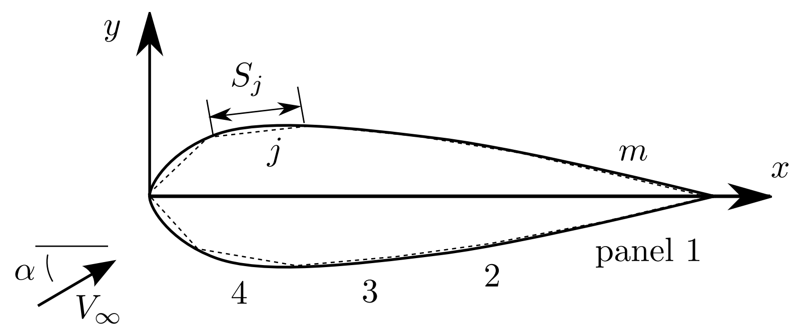
THE VORTEX PANEL METHOD

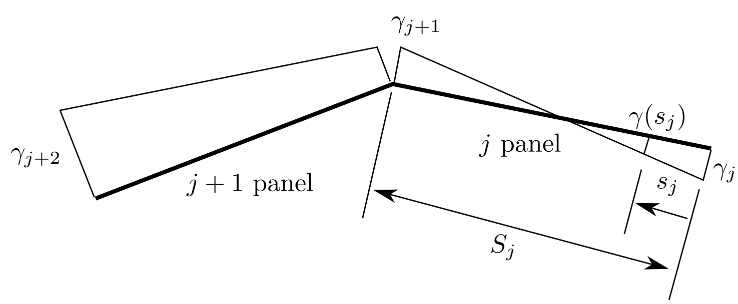


a) Approximate the contour of the airfoil by an inscribed polygon with m sides, called "**panels**". Number the panels clockwise with panel #1 starting on the lower (pressure) side, at the trailing edge and panel #m ending on the trailing edge from the upper (suction) side.

b) Assume that each panel represents a planar vortex sheet with linearly varying strength and such that the end strength of each panel is the same as the starting strength of the next panel:

$$\gamma(s_j) = \gamma_j + \frac{s_j}{S_j}(\gamma_{j+1} - \gamma_j)$$

Exception: $\gamma_1 \neq \gamma_{m+1}$



Instead of the dimensional strength γ (in velocity units), it is more convenient to use the **dimensionless** strength γ', defined as

$$\gamma' = \frac{\gamma}{2\pi V_\infty}$$

Then,

$$\gamma'(s_j) = \gamma'_j + \frac{s_j}{S_j}(\gamma'_{j+1} - \gamma'_j)$$

c) The only unknowns of this problem are the **end strengths** γ'_j, $j = 1, 2, ..., m+1$. They can be found by solving $m+1$ equations consisting of

i) m-equations, expressing the **no penetration** condition at the m mid-points of the panels (called "control points")

ii) one equation, expressing the **Kutta condition**.

Once γ_j are found, the velocity can be found by superposition and the pressure can be computed from Bernoulli's equation.

d) Apply the **Kutta condition** at the trailing edge. Note that strict application of K-c would require that

$$(V_u)_{\text{t.e.}} = (V_l)_{\text{t.e.}} = 0$$

Instead, apply the weaker condition

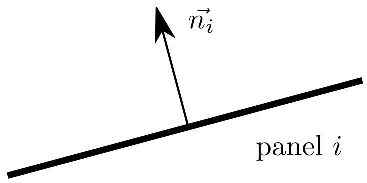
$$(V_u - V_l)_{\text{t.e.}} = 0$$

which implies that

$$\gamma'_1 + \gamma'_{m+1} = 0 \qquad \text{(A)}$$

e) Apply the **no penetration** condition at the m **control points** (mid-panels). Notice that all panels are assumed to be parts of the same streamline. The equipotential lines are normal to streamlines and, therefore, to panels. Let ϕ be the velocity potential. Then,

$$\nabla\phi = \vec{V}, \text{ which implies that } V_{ni} = \frac{\partial \phi}{\partial n_i}$$



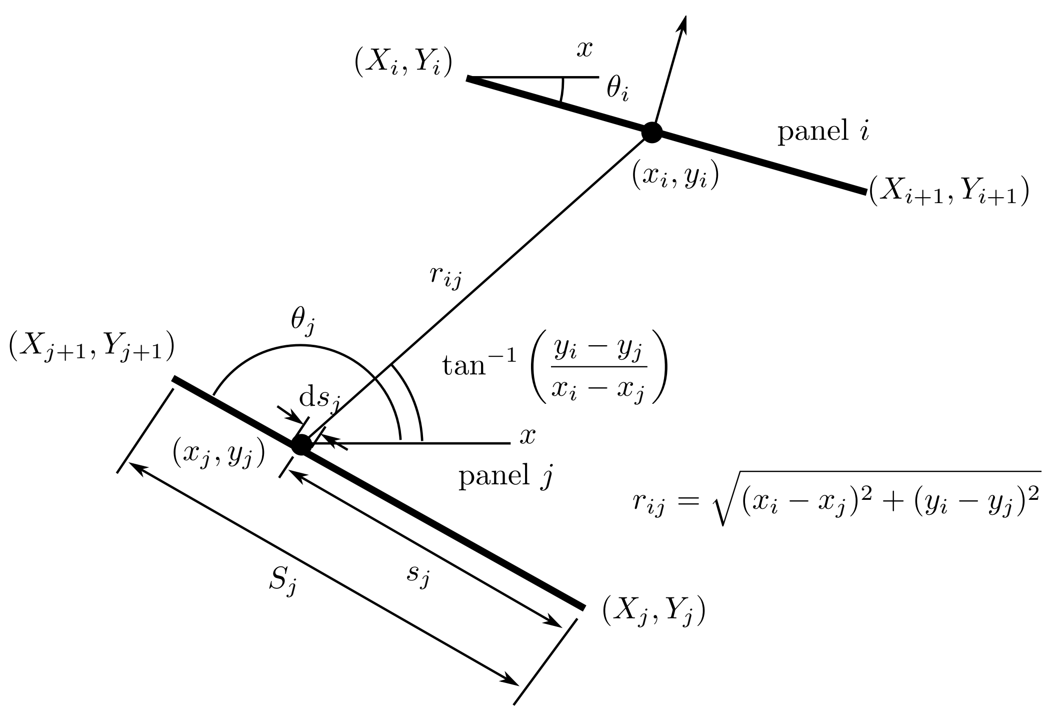
where V_{ni} is the velocity component normal to the panel i, and $\partial\phi/\partial n_i$ is the derivative of ϕ in the direction of the normal vector, $\vec{n_i}$. Therefore, the n-equation of no penetration becomes

$$\frac{\partial\phi(x_i, y_i)}{\partial n_i} = 0, \quad i = 1, 2, \ldots, m \tag{B}$$

f) The velocity potential $\phi(x_i, y_i)$ at a control point can be found by superposition of the potential due to the uniform stream and the potentials due to the m vortex panels, as

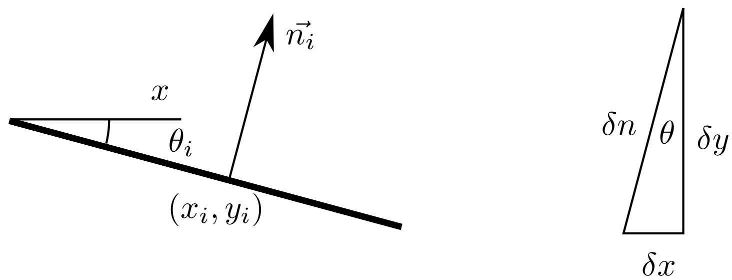
$$\phi(x_i, y_i) = V_\infty(x_i \cos\alpha + y_i \sin\alpha) + \sum_{j=1}^{m} \int_{\text{panel } j} \frac{-\gamma(s_j)}{2\pi} \tan\frac{y_i - y_j}{x_i - x_j} \mathrm{d}s_j \tag{C}$$

The remaining work is only algebraic and trigonometric manipulations.



g) Notice that

$$\frac{\partial x_i}{\partial n_i} \approx \frac{\delta x_i}{\delta n_i} = \sin\theta_i$$
$$\frac{\partial y_i}{\partial n_i} \approx \frac{\delta y_i}{\delta n_i} = \cos\theta_i$$



Then, by chain rule,

$$\frac{\partial \phi}{\partial n_i} = \frac{\partial \phi}{\partial x_i}\frac{\partial x_i}{\partial n_i} + \frac{\partial \phi}{\partial y_i}\frac{\partial y_i}{\partial n_i} = \frac{\partial \phi}{\partial x_i}\sin\theta_i + \frac{\partial \phi}{\partial y_i}\cos\theta_i$$

In expression (C), everything has already been expressed in terms of x_i, y_i, so it is straightforward to apply the chain rule to the various derivatives.

Reminder
$$\frac{\partial}{\partial t}\tan^{-1}[f(t)] = \frac{1}{1+f^2(t)}\frac{\mathrm{d}f}{\mathrm{d}t}$$

Then
$$\frac{\partial}{\partial x_i}\left[\tan^{-1}\frac{y_i-y_j}{x_i-x_j}\right] = \frac{-1}{1+\left(\frac{y_i-y_j}{x_i-x_j}\right)^2}\frac{y_i-y_j}{(x_i-x_j)^2}$$
$$\frac{\partial}{\partial y_i}\left[\tan^{-1}\frac{y_i-y_j}{x_i-x_j}\right] = \frac{1}{1+\left(\frac{y_i-y_j}{x_i-x_j}\right)^2}\frac{1}{x_i-x_j}$$

h) The integral in eq. (C) is w.r.t. the j panel, therefore x_i, y_i are constant during integration. Because x_j, y_j, s_j vary, it is necessary to express all of them in terms of one variable, e.g. s_j.

$$x_j = X_j - s_j\cos\theta_j$$

$$y_j = Y_j + s_j \sin\theta_j$$

Also $\gamma(s_j)$ has been expressed in terms of s_j. Then the integral for panel j becomes

$$\gamma_j' \int_0^{S_j} f_{1j}(s_j)\mathrm{d}s_j + \gamma_{j+1}' \int_0^{S_j} f_{2j}(s_j)\mathrm{d}s_j$$

which can be computed numerically.

i) Following various manipulations (see textbook for some details) one gets the system of $m+1$ linear, algebraic equations

$$\sum A_{n_{ij}}\gamma_j' = \sin(\theta_i - \alpha), \quad i = 1, 2, \ldots, m$$

$$\gamma_1' + \gamma_{m+1}' = 0$$

where the constants $A_{n_{ij}}$ depend only on the coordinates of the end points of panels i and j and can be computed immediately once the panels have been selected. Notice that the effect of panel i on itself has also been taken into account. The system of the above equations is solved numerically to provide the end strengths.

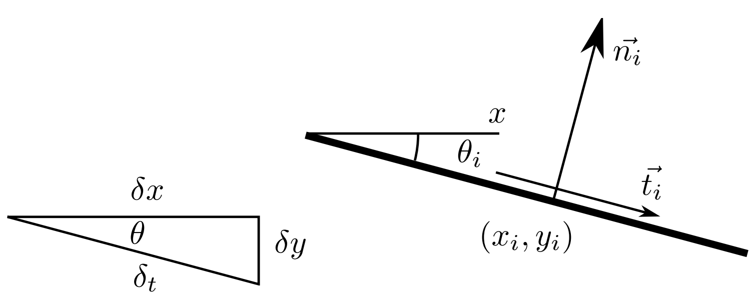
$$\gamma_j', \quad j = 1, 2, \ldots, m+1$$

j) The **velocity at the control points is tangential** to the panel and can be found as

$$V_i = V_{t_i} = \frac{\partial\phi}{\partial t_i} = \frac{\partial\phi}{\partial x_i}\frac{\partial x_i}{\partial t_i} + \frac{\partial\phi}{\partial y_i}\frac{\partial y_i}{\partial t_i}$$

or

$$V_i = \frac{\partial\phi}{\partial x_i}\cos\theta_i + \frac{\partial\phi}{\partial y_i}\sin\theta_i$$



Instead of the dimensional velocity V_i, the program computes a **dimensionless** velocity as

$$\forall_i = \frac{V_i}{V_\infty} = \frac{1}{V_\infty}\frac{\partial \phi}{\partial t_i}$$

Following various manipulations, this is found as

$$\forall_i = \sum_{j=1}^{m+1} = A_{t_{ij}}\gamma'_j + \cos(\theta_i - \alpha), \quad i = 1, 2, \ldots, m$$

where the coefficients $A_{t_{ij}}$ depend on the end coordinates of panels i and j alone.

k) The pressure **at the control points** can be found from Bernoulli's eq. as

$$C_{P_i} = \frac{P_i - P_\infty}{\frac{1}{2}\rho V_\infty^2} = 1 - \forall_i^2$$

l) The lift can be found in two ways:

   i) By integrating all vortex strengths and using the **Kutta-Joukowsky** theorem.

   ii) By integrating the surface pressure.