Bézier-Splines

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Wann ist ein kubischer Bezier-Splines $S(u) = [F(2 \cdot u), G(2(u-1))]$, welche Kontrollpunkte in $\mathbb{R}^2$ haben, C^2-stetig?

Sei im Folgenden $a = P_0$, $b = P_1$, $c = P_2$, $d = P_3$. Dann gilt:

$$
\begin{align}
F(x) &= \sum_{i=0}^{3} \mathbf{b}_i^n P_i \tag{1}\\
&= \sum_{i=0}^{3} \binom{n}{i} x^i (1-x)^{n-i} P_i \tag{2}\\
&= (1-x)^3 P_0 + 3x(1-x)^2 P_1 + 3x^2(1-x)P_2 + x^3 P_3 \tag{3}\\
&= -ax^3 + 3ax^2 - 3ax + a + 3bx^3 - 6bx^2 + 3bx - 3cx^3 + 3cx^2 + dx^3 \tag{4}\\
&= (-a + 3b - 3c + d)x^3 + (3a - 6b + 3c)x^2 + (-3a + 3b)x + a \tag{5}\\
F'(x) &= 3(-a + 3b - 3c + d)x^2 + 2(3a - 6b + 3c)x + (-3a + 3b) \tag{6}\\
F''(x) &= 6(-a + 3b - 3c + d)x + 2(3a - 6b + 3c) \tag{7}
\end{align}
$$

1 C^0-Stetigkeit

Damit S nun C^0-stetig ist, muss $F(1) = G(0)$ gelten. Also:

$$
\begin{align}
&G(0) = F(1) \tag{8}\\
\Leftrightarrow\ &a_G = (-a_F + 3b_F - 3c_F + d_F) + (3a_F - 6b_F + 3c_F) + (-3a_F + 3b_F) + (2 + a_F) \tag{9}\\
\Leftrightarrow\ &a_G = d_F \tag{10}
\end{align}
$$

2 C^1-Stetigkeit

Damit S nun C^1-stetig ist, muss zusätzlich $F'(1) = G'(0)$ gelten. Also:

$$G'(0) = F'(1) \tag{11}$$
$$\Leftrightarrow -3a_G + 3b_G = 3(-a_F + 3b_F - 3c_F + d_F) + 2(3a_F - 6b_F + 3c_F) + (-3a_F + 3b_F) \tag{12}$$
$$\Leftrightarrow -3(a_G - b_G) = 3(-c_F + d_F) \tag{13}$$
$$\Leftrightarrow -a_G + b_G = -c_F + d_F \tag{14}$$

3 C^2-Stetigkeit

Damit S nun C^2-stetig ist, muss zusätzlich $F''(1) = G''(0)$ gelten. Also:

$$G''(0) = F''(1) \tag{15}$$
$$\Leftrightarrow 2(3a_G - 6b_G + 3c_G) = 6(-a_F + 3b_F - 3c_F + d_F) + 2(3a_F - 6b_F + 3c_F) \tag{16}$$
$$\Leftrightarrow 6(a_G - 2b_G + c_G) = 6(b_F - 2c_F + d_F) \tag{17}$$
$$\Leftrightarrow a_G - 2b_G + c_G = b_F - 2c_F + d_F \tag{18}$$