Nonregular Languages

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Recall 1

Lemma 1: The Pumping Lemma

If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$
2. $|y| > 0$
3. $|xy| \leq p$

[Sip96]

1 Using the Pumping Lemma to Prove A Language is Nonregular

1.1 a^nb^n

Example 1

We want to prove that $B = \{a^nb^n \mid n \geq 0\}$ is nonregular. We will start by assuming it is regular and finding a contradiction.

$B = \{a^nb^n \mid n \geq 0\}$ is a regular language (Assumption) (1)

$s = a^pb^p \in B$ (Plugged p into line 1) (2)

s can be split into $s = xyz$, where for any $i \geq 0$ the string xy^iz is in B (Pumping lemma) (3)

It is impossible to split s into $s = xyz$, where for any $i \geq 0$ the string xy^iz is in B (Shown below.) (4)

Lines 3 and 4 contradict. $\rightarrow\leftarrow$ (5)

We can prove line 4 with a proof by cases.

There are three possible situations we can consider for our string y in $s = xyz$, and we can see why for each of them $xyz \notin B$.

1. The string y consists only of as. In this case, we would have a situation like the one above. y can't be "deleted" because then there will be more bs than as, and "pumping" y (e.g. the string $xyyz$ would result in more as than bs and so is not a member of B.
2. The string y consists only of bs. This can be shown with the same reasoning as 1.

3. The string y consists of both as and bs. In this case, pumping y (e.g. the string $xyyz$) may have the same number of as and bs, but they will be out of order with some bs before as. Hence it is not a member of B.

1.2 www

Example 2

We want to prove that $B = \{www|w \in \{a,b\}^*\}$ is nonregular.

We can do this as a proof by contradiction same as before, and with similar reasoning!

So assume $B = \{www|w \in \{a,b\}^*\}$ is regular.

$$B = \{www|w \in \{a,b\}^*\} \text{ is regular} \quad \text{(Assumption)} \tag{1}$$
$$a^p a^p a^p = a^{3p} \in B \quad \text{(By definition of } B\text{)} \tag{2}$$
$$a^{3p} \text{can be split such that } a^{3p} = xyz, \text{ where for any } i \geq 0,\ xy^iz \in B \quad \text{(Pumping Lemma)} \tag{3}$$
$$\text{Let } x = a^k, y = a^m, \text{ and } z = a^{3p-k-m} \quad \text{(Rewrite of } a^{3p} = xyz\text{)} \tag{4}$$
$$xy^2z \in B \quad \text{(Used the pumping lemma to "pump" in a copy of } y\text{)} \tag{5}$$
$$a^k a^{2m} a^{3p-k-m} \in B \quad \text{(Plugged line 4 into line 5)} \tag{6}$$
$$a^{3p+m} \in B \quad \text{(Simplified line 6)} \tag{7}$$
$$k + m < p \quad \text{(Pumping Lemma)} \tag{8}$$
$$m < p \quad \text{(Line 8)} \tag{9}$$
$$3p + m \text{ is } \textit{not} \text{ a multiple of 3} \quad \text{(Line 9)} \tag{10}$$
$$a^{3p+m} \notin B \quad \text{(Lines 1 and 10)} \tag{11}$$
$$\text{Lines 7 and 11 contradict. } \rightarrow\leftarrow \tag{12}$$

Why did we have to do a proof by cases for example 1 by not example 2? For both of these examples, we were able to choose a string that is accepted by the language but demonstrates a contradiction of the pumping lemma. For example 1, we chose a^pb^p, and for example 2, we chose a^{3p}. The reason we had to do a proof by cases for number 1 is that we had to consider three possible outcomes of assigning $a^pb^p = xyz$ where $|y| > 0$ and $|xy| \leq p$. However, since a^{3p} is string of only as, we were able to reason about it as a whole and not worry about different cases. (In other words, the only possibility was that x and y were going to be made entirely of as.)

1.3 Proof Strategy

The basic steps of this proof strategy is as follows:

1. Assume the language A *is* regular.
2. Choose an input string s that would be accepted by this language. (Usually it's helpful to make it a string whose length is a multiple of pumping length p.)
3. Apply the pumping lemma to s.
4. Consider how you would assign $s = xyz$, where $|y| > 0$ and $|xy| \leq p$. (This is where in example 1 we had to do a proof by cases.)
5. Display where the pumping lemma causes a contradiction. (e.g. $xy^2z \notin A$)

References

[Sip96] Michael Sipser. Introduction to the theory of computation. *ACM Sigact News*, 27(1):27–29, 1996.