Proofs with Regular Expressions

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Recall

In general, we also discussed how some patterns are redundant, and therefore we can reduce our "set" of notations for patterns to be:

- Atomic patterns: $\{a, \boldsymbol{\epsilon}, \boldsymbol{\emptyset}\}$
- Compound patterns: $\{\cup, \circ, *\}$

Another way to say this is that every pattern α can be expressed in the following form. (Where β and γ are patterns.)

- $a \in \Sigma$
- $\boldsymbol{\epsilon}$
- $\boldsymbol{\emptyset}$
- $\beta \cup \gamma$
- $\beta \circ \gamma$
- β^*

Which lead us to the inductive definition of regular expressions.

Definition 1

The set of regular expressions can be defined *inductively* using atomic patterns and operators.

R is a regular expression if R is

1. $a \in \Sigma$
2. $\boldsymbol{\epsilon}$
3. $\boldsymbol{\emptyset}$
4. $R_1 \cup R_2$ where R_1 and R_2 are regular expressions
5. $R_1 \circ R_2$ where R_1 and R_2 are regular expressions
6. R_1^* where R_1 is a regular expression

Theorem 1

Let $A \subseteq \Sigma^*$. The following three statements are equivalent:

1. A is a regular language
2. $A = L(\alpha)$ for some pattern α
3. $A = L(\alpha)$ for some regular expression α

1 Proving Theorem 1

Since there are three equivalent statements, to prove this theorem, do you have to prove every combination of statements?

Not exactly! You can prove them in a "chain" if you will.

So we are going to prove $3 \to 2$, $2 \to 1$, and $1 \to 3$!

- $3 \to 2$: If $A = L(\alpha)$ for some regular expression α, then $A = L(\alpha)$ for some pattern α
- $2 \to 1$: If $A = L(\alpha)$ for some pattern α, then A is a regular language
- $1 \to 3$: If A is a regular language, then $A = L(\alpha)$ for some regular expression α

1.1 Proving $3 \to 2$

$3 \to 2$ is trivial because every regular expression *is* a pattern!

1.2 Proving $2 \to 1$

Want to Prove: If $A = L(\alpha)$ for some pattern α, then A is a regular language.

We can do this with a proof by induction!

We can use the (non redundant) atomic patterns for our base case a, $\boldsymbol{\epsilon}$, and $\boldsymbol{\emptyset}$

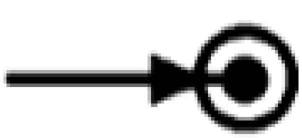
and the (non redundant) compound patterns $\cup$, $\circ$, and $*$

Base Cases: $\{\{a\}, \boldsymbol{\epsilon}, \boldsymbol{\emptyset}\}$

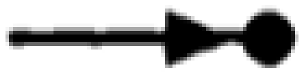
For each of these, we can just define an automaton that accepts each one!

- $a, a \in \Sigma$

- $\boldsymbol{\epsilon}$



- $\emptyset$



Inductive Step: $\{\cup, \circ, *\}$

Let our statement hold for smaller patterns β and γ. (Our inductive hypothesis)

More specifically, we use the IH that if $B = L(\beta)$ and $C = L(\gamma)$ for some pattern α, then B and C are regular languages.

Since every pattern α can be expressed in the following way:

- $\beta \cup \gamma$
- $\beta \circ \gamma$
- β^*

So, we can show that it holds for α by showing that it holds for $\beta \cup \gamma$, $\beta \circ \gamma$, and β^*. (Our WTP.)

More specifically, our WTP is in three parts:

- $L(\beta \cup \gamma)$ is a regular language
- $L(\beta \circ \gamma)$ is a regular language
- $L(\beta^*)$ is a regular language

As you can see this is a *proof by cases.* For the sake of time, we will not prove each of these, but for each case, you will do a *proof by construction.*

Let's prove the first case: $L(\beta \cup \gamma)$ is a regular language.

1. $B = L(\beta)$ and $C = L(\gamma)$ are regular languages (Inductive Hypothesis)
2. $L(\beta \cup \gamma) = L(\beta) \cup L(\gamma)$ (Compound Pattern Definition)
3. $L(\beta) \cup L(\gamma)$ is a regular language. (Needs to be proved!)
4. $L(\beta \cup \gamma)$ is a regular language. (Plugged in equivalence from line 2.)

As you can see, we need to prove if B and C are regular languages $B \cup C$ is a regular language. This would be your proof by construction. We already did a proof like this for $B \cap C$!

2 Proving $1 \rightarrow 3$

Want to Prove: If A is a regular language, then $A = L(\alpha)$ for some regular expression α

This is essentially saying each language accepted by a finite automaton can be represented by an equivalent regular expression.

So, we can prove this by converting a general finite automaton N to an equivalent regular expression! (A proof by construction!)