Nonregular Languages

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Recall

Lemma 1: The Pumping Lemma

If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$
2. $|y| > 0$
3. $|xy| \leq p$

[Sip96]

1 Using the Pumping Lemma to Prove A Language is Nonregular

1.1 a^nb^n

Example 1

We want to prove that $B = \{a^nb^n \mid n \geq 0\}$ is nonregular. We will start by assuming it is regular and finding a contradiction.

$B = \{a^nb^n \mid n \geq 0\}$ is a regular language (Assumption) (1)

$s = a^pb^p \in B$ (Plugged p into line 1) (2)

s can be split into $s = xyz$, where for any $i \geq 0$ the string xy^iz is in B (Pumping lemma) (3)

It is impossible to split s into $s = xyz$, where for any $i \geq 0$ the string xy^iz is in B (Shown below.) (4)

Lines 3 and 4 contradict. $\rightarrow\leftarrow$ (5)

We can prove line 4 with a proof by cases.

There are three possible situations we can consider for our string y in $s = xyz$, and we can see why for each of them $xyz \notin B$.

1. The string y consists only of as. In this case, we would have a situation like the one above. y can't be "deleted" because then there will be more bs than as, and "pumping" y (e.g. the string $xyyz$ would result in more as than bs and so is not a member of B.
2. The string y consists only of bs. This can be shown with the same reasoning as 1.

3. The string y consists of both as and bs. In this case, pumping y (e.g. the string $xyyz$) may have the same number of as and bs, but they will be out of order with some bs before as. Hence it is not a member of B.

1.2 Proof Strategy

The basic steps of this proof strategy is as follows:

1. Assume the language A *is* regular.
2. Choose an input string s that would be accepted by this language. (Usually it's helpful to make it a string whose length is a multiple of pumping length p.)
3. Apply the pumping lemma to s.
4. Consider how you would assign $s = xyz$, where $|y| > 0$ and $|xy| \leq p$. (This is where in example 1 we had to do a proof by cases.)
5. Display where the pumping lemma causes a contradiction. (e.g. $xy^iz \notin A$)

References

[Sip96] Michael Sipser. Introduction to the theory of computation. *ACM Sigact News*, 27(1):27–29, 1996.