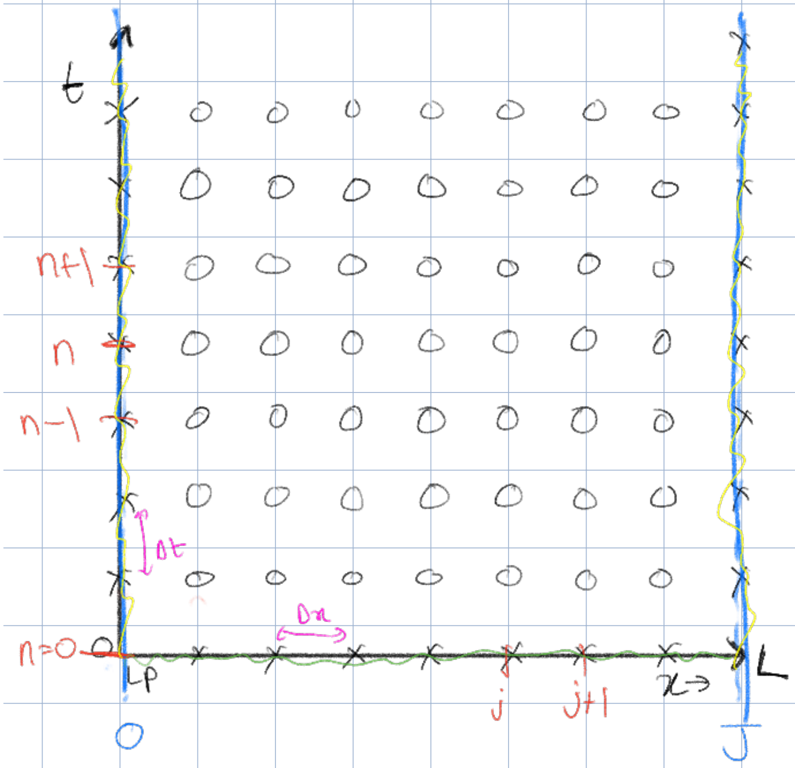
Question 3



x = knowns from BCs

O = unknown vals

$x_j = j\Delta x$; $j = 0, 1, 2 \dots J$

$t_n = n\Delta t$; $n = 0, 1, \dots$

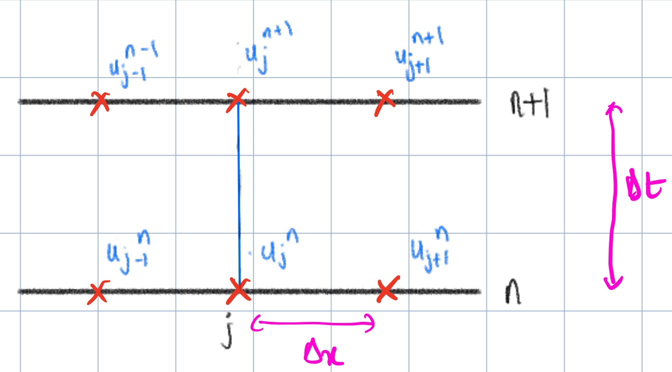
$u_j^n = U(x_j, t_n)$

Boundary conditions

$u(x, 0) = u_0(x)$

$u(L_p, t) = U(L, t) = 0$

drawing the six point stencil :-



$$u_t = \frac{u_j^{n+1} - u_j^n}{\Delta t}$$

applying the θ method for the advection term using upwind discretisation

* upwind discretisation means previous values are utilised to calculate erivative

for $a > 0$ flow

$$\theta\left(\frac{u_{j+1}^{n+1} - u_j^{n+1}}{\Delta x}\right) + (1-\theta)\left(\frac{u_{j+1}^{n} - u_j^{n}}{\Delta x}\right)$$

for $a < 0$ flow

$$\theta\left(\frac{u_j^{n+1} - u_{j-1}^{n+1}}{\Delta x}\right) + (1-\theta)\left(\frac{u_j^{n} - U_{n-1}^{n}}{\Delta x}\right)$$

* assuming $a(t)$ is always positive to simplify:

$$\therefore \quad u_x = \theta\left(\frac{u_{j+1}^{n+1} - u_j^{n+1}}{\Delta x}\right) + (1-\theta)\left(\frac{u_{j+1}^{n} - u_j^{n}}{\Delta x}\right)$$

$$u_{xx} = \theta\left(\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x}\right) + (1-\theta)\left(\frac{u_{j+1}^{n} - 2u_j^{n} + u_{j-1}^{n}}{\Delta x}\right)$$

final form :-

$$\frac{u_j^{n+1} - u_j^{n}}{\Delta t} - a(t)\left[\theta\left(\frac{u_{j+1}^{n+1} - u_j^{n+1}}{\Delta x}\right) + (1-\theta)\left(\frac{u_{j+1}^{n} - u_j^{n}}{\Delta x}\right)\right]$$
$$- \epsilon\left\{\theta\left(\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x}\right) + (1-\theta)\left(\frac{u_{j+1}^{n} - 2u_j^{n} + u_{j-1}^{n}}{\Delta x}\right)\right\} = 0$$

Assuming that $\alpha(t)$ is always a constant:

define $A = \frac{\alpha(t)\Delta t}{\Delta x}$ $\quad B = \frac{\epsilon \Delta t}{\Delta x^2}$

$$u_j^{n+1} - u_j^n = A\theta U_{j+1}^{n+1} - A\theta U_j^{n+1} + (1-\theta)A U_{j+1}^n - (1-\theta)A U_j^n$$

$$+ B\theta U_{j+1}^{n+1} - 2B\theta U_j^{n+1} + B\theta U_{j-1}^{n+1}$$

$$+ (1-\theta)BU_{j+1}^n - 2B(1-\theta)U_j^n + B(1-\theta)U_{j-1}^n$$

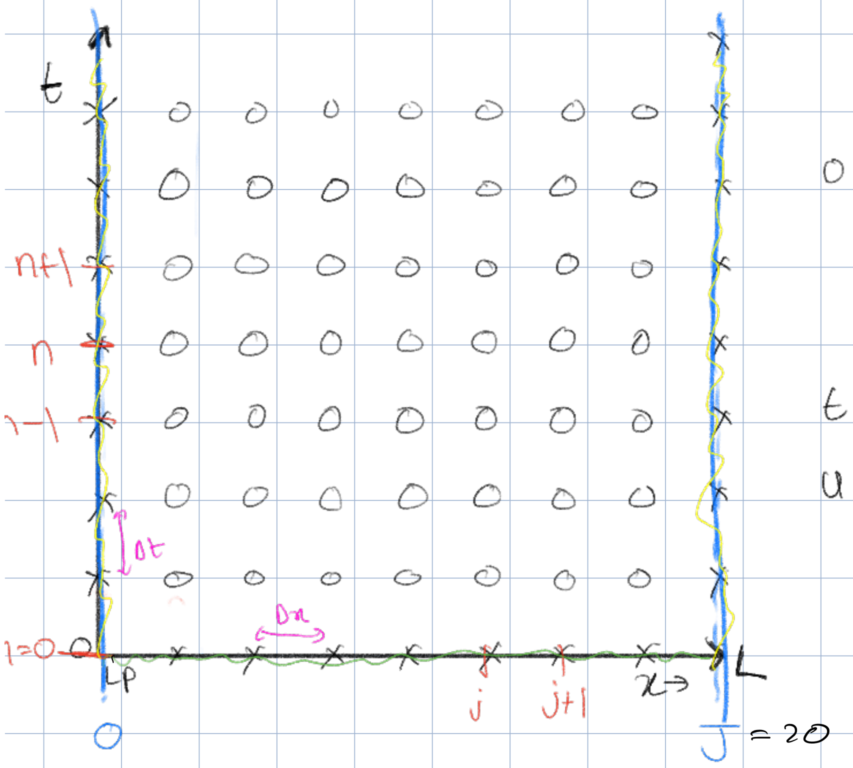
$$U_j^{n+1}\left(1 + A\theta + 2B\right) - U_{j-1}^{n+1}(B\theta) - U_{j+1}^{n+1}(A\theta + B\theta)$$

$$=$$

$$U_j^n\left(1 - A(1-\theta) - 2B(1-\theta)\right) + U_{j+1}^n\left(A(1-\theta) + B(1-\theta)\right)$$

$$+ U_{j-1}^n\left(B(1-\theta)\right) \qquad \text{(eq 3)}$$

near the boundary



for $t=0$ $\quad u(x,0) = U_0(x) = U_j^0$

LHS

$$U_j^{n+1}\left(1 + A\theta + 2B\right) - U_{j-1}^{n+1}\left(B\theta\right) - U_{j+1}^{n+1}\left(A\theta + B\theta\right) \quad \text{(unknowns)}$$

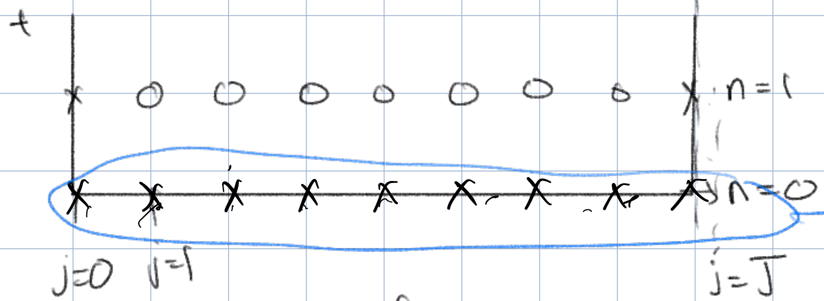
RH

$$A - A\theta + B - B\theta$$

$$U_j^{n}\left(1 - A(1-\theta) - 2B(1-\theta)\right) + U_{j+1}^{n}\left(A(1-\theta) + B(1-\theta)\right) \quad \text{(knowns)}$$

$$+\ U_{j-1}^{n}\left(B(1-\theta)\right)$$

dealing with the point close to the boundary



known as $U_j^0 = U_0(x)$

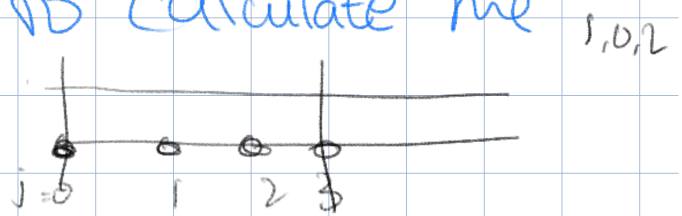
considering first row

$$
\begin{array}{c}
j=0 \\ j=1 \\ j=2
\end{array}
\underbrace{\begin{pmatrix}
(1+A\theta+B\theta) & (A\theta+B\theta) & \cdots & 0 \\
(-B\theta) & (1+A\theta+B\theta) & -(A\theta+B\theta) & \\
0 & (-B\theta) & (1+A\theta+B\theta) & -(A\theta+B\theta) \\
 & & \cdots &
\end{pmatrix}}_{J}
\begin{pmatrix}
U_1^{n+1} \\ U_2^{n+1} \\ U_3^{n+1} \\ \vdots \\ U_{J-1}^{n+1}
\end{pmatrix} J
=
\begin{pmatrix}
0 \\
U_n^0 (B(1-\theta)) + U_1^0(1-A(1-\theta)-2B(1-\theta)) + U_2^0(B(1-\theta)) \\
U_3^0 \\
\vdots \\
0
\end{pmatrix}
$$

tridiagonal J matrix

all values of U_j^0 are known

→ since all values of U_j^0 are known from the Boundary conditions, we can use this to calculate the next row.



$$\left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & & \\ -B\theta & (1+A\theta+2B\theta) & (-A\theta-B\theta) & \cdots & & \\ \vdots & & \ddots & & & \\ & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{array}\right) \begin{array}{l} j=0 \\ j=1 \\ j=2 \\ \\ j=J \end{array} \begin{pmatrix} U_0^{n+1} \\ U_1^{n+1} \\ \vdots \\ U_J^{n+1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

tridiagonal