Toolbox

For this homework, you can use the following problem set definitions. Assume you already know these problems are NP-Complete:

- $SAT = \{\langle\phi\rangle \mid \phi$ is a satisfiable boolean formula $\}$.
- $3SAT = \{\langle\phi\rangle \mid \phi$ is a satisfiable 3cnf-formula $\}$.
- $NAE3SAT = \{\langle\phi\rangle \mid \phi$ is a satisfiable 3cnf-formula *and* the three values in each clause are not all equal to each other (at least one is true, and at least one is false)$\}$
- $HAMPATH = \{\langle G, s, t\rangle \mid G$ is a directed graph with a Hamiltonian path from s to $t\}$
- $UNHAMPATH = \{\langle G, s, t\rangle \mid G$ is an undirected graph with a Hamiltonian path from s to $t\}$
- $CLIQUE = \{\langle G, k\rangle \mid G$ is an undirected graph with a k-CLIQUE $\}$

Problem 1

Let

$LPATH = \{\langle G, a, b, k\rangle \mid$ Directed graph G contains a path of length at least k from a to $b\}$.

1. Show $LPATH \in$ NP.
2. Show $LPATH$ is NP-Complete by demonstrating $HAMPATH \leq_{\mathrm{P}} LPATH$

(Both individual parts are eligible for credit.)

Problem 2

Let

$$\begin{aligned} SET-SPLITTING = &\{\langle S, C\rangle \mid S \text{ is a finite set and } C = \{C_1, \ldots, C_k\} \text{ is a collection of subsets of } S, \\ &\text{for some } k > 0, \text{ such that elements of } S \text{ can be colored red } \textit{or} \text{ blue} \\ &\text{so that no } C_i \text{ has all its elements colored with the same color}\} \end{aligned}$$

1. Show $LPATH \in$ NP.
2. Show that $SET-SPLITTING$ is NP-Complete by demonstrating $NAE3SAT \leq_P SET-SPLITTING$.

(Both individual parts are eligible for credit.)

Problem 3

The **Traveling Salesperson Problem** is a famous NP-Complete problem, as it acts as a very straightforward real-world application of the types of problems we're discussing.

> The traveling salesperson problem is an optimization problem that seeks the *shortest possible route* for a salesperson to visit a given set of cities and return to the origin city.

This is represented via a graph, where (as you may assume) the cities are nodes and the edges represent connecting roads to each city. To represent the time to travel from city to city, the edges are given weights. (Let's assume that if you can travel from city a to b in 5 minutes, you can also travel from city b to a in 5 minutes. That allows us to make our graph undirected.)

To make this a *decision problem* rather than an optimization problem, we need to fix a max amount of time t and see if a path exists that stays below it.

$$\begin{aligned} TSP = \{\langle G, t\rangle \mid\ & G \text{ is an undirected weighted graph, } t > 0\text{, and} \\ & \text{there exists a path } \pi \text{ that starts and ends at some node } s \in G \\ & \text{and visits every node in } G \\ & \text{where the sum of the weight of the edges in } \pi \text{ is } \leq t\} \end{aligned}$$

1. Show $TSP \in$ NP.
2. Show TSP is NP-Complete by demonstrating $UNHAMPATH \leq_{\mathrm{P}} TSP$

(Both individual parts are eligible for credit.)

Problem 4

(Only part of this problem is for credit since we will do these solutions in class.)

Let

$$\begin{aligned} SUBSETSUM = \{\langle S, t\rangle \mid & S = \{x_1, x_2, \ldots, x_k\}, \\ & \text{and for some } \{y_1, \ldots, y_l\} \subseteq S, \\ & \sum y_i = t\} \end{aligned}$$

1. Show $SUBSETSUM \in$ NP.
2. Show $SUBSETSUM$ is NP-Complete by demonstrating $3SAT \leq_{\mathrm{P}} SUBSETSUM$

For credit: We proved in class $\phi \in 3SAT \implies f(\phi) = \langle S, t\rangle \in SUBSETSUM$.
However, we did not show $\phi \notin 3SAT \implies f(\phi) = \langle S, t\rangle \notin SUBSETSUM$
Prove this.

Problem 5

(This one is NOT for credit since we will do these solutions in class.)

The **Knapsack Problem** comes from the following idea:

> Given a set of items, each with a weight and a value, determine which items to include in the collection so that the total *weight* is less than or equal to a given limit and the total *value* is as large as possible.

This is a famous optimization problem. Let's formalize it a little bit to turn it into a decision problem.

Let X be a collection of objects $X = \{x_0, \ldots, x_n\}$ with corresponding weights $W = \{w_0, \ldots, w_n\}$ and values $V = \{v_0, \ldots, v_n\}$. In other words, object x_i would have a weight of w_i and a value of v_i.

Now let

$$\begin{aligned} KNAPSACK = \{\langle X, W, V, w', v' \rangle \mid\ & X \text{ is a collection of objects } X = \{x_0, \ldots, x_n\} \\ & \text{with corresponding weights } W = \{w_0, \ldots, w_n\} \\ & \text{and values } V = \{v_0, \ldots, v_n\} \text{ for } n > 0 \text{ such that} \\ & \text{a subset } Y \text{ of } X \text{ can be chosen such that} \\ & \ \text{the sum of the values of } Y \text{ equals } v' \text{ and} \\ & \ \text{the sum of the weights of } Y \text{ is } \leq w' \ \} \end{aligned}$$

1. Show $KNAPSACK \in$ NP.
2. Show $KNAPSACK$ is NP-Complete.