Literature Review Summary

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Important Papers

Brons et at 2014

Lundgren & Koumotsakos (1998), 'On the generation of vorticity at a free surface'

Discuss the generation of vorticity at free surfaces, considering the idealised interface between a viscous fluid, and an inviscid, zero-density fluid. They present an equivilent to Lighthill's numerical method, whereby a vortex sheet at the free surface boundary is introduced to ensure the boundary conditions are satisfied. The flow is first evolved by solving the for vorticity distribution at a time $n + 1$, assuming that vorticity at the boundary is purely convected. This will not, in general, satisfy the correct boundary conditions, and will have to be modified by the inclusion of additional vorticity into the flow. The effect of this additional bounary layer vorticity on the flow field is neglected. The velocity field is computed for the entire flow, applying the Biot-Savart integral to this vorticity distribution. An interface vortex sheet is applied to the flow, in order to modify the flow field such that the pressure boundary condition is satisfied.

An updated boundary condition for the surface vorticity is subsequently enforced (having determined tangential and normal velocities in the previous fractional step). The boundary layer vorticity is computed solving the vorticity transport equation, with the surface boundary condition applied, and a zero initial condition. The boundary layer vorticity distribution is subsequently added to the vorticity distributaion computed in the first half-step, to obtain the completely updated vorticity field.

The vorticity boundary condition comes from the requirement of zero shear stress at the boundary. This can be expressed as a relationship between the surface vorticity and the surface velocity and curvature:

$$\omega_1 = -2\frac{\partial \mathbf{u} \cdot \hat{\mathbf{n}}}{\partial s} + 2\kappa(\mathbf{u} \cdot \hat{\mathbf{t}}) \tag{1}$$

The pressure boundary arises due to continuity of normal stress across the interface. Pressure can be expressed in terms of primary variables by a rearrangement of the momentum equation.

$$p_1 = p_2 + T\kappa - 2\rho\nu(\frac{\partial \mathbf{u}\cdot\hat{\mathbf{t}}}{\partial s} + \kappa(\mathbf{u}\cdot\hat{\mathbf{n}})) \tag{2}$$

$$\frac{\mathrm{d}\mathbf{u}\cdot\hat{\mathbf{t}}}{\mathrm{d}t} = (\mathbf{u}_1\cdot\hat{\mathbf{n}})\frac{\partial \mathbf{u}_1\cdot\hat{\mathbf{n}}}{\partial s} - \kappa(\mathbf{u}_1\cdot\hat{\mathbf{t}})(\mathbf{u}_1\cdot\hat{\mathbf{n}}) - \frac{1}{\rho}\frac{\partial p_1}{\partial s} + \nu\nabla\omega\cdot\hat{\mathbf{n}} - \mathbf{g}\cdot\hat{\mathbf{t}} \tag{3}$$

Lundgren and Koumotsakos also discuss the conservation of vorticity within their system. If the domain extends to a uniform, irrotational far-field, the total ammount of vorticity/circulation in the flow is conserved, if the interface circulation is included. Vorticity which appears to 'dissapear' from the domain may be found as interface circulation in the vortex sheet. Conservation of vorticity is also shown for 3D free surface flows, as well as for the interface between two viscous fluids.

lighthill M. J. (1963), 'Introduction. Boundary Layer Theory', from 'Laminar Boundary Layers' *Fluid Dynamics Memoirs, Oxford University Press*

Lighthill is one of the first authors to provide a discussion on the creation of vorticity at fluid boundaries. He notes that there is no mechanism of vorticity generation within a fluid, so that any vorticity introduced to a fluid must occur so at the domain boundaries. He considers the case of a staionary flat plate, where production of tangential vorticity depends on the pressure gradient alone. He notes that even for an inviscid flow (as $\nu \to 0$), the vorticity generation is not zero (although no diffusion into the fluid body can occur).

Lighthill then presents a numerical methodology for computing vorticity generation for vortex-dynamics approaches. This approach involves evolving the flow in an inviscid manner (i.e using potential flow, and the Biot-Savart law to account for vorticity already within the flow). Vorticity is generated at the surface to ensure the no-slip condition holds. If the timestep is small, the created vorticity is confined to a thin 'vortex sheet' at the surface, which is then diffused into the flow by viscosity.

Batchelor G K (1963), 'An introduction to fluid dynamics'

One of the first authors to discuss boundary conditions to the vorticity dynamics equation. Notes that vorticity is not created in the interior of a fluid, so must be introduced at the boundaries. A flux of vorticity into the fluid may occur at solid boundaries, where it is then diffused and convected into the fluid interior. Batchelor states that the boundary condition on vorticity must come from the velocity boundary condition, with zero slip velocity required at solid walls. Batchelor first introduces an irrotational flow field, which satisfies the normal velocity boundary condition ($\mathbf{u}\cdot\hat{\mathbf{n}} = 0$). He then argues that modification of this irrotational flow field to satisfy the no-slip condition results in a velocity discontinuity at the wall, which is interpereted as a thin layer of infinite vorticity (a sheet vortex). Batchelor then postulates that this is the mechanism behind the introduction of vorticity into the fluid, which then diffuses into the fluid interior under the action of viscosity. Batchelor's analysis is largely descriptive, and he does not develop an expression to quantify the vorticity flux at the surface.

Batchelor also discusses the boundary condition for vorticity at a free surface. Velocity gradients are constrained by the zero tangential stress condition, as well as by a constant pressure (normal stress) constraint. An initial irrotational flow which satisfies the pressure condition will not, in general, satisfy the zero tangential stress condition. Fluid will be accelerated instantaneously under the action of the unbananced stresses, such that the surface velocity will satisfy the boundary conditions. This proces introduces vorticity into the fluid. Unlike the solid wall case, a discontinuity in velocity gradient, rather than the velocity magnitude, exists at the free surface, producing a layer of finite vorticity. The magnitude of this vorticity is expressed in terms of velocity derivatives at the surface. This expression simplifies to $\omega = 2\kappa q$ for stationary free surfaces.

Morton, B. R. 1984 The generation and decay of vorticity. Geophys. Astrophys. Fluid Dyn. 28, 277-308.

Presents the first detailed description of vorticity generation at solid boundaries. Restricts his analysis to two-dimensional planar boundaries, moving in their own plane. Reiterates that since there is no true source term in the vorticity equation, vorticity can only be generated at fluid boundaries. He finds, for 2D flat plate boundaries that vorticity is generated by tangential pressure gradients along the boundary, and by tangential acceleration of the boundary. He separates the creation of vorticity at solid boundaries from its subsequent diffusion away from the wall, essentially arguing that in the limit of an impulsive acceleration of the plate, no time has elapsed for viscous effects to have occured, but a vortex sheet occurs at the solid boundary, due to the discontinuity in velocities at the wall. Thus he argues that creation of vorticity is not a viscous process, but that the transport of vorticity into the mean flow is. He argues, contrary to what some authors had written, that vorticity does not diffuse into solid boundaries, rather a negative vorticity flux implies the generation of opposite signed vorticity. Vorticity of opposite sign may cross-annihilate in the bulk flow, and this represents the only means of vorticity decay within the fluid.

Cresswell, R.W. & Morton, B.R., 1995 "Drop-formed vortex rings - The generation of vorticity", Phys. Fluids. (7)

A discussion on the vorticity generation mechansims in Drop-formed vortex rings. Asserts that it is the zero-shear boundary condition responsible for the creation of vorticity, initially as an infinitessimal sheet of finite vorticity, which satisfies boundary conditions. No net vorticity has entered the flow (since the integral of vorticity is zero), however, gradients of stress and vorticity are infinite, so produce an immediate diffusion of both momentum and vorticity within the fluid. It is therefore the zero shear boundary condition which produces vorticity, at the rate at which it is diffused away from the boundary (whereby more vorticity appears at the boundary to satisfy the free-surface conditions).

This theory is then specifically applied to drop formed vortex rings, where the mechanism of vorticity generation is postulated to be surface curvature between the drop and the initial surface.

Morino, L. 1986 "Helmholtz decomposition revisited: Vorticity generation and trailing edge condition", Computational Mechanics

Uses a Helmholdtz decomposition for the velocity field, extended over an infinite domain, to discuss the flow of inviscid and viscous fluid around a solid body. The velocity field therefore includes a velocity inside the solid body, as well as in the fluid domain. Is interested in the vorticity generation mechanisms, as well as an appropriate trailing edge condition. In terms of vorticity creation, he argues that this is an inherently inviscid process. Along similar lines of reasoning to Morton, Batchelor and Lighthill, a slip velocity is created by the boundary conditions on normal velocity, under impulsive acellerations of the solid body. This slip velocity is interpereted as a vortex layer which lies inside the fluid domain. He argues that continuous generation in a smooth acellerations can be understood as the limit of infinitesimal impulsive acellerations.

Discussion on Conservation of Vorticity

Rood, E. P. (1994), Myths, math, and physics of free-surface vorticity, in A. S. Kobayashi, ed., 'Proceedings of the Twelfth US National Congress of Applied Mechanics, June 1994, Seattle, Washington (Applied Mechanics Reviews)', Vol. 47, pp. 152–156.

Rood presents a discussion of the source of vorticity at a free surface. He finds the vorticity flux is purely due to the tangential acceleration of the fluid with respect to the surface, and this acceleration is as a result of the flow satisfying the requirement of zero viscous stress at the boundary. In 3D flows, a flux of surface normal vorticity may also result in a flux of surface tangential vorticity. He considers the example of the flux of vorticity at a small amplitude surface wave – the bulk flow physics may be described by linear inviscid and irrational flow theory, but the boundary conditions are not satisfied at the free surface. Vorticity is generated at the free surface, as acceleration of the fluid will result from any unbalanced shear stress, producing exactly the required surface vorticity to satisfy the zero shear stress condition. A flux of vorticity at the free surface also occurs as a result of this fluid acceleration. Other examples display similar features, with a flux of vorticity occurring into or out of a free surface. Rood argues that this is a violation of the conservation of vorticity, which is ok since there is no universal law mandating its conservation across boundaries or interfaces, but only a mathematical basis for conservation of vorticity within a fluid volume, arising from Gauss' theorem. This is contrasted with momentum or energy, where Newton's 3rd law results in a universal conservation principle for these quantities. Roods argument essentially summarised by his statement: "*Notice that there is no requirement that a* *second fluid be present above the channel fluid to generate or absorb vorticity*". This differs to the analysis of Lundgren & Koumotsakos (1999) and Brons et al. (2014), wich both consider the presence of a seccondary fluid above the interface. Rood argues mathematically that vorticity doesn't have to travel into another fluid, but can just dissapear out of the domain, and is not generally conserved at a free surface. He also discusses the breaking of vortex lines at a free surface, and that image vortex interactions do not hold for deformable free surfaces in general.

Further discussion on these topics can also be found in Rood's chapter of Green (1995).

Zheng P. & Mohseni K "Dipole Vorticity Sources at Sharp Corners along a fluid interface"

Presents sharp corners at fluid interfaces as another source of vorticity that has not been considered previously. Argues that this can help explain certain phenomena such as extra vorticity in ships wakes that cannot otherwise be explained - the trepple point between air, water and the ships hull can act as a dipole source of vorticity.

Vorticity at a free surface

Lugt H. J. 1987 Local flow properties at a viscous free surface

Lugt discusses a boundary condition for the vorticity at a free surface, using series expansions and polar coordinates. Only considers stationary surfaces in two-dimensions, and argues that vorticity is created by free surface curvature in order to satisfy the boundary conditions. Has some discussion about dividing streamlines at a free surface, which leads to a discussion of the properties of stagnation points on a free surface. His findings suggested that vorticity generation is due to the change in hydrodynamic head along the surface. Since the analysis was steady, no consideration was made of of boundary accelerations.

Brøns, M "Topological fluid dynamics of interfacial flows"

Extend's Lugdt's discussion of the flow at a free surface to two-dimensional interfacial or free surface flows, where the interface curvature and surface tension are not constant. Uses Taylor series expansion about a point on the surface, and is largely interested in flow topology. Predicts separation streamlines and flow structures, and unfoldings of the degenerate singular points, in terms of the taylor series coeficients.

Three Dimensional Vorticity

Peck, B. & Sigurdson, L 1998 "On the kinematics at a free surface" IMA J. Applied Math.

Present a discussion on the shear free boundary condition for surface vorticity at free surfaces and interfaces. They give a brief discussion on jump conditions at fluid-fluid interfaces, but quickly make the simplification that the seccondary fluid is inviscid. Furthermore, they assume a clean interface and ignore spatial gradients of surface tension. These conditions imply that one principle axis of strain is normal to the surface. They stress that fluid elements are not in pure solid body rotation, in that the principal rate-of-strains will not, in general, be zero. The surface vorticity will be given by the rate of rotation of the surface normal. However, for cases where the surface tension gradients are significant, the principle rate of strain axis may not be aligned with the surface normal, and the surface vorticity will be:

$$\boldsymbol{\omega}_t = 2\hat{\mathbf{n}} \times \dot{\hat{\mathbf{n}}} + \frac{1}{\mu}\hat{\mathbf{n}} \times \nabla_s \gamma \tag{4}$$

Neglecting gradients in the surface tension, the surface vorticity may be written as:

$$\boldsymbol{\omega}_t = 2\nabla_s(u_n) \times \hat{\mathbf{n}} + 2\mathbf{K} \cdot \mathbf{u}_t \times \hat{\mathbf{n}} \tag{5}$$

Where $\mathbf{K}$ is the surface curvature tensor. The surface vorticity therefore contains two components- a rotation of the surface element due to rotation of the surface (the gradient of normal velocity), and rotation of the fluid element as it follows the curvature of the surface. This seccond term can also be written in terms of curvature and surface twist in the direction of fluid flow. Specifically, the component of $\boldsymbol{\omega}$ tangent to the velocity vector is:

$$\omega_{t,t} = -2q(\kappa_1 \cos^2\alpha + \kappa_2 \sin^2\alpha) = -2q\kappa_v \tag{6}$$

Which is the surface curvature in the direction of fluid motion. The component of vorticity normal to the velocity vector is:

$$\omega_{t,n} = q(\kappa_2 - \kappa_1)\sin(2\alpha) \tag{7}$$

Which is the effect of surface twist. There is no component of vorticity normal to the fluid velocity when the velocity is alligned with the principal axis, or at umbilical points.

LONGUET-HIGGINS, M. S. 1998 Vorticity and curvature at a free surface. J. Fluid Mech. 356,

Breaks down free surface vorticity into the component normal to the flow direction ($\omega_\perp = 2\kappa_n u$) and a component parallel to the flow ($\omega_\| = 2q(\kappa_b - \kappa_a)\cos\alpha\sin\alpha$) This first term is identical to the two-dimensional case, and is equivelent to solid body rotation. The seccond term exists when the flow is not locally directed along a line of principal curvature, and is essentially vorticity associated with surface twist along the flow path. This analysis only holds for a static free surface boundary, and no mention is made of the vorticity flux from this boundary.

Peck, B. & Sigurdson, L. 1999 "Geometry Effects on Free Surface Vorticity Flux" J. Fluids Engineering Vol. 121(3) pp. 678-683

Discusses the flux of vorticity at free surfaces. Briefly reviews the boundary conditions given in the earlier article, and follows by deriving a full surface vorticity flux equation for arbitrary free surfaces. These equations are identical to those in Wu (1995), although with different notation. They then proceed to elaborate on the surface curvature term ($\mathbf{\Phi}_\kappa = \nu \mathbf{K} \cdot \boldsymbol{\omega}_t$) , where $\mathbf{K}$ is the surface curvature tensor. They demonstrate that it may be expressed in terms of principal curvatures:

$$2\nu|\boldsymbol{\omega}_t|[\kappa_1(\hat{\mathbf{x}}_1 \cdot \hat{\mathbf{t}})\hat{\mathbf{x}}_1 + \kappa_2(\hat{\mathbf{x}}_2 \cdot \hat{\mathbf{t}})\hat{\mathbf{x}}_2] \tag{8}$$

Where $\hat{\mathbf{t}}$ is the unit vector in the direction of the surface tangential vorticity. $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$ are the surface tangential unit vectors in the directions of principal curvature. This equation shows that the flux due to curvature will be zero when the surface tangential vortex lines are alligned with a principal axis of zero curvature. If the flow is not alligned with a principal axis, there is:

$$2\nu|\boldsymbol{\omega}|[\kappa_\zeta(\hat{\zeta} \cdot \hat{\mathbf{t}})\hat{\zeta} + \tau(\hat{\xi} \cdot \hat{\mathbf{t}})\hat{\xi}] \tag{9}$$

Which demonstrates that the vorticity flux may exist on a vortex line with no normal curvature, when the surface twist is non-zero. For a steady free surface, the vorticity flux to surface curvature becomes:

$$\mathbf{\Phi}_\kappa = 2\nu K q \hat{\mathbf{t}} \tag{10}$$

Where $\hat{\mathbf{t}}$ is a unit vector tangent to the surface, and normal to the surface velocity and K is the Gaussian curvature, $K = \kappa_1\kappa_2$. The curvature flux is only zero if the Gaussian curvature is zero, or if the velocity is zero. The Gaussian curvature may be used to predict the sign of the vorticity generated. The vorticity generated will always be normal to the velocity vector, and the magnitude of the vorticity flux does not depend on the direction of flow.

Herrera, B (2010) "Vorticity and curvature at a general material surface", Phys. Fluids 22, 042104

Presents a mathematical analysis simmilar to that of Peck and Sidgurston, except for an arbitrary material surface in the flow. This reduces to Peck and Sidgurston's equations under the assumption of a free surface, however, in general, includes the symmetric part of the strane rate tensor in adition to the usual terms. He also includes an expression for the normal vorticity, however the rate of strain tensor is non-zero for this component. He addresses the vorticity flux, but only in regards to the curvature term, in which he includes the specified vorticity to write a fully expanded equation for the vorticity flux in terms of velocities, surface curvatures, and the rate of strain tensor.

Waves

Andre M. A. & Bardet P. M. 2017 Free surface over a horizontal shear layer: Vorticity generation and air entrapment mechanisms

Discusses vorticity generation and air entrapment for gravity-capilliary waves generated by shear layer roll-up just beneath a free surface. This problem is investigated experimentally, with PIV in a wave tank. The boundary layer at the jet becomes a shear layer. Vorticity in this layer rolls-up into a vortex street (of positive vorticity only). These create small waves on the water surface, where negative surface vorticity is 'generated' due to curvature and other effects. The negative vorticity may be shed from the wave troughs at a separation point, producing CRVP's (although the secondary vortex is much smaller). Air entrapment may occur during this process although this is not relevent for my research. Analysed the vorticity flux into the wave system as it travels downstream, largely based of Rood's non-conservation approach, observing a significant change in net circulation downstream.

Dabiri D & Gharib M 1997 Experimental investigation of the vorticity generation within a spilling water wave

Uses PIV techniques to resolve the velocity vectors and vorticity field for a stationary spilling wave. Consider 2D generation of vorticity based on Rood's equations (which don't consider the 'conservation' of vorticity). Find that vorticity is produced due to the deceleration of the free surface at the trough of the wave. They note that this occurs where the wave curvature changes, but they don't think this causes vorticity generation, prefering to state the surface deceleration as the vorticity generation source.

Under a conservation perspective, as per Brons et al. (2014), the surface deceleration could be interpereted as a loss (or gain) of vorticity from the free surface vortex sheet. This may be since the surface vorticity must rapidly change to satisfy the free surface boundary condition. Large ammounts of negative vorticity are thus shed into the wake, and the surface velocity changes due to conservation of vorticity.

Belden j & Techet A H "Simultaneous quantitative flow measurement using PIV on both sides of the air-water interface for breaking waves".

Discusses a novel method to generate simultaneous PIV measurements in both air and water phases during free surface experiments. This is applied to spilling breaking waves, where vorticity generation is found to occur in the air as it flows over the crest, while significant vorticity generation mostly occurs near the wave 'toe', much later than the air vorticity. (Theoretically at a viscous interface, vorticity fluxes should be equal, but perhaps viscous diffusion of vorticity occurs more quickly in air, due to the much lower density. This would be a subject worthy of investigation). Their discussion on the sources of vorticity was largely speculative, and could use some more thorough analysis. They allude to the analysis of Dabiri & Gharib for stationary waves, and comment that simmilar analysis is lacking for the traveling wave case.

Fenton J. D. (1988) "The Numerical Solution of Steady Water Wave Problems"

Presents a numerical scheme to solve surface shapes, velocity fields and pressure distributions for steady traveling water waves.

Raval A, Wen X & Smith M H (2009), 'Numerical simulation of viscous, nonlinear and progressive water waves'

A VOF (Volume of fluid) method is used to compute the surface shape, velocity fields and the vorticity and shear stress distributions in travelling water waves. Uses a stationary frame of reference, using inlet and outlet conditions (as opposed to a periodic simulation). An inviscid potential flow solution (assuming a sinusoidal wave shape) is used for both initial conditions, and for the inlet boundary. A constant outled velocity is used to accound for volume flow due to stokes drift.

Compare nonlinear solutions to potential, inviscid or low-Re theories, and find that these analytical solutions are not able to provide accurate representations of the flow. Nonlinear viscous effects result in larger vorticity layers at the free surface, as well as producing wave profiles with larger crests and sharper peaks. Streamlines and shear-stress distribuitons are also not accurately predicted by either linear or low-Re theory.

Klettner, C A & Eames, I (2012), "The laminar free surface boundary layer of a solitary wave"

Use an arbitrary Lagrangian-Eularian (ALE) method to numerically investigate the laminar boundary layer at the free surface of a solitary traveling wave. This method directly tracks the free surface, and can apply the required boundary conditions directly. A third-order accurate solution is used for the initial wave profile.

They investigate the surface vorticity, as well as the distribution of vorticity and velocity gradients in the boundary layer, and compare to an approximate boundary layer vorticity solution obtained by mapping the vorticity transport equation onto the potential flow solution.

Monsimith S G, Cowen E A, Neph H M, Magnaudet J & Thais L (2007) "Laboratory observations of mean flows under surface gravity waves"

Summarises results of a range of wave tank experiments, finding that in deep water flows, the Eulerian mean flow induced by wave motion exactly cancels the Stokes drift induced by the wave. Thus, mass transport is largely unnafected by the presence of waves.

Colagrossi A, Souto-Iglesias, A. Antuono M & Marrone, S. (2013), "Smoothed-particle-hydrodynamics modeling of dissipation mechanisms in gravity waves"

More a summary of the SPH method, and how insufficient resolution results in a spurious vorticity field occuring, which overpredicts viscous damping of wave energy. Give some good formulas for wave dissipation, and find that the strongest dissipation contributions are terms expressed as surface integrals of pressure and surface vorticity, respectively.

Gerstner's Wave

Stuhlmeier R (2015), "Gerstner's Water Wave and Mass Transport"

Presents a general review outlining some recent developments into the applicability of Gerstner's wave to real flows. Gertner's wave is the 'only known explicit solution to the inviscid incompressible gravity water wave problem with a non-flat free surface.' Arguments against its use include that it is not irrotational, so such a wave cannot be introduced in an inviscid fluid under the action of potential forces, and that it predicts no stokes drift, which is in contrast to experemental observations of water waves.

Henry D (2008), "On Gerstner's Water Wave"

Confirms that Gerstner's wave is a physical solution to the inviscid water wave equations. The wave form is shown to satisfy continuity, the existance of a pressure field which satisfies the governing equations and appropriate boundary conditions is confirmed, particles on the free surface remain on the surface, and the deep water boundary condition ($\lim_{z\to\infty}(v) = 0$) is satisfied.

The wave is shown to be rotational, with vorticity depending only on 'b':

$$\omega_z = \frac{-2\sqrt{mg}\mathrm{e}^{2mb}}{1-\mathrm{e}^{2mb}} \tag{11}$$

Weber J E H (2011) "Do we observe Gerstner waves in wave tank experiments"

Presents a discussion of the problem of stokes drift in gertner waves. Constructs a small amplitude pertubation solution to the viscous water wave problem, finding a trochoidal solution, resembling Gertner waves, but modified under the action of vorticity. Particle paths are not circular, but inwards spirals, as a result of viscous damping. Drift velocity is found to be high near the surface, but negative deeper in the water with a zero net drift. These results suggest the viscosity modified Gerstner wave may explain some wave tank experimental observations.

Alternative numerical schemes

VOF (volume of fluid)

-Need some references here (look up Reichl's references)

Arbitrary Lagrangian-Eularian (ALE)

-See Klettner & Eames (2012), read references for more details

Meshless RBF (Radial Basis Function) method

Wu N, Tsay T & Young D L (2005) "Meshless numerical simulation for fully nonlinear water waves"

Present a new meshless technique for calculating irrotational free surface flows, using radial basis functions to describe the flow field, and ensuring the boundary conditions are satisfied at discrete boundary nodes.

Moving Particle Semi-Implicit

Imanian H, Kolahdoozan M & Zarrati A R (2011), "Waves Simulation in Viscous Waters Using MPS"

Describe an implimentation of Moving-Particle-semi-implicit (MPS) model which includes viscous effects. MPS uses a set of particles, and characterises their interactions by approximating the gradient operator using kernel functions to describe the dependance on particle interactions with radial displacement.

consistent Particle Method (CPM)

Koh C G, Gao M & Luo C (2012), "A new particle method for simulation of incompressible free surface flow problems"

Present a new particle method, similar to SPH and MPS, but uses Taylor Series based functions for particle effects, as opposed to the somewhat arbitrary kernal functions used in other methods. This prevents interpolation errors in the pressure field that may occur due to large particle spacings.