Question 2

eq. 2.80

$$u_j^{n+1} = \left[u + \frac{1}{2}\Delta t\, u_t + \frac{1}{2}\left(\frac{1}{2}\Delta t\right)^2 u_{tt} + \frac{1}{6}\left(\frac{1}{2}\Delta t\right)^3 u_{ttt} + \dots\right]_j^{n+1/2}$$

$$u_j^n = \left[u - \frac{1}{2}\Delta t\, u_t + \frac{1}{2}\left(\frac{1}{2}\Delta t\right)^2 u_{tt} - \frac{1}{6}\left(\frac{1}{2}\Delta t\right)^3 u_{ttt} + \dots\right]$$

if w subtract the two:

$u_j^{n+1} - u_j^n$

first term: $u - u = 0$

second term: $+\frac{1}{2}\Delta t\, u_t - \left(-\frac{1}{2}\Delta t\, u_t\right) = \Delta t\, u_t$

third term: $\frac{1}{2}\left(\frac{1}{2}\Delta t\right)^2 u_{tt} - \frac{1}{2}\left(\frac{1}{2}\Delta t\right)^2 u_{tt} = 0$

fourth term: $+\frac{1}{6}\left(\frac{1}{2}\Delta t\right)^3 u_{ttt} - \left[-\frac{1}{6}\left(\frac{1}{2}\Delta t\right)^3 u_{ttt}\right]$

$$= \frac{1}{3}\left(\frac{1}{2}\Delta t\right)^3 u_{ttt}$$

$$= \frac{1}{3} \times \frac{1}{8}(\Delta t)^3 u_{ttt}$$

$$= \frac{1}{24}(\Delta t)^3 u_{ttt} \dots$$

$$\therefore \; \delta_t u_j^{n+1/2} = u_j^{n+1} - u_j^{n} = \left[t\Delta u_t + \frac{1}{24}(\Delta t)^3 u_{ttt} \cdots \right]_j^{n+1}$$

central difference term

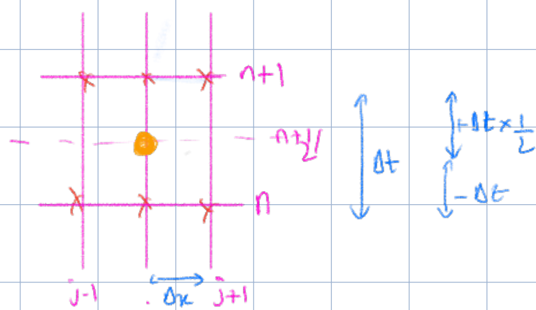
(2.80)

Eq 2.81

note: $U_j^n = u(x_j, t_n)$

$$(+) \; \delta_x^2 u(x,t) = \left(u(x+\Delta x, t) - 2u(x,t) + u(x-\Delta x) \right)$$

$\Rightarrow$ A taylor expansion on (+) using the stencil drawn is preformed below



1st term: $u(x+\Delta x, t)$

$$\left[u(x,t) + u_x \Delta x + \frac{1}{2} u_{xx} (\Delta x)^2 + \frac{1}{3} u_{xxx} (\Delta x)^3 + \frac{1}{24} u_{xxxx} (\Delta x)^4 \right]_j^{n+1}$$

... hot

last term: $u(x - \Delta x, t)$

$$\left(u(x,t) - u_x \Delta x + \frac{1}{2} u_{xx}(\Delta x)^2 - \frac{1}{3} u_{xxx}(\Delta x)^3 + \frac{1}{24} u_{xxxx}(\Delta x)^4\right)_j^{n+1}$$

... h.o.t

* where h.o.t means higher order terms.

adding them together gives $\Rightarrow$

$$\therefore \left(\cancel{2u(x,t)} + u_{xx}(\Delta x)^2 + \frac{1}{12} u_{xxxx}(\Delta x)^4 - \cancel{2u(x,t)}\right)_j^{n+1}$$

$$\therefore \delta_x^2 u_j^{n+1} = \left\{u_{xx}(\Delta x)^2 + \frac{1}{12} u_{xxxx}(\Delta x)^4 \ldots \text{h.o.t}\right\}_j^{n+1}$$

(2.81)

Eq 2.82

taylor series expansion of each term in Δt around $(x_n, t_{n+1/2})$

$(\Delta x)^2 u_{xx} \Rightarrow (\Delta x)^2 u_{xx} + (\Delta x)^2 u_{xxt}\left(\frac{1}{2}\Delta t\right) + (\Delta x)^2 u_{xxtt}\left(\frac{1}{2}\Delta t\right)^2$

$= (\Delta x)^2 u_{xx} + (\Delta x)^2 u_{xxt}\left(\frac{1}{2}\Delta t\right) + (\Delta x)^2 u_{xxtt}\left(\frac{1}{4}\Delta t\right)^2$

$(\Delta x)^4 \frac{1}{12} u_{xxxx} \Rightarrow \frac{1}{12}(\Delta x)^4 u_{xxxx} + (\Delta x)^2 u_{xxxxt}\left(\frac{1}{2}\Delta t\right) \ldots$

for $\delta_x^2 u_j^n$

$\Rightarrow$ $\Delta(x)^2 u_{xx} = (\Delta x)^2 u_{xx} - (\Delta x)^2 u_{xxt}\left(\frac{1}{2}\Delta t\right) + \Delta x\, u_{xxtt}\left(\frac{1}{2}\Delta t\right)^2$

$\frac{1}{12}(\Delta x)^4 u_{xxxx} \Rightarrow \frac{1}{12}(\Delta x)^4 u_{xxxx} - (\Delta x)^2 u_{xxt}\left(\frac{1}{2}\Delta t\right) \ldots$ hot

combining them by adding together:

$\theta \delta_x^2 u_j^{n+1} + (1-\theta)\delta_x^2 u_j^n \Rightarrow$

first terms $(u_{xx}) \Rightarrow \theta(\Delta x^2)u_{xx} + (1-\theta)(\Delta x^2)u_{xx}$
$= (\Delta x)^2 u_{xx}$ .. same

second terms (u_{xxt}) $\Rightarrow \theta(\Delta x^2)u_{xxt}\left(\frac{1}{2}\Delta t\right) - (1-\theta)(\Delta x)^2 u_{xxt}\left(\frac{1}{2}\Delta t\right)$

$$\theta\delta_x^2 u_j^{n+1} + (1-\theta)\delta_x^2 u_j^n = \left[(\Delta x)^2 u_{xx} + \frac{1}{12}(\Delta x)^4 u_{xxxx} \cdots\right]$$
$$+ \left(\theta - \frac{1}{2}\right)\Delta t\left[(\Delta x)^2 u_{xxt} + \frac{1}{12}(\Delta x)^4 u_{xxxxt} \cdots\right]$$
$$+ \frac{1}{8}(\Delta t)^2(\Delta x)^2\left[u_{xxtt}\right] + \cdots \quad (2\text{-}82)$$

Eq 2.83-2.84 Truncation error in defined as:

$$u_j^{n+1} - u_j^n = \mu\left[\theta \delta_x^2 u_j^{n+1} + (1-\theta)\, \delta_x^2 u_j^n\right]$$

$$\mu = \Delta t / \Delta x^2$$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{\theta \delta_x^2 u_j^{n+1} + (1-\theta)\delta_x^2 u_j^n}{\Delta x^2}$$

2.83

$$\approx \frac{\delta_t u_j^{n+1/2}}{\Delta t} - \left(\frac{\theta \delta_x^2 u_j^{n+1} + (1-\theta)\delta_x^2 u_j^n}{\Delta x^2}\right) = T_j^{n+1/2}$$

2.84

$$\frac{\Delta t\, u_t + \frac{1}{24}(\Delta t)^3 u_{tttt}}{\Delta t} - \frac{\left[\Delta x^2 u_{xx} + \frac{1}{12}(\Delta x)^4 u_{xxxx} \ldots\right] + \left[(\theta - \tfrac{1}{2})\Delta t(\Delta x^2) u_{xxt} + \frac{1}{12}(\Delta x)^4 u_{xxxxt}\right] + \left[\frac{1}{8}(\Delta t)^2 (\Delta x)^2 (u_{xxtt} \ldots\right]}{\Delta x^2}$$

$$= \left(u_t - u_{xx}\right) + \left[\frac{1}{24}(\Delta t)^2 u_{tttt} - \frac{1}{8}(\Delta t)^2 u_{xxtt}\right]$$

$$+ \left[\left(\frac{1}{2} - \theta\right)\Delta t\, u_{xxt} - \frac{1}{12}(\Delta x^2) u_{xxxx}\right]$$

$$+ \left[\left[\frac{1}{2} - \theta\right](\Delta t)\frac{1}{12} u_{xxxxt} - \frac{2}{6!}(\Delta x^4) u_{xxxxxx}\right]$$