P1

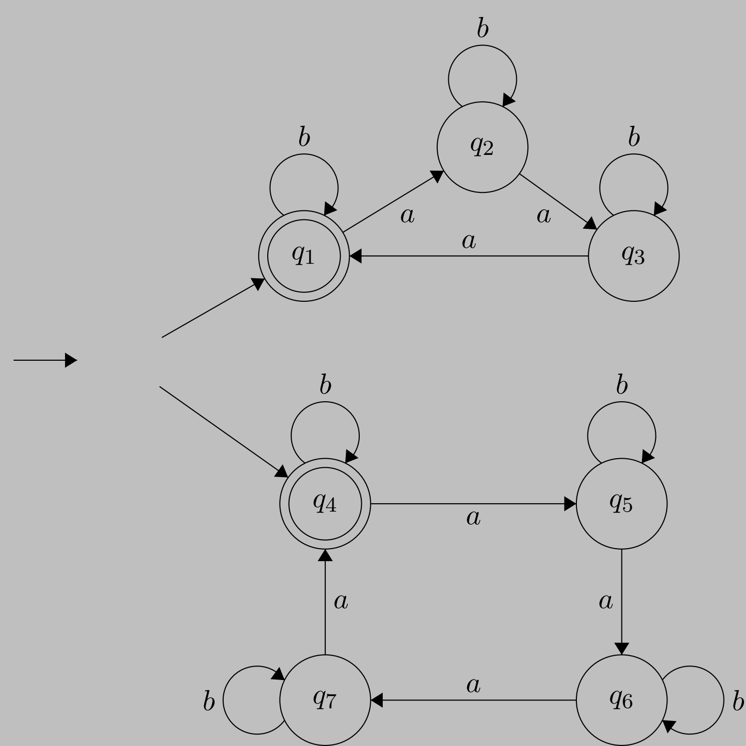
Problem 1

Design an NFA for the following languages over their respective alphabets:

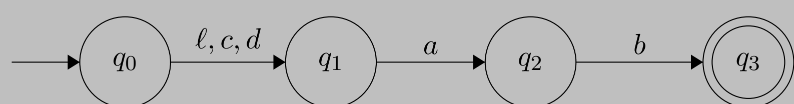
- $L = \{w|w$ has $3m$ or $4m$ $a's$ for some $m \in \mathbb{N}\}$. L is over the alphabet $\Sigma = \{a, b\}$. [Tug22]
- $L = \{lab, cab, dab\}$. L is over the alphabet $\Sigma = \{a, b, c, d, l\}$ [HMU01]
- $L = \{1101, 101, 111\}$. L is over the alphabet $\Sigma = \{0, 1\}$. [HMU01]
- $L = \{w|w$ ends with $00\}$. L is over the alphabet $\Sigma = \{0, 1\}$. (For this one, your NFA can only have 3 states.) [Sip96]

Solution to Problem 1

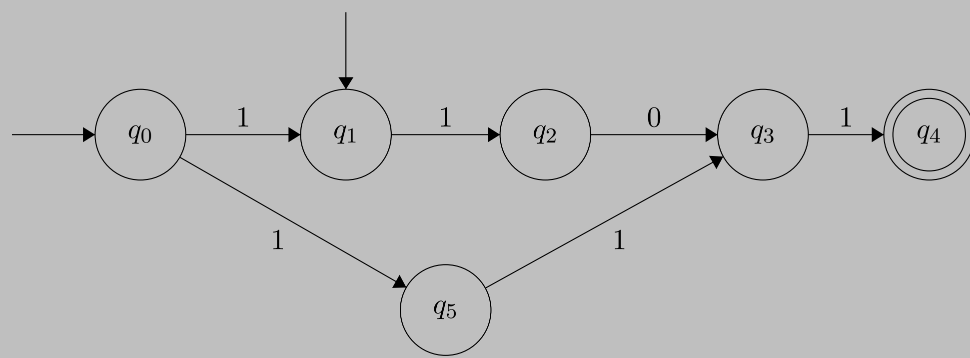
1.



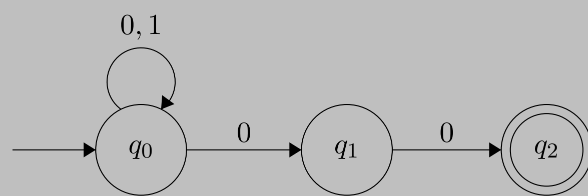
2.



3.



4.



P2

Problem 2

In the previous problem, you defined an N such that $L(N) = \{w|w \text{ ends with } 00\}$, Convert this NFA into a DFA.

δ	0	1
$\emptyset$	$\emptyset$	$\emptyset$
$\rightarrow \{q0\}$	$\{q0, q1\}$	$\{q0\}$
$\{q0, q1\}$	$\{q0, q1, q2\}$	$\{q0\}$
$*\{q0, q1, q2\}$	$\{q0, q1, q2\}$	$\{q0\}$

Table 1: Converted DFA Transition Table for Reachable States (δ)

δ	0	1
$\emptyset$	$\emptyset$	$\emptyset$
$\rightarrow \{q0\}$	$\{q0, q1\}$	$\{q0\}$
$\{q1\}$	$\{q2\}$	$\emptyset$
$*\{q2\}$	$\emptyset$	$\emptyset$
$\{q0, q1\}$	$\{q0, q1, q2\}$	$\{q0\}$
$*\{q0, q2\}$	$\{q0, q1\}$	$\{q0\}$
$*\{q1, q2\}$	$\{q2\}$	$\emptyset$
$*\{q0, q1, q2\}$	$\{q0, q1, q2\}$	$\{q0\}$

Table 2: Converted DFA Transition Table for All States (δ)

Solution to Problem 2

Let $(Q_N, \Sigma, \Delta_N, S_N, F_N)$ be the tuple representation of the NFA.

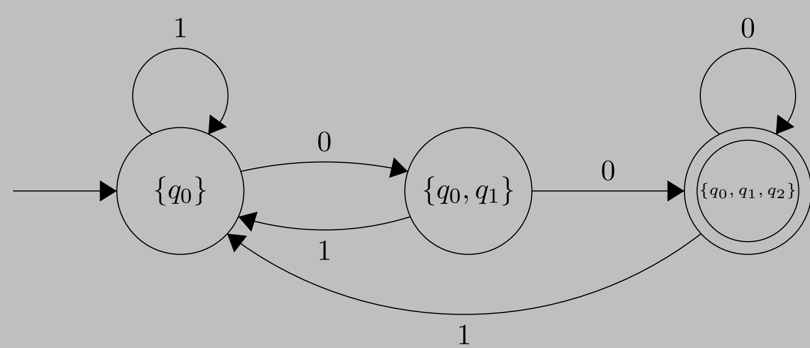
Doing the Tuple definition for a DFA with reachable states:

- $Q = 2^Q = \{\emptyset, \{q0\}, \{q0, q1\}, \{q0, q1, q2\}\}$
- $\Sigma = \{0, 1\}$
- δ can be defined using the transition table in Table 1
- $s = \{q0\}$
- $F = \{\{q0, q1, q2\}\}$

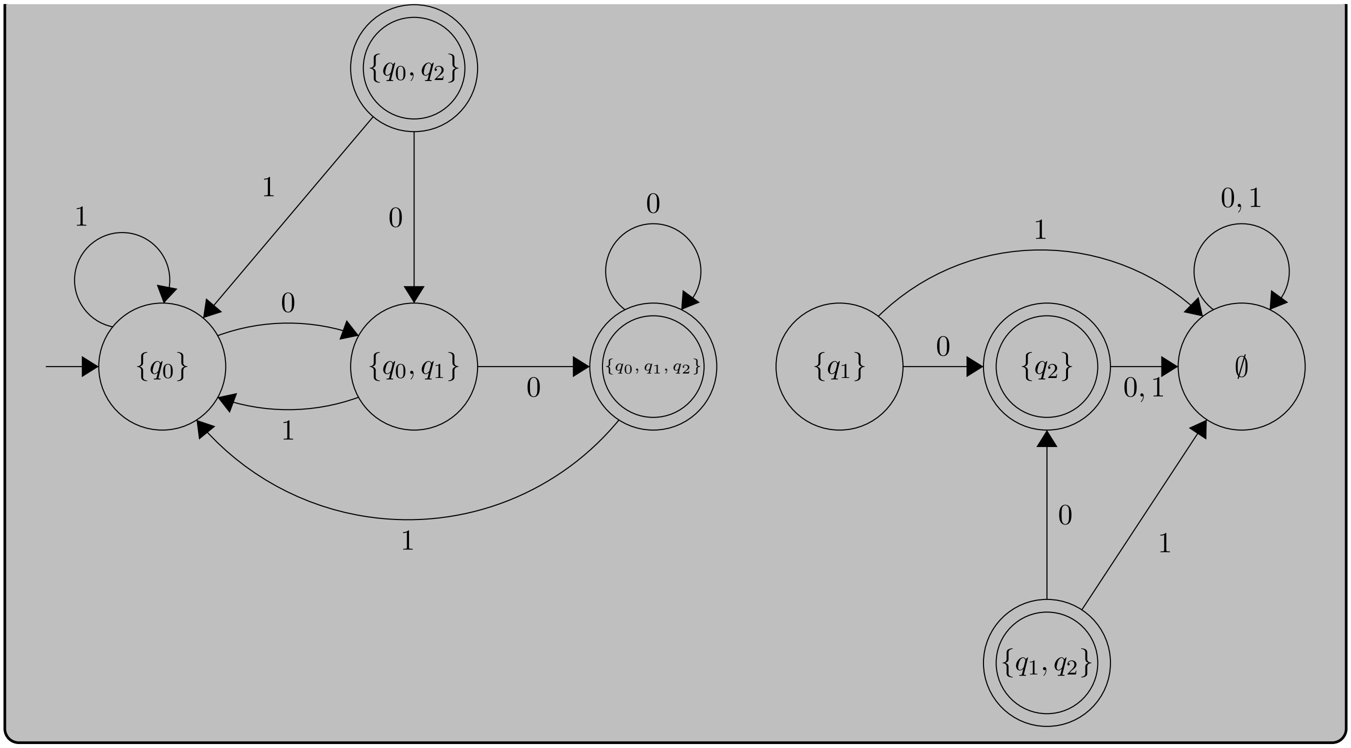
Doing the Tuple definition for a DFA with all states:

- $Q = 2^Q = \{\emptyset, \{q0\}, \{q1\}, \{q2\}, \{q0, q1\}, \{q0, q2\}, \{q1, q2\}, \{q0, q1, q2\}\}$
- $\Sigma = \{0, 1\}$
- δ can be defined using the transition table in Table 2
- $s = \{q0\}$
- $F = \{\{q2\}, \{q0, q2\}, \{q1, q2\}, \{q0, q1, q2\}\}$

DFA for reachable states:



DFA with unreachable states:


$\{q_0, q_2\}$
1
0
1
0
0, 1
1
$\{q_0\}$
$\{q_0, q_1\}$
$\{q_0, q_1, q_2\}$
$\{q_1\}$
0
$\{q_2\}$
0, 1
$\emptyset$
1
0
0
1
1
$\{q_1, q_2\}$

Δ	0	1
$\rightarrow p$	$\{p, q\}$	$\{p\}$
q	$\emptyset$	$\{r\}$
$*r$	$\{p, r\}$	$\{q\}$

Table 3: NFA

P3

Problem 3

Convert the NFA defined in Table 3 to a DFA. [HMU01]

δ	0	1
$\emptyset$	$\emptyset$	$\emptyset$
$\rightarrow \{p\}$	$\{p, q\}$	$\{p\}$
$\{q\}$	$\emptyset$	$\{r\}$
$*\{r\}$	$\{p, r\}$	$\{q\}$
$\{p, q\}$	$\{p, q\}$	$\{p, r\}$
$*\{p, r\}$	$\{p, q, r\}$	$\{p, q\}$
$*\{q, r\}$	$\{p, r\}$	$\{q, r\}$
$*\{p, q, r\}$	$\{p, q, r\}$	$\{p, q, r\}$

Table 4: Converted DFA Transition Table with All States (δ)

δ	0	1
$\emptyset$	$\emptyset$	$\emptyset$
$\rightarrow \{p\}$	$\{p, q\}$	$\{p\}$
$\{p, q\}$	$\{p, q\}$	$\{p, r\}$
$*\{p, r\}$	$\{p, q, r\}$	$\{p, q\}$
$*\{p, q, r\}$	$\{p, q, r\}$	$\{p, q, r\}$

Table 5: Converted DFA Transition Table with Reachable States (δ)

Solution to Problem 3

Let $(Q_N, \Sigma, \Delta_N, S_N, F_N)$ be the tuple representation of the NFA.

Doing the Tuple definition for all states:

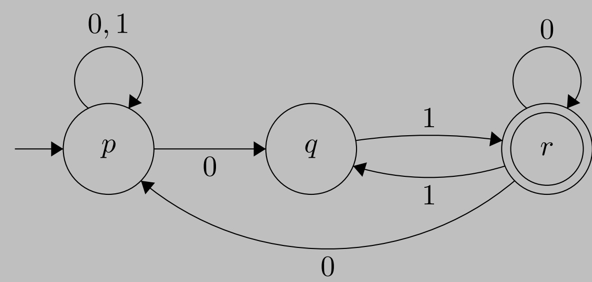
- $Q = 2^Q = \{\emptyset, \{p\}, \{q\}, \{r\}, \{p, q\}, \{p, r\}, \{q, r\}, \{p, q, r\}\}$
- $\Sigma = \{0, 1\}$
- δ can be defined using the transition table in Table 4
- $s = p$
- $F = \{\{r\}, \{p, r\}, \{q, r\}, \{p, q, r\}\}$

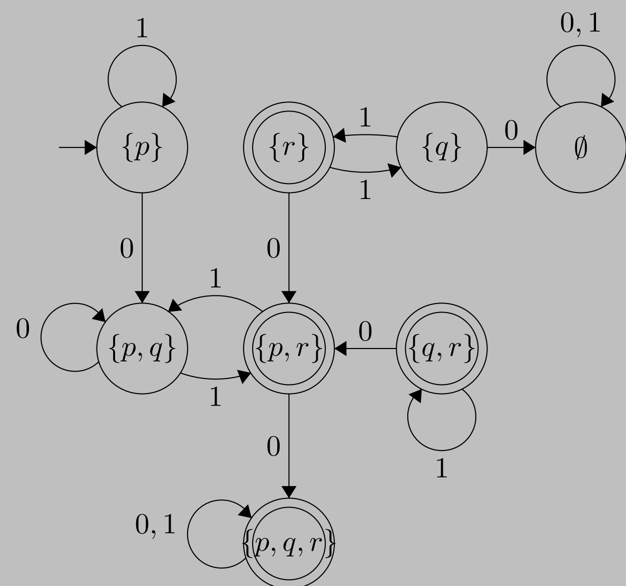
Doing the Tuple definition for reachable states:

- $Q = 2^Q = \{\emptyset, \{p\}, \{p, q\}, \{p, r\}, \{p, q, r\}\}$

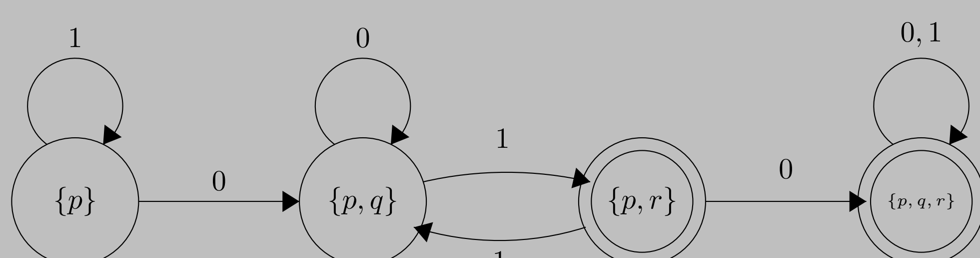
- $\Sigma = \{0, 1\}$
- δ can be defined using the transition table in Table 5
- $s = p$
- $F = \{\{p, r\}, \{p, q, r\}\}$

You don't need to do this next part, but for clarity, we also drew out the original NFA and the resulting DFA:





DFA with reachable states only:



P4

Problem 4

In class we used the following lemma based on our definition of $\hat{\Delta}$. Prove it by induction on $|y|$.

For any $x, y \in \Sigma^*$ and $A \subseteq Q$,

$$\hat{\Delta}(A, xy) = \hat{\Delta}(\hat{\Delta}(A, x), y)$$

$\hat{\Delta}(A, x\epsilon)$	Given
$= \hat{\Delta}(A, x)$	Simplifying concatenation.
$= \hat{\Delta}(\hat{\Delta}(A, x), \epsilon)$	Base case definition of $\hat{\Delta}$.

Table 6: Base case for Problem 4

$\hat{\Delta}(A, xy) = \hat{\Delta}(\hat{\Delta}(A, x), y)$	Inductive Hypothesis
$\hat{\Delta}(A, xya) = \bigcup_{q \in \hat{\Delta}(A, xy)} \Delta(q, a)$	Definition of $\hat{\Delta}$
$\hat{\Delta}(A, xya) = \bigcup_{q \in \hat{\Delta}(\hat{\Delta}(A,x),y)} \Delta(q, a)$	Applied Ind. Hypothesis to substitute subscription.
$\hat{\Delta}(A, xya) = \hat{\Delta}(\hat{\Delta}(A, x), ya)$	Applied Definition of $\hat{\Delta}$ to previous line.

Table 7: Inductive Step for Problem 4

Solution to Problem 4

Proof by induction on $|y|$.

Base case:

Let $y = \epsilon$, we want to show $\hat{\Delta}(A, x\epsilon) = \hat{\Delta}(\hat{\Delta}(A, x), \epsilon)$.

See Table 6 for solution.

(Doesn't have to be written exactly this way. Also a base case where $y = a$, $a \in \Sigma$ is acceptable.)

Inductive Hypothesis: For a y of arbitrary length, $\hat{\Delta}(A, xy) = \hat{\Delta}(\hat{\Delta}(A, x), y)$

We want to show: $\hat{\Delta}(A, xya) = \hat{\Delta}(\hat{\Delta}(A, x), ya)$

See Table 7 for solution.

Problem 5

Prove that every NFA can be converted to an equivalent one that has a single accept state. [Sip96]

References

[HMU01] John E Hopcroft, Rajeev Motwani, and Jeffrey D Ullman. Introduction to automata theory, languages, and computation. *Acm Sigact News*, 32(1):60–65, 2001.

[Sip96] Michael Sipser. Introduction to the theory of computation. *ACM Sigact News*, 27(1):27–29, 1996.

[Tug22] Randal Tuggle. Homework problem for comp 455. HW2, 2022.