

Practice Exercise 1

Exercise

We write $X \sim \text{Poisson}(\theta)$ if X has the Poisson distribution with rate $\theta > 0$, that is, its p.m.f. is

$$p(x|\theta) = \text{Poisson}(x|\theta) = e^{-\theta} \theta^x / x!$$

for $x \in \{0, 1, 2, \dots\}$ (and is 0 otherwise). Suppose $X_1, \dots, X_n \stackrel{\text{iid}}{\sim} \text{Poisson}(\theta)$ given θ , and your prior is

$$p(\theta) = \text{Gamma}(\theta|a, b) = \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta} \mathbf{1}(\theta > 0).$$

What is the posterior distribution on θ ?

Solution

Since the data is independent given θ , the likelihood factors and we get

$$\begin{aligned} d(x_{1:n}|\theta) &= \prod_{i=1}^n d(x_i|\theta) \\ &= \prod_{i=1}^n e^{-\theta x_i} \theta^{x_i} \\ &\propto e^{-\sum_{i=1}^n \theta x_i} \theta^{\sum_{i=1}^n x_i} \end{aligned}$$

Thus, using Bayes' theorem,

$$\begin{aligned} d(\theta|x_{1:n}) &\propto d(x_{1:n}|\theta) d(\theta) \\ &\propto e^{-\sum_{i=1}^n \theta x_i} \theta^{\sum_{i=1}^n x_i} \mathbb{I}_{(0 < \theta)} \\ &\propto \text{Gamma}(\theta | a + \sum_{i=1}^n x_i, b + n). \end{aligned}$$

Therefore, since the posterior density must integrate to 1, we have