Information-Geometric Engineering: Constructing Spacetime

A Variational Principle for the Einstein Equations

Based on: "Einstein Equations from Information-Geometric Variational Principle," Haobo Ma & Wenlin Zhang.

JHEP Submission.

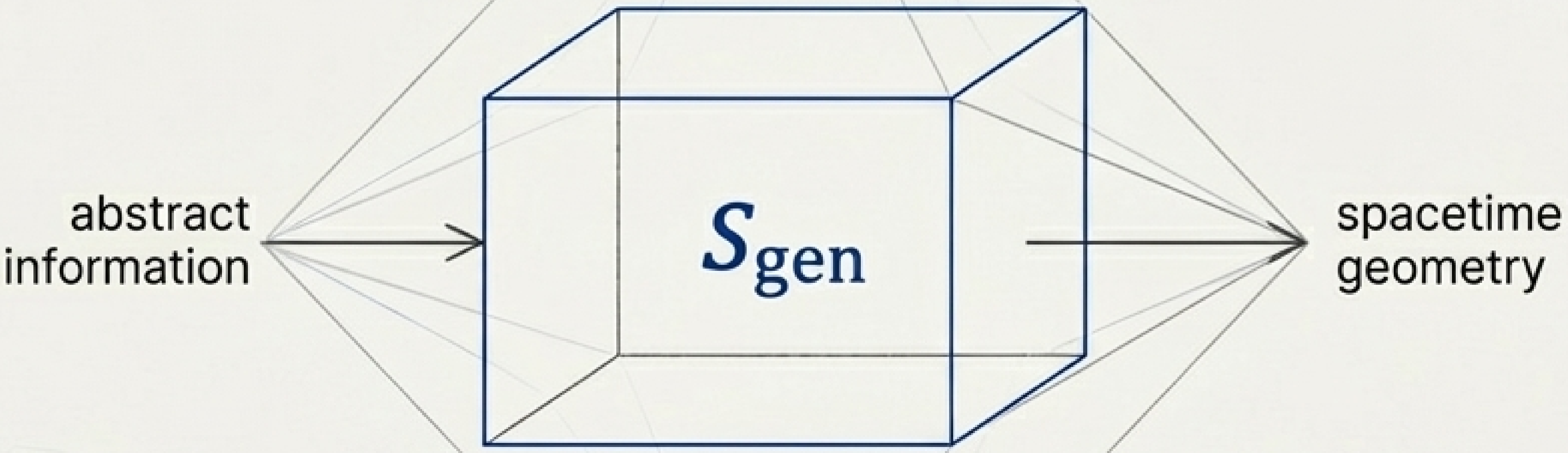
$G_{ab} + \Lambda g_{ab} = 8\pi G T_{ab}$

An equation of state, or an emergent architectural law?

This presentation outlines a rigorous derivation of the local Einstein equations (for $d \geq 3$) from a more fundamental principle. We will demonstrate how spacetime geometry is constructed, piece by piece, from the thermodynamics of information.

The Foundation: The Information-Geometric Variational Principle (IGVP)

Spacetime dynamics arise from a single instruction: Extremize the generalized entropy within a causal diamond of fixed volume.



$$S_{\text{gen}} = \frac{A}{4G\hbar} + S_{\text{ren\_out}} - \frac{\Lambda}{8\pi G} * \frac{V}{T}$$

δ The entire structure is derived from the first-order stationarity condition $\delta S_{\text{gen}} = 0$ under the constraint $\delta V = 0$.

The Materials: Deconstructing Generalized Entropy (S_{gen})

$$S_{\text{gen}} = \frac{A}{4G\hbar} + S_{\text{ren\_out}} - \frac{\Lambda}{8\pi G} * \frac{V}{T}$$

Geometric Entropy

The Bekenstein-Hawking area term, linking geometry to entropy.

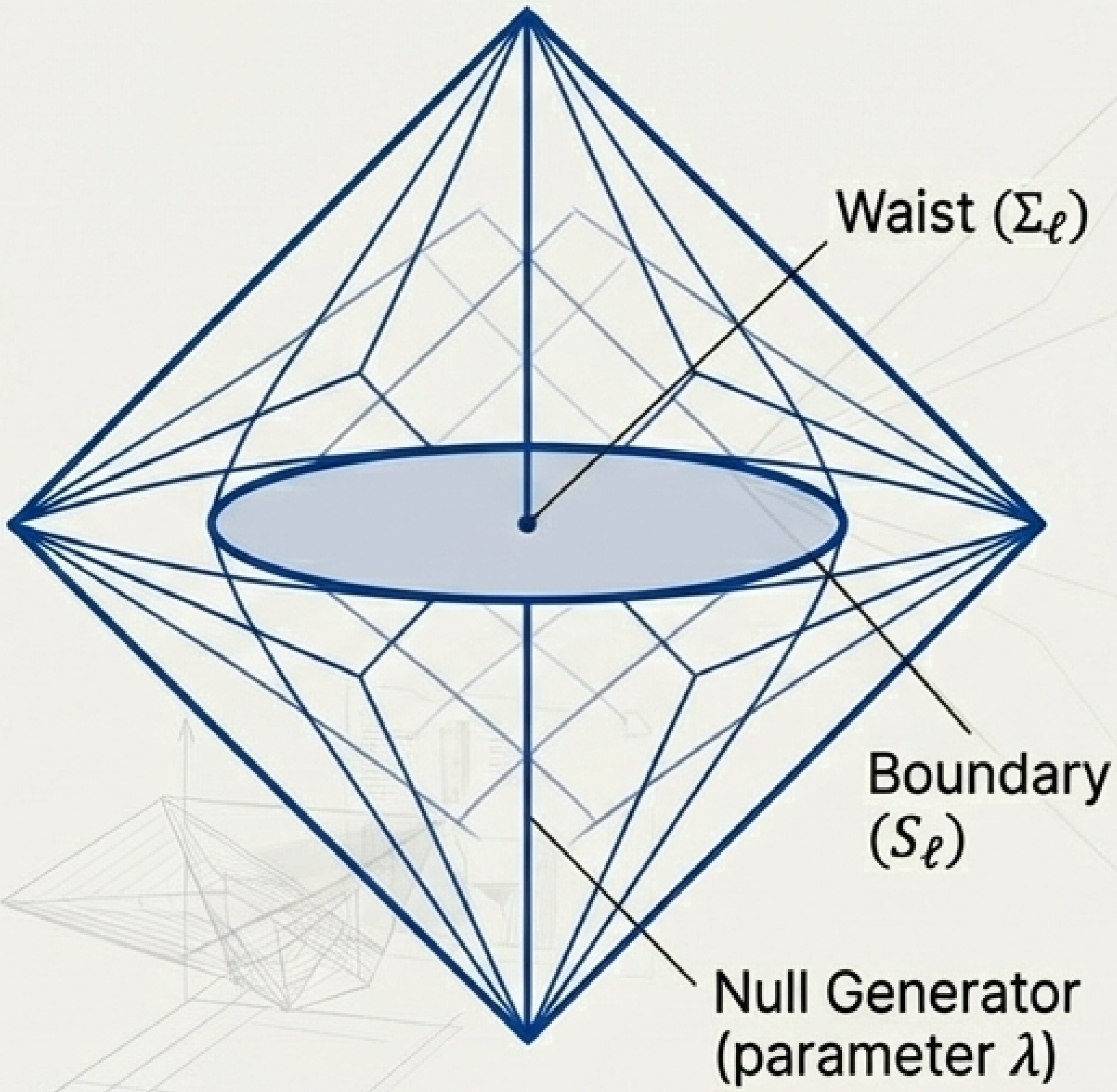
Matter & Entanglement Entropy

The renormalized entropy of quantum fields outside the causal diamond's boundary. For a KMS state, $\delta S_{\text{ren\_out}}$ is the variation of the modular Hamiltonian, $\delta\langle K\chi\rangle$.

Volume Constraint

A dual term incorporating the cosmological constant Λ as a Lagrange multiplier that enforces a fixed spacetime volume V at temperature T.

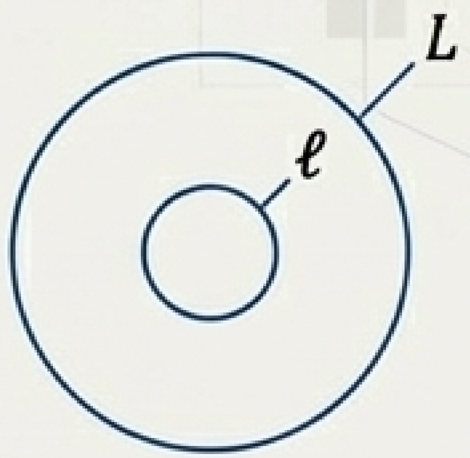
The Worksite: The Small Causal Diamond



Prerequisites: Our construction requires a controlled environment defined by two key conditions:

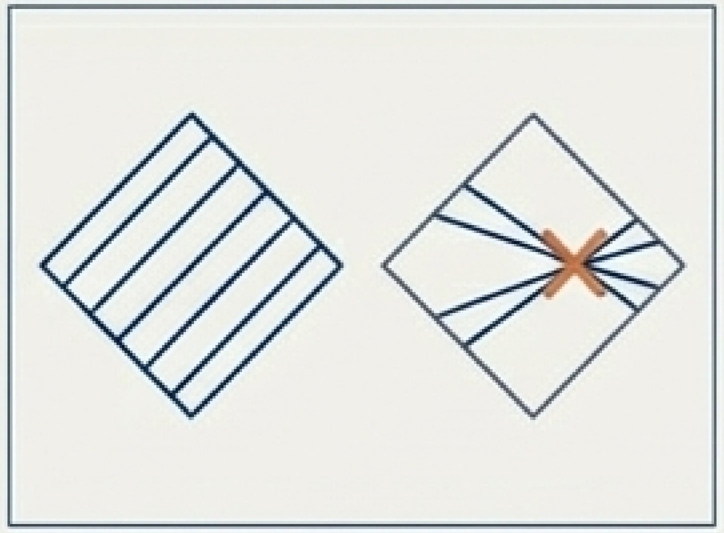
1. Scale Separation

The diamond's size ℓ is much smaller than the characteristic scale of curvature and matter variation, L.
$\varepsilon = \ell/L \ll 1$.



2. No Conjugate Points

Within the diamond, null geodesics do not refocus. This ensures the Raychaudhuri equation is controllable and geometric transforms are locally invertible.



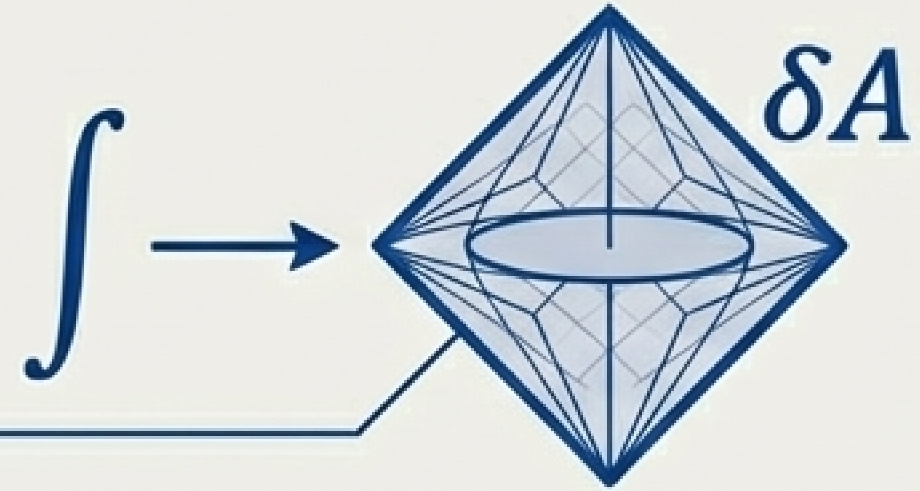
The Primary Tool: Raychaudhuri's Equation

The expansion (θ) of a congruence of null rays is governed by three factors: its own focusing, the shear (σ), and the spacetime curvature, encoded in the null-contracted Ricci tensor ($R_{kk} := R_{abka}k_b$).

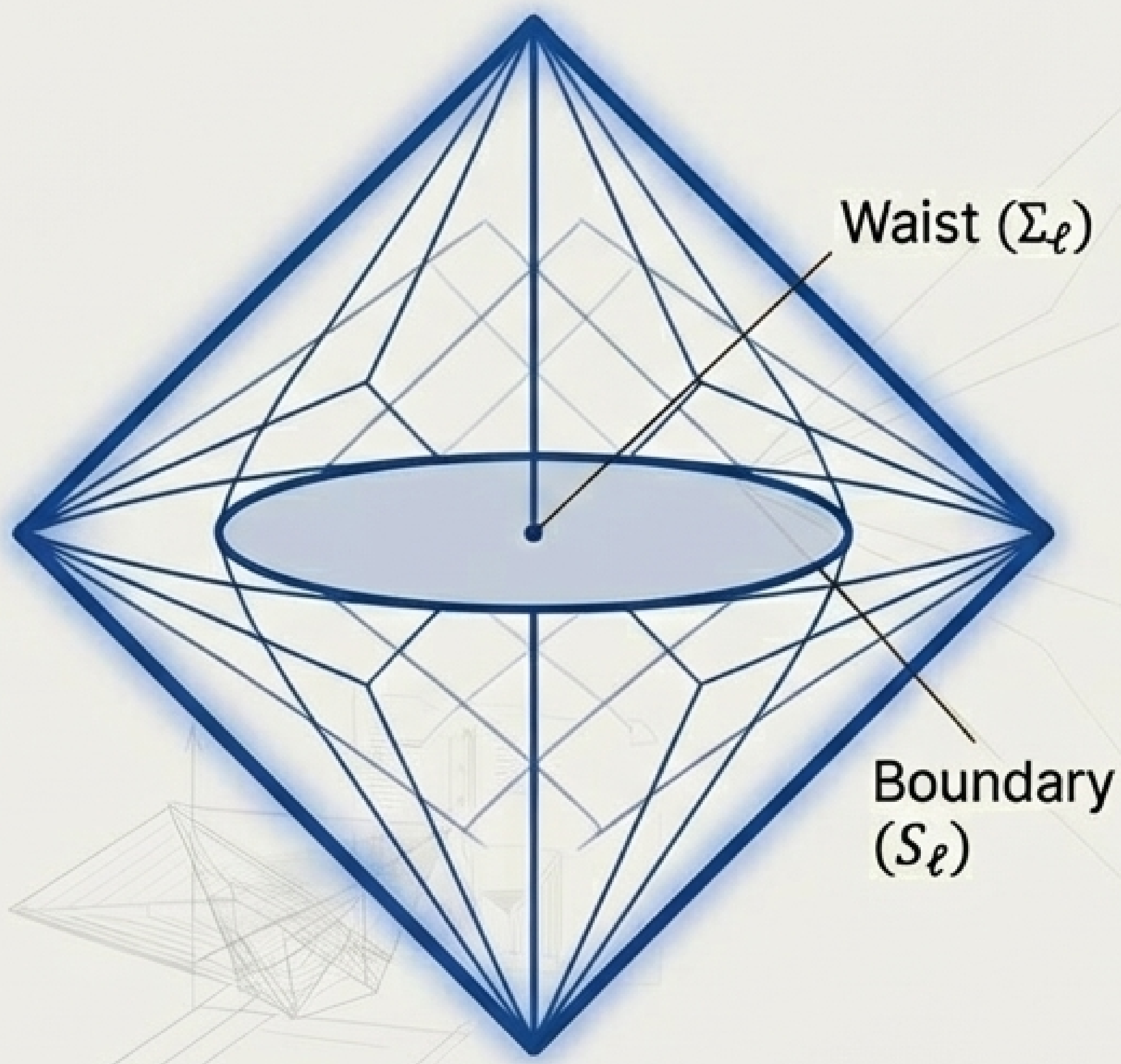
$$\theta' = -\frac{1}{d-2}\theta^2 - \boldsymbol{\sigma}^2 - \boldsymbol{R_{kk}}$$

(Note: With twist ω=0 due to hypersurface orthogonality at the diamond's waist.)

By integrating this equation along the null generators of the diamond, we can relate the change in the boundary's area to the curvature inside.



First Structural Result: The Area-Curvature Balance



The Core Identity

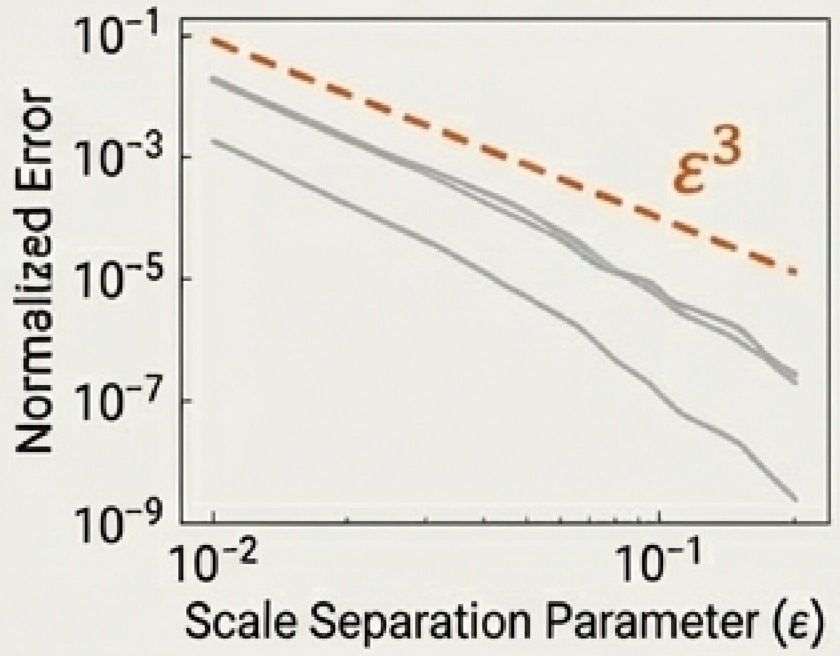
In the small diamond limit, the variation of the boundary area (δA) is directly dictated by the first moment of the integrated null curvature:

$$\delta A \approx -\int_H \lambda R_{kk}\, d\lambda\, dA$$

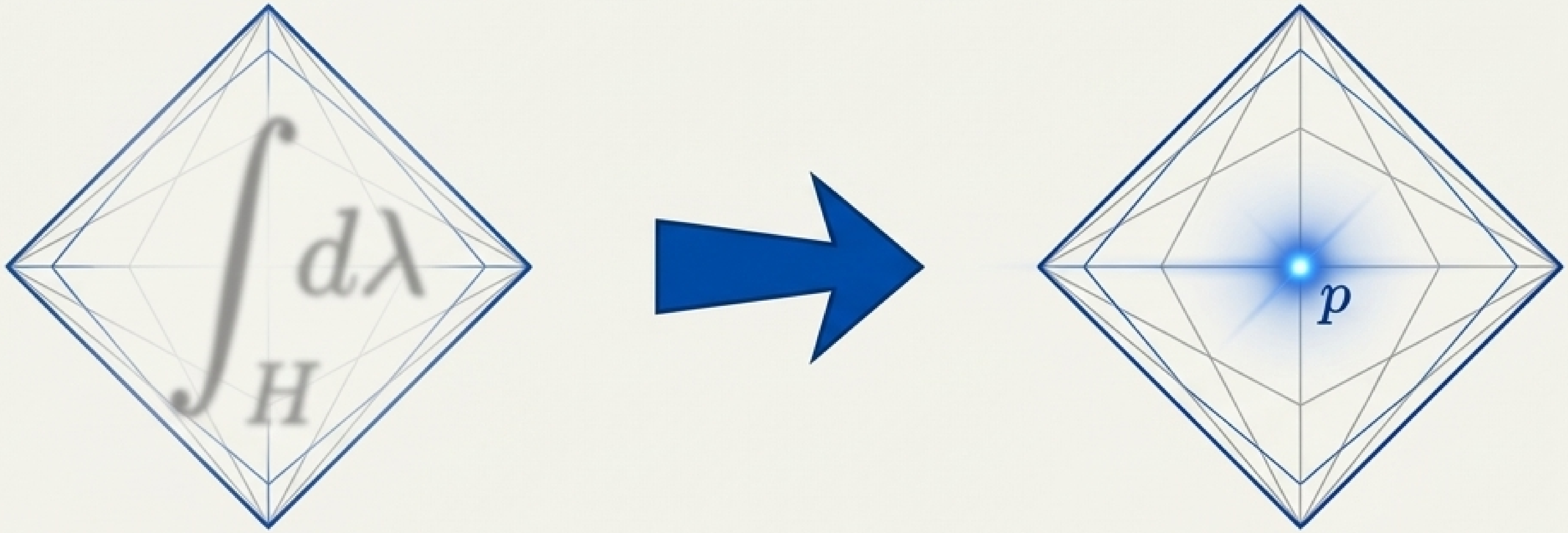
Precision

This is a high-precision result. For 'weak-shear' families ($|\sigma(0)| \leq c_s \varepsilon$), the normalized error scales as ε^3, ensuring it is a sub-dominant effect.

Numerical Verification: Error Scaling



From Global Constraint to Pointwise Law



The Challenge

The stationarity of entropy ($\delta S_{\text{gen}} = 0$) combines with the area-curvature balance to yield an integral constraint over all null directions:

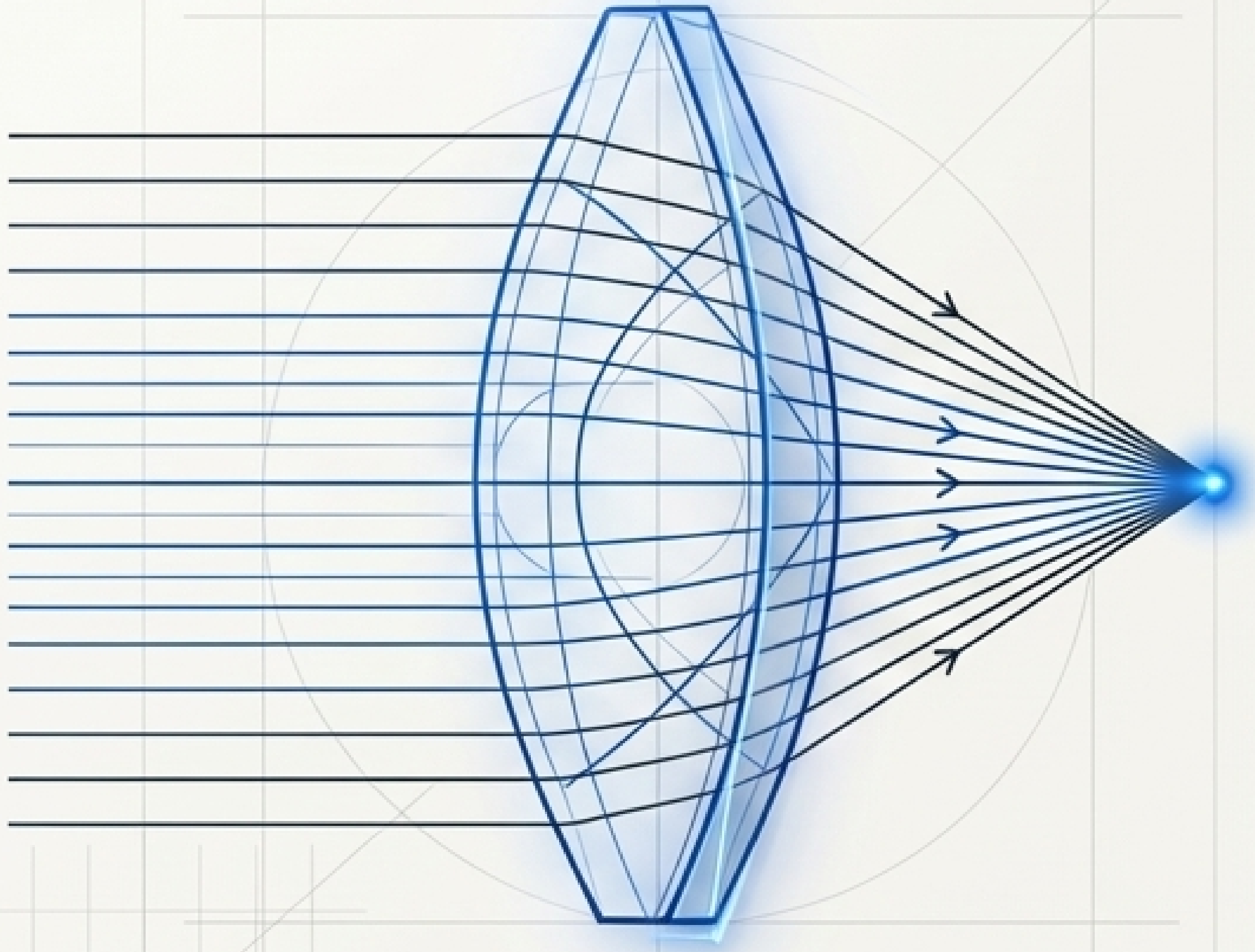
$$\int_H \lambda \left(R_{kk} - 8\pi G T_{kk}\right) d\lambda\, dA = \underline{o(\ell^2)}$$

This tells us that a weighted average is zero, but not that the quantity itself is zero everywhere.

The Solution

We need a mathematical tool to 'focus' this integral information onto the central point *p*. This is achieved with a **weighted null ray transform**.

The Focusing Lens: Weighted Null Ray Transform



Definition

For a function f along a null ray γ through point p, we define the transform:

$$L_\lambda[f](p, \hat{k}) := \int_0^{\ell_*} \lambda f(\gamma(\lambda))\, d\lambda$$

Key Property (Local Stability)

Under the small diamond assumptions, this transform has a non-degenerate principal part. It effectively isolates the function's value at the origin:

$$L_\lambda[f](p, \hat{k}) = \frac{1}{2}\lambda^{*2} f(p) + R(p, \hat{k})$$

where the remainder R is of a higher order in ℓ.

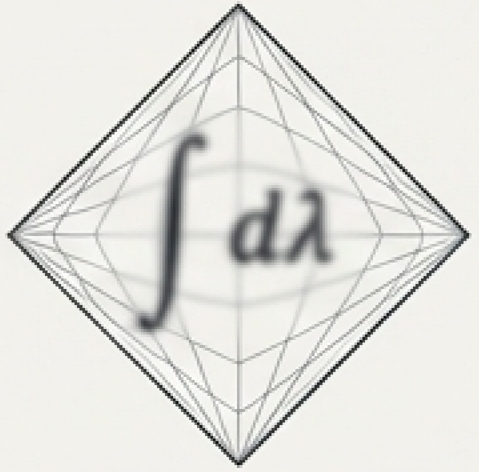
Invertibility Corollary

If $\sup_{\hat{k}} |L_\lambda[f](p, \hat{k})| = o(\ell^2)$, then it must be that $f(p) = 0$.

Resolution Achieved: The Null-Contracted Equation

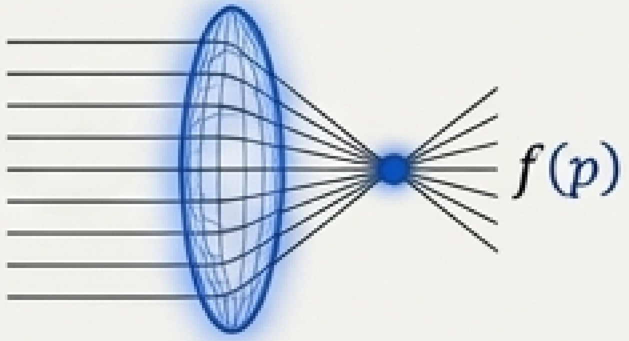
Start with the Integral Constraint

$$\int_H \lambda \left(R_{kk} - 8\pi G T_{kk} \right) d\lambda \, dA = o(\ell^2)$$



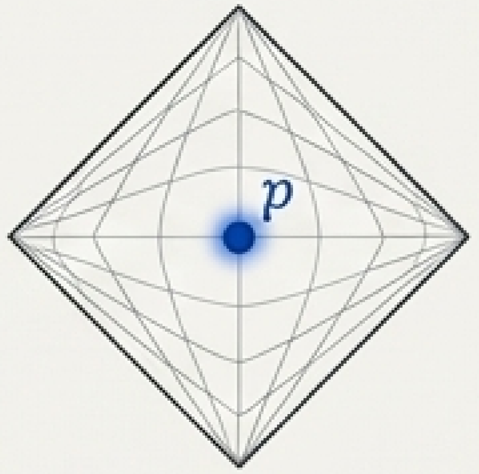
Define the Function

Let $f = R_{kk} - 8\pi G T_{kk}$. The constraint means $\mathcal{L}_\lambda[f] = o(\ell^2)$.



Apply Invertibility

The null ray transform's stability (Corollary 4.2) implies that if the integral is $o(\ell^2)$, the function itself must be zero at the central point p.



$$R_{kk} = 8\pi G T_{kk}$$

(for all null vectors **k**)

Final Assembly: Tensorial Closure (for d ≥ 3)

The Geometric Lemma

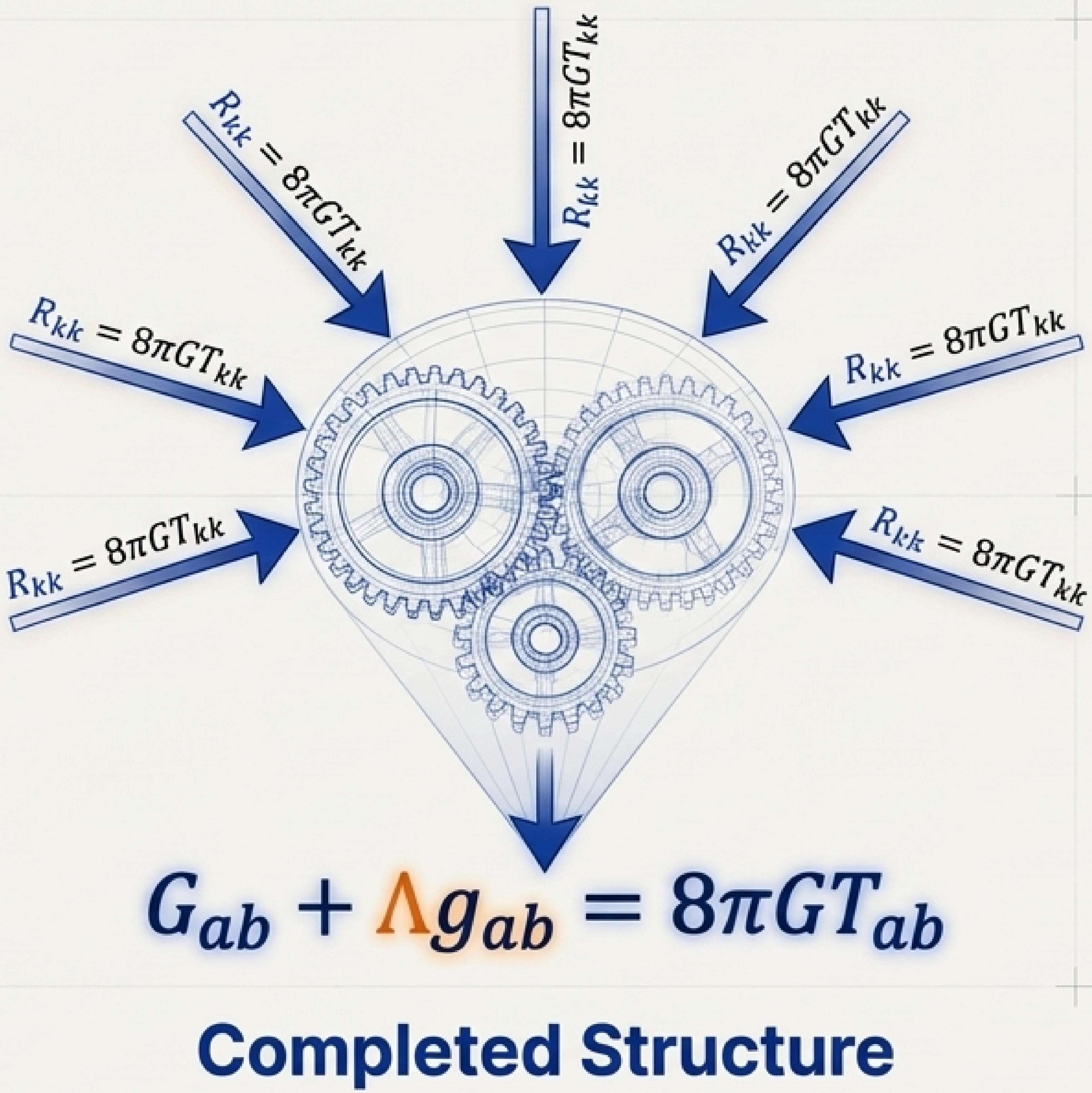
In dimensions $d \geq 3$, the null cone uniquely determines the spacetime metric up to conformal scale. This leads to a powerful result:

> If a symmetric tensor X_{ab} satisfies $X_{ab}{}^{a}k^{b} = 0$ for all null vectors k, then X_{ab} must be proportional to the metric, $X_{ab} = \Phi g_{ab}$.

Application

1. Let $X_{ab} = R_{ab} - 8\pi G T_{ab}$. Our result $R_{kk} = 8\pi G T_{kk}$ means $X_{ab}{}^{a}k^{b} = 0$.
2. Therefore, $R_{ab} - 8\pi G T_{ab} = \Phi g_{ab}$.

   Applying the conservation ($G_{ab} = \frac{1R}{2s} G T_{k}$).
3. Applying the contracted Bianchi identity ($\nabla^{a} G_{ab} = 0$) and matter conservation ($\nabla^{a} T_{ab} = 0$) fixes the proportionality function Φ to be $\frac{1}{2}R - \Lambda$, where Λ is a constant of integration.



Structural Integrity: The Second-Order Stability Check

The Test
Is the $\delta S_{\text{gen}} = 0$ solution a stable extremum? We examine the second-order variation of relative entropy, $\delta^2 S_{\text{rel}}$.

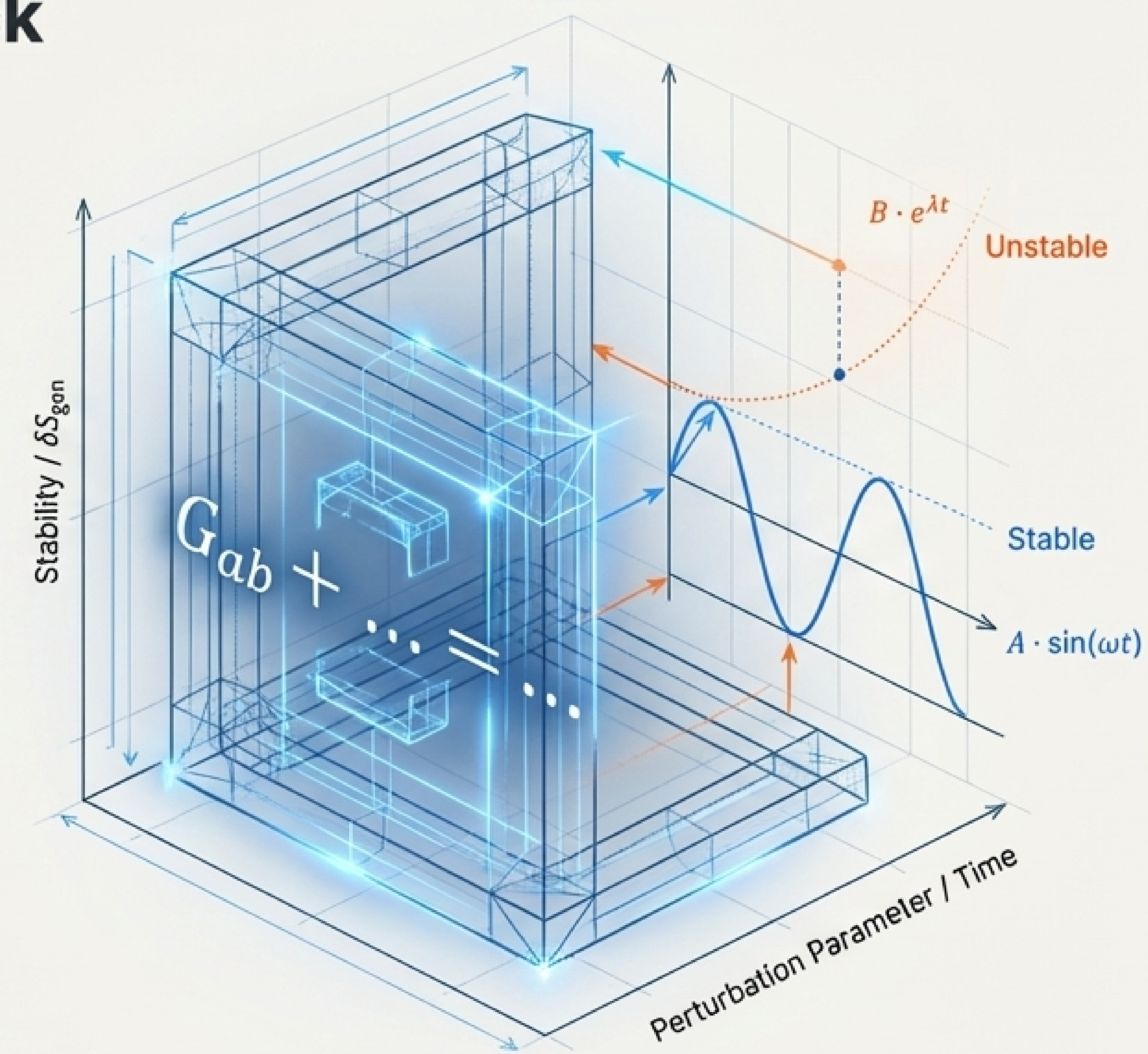
The Condition
Physical stability requires $\delta^2 S_{\text{rel}} \geq 0$.

The Physical Identification
Under the JLMS/FQ identification (known to hold in certain code subspaces), the second-order relative entropy is equivalent to well-known physical quantities:

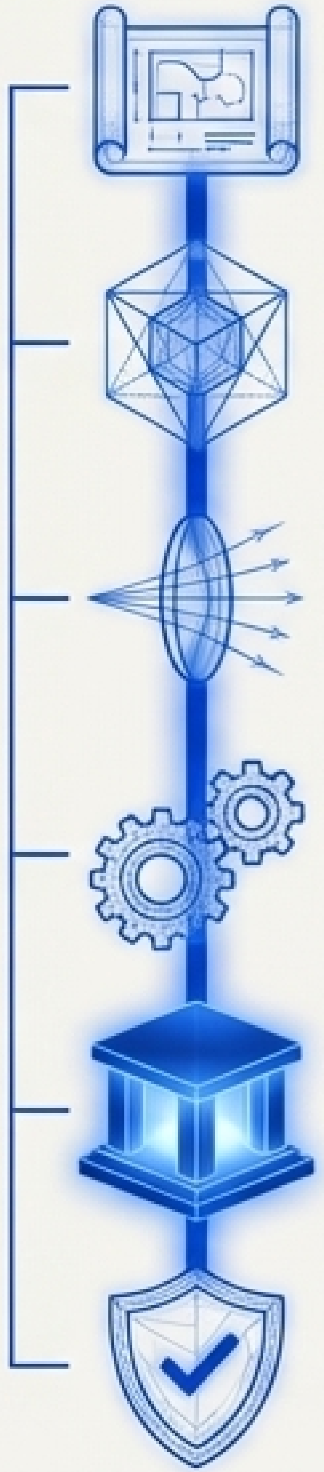
$$\delta^2 S_{\text{rel}} = FQ = E_{\text{can}}[h, h] \geq 0$$

(Relative Entropy = Quantum Fisher Information = Hollands-Wald Canonical Energy of metric perturbations h)

The Verdict
The solution is stable, provided appropriate boundary conditions are met. An alternative 'no-duality' route using the Quantum Null Energy Condition (QNEC) provides a universal criterion, reinforcing the result's robustness.



Blueprint Realized: The Chain of Derivation



Foundation
IGVP Principle ($\delta S_{\mathbf{gen}} = \mathbf{0}$)

Scaffolding
Small Diamond Limit & Raychaudhuri Eq. ($\delta A \approx -\int \lambda R_{kk} \ldots$)

Resolution
Weighted Null Ray Transform ($L_{\lambda}[f] \approx \frac{1}{2}\lambda^{*2} f(p)$)

Assembly
Tensorial Closure ($X_{abk}{}^{a}{}_{k}{}^{b}=0 \Rightarrow X_{ab} \propto g_{ab}$)

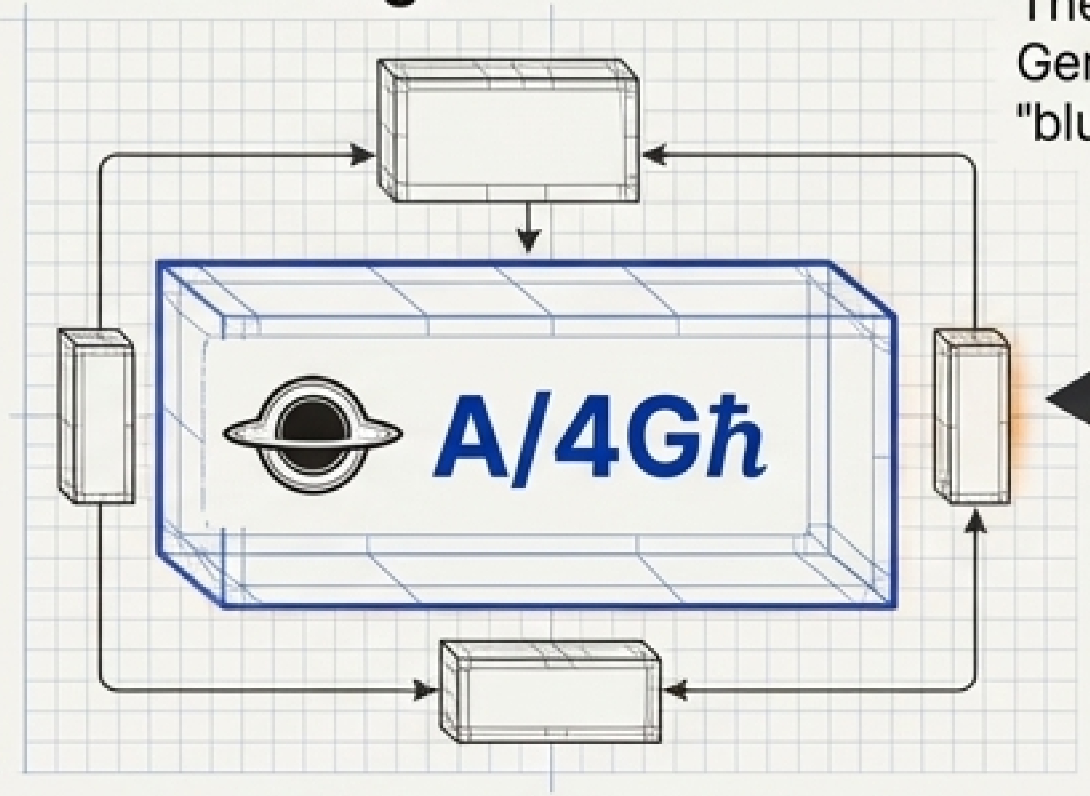
Final Edifice
Einstein Field Equations ($\boldsymbol{G_{ab} + \Lambda g_{ab} = 8\pi G T_{ab}}$)

Integrity Check
Second-Order Stability ($\delta^2 S_{rel} = \boldsymbol{E_{can} \geq 0}$)

An Extensible Framework: Beyond Einstein

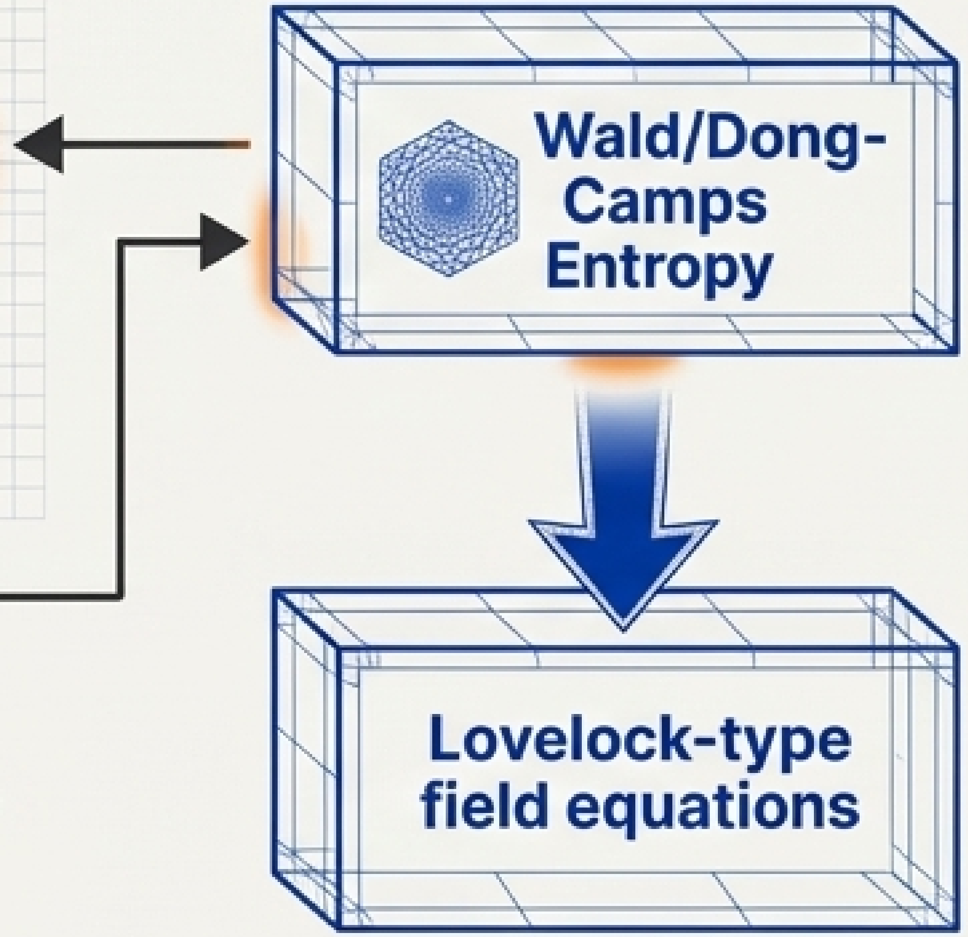
- **Foundation**
- Scaffolding
- Resolution
- Assembly

Modular S_{gen} Functional



The Power of the Principle

The IGVP construction is not limited to General Relativity. The foundational "blueprint" remains the same.



Higher-Order Gravity

By replacing the geometric entropy term **$A/4G\hbar$** with more general functionals, the same machinery can be used to derive higher-order theories of gravity.

This **information-geometric approach** provides a general framework for understanding the emergence of gravitational dynamics from entanglement entropy.

Technical Foundations & Reproducibility

Deep Dive Appendices

The full rigor of this derivation rests on proofs for three core technical challenges:

M1 Uniform bound for the modular Hamiltonian approximation.

M2 Local invertibility and stability of the first-moment null ray transform.

M3 Constructive existence and stability of the required weak-shear diamond families. ⚠

Reproducible Operations

All numerical demonstrations are reproducible. The ancillary files of the paper contain:

- Parameter tables and normalization conventions.
- Python scripts (`` `scripts/generate_igvp_figure1.py` ``, etc.).
- Complete datasets and execution environment details.