Assume A and B are regular languages.

Is $A \cap B$ regular?

Regular - we can define DFA that accepts it

Proof by <u>construction</u>.

$\Sigma = \{a, b\}$

$A = \{x_0 x_1 \ldots x_n \mid x_i \in \{a\} \wedge n \geq 0\} \cup \{\epsilon\}$

$\{a^n \mid n \geq 0\}$

$B = \{x_0 x_1 \ldots x_n \mid x_i \in \{a, b\} \wedge n \geq 0\}$

What is $A \cap B$?

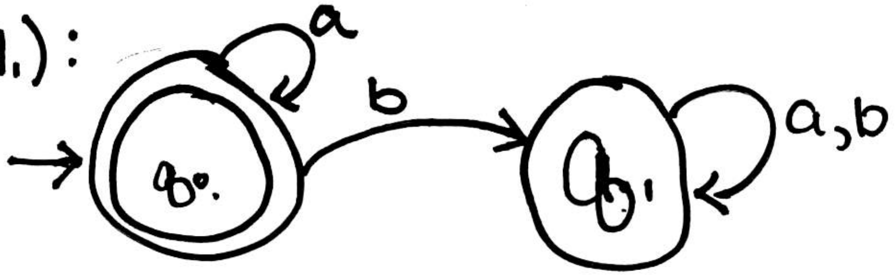
* Refresher:

$P \cap Q = \{x \mid x \in P \wedge x \in Q\}$

$P \cup Q = \{x \mid x \in P \vee x \in Q\}$

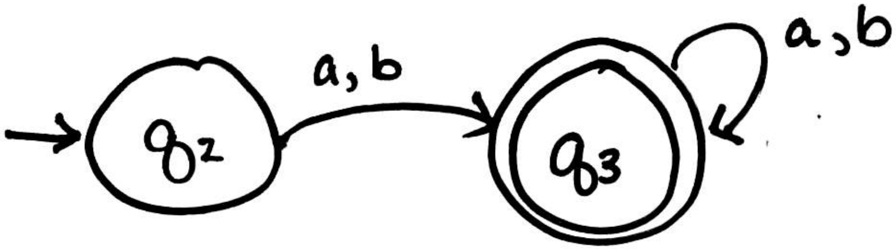
$A \cap B = \{a^n \mid n > 0\}$

$A = L(M_1)$:



$\leftarrow M_1$

$B = L(M_2)$



$\leftarrow M_2$

$A \cap B = L(M_3)$

$M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$

$M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$

$Q_1 = \{q_0, q_1\}$

$Q_2 = \{q_2, q_3\}$

$s_1 = q_0 \quad s_2 = q_2$

$F_1 = \{q_0\} \quad F_2 = \{q_3\}$

$M_3 = (Q_3, \Sigma, \delta_3, s_3, F_3)$

$Q_3 = Q_1 \times Q_2$

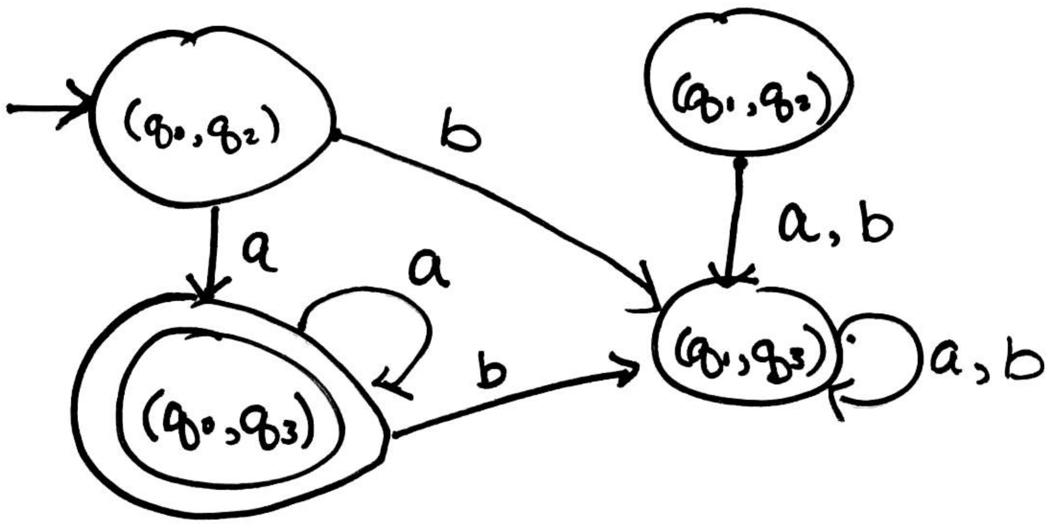
$Q_3 = \{(q_0, q_2), (q_1, q_2), (q_0, q_3), (q_1, q_3)\}$

$s_3 = (q_0, q_2)$

$F_3 = F_1 \times F_2$

$F_3 = \{(q_0, q_3)\}$

M_3:



$$\delta_1(q_0, a) = q_0 \qquad \delta_2(q_2, a) = q_3$$

$$\delta_3((q_0, q_2), a) = (q_0, q_3)$$

$$L(M_3) = A \cap B = L(M_1) \cap L(M_2) \quad \square$$

If A and B are regular languages, then $A \cap B$ is a regular language.	WTP
$\exists M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$ s.t. $A = L(M_1)$	Reg. Lang. Def.
$\exists M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$ s.t. $B = L(M_2)$	Reg. Lang. Def
We can construct $M_3 = (Q_3, \Sigma, \delta_3, s_3, F_3)$ s.t. $L(M_3) = L(M_1) \cap L(M_2) = A \cap B$ $Q_3 = Q_1 \times Q_2 = \{(p,q) \mid p \in Q_1 \wedge q \in Q_2\}$ $F_3 = F_1 \times F_2$ $s_3 = (s_1, s_2)$ $\delta_3 : Q_3 \times \Sigma \to Q_3$ $\delta_3((p,q), d) = (\delta_1(p,d), \delta_2(q,d))$	Construction
$\forall x \in \Sigma^*, x \in L(M_3) \Longleftrightarrow x \in L(M_1) \cap L(M_2)$	*
$\square$	

Proof of *

(From Kozen.)

Theorem 4.2 $L(M_3) = L(M_1) \cap L(M_2)$.

Proof. For all $x \in \Sigma^*$,

$$
\begin{array}{ll}
x \in L(M_3) & \\
\iff \widehat{\delta}_3(s_3, x) \in F_3 & \text{definition of acceptance} \\
\iff \widehat{\delta}_3((s_1, s_2), x) \in F_1 \times F_2 & \text{definition of } s_3 \text{ and } F_3 \\
\iff (\widehat{\delta}_1(s_1, x), \widehat{\delta}_2(s_2, x)) \in F_1 \times F_2 & \text{Lemma 4.1} \\
\iff \widehat{\delta}_1(s_1, x) \in F_1 \text{ and } \widehat{\delta}_2(s_2, x) \in F_2 & \text{definition of set product} \\
\iff x \in L(M_1) \text{ and } x \in L(M_2) & \text{definition of acceptance} \\
\iff x \in L(M_1) \cap L(M_2) & \text{definition of intersection.} \quad \square
\end{array}
$$