Noninformative (“Default”) Bayes

Rebecca C. Steorts

Bayesian Methods and Modern Statistics: STA 360/601

Lecture 6

Exam I

- Exam Thursday, Feb 11th in class. Be early to class so that you can start you exam on time.
- You will need pencil and paper. No calculators, no computers, no cell phones, etc permitted. No notes permitted.
- The exam will cover material through Module 4. This includes all readings.
- Assignment 2 solutions will be posted shortly.
- Assignment 3 has been posted with 2 suggested problems to work on (and you will get credit for them).
- There was an optional homework problem with Module 3, Part I. The solutions have been posted.
- Lab next week: Review sessions to prepare for the exam.

Exam I

- Intro to Bayes. What is it and why do we use it?
- Decision theory - loss, risk (all three of them).
- Hierarchical modeling - conjugacy, priors, posteriors, likelihood.
- Consistency, posterior predictive, credible intervals.
- Objective Bayes

Exam I: Expect 4 – 6 problems. You will need to really know the material to get through this exam.

Today's menu

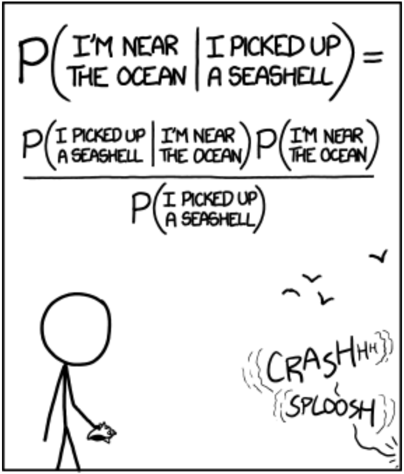
- Subjective prior
- Default prior
- Are they really noninformative?
- Invariance property
- Jeffreys' prior

- Ideally, we would like a *subjective prior*: a prior reflecting our beliefs about the unknown parameter of interest.
- What are some examples in practice when we have subjective information?
- When may we not have subjective information?

In dealing with real-life problems you may run into problems such as

- not having past historical data,
- not having an expert opinion to base your prior knowledge on (perhaps your research is cutting-edge and new), or
- as your model becomes more complicated, it becomes hard to know what priors to put on each unknown parameter.
- What do we do in such situations?

That Rule Bayes



STATISTICALLY SPEAKING, IF YOU PICK UP A SEASHELL AND *DON'T* HOLD IT TO YOUR EAR, YOU CAN PROBABLY HEAR THE OCEAN.

What did Bayes say exactly?

PROBLEM.

Given the number of times in which an unknown event has happened and failed: *Required* the chance that the probability of its happening in a ſingle trial lies ſomewhere between any two degrees of probability that can be named.

Translation (courtesy of Christian Robert)!

Billiard ball W rolled on a line of length one, with a uniform probability of stopping anywhere:

W stops at p

Second ball O then rolled n times under the same assumptions.

X denotes the number of times the ball O stopped on the left of W

Derive the posterior distribution of p given X, when $p \sim U[0,1]$ and $X \mid p \sim \text{Binomial}(n, p)$

Such priors on p are said to be uniform or flat.

Comment: Since many of the objective priors are improper, so we must check that the posterior is proper.
Propriety of the Posterior

- If the prior is proper, then the posterior will *always* be proper.
- If the prior is improper, you must check that the posterior is proper.

A flat prior (my longer translation....)

Let's talk about what people really mean when they use the term "flat," since it can have different meanings.

Often statisticians will refer to a prior as being flat, when a plot of its density actually looks flat, i.e., uniform.

$$\theta \sim \text{Unif}(0, 1).$$

Why do we call it flat? It's assigning equal weight to each parameter value. Does it always do this?

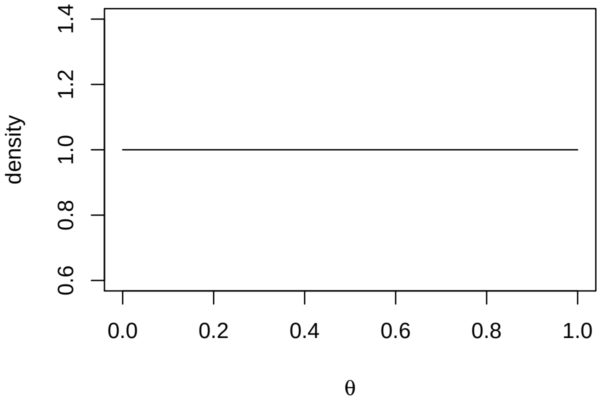


Figure 1: Unif(0,1) prior

What happens if we consider though the transformation to $1/\theta$. Is our prior still flat (does it place equal weight at every parameter value)?

Suppose we consider Jeffreys' prior, $p_J(\theta)$, where $X \sim \text{Bin}(n, \theta)$.

We calculate Jeffreys' prior by finding the Fisher information. The Fisher information tells us how much information the data gives us for certain parameter values.

- Here, $p_J(\theta) \propto \text{Beta}(1/2, 1/2)$.
- Let's consider the plot of this prior. Flat here is a purely abstract idea.
- In order to achieve objective inference, we need to compensate more for values on the boundary than values in the middle.

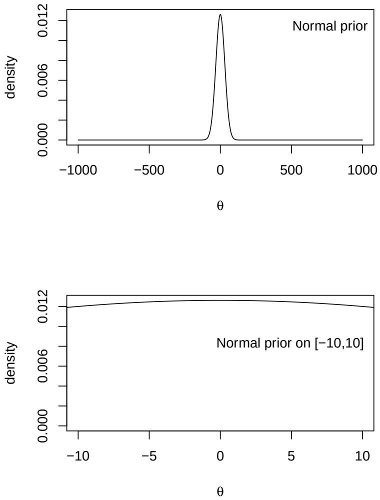


Figure 2: Normal priors

The Frenchmen, Laplace

(Laplace) In 1814, Pierre-Simon Laplace wanted to know the probability that the sun will rise tomorrow. He answered this question using the following Bayesian analysis:

- Let X represent the number of days the sun rises. Let p be the probability the sun will rise tomorrow.
- Let $X|p \sim \text{Bin}(n, p)$.
- Suppose $p \sim \text{Uniform}(0, 1)$.
- Based on reading the Bible, Laplace computed the total number of days n in recorded history, and the number of days x on which the sun rose. Clearly, $x = n$.

Then

$$\pi(p|x) \propto \binom{n}{x} p^x (1-p)^{n-x} \cdot 1$$
$$\propto p^{x+1-1}(1-p)^{n-x+1-1}$$

This implies

$$p|x \sim \mathsf{Beta}(x+1, n-x+1)$$

Then

$$\hat{p} = E[p|x] = \frac{x+1}{x+1+n-x+1} = \frac{x+1}{n+2} = \frac{n+1}{n+2}.$$

- Thus, Laplace's estimate for the probability that the sun rises tomorrow is $(n+1)/(n+2)$, where n is the total number of days recorded in history.
- For instance, if so far we have encountered 100 days in the history of our universe, this would say that the probability the sun will rise tomorrow is $101/102 \approx 0.9902$.
- However, we know that this calculation is ridiculous.
- Here, we have extremely strong subjective information (the laws of physics) that says it is extremely likely that the sun will rise tomorrow.
- Thus, objective Bayesian methods shouldn't be recklessly applied to every problem we study—especially when subjective information this strong is available.
- If you have a philosophical question or debate, please come see me in office hours!

Criticism of the Uniform Prior

- The Uniform prior of Bayes and Laplace and has been criticized for many different reasons.
- We will discuss one important reason for criticism and not go into the other reasons since they go beyond the scope of this course.
- In statistics, it is often a good property when a rule for choosing a prior is *invariant* under what are called one-to-one transformations. Invariant basically means unchanging in some sense.
- The invariance principle means that a rule for choosing a prior should provide equivalent beliefs even if we consider a transformed version of our parameter, like p^2 or $\log p$ instead of p.

Jeffreys' Prior

One prior that is invariant under one-to-one transformations is Jeffreys' prior.

What does the invariance principle mean?

Suppose our prior parameter is θ, however we would like to transform to ϕ.

Define $\phi = f(\theta)$, where f is a one-to-one function.

Jeffreys' prior says that if θ has the distribution specified by Jeffreys' prior for θ, then $f(\theta)$ will have the distribution specified by Jeffreys' prior for ϕ. We will clarify by going over two examples to illustrate this idea.

Example: Uniform

Note, for example, that if θ has a Uniform prior, Then one can show $\phi = f(\theta)$ will not have a Uniform prior (unless f is the identity function).

Example: Jeffreys'

Define

$$I(\theta) = -E\left[\frac{\partial^2 \log p(y|\theta)}{\partial\theta^2}\right],$$

where $I(\theta)$ is called the Fisher information. Then *Jeffreys' prior* is defined to be

$$p_J(\theta) = \sqrt{I(\theta)}.$$

For homework you will prove that the uniform prior in not invariant to transformation but that Jeffrey's is.

Example: Jeffreys'

Suppose

$$X|\theta \sim \text{Binomial}(n, \theta).$$

Let's calculate the posterior using Jeffreys' prior. To do so we need to calculate $I(\theta)$. Ignoring terms that don't depend on θ, we find

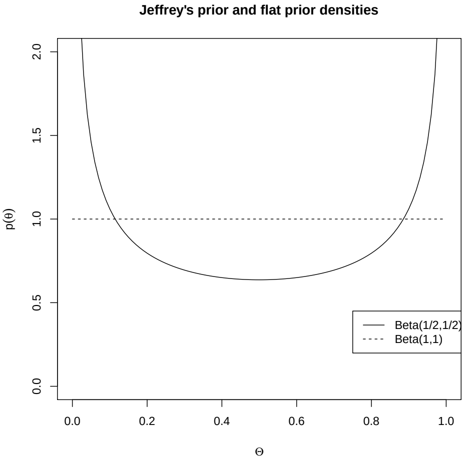


Figure 3: Jeffreys' prior and flat prior densities

Figure **??** compares the prior density $\pi_J(\theta)$ with that for a flat prior, which is equivalent to a Beta(1,1) distribution.

- We see that the data has the least effect on the posterior when the true $\theta = 1$, and has the greatest effect near the extremes, $\theta = 0$ or 1.
- Jeffreys' prior compensates for this by placing more mass near the extremes of the range, where the data has the strongest effect.
- We could get the same effect by (for example) letting the prior be $\pi(\theta) \propto \dfrac{1}{\text{Var}\theta}$ instead of $\pi(\theta) \propto \dfrac{1}{[\text{Var}\theta]^{1/2}}$.
- However, the former prior is not invariant under reparameterization, as we would prefer.

We then find that

$$\begin{aligned}
p(\theta \mid x) &\propto \theta^{x}(1-\theta)^{n-x}\theta^{1/2-1}(1-\theta)^{1/2-1} \\
&= \theta^{x-1/2}(1-\theta)^{n-x-1/2} \\
&= \theta^{x-1/2+1-1}(1-\theta)^{n-x-1/2+1-1}.
\end{aligned}$$

Thus, $\theta|x \sim \text{Beta}(x+1/2, n-x+1/2)$, which is a proper posterior since the prior is proper.

Jeffreys' and Conjugacy

- In general, they are not conjugate priors.
- For example, with a Gaussian model $X \sim N(\mu, \sigma^2)$, it can be shown that $\pi_J(\mu) = 1$ and $\pi_J(\sigma) = \frac{1}{\sigma}$, which do not look anything like a Gaussian or an inverse gamma, respectively.
- However, it can be shown that Jeffreys priors are limits of conjugate prior densities.
- For example, a Gaussian density $N(\mu_o, \sigma_o^2)$ approaches a flat prior as $\sigma_o^2 \to \infty$, while the inverse gamma $\sigma^{-(a+1)} e^{-b/\sigma} \to \sigma^{-1}$ as $a, b \to 0$.

Limitations of Jeffreys'

Jeffreys' priors work well for single-parameter models, but not for models with multidimensional parameters. By analogy with the one-dimensional case, one might construct a naive Jeffreys prior as the joint density:

$$\pi_J(\theta) = |I(\theta)|^{1/2},$$

where $|\cdot|$ denotes the determinant and the (i,j)th element of the Fisher information matrix is given by

$$I(\theta)_{ij} = -E\left[\frac{\partial^2 \log p(X|\theta)}{\partial\theta_i \partial\theta_j}\right].$$

[For more reading: See PhD notes: Objective Bayes Chapter on reference priors, Gelman, et al. (2013)]

Let's see what happens when we apply a Jeffreys' prior for θ to a multivariate Gaussian location model. Suppose

$$X \sim N_p(\theta, I),$$

and we are interested in performing inference on $||\theta||^2$.

- In this case the Jeffreys' prior for θ is flat.
- It turns out that the posterior has the form of a non-central χ^2 distribution with p degrees of freedom.
- The posterior mean given one observation of X is $E(||\theta||^2 \mid X) = ||X||^2 + p$.
- This is not a good estimate because it adds p to the square of the norm of X, whereas we might normally want to shrink our estimate towards zero.
- By contrast, the minimum variance frequentist estimate of $||\theta||^2$ is $||X||^2 - p$.

[To learn more, Decision theory offered this fall, Read TPE, Lehmann and Casella, 2nd Ed.]