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Airfoil Theory for Non-Uniform Motion

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ABSTRACT

The basic conceptions of the circulation theory of airfoils are reviewed briefly, and the mechanism by which a "wake" of vorticity is produced by an airfoil in non-uniform motion is pointed out. It is shown how the lift and moment acting upon an airfoil in the two-dimensional case may be calculated directly from simple physical considerations of momentum and moment of momentum. After a calculation of the induction effects of a wake vortex, formulae for the lift and moment are obtained which are applicable to all cases of motion of a two-dimensional thin airfoil in which the wake produced is approximately flat; *i.e.*, in which the movement of the airfoil normal to its mean path is small.

The general results are applied first to the case of an oscillating airfoil and then to the problem of a plane airfoil entering a "sharp-edged" gust. In the latter case the rate of increase of the lift after the entrance of the airfoil into the gust boundary is determined, and it is shown that during the entire process the lift acts at the quarter-chord point of the airfoil.

The intention of the authors has been to make the airfoil theory of non-uniform motion more accessible to engineers by showing the physical significance of the various steps of the mathematical deductions, and to present the results of the theory in a form suitable for immediate application to certain flutter and gust problems.

I. INTRODUCTION

THE theory of non-stationary motion around airfoils has important applications, perhaps the most significant problems involved being flutter and the forces experienced by airplanes flying through gusts. The theory has been developed by a number of authors, in particular Birnbaum, Wagner, Küssner, Glauert, and Theodorsen. However, the equations obtained by most of these authors are rather complicated, and, in addition, the methods applied to the calculation of the forces are not especially transparent in the light of physical evidence. One of the English authors who recently dealt with the problem in a rather general manner declared that "the general formulae we have now obtained are rather too complex to convey directly any idea of their physical significance."[1] The idea of this paper is to eliminate unnecessary mathematical complications and to try to use only the basic conceptions of the vortex theory familiar to the modern aeronautical engineer.

It is advisable to review the fundamental conceptions of the circulation theory of airfoils for the case of two-dimensional motion, *i.e.*, of infinite aspect ratio. The airfoil put into motion creates a circulation around itself due to the presence of the sharp trailing edge; however, according to the theorem of conservation of moment of momentum, a counter-circulation develops in the fluid, called the "starting vortex." The airfoil and the starting vortex constitute a vortex-pair with an increasing distance between the two vortex lines, because the starting vortex remains at the place where it was created (neglecting its slow displacement perpendicular to the flight direction). Hence the momentum of the vortex pair, being proportional to the product of the circulation and the distance between the two vortices, increases continuously. The rate of this increase of the momentum is equal to the lift.

The first refinement of this simplified picture is the representation of the airfoil by a vortex sheet, *i.e.*, by a system of vortex lines with a continuous distribution of vorticity. Combined with the assumption that the starting vortex is so far behind the airfoil that it does not influence the flow at the airfoil, this leads to the so-

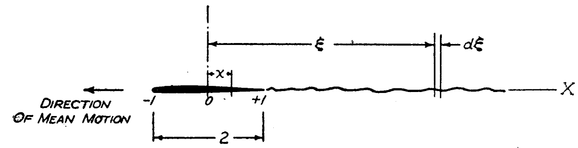

FIG. 1. Diagram showing notation employed.

called "stationary" theory of thin airfoils. However, if the total circulation around the airfoil is variable, because of non-uniformity of the motion, the airfoil leaves a wake composed of continuously distributed vortex lines behind itself, and the influence of this wake must be taken into account. The intensity of the vortex lines which constitute the wake is determined in this case by two conditions: (a) that the total circulation of the whole system is invariably equal to zero, and (b) that the wake vortices move with the fluid, *e.g.*, if the undisturbed fluid is at rest they keep their positions in space while the airfoil proceeds.

The method presented in this paper is based on the idea that the momentum per unit span of the system can be expressed by the sum of the momentum of the vortex pairs which constitute the system, where the momentum of each particular vortex pair is given by the product of the circulation and the distance between the individual vortices. It is assumed that all vortices are located along the X-axis; the circulation of a particular vortex is denoted by Γ_i, its X-coordinate by x_i, and the density of the fluid by ρ. Then the total momentum perpendicular to the X-axis is equal to $\rho\Sigma\Gamma_i x_i$, while the condition $\Sigma\Gamma_i = 0$ expresses the fact that the total circulation vanishes. The rate of change of the total momentum at any instant determines the magnitude of the lift. In a similar way the total moment of momentum of the fluid with respect to a suitably chosen point may be expressed. If the strengths of the two vortices of a particular vortex pair are denoted by $\pm\Gamma$, and the X-coordinates of the two vortices by x_1 and x_2, then the magnitude of the momentum is $\rho\Gamma(x_2 - x_1)$ and the line of action of the momentum, due to symmetry, is given by $x = (x_1 + x_2)/2$. Consequently the moment of momentum with respect to the origin of the coordinate system is equal to $\rho\Gamma(x_2^2 - x_1^2)/2$, and it is seen that the total moment of momentum of the system is given by the sum $(^1/_2)\rho\Sigma\Gamma_i x_i^2$. Hence the two equations

$$L = -\rho(d/dt)\Sigma\Gamma_i x_i \qquad (1)$$

and

$$M = -(\rho/2)(d/dt)\Sigma\Gamma_i x_i^2 \qquad (2)$$

determine the lift and the moment acting on the airfoil per unit span.

In the following sections this simple physical idea is carried out for the two-dimensional case of a thin airfoil with a wake composed of a plane vortex sheet. The general results are first applied to determine the forces and moments acting on an oscillating airfoil. This section (Section IV) is essentially a shortcut deduction of Glauert's, Küssner's, and Theodorsen's results. The next application refers to the penetration of an airfoil into a so-called "sharp-edged" gust. A simple proof of Küssner's theorem, previously not proved, concerning the line of action of the lift produced by the gust, is given. Then, in addition to the correction of some erroneous results of Küssner, a general method is developed for computing the lift produced by a gust of arbitrary shape.

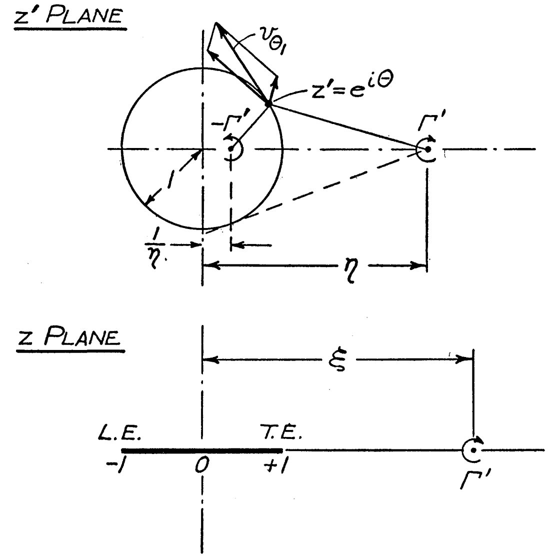

FIG. 2. Conformal representation of the airfoil and a wake vortex.

Before the deduction of the general formulae for lift and moment, it appears desirable to recapitulate some simple calculations concerning the effect of a single vortex on a thin airfoil.

II. Calculation of the Effects of the Vortices in the Wake

The effects of the vortices in the wake are here calculated for the simple case shown in Fig. 1, under the following assumptions:

(a) that the flow about the airfoil is two-dimensional;

(b) that the vertical movement of any part of the airfoil is small, so that the airfoil and every point of the trail of vortices which it leaves behind may be considered to lie upon the X-axis;

(c) that the theory of thin airfoils can be applied to the calculation of the forces; in particular that the total circulation about the airfoil at any instant is such as to produce tangential flow at the trailing edge.

The effect of an element of vorticity, Γ', at a point ξ in the wake may be evaluated by the method of conformal transformation as pictured in Fig. 2. The transformation relating the two planes is

$$2z = z' + 1/z' \qquad (3)$$

In the z'-plane the vortex $-\Gamma'$ is placed at $1/\eta$ to make the unit circle a streamline, by the usual method of "images." This means that the resultant velocity is tangential at all points on the circle. Its magnitude is

$$v_{\theta_1} = \frac{\Gamma'}{2\pi}\left|\frac{1}{z'-\eta} - \frac{1}{z'-1/\eta}\right|_{z' = e^{i\theta}} \tag{4}$$

By simple algebra, using the relations, from Eq. (3), $\eta + 1/\eta = 2\xi$ and $\eta - 1/\eta = 2\sqrt{\xi^2 - 1}$, this becomes

$$v_{\theta_1} = \frac{\Gamma'}{2\pi}\frac{\sqrt{\xi^2-1}}{\xi - \cos\theta} \tag{5}$$

In particular, the velocity at the trailing edge, where $\cos\theta = 1$, is equal to $(\Gamma'/2\pi)\sqrt{\xi+1}/\sqrt{\xi-1}$. In accordance with assumption (c) above, a circulation about the airfoil arises which is just great enough to cancel this velocity. Hence a second, uniform velocity $v_{\theta_2} = -(\Gamma'/2\pi)\sqrt{\xi+1}/\sqrt{\xi-1}$ is added to v_{θ_1}. Then the total tangential velocity becomes

$$v_\theta = \frac{\Gamma'}{2\pi}\left\{\frac{\sqrt{\xi^2-1}}{\xi-\cos\theta} - \sqrt{\frac{\xi+1}{\xi-1}}\right\} = -\frac{\Gamma'}{2\pi}\frac{1-\cos\theta}{\xi-\cos\theta}\sqrt{\frac{\xi+1}{\xi-1}} \tag{6}$$

The relation between the velocity v_θ and the vorticity distribution over the airfoil, $\gamma(x)$, is given by the formula $\gamma(x) = -2v_\theta/\sin\theta$ (cf. von Kármán and Burgers,[2] page 46, noting that their transformation differs by a factor of 2 from the one given in Eq. (3)). Thus, from Eq. (6), it follows that

$$\gamma(x) = \frac{1}{\pi}\frac{\Gamma'}{\xi - x}\sqrt{\frac{1-x}{1+x}}\sqrt{\frac{\xi+1}{\xi-1}} \tag{7}$$

This vorticity distribution on the airfoil is plotted in Fig. 3 for several values of ξ. It is seen that a wake vortex located one half-chord length or more behind the trailing edge induces a vorticity distribution which is similar to the well-known one produced by a small angle of attack, while a vortex placed very close to the airfoil induces a much stronger vorticity over the chord, with a definite peak near the trailing edge in addition to that at the leading edge.

The total circulation is obtained by integration of Eq. (7), and is

$$\Gamma = \int_{-1}^{1}\gamma(x)dx = \Gamma'\{\sqrt{(\xi+1)/(\xi-1)} - 1\} \tag{8}$$

III. General Expressions for the Lift and Moment

In the following calculations the chord of the airfoil is again taken as 2, so that all distances are measured relative to the half-chord length. All forces are calculated for a unit length in the spanwise direction. The

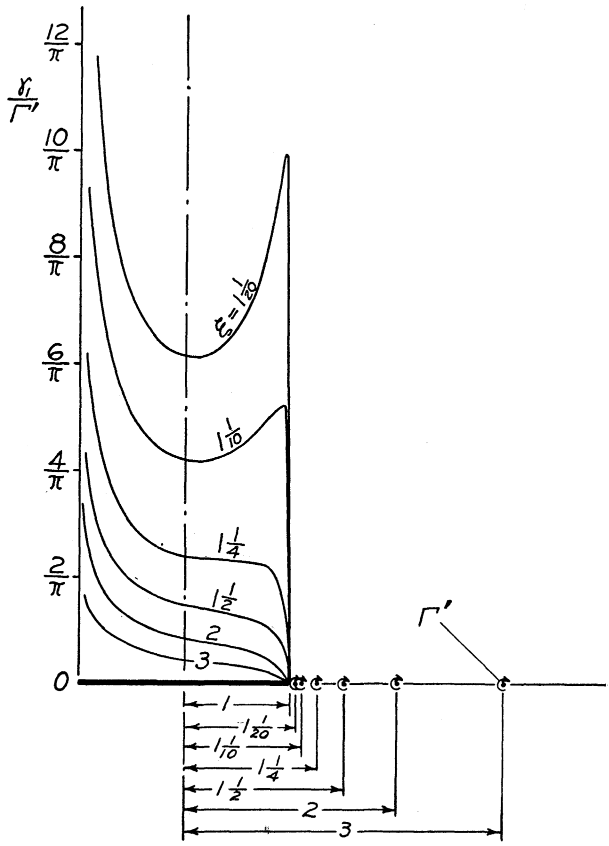

FIG. 3. Vorticity distributions induced by a wake vortex at various distances from the midpoint of the airfoil.

symbol x is used for the X-coordinate between the leading edge ($x = -1$) and the trailing edge ($x = 1$), and the symbol ξ is used in the wake. Hence the vorticity bound to the airfoil is denoted by $\gamma(x)$ and that in the wake by $\gamma(\xi)$. The vorticity $\gamma(x)$ is composed of two parts:

(a) the vorticity, $\gamma_0(x)$, which would be produced, according to the thin airfoil theory, by the motion of the airfoil or the given velocity distribution (gust) in the air, if the wake had no effect. $\gamma_0(x)$ is called the "quasi-steady" vorticity distribution;

(b) the vorticity, $\gamma_1(x)$, which is induced by the wake, as calculated in the preceding section.

The circulation resulting from (a) is denoted by Γ_0, and that from (b) by Γ_1; the total circulation about the airfoil is then $\Gamma = \Gamma_0 + \Gamma_1$. According to the basic conceptions explained in Section I, the total circulation of the whole system must be zero, hence

$$\Gamma + \int_1^\infty \gamma(\xi)d\xi = 0 \tag{9}$$

The contribution Γ_1 of the wake can be calculated using Eq. (8). Putting $\Gamma' = \gamma(\xi)d\xi$ and integrating over the whole length of the wake,

$$\Gamma_1 = \int_1^\infty (\sqrt{(\xi+1)/(\xi-1)} - 1)\gamma(\xi)d\xi \tag{10}$$

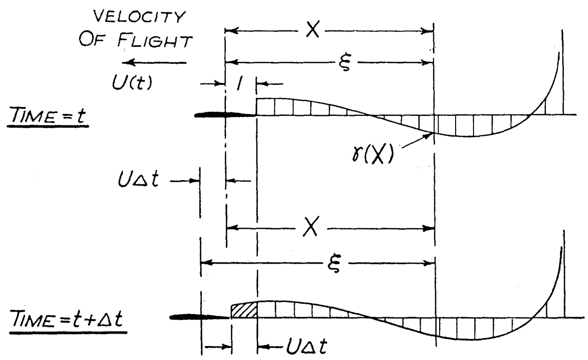

FIG. 4. Auxiliary diagram used in the calculation of the time derivatives of integrals over the wake.

Hence the circulation about the airfoil at any instant is given by

$$\Gamma = \Gamma_0 + \int_1^\infty (\sqrt{(\xi + 1)/(\xi - 1)} - 1)\gamma(\xi)d\xi \quad (10a)$$

From Eq. (9) it is seen that the following relation holds between the quasi-steady circulation and the vorticity in the wake:

$$\Gamma_0 + \int_1^\infty \gamma(\xi)\sqrt{(\xi + 1)/(\xi - 1)}\, d\xi = 0 \quad (11)$$

This relation will be employed later on in the paper.

The total circulation of the system being equal to zero, according to Eq. (9), the system can be considered as composed of vortex pairs and apply Eqs. (1) and (2) for the computation of lift and moment.

As was mentioned in Section I, the total momentum per unit length of a two-dimensional system of vortex pairs lying along the X-axis is equal to $\rho\Sigma\Gamma_i x_i$. Hence the total momentum of the system of continuously distributed vortices consisting of the airfoil and its wake is

$$I = \rho\int_{-1}^{1}\gamma(x)xdx + \rho\int_1^\infty \gamma(\xi)\xi d\xi \quad (12)$$

Substituting $\gamma(x) = \gamma_0(x) + \gamma_1(x)$ and using Eq. (7) for $\gamma_1(x)$, replacing Γ' by $\gamma(\xi)d\xi$,

$$\int_{-1}^{1}\gamma(x)xdx = \int_{-1}^{1}\gamma_0(x)xdx + \frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\,xdx\int_1^\infty \frac{\gamma(\xi)}{\xi - x}\sqrt{\frac{\xi+1}{\xi-1}}\,d\xi \quad (13)$$

When the integration with respect to x in the last term is carried out, the double integral is reduced to $\pi\int_1^\infty \gamma(\xi)(\sqrt{\xi^2 - 1} - \xi)d\xi$ and therefore

$$I = \rho\int_{-1}^{1}\gamma_1(x)xdx + \rho\int_1^\infty \gamma(\xi)\sqrt{\xi^2 - 1}\, d\xi \quad (14)$$

Now it is desired to differentiate this expression to obtain the lift, but since $\partial\gamma(\xi)/\partial\xi$ may be discontinuous at certain points in the wake (*e.g.*, the case considered in Section VI of this paper), the use of an integration by parts (as employed in reference 2, page 301) is not allowable in the evaluation of the second term. Since a similar problem is encountered later in calculating the moment, it is desirable to consider a general integral of the form

$$A = \int_1^\infty \gamma(\xi)f(\xi)d\xi$$

The wake vorticity, γ, according to the assumptions already made, is stationary relative to the fluid. Hence if X is the distance of an arbitrary wake vortex from a fixed origin, say from the location of the center of the airfoil at the instant $t = t$, then γ is a function of X only. The integral considered can therefore be written

$$A = \int_1^\infty \gamma(X)f(\xi)d\xi$$

If A is the value of this integral at the time t and $A + \Delta A$ the value at the time $t + \Delta t$, and if account is taken of the fact that the airfoil has moved through a distance $U\Delta t$ during this interval, where U is the velocity of flight, so that $\xi = X + U\Delta t$ (cf. Fig. 4), it is seen that

$$A + \Delta A = \int_{1-U\Delta t}^\infty \gamma(X)f(X + U\Delta t)dX$$

Neglecting terms of second order and higher,

$$\Delta A = \int_{1-U\Delta t}^{1}\gamma(X)f(X)dX + U\Delta t\int_1^\infty \gamma(X)f'(X)dX$$

Now if $\gamma(X)$ is finite in the interval and if $f(1) = 0$, then in the limit $\Delta t \longrightarrow 0$ the first term vanishes, and, replacing X by ξ in the second term,

$$dA/dt = U\int_1^\infty \gamma(\xi)f'(\xi)d\xi \quad (15)$$

Applying this result in the differentiation of the second term of Eq. (14), the lift becomes

$$L = -dI/dt = -\rho\frac{d}{dt}\int_{-1}^{1}\gamma_0(x)xdx - \rho U\int_1^\infty \gamma(\xi)\xi d\xi/\sqrt{\xi^2 - 1} \quad (16)$$

Using Eq. (11) the lift may also be written in the form

$$L = -\rho\frac{d}{dt}\int_{-1}^{1}\gamma_0(x)xdx + \rho U\Gamma_0 + \rho U\int_1^\infty \gamma(\xi)d\xi/\sqrt{\xi^2 - 1} \quad (16a)$$

Thus the lift consists of three parts:

(a) $L_1 = -\rho(d/dt)\int_{-1}^{1}\gamma_0(x)xdx$, which will be called the contribution of the apparent mass;

(b) $L_0 = \rho U\Gamma_0$, the quasi-steady lift;

(c) $L_2 = \rho U \int_1^\infty \gamma(\xi)d\xi/\sqrt{\xi^2 - 1}$. This is the only contribution which depends explicitly on $\gamma(\xi)$, the vorticity distribution in the wake.

In a similar manner, the total moment of momentum per unit span of the system, referred to a fixed axis, may be calculated. It is $M_m = (^1/_2)\rho\Sigma\Gamma_i x_i^2$. If the center of the airfoil is imagined to be at a distance s from the fixed axis,

$$M_m = \frac{1}{2}\rho\int_{-1}^{1}\gamma(x)(x+s)^2dx + \frac{1}{2}\rho\int_1^\infty \gamma(\xi)(\xi+s)^2d\xi \quad (17)$$

where x and ξ are again measured from the center of the airfoil. The moment M acting on the airfoil, referred to its midpoint, is then given by the value of dM_m/dt for $s = 0$. Carrying out the differentiation and taking into account that $ds/dt = -U$, where U is the velocity of flight of the airfoil, this becomes

$$M = -\frac{1}{2}\rho\frac{d}{dt}\left\{\int_{-1}^{1}\gamma(x)x^2dx + \int_1^\infty \gamma(\xi)\xi^2d\xi\right\} + UI \quad (18)$$

where I is the total momentum as calculated in Eq. (14). A diving moment is here considered positive.

Now substituting again $\gamma(x) = \gamma_0(x) + \gamma_1(x)$, and using Eq. (7), an analysis similar to that leading to Eq. (14) gives

$$M = -\frac{1}{2}\rho\frac{d}{dt}\left\{\int_{-1}^{1}\gamma_1(x)x^2dx + \frac{1}{2}\int_1^\infty \gamma(\xi)\sqrt{\frac{\xi+1}{\xi-1}}\,d\xi + \int_1^\infty \gamma(\xi)\xi\sqrt{\xi^2-1}\,d\xi\right\} + UI \quad (19)$$

If the second integral in the bracket is replaced according to Eq. (11) and the differentiation of the third integral is carried out by the method of Eq. (15), then, by substitution for I from Eq. (14), the moment becomes

$$M = -\frac{1}{2}\rho\frac{d}{dt}\int_{-1}^{1}\gamma_0(x)\,[x^2 - (^1/_2)]dx + \rho U\int_{-1}^{1}\gamma_0(x)xdx - \frac{1}{2}\rho U\int_1^\infty \gamma(\xi)d\xi/\sqrt{\xi^2-1} \quad (20)$$

Therefore, the moment also consists of three parts:

(a) $M_1 = -(^1/_2)\rho(d/dt)\int_{-1}^{1}\gamma_0(x)\,[x^2-(^1/_2)]dx$, analogous to L_1;

(b) M_0, the quasi-steady moment;

(c) M_2, the last term, depending explicitly upon the vorticity in the wake. Since $M_2 = -L_2/2$, the lift L_2 always acts through the quarter-chord point of the airfoil ($x = -^1/_2$).

According to Eqs. (16a) and (20), both lift and moment consist of three parts, itemized under (a), (b), and (c) above. The physical significance of these contributions will now be explained briefly.

Considering first the lift, let it be assumed that the airfoil carries out its motion without producing circulation. Then the quasi-steady lift is zero, and, because obviously no wake is produced, the part called L_2 also vanishes. It is known from general principles that in such a case the only forces acting on a body moving in an ideal fluid are those corresponding to the apparent mass of the body. These can be obtained by integrating over the surface of the airfoil the so-called "impulsive pressures," $\rho(\partial\varphi/\partial t)$, where φ is the velocity potential of the circulationless flow. Hence, if C indicates a path of integration starting at some point A on the airfoil and going completely around the airfoil profile back to A, the lift is

$$\rho\int_C \frac{\partial\varphi}{\partial t}ds = \rho\frac{\partial}{\partial t}\int_C \varphi ds = \rho\frac{\partial}{\partial t}\left\{\varphi s\Big|_A^A - \int_C \frac{\partial\varphi}{\partial s}sds\right\}$$

where s represents the distance along the surface. Now the first term obtained in the integration by parts is obviously equal to zero, since there is no circulation and therefore φ is single-valued. Also, for a thin airfoil, $\partial\varphi/\partial s$, which is the velocity along the surface of the airfoil, is equal to $\gamma_0(s)/2$ because the surface velocities on opposite sides of the airfoil at any point are equal and opposite and $\gamma_0(s)$, the vorticity at s, is their difference. Therefore, the lift for this case is simply

$$-\rho(d/dt)\int_C \gamma_0(s)sds/2 = -\rho(d/dt)\int_{-1}^{1}\gamma_0(x)xdx$$

which is exactly the term called L_1 above.

Therefore, the term L_1 gives correctly the lift due to the apparent mass in the case of the airfoil without circulation. Now the addition of circulation increases the function $\gamma_0(x)$ by a term equal to $(\Gamma_0/\pi)/\sqrt{1 - x^2}$, which is the vorticity distribution of a pure circulation about a plane airfoil. It is seen by considerations of symmetry that this term does not contribute to the integral L_1. As far as the corresponding part of the moment is concerned, in the case of flow without circulation the integral $-(^1/_2)\rho(d/dt)\int_{-1}^{1}\gamma_0(x)x^2dx$ gives the moment due to the apparent mass distribution, as can easily be verified by integrating the moments of the impulsive pressures in a manner similar to that used above for the lift. The rest of the integral called M_1 above vanishes because $\int_{-1}^{1}\gamma_0(x)dx = 0$. Again the addition of the circulation term to $\gamma_0(x)$ does not change the value of the whole integral, for this term is $(\Gamma_0/\pi)/\sqrt{1 - x^2}$, and

$$\frac{\Gamma_0}{\pi}\int_{-1}^{1}\frac{x^2 - (^1/_2)}{\sqrt{1 - x^2}}\,dx = 0$$

Hence it is found that the contributions under (a), above, L_1 and M_1, are equal to the force and moment which the airfoil would encounter in a flow without circulation, due to the reaction of the accelerated fluid masses. These contributions in both cases are called the "apparent mass contributions." It should be noted that the determination of $\gamma_0(x)$ involves only the solution of steady-motion problems and can be done in any given case by the use of known formulae of the stationary airfoil theory.

The contributions under (b), above, the lift L_0 and moment M_0, are easily interpreted. They represent the force and moment which would be produced if the instantaneous velocity and angle of attack of the airfoil were permanently maintained. The calculation of L_0 and M_0 also requires only the solution of stationary problems by the usual methods.

The third contributions, L_2 and M_2, represent the influence of the wake. Their interpretation is simplified by considering a case in which quasi-steady lift and moment (*i.e.*, the angle of attack or speed) undergo a sudden change at the instant $t = 0$ and are kept constant for $t > 0$. In this case $L_1 = M_1 = 0$ for $t > 0$, and the lift and moment are given by $L_0 + L_2$ and $M_0 + M_2$. For $t = \infty$ the final values of lift and moment will be L_0 and M_0 because the conditions of the "stationary" case are approached. It is seen that L_2 and M_2 give the difference between the instantaneous and final values of lift and moment. Hence $-L_2$ and $-M_2$ can be called the "deficiencies" caused by the non-uniformity of the motion of the airfoil or of the wind velocity encountered by it.

Before proceeding to the next section, it should be pointed out that the general formulae, Eqs. (16a) and (20), developed in this section, can be applied to the case of any thin airfoil with arbitrary shape, performing an arbitrary (accelerated, oscillatory, or uniform) motion, provided only that its deviation from a straight path is small, so that the assumption of a wake distributed along a straight line is justified.

It may be pointed out that the momentum of the fluid can again be expressed in terms of the characteristics of the vortex system in the case of a wing of finite span. The authors hope to extend the methods used here to that case and to obtain analogous results.

IV. Application to the Case of Steady-State Oscillations

The theory of steady-state oscillations of an airfoil has a bearing on many practically important problems. Actually it was first developed in anticipation of flying machines with flapping wings. Later it was found to be closely connected with the theory of wing flutter. In the earliest flutter theories, it was assumed that the forces acting on the oscillating airfoil could be approximated by those referred to in the preceding section as quasi-steady forces plus some damping forces assumed more or less arbitrarily on the basis of a few wind-tunnel experiments. However, the lag between the actual forces and the quasi-steady forces and the influence of the apparent mass could not be taken into account in this way. The theory of the oscillating airfoil now opens the way to a more systematic estimate of the forces. To be sure, the assumption of infinite aspect ratio restricts the accuracy of purely theoretical predictions for the flutter of wings of finite spans, but in any case the results obtained by the theory aid in scientific analysis of experimental data obtained in the laboratory and with actual airplanes in flight. The experimental work done by Küssner and others has shown that the so-called "reduced frequency" (*i.e.*, the product of half-chord and vibration frequency divided by the flying speed) is the most adequate parameter for the discussion of flutter data. The following analysis leads to simple formulae and diagrams showing how the magnitude and the phase of the actual lift and moment depend on the reduced frequency.

If the motion of the airfoil is periodic, the resulting quasi-steady circulation is also periodic and, using the complex variable notation, may be written

$$\Gamma_0 = G_0 e^{i\nu t} \tag{21}$$

where G_0 is a constant.

If the motion has been occurring so long that transient phenomena have disappeared, it may be assumed that the vortex strength in the wake can be expressed as

$$\gamma(\xi) = g e^{i\nu[t-(\xi/U)]} \tag{22}$$

where g is also a constant and U is the constant mean horizontal velocity.

Then the total circulation about the airfoil is given (from Eq. (10a)) by

$$\Gamma = e^{i\nu t}\left[G_0 + g\int_1^\infty \left(\sqrt{\frac{\xi+1}{\xi-1}} - 1\right) e^{-i\nu\xi/U} d\xi\right] \tag{23}$$

Hence Γ is also a periodic function of the time, and, because the wake vorticity is produced by the changes of circulation of the airfoil, the increment of circulation, $(d\Gamma/dt)dt$, must be equal and opposite to the circulation in the wake between $\xi = 1$ and $\xi = 1 + Udt$. Consequently $(d\Gamma/dt)dt = -\gamma(1)Udt$. By differentiation of Eq. (23) and substitution of $d\Gamma/dt = -\gamma(1)U$, the following relation between G_0 and g is obtained:

$$-\frac{G_0}{g} = \int_1^\infty \left(\frac{1+\xi}{\sqrt{\xi^2-1}} - 1\right) e^{-i\nu\xi/U} d\xi + \frac{U}{i\nu} e^{-i\nu/U} \tag{24}$$

The right side of Eq. (24) can be expressed as the sum of two modified Bessel functions of the second kind of the argument $iz = i\nu/U$, namely

$$K_0(iz) = \int_1^\infty e^{-iz\xi} d\xi/\sqrt{\xi^2-1} \text{ and } K_1(iz) = -K_0'(iz)$$

(See reference 3; Eq. (29), page 50, and Eq. (19), page 22.) Using K_0 and K_1 as abbreviations for $K_0(i\nu/U)$ and $K_1(i\nu/U)$, the result obtained* is

* The integral $\int_1^\infty e^{-i\nu\xi/U} d\xi/\sqrt{\xi^2 - 1}$ is easily identified as $K_0(i\nu/U)$. That the rest of the right side of Eq. (24) is equal to

$$-G_0/g = K_0 + K_1 \tag{25}$$

In any given case of periodic motion $g_0(x)$ and G_0 can be easily calculated, and they determine directly the first two terms of the expressions for lift and moment, Eqs. (16a) and (20). After substitution of $\gamma(\xi)$ from Eq. (22) and g from Eq. (25), the third term of Eq. (16a) becomes

$$L_2 = -\rho U G_0 e^{i\nu t}(K_0 + K_1)^{-1}\int_1^\infty e^{-i\nu\xi/U}d\xi/\sqrt{\xi^2 - 1}$$

or

$$L_2 = -\rho U G_0 e^{i\nu t} K_0/(K_0 + K_1) \tag{26}$$

The corresponding moment is $M_2 = -L_2/2$.

These results will now be applied to the case of an airfoil performing (1) translatory oscillations normal to the flight direction, and (2) rotational oscillations around its midpoint.

Case 1. Translatory Oscillation

For this case the vertical velocity of every point of the airfoil can be written as

$$w = A_0 U e^{i\nu t} \tag{27}$$

where A_0 is a constant and w is taken as positive downward. The quasi-steady portion of the vorticity depends only on the instantaneous relative velocity of the air and the airfoil, and therefore the quasi-steady quantities can be calculated by the formulae of reference 2, Chapter 2, by replacing v_y by $-w$. For the present case (from reference 2, page 38, putting $c = 2$)

$$G_0 = \Gamma_0/e^{i\nu t} = 2\pi U A_0$$

$$g_0(x) = \gamma_0(x)/e^{i\nu t} = 2UA_0(1 - \cos\theta)/\sin\theta \tag{28}$$

where again $x = \cos\theta$

The three parts of the lift, as in Eq. (16a), are therefore

$$L_0 = 2\pi\rho U^2 A_0 e^{i\nu t}$$

$$L_1 = \pi\rho i\nu U A_0 e^{i\nu t}$$

and, from Eq. (26),

$$L_2 = -2\pi\rho U^2 A_0 e^{i\nu t} K_0/(K_0 + K_1)$$

The total lift is therefore

$$L = 2\pi\rho U^2 e^{i\nu t} A_0 \left\{ \frac{K_1(i\nu/U)}{K_0(i\nu/U) + K_1(i\nu/U)} + \frac{i\nu}{2U} \right\} \tag{29}$$

$-K_0'(iz)$ can be shown in the following way, which avoids convergency difficulties: Write

$$K_0(iz) = \int_1^\infty e^{-iz\xi}d\xi/\sqrt{\xi^2 - 1} - \int_1^\infty e^{-iz\xi}d\xi/\xi + \int_z^\infty e^{-it}dt/t$$

which is obviously correct because the difference between the second and third integrals vanishes identically. By differentiation with respect to iz and substitution of $z = \nu/U$, the identification is completed.

It is seen that L_1 is equal to the product of the acceleration, $A_0 U i\nu e^{i\nu t}$, and the apparent mass of a flat plate of length $= 2$, which is equal to $\pi\rho$.

The apparent-mass contribution to the moment, M_1, vanishes, since the motion is purely translatory and the apparent-mass lift acts through the center of the airfoil. This can be verified by substituting $g_0(x)$ from Eq. (28) into the first term of Eq. (20) and integrating. The quasi-steady lift in this case acts at the quarter-chord point ($x = -1/2$), as does L_2. The total moment is therefore

$$\begin{aligned} M &= -(L_0 + L_2)/2 \\ &= -\pi\rho U^2 e^{i\nu t} A_0 \frac{K_1(i\nu/U)}{K_0(i\nu/U) + K_1(i\nu/U)} \end{aligned} \tag{30}$$

Case 2. Rotational Oscillation

In this case the vertical velocities of the various points of the airfoil are given by

$$w = 2Ue^{i\nu t}A_1 \cos\theta \tag{31}$$

In this case the formulae of reference 2 lead to the following expressions for the quasi-steady quantities:

$$\begin{aligned} G_0 &= 2\pi U A_1 \\ g_0 &= 4UA_1 \sin\theta \end{aligned} \tag{32}$$

By substitution into Eqs. (16a) and (26), then,

$$L_0 = 2\pi\rho U^2 e^{i\nu t} A_1$$

$$L_1 = 0$$

$$L_2 = -2\pi\rho U^2 e^{i\nu t} A_1 K_0/(K_0 + K_1)$$

so that the total lift in this case becomes*

$$L = 2\pi\rho U^2 e^{i\nu t} A_1 \frac{K_1(i\nu/U)}{K_0(i\nu/U) + K_1(i\nu/U)} \tag{33}$$

Also, by substitution into Eq. (20),

$$M_1 = \pi\rho U A_1 e^{i\nu t} i\nu/4$$

$$M_0 = 0$$

and

$$M_2 = -L_2/2$$

Therefore, the total moment is*

$$M = \pi\rho U^2 e^{i\nu t} A_1 \left\{ \frac{K_0(i\nu/U)}{K_0(i\nu/U) + K_1(i\nu/U)} + \frac{i\nu}{4U} \right\} \tag{34}$$

* It should be noted that Eqs. (33) and (34) give only the portions of lift and moment which are due to the angular velocity of the airfoil. The effects of the periodic changes in angle of attack which actually accompany the rotation are to be obtained from the results of Case 1. Since the instantaneous angle of attack is equal to $2UA_1e^{i\nu t}/i\nu$, these contributions are evaluated by taking A_0 of Case 1 equal to $2UA_1/i\nu$.

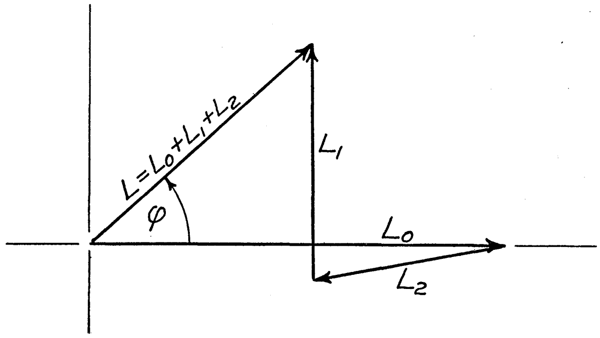

FIG. 5. Typical vector diagram for the lift of an oscillating airfoil.

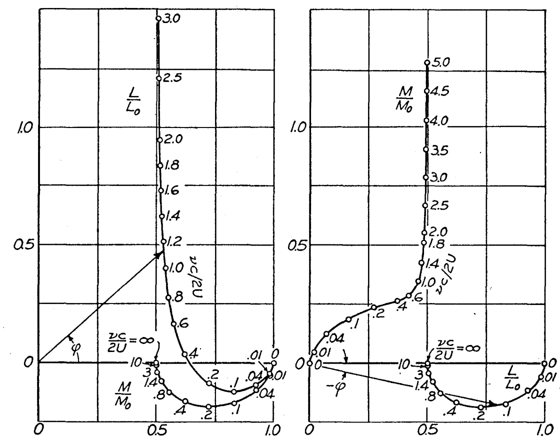

FIG. 6. Vector diagrams for the lift and moment of oscillating airfoils, as functions of the reduced frequency. L_0 and M_0 are the respective quasi-steady values. (In the right-hand diagram, M_0 should be replaced by $\pi\rho U^2 e^{i\nu t} A_1$, as defined in the text.)

In a similar manner, the lift and moment on the airfoil may be calculated for the cases of periodic deformations represented by higher values of n in the general expression for the vertical velocity of any point of the airfoil:

$$w(\theta) = Ue^{i\nu t}\left\{A_0 + 2\sum_{n=1}^{\infty} A_n \cos n\theta\right\} \qquad (35)$$

The results, including those of Cases 1 and 2 above, agree with those obtained by Küssner.[4*]

The physical significance of the complex forms of Eqs. (29), (30), (33), and (34) may be clarified by means of "vector diagrams" which show the phase relationships of the quantities involved as well as their magnitudes. Each of these results may be abbreviated as

$$f(t) = Fe^{i\nu t}\{f_1(\nu/U) + if_2(\nu/U)\} \qquad (36)$$

where $f(t)$ represents the lift or moment; F is a constant involving only the dynamic pressure, $\rho U^2/2$, and the amplitude of the oscillation; and f_1 and f_2 are real functions. The real part of this expression (which is the actual force or moment) may be written as

$$R[f(t)] = F\{f_1 \cos \nu t - f_2 \sin \nu t\}$$
$$= F\sqrt{f_1^2 + f_2^2} \cos(\nu t + \varphi) \qquad (37)$$

where $\varphi = \tan^{-1}(f_2/f_1)$. Thus, in vector representation, the lift or moment vector has the magnitude $F\sqrt{f_1^2 + f_2^2}$ and leads the vector of the oscillating velocity, w, by a phase angle, φ. In Fig. 5 is given an example, taken from Case 1, above, which shows schematically how the total lift vector is composed of the vectors L_0, L_1, and L_2 for a certain value of ν/U. The quasi-steady part, L_0, being in phase with the velocity, appears as a horizontal vector, while the vector, L_2, tends to diminish the lift and cause it to lag behind the velocity. The apparent-mass lift, L_1, being proportional to the acceleration, is directed vertically, *i.e.*, leads the velocity by 90°. The total lift, L, is the sum of these three vectors, and has the phase angle φ.

In Fig. 6 are plotted "vector diagrams" which give the magnitudes of the lift and moment together with their phase angles for various values of the reduced frequency ν/U (or $c\nu/2U$ for an airfoil of chord $= c$).† In these diagrams the length of the vector drawn from the origin to the appropriate value of $c\nu/2U$ on the curve gives the maximum value of the total lift or moment (referred to that of the corresponding quasi-steady quantity, L_0 or M_0), and its angle with the horizontal axis gives the phase angle relative to w. It is seen that as the frequency of the translatory oscillation (Case 1) is increased from $c\nu/2U = 0$ (uniform motion) the maximum value or amplitude of the lift at first steadily decreases, and the lift vector lags slightly behind the vertical-velocity vector, w. These effects are produced by the wake contribution, L_2. With further increase of the frequency, however, the apparent-mass contribution, L_1, which is proportional to the acceleration, becomes very large, and the lift vector leads the velocity vector. In the limit $c\nu/2U \longrightarrow \infty$ the sum $(L_0 + L_2)$ is equal to half of L_0, but $L_1 \longrightarrow \infty$, and the lift vector leads w by 90°. Since the apparent-mass lift, L_1, acts through the midpoint of the airfoil in this case, the limiting value of the moment is also half of its steady-motion value. In the rotational case (Case 2) the total lift behaves exactly like the moment of Case 1 as the frequency is increased, while the apparent-mass moment contribution, M_1, increases proportionally to the frequency.

It is believed that the method of representation of lift and moment by vector diagrams will be useful in the discussion of flutter problems because both the elastic restoring forces and the inertia forces of the wing can be introduced in such diagrams. It is intended to extend the work in this direction.

* In comparing Küssner's results, it should be noted that

$$\pi H_n^{(2)}(z) = 2i^{(n+1)}K_n(iz)$$

Also Küssner's $(-i\omega)$ is the same as (ν/U) here, and, because of the difference in definition of θ, his P_n is $(-)^n A_n$ of the present paper.

† Küssner[4] (page 416) gives a table of Bessel functions involved in the calculation of Fig. 6.

The results presented in Fig. 6 are applicable to cases of bending-torsional wing flutter. The calculation of aileron or rudder flutter requires the expression of the non-steady forces acting on the aileron or rudder in a similar way. The aileron or rudder constitutes one portion of a wing or a fin, while the equations presented in this paper give only the lift and moment acting on the wing as a whole. However, by determining the vorticity distribution produced by the wake, similar equations can be deduced for the non-steady normal force on the aileron and for the non-steady hinge moment. Both Küssner and Theodorsen have published such calculations, but it is felt that extension of the methods presented in this paper to the aileron case might facilitate understanding of the physical principles involved and reduce the amount of mathematical work required.

V. General Method for Transient Phenomena

There are many cases in which knowledge of the forces produced by a transient phenomenon is of practical interest. Examples of such cases are the reaction of an airplane to certain control operations (aileron or rudder deflection, etc.), and the behavior of an airplane encountering gusts. In the second case, an estimate of the forces acting on the wing is of importance, as well as the reaction of the airplane as a whole, in view of strength requirements. It is believed that the general method developed in the following paragraphs will be useful in many such transient problems.*

In order to determine the response of the system to a "unit disturbance," it is first assumed that the velocity of flight, U, is constant, and that the quasi-steady circulation, Γ_0, changes suddenly from zero to unity at the time $t = 0$ and remains constant for $t > 0$. Then the wake extends from $\xi = 1$ to $\xi = 1 + Ut$, and, according to Eq. (11),

$$\Gamma_0 = 1 = -\int_1^{1+Ut} \gamma(\xi)\sqrt{(\xi+1)/(\xi-1)}\,d\xi \quad (38)$$

The vortex strength in the wake is a function of the distance s from the endpoint of the wake, $s = 1 + Ut - \xi$, hence it may be written as $\gamma(\xi) = \mu(1 + Ut - \xi)$. Eq. (38) represents an integral equation for μ in the form

$$-\int_1^{1+Ut} \mu(1 + Ut - \xi)\sqrt{(\xi+1)/(\xi-1)}\,d\xi = 1$$

or, introducing the variable s,

$$-\int_0^{Ut} \mu(s)\frac{\sqrt{2 + Ut - s}}{\sqrt{Ut - s}}ds = 1 \quad (39)$$

* The elementary cases, *e.g.*, the airfoil put suddenly into motion or suddenly given a certain angle of attack, have been treated by H. Wagner.[5] Küssner[4] has developed a more general method using the conception of the Fourier integral.

The function $\mu(s)$ has been determined by Wagner.[5]

For the following applications, the main problem is the calculation of L_2, the contribution of the wake. For the case being considered, L_2 is equal to (from Eq. (16a))

$$\rho U \int_1^{1+Ut} \gamma(\xi)d\xi/\sqrt{\xi^2 - 1} = \rho U \int_1^{1+Ut} \mu(1 + Ut - \xi)\, d\xi/\sqrt{\xi^2 - 1} \quad (40)$$

where the function $\mu(s)$ is again the solution of the integral equation (Eq. (39)). The function

$$\Phi(Ut) = -\int_1^{1+Ut} \mu(1 + Ut - \xi)d\xi/\sqrt{\xi^2 - 1} \quad (41)$$

will be called the lift-deficiency function. It represents the difference between the instantaneous and final values of the lift in the case of sudden unit increase of the quasi-steady circulation, and is a function of the distance Ut traveled by the airfoil since the change of circulation took place.

It is evident that the function Φ can be used to calculate the lift acting on an airfoil which is subjected to an arbitrary transient variation of the quasi-steady circulation Γ_0. Assuming that Γ_0 changes at the instant τ by the increment $\Delta\Gamma_0 = \Gamma_0'(\tau)\Delta\tau$, the deficiency in lift at $t = t$, *i.e.*, after the airfoil has traveled a distance $U(t - \tau)$ will be $\Gamma_0'(\tau)\Phi[U(t - \tau)]\Delta\tau$ and the total deficiency in lift will be given by

$$-L_2 = \rho U \int_0^t \Gamma_0'(\tau)\Phi[U(t - \tau)]d\tau \quad (42)$$

In this equation it is assumed that $\Gamma_0(0) = 0$. If, at the instant $t = 0$, Γ_0 is suddenly increased from 0 to $\Gamma_0(0)$, a term equal to $\Gamma_0(0)\ \Phi(Ut)$ is to be added to the right-hand side of Eq. (42).

The elementary case of a sudden unit increment in Γ_0 has been treated by Wagner, and the function $1 - \Phi$ is equal to his $A/2b\pi\rho v^2 \sin\beta$ (see page 31 of reference 5; also Fig. 9 and Table 2). Thus the lift and moment can be calculated by Eqs. (16a) and (20), for a given $\gamma_0(x,t)$ distribution, because L_0, L_1, M_0, M_1 are determined by γ_0 and its time derivative, L_2 is given by (42), and $M_2 = -L_2/2$.

The function $\Phi(Ut)$ is not given in analytical form because it requires the solution of the integral Eq. (38). However, it can be fairly closely approximated by the following relatively simple formulae which are chosen so as to facilitate subsequent calculations.

(a) for $0 \leqq \sigma \leqq 2$ the following power series can be used:

$$\Phi(\sigma) = (1/2) - (\sigma/8) + (\sigma^2/32) - 0.00554\,\sigma^3 \quad (43)$$

(b) for $0 \leqq \sigma \leqq 10$

$$\Phi(\sigma) = (e^{-\sigma/2}/4) + (1 + 0.185\,\sigma)e^{-0.185\sigma}/4 \quad (44)$$

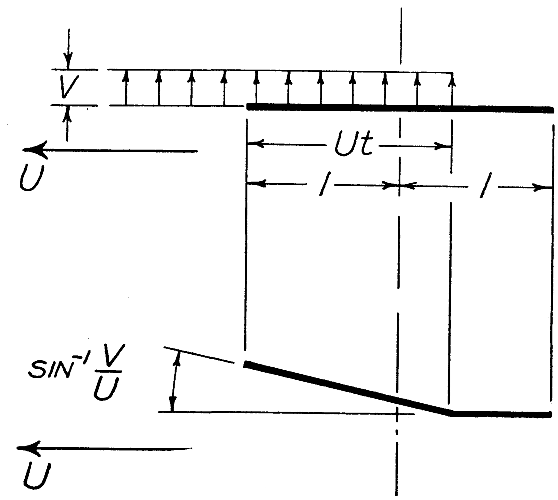

FIG. 7. The analogy between an airfoil entering a sharp-edged gust and a broken-line airfoil.

However, $1 - \Phi$ enters in the final result, and since $\Phi(\sigma)$ is very small for $\sigma > 10$, Eq. (44) can be applied for $0 \leqq \sigma \leqq \infty$ without introducing important errors.

In Table 1 the approximate values for $\Phi(\sigma)$ are shown in comparison with those given by Wagner, which are assertedly correct in the first four digits for $0 \leqq \sigma \leqq 10$. For large values of σ Wagner's calculations are not accurate, since his curve tends to a false asymptote.

TABLE 1
Approximations to Φ

σ	Wagner	Eq. (43)	Eq. (44)
0	1/2	1/2	1/2
1/2	0.4443	0.4446	0.444
1	0.3994	0.4008	0.398
2	0.3307	0.3307	0.329
4	0.2418	..	0.242
10	0.1255	..	0.114
20	0.0679	..	0.029

The approximations given in Eqs. (43) and (44) will be used in the application of the next section.

VI. Application to the Case of a Sharp-Edged Gust

The results of the preceding section may be applied to the problem of a flat-plate airfoil entering a sharp-edged vertical gust. The flow in this case is not a potential one because the gust boundary itself constitutes a vortex sheet. It will be assumed that in spite of this fact the thin-airfoil theory can be applied to the calculation of the quasi-steady quantities $\gamma_0(x)$ and Γ_0. This corresponds to the assumption that the principle of superposition of flow patterns is applicable. Strictly speaking, this is only the case for potential motions; however, the method yields results which are probably sufficiently exact in the present case, provided that all additional velocities are small so that the deformation of the vortex sheet can be neglected.

Suppose that the leading edge of the airfoil reaches the gust boundary at the instant $t = 0$. Then at the time, t, the relative transverse velocity is equal to V (cf. Fig. 7) between $x = -1$ and $x = -1 + Ut$, and it is equal to zero for $x > -1 + Ut$. The vorticity distribution $\gamma_0(x)$ produced by these velocities is the same as that of the broken-line airfoil represented in Fig. 7 in a parallel stream. Applying the equations of the thin-airfoil theory to this case (see reference 2, Chapter 2), the following formulae are obtained:

$$\gamma_0(x,t) = \frac{\Gamma_0}{\pi \sin\theta} + \sum_{k=1}^{\infty} a_k \frac{\cos k\theta}{\sin\theta} \tag{45}$$

where $\cos\theta = x$ and

$$a_k = -\frac{4V}{\pi}\int_{\cos^{-1}(Ut-1)}^{\pi} \sin\theta \sin k\theta\, d\theta \tag{46}$$

and

$$\Gamma_0(t) = 2V[\cos^{-1}(1 - Ut) - \sqrt{2Ut - U^2t^2}] \tag{47}$$

These formulae apply to the interval $0 \leqq Ut \leqq 2$ during which the airfoil crosses the boundary of the gust. For $Ut > 2$ the airfoil is entirely within the gust, the transverse velocity is constant, and $\Gamma_0 = 2\pi V$ and $\gamma_0(x) = (\Gamma_0/\pi)\sqrt{1-x}/\sqrt{1+x}$, both independent of the time. The two ranges are now considered separately:

(a) $0 \leqq Ut \leqq 2$

The apparent-mass terms are readily obtained using Eqs. (45–47). The integrals are

$$\int_{-1}^{1} \gamma_0(x)x dx = \pi a_1/2 = - V[\cos^{-1}(1 - Ut) + (Ut - 1)\sqrt{2Ut - U^2t^2}] \tag{48}$$

$$\int_{-1}^{1} \gamma_0(x)(x^2 - {}^1/_2)dx = \pi a_2/4 = 2V(2Ut - U^2t^2)^{3/2}/3 \tag{49}$$

Hence, from Eqs. (47) and (48), the sum of the first two terms of the lift, Eq. (16a), is

$$L_1 + L_0 = -\rho(d/dt)\int_{-1}^{1}\gamma_0(x)x dx + \rho U\Gamma_0 = 2\rho UV\cos^{-1}(1 - Ut) \tag{50}$$

and, from Eqs. (48) and (49), the sum of the first two terms of the moment, Eq. (20), is

$$M_1 + M_0 = -(\rho/2)(d/dt)\int_{-1}^{1}\gamma_1(x)[x^2 - ({}^1/_2)]dx + \rho U\int_{-1}^{1}\gamma_0(x)x dx = -\rho UV\cos^{-1}(1 - Ut) \tag{51}$$

It is seen immediately that $M_0 + M_1 = -(L_0 + L_1)/2$, and since it has already been proved that $M_2 =$

$-L_2/2$, this means that the total lift acts at the forward quarter-chord ($x = -1/2$) at every instant. This result was predicted by Küssner[4] and verified experimentally by him,[6] but he was unable to prove it theoretically because of an error in his fundamental equation for waves progressing over the airfoil. Because of an error in sign (see reference 4, page 420, Eq. (60)), these disturbances move over his airfoil from rear to front, which, of course, confuses the results.

The calculation of L_2 is carried out by introducing Φ from Eq. (43) and $\Gamma_0(t)$ from Eq. (47) into Eq. (42). This leads to the elementary integrals

$$\int_0^{Ut} \sqrt{(\tau/2 - \tau)}\, \tau^n d\tau \text{ for } n = 0, 1, 2, \text{ and } 3$$

Hence L_2 is easily calculated, and when the result is combined with $L_0 + L_1$ from Eq. (50) the total lift becomes

$$L = 2\rho UV\{[0.2103 + 0.2603(Ut) - 0.0562(Ut)^2 + 0.0055(Ut)^3]\cos^{-1}(1 - Ut) + [0.7897 - 0.1637(Ut) + 0.0247(Ut)^2 - 0.0014(Ut)^3]\sqrt{2Ut - U^2t^2}\} \quad (52)$$

where $0 \leqq \cos^{-1}(1 - Ut) \leqq \pi$. The result, Eq. (52), is plotted in Fig. 8.

(*b*) *Ut* > 2

In this regime, since the airfoil is subjected to a constant transverse velocity, V, it is seen immediately that $L_1 = M_1 = 0$, $L_0 = 2\pi\rho UV$, and $M_0 = -L_0/2$. Hence the lift again acts at the quarter-chord point. In the calculation of L_2 by means of Eq. (42) the function Φ is taken from Eq. (44) and the value of $\Gamma_0'(\tau)$ is that obtained from Eq. (47) for $0 \leqq Ut \leqq 2$ and is equal to zero for greater values of Ut. Hence the integrals which arise in this case are the following:

$$\int_0^2 \sqrt{(\tau/2 - \tau)}\, e^{a\tau} d\tau = \pi e^a\{I_0(a) + I_1(a)\}$$

and

$$\int_0^2 \sqrt{(\tau/2 - \tau)}\, e^{a\tau}\tau d\tau = \pi e^a\{2I_0(a) + [2 - (1/a)]I_1(a)\}$$

where $I_n(a)$ is a modified Bessel function of the first kind (see reference 3, page 46), and the exponent a has the values ($1/2$) and 0.185. The total lift in this range is $L_0 + L_2$, which finally becomes

$$L = 2\pi\rho UV\{1 - 0.3304e^{-(Ut-1)/2} - (0.1917 + 0.0510t)e^{-0.185(Ut-1)}\} \quad (53)$$

This result is also plotted in Fig. 8. It is seen that the lift on the airfoil increases rapidly after the entrance of the leading edge into the gust ($Ut = 0$), and is equal to 55 percent of its final value when the trailing edge reaches the gust boundary ($Ut = 2$). The increase then becomes progressively slower, and when the leading edge has progressed five chord-lengths into the gust ($Ut = 10$) the lift is 86 percent of its final value.

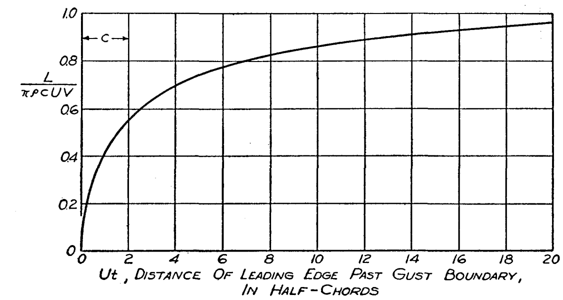

FIG. 8. The lift on an airfoil during and following its entrance into a sharp-edged gust. (Airfoil chord = c.)

Thus, for a wing of chord equal to 20 ft., flying at 200 m.p.h., the lift would reach 55 percent of its final value in 0.07 sec. and 86 percent of its final value in 0.34 sec.

It is known that the vertical gusts which actually occur in the atmosphere are not exactly sharp edged, but consist of a smooth, although rapid, transition of vertical velocity. The rate of increase of lift on an airfoil entering such a smoothly-graded gust can easily be calculated by integrating the effects of a sequence of infinitesimal sharp gusts, using the curve of Fig. 8 for the response of the wing to a sharp gust of unit intensity. A useful graphical method of carrying out such a calculation is given in reference 7. It is obvious that the lift will always act at the quarter-chord point, regardless of the shape of the gust.

The most important effects which have been neglected in the present section, in addition to the effects of finite span, are those resulting from the elastic deflections of the wing and the motion of the airplane as a whole (due to its stability) after entering the gust. Calculations including both effects, but using Wagner's elementary function, $1 - \Phi$, for the rate of build-up of lift (*i.e.*, neglecting the fact that the entire chord does not strike the gust boundary at once), have been published in reference 8. The results of the present paper now provide a better foundation for such calculations.

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Letters to the Editor

June 26, 1938

Dear Sir:

In the June, 1938, issue of the Journal of the Aeronautical Sciences, Richard C. Molloy presented a paper entitled "A Study of Available Flap Data." As the paper states, it is true that as among the usual types of flap there is little to be gained in $C_{L_{max}}$. where the flap chord exceeds 20 percent. But this is not true for the Fowler flap where $\Delta C_{L_{max}}$ reaches 1.70 for a 40 percent chord on a 23012 or a Clark Y airfoil, the flap angle being 30°. For 40° this value rises to 1.80.

Furthermore, scale effect does exist with the Fowler flap and is undoubtedly due to the gap inherent with this device.

All tests conducted by the N.A.C.A. to date show that for take-off purposes the Fowler flap is also the best because the maximum lift can be used and the L/D ratio is higher than for any other type of flap. The most spectacular proof of this is demonstrated by the Lockheed 14.

It is felt that Mr. Molloy has not presented the true story of flaps, nor is the subject so devoid of facts as he would lead the uninformed reader to believe.

HARLAN D. FOWLER

June 7, 1938

Dear Sir:

I have read with interest Richard H. Smith's paper "Laminar Boundary Layer Based on a Minimum Theorem," published in the May, 1938, issue of the Journal of the Aeronautical Sciences, but I must remark that the statement in the footnote to p. 271, that the Blasius profile is out of balance with respect to force or energy or both, cannot be true. The Blasius profile is obtained by an exact solution of the system of equations:

$$\rho[u\,(\partial u/\partial x) + v\,(\partial u/\partial y)] = \mu\,(\partial^2 u/\partial y^2) \qquad (1)$$

$$(\partial u/\partial x) + (\partial v/\partial y) = 0 \qquad (2)$$

u representing here what is called the "complementary velocity" u' by Mr. Smith.

From (1) the following relations can be deduced (by applying partial integrations):

$$\text{(a)}\ \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = -\mu\int_0^\infty \frac{\partial^2 u}{\partial y^2}dy = -\int_0^\infty \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)dy$$

$$= -2\int_0^\infty \rho u(\partial u/\partial x)dy - \rho U v_\infty \qquad (3)$$

$$\text{(b)}\ \mu\int_0^\infty\left(\frac{\partial u}{\partial y}\right)^2 dy = -\mu\int_0^\infty u\frac{\partial^2 u}{\partial y^2}dy =$$

$$-\int_0^\infty \rho\left(u^2\frac{\partial u}{\partial x} + uv\frac{\partial u}{\partial y}\right)dy$$

$$= -\int_0^\infty (3/2)\rho u^2(\partial u/\partial x)dy - (\rho/2)U^2 v_\infty \qquad (4)$$

When these results are substituted into the equations of momentum and of energy, respectively:

$$\frac{d}{dx}\int_0^\infty \rho u^2 dy - \rho U\frac{d}{dx}\int_0^\infty u\,dy = -\mu\left(\frac{\partial u}{\partial y}\right)_{y=0} \qquad (5)$$

$$\frac{d}{dx}\int_0^\infty \frac{\rho}{2}u^3 dy - \frac{\rho}{2}U^2\frac{d}{dx}\int_0^\infty u\,dy = -\mu\int_0^\infty\left(\frac{\partial u}{\partial y}\right)^2 dy \qquad (6)$$

and when it is noticed that: $\int_0^\infty (\partial u/\partial x)dy = -v_\infty$ then it will be seen that both of them are satisfied completely.

The fact that Blasius' profile is a particular solution of Eqs. (1) and (2), obtained by assuming u to be a function of $y/\sqrt{x}$ only, is of no weight in this connection, so long as it is an exact solution. It can be expected on the other hand that expressions which represent more or less satisfactory numerical approximations to this solution, will not exactly fulfill the equations of momentum and of energy, the discrepancy being dependent upon the margin of inaccuracy tolerated in the expressions used.

In connection with this remark it would appear to the undersigned that the considerations developed by Mr. Smith on the origin of turbulence as due to a failure of balance in the energy relations (p. 270, first column) do not appear convincing.

Moreover, the application of the term "separation profile," etc., to the case of boundary layer flow under constant pressure, would give rise to questions.

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July 12, 1938

Dear Sir:

In a footnote in my recent paper "Laminar Boundary Layer Based on a Minimum Theorem," published in the May, 1938, issue of the Journal of the Aeronautical Sciences, I gave some figures which I computed a few years ago which indicated that the Blasius' velocity profile is out of balance with respect to energy. In a letter from Dr. Burgers at Delft, which appears also in this issue, he shows beautifully and correctly that this conclusion cannot be correct. After agreeing with Burgers' proof, I have made a complete re-examination of the Blasius' development with a view to discovering the discrepancy in my early computations. This analysis, just completed, shows that the Blasius' velocity profile is in force and energy balance, as Dr. Burgers shows it should be, provided one takes as the layer thickness factor the theoretical value $R_x^{1/2}(\delta/x) = 6.28$ as found in the new analysis, instead of the value 4.5 or thereabouts, which has been commonly accepted and which was used in my early computations for energy balance.

I wish, therefore, to retract the remark in the footnote cited that the Blasius profile is unbalanced with respect to energy and replace it with the above statement. Also I wish to thank the Editor for giving me an opportunity to read Dr. Burgers' letter before its publication and to prepare an answer that could appear with it.

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