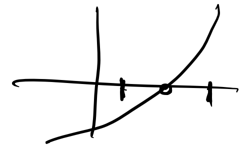
Recap:

root-finding:



- Bisection
- Newton-Raphson
- Secant Method

$\boxed{\text{Convex}}$ ✓

concave

f is concave $(-f)$ is convex

Convex optimization

Agenda

- Taylor Expansion

  $\sin(x)$ or $\sqrt{x}$, ...

- Reverse Engineering (soft intro)

  Curve fitting

Taylor Expansion

$f(x)$ f is "very" smooth

you can take infinitely many derivatives w/o any issue

$$f \in C^{\infty}$$

$\boxed{\text{Science}}$ vs. $\boxed{\text{Art}}$

$$\boxed{f(x+h)} = f(x) + h\frac{f'(x)}{1} + h^2\frac{f''(x)}{2} + h^3\frac{f^{(3)}(x)}{3!} + h^4\frac{f^{(4)}(x)}{4!} + \cdots$$

expansion around x

$$\boxed{f(x+h) = \sum_{n=0}^{\infty} \frac{h^n}{n!} f^{(n)}(x)}$$

$$\frac{h^n f^{(n)}(x)}{n!}$$

Example $\boxed{\sin(1.5) = ?}$

$f(x+h) = f(x) + \cdots$ $\boxed{f^{(n)}(1)}$

$\boxed{x = 0}$ $x = 1$

$h = 1.5$ $h = 0.5$

$$\frac{(1.5)^4}{4!} f^{(n)}(x)$$

$f(x) = \sin x$ 0

$f'(x) = \cos x$ 1

$f^{(2)}(x) = -\sin(x)$ 0

$f^{(3)}(x) = -\cos x$ -1

$f^{(4)}(x) = \sin x$ 0

$\vdots$

$\underset{x}{0} + \underset{h}{1.5}$

$$\sin(1.5) = \underline{\sin(0)} + 1.5\cos(0) + \frac{(1.5)^2}{2}(-\sin(0))$$

$$-\frac{(1.5)^3}{3!}\cos(0) + \cancel{\frac{(1.5)^4}{4!}\sin(0)} + \cdots$$

$+ \cdots$

$$10! = 1 \times 2 \times 3 \times 4 \times \cdots \times 10$$

1×2

$1 \times 2 \times 3 \times 4$

$1 \times 2 \times 3 \times 4 \times 5 \times 6$

$\vdots$

recursive

Example2 $\sqrt{5} = ?$

$$f(x+h) = f(x) + \frac{h}{1} f'(x) + \frac{h^2}{2!} f''(x)$$

$$+ \frac{h^3}{3!} f^{(3)}(x) + \frac{h^4}{4!} \boxed{f^{(4)}(x)}$$

demon should go much faster than

numer $f^{(n)}(x)$ vs. $n!$

$$f(x) = \sqrt{x} = x^{1/2}$$

$$f'(x) = \frac{1}{2} x^{\frac{-1}{2}} = \frac{1}{2} \frac{1}{x^{1/2}}$$

pattern!

$$f''(x) = \frac{-1}{4} x^{\frac{-3}{2}} = \frac{-1}{4} \frac{1}{x^{3/2}} = \frac{-1}{2^2} \frac{1}{x^{3/2}}$$

$$f^{(3)}(x) = \frac{+3}{8} x^{-5/2} = \frac{+3}{8} \frac{1}{x^{5/2}} = \frac{3}{2^3} \cdot \frac{1}{x^{5/2}}$$

$$f^{(4)}(x) = \frac{-3 \times 5}{16} x^{-7/2}$$

$$= \frac{-3 \times 5}{2^4} \frac{1}{x^{7/2}}$$

$$\boxed{f^{(k)}(x) = \frac{(-1)^{k+1} \prod_{j=1}^{k} (2j-3)}{2^k} \cdot \frac{1}{x^{(2k-1)/2}}}$$

$$f^{(k)}(x) = \frac{(-1)^{k+1} \prod_{j=1}^{k} (2j-3)}{2^k} \cdot \frac{1}{x^{(2k-1)/2}}$$

k^{th} derivative of $x^{1/2}$

$$\sum_{j=1}^{k} j \qquad \prod_{j=1}^{k} j$$

$1+2+\cdots+k \qquad 1\times 2\times 3\times\cdots\times k$

$$\frac{h^n}{n!} \times \frac{(-1)^{k+1} \prod_{j=1}^{k} (2j-3)}{2k} \qquad \frac{1}{x^{(2k-1)/2}} \qquad 2^{2k-1}$$

$$\sqrt{5} = \sqrt{4+1}$$

$x \qquad h$

$$\sqrt{2} = \sqrt{4-2}$$

$$\sqrt{3} = \sqrt{4-1}$$

$$\sqrt{4} = \sqrt{4-0}$$

$$\sqrt{5} = \sqrt{4+1}$$

$\vdots$

$$\sqrt{5} = \sqrt{1+4}$$

$$\cancel{\sqrt{5} = \sqrt{16-11}}$$

$$f^{(k)}(4) = \frac{(-1)^{k+1}\prod_{j=1}^{k}(2j-3)}{2^k} \cdot \frac{1}{2^{2k-1}}$$

$$\sqrt{5} = \sqrt{4+1}$$

$$= \sqrt{4}$$

$$+ \frac{1}{1} \times \frac{1}{2} \times \frac{1}{2}$$

$$+ \frac{1^2}{2!} \times \frac{-1}{2^2} \times \frac{1}{2^3}$$

$$+ \frac{1^3}{3!} \times \frac{+1 \times 3}{2^3} \times \frac{1}{2^5}$$

$$+ \frac{1^4}{4!} \times \frac{-1 \times 3 \times 5}{2^4} \times \frac{1}{2^7}$$

$$+ \frac{1^5}{5!} \times \frac{1 \times 3 \times 5 \times 7}{2^5} \times \frac{1}{2^9}$$

$$\vdots$$

why 4?

Art

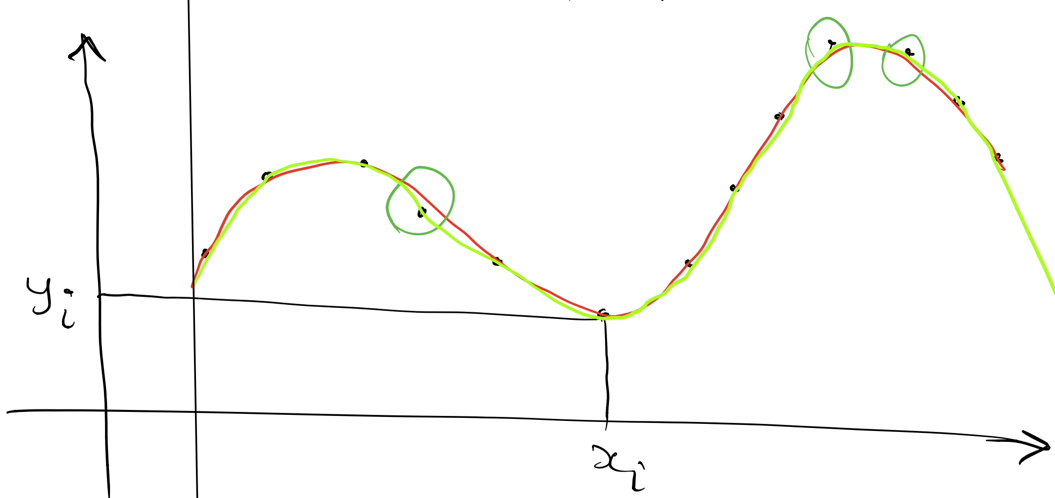
$\sqrt{5}$ $\quad x^2 - 25 = 0$

Tailor-made for Taylor

§ Gentle Introduction to Reverse Engineering

- Curve fitting

- Calibration



$\{(x_i, y_i)\}_{i=1,2,\dots,13}$ WLOG

$$\boxed{f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_d x^d}$$

$a_0, a_1, \ldots, a_d$

Perfect fit

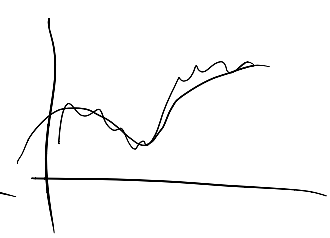
Example

$$\checkmark \quad \boxed{f(x) = x^2 + 1}$$

$$\checkmark \quad \boxed{(-2,5) \quad (-1,2) \quad (0,1) \quad (1,2) \quad (2,5)}$$

real-life

economist traders ...



A-priori

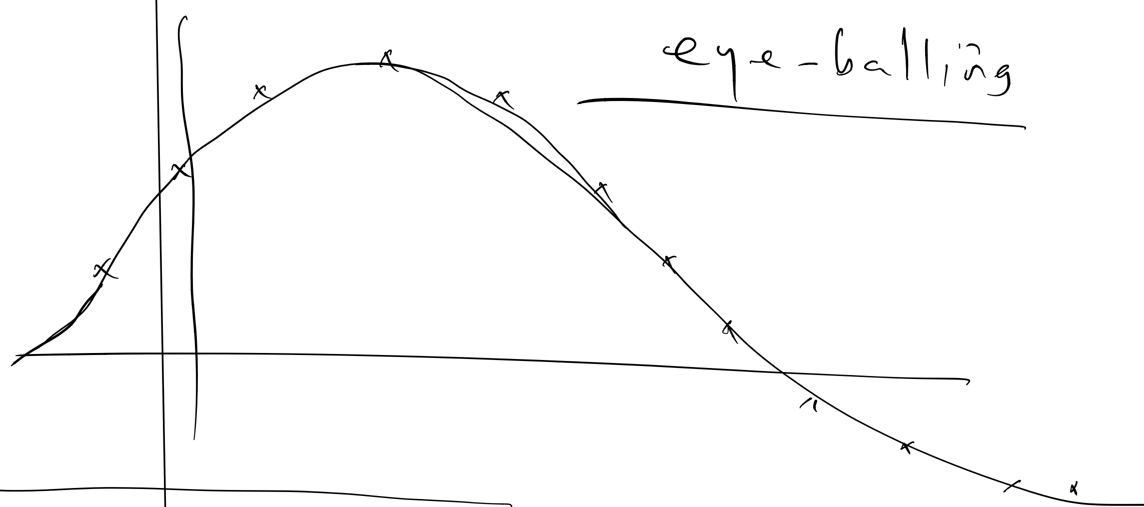
$$f(x) = a_0 + a_3 x^3$$

a_0, a_3

$(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$

find a_0 & a_3 ?

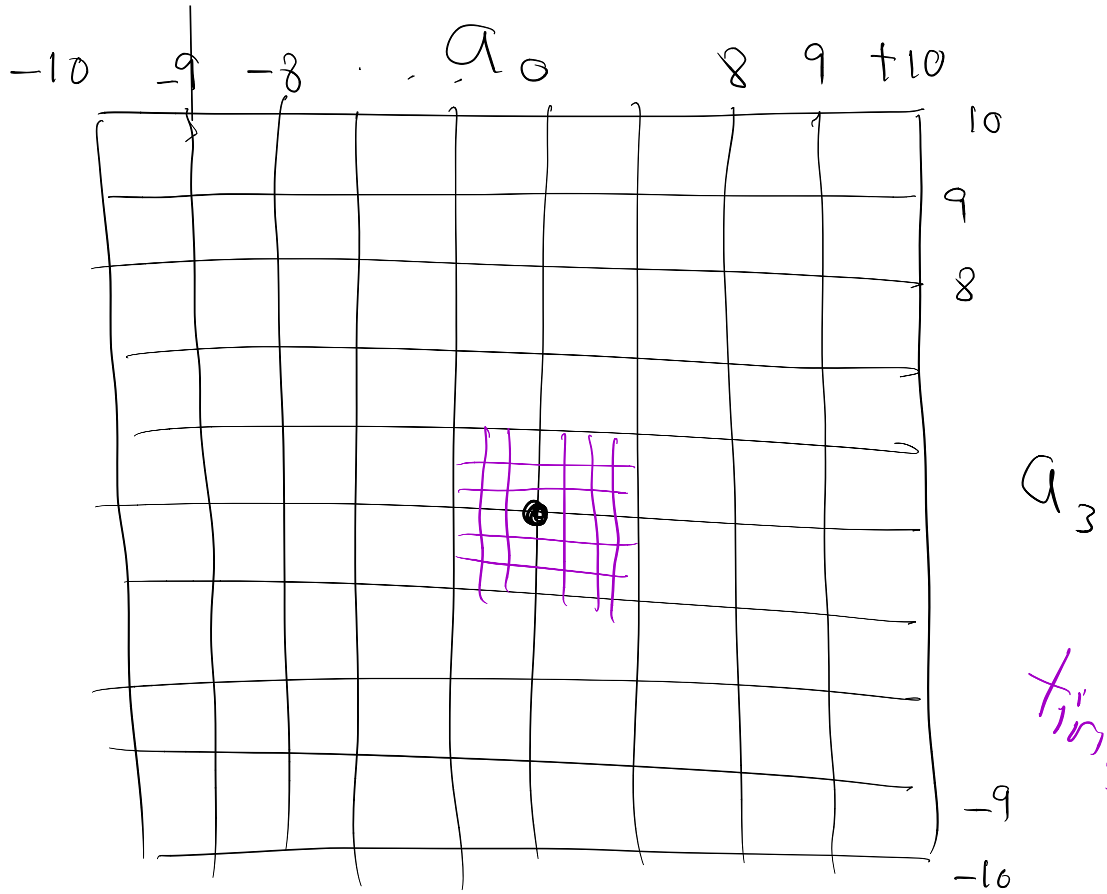
Curse of dimensionality



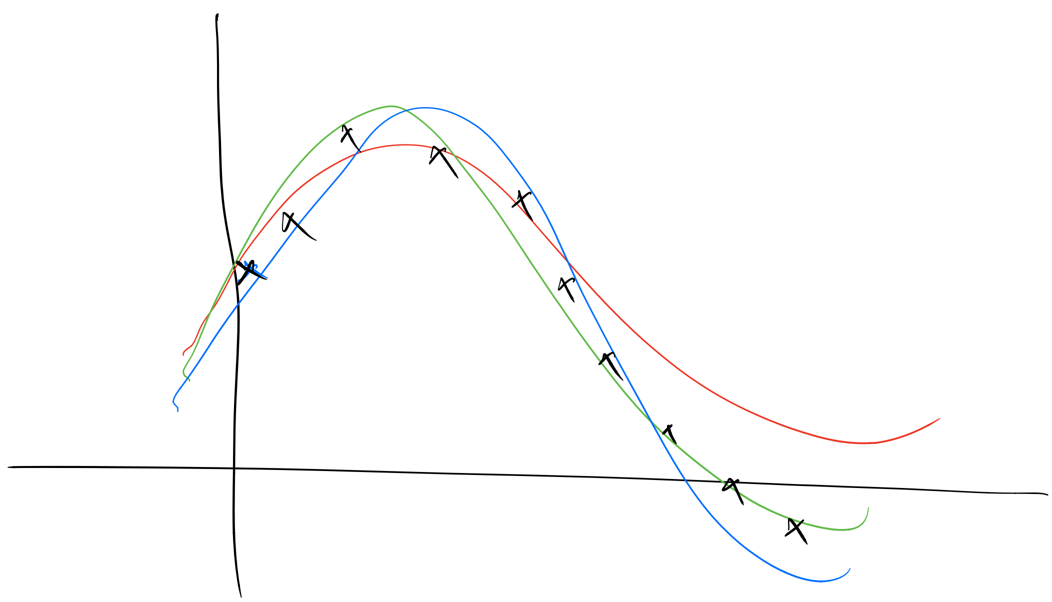
$\boxed{\text{Grid Search}}$ a_0 & a_3

Brute-Force

Fine vs. Coarse



tiring



• Measure of goodness
tightness

- Loss function
- Penalty function

(x_i, y_i)

$(x_0, y_0) \quad (x_1, y_1) \quad (x_2, y_2), \ldots, (x_n, y_n)$

$(x_0, \hat{y}_0^{(k)}) \quad (x_1, \hat{y}_1^{(k)}) \quad (x_2, \hat{y}_2^{(k)}), \ldots (x_n, \hat{y}_n^{(k)})$

distance $\quad |y_0 - \hat{y}_0^{(k)}| + |y_1 - \hat{y}_1^{(k)}| + \cdots + |y_n - \hat{y}_n^{(k)}|$

$$(y_0 - \hat{y}_0^{(k)})^2 + (y_1 - y_1^{(k)})^2 + \cdots + (y_n - \hat{y}_n^{(k)})^2$$

$$\sqrt{L_2 = \sum_{i=0}^{n} (y_j - \hat{y}_j)^2 \quad \text{RMS}}$$

$L_2 \to 0$

$x = (1, 4, 7);$

$y = x.\hat{}\,2 = (1, 16, 49);$

$$\begin{array}{r} 17+ \\ 49 \\ \hline 66 \end{array}$$

sum(y) = 66

Pros.

$\boxed{\text{Bisection}}$ To find some starting point! Yet could be super expensive

Cons.

- Knowing the range a-priori is a MUST.
- Curse of dimensionality

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$$

$$100 \times 100 = 10{,}000$$

$$100^6 = 1{,}000{,}000{,}000{,}000$$

$$10^6 = 1{,}000{,}000$$

Very expensive computationally as number of parameters grows.

"Curse of Dimensionality"

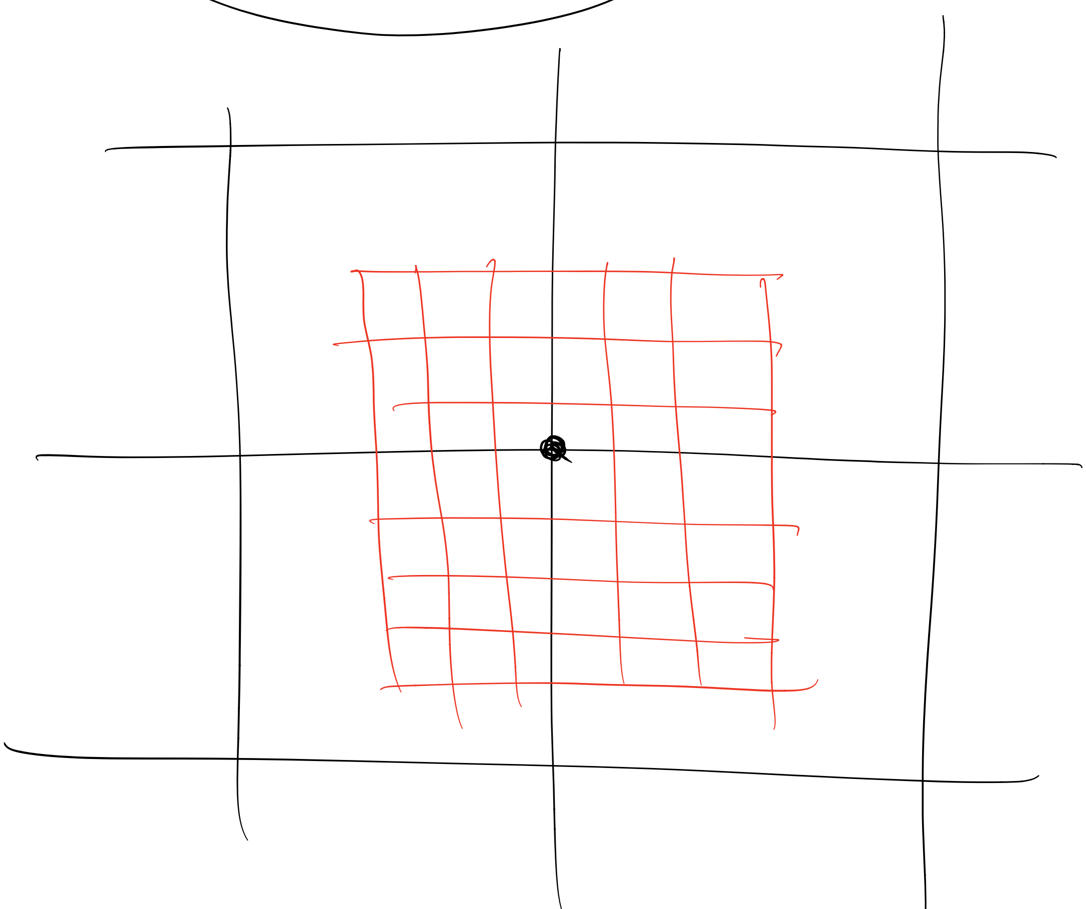
2004

28

6 weeks

4^{28}

Starting point



Brute-Force
Grid Search

$\boxed{\text{101}}$

fminsearch

Simplex method

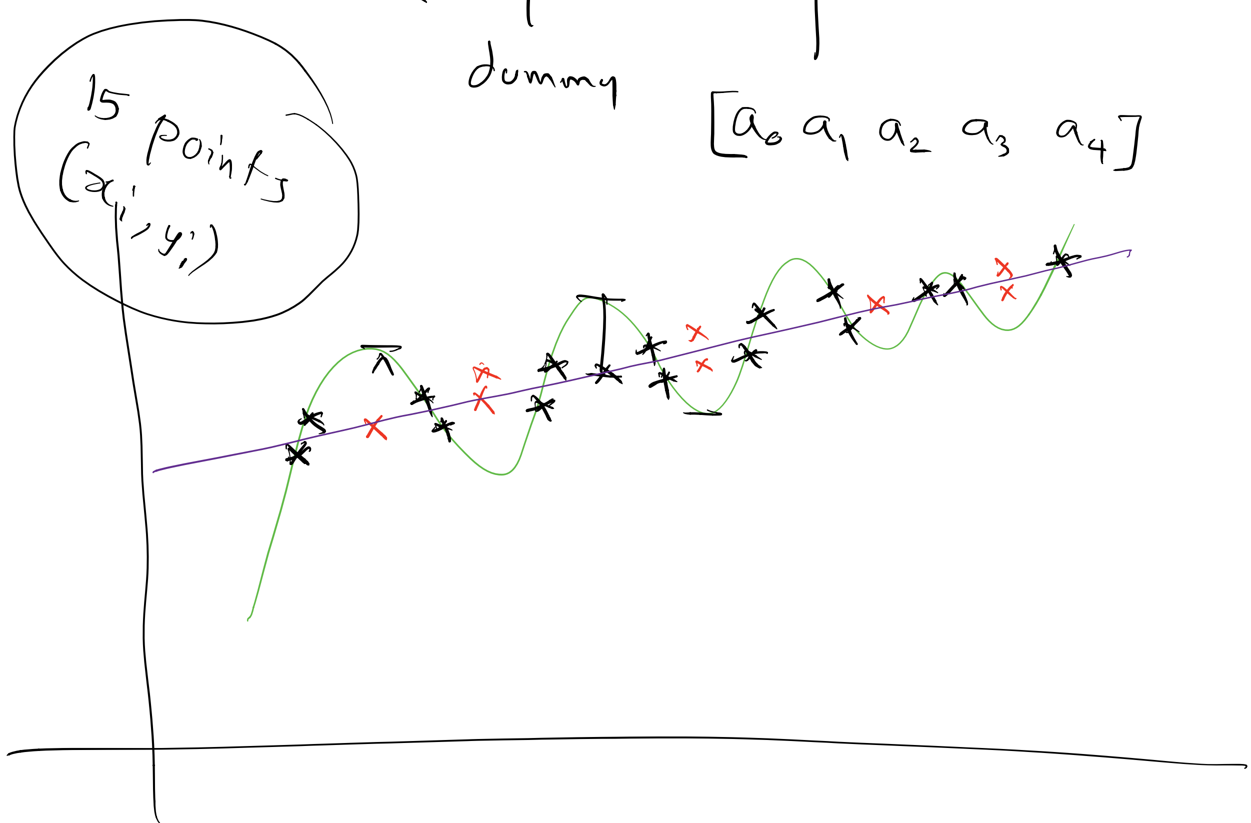
Nelder-Mead

last resort

fminsearch('myObjFunc', params, myOption, data)

dummy

$[a_0 \; a_1 \; a_2 \; a_3 \; a_4]$



Over-fitting

data set

= Part 1 + Part 2

Overkilled it + you completely miss

Machine Learning

NLP

⋮

$a_0 + \cdots + a_9 x^9$

dataset

Part 1

Part 2

lastweek

this week

PM

-$$$$

Reverse Engineering

+

Machine Learning

Gradient Descent

3 lectures

- Batch - Vanilla
- Stochastic
- Mini-Batch