \frac{1}{4\pi} \int_{-b/2}^{b/2} \frac{\gamma(y_o)}{y - y_o} dy_o \tag{6.23}$$

It is worth noting that in a stationary reference frame, the starting vortex is not at rest, but is continually moving downward. In other words, the starting vortex does not reside in the freestream—the flow undisturbed by the solid motion. Observed in the stationary reference system, the downward motion of the starting vortex is accompanied by a growth of both the x- and the z-components of trailing vortices. The downward motion of the starting vortex and the growth of the z-component of the trailing vortices together give rise to the induced drag stated by (6.21). It is also worth noting that the velocity field, and hence also the associated momentum and kinetic energy of the fluid, etc., is not invariant in a Galilean transformation. In contrast, the vorticity field is invariant to a Galilean transformation. Moreover, the total vorticity in R_∞ is zero and hence, as shown in Sect. 7.4, the total vorticity moment is independent of the selection of a reference point. Thus, (6.2) is invariant in a Galilean transformation and this fact facilitates aerodynamic analyses using the vorticity-moment theorem.

6.8 Wake-Integral Expressions for Lift and Drag

The induced drag is often said to be the drag due to lift because, if there is a non-zero lift, then the induced drag is non-zero. The vorticity-moment theorem gives results identical to those of the lifting-line theory, but provides a different interpretation. Specifically, the lift and the induced drag are both related to the change of vorticity moments due to the movement of the vorticity in the ultimate wake (the starting vortex) and the concurrent growth (because of Helmholtz' first vortex theorem) of the trailing vortices (in R_n). With a non-zero steady lift, the ambient zone has a non-zero circulation. This means a starting vortex with an opposite circulation must be present. Therefore, if there is a non-zero lift, there is also an induced drag. However, a zero lift does not necessarily mean the induced drag is zero. If the lift is positive over certain sections of the wing and negative elsewhere, then it is possible that the total lift on the wing is zero, but a starting vortex is present and the induced drag is non-zero.

Using (6.20), it can be shown that

$$\Gamma(y) = d(y\Gamma)/dy + y\gamma \tag{6.24}$$

Placing (6.24) into (6.16), one obtains

$$L = \rho U_\infty \left[y\Gamma\big|_{-b/2}^{b/2} + \int_{-b/2}^{b/2} y\gamma(y)dy \right] \tag{6.25}$$

Prandtl observed that *the density of the lift …in general falls off gradually from a maximum at the middle nearly to zero at the end.* With zero lift at wing tips, the circulation is zero there and the first term on the right-hand side of (6.25) vanishes. It is known from vorticity kinetics (Chap. 4) that the circulation is, in any event, zero a short distance beyond the wing tips in the span direction. Thus, one obtains:

$$L = \rho U_\infty \int_{-b/2}^{b/2} y\gamma(y)dy \tag{6.26}$$

Equation (6.26) expresses the lift in terms of the strength of the trailing vortex sheet. Note that for a symmetric airfoil, $\Gamma(-y) = \Gamma(y)$ and the first term in (6.25) is always zero. The integral in (6.25) and (6.26) however, is not zero since $\gamma(-y) = -\gamma(y)$. In fact, (6.26) can be restated for the symmetric wing as

$$L = 2\rho U_\infty \int_{0}^{b/2} y\gamma(y)dy \tag{6.27}$$

For the induced drag, (6.21) can be restated using (6.24) as

$$D_i = -\rho \int_{-b/2}^{b/2} y\gamma(y)w(y)dy - \rho \int_{-b/2}^{b/2} \frac{d(y\Gamma)}{dy} w(y)dy \tag{6.28}$$

Omitting the last integral in (6.28), one obtains the following expression for the induce drag:

$$D_i = -\rho \int_{-b/2}^{b/2} y\gamma(y)w(y)dy \tag{6.29}$$

Equation (6.29) is equivalent to a known wake integral expression for the induced drag on the wing derived using the momentum theorem (Wu et al. 1979, 2002). In reality, the trailing vorticity sheet is an approximation of a vorticity layer trailing the wing. This layer contains mostly ξ, the x-component of the vorticity $\boldsymbol{\omega}$. The strength γ of the sheet is the integrated ξ value across the layer. Thus, (6.27) and (6.29) can be restated as

$$L = \rho U_\infty \iint_{W_a} y\xi dydz \tag{6.30}$$

$$D_i = -\rho \iint_{W_a} yw\xi dydz, \tag{6.31}$$

where w_a is a cross-sectional area of the vertical wake trailing the wing.

Equations (6.30) and (6.31) express the steady lift and the steady induced drag on the wing in the form of wake integrals. In a stationary reference frame, as the wing moves, it leaves behind footprints in the form of wakes containing vorticity. Wake integrals expressions such as (6.30) and (6.31) are mathematical statements connecting aerodynamic forces on the wing to the footprints. Specifically, the lift and the induced drag are both connected to ξ, the stream-wise component of vorticity, in the footprints.

The profile drag is connected to η, the y-component of $\boldsymbol{\omega}$. As the wing moves forward, the vorticity in the boundary layers covering the wing are continually left behind as footprints. The vorticity in the boundary layers is essentially in the y-direction. In the case of the streamlined flow, the boundary-layer footprint is a thin wake layer containing η. This wake layer is composed of two sub-layers connected to the boundary layers on the upper the lower surfaces of the wing. As discussed in Sect. 6.2, in steady flows, the strengths of the two sub-layers are equal and opposite. That is, the integration of η with respect to z across the wake layer is zero. The structure of the wake layer, with its sub-layers containing positive and negative η, however, means that $z\eta$, an x-component of vorticity moment, of the wake layer, is not zero. In principle, the profile drag in both streamlined and non-streamlined flows can be determined by analyzing the rate at which the boundary layers feed the x-component vorticity moment ($-z\eta$) into the wake layer.

Betz (1925) pioneered the concept of wake integrals and presented two integral expressions, one for the induced drag and the other for the profile drag. Betz' integral for the induced drag does not limit the integration to the vortical region of the wake. To evaluate the induced drag using Betz's integral requires velocity values in potential flow regions where disturbance velocities are very small and difficult to measure with precision. Maskell (1973) used vortex filaments and source–sink singularity to model flows in wind tunnels and showed that it is possible to express the induced drag as a wake integral involving the vortical wake region only. With the vorticity-moment theorem, wake integrals for the lift and the induced drag are established in the form (6.30) and (6.31). The integrals in (6.30) and (6.31) can be evaluated through wake surveys in which values of **v** are measured and used in computing values of ξ. These wake surveys are needed only over a single wake plane and are limited to the small vortical region. Experimental efforts required for such surveys are not excessive using modern experimental methods such as the particle image velocimetry (PIV). Computations limiting the computational domain to regions of non-zero vorticity are ideally suited for the wake integral calculations. The wake-integral method for the steady lift has reached a reasonable stage of maturity at the present. The simplifying assumptions underlying (6.30) are well understood. Readers are referred to Wu et al. (2002) for relatively recent developments of this method.

Betz's wake integral for the profile drag limits the integral to the region of the viscous wake, but does not relate the profile drag to the vorticity moment. The development of the wake integral method for the profile drag remains at the present a topic of research interest. For streamlined flows, the boundary layer theory provides a means for the computation of the profile drag. Many open issues, however, remain to be resolved relating to boundary layers in three-dimensional space and to non-streamlined flows in two- and three-dimensional space.

The wake-integral expression (6.31) for the induced drag is based on (6.29), which contains only the first term on the right-hand side of(6.28).Through the vorticity-moment theorem, assumptions adopted for the derivation of (6.28) are clearly identified and justified as approximations of physical reality (Sect. 1.3 and 5.5). The presence, significance, and practical importance of the last integral in (6.28) require further research.

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Chapter 7
Unsteady Aerodynamics

7.1 Aircraft and Streamlined Flows

Man's dream of flight became a reality when the Wright brothers lifted off in their first flyer in 1903. During the subsequent 100 years, advancements in manned flying machines were amazingly rapid. The speed, flight range and height achievable with modern airplanes far exceed those of the birds whose flights inspired mankind since ancient times. With air travel so commonplace today, the flying public generally deems aerodynamics a mature discipline of science. The popular view is that all major discoveries in aerodynamics were made long before the end of the past century. The remaining tasks today are not so much the search for improved physical understanding, but the collection of additional aerodynamic data for design refinements, tasks that require mostly testing and computation. It is informative, however, to pause a moment and review briefly a fundamental concept of aerodynamics based on which manned flying machines were designed and built —the idea of streamlined steady flows around fixed wings—and to compare the flights of these machines with those of birds.

The Wright brothers designed and built their flyers as fixed-wing airplanes. Their success is followed quickly by discoveries of principles of aerodynamics of steady and streamlined flows. These discoveries made it possible to predict aerodynamic forces quantitatively and to design and build successive generations of fixed-wing airplanes, each surpassing the previous in speed, height, range, levels of comfort and safety. Aerial animals, especially large birds, fly in the fixed-wing mode at times. All aerial animals, however, flap their wings in all phases of their flights. Flapping-wing flyers are mechanically much more demanding to build than fixed-wing airplanes. In part because of mechanical difficulties, and surely inspired by Wright brothers' success, the aeronautical engineers focused almost entirely on the fixed-wing concept in the twentieth century. Aeronautical designs of the past century relied primarily on steady streamlined aerodynamics. The building of

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ornithopters that emulate the flapping-wing flights of the birds was relegated to a hobby status.

At Reynolds' numbers of interest in aeronautics, airflows about streamline-shaped wings at low angles of attack are typically attached. In other words, except in small regions near the trailing edges and wing tips, flows near the wing—boundary layers—are essentially parallel to the surface of the wing. A most important attribute of such streamlined flows is that the aerodynamic load on the wing is directed nearly perpendicular to the plane of the wing. This means that, since the angle of attack is low, the drag component of the force is much smaller than the lift component. This attribute—large lift and small drag—is obviously critically important for efficient flying.

The aerodynamic load on the wing in a streamlined flow is nearly proportional to the square of the flight speed. This load also increases nearly linearly with increasing angle of attack. There is a combination of angle of attack and flight speed with which the lift is sufficient to keep the airplane airborne. An excessively large angle of attack, however, leads to massive flow separation and large drags. Therefore, to produce efficiently a specific amount of lift to stay airborne, a minimum speed (corresponding to the maximum angle of attack for streamlined flow) is required. During forward flight, insufficient airspeed or excessively large angle of attack causes the wing to stall: The flow around the wing does not remain streamlined and the resulting large drag leads to a loss of speed and control of the flight.

Conventional airplanes fly in the fixed-wing streamlined mode in all phases of their flights. A propulsion system is used to provide the thrust while the airplane cruises. For takeoff, the same propulsion system produces the thrust to accelerate the airplane on the ground. The airplane lifts off when its speed exceeds the critical takeoff speed and the lift on the airplane exceeds the airplane's takeoff weight. At takeoff, the angle of attack of the airplane wing is typically less than the stall angle, but greater than the design angle of attack for cruising. Thus, the takeoff speed of the airplane is substantially smaller than the cruising speed. For example, the Boeing 747-400 airplane typically takes off at about 290 km (180 miles) per hour and cruises at about 910 km/h.

While cruising, the thrust produced by the propulsion system balances the drag that the airplane encounters. During the landing phase, the thrust is set marginally smaller than the drag. The airplane slows and descends gradually while continuing to fly in the streamlined fixed-wing mode. Brakes, including airbrakes and thrust reversers, are applied to bring the airplane to a stop after the airplane touches down on the runway.

By streamlining shapes of airplanes and their wings and by minimizing the drag, modern transport airplanes—airliners and cargo planes—achieve a lift-to-drag ratio close to 20 in cruising. Such large lift-to-drag ratios mean that the flights of modern airplanes are fuel-efficient. For example, the average fuel for each passenger flying 6000 km in a fully loaded Boeing 747-400 is less than 200 L.

Several types of power plants are in common use today: propeller-reciprocating engines, turboprops, turbofans, and turbojets. An airplane with a lift-to-drag ratio of

20 needs a small thrust (5% of the airplane weight) while cruising. In general, a larger power plant, one capable of delivering a larger thrust, weighs more than a smaller one. The size of the power plant of the modern airplane, however, is not determined by cruising or in-flight maneuvering, but by the length of runways. The smaller the thrust, the less rapid is the acceleration rate of the airplane and the longer is the runway length needed for takeoff. Earlier airplanes and today's light aircraft are relatively slow flyers capable of taking off and landing on short landing strips. As the size and speed of airplanes grew, longer and longer runways are needed. To accommodate modern airliners such as the Boeing 747-400, modern commercial airports have runways about 4 km long. Even with such long runways, the takeoff thrust required by large modern airplanes is typically many times the cruising thrust. The total maximum thrust producible by the jet engines on a Boeing 747-400, for example, is about 29% of the maximum takeoff weight of the airplane and is nearly 6 times the thrust needed for cruising. The aeronautical engineer accepts the parasitic weight of the large power plants for cruising as a compromise to keep the runway length within practical limits. Even with this compromise, very long runways are required. Land use and environmental concerns such as noise pollution then make it impractical to construct commercial airports in the midst of urban population centers. Ground transportations to and from distant airports add significantly to the time and cost and take away from the convenience of air travel.

A vertical takeoff and landing (VTOL) airplane has a propulsion system capable of producing a thrust larger than the weight of the airplane. The Harrier, for example, has a specially designed jet engine with thrust vectoring capability. By directing the thrust vertically upward, the Harrier is capable of vertical takeoff, vertical landing, and hover. The Harrier can reach a forward flight speed of about 300 km/h using mostly fixed wings to provide the lift and the jet to propel the aircraft. During low speed flights, either forward or backward, the wings and the jet together provide sufficient lift to keep the Harrier airborne. The airflow around the wings during backward flights is not adequately streamlined. The drag on the Harrier is clearly substantially larger in backward flight than in forward flight. As a result, the backward flight speed of the Harrier is limited to about 100 km/h.

A short takeoff and landing (STOL) aircraft is required to clear a 15 m high obstacle within 450 m from the point the aircraft commences takeoff. For an airplane with a stall speed of 150 km/h, the propulsion system needs to deliver a thrust about 2/3 of the airplane's weight to meet these requirements. Because of the parasitic weight of the large propulsion system, the Harrier and other V/STOL aircraft have limited flight range and payload capability. They are not designed for civil aviation.

Helicopters take off and land vertically. They are also called rotorcraft because they use rotors in place of fixed wings to produce lift. A helicopter rotor is similar to a propeller: It has two or more blades rotating around the rotor shaft and produces a force in the axial direction. Helicopter blades, however, have higher *aspect ratios* (about 15–20) then propeller blades and, in this respect, are more similar to airplane wings. For this reason, helicopters are often called rotating-wing aircraft. Non-streamlined flows around blades produce large aerodynamic drags. For the rotor, this means large torques requiring excessively large power input to overcome. Thus, streamlining of blades is of decisive importance in the rotor and propeller designs. During takeoff and landing, the rotor axis of the helicopter is vertical and the thrust produced by the rotor is a lift force. In forward flights, the rotor is tilted a small angle to produce a thrust component in the flight direction. The flow around the rotor blade in forward flight is unsteady.

Unsteady flows about rotor blades in forward flight are exceedingly complicated and aerodynamic analyses of these flows are exceedingly difficult. Since the first practical helicopters were built in the 1930s, extensive research programs in unsteady aerodynamics have been conducted at research institutes and universities at various points of the world. The vast majority of these programs are aimed at alleviating adverse effects of flow unsteadiness rather than producing useful unsteady aerodynamics forces. Modern helicopters are limited to about 320 km/h in flight speed and about 1000 km in range without refueling. To a large extent, these limitations are due to the adverse effects of flow unsteadiness. Helicopters are used for a multitude of tasks that airplanes, because of their requirement of runways, are incapable of performing.

During the past two decades, tilt-rotor airplanes have received attention in the aeronautics community. Tilt-rotor airplanes use both rotors and fixed wings in their flights. The rotors produce lift during takeoff and landing, as the helicopter rotor does. After lifting off, the rotors are pivoted gradually forward to produce a horizontal thrust. As the flight speed increases, the wing lift increases. While cruising, the tilt-rotor airplane flies like a conventional airplane with large propellers (rotors) providing thrust to counter the drag and the fixed wings produce the lift. The development of tilt-rotor airplanes is currently in progress. They are expected to be more efficient than helicopter, but less efficient in forward flight than modern conventional fixed-wing airplanes.

Aerodynamic forces on rotor and propeller blades, like those on fixed wings, are produced by streamlined flows around the blades. These forces are nearly perpendicular to the direction of motion of the blades, i.e., the plane of the rotor. The well-known blade element theory for analyzing the aerodynamics of rotating blades was established on the basis of this similarity between rotor blades and the fixed wings. Blades and vanes of turbo-machineries, including jet engines, are also similar to fixed wings. Pumps and compressors are propellers installed in casings that guide the flows created by rotating blades. Turbines are compressors operating in reverse. In short, the overwhelming majority of today's fluid machineries, including the modern flying machines, are designed to operate in the streamlined fixed-wing mode.

7.2 Animal Flight and Flapping Wings

In his groundbreaking book on bio-fluid-dynamics, Lighthill (1975) remarked: *The vast majority of hovering animals support their weight by motions that lend themselves to analysis on classical aerodynamic principles. There are important* *exceptions, however, where substantially better performance is achieved by use of mechanisms of lift generation never discovered by engineers*. In a relatively recent article, Templin (2000) summarized the geometry and flight characteristics of many types of winged animals. It is informative to review briefly the most perceptible characteristics of animal flight, especially those of birds, before attempting to establish a rational understanding of the undiscovered mechanisms referred to by Lighthill.

All flights of course must be comprised of takeoff, forward flight, and landing phases. The flight phases of some birds and insects also include hovering. A bird in forward flight must produce a lift countering its body weight. The bird also experiences a drag and loses speed unless it produces a thrust to counter the drag. A large bird in forward flight at times spreads and holds fixed its wings at a small angle of attack. Many large birds are capable of maintaining nearly level flight paths in the fixed-wing mode for long intervals of time, apparently experiencing little deceleration. Based on this observation, it is clear that fixed wings produce a lift that is sufficiently large to support the weight of the bird. It is also clear that, because the flight speed of the bird does not decrease rapidly, the drag experienced by the bird with its wings held fixed is small. This fact—large lift and small drag—is, as discussed earlier, attractive for flying. The emulation of the forward flight of large birds with fixed wings is obviously the pivotal idea that led to the successes of manned flights in the past century.

The birds, however, do not fly only in the fixed-wing mode. Rather, they flap their wings up and down in all phases of their flights. With fixed wings, a bird needs a critical flight speed to produce a sufficiently large lift to stay airborne. The large bird at times maintains a sufficiently high flight speed and utilizes only the fixed-wing lift to stay airborne. Unlike the fixed-wing force, which needs a non-zero flight speed, the flapping-wing force does not require a nonzero speed and can point in an arbitrary direction. To take off from the ground or a perch, the bird flaps its wings and produces an aerodynamic force with both a lift component and a thrust component. The lift is sufficiently large to heave the bird skyward; and no runway is needed. The thrust accelerates the bird, giving it a forward speed. As the flight progresses, a large bird soon reaches a critical speed that allows it to fly in the fixed-wing mode. In this mode, the large bird typically flaps its wings intermittently. Flapping wings produce mainly the thrust needed to counter the drag and thereby maintain the bird's flight speed. They contribute little to the lift.

The large bird has at its disposal several techniques to stay airborne with fixed wings. The bird may *glide*, that is, descend slowly, utilizing gravitational energy to maintain the critical speed. It may use an updraft in the atmosphere to *soar*, i.e., to glide without a loss of altitude, or even to rise. The critical speed for the needed lift decreases with increasing angle of attack. Therefore the bird may gradually increase the angle of attack of its wings so as to stay airborne while its flight speed is gradually reduced. This last option, however, has its limitations because above a certain limiting angle of attack—the *stall angle*—the flow about the wings is not streamlined, but is separated. The bird then experiences a large drag and decelerates rapidly. In aeronautics, the term "*to feather*" means to change the pitch on a propeller or a rotary wing. In a more general context, it describes the change of the angle of attack of a wing in forward flight while retaining control of the flight. The flow surrounding the wing is streamlined and nearly steady.

Smaller and slower birds flap their wings more frequently in forward flight than do their larger and faster cousins. Many small and slow aerial animals, birds and insects alike, flap their wings continually. This observation suggests that fixed-wing lift alone is insufficient to keep small and slow animals airborne. Flapping-wing lift provides a portion of the lift needed for small animals to stay airborne. Many insects and birds, for example the hummingbirds, are capable of hovering. They stay airborne without a forward flight speed. During hover, flapping-wing lift is obviously large enough to support the animal's body weight.

During the initial landing phase of the flight, the bird glides and often also flaps its wings. With decreasing flight speed, flapping wings produce a lift that keeps the bird from plummeting. Near the completion of landing, the bird may briefly holds it wings at an angle of attack much higher than the stall angle. The wings experience a large drag and serve as an airbrake, stopping the flight quickly to enable a soft landing. In this brief instant, non-streamlined flow is not detrimental, but is put to use by the bird.

In short, large birds are capable of generating sufficient fixed-wing lift. They exploit a variety of aerodynamic opportunities to stay airborne: soaring, gliding, and feathering. They utilize flapping-wing lift in forward flight when these opportunities are unavailable. In other phases of their flight, sufficiently large fixed-wing forces are not always available and large birds, like small birds, uses flapping-wing forces. Very small birds and most insects are apparently incapable of generating sufficient fixed-wing lift during forward flight. They utilize flapping-wing forces for the lift in all phases of their flights.

A swimming fish typically flaps its tail (caudal) fin to produce a thrust. Unlike the bird, the fish does not need to produce a lift. The axis of the flapping motion of the tail is perpendicular to the direction of the fish motion. In contrast, the axes of the flapping motion of the bird's wings are generally aligned with the bird's flight direction.

During the past few years, emphases on unmanned air vehicle (UAV) and micro air vehicle (MAV) have revived aeronautical interests in flapping wings (see, e.g. Mueller 2001). Successes in building small prototype ornithopters quickly followed (e.g. Jones and Platzer 2003). These successes indicate that, with modern material and control technology, mechanical difficulties attendant to the building of flapping-wing flyers, at least unmanned small ones, are surmountable. The question concerning the practicality of large flapping-wing aircraft cannot be answered today on a rational basis because principles of unsteady and non-streamlined aerodynamics remain unknown. One purpose of the present study is to demonstrate that vorticity dynamics, used a century ago by pioneering aerodynamicists as the key to unlock the mysteries of steady fixed-wing aerodynamics, once again provides opportunities in unsteady non-streamlined aerodynamic research. The goal of establishing a rational understanding of the enormously complex subject of non-streamlined and fully unsteady aerodynamics is clearly exceedingly challenging. At the same time, there are also unmistakable indications that the dynamics of the vorticity field is the underpinnings of aerodynamic principles whether the flow is non-streamlined or streamlined, fully unsteady or steady.

Basic research aimed at understanding important natural phenomena often brings forth unforeseen applications as valuable byproducts. With flapping wings and fins used so pervasively in the animal kingdom, there can be little doubt as to the usefulness of unsteady aerodynamic forces, in some forms and shapes, in aeronautics and in other engineering applications. If man is destined for flapping-wing flights, an understanding of unsteady aerodynamic principles is obviously indispensable in his attempt to build flapping wing flying machines. If man is not so destined, discoveries of unsteady aerodynamic principles will certainly bring forth new and important applications in aeronautics and in other fields of fluid engineering. Such applications may, in fact, go beyond flapping wings that represent a focus of unsteady aerodynamic research today.

7.3 Classical Versus Vorticity-Dynamic Approach

Von Karman and Burgers (1934) reviewed earlier studies of unsteady aerodynamics under the heading *Problems of Non-Uniform and of Curvilinear Motion*. They prefaced their review with the proviso: *……these subjects are in the midst, or even at the beginning of their development, so no final results are to be expected.* At the time of these studies, modern computational and experimental methods were unavailable and the aerodynamicist's ability to learn unsteady flow details was rather limited. Much of the pioneering studies were aimed at the extension of the circulation theory to unsteady aerodynamics. The strategy at that time was to investigate, using theoretical and experimental tools then available, the most meaningful unsteady motion parameters involving simple airfoil shapes.

There exists a bedazzling diversity of shapes and motion parameters of wings and fins in the animal kingdom. Laboratory measurements of flows surrounding (and forces on) flapping models and live animals have progressed rapidly in recent years at a number of universities and research institutes. Computational simulations of animal flights have advanced remarkably in the past three decades. Many worthy articles and books have been published on flapping-wing and flapping-fin aerodynamics (e.g. Childress 1981; Hu et al. 2004; Sun and Wu 2004; Wu et al. 1975). At this point in time, the aerodynamicist has at his disposal unprecedented experimental and computational abilities to simulate, observe, and measure unsteady flow details. With regards to the understanding of principles of unsteady non-streamlined aerodynamics, it is evident that, while the classical approach to aerodynamics was remarkably effective for streamlined steady flows, one must now venture beyond the bounds of the circulation theory. The strategy of treating the most meaningful unsteady motion parameters involving relatively simple shapes, nevertheless, remains imperative today.

It is informative at this point to examine the reference problem defined in Chap. 1 that of a wing initially at rest, from both the vorticity-dynamic and the classical viewpoints. Suppose the wing is set into rectilinear motion, accelerating at a constant rate $\mathbf{a}$ during the time interval $0 < t < \delta t$. At the time level $t = \delta t$, the velocity of the wing is $\delta u = \mathbf{a}\delta t$. The aerodynamic force experienced by the wing is to be evaluated. Prior to the start of the motion, the velocity, and hence also the vorticity, is zero at all points in the fluid region R_f. During the time interval δt, vorticity is introduced on the surface S of the wing to satisfy the new velocity boundary condition on S. Suppose δt is so small that the vorticity appearing on S is transported only a small distance into the fluid during this time interval. Since vorticity is neither created nor destroyed in the interior of the fluid, there is immediately adjacent to S a thin layer of vorticity at the time level $t = \delta t$. No vorticity exists outside the layer. Therefore the total vorticity in the layer is zero according to the principle of total vorticity conservation. Denote the layer region by R^*, the region of zero vorticity by R^-, and the boundary between R^* and R^- by S^-. The vorticity content in R^* can be, at the time level $t = \delta t$, approximately a vorticity sheet. In the limit as $\delta t \rightarrow 0$, the vorticity-sheet approximation is precise.

With the vorticity-dynamic approach, the strength $\boldsymbol{\gamma}$ in R^* and the flow in R_f, including the potential flow in R^-, are determined uniquely by Cauchy's condition and the pair of kinematic equations $\nabla \cdot \mathbf{v} = 0$ and $\nabla \times \mathbf{v} = \boldsymbol{\omega}$, with $\boldsymbol{\omega}$ known in R^-. The generalized Biot–Savart Law then provides a means to evaluate the flow in R_f explicitly whenever needed. With the classical approach, the surface S^- is typically viewed as coinciding with S. Neumann's condition is imposed on S (or S^-) to obtain a potential flow solution in R^-. Thus, the tangential velocity components on S are discontinuous. This discontinuity, however, is in reality is equivalent to the vortex sheet in the vorticity-dynamic approach. The discontinuity has two sides: the fluid side and the solid side. With the tangential velocity on the fluid side determined from the classical potential flow solution, the strength of the discontinuity $\boldsymbol{\gamma}$ is determined uniquely only if values of the tangential velocity components at all points on the solid side are prescribed. The prescription of tangential velocity on the solid side, however, is equivalent to the prescription of Dirichlet's condition. In summary, with the classical approach, the determination of the potential flow requires only Neumann's condition. However, if one is interested beyond the potential flow region, the classical approach, like the vorticity-dynamic approach, requires Cauchy's condition. As shown in Sects. 7.4 and 7.5, if $\boldsymbol{\gamma}$ is known, then the aerodynamic force experience by the wing during the time interval $0 < t < \delta t$ is determinate through the vorticity-moment theorem. No information about the potential flow in R^- is needed. The evaluation of the potential flow in R^- is unnecessary and the vorticity-dynamic approach bypasses the need to know the potential flow.

A layer of vorticity adjacent to the wing can be approximated by a vortex sheet in vorticity kinematics simply because they are thin. The existence of thin vorticity layers next to S (in R^*), on the other hand, is understandable from vorticity kinetics. Thin vorticity layers exist under at least two circumstances. If the flow Reynolds number is very large, say much greater than 1000, then the speed of convection is much grater then that of diffusion. In streamlined, or attached, flows, the direction of the flow adjacent to S is nearly tangential to S. This means that the diffusion process spreads the vorticity, introduced on S, only a short distance into the interior of the fluid domain before it is carried downstream by convection. Under this circumstance, vorticity layers are in reality familiar boundary layers. The second circumstance involves unsteady flows in response to accelerations of wings. In this situation, the vorticity layer is thin not because the flow is streamlined, but because the time interval δt is so small that the kinetic processes has no time to transport vorticity away from the S, where it originates.

In the flow containing a thin vorticity layer adjacent to S, the strength of the vortex sheet representing the layer is uniquely determined by Cauchy's condition on S provided the vorticity distribution outside the layer is known. In boundary layer flows, if the first scenario discussed in Sect. 4.8 is used, i.e., if the starting vortex is excluded from the region of analysis, then trailing vortices cross over the external boundary of the region, causing uncertainties in the boundary conditions to prescribe there. For these reasons, the second scenario discussed in Sect. 4.8 is used in Chap. 6. In this scenario, the region of study contains the starting and the trailing vortices. In comparison, the vorticity layer strength next to S short after the motion's onset is simple to determine.

Shortly after the motion's onset, there is no vorticity outside R^* at all. Since the total vorticity in R^- is zero, the circulation (total vorticity) of the vorticity in the layer R^* surrounding the non-rotating wing is zero. Therefore the vortex sheet strength $\boldsymbol{\gamma}$ in R^* is uniquely determined by values of the velocity vector (all three components in three-dimensions), i.e., Cauchy's condition. In two-dimensional flows, the flow region surrounding the airfoil is doubly connected. The potential flow around the airfoil, however, is acyclic. In three-dimensional flows, if the wing is not rotating, the vorticity tubes are all closed within the layer R^*. In short, the vortex sheet strength $\boldsymbol{\gamma}$ is determined uniquely by vorticity kinematics alone. As a result, unsteady vorticity layers are remarkably amenable to mathematical analyses using the vorticity-dynamic approach.

Von Karman and Burgers (1934) discussed the two-dimensional flow surrounding an airfoil originally at rest and concluded: *This initial motion nowhere presents a circulation around the airfoil.* Von Karman and Burgers used the term *Dirichlet-motion* to denote *an irrotational motion of an ideal fluid along a body* and considered the no-penetration condition to be a part of the Dirichlet motion. In the present study, for reasons discussed in Chap. 3, the no-penetration condition is referred to as Neumann's condition and the no-slip condition as Dirichlet's condition.

While the circulation of the vortex sheet surrounding the wing is zero at the time level δt, the moment of this vortex sheet is in general not zero. According to the vorticity-moment theorem, because the vorticity moment in the fluid undergoes a change during the interval $0 < t < \delta t$, the wing experiences an aerodynamic force during this time interval. This force is a reaction to the acceleration of the wing (from 0 to δU in the present case) and is said to be a consequence of the apparent mass, or virtual mass, of the wing. In classical aerodynamics, the evaluation of apparent mass properties is an exceedingly difficult task for most wing shapes. With the vorticity-dynamic approach, the determination of apparent mass properties is relatively simple, as is shown in Sect. 7.5.

The vorticity distribution in R* is changed by convection and diffusion as the wing continues to accelerate. As time progresses, the thin layers of vorticity in R* develops into two boundary layers covering the upper and lower surfaces of the wing. Concurrent with the development of boundary layers, the vorticity is shed into the wake. The starting vortex and the trailing vortices are formed, and a circulation appears around the wing. Von Karman and Burgers (1934) discussed the *Origin of the Circulation around the Airfoil* in connection with these occurrences. The formation of the trailing vortices is not discussed since these vortices are not a part of the two-dimensional flow. Suppose the acceleration of the wing continues until the wing reaches a constant velocity and is kept moving at this velocity thereafter, then in time a steady flow is established near the wing, provided the flow is streamlined. The steady aerodynamic force on the wing is reasonably predictable using the classical circulation and the boundary layer theories.

From the viewpoint of the vorticity-moment theorem, the unsteady force on the wing is related to two major events that cause the vorticity moment in the fluid to change with time: (i) the appearance of the vorticity layer over the wing's surface in response to the wing's acceleration and (ii) the subsequent development of the circulation around the wing as boundary layers, the starting vortex, and the associated trailing vortices are formed. Motions starting from rest are special situations where the wing experiences unsteady forces. Suppose the wing is initially not at rest, but is undergoing a steady rectilinear motion. At the time level, say $t = t_o$, the wing starts accelerating rectilinearly. This acceleration introduces a new layer of vorticity in R* to satisfy the changed Cauchy's condition on the wing surface. Again, this layer of vorticity has a zero circulation but a non-zero vorticity moment. This layer of vorticity merges into the existing boundary layers and causes new starting vorticity and new trailing vortices to form. With changes of the vorticity moment in the fluid, the accelerating wing continually experiences an unsteady force. This force is again related to the two major events described earlier.

A finite wing may heave, pitch, and roll and yaw about certain axes. The wing may undergo accelerations that are various combinations of these modes of motion. For aerial animals, there may be one or more pairs of flapping, clapping, and flinging wings. For aquatic animals, there are caudal fins that yaw in the case of most fishes and pitch in the case of most marine mammals. In many situations, the motions of the wings and fins are periodical, leaving a sequence of starting vortices as Karman vortices in reverse. For hovering animals, the starting vortices often remain near the wings. These wings and fins, moreover, have a diversity of planform, camber, twist, sectional profile, etc. Without exception, however, the unsteady forces on wings and fins are connected to the two major events just outlined.

Starting vortices and their associated tip vortices are often conspicuous. In classical aerodynamics, considerable efforts have been devoted to the understanding of the relation between these visible wake features and the unsteady forces on the wing. The two major events described above, specially the first event, however, deserves added attention.

There are many open issues related to the evolution of circulation around wings (and fins). This evolution follows immediately the first event: the appearance of the thin vorticity layer on wing surfaces in response to the acceleration of the wing. The question as to the quantitative importance of viscosity as related to this event is at the present not fully answered. For hovering animals, using for example the Weis-Fogh lift generation mechanism, the meaning of a flow Reynolds number defined based on a representative velocity values is uncertain. These issues involve vorticity kinetics and are difficult to treat quantitatively because of the non-linearity of the problem. There are, however, clear opportunities for the aerodynamicist to contribute to the understanding of these issues through extensive experimental, computational, and theoretical studies.

The force brought about by the first and seminal event—the apparent mass force due to the appearance of the vortex sheet—is controllable: It is directly proportional to the rate of acceleration of the wing or the fin. This fact suggests possibilities of its effective use in manmade flyers and fluid machineries.

7.4 Apparent Mass of Sphere and Circular Disk

In the unsteady motion of a wing, the force required to accelerate the wing is different from the force required to overcome the wing's inertia. Even if the mass of the wing is zero, as in the case of an infinitesimally thin wing, a force is required to accelerate the wing in the fluid in the direction perpendicular to the plane of the wing. A motion of the wing induces a corresponding motion of the fluid. This means that the kinetic energy is imparted to the fluid. The wing must performed work to impart this energy and therefore experiences a resistance to its acceleration. This resistance is commonly said to be the effect of the *apparent mass* or the *virtual mass*.

As an example, consider a sphere of radius *a* set into rectilinear motion from rest in the x-direction, accelerating at a constant rate $\boldsymbol{\alpha} = \alpha\mathbf{i}$ during the time interval $0 < t < \delta t$. The velocity of the sphere at the time level $t = \delta t$ is $\delta u\ \mathbf{i}$, where $\delta u = \alpha\delta t$. Consider a function $\phi = -\frac{1}{2}(\delta u)a^3r^{-2}\cos\theta$, where θ is the angle between the x-axis and the position vector $\mathbf{r}$. The potential flow velocity field $\mathbf{v}$ corresponding to ϕ is:

$$\mathbf{v} = \Delta\phi = (\delta u)a^3r^{-3}(\cos\theta\mathbf{e}_r + 1/2\ \sin\theta\mathbf{e}_\theta), \tag{7.1}$$

where $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are unit vectors in the spherical coordinate system (r, θ, φ).

As $\mathbf{r} \to \infty$, the velocity given by (7.1) goes to zero. On the surface S of a sphere of radius a centered at the point of origin ($\mathbf{r} = 0$), (7.1) gives

$$\mathbf{v} \cdot \mathbf{n} = (\delta\, u) \cos\theta \tag{7.2}$$

Thus, the velocity field (7.1) satisfies Neumann's condition on S, the surface of the sphere moving at the velocity $\delta u\mathbf{i}$. It can be shown that, corresponding to (7.1), the total kinetic energy T in the fluid at the time level $t = \delta t$ is

$$T = \frac{1}{3}\rho\pi a^3 (\delta u)^2 \tag{7.3}$$

During the time interval δt, the distance $\delta x\mathbf{i}$ traveled by the sphere is $\frac{1}{2}\alpha\delta t^2\mathbf{i}$. Therefore the work done by the sphere on the fluid is $-\frac{1}{2}\alpha\delta t^2 F_x = -\frac{1}{2}\delta u^2 F_x/\alpha$, where $-F_x$ is the x-component of the aerodynamic force exerted by the fluid on the sphere. Since the work done by the sphere during the time interval $0 < t < \delta t$ is equal to the kinetic energy of the fluid at the time level $t = \delta t$, one obtains:

$$F_x = -\frac{2}{3}\rho\pi\, a^3 \alpha \tag{7.4a}$$

$$= -(1/2)\, m_s \alpha, \tag{7.4b}$$

where m_s is the mass of the fluid displaced by the sphere.

Equation (7.4b) states that, as the sphere accelerates, it overcomes not only its own inertia, but also an additional resistance. This additional resistance is equivalent to the sphere carrying along an additional mass, the apparent mass, in the amount $(2/3)\rho\pi a^3$ as it accelerates. The apparent mass is often used as a gauge to measure the importance of the unsteady aerodynamic force due to the acceleration of the solid. For the sphere, the apparent mass is one half of the mass of the fluid displaced by the sphere.

Lighthill (1986) noted that *for a solid sphere immersed in air, the added mass may represent a negligible addition to the normal* (solid) *mass of the sphere. If the sphere is immersed in water, however, the addition becomes significant, although our formulas ……apply only in the very early stages of the sphere's motion before the drag associated with the formation of a vortex wake becomes significant*. In this regard, it is noted that the apparent mass properties of a solid body depend on the shape and the direction of acceleration of the solid body. For an infinitesimally thin circular disk of radius *a* moving at the velocity δu perpendicular to the plane of the disk, the total kinetic energy of the fluid, for a potential flow, is $(4/3)\rho a^3(\delta u)^2$ (Batchelor 1967). In this case, the disk has an infinitesimal normal mass, but its apparent mass, $(8/3)\rho a^3$, is comparable to that of a sphere of the same radius if the disk accelerates in the direction perpendicular to its plane. In contrast, the kinetic energy associated with a circular disk moving and accelerating in its plane is zero, assuming potential flow. The wing of the aerial animal is typically thin and typically accelerates more or less in the direction perpendicular to the plane of the wing. These wings are more like the circular disk than the sphere. Their apparent mass can be several orders of magnitude greater than the true mass of the wing. With the unsteady force associated with the apparent mass proportional to the acceleration rate, large forces are obviously obtainable by very rapid accelerations.

Apparent mass properties are tensorial. A wing accelerating in the x-directions can experience forces in the y- and z-directions as well as in the x-directions. The sphere and the circular disk are among the very few three-dimensional shapes with known analytical expressions of apparent mass. For the sphere, the apparent mass tensor contracts to a scalar. As discussed, the appearance of the vorticity layer over the wing and fin surfaces is the seminal event in the production of unsteady aerodynamic forces by flapping, clapping, flinging, etc. New vorticity layers continue to appear in R* subsequent to the formulation of wakes as long as the wing continues accelerating. Rotation is a form of acceleration (centripetal) even at a constant angular velocity. It is important to recognize that animals use apparent mass effects to produce unsteady thrust and lift for locomotion. These forces are resistance forces. They are, however, not drags in cruising flight: They are forces produced for good purposes, not forces to be alleviated.

The apparent mass properties can be determined using the vorticity-moment equation:

$$F = -\frac{\rho}{2}\frac{d}{dt}\iiint_{R^{\infty}} \mathbf{r}\times\omega dR + \rho\frac{d}{dt}\iiint_{R_S} \mathbf{v}dR \tag{7.5}$$

Consider again the accelerating sphere. The last integral in (7.5) yields

$$\rho\frac{d}{dt}\iiint_{R_S} \mathbf{v}dR = \frac{4}{3}\rho\pi\mathbf{a}^3\alpha\mathbf{i} \tag{7.6}$$

Subtracting $\delta u\mathbf{i}$ from (7.1) gives the velocity around the sphere at the time level $t = \delta t$ observed in a reference frame moving at the velocity $\delta u\mathbf{i}$:

$$\mathbf{v} = (\delta u)[(a^3r^{-3} - 1)\cos\theta\mathbf{e}_r + (a^3r^{-3} + 1/2)\sin\theta\mathbf{e}_\theta] \tag{7.7}$$

Equation (7.7) satisfies Neumann's condition $\mathbf{v} \cdot \mathbf{n} = 0$ on the surface S of the sphere and gives a tangential velocity $3(\delta u)\sin\theta/2$ adjacent to S. With the no-slip condition, the tangential component of the fluid velocity at the inner boundary S is zero as observed in the reference moving with the sphere. This difference between zero and $3\delta u\sin\theta/2$, the θ-component of the velocity given by (7.7) as $r \to a$, means the presence of a thin layer of vorticity of strength (integrated value of vorticity across the layer) $3(\delta u)\sin\theta/2$ in the fluid immediately adjacent to the surface S of the sphere at the time level $t = \delta t$.

Specifically, during the time interval δt, a layer of vorticity appears in the fluid immediately adjacent to the surface S of the wing to satisfy the new velocity boundary condition on S (see discussions in Chap. 3). Suppose δt is very small and the transport of vorticity in the layer by convection and diffusion is insignificant, then this layer is very thin and can be approximated by a sheet of vorticity on S*, a surface surrounding the sphere. This surface is infinitesimally close to S, but is distinct from S: It is a part of the fluid domain and not the boundary of the fluid. The thickness of the vortex sheet is infinitesimally small as an approximation, but not zero. In the limit as $\delta t \to 0$, the approximation becomes precise. The vorticity at all points outside this sheet is zero and the strength γ_x of this sheet (the subscript x indicates the vorticity layer appears as a result of the acceleration in the x-direction), as shown in Chap. 3, is uniquely determined by the known velocity on S. For the sphere at the time level $t = \delta t$, the strength of the vortex sheet is given by

$$\boldsymbol{\gamma}_x = \frac{3}{2}(\delta u)\sin\theta\,\mathbf{e}_\varphi \tag{7.8}$$

In (7.8), $\mathbf{e}_\varphi$ is the unit vector in the φ-direction of the spherical coordinate system defined earlier. With $\mathbf{e}_\varphi = -\sin\varphi\,\mathbf{j} + \cos\varphi\,\mathbf{k}$, it is not difficult to show that the integration of $\boldsymbol{\gamma}_x$ over the surface S* is zero. According to the principle of conservation of total vorticity (Chap. 5), the total vorticity in R_∞ is zero. Since at the time level $t = \delta t$, the only vorticity present in R_∞ is contained in the layer of vorticity represented by the vortex sheet $\boldsymbol{\gamma}_x$, (7.8) satisfies the principle of total vorticity conservation. The first integral in (7.5) is the total first moment of vorticity in R_∞. At the time level $t = \delta t$, this integral is approximately:

$$\oiint_{S^*} \mathbf{r}\mathbf{x}\boldsymbol{\gamma}_x\,dS = (\delta u)\oiint_{S^*} \mathbf{r}\mathbf{x}\boldsymbol{\beta}_x\,dS \tag{7.9}$$

where $\boldsymbol{\beta}_x = \boldsymbol{\gamma}_x/(\delta u)$ is the strength of the vortex sheet that appear on S* in response to a unit change of velocity in the x-direction. From (7.8), one obtains

$$\oiint_{S} \mathbf{r}\mathbf{x}\boldsymbol{\beta}_x\,dS = -\frac{3}{2}a^3\int_0^\pi\left(\int_0^{2\pi}\mathbf{e}_\theta d\varphi\right)\sin^2\theta\,d\theta \tag{7.10}$$

With $\mathbf{e}_\theta = -\sin\theta\ \mathbf{i} + \cos\theta\ \cos\varphi\ \mathbf{j} + \cos\theta\ \sin\varphi\ \mathbf{k}$, the integration of $\mathbf{e}_\theta$ over $0 < \varphi < 2\pi$ gives $-2\pi\sin\theta\,\mathbf{i}$. Therefore, upon carrying out the integration over θ in (7.10) and placing the result in (7.9), one finds that the total first moment of vorticity at the time level $t = \delta t$ is $4\pi a^3(\delta u)\mathbf{i} = 4\pi a^3\alpha(\delta t)\mathbf{i}$. In (7.10), the asterisk for S* is dropped for convenience. This is permissible for the present purpose since, although the surfaces S and S* are conceptually different, they are infinitesimally close. Because the total vorticity in R_∞ is zero, the reference point for the evaluation of the total vorticity moment can be arbitrarily selected. In (7.10), for convenience, the reference point is selected to be the center of the sphere at the time level $t = \delta t$. The total first moment of vorticity at the time level $t = 0$ is zero. Therefore the total first moment of vorticity changes by the amount $4\pi a^3\alpha(\delta t)\mathbf{i}$ during the period $0 < t < \delta t$. Placing this expression and (7.6) into (7.5) yields, in the limit as $\delta t \to 0$,

$$\mathbf{F} = -\frac{2}{3}\rho\pi a^3\alpha\mathbf{i} \tag{7.11}$$

Equation (7.11) is obtained using the vorticity-moment theorem. This result is identical to (7.4a). Thus, the vorticity-moment theorem and the kinetic energy method yield the same apparent mass, one-half of the mass of the fluid displaced by the sphere, for the sphere.

7.5 Apparent Mass Coefficients

As mentioned earlier, apparent mass properties are tensorial. Consider a nonrotating right cylinder of arbitrary shape with its axis (span) in the y-direction. The cylinder is moving in the z-x plane and accelerating at the rate $\boldsymbol{\alpha} = \alpha_z\,\mathbf{k} + \alpha_x\,\mathbf{i}$. According to (5.69), the vorticity-moment equation for $\mathbf{f}$, the force per unit span, on the cylinder is given by

$$\mathbf{f} = -\rho\frac{d}{dt}\iint_{R_\infty}\mathbf{r}\times\boldsymbol{\omega}dR + \rho\frac{d}{dt}\iint_{R_s}\mathbf{v}dR \tag{7.12}$$

The last integral in (7.12) times the density ρ of the fluid is the mass per unit span of the cylinder of the fluid displaced by the airfoil. Designating this mass per unit span by m, the last term in (7.12) is then, for the cylinder accelerating at the rate $\boldsymbol{\alpha}$, the rate of change of momentum of m. That is,

$$\rho\frac{d}{dt}\iint_{R_s}\mathbf{v}dR = m\alpha \tag{7.13}$$

Consider the case $\boldsymbol{\alpha} = \alpha_x\mathbf{i}$. In response to a change of velocity δu during the time interval δt, a vortex sheet of strength $\boldsymbol{\gamma}_x$, where the subscript x identifies the strength of the vortex sheet with the acceleration $\alpha_x\mathbf{i}$, appears on s*. Following the discussions of Sect. 7.4, the first integral in (7.12) is restated as:

$$-\rho\frac{d}{dt}\oint_s \mathbf{r}\,\mathbf{x}\,\boldsymbol{\gamma}_x ds = -\rho\lim_{\delta t\to 0}\left[\frac{\delta u}{\delta t}\left(\mathbf{k}\oint_s(x\,\beta_x)ds - \mathbf{i}\oint_s(z\beta_x)ds\right)\right] \tag{7.14}$$

In (7.14), the asterisk for s* is dropped for convenience because all points on s* are infinitesimally close to s. The vector $\boldsymbol{\beta}_x = \beta_x\,\mathbf{j}$ is the strength of the vortex sheet that appears on s* in response to a unit change of velocity in the x-direction. That is, $\boldsymbol{\beta}_x = \gamma_x/\delta u$.

As noted in Sect. 7.4, the total vorticity in R_∞ is zero. Since vorticity is non-zero only over s*, the integrals of $\boldsymbol{\gamma}_x$, or $\boldsymbol{\beta}_x$, over s* are zero because of the principle of total vorticity conservation. The reference point about which the vorticity moment is evaluated in (7.14) can therefore be chosen arbitrarily. This fact is readily shown using the identity:

$$\oint_s (\mathbf{r}-\mathbf{r}_a)\times\gamma_x ds = \oint_s \mathbf{r}\times\gamma_x ds - \mathbf{r}_a\times\oint_s \gamma_x ds, \tag{7.15}$$

where $\mathbf{r}_a$ is a fixed point in space.

Since the last term in (7.15) vanishes, the value of the total vorticity moment does not depend on the reference point. In consequence, since the integrals in (7.14) are independent of the reference point, they are invariant to Galilean coordinate transformations. It is important to note, however, that over a region in which the total vorticity is not zero, the total vorticity moment is dependent on the reference point.

Placing (7.14) and (7.13) into (7.12) and noting that, $\delta u/\delta t$ goes to α_x in the limit as $\delta t \to 0$, one obtains the following expression for $\mathbf{f}_x$, the unsteady force on the airfoil due to its acceleration in the x-direction at the rate $\alpha_x\mathbf{i}$:

$$\mathbf{f}_x = -\rho\alpha_x \oint_s (x\,\beta_X) ds\mathbf{k} + \rho\alpha_x \oint_s (\mathbf{z}\,\boldsymbol{\beta}_x) ds\mathbf{i} + m\alpha_x\mathbf{i} \tag{7.16}$$

According to (7.16), as the cylinder accelerates in the x-direction, it experiences a force $\mathbf{f}_x$ that has non-zero components in both the x- and the z-directions. The x-component force, as a resistance to acceleration, is conveniently interpreted as an apparent mass effect. The presence of a force perpendicular to the direction of acceleration, however, needs proper recognition. If the idea of the apparent mass is retained and one writes the apparent mass in the vector form $\mathbf{m}_x = m_{xz}\mathbf{k} + m_{xx}\mathbf{i}$ and defines $\mathbf{m}_x$ by

$$\mathbf{f}_x = -\mathbf{m}_x\boldsymbol{\alpha}_x \tag{7.17}$$

then (7.16) yields

$$m_{xx} = -\rho\oint_s (z\,\beta_x) ds - m \tag{7.18}$$

$$m_{xz} = \rho\oint_s (x\,\beta_x) ds \tag{7.19}$$

Consider the case $\boldsymbol{\alpha} = \alpha_z \mathbf{k}$. Parallel to the analysis that yields (7.16), one obtains,

$$\mathbf{f}_z = -\rho\alpha_z \oint_s (x\beta_z)ds\mathbf{k} + \rho\alpha_z \oint_s (z\beta_z)ds\mathbf{i} + m\alpha_z\mathbf{k} \tag{7.20}$$

For this case, again the acceleration produces a force with two components, one in the direction of the acceleration and one perpendicular to it. One obtains, by defining the apparent mass in the vector form $\mathbf{m}_z = m_{zz}\mathbf{k} + m_{zx}\mathbf{i}$ and writing $\mathbf{f}_z = -\mathbf{m}_z\alpha_z$:

$$m_{zx} = -\rho \oint_s (z\,\beta_z)ds \tag{7.21}$$

$$m_{zz} = \rho \oint_s (x\,\beta_z)ds - m \tag{7.22}$$

The force $\mathbf{f}$ per unit span on the airfoil due to the rectilinear acceleration $\boldsymbol{\alpha} = \alpha_z\mathbf{k} + \alpha_x\mathbf{i}$ of the airfoil is the sum of $\mathbf{f}_z$ and $\mathbf{f}_x$. Therefore, one obtains the following matrix expression for $\mathbf{f}$:

$$\begin{Bmatrix} f_z \\ f_x \end{Bmatrix} = -\begin{bmatrix} m_{zz} & m_{xz} \\ m_{zx} & m_{xx} \end{bmatrix} \begin{Bmatrix} \alpha_z \\ \alpha_x \end{Bmatrix} \tag{7.23}$$

where the apparent mass coefficients m_{xx}, m_{xz}, m_{zx}, m_{zz}, are given by (7.18), (7.19), (7.21), and (7.22) respectively.

Using procedures similar to those leading to (7.16) and (7.20), (7.5) yields the following expression $\mathbf{F}_x$ for the force on a finite wing accelerating at the rate $\boldsymbol{\alpha} = \alpha_x\mathbf{i}$:

$$\mathbf{F}_x = -\frac{\rho}{2}\alpha_x \oiint_S \left[(y\beta_{xz} - z\beta_{xy})\mathbf{i} + (z\beta_{xx} - x\beta_{xz})\mathbf{j} + (x\beta_{xy} - y\beta_{xx})\mathbf{k}\right]dS + M\alpha_x\mathbf{i} \tag{7.24}$$

In (7.24), M is the mass of the fluid displaced by the wing. β_{xx}, β_{xy}, and β_{xz} are, respectively, the x-, y-, and z-components of the vector $\boldsymbol{\beta}_x$, the strength of the vortex sheet on S* which surrounds and is immediately adjacent to S (see Sect. 7.4 and note that the asterisk in S* is dropped in (7.24) for convenience). This vortex sheet appears in response to the acceleration of the wing in the x-direction. $\boldsymbol{\beta}_x$ is interpretable as the strength of the vortex sheet surrounding the wing that moves at unit speed in the x-direction. The flow outside the sheet is irrotational and acyclic. Therefore $\boldsymbol{\beta}_x$ can be determined readily using a suitable computer code for three-dimensional potential flows around wings.

Writing $\mathbf{M}_x = M_{xx}\mathbf{i} + M_{xy}\mathbf{j} + M_{xz}\mathbf{k}$ and $\mathbf{F}_x = -\mathbf{M}_x\alpha_x$, one obtains:

$$M_{xx} = \frac{\rho}{2}\oiint_S (y\beta_{xz} - z\beta_{xy})dS - M \tag{7.25}$$

$$M_{xy} = \frac{\rho}{2}\oiint_S (z\beta_{xy} - x\beta_{xz})dS \tag{7.26}$$

$$M_{xz} = \frac{\rho}{2}\oiint_S (x\beta_{xy} - y\beta_{xx})dS \tag{7.27}$$

Apparent mass coefficients for the wing undergoing rectilinear accelerations $\alpha_y\mathbf{j}$ and $\alpha_z\mathbf{k}$ are similarly subject to the forces $\mathbf{F}_y = -\mathbf{M}_y\alpha_y$ and $\mathbf{F}_z = -\mathbf{M}_z\alpha_z$. One thus writes

$$\begin{Bmatrix} \mathbf{F}_x \\ \mathbf{F}_y \\ \mathbf{F}_z \end{Bmatrix} = \begin{bmatrix} \mathbf{M}_{xx} & \mathbf{M}_{yx} & \mathbf{M}_{zx} \\ \mathbf{M}_{xy} & \mathbf{M}_{yy} & \mathbf{M}_{zy} \\ \mathbf{M}_{xz} & \mathbf{M}_{yz} & \mathbf{M}_{zz} \end{bmatrix} \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{Bmatrix} \tag{7.28}$$

Using the indicial notation, with $x = x_1$, $y = x_2$, and $z = x_3$ and the summation convention, (7.28) can be stated as:

$$\mathbf{F}_i = -\mathbf{M}_{ni}a_n \tag{7.29}$$

The apparent mass coefficients $\mathbf{M}_{ni}$ in (7.29) are given by:

$$\mathbf{M}_{ni} = \frac{\rho}{2}\oiint_S \varepsilon_{ijk}x_j\beta_{nk}dS - \delta_{ni}m, \tag{7.30}$$

where ε_{ijk} is the permutation symbol, δ_{ni} is the Kronecker delta, and β_{nk} is the strength of the x_k component of the vortex sheet $\boldsymbol{\beta}_n$. The flow outside S* is irrotational and acyclic.

The apparent mass coefficients $\mathbf{M}_{ni}$ are dependent only on the shape of the wing's surface S. Suppose the geometry of S is specified, then the vectors $\boldsymbol{\beta}_n$ can be computed using the generalized Biot–Savart law (see Sect. 3.8). These vectors are unique functions of the coordinates of S. Placing their values into (7.30) gives a set of coefficients $\mathbf{M}_{ni}$ that are constants for any specific S. The unsteady forces due to the acceleration of the rigid body bounded by S are equal to the product of these coefficients and the acceleration rates. Once the coefficients are obtained, they can be stored and used to evaluate the unsteady forces produced with various unsteady motion parameters. Different geometric parameters that make up the shape of the surface S can be researched with relative ease.

For two-dimensional flows, the computations of the apparent mass coefficients are remarkably simple. Airfoils of all shapes can in general be mapped onto a unit circle using conformal transformation. The vortex strength functions β_{zz}, β_{xx}, β_{xz}, and β_{zx} can be shown to be the scale factor of the transformation times the corresponding functions in the transformed plane. Therefore, the task of evaluating the functions β_{zz}, β_{xx}, β_{xz}, and β_{zx} for an airfoil is equivalent to the finding of a conformal transformation that maps the airfoil onto the circle. For many airfoil and surface shapes, for example, the Joukowski airfoil, closed-form expressions for these functions are obtainable. The values of the apparent mass coefficients are then immediately available. For other airfoil shapes, the numerical evaluation of these functions follows immediately from the numerical conformal mapping of the airfoil onto the circle.

For a few very simple three-dimensional surface shapes, for example the sphere, the circular disk and Rankine ovoid, closed-form expressions for the vortex vectors $\boldsymbol{\beta}_n$ on S are obtainable. For more complex three-dimensional shapes, these vectors are in generally obtainable numerically using a suitable panel code. Many currently available codes are known to be capable of routine, accurate, and efficient computation of three-dimensional potential flows. They can be used in computing the vortex vectors $\boldsymbol{\beta}_{n.}$ It is important to note that, with the generalized law of Biot–Savart, the computation of these vortex vectors requires only the solution of a Fredholm's integral equation over S, and not the potential flow-solution. This means only the strengths of the vortex panels (or equivalently the sour-sink panels) need to be computed. Flow computations are not needed and can be bypassed completely in computing $\boldsymbol{\beta}_{n.}$

The procedure outlined above can be generalized for the determination of the apparent mass forces related to angular accelerations. Several additional issues, however, need to be incorporated into aerodynamic analyses. For axisymmetric bodies rotating about their axes of symmetry, if the effects of viscosity are ignored, the rotation does not perform work and does not add to the kinetic energy of the fluid. For all other shapes and for all axisymmetric shapes rotating about axes that are not axes of symmetry, work is performed and kinetic energy is continually added to the fluid. This energy input needs to be accounted for in aerodynamic analysis, even if the bodies are not accelerating. With the vorticity-moment theorem, the vorticity layer appearing on the surface of the wing due to angular acceleration needs to be properly demarcated from that associated with a constant rotation. Also important is the fact that, with a wing undergoing rotation, the vorticity in the solid region is not zero. Therefore, the total vorticity in the fluid region, according to the principle of total vorticity conservation, is also not zero. In consequence, the value of the total vorticity moment in the fluid region is dependent on the reference point about which the vorticity moment is evaluated. As discussed in Chap. 2, the vorticity field in the infinite unlimited region jointly occupied by the fluid and the solid can be portrayed by a set of closed vorticity loops. With non-rotating solids, the vorticity in the solid region is zero and all the vorticity loops are closed in the fluid region. With rotating solid, however, some of the vorticity loops must pass through the solid region. This situation must be properly dealt with in aerodynamic analyses using the vorticity-moment theorem.

The issues just described do not present conceptual difficulties. They need, however, to be properly included in studies of wings that are rotating. In studies of flapping wings, especially those of insects, the assumption that the wings are infinitesimally thin is appropriate in many applications. With such thin wings, the vorticity content of the wing is negligible. Therefore the study of thin flapping wings simplified the proper accounting of the total vorticity in the fluid. The task of determining the work done by a rotating wing on the fluid is also simpler for the thin wing than for the thick wing, especially if the wing is flat. Studies of thin and flat wings with relatively simple planform undergoing simple accelerations—plunging, heaving, rolling, yawing, pitching but not overly complex combinations of these modes—are feasible and worthy of sustained efforts.

Being the seminal event in the chain of proceedings responsible for unsteady aerodynamic forces, the appearance of the vorticity layer in response to the acceleration of wings has a special physical significance that cannot be overemphasized. The initial distribution of vorticity within this layer is dependent only the shape and the acceleration mode. The strength of the layer is proportional to the acceleration rate. Subsequent changes of the vorticity distribution in the ambient zone R_b are dependent not only on the shapes and the motion parameters but also on the initial distribution. This suggests that apparent mass-properties are intrinsic to the principle of unsteady aerodynamics. There is a possibility that studies of events subsequent to the appearance of the vorticity layer can take advantage of simplifications proffered by, for example, the unsteady boundary layer theory and the development of appropriate unsteady flow models. Such possibilities cannot be quantitatively explored without knowing first the initial vorticity distributions.

As noted earlier, historically, successes in theoretical aerodynamics were in general linked to discoveries that permit the prediction of aerodynamic force without requiring the knowledge of complete flow details. Much of the recent advances in aerodynamics, in contrast, are in the arena of experimental and computational methods dealing with flow details. Saffman (1992) prefaced his book on vortex dynamics with the following remark about modern and amazingly rapid advances in experiments and computations: *Of course, this is not an unmixed blessing as data and information are produced at a rate greater than the capability of the average scientist to absorb it.* Lamb's (1932) earlier remark: *The motion of a solid in a liquid (incompressible flow) endowed with vorticity is a problem of considerable interest, but is unfortunately not very tractable* still rings true today. Modern developments in vorticity dynamics, however, offers a ray of hope that, the powerful strategy of bypassing flow details may still be useful, this time in our quest for the principles of unsteady and non-streamlined aerodynamics.

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