Exercise 3

Exercise

Suppose $\{p_\alpha(\theta) : \alpha \in H\}$ is a conjugate family for some generator family $\{p(x|\theta) : \theta \in \Theta\}$. Let $g(\theta)$ be a nonnegative function, and define

$$z(\alpha) = \int p_\alpha(\theta) g(\theta) d\theta.$$

Show that if $0 < z(\alpha) < \infty$ for all $\alpha \in H$, then

$$\left\{ p_\alpha(\theta) g(\theta) / z(\alpha) : \alpha \in H \right\}$$

is also a conjugate family.

Solution

For $\alpha \in H$, define the p.d.f.

$$\pi_\alpha(\theta) = \frac{p_\alpha(\theta)g(\theta)}{z(\alpha)}.$$

Consider data $x_1, \dots, x_n$. Since $\{p_\alpha : \alpha \in H\}$ is a conjugate prior family, then for any $\alpha \in H$, there is an $\alpha' \in H$ such that $p(x_{1:n}|\theta)p_\alpha(\theta) \propto p_{\alpha'}(\theta)$. Thus, using $\pi_\alpha(\theta)$ as the prior results in the posterior

$$\begin{aligned}
p(\theta|x_{1:n}) &\propto p(x_{1:n}|\theta)\pi_\alpha(\theta) \\
&= p(x_{1:n}|\theta)p_\alpha(\theta)\frac{g(\theta)}{z(\alpha)} \\
&\propto p_{\alpha'}(\theta)\frac{g(\theta)}{z(\alpha)} \\
&\propto p_{\alpha'}(\theta)\frac{g(\theta)}{z(\alpha')} \\
&= \pi_{\alpha'}(\theta).
\end{aligned}$$

Therefore, $\{\pi_\alpha : \alpha \in H\}$ is a conjugate prior family.