Intro to Monte Carlo, Part II

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Bayesian Methods and Modern Statistics: STA 360/601

Module 5

Rejection Sampling

Rejection sampling is a method for drawing random samples from a distribution whose p.d.f. can be evaluated up to a constant of proportionality.

Difficulties? You must design a good proposal distribution (which can be difficult, especially in high-dimensional settings).

Uniform Sampler

Goal: Generate samples from Uniform(A), where A is complicated.

Example: $X \sim$ Uniform(Mandelbrot).

How? Consider $I_X(A)$.

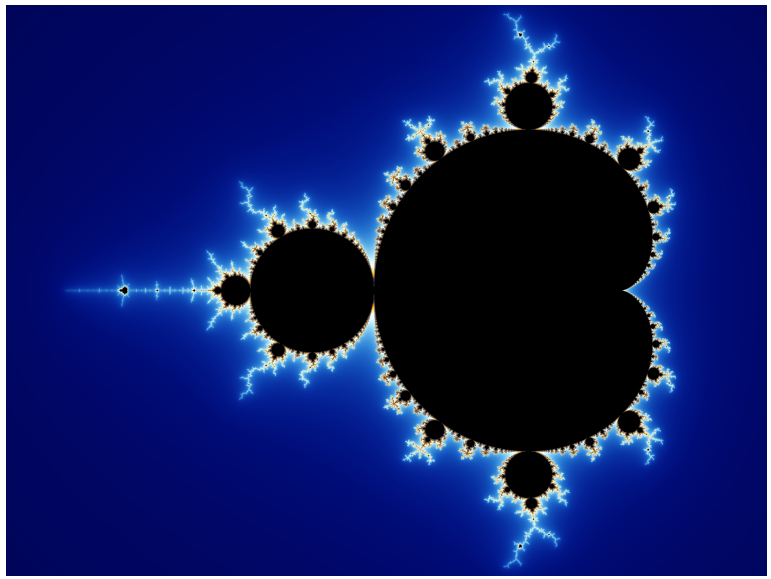

Figure 1: A complicated function A, called the Mandelbrot!

Proposition

- Suppose $A \subset B$.
- Let $Y_1, Y_2, \ldots \sim$ Uniform(B) iid and
- $X = Y_k$ where $k = \min\{k : Y_k \in A\}$,

Then it follows that

$$X \sim \text{Uniform}(A).$$

Proof: Exercise. Hint: Try the discrete case first and use a geometric series.

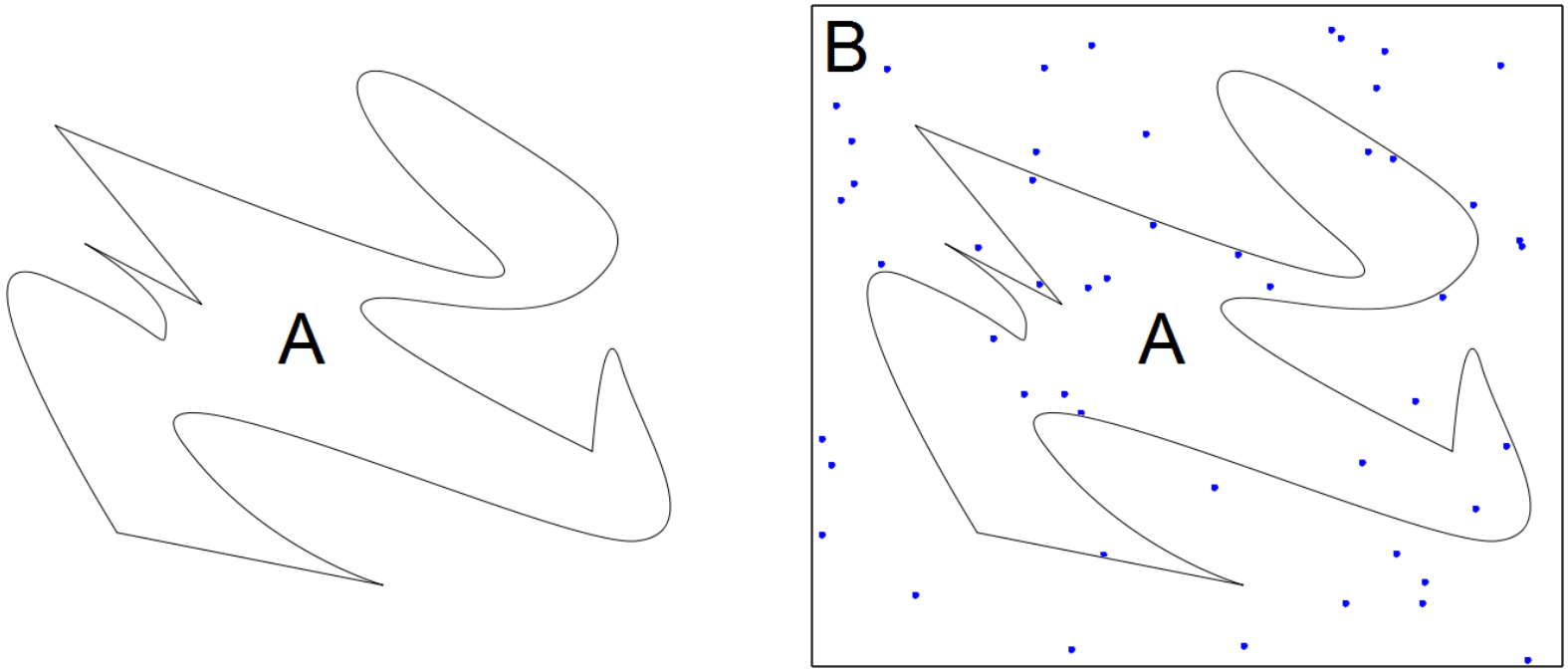

Figure 2: (Left) How to draw uniform samples from region A? (Right) Draw uniform samples from B and keep only those that are in A.

General Rejection Sampling Algorithm

Goal: Sample from a complicated pdf $f(x)$.

Suppose that

$$f(x) = \tilde{f}(x)/\alpha, \alpha > 0$$

.

Algorithm:

1. Choose a proposal distribution q such that $c > 0$ with

$$cq(x) \geq \tilde{f}(x).$$

2. Sample $X \sim q$, sample $Y \sim \text{Unif}(0, c\, q(X))$ (given X)
3. If $Y \leq \tilde{f}(X), Z = X,$
   Otherwise we reject and return to step (2).

Output: $Z \sim f$

Proof: Exercise.

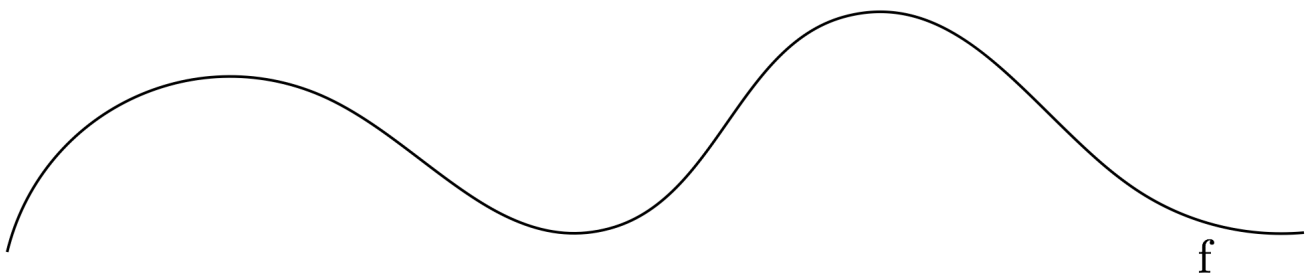

Figure 3: Visualizing just f (hard to sample from).

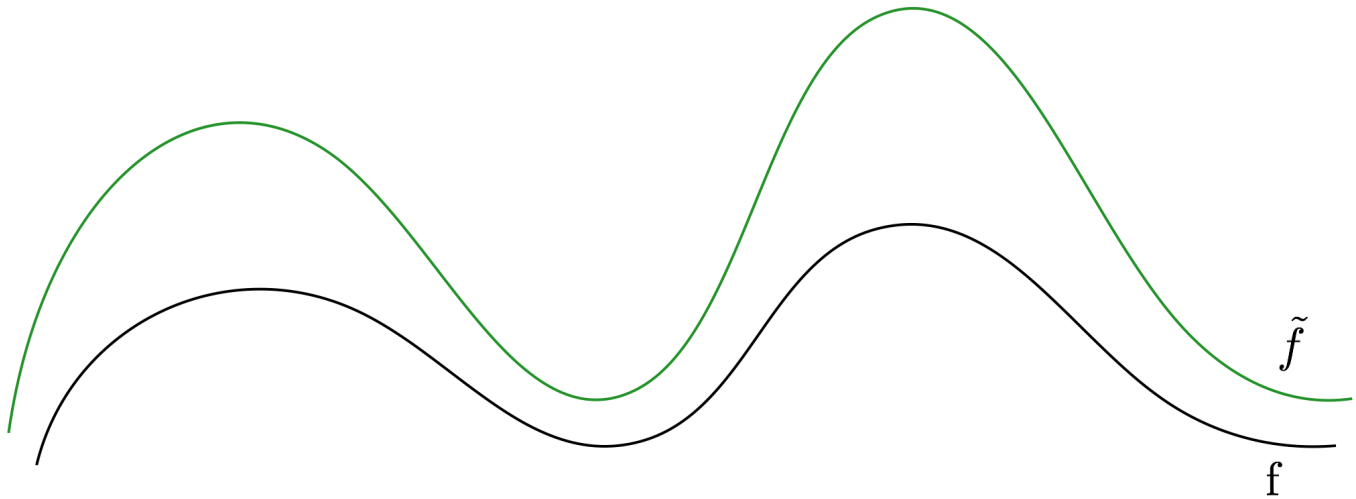

Figure 4: Visualizing just f and $\tilde{f}$.

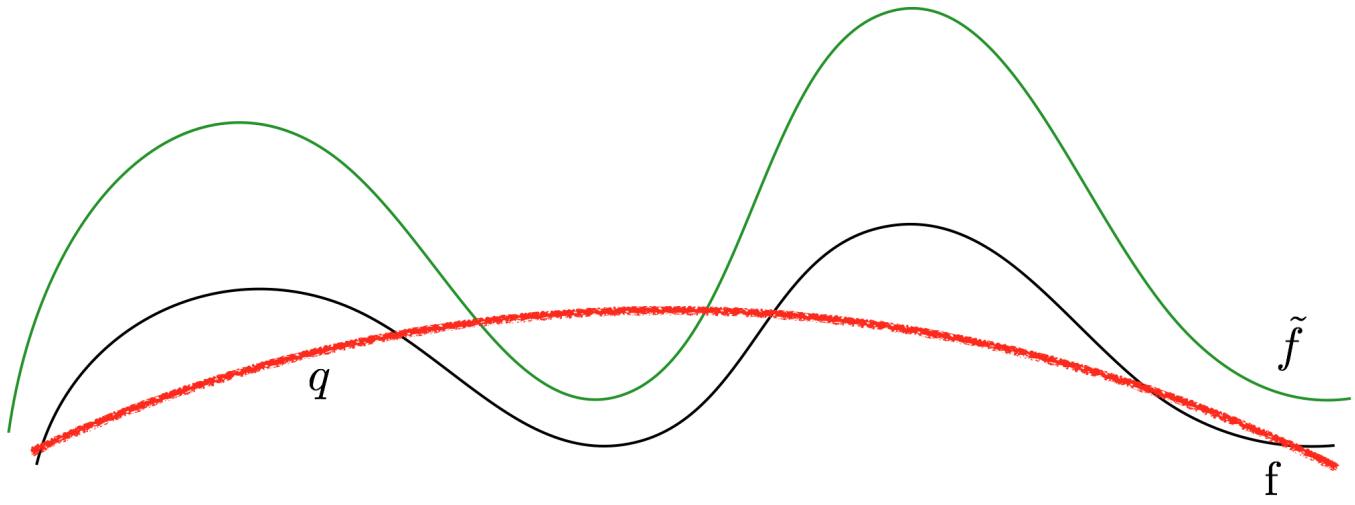

Figure 5: Visualizing f and $\tilde{f}$. Now we look at enveloping q over f.

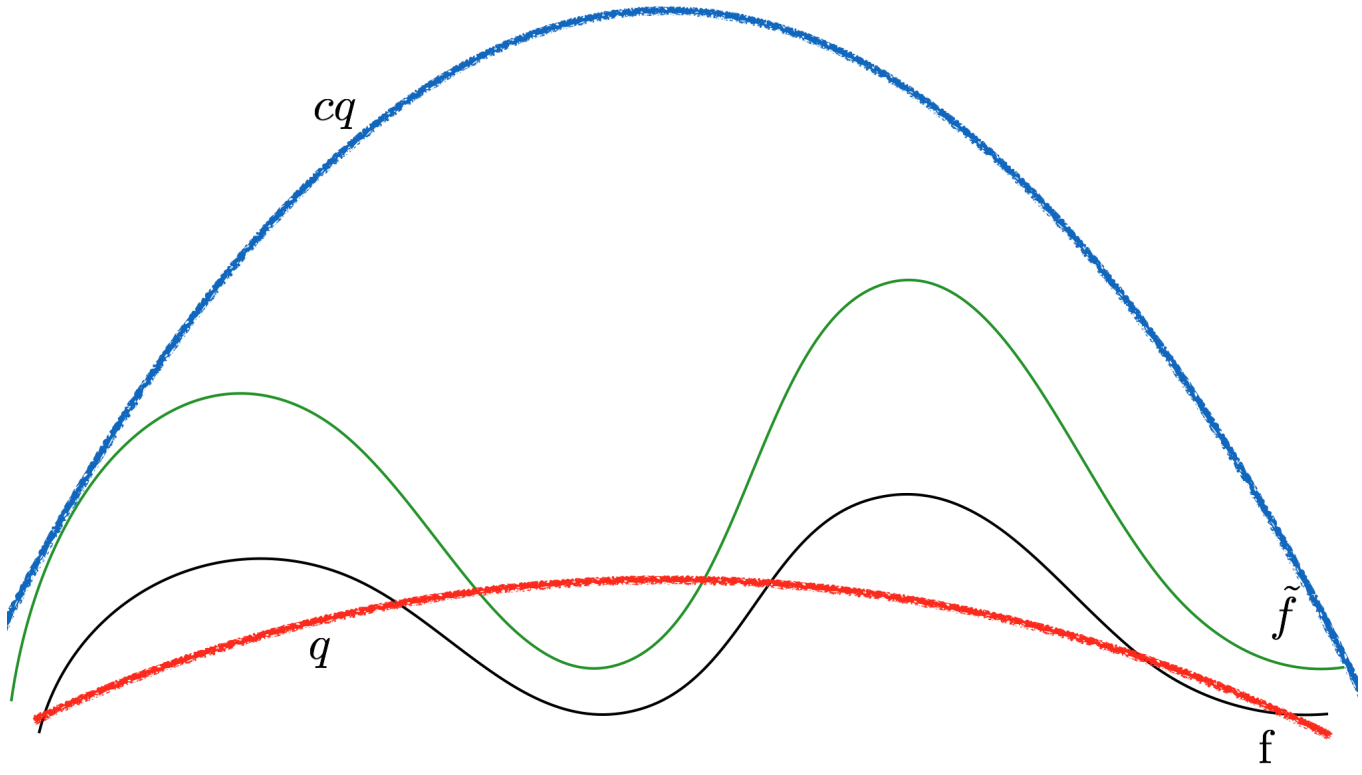

Figure 6: Visualizing f and $\tilde{f}$. Now we look at enveloping cq over $\tilde{f}$.

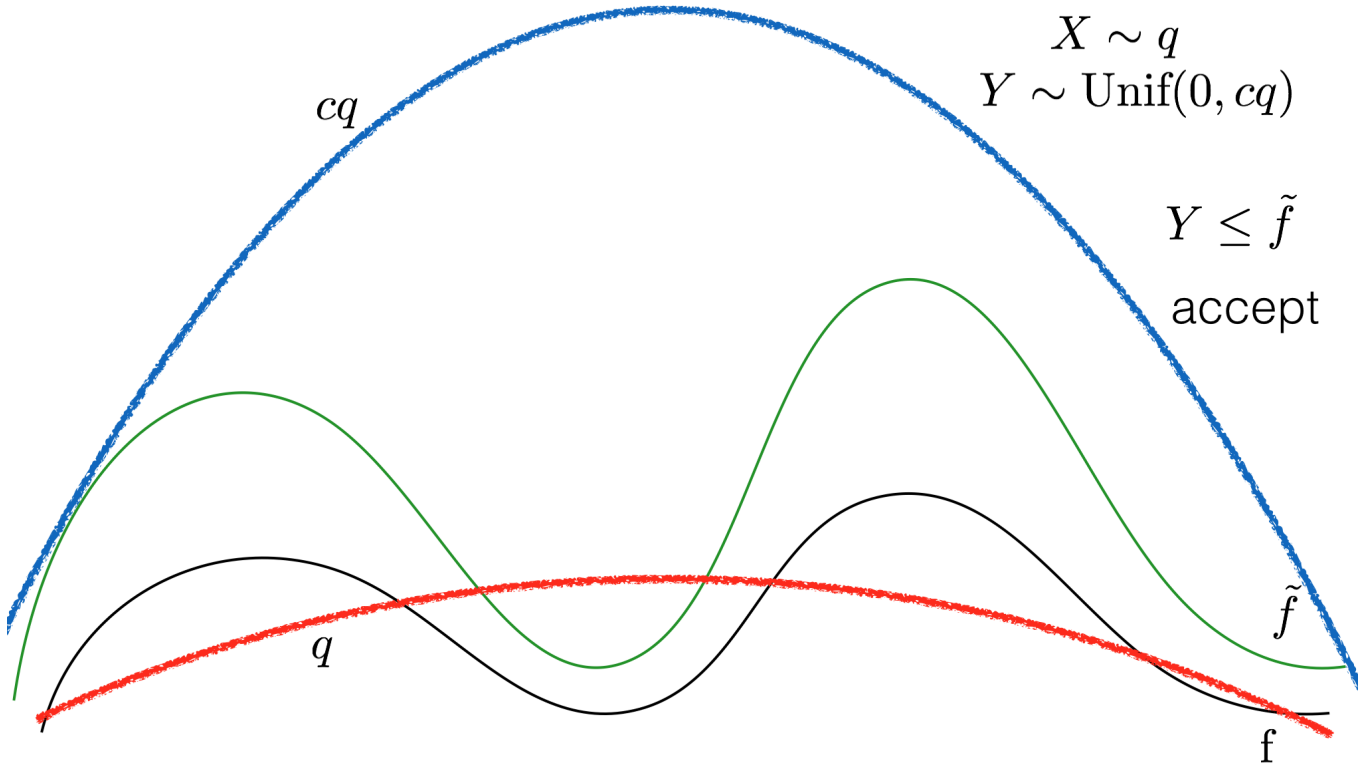

Figure 7: Recalling the sampling method and accept/reject step.

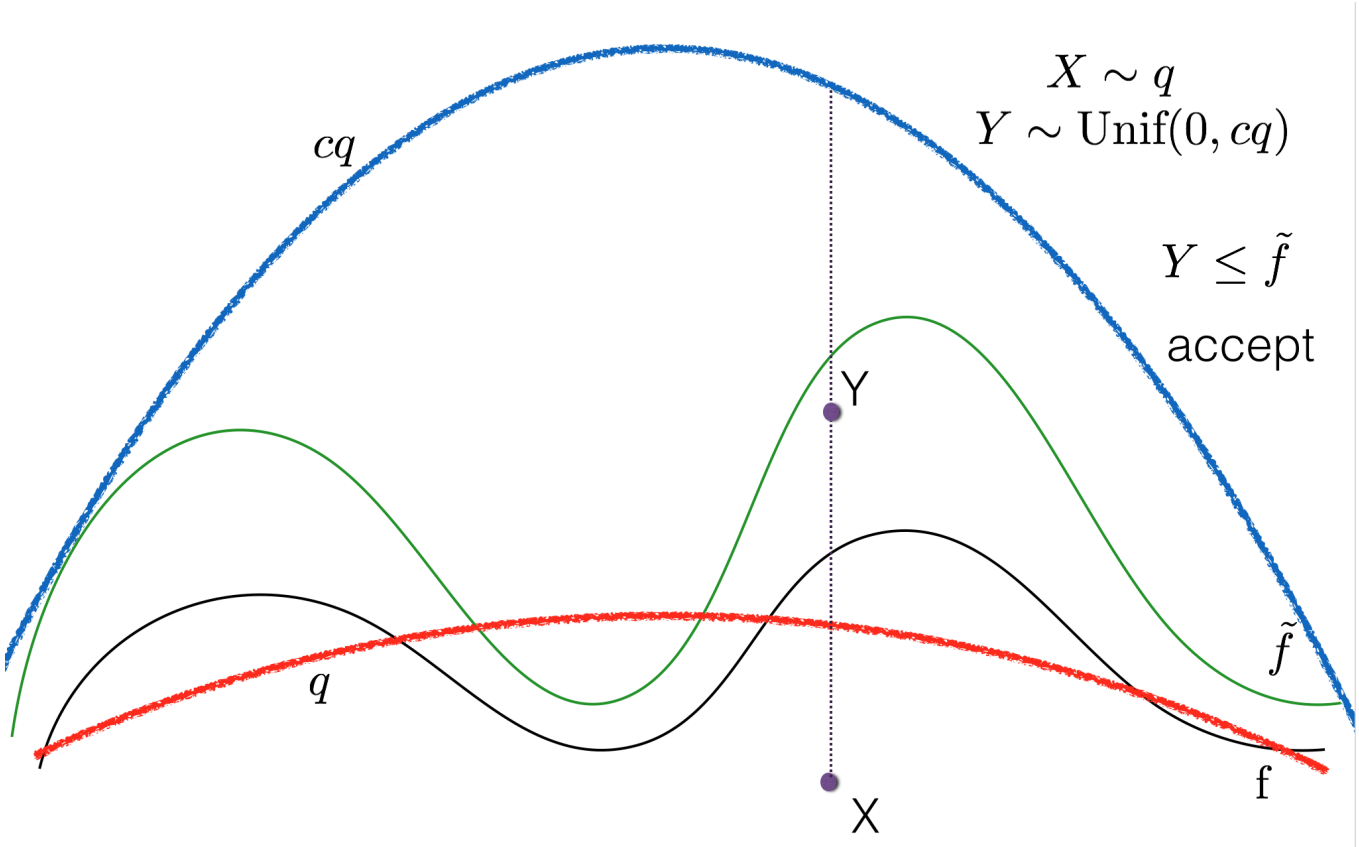

Figure 8: Entire picture and an example point X and Y.

- Suppose we want to generate random variables from the Beta(5.5,5.5) distribution.
- There are no direct methods for generating from Beta(a,b) if a,b are not integers.
- One possibility is to use a Uniform(0,1) as the trial distribution. A better idea is to use an approximating normal distribution.
- Do this as an exercise on your own.
- In lab: you'll go through both importance sampling and rejection sampling.