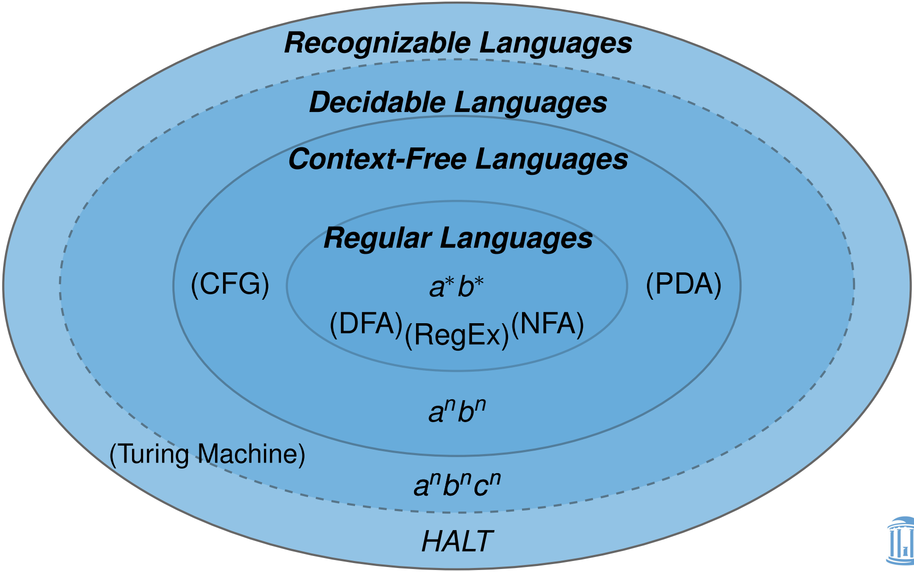
(Un)decidability, (Un)recognizability, and Reductions

Models of Languages and Computation

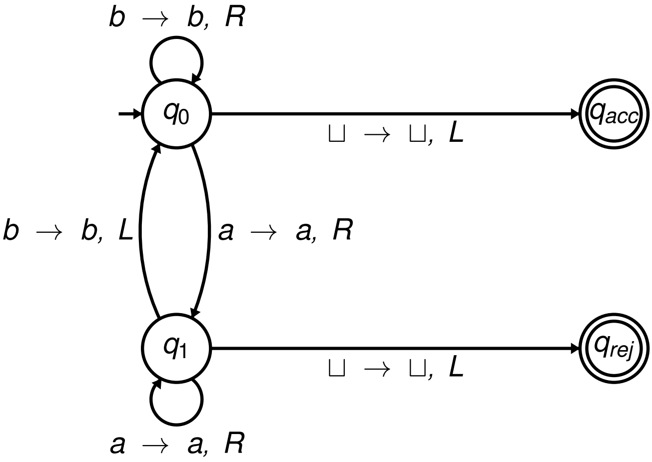
Overview

1. **Clarifying Terms**
   - Decides vs Recognizes (Machines)
   - Decidable vs Recognizable (Languages)
   - Mapping Reduction
2. **Example Proofs**
   - Reduce One Decidable Language to Another
   - $A_{TM} \leq_m REGULAR_{TM}$
   - $HALT \leq_m VST$
   - $A_{TM} \leq_m NE_{TM}$
   - L_d is not recognizable
   - $L_d \leq_m E_{TM}$

Languages So Far



Decides vs Recognizes (Machines)



Decides vs Recognizes (Machines)

Definition (Decides)

A Turing machine M ***decides*** a language L if

- For every string $w \in L$, M accepts w, and
- For every string $w \notin L$, M rejects w.

Decides vs Recognizes (Machines)

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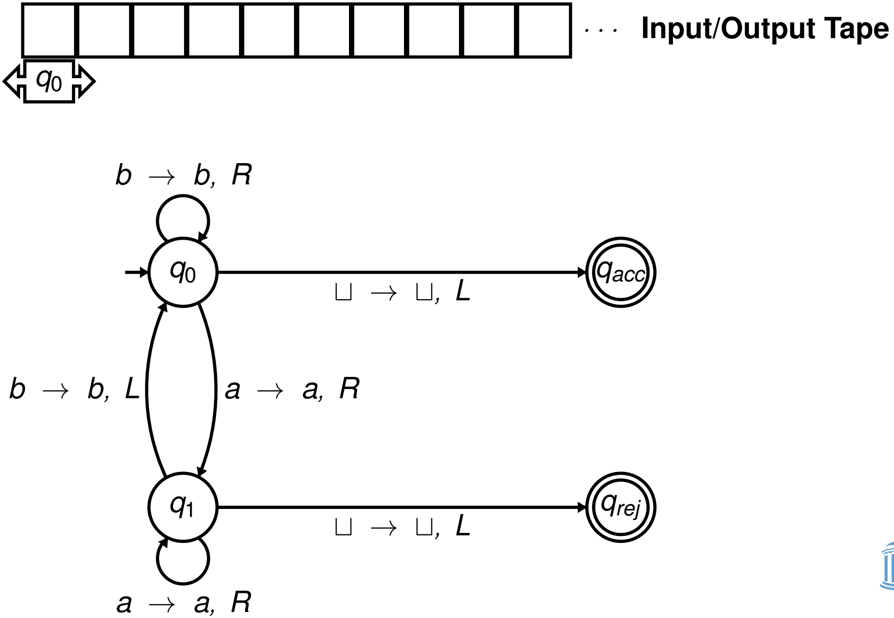
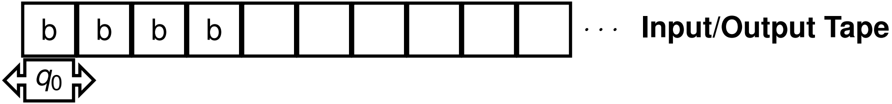
Definition (Recognizes)

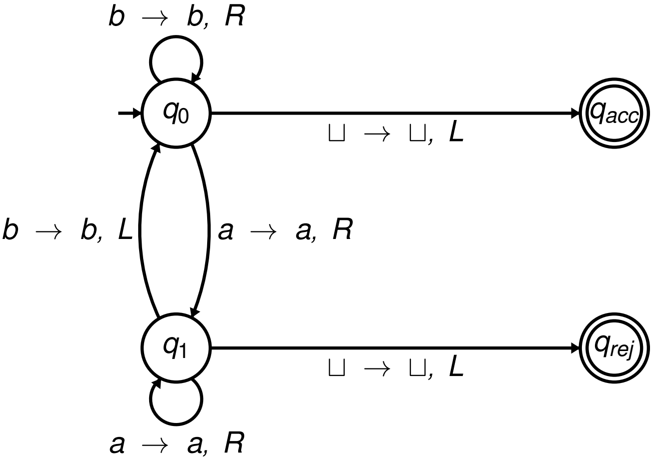
A Turing machine M ***recognizes*** a language L if

- For every string $w \in L$, M accepts w, and
- For every string $w \notin L$, M either rejects or goes into an infinite loop on w.


Input/Output Tape
q0
b → b, R
q0
⊔ → ⊔, L
qacc
b → b, L
a → a, R
q1
⊔ → ⊔, L
qrej
a → a, R

- What happens to the input bbbb?





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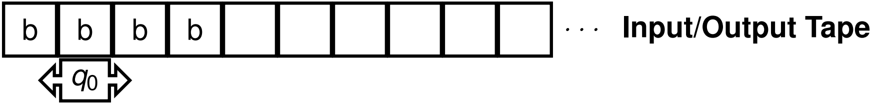

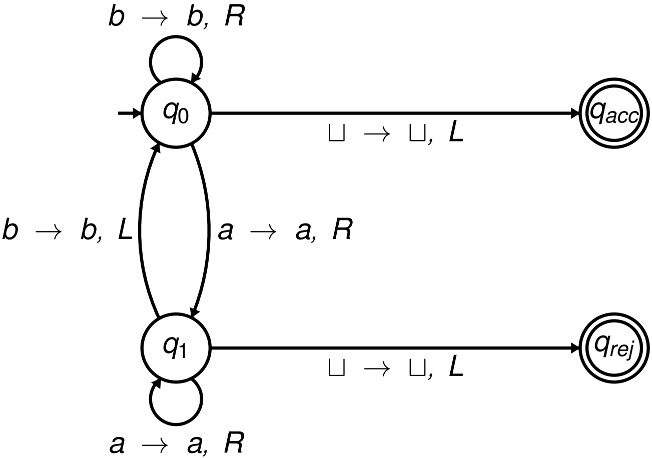

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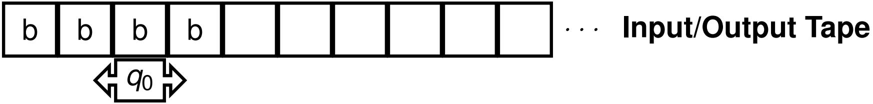

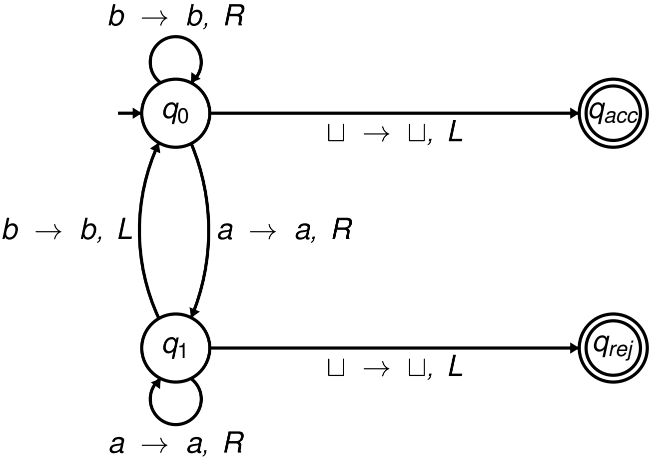

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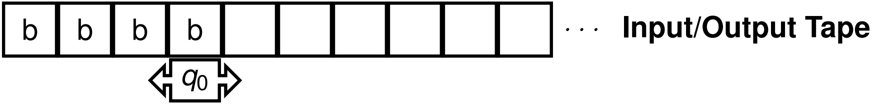

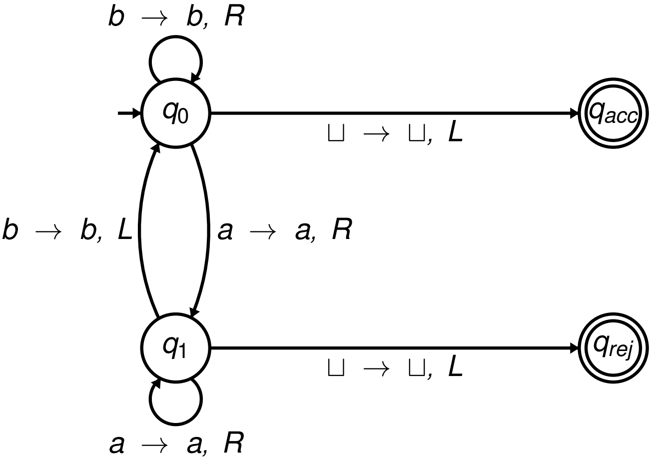

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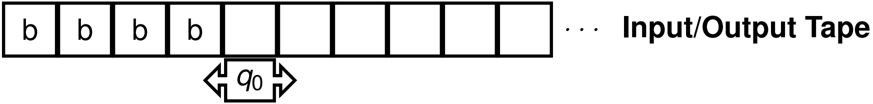

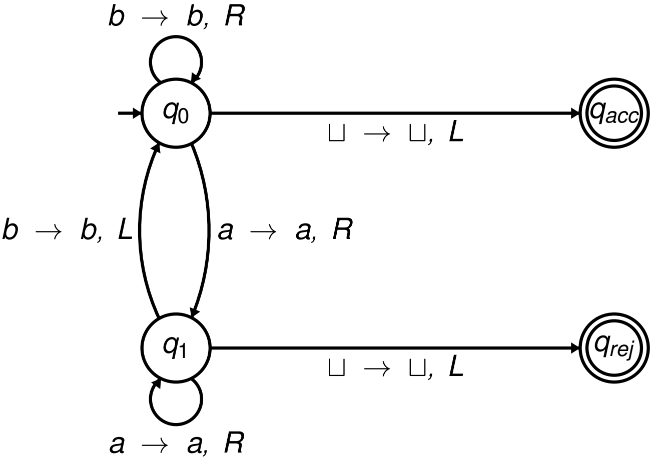

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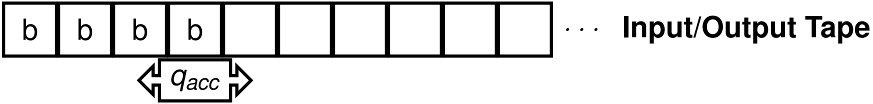
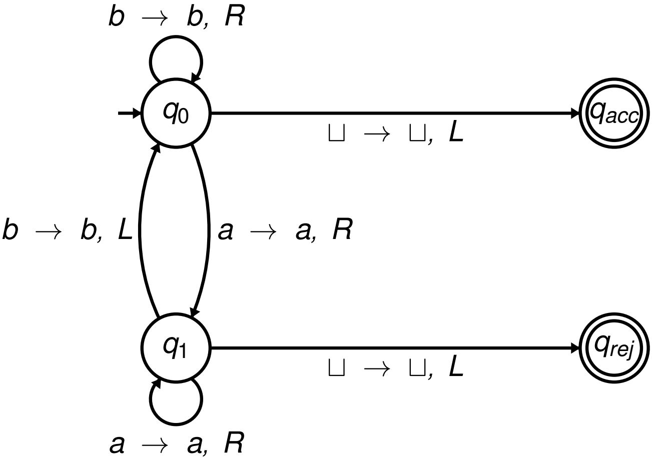
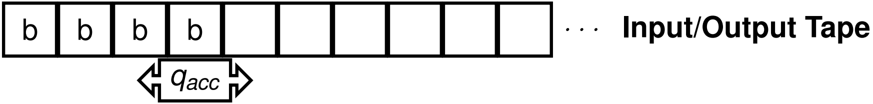
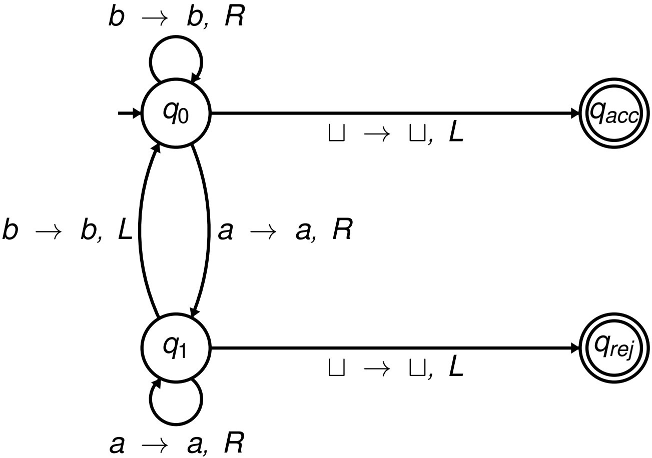


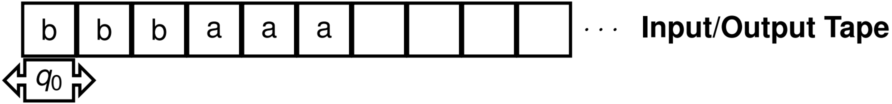


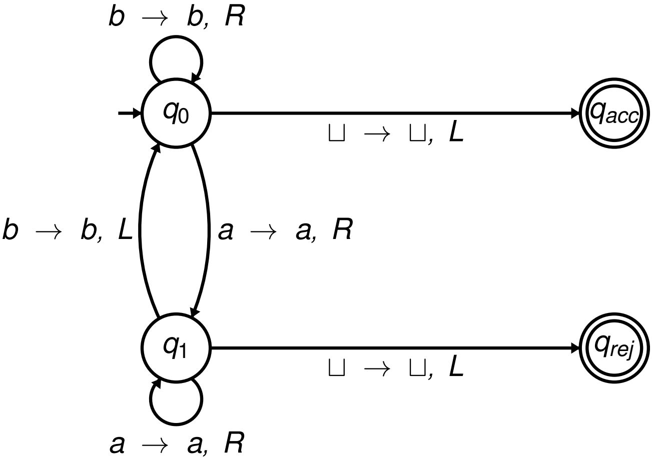
- What happens to the input bbbb? We accept it





- What happens to the input bbbaaa?





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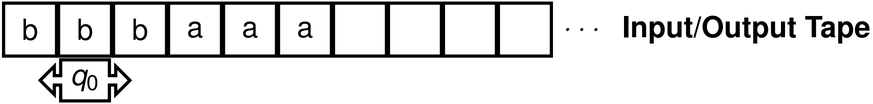

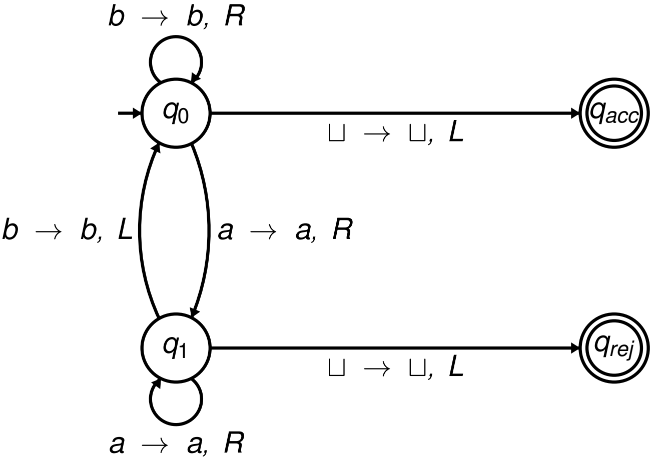

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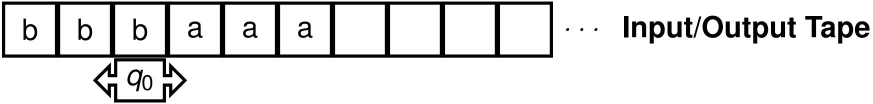

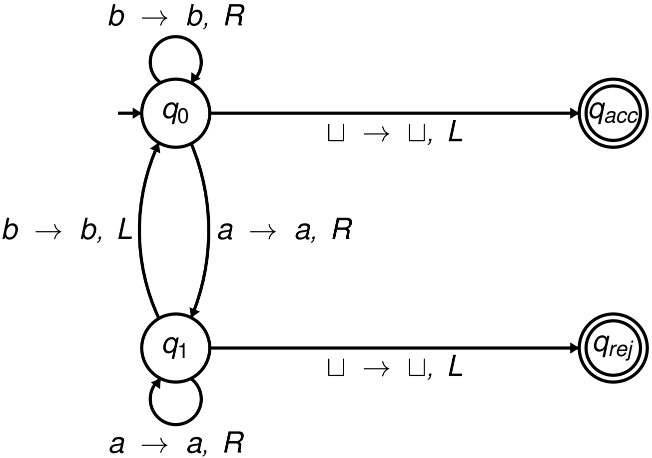

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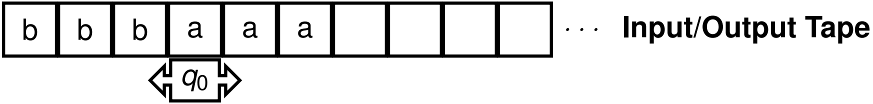

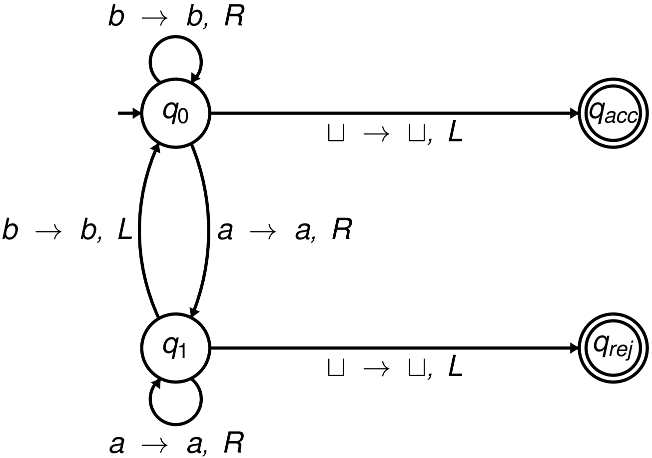

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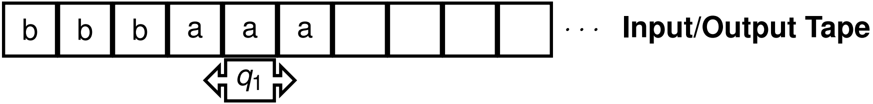

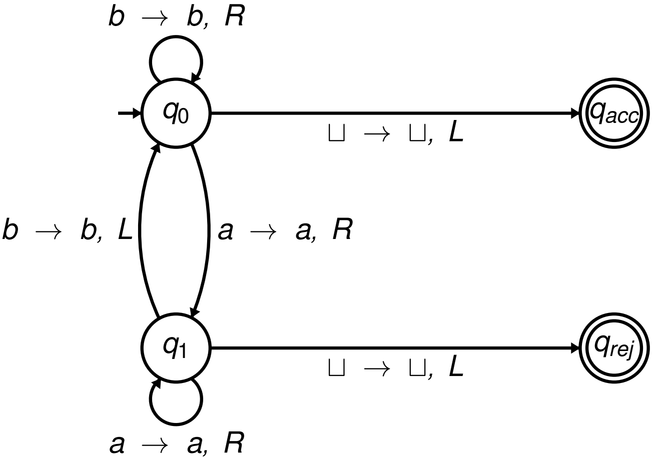

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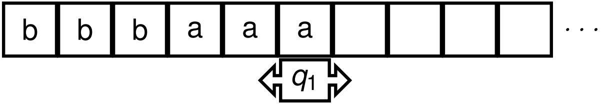
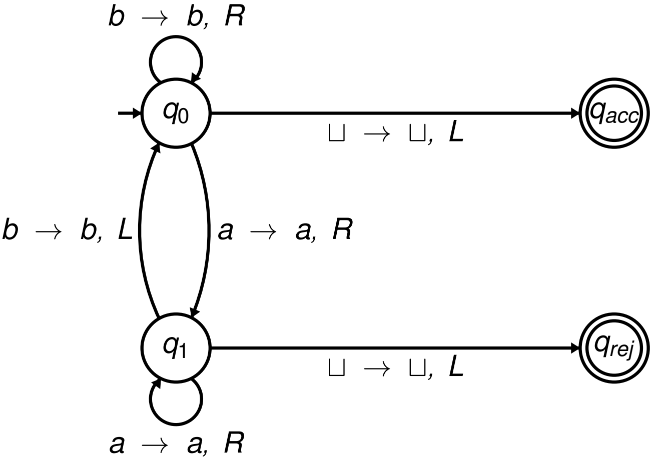


Input/Output Tape



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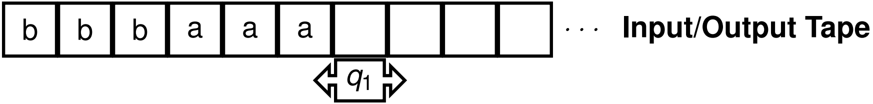

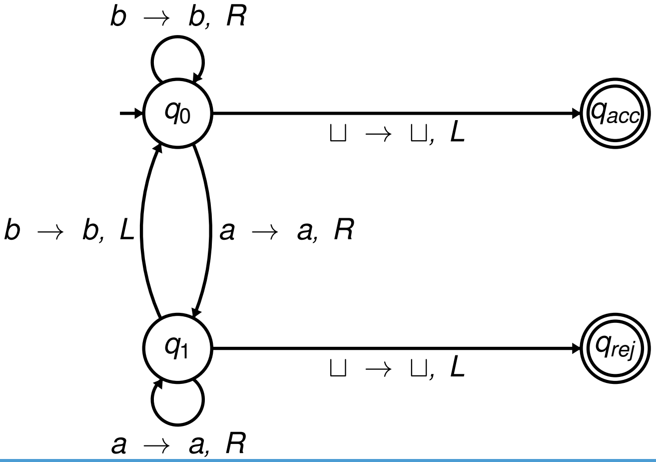

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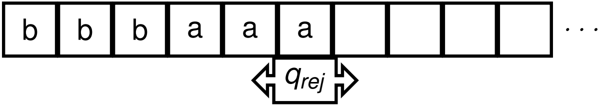

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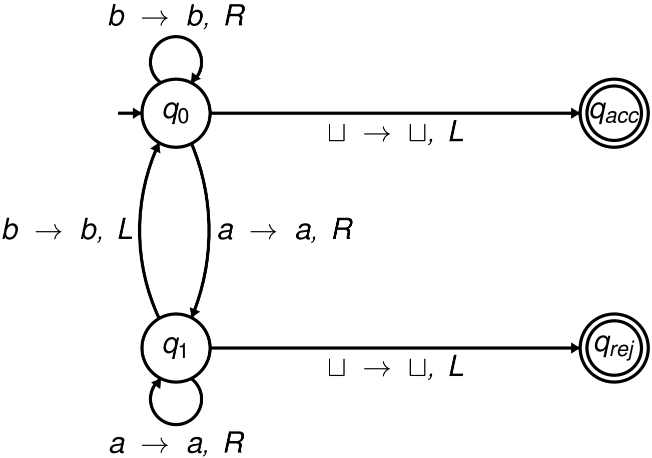
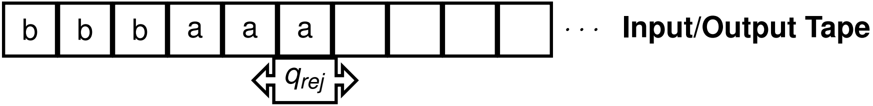
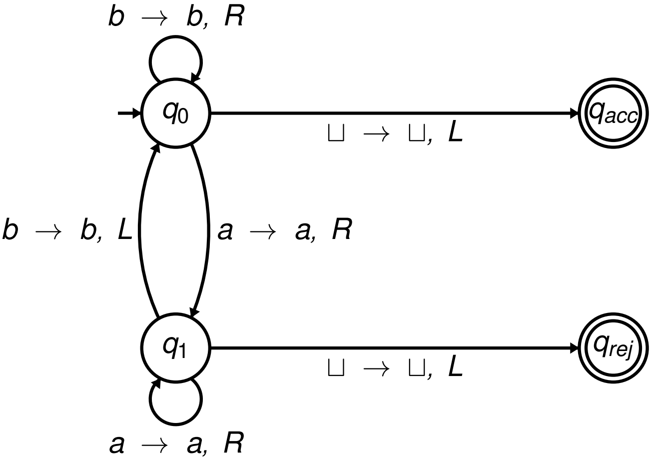
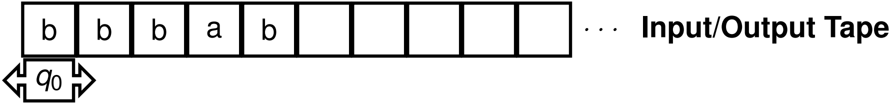


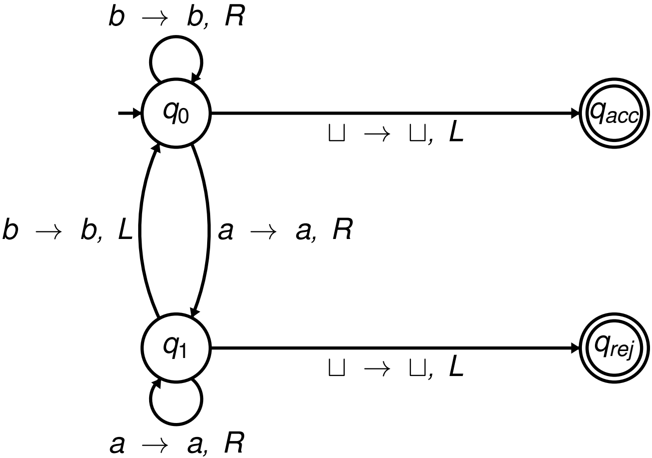
- What happens to the input bbbaaa? We reject it





- What happens to the input bbbab?





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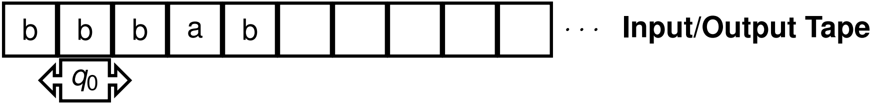

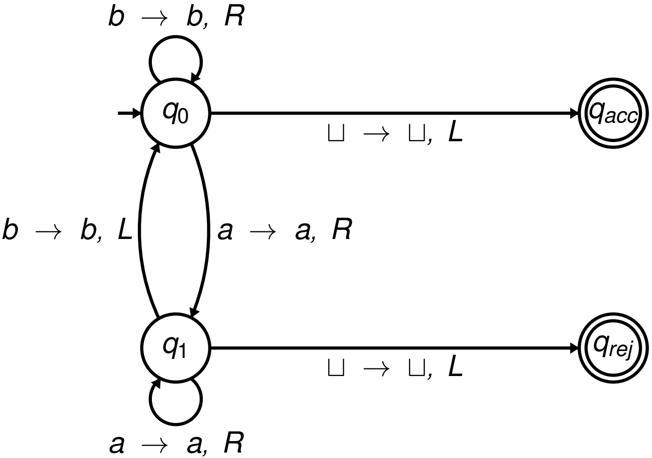

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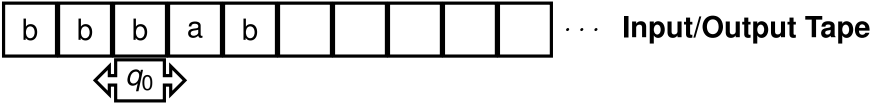

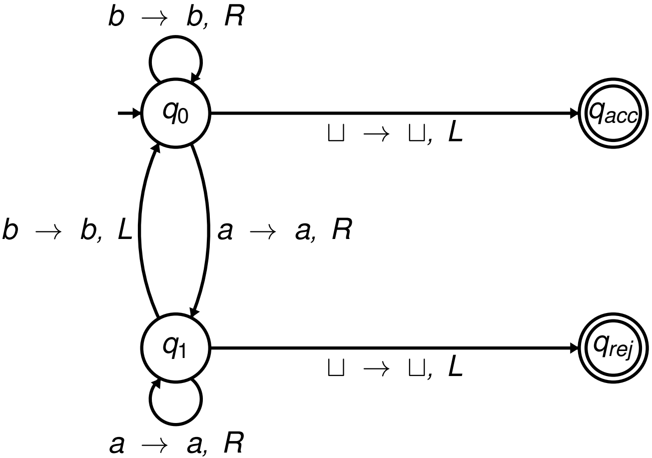

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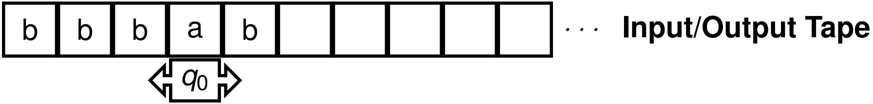

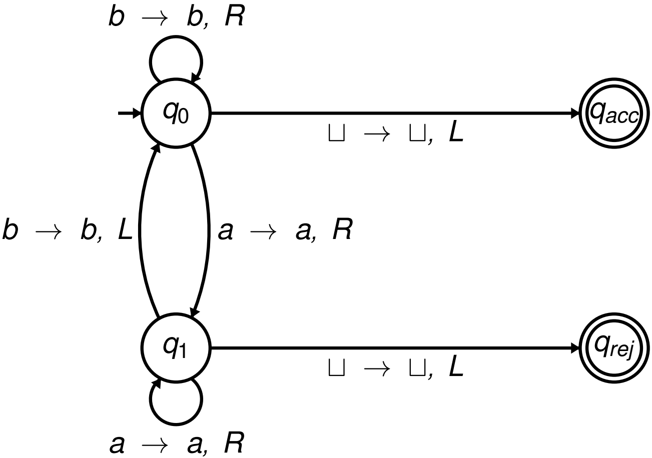

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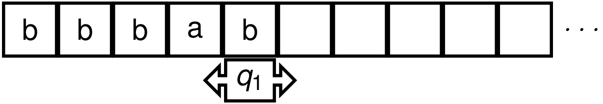

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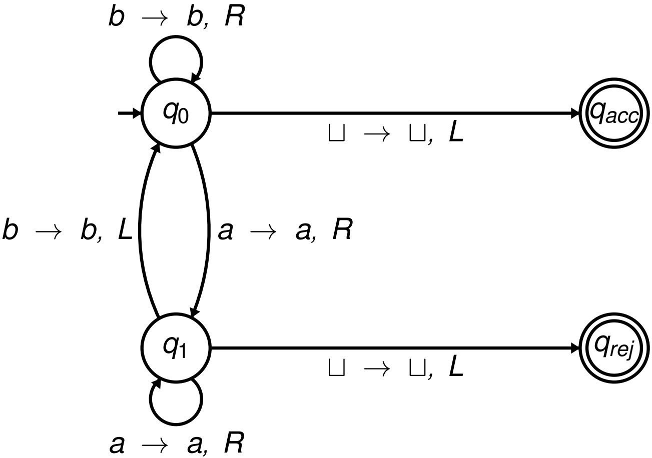

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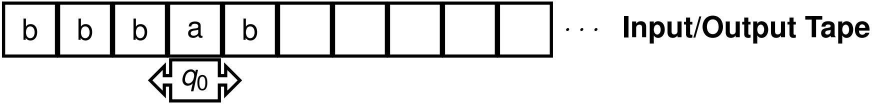

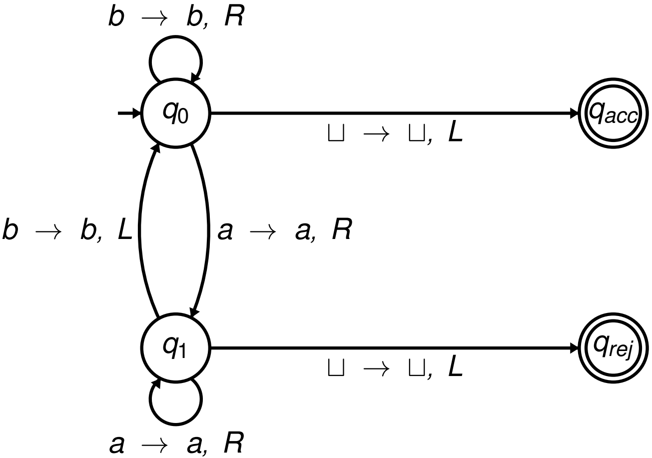

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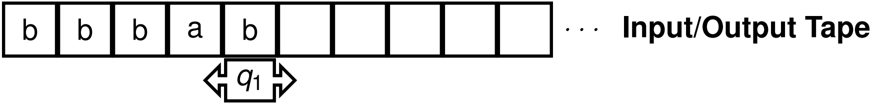

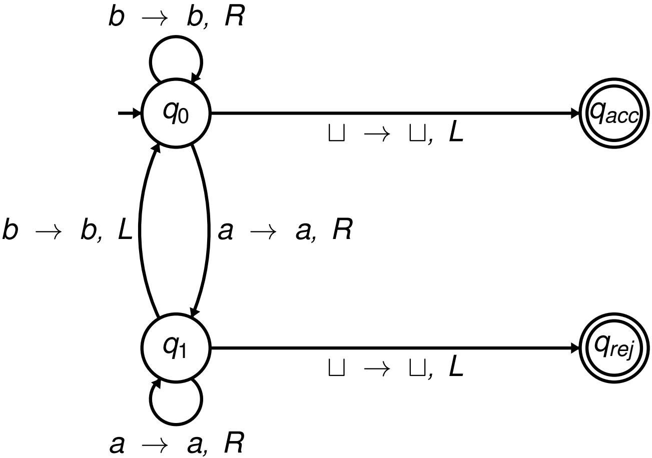

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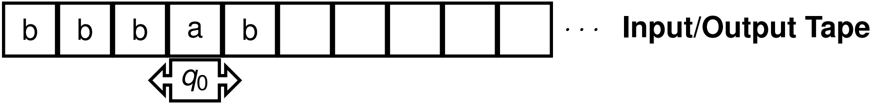

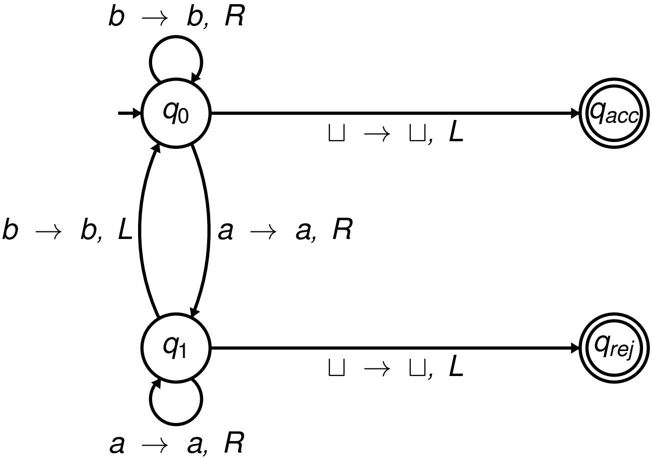

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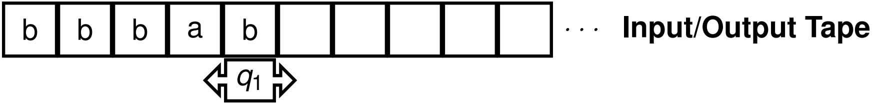

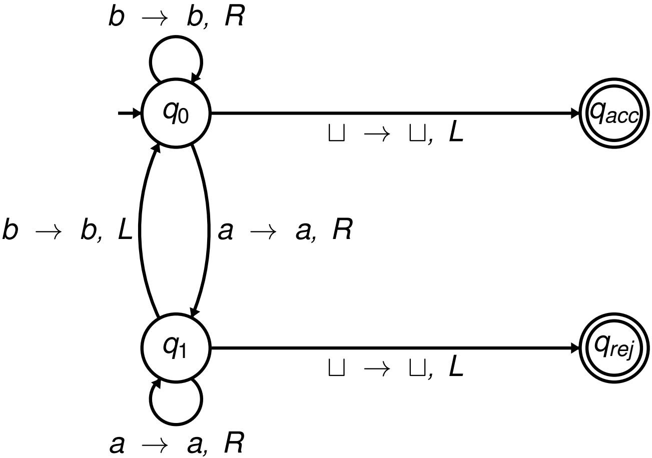

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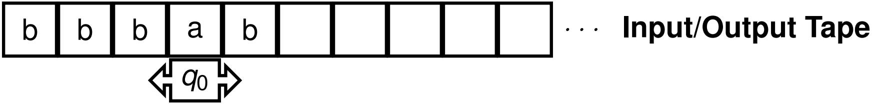
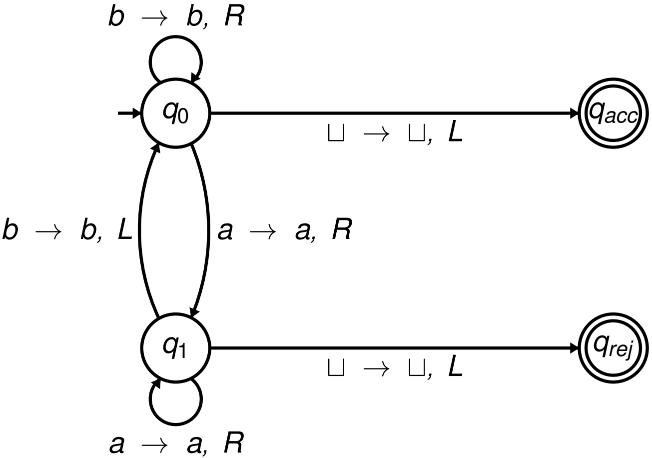
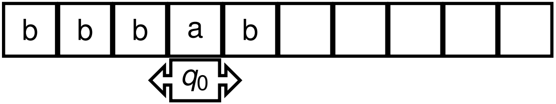
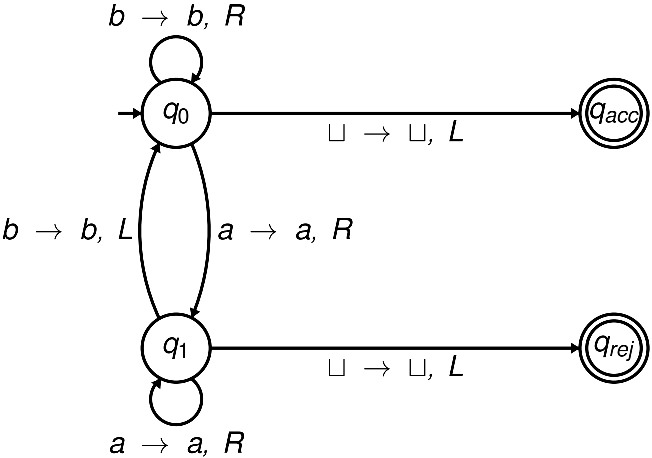




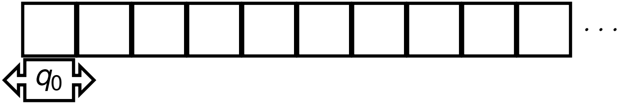
- What happens to the input bbbab? We get stuck in an infinite loop



. . . **Input/Output Tape**



- What language does this machine recognize?



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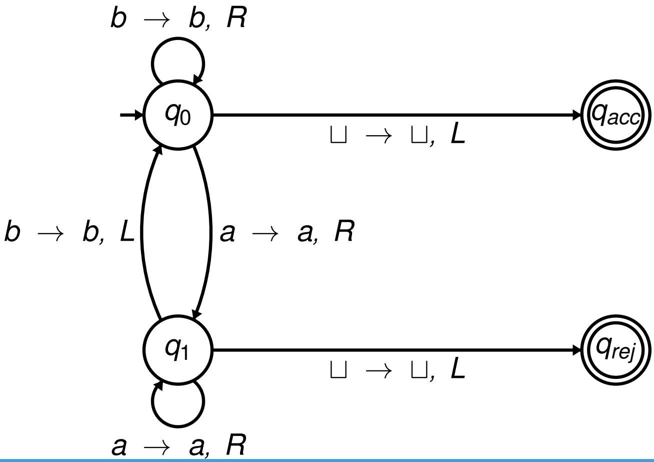
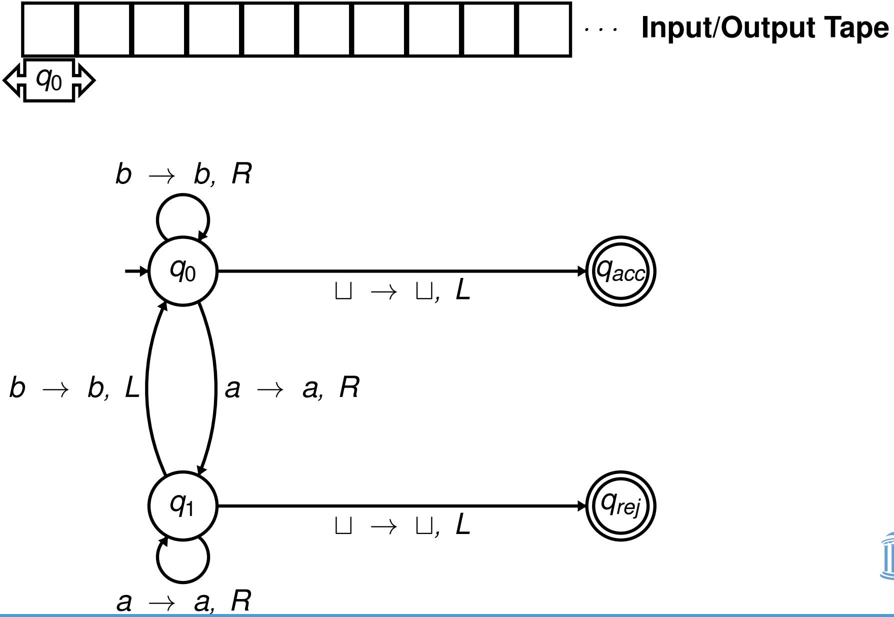
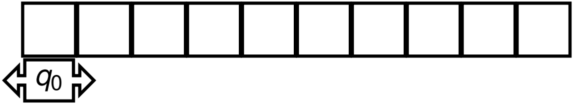


- What language does this machine recognize? $\{b^n \mid n \geq 0\}$



- Is this machine a decider?



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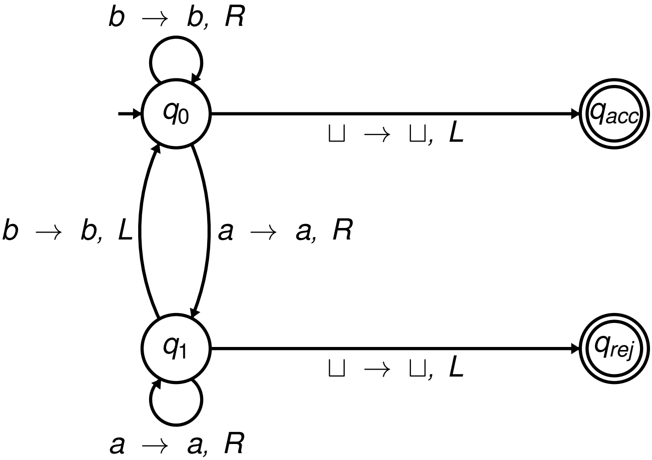
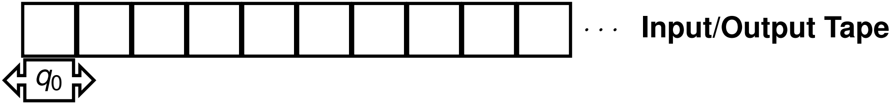
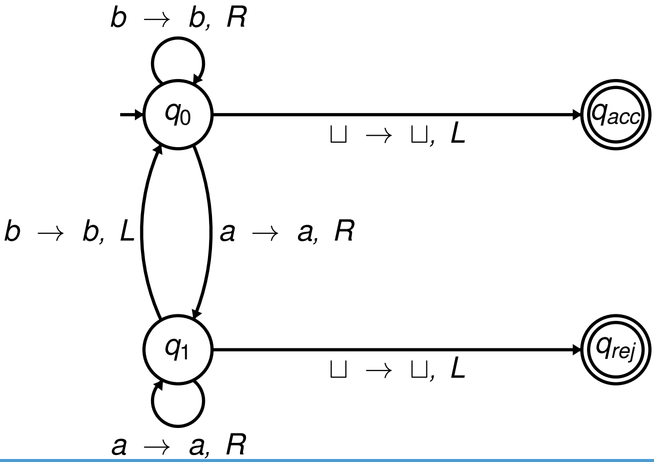
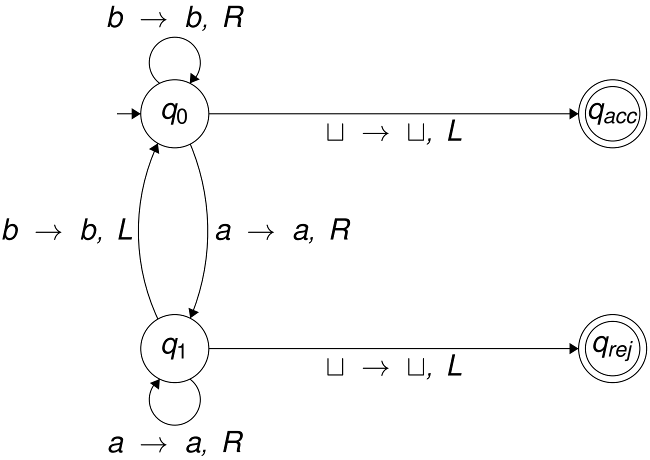


- Is this machine a decider? No, since it loops infinitely on bbbab

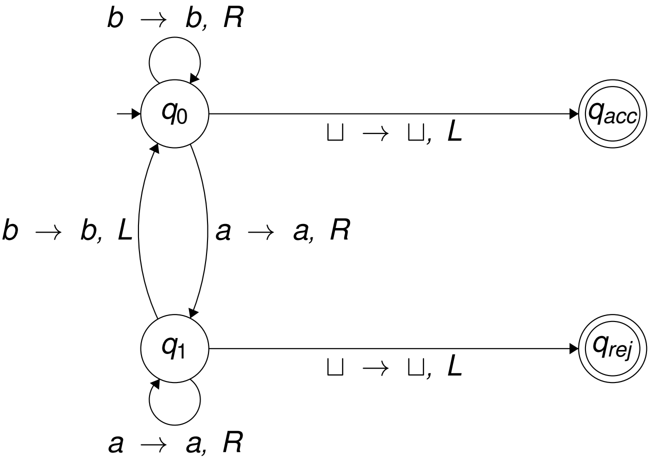




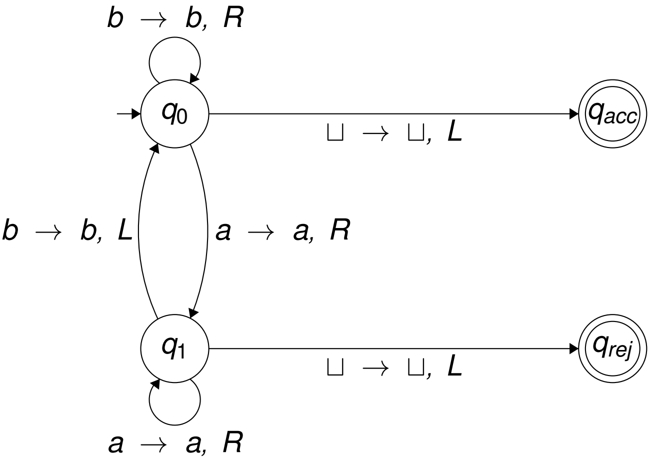
- This machine does ***not decide*** $\{b^n \mid n \geq 0\}$



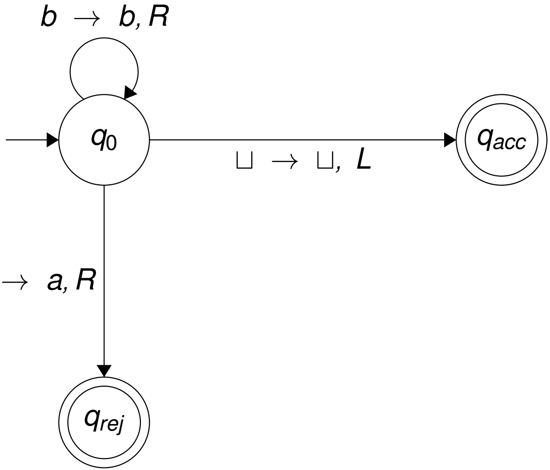
- This machine does ***not decide*** $\{b^n \mid n \geq 0\}$
- This machine does ***recognize*** $\{b^n \mid n \geq 0\}$



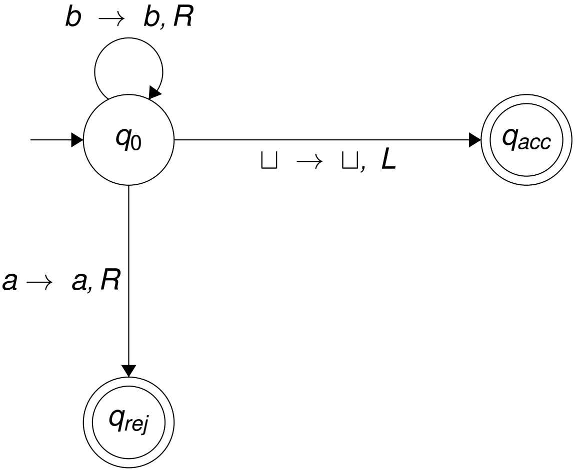
- This machine does ***not decide*** $\{b^n \mid n \geq 0\}$
- This machine does ***recognize*** $\{b^n \mid n \geq 0\}$
- IMPORTANT: Even though this particular machine does not decide $\{b^n \mid n \geq 0\}$, some other machine might decide it



- The following machine ***decides*** $\{b^n \mid n \geq 0\}$



- The following machine ***decides*** $\{b^n \mid n \geq 0\}$
- So, $\{b^n \mid n \geq 0\}$ is a ***decidable language***



Decidable vs Recognizable (Languages)

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Definition (Decidable)

A language L is ***decidable*** if and only if <u>there exists</u> a Turing machine M that ***decides*** L.

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Definition (Recognizable)

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Mapping Reduction

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What $A \leq_m B$ means conceptually:

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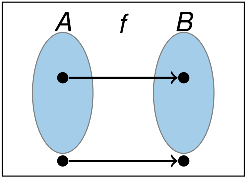

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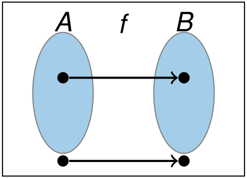
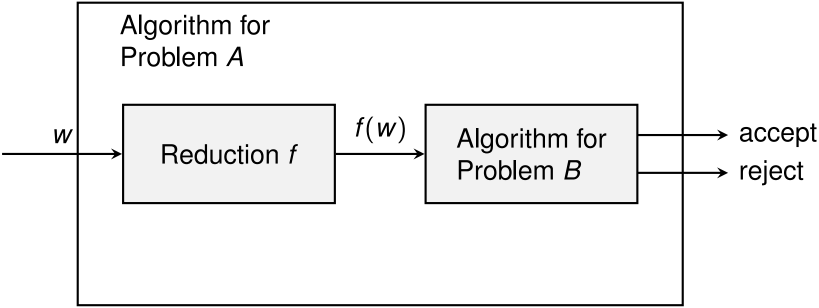
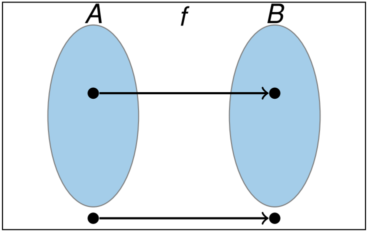




Mapping Reduction

Definition

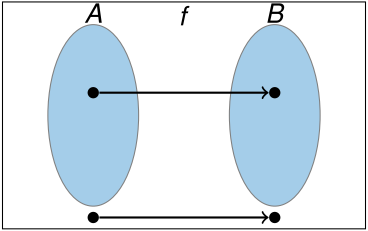
A function $f : \Sigma^* \to \Sigma^*$ is ***computable*** if there exists a Turing machine M that on every input w, halts with $f(w)$ on the tape.



Mapping Reduction

Definition

A ***mapping reduction*** from a language A to a language B is a computable function $f : \Sigma^* \to \Sigma^*$ such that $w \in A \iff f(w) \in B$. We say that A is **reducible to** B, and we denote it by $A \leq_m B$.



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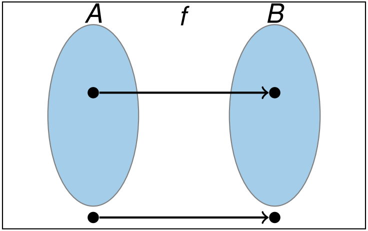
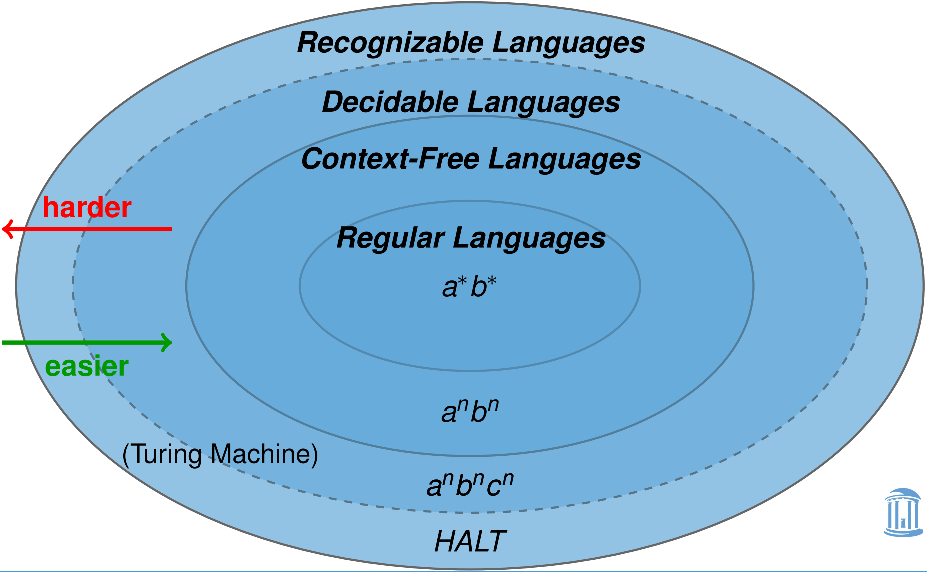
- Informally, $A \leq_m B$ means “Problem A is no harder than B," and all decidable problems are equally difficult under mapping reductions (e.g., regular vs. context-free languages).



Mapping Reductions



Suppose we have $X \leq_m Y$

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Conclusion (choose the smallest one that must be true):

Options:

A: Decidable
B: Undecidable
C: Recognizable
D: Unrecognizable
W: Cannot tell

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1. X is decidable. Y is ___

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Suppose we have $X \leq_m Y$

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1. X is decidable. Y is **(W)**
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Decidable Languages: Mapping Reduction Example

Theorem

Let

$$X = \{w \in \{0,1\}^* \mid \textit{\#0s in } w \textit{ is even}\}$$

and

$$Y = \{w \in \{0,1\}^* \mid \textit{\#0s in } w \textit{ is divisible by 6}\}.$$

Then $X \leq_m Y$.

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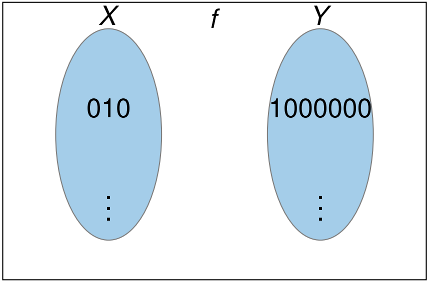
Hence, $w \in X \iff f(w) \in Y$.

Some Properties of the Reduction f

The previous proof is complete, but I just want to emphasize some things about f:



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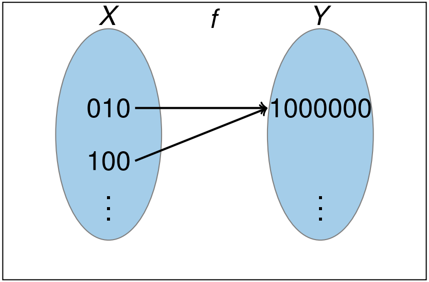

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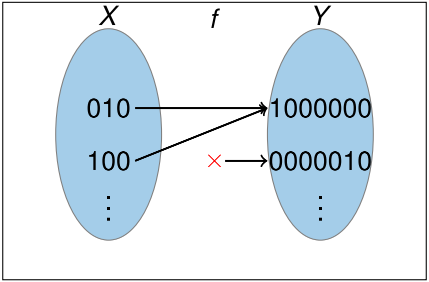

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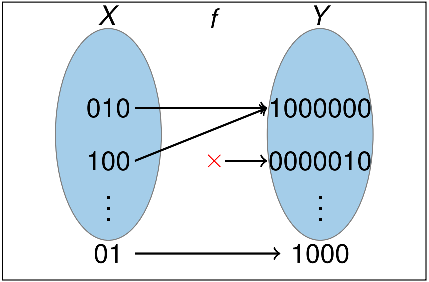
3. We need: All elements in X map to elements in Y, and all elements not in X map outside Y



Undecidability of `REGULAR`

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$$REGULAR = \{\langle M \rangle \mid L(M) \textit{ is regular}\}$$

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- We want the outputs of f to be of the form:
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- Again, we want $\langle M, w\rangle \in A_{TM} \iff f(\langle M, w\rangle) \in REGULAR$

Construction of N for $f(\langle M, w \rangle)$

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- If $\langle M, w\rangle \in A_{TM}$

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- If $\langle M, w \rangle \in A_{TM}$
  - Then M accepts w.

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We will map the input $\langle M, w \rangle$ to the following Turing machine N:

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- If $\langle M, w \rangle \in A_{TM}$
  - Then M accepts w.
  - Then N accepts everything.

Construction of N for $f(\langle M, w \rangle)$

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Theorem

The language

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(Temporary Detour from Reductions)

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The language $L_d = \{\langle M \rangle \mid \langle M \rangle \notin L(M)\}$ is not recognizable.

- Suppose for contradiction that some machine D recognizes L_d
- If D accepts $\langle D \rangle$, then $\langle D \rangle \notin L_d$.
  - Since D recognizes L_d, D shouldn't accept encodings of machines that accept their own encoding.
  - So D doesn't accept $\langle D \rangle$.
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    - This is a contradiction.
- So, no machine D can recognize L_d.

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- Again, we want $\langle M \rangle \in L_d \iff f(\langle M \rangle) \in E_{TM}$

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  - So, $\langle N \rangle \notin E_{TM}$.
- Hence, $\langle M \rangle \in L_d \iff \langle N \rangle \in E_{TM}$