P1

Problem 1

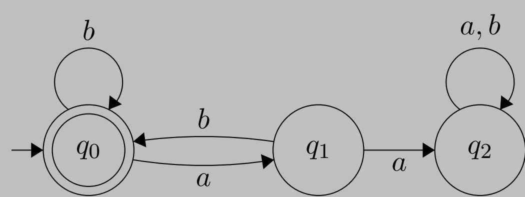
Design a DFA for the following languages over the alphabet $\Sigma = \{a, b\}$:

- $L_1 = \{w : \text{ every } a \text{ is immediately followed by a } b\}$
- $L_2 = \{w : w \text{ has an even number of } a's \text{ and an odd number of } b's\}$
- $L_3 = \{w : w \text{ contains the substring } baa\}$
- $L_4 = \{w : w \text{ every odd position of } w \text{ is an } a\}$
- $L_5 = \{\epsilon, a\}$

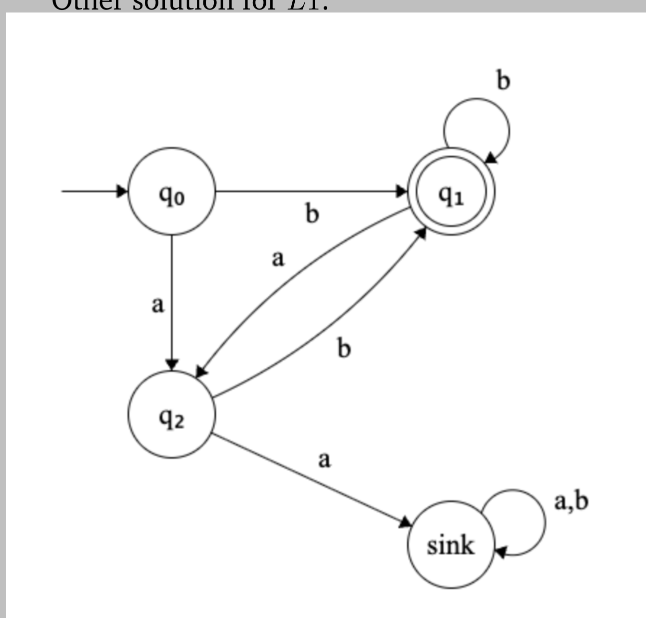
[Tug22]

Solution to Problem 1

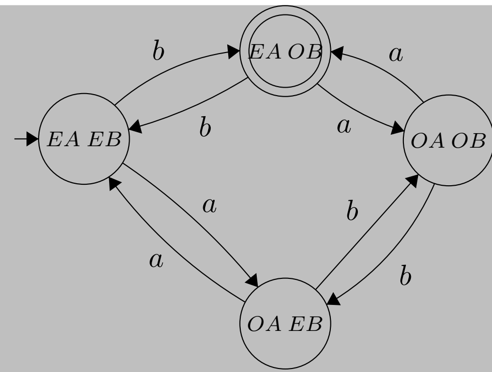
L_1 :



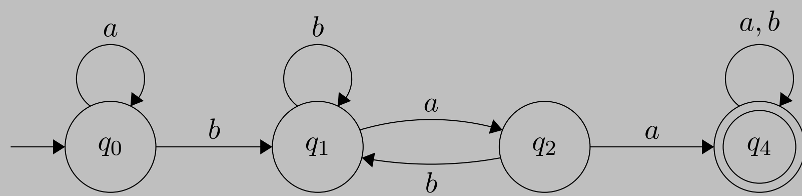
Other solution for $L1$:



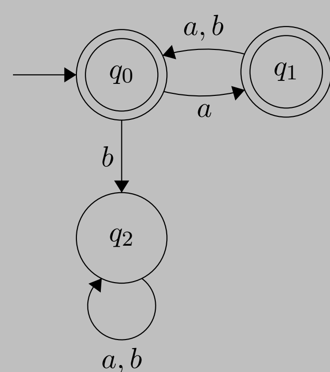
L_2 :


EA OB
EA EB
OA OB
OA EB
b
b
a
a
a
a
b
b

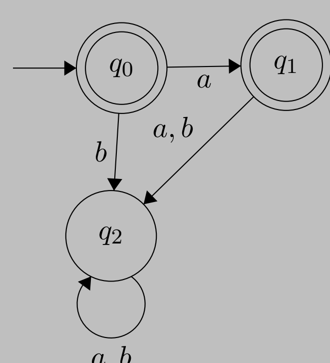
L_3 :


a
b
a, b
q0
q1
q2
q4
b
a
b
a

L_4 :

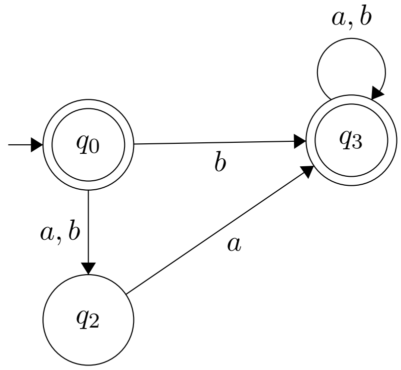

a, b
q0
q1
a
b
q2
a, b

L_5 :


q0
q1
a
a, b
b
q2
a, b

P2

Problem 2



Provide a 5-tuple definition for the NFA above. [Tug22]

Solution to Problem 2

A 5-tuple (without ϵ transitions) for the NFA pictured is:

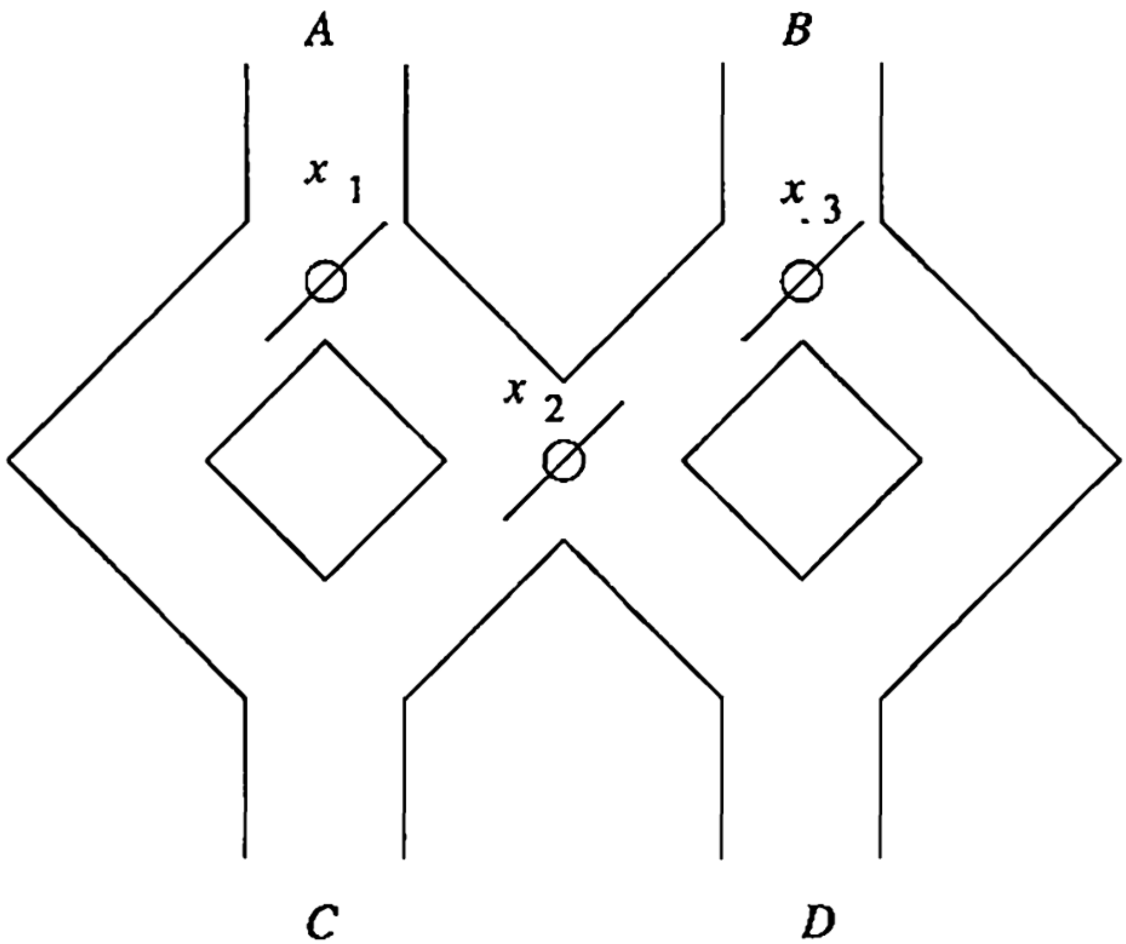
$$(\{q_0, q_2, q_3\}, \{a, b\}, \Delta = \begin{array}{c|cc} & \text{a} & \text{b} \\ \hline q_0 & \{q_2\} & \{q_2, q_3\} \\ q_2 & \{q_3\} & \{\} \\ q_3 & \{q_3\} & \{q_3\} \end{array}, \{q_0\}, \{q_0, q_3\}).$$

A 5-tuple (with ϵ transitions) for the NFA pictured is:

$$(\{q_0, q_2, q_3\}, \{a, b\}, \Delta = \begin{array}{c|ccc} & \text{a} & \text{b} & \epsilon \\ \hline q_0 & \{q_2\} & \{q_2, q_3\} & \{\} \\ q_2 & \{q_3\} & \{\} & \{\} \\ q_3 & \{q_3\} & \{q_3\} & \{\} \end{array}, \{q_0\}, \{q_0, q_3\}).$$

P3

Problem 3



In the figure is a marble-rolling toy. A marble is dropped at A or B. Levers x_1, x_2, x_3 cause the marble to fall either to the left or to the right. Whenever a marble encounters a lever, it causes the lever to reverse after the marble passes, so the next marble will take the opposite branch.

Model this toy by a finite automaton. Let the inputs A and B represent the input into which the marble is dropped. Let acceptance correspond to the marble exiting at D; nonacceptance represents a marble exiting at C.

[HMU01]

**Hint: think of your "states" in terms of lever positions.*

Solution to Problem 3

Let us model the toy so that each state is a four-character sequence. The first three characters of the sequence are each either L or R, representing the position of the first, second, and third levers (left or right). The fourth character in the sequence is either a or r, indicating whether the previous input was accepted or rejected.

With this encoding, we define the transition table for our automaton as follows:

	A	B
$\rightarrow LLLr$	$RLLr$	$LRRr$
$*LLLa$	$RLLr$	$LRRr$
$*LLRa$	$RLRr$	$LLLa$
$LRLr$	$RRLr$	$LLRa$
$*LRLa$	$RRLr$	$LLRa$
$LRRr$	$RRRr$	$LRLa$
$RLLr$	$LRLr$	$RRRr$
$*RLLa$	$LRLr$	$RRRr$
$RLRr$	$LRRr$	$RLLa$
$*RLRa$	$LRRr$	$RLLa$
$RRLr$	$LLLa$	$RLRa$
$*RRLa$	$LLLa$	$RLRa$
$RRRr$	$LLRa$	$RRLa$

P4

Problem 4

$L_{\text{float}} = \{w : w$ is the string representation of a floating point number$\}$ Assume the following syntax for floating point numbers:

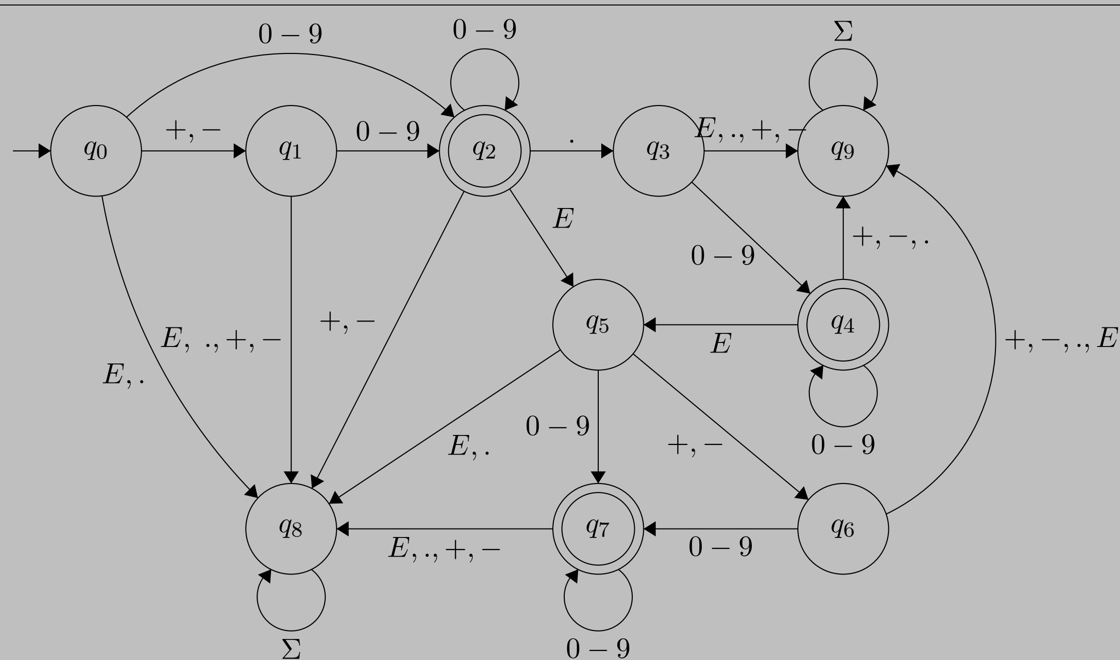
- A floating point number is an optional sign, followed by a decimal number, followed by an optional exponent.
- A decimal number may be of the form x or $x.y$, where x and y are nonempty strings of decimal digits.
- An exponent beings with `E` and is followed by an optional sign and then an integer.
- An integer is a nonempty string of decimal digits.

The following strings are examples of floating point numbers:

$$+3.0, 3.0, 0.3\text{E}1, 0.3\text{E}+1, -0.3\text{E}+1, -3\text{E}8, 7$$

Show that L_{float} is regular by constructing a DFA that recognizes L_{float}. [Tug22]

Solution to Problem 4



P5

Problem 5

In our lecture, we defined three automata:

- $M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$
- $M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$
- $M_3 = (Q_3, \Sigma, \delta_3, s_3, F_3)$

with

- $Q_3 = Q_1 \times Q_2$
- $\delta_3((p, q), a) = \big(\delta_1(p, a), \delta_2(q, a)\big)$
- $s_3 = (s_1, s_2)$
- $F_3 = F_1 \times F_2$

Prove $\hat{\delta_3}((p, q), x) = (\hat{\delta_1}(p, x), \hat{\delta_2}(q, x))$.
Hint: Prove by induction on the length of x.

Solution to Problem 5

For all $x \in \Sigma^*$, define $P(x)$ to be the statement:

$$\forall (p, q) \in Q_1 \times Q_2,\ \hat{\delta}_3((p, q), x) = \big(\hat{\delta}_1(p, x), \hat{\delta}_2(q, x)\big).$$

We prove this by induction on $|x|$.
(For base case, you can do $|x| = 0$ or $|x| = 1$. Either is fine!)
Base case ($|x| = 0$). Let $x = \epsilon$. Then, by the definition of $\hat{\delta}$:
$\hat{\delta}_3((p, q), \epsilon) = (p, q)$ and $(\hat{\delta}_1(p, \epsilon), \hat{\delta}_2(q, \epsilon)) = (p, q)$,
so $\hat{\delta}_3((p, q), \epsilon) = (\hat{\delta}_1(p, \epsilon), \hat{\delta}_2(q, \epsilon))$.
Base case ($|x| = 1$). We have $x = a$ for some character $a \in \Sigma$. Let $(p, q) \in Q_1 \times Q_2$. Then

$$\begin{aligned}
\hat{\delta}_3((p, q), x) &= \hat{\delta}_3((p, q), a) \\
&= \delta_3((p, q), a) \quad \text{(definition of extended transition on one symbol)} \\
&= (\delta_1(p, a), \delta_2(q, a)) \quad \text{(definition of } \delta_3) \\
&= (\hat{\delta}_1(p, a), \hat{\delta}_2(q, a)) \quad \text{(definition of extended transition on one symbol)} \\
&= (\hat{\delta}_1(p, x), \hat{\delta}_2(q, x)).
\end{aligned}$$

Thus, $P(x)$ holds when $|x| = 1$.

Induction step ($|x| > 1$).
Inductive Hypothesis:

Given an $x \in \Sigma^*$ with $|x| > 1$, assume $P(y)$ holds for all strings $y \in \Sigma^*$ with $1 \leq |y| < |x|$.

In other words, it holds for all strings y *smaller than* x.

Another way of saying this:

For all strings $y \in \Sigma^*$ with $1 \leq |y| < |x|$, $\hat{\delta_3}((p,q),y) = (\hat{\delta_1}(p,y), \hat{\delta_2}(q,y))$. $\leftarrow$ **Inductive Hypothesis**

Want to Prove:

We wish to show $P(x)$ for the given $x \in \Sigma^*$.

By the recursive definition of strings, there exists $y \in \Sigma^*$ with $|y| = |x| - 1$ and $a \in \Sigma$ such that $x = ya$.

So, for $x = ya$, $\hat{\delta_3}((p,q),x) = (\hat{\delta_1}(p,x), \hat{\delta_2}(q,x))$. $\leftarrow$ **WTP**

Proof:

An easy trick for writing a proof is starting with the inductive hypothesis on the first line and ending with what you want to prove (WTP) on the last line. Then fill in your proof to get from the first line to the last line!

For $x = ya$,

$$
\begin{aligned}
\hat{\delta_3}((p,q),y) &= (\hat{\delta_1}(p,y), \hat{\delta_2}(q,y)) \quad \text{(Inductive Hypothesis)}\\
\hat{\delta_3}((p,q),ya) &= \delta_3(\hat{\delta_3}((p,q),y),a) \quad \text{(Introduce } x = ya\text{, using definition of } \hat{\delta}\text{)}\\
\hat{\delta_3}((p,q),ya) &= \delta_3\big((\hat{\delta_1}(p,y), \hat{\delta_2}(q,y)),a\big) \quad \text{(Apply IH to right side)}\\
\hat{\delta_3}((p,q),ya) &= (\delta_1(\hat{\delta_1}(p,y),a), \delta_2(\hat{\delta_2}(q,y),a)) \quad \text{(Apply definition of } \delta_3 \text{ to right side)}\\
\hat{\delta_3}((p,q),ya) &= (\hat{\delta_1}(p,ya), \hat{\delta_2}(q,ya)) \quad \text{(Apply definition of } \hat{\delta} \text{ to right side)}\\
\hat{\delta_3}((p,q),ya) &= (\hat{\delta_1}(p,x), \hat{\delta_2}(q,x)) \quad \text{(Substitute } x = ya\text{)}
\end{aligned}
$$

Other acceptable proof:

Let $(p,q) \in Q_1 \times Q_2$. Then

$$
\begin{aligned}
\hat{\delta}_3((p,q),x) &= \hat{\delta}_3((p,q),ya)\\
&= \delta_3(\hat{\delta}_3((p,q),y),a) \quad \text{(definition of extended transition)}\\
&= \delta_3\big((\hat{\delta}_1(p,y), \hat{\delta}_2(q,y)),a\big) \quad \text{(inductive hypothesis)}\\
&= (\delta_1(\hat{\delta}_1(p,y),a), \delta_2(\hat{\delta}_2(q,y),a)) \quad \text{(definition of } \delta_3\text{)}\\
&= (\hat{\delta}_1(p,ya), \hat{\delta}_2(q,ya)) \quad \text{(definition of extended transition for } \delta_1, \delta_2\text{)}\\
&= (\hat{\delta}_1(p,x), \hat{\delta}_2(q,x)).
\end{aligned}
$$

Thus $P(x)$ holds when $|x| > 1$.

Therefore, by induction on $|x|$, $P(x)$ holds for all $x \in \Sigma^*$ with $|x| \geq 1$.

References

[HMU01] John E Hopcroft, Rajeev Motwani, and Jeffrey D Ullman. Introduction to automata theory, languages, and computation. *Acm Sigact News*, 32(1):60–65, 2001.

[Tug22] Randal Tuggle. Homework problem for comp 455. HW2, 2022.