Lab 7 Solutions, STA 360/602

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1. Consider the following Exponential model for observation(s) $x = (x_1, \ldots, x_n)$.[1]:

$$p(x|a, b) = ab \exp(-abx) I(x > 0)$$

and suppose the prior is

$$p(a, b) = \exp(-a - b) I(a, b > 0).$$

You want to sample from the posterior $p(a, b|x)$.

It is easy to show that the posterior distribution is intractable, hence, we derive the conditional distributions:

$$\begin{aligned} p(\boldsymbol{x}|a, b) &= \prod_{i=1}^{n} p(x_i|a, b) \\ &= \prod_{i=1}^{n} ab \exp(-abx_i) \\ &= (ab)^n \exp\left(-ab \sum_{i=1}^{n} x_i\right). \end{aligned}$$

The function is symmetric for a and b, so we only need to derive $p(a|\boldsymbol{x}, b)$.

This conditional distribution satisfies

$$\begin{aligned} p(a|\boldsymbol{x}, b) &\propto_a p(a, b, \boldsymbol{x}) \\ &= p(\boldsymbol{x}|a, b) p(a, b) \\ &= (ab)^n \exp\left(-ab \sum_{i=1}^{n} x_i\right) \times \exp(-a - b) I(a, b > 0) \\ &\underset{a}{\propto} p(x, a, b) \underset{a}{\propto} a^n \exp(-abn\bar{x} - a) \mathbb{1}(a > 0) = a^{n+1-1} \exp(-(bn\bar{x} + 1)a) \mathbb{1}(a > 0) \underset{a}{\propto} \text{Gamma}(a \mid n + 1,\, bn\bar{x} + 1). \end{aligned}$$

Therefore, $p(a|b, x) = \text{Gamma}(a \mid n + 1,\, bn\bar{x} + 1)$ and by symmetry, $p(b|a, x) = \text{Gamma}(b \mid n + 1,\, an\bar{x} + 1)$.

We now give the Gibbs sampling code

```
knitr::opts_chunk$set(cache=TRUE)
library(MASS)
data <- read.csv("data-exponential.csv", header = FALSE)
#######################################
# This function is a Gibbs sampler
#
# Args
#   start.a: initial value for a
#   start.b: initial value for b
#   n.sims: number of iterations to run
#   data:  observed data, should be in a
           # data frame with one column
```

[1] Please note that in the attached data there are 40 observations, which can be found in data-exponential.csv.

```
#
# Returns:
#   A two column matrix with samples
     #   for a in first column and
# samples for b in second column
########################################
```

```
knitr::opts_chunk$set(cache=TRUE)
sampleGibbs <- function(start.a, start.b, n.sims, data){
  # get sum, which is sufficient statistic. note: sum(x) = n*x_bar.
  x_sum <- sum(data)
  # get n
  n <- nrow(data)
  # create empty matrix, allocate memory for efficiency
  res <- matrix(NA, nrow = n.sims, ncol = 2)
  res[1,] <- c(start.a,start.b)
  for (i in 2:n.sims){
    # sample the values
    res[i,1] <- rgamma(1, shape = n+1,
                       rate = res[i-1,2]*x_sum+1)
    res[i,2] <- rgamma(1, shape = n+1,
                       rate = res[i,1]*x_sum+1)
  }
  return(res)
}
```

We now run the Gibbs sampler and produce some results. In addition to traceplots, running averages such as the one below are a useful heuristic for visually assessing the convergence of the Markov chain.

```
# run Gibbs sampler
n.sims <- 10000
res <- sampleGibbs(.25,.25,n.sims,data)
head(res)
```

```
##             [,1]      [,2]
## [1,] 0.25000000 0.2500000
## [2,] 0.06909933 0.2189269
## [3,] 0.06711379 0.3036429
## [4,] 0.04038261 0.4300210
## [5,] 0.04442655 0.3703045
## [6,] 0.04233643 0.4910783
```

```
dim(res)
```

```
## [1] 10000     2
```

```
res[1,1]
```

```
## [1] 0.25
```