

Module 2: Introduction to Decision Theory

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Agenda

- ▶ What is decision theory?
- ▶ General setup
- ▶ Bayesian approach
- ▶ Frequentist and Integrated Risk
- ▶ Examples

General setup

Assume an unknown state S (a.k.a. the state of nature). Assume

- ▶ we receive an observation x ,
- ▶ we take an action a , and
- ▶ we incur a real-valued loss $\ell(S, a)$.

| | |
|--------------|---------------------|
| S | state (unknown) |
| x | observation (known) |
| a | action |
| $\ell(s, a)$ | loss |

Bayesian approach

- ▶ S is a random variable,
- ▶ the distribution of x depends on S ,
- ▶ and the optimal decision is to choose an action a that minimizes the ***posterior expected loss***,

$$\rho(a, x) = \mathbb{E}(\ell(S, a)|x).$$

In other words, $\rho(a, x) = \sum_s \ell(s, a)p(s|x)$ if S is a discrete random variable, while if S is continuous then the sum is replaced by an integral.

Bayesian approach (continued)

1. A **decision procedure** δ is a systematic way of choosing actions a based on observations x . Typically, this is a deterministic function $a = \delta(x)$ (but sometimes introducing some randomness into a can be useful).
2. A **Bayes procedure** is a decision procedure that chooses an a minimizing the posterior expected loss $\rho(a, x)$, for each x .
3. Note: Sometimes the loss is restricted to be nonnegative, to avoid certain pathologies.

Example 1

1. State: $S = \theta$
2. Observation: $x = x_{1:n}$
3. Action: $a = \hat{\theta}$
4. Loss: $\ell(\theta, \hat{\theta}) = (\theta - \hat{\theta})^2$ (quadratic loss, a.k.a. square loss)

What is the optimal decision rule?

- ▶ Goal: Minimize the posterior risk
- ▶ First note that

$$\ell(\theta, \hat{\theta}) = \theta^2 - 2\theta\hat{\theta} + \hat{\theta}^2$$

- ▶ It then follows that

$$\rho(\hat{\theta}, x_{1:n}) = \mathbb{E}(\ell(\theta, \hat{\theta}) | x_{1:n}) = \mathbb{E}(\theta^2 | x_{1:n}) - 2\hat{\theta}\mathbb{E}(\theta | x_{1:n}) + \hat{\theta}^2,$$

which is a convex function of $\hat{\theta}$.

Setting the derivative with respect to $\hat{\theta}$ equal to 0, and solving, we find that the minimum occurs at $\hat{\theta} = \mathbb{E}(\theta | x_{1:n})$, **the posterior mean**.

Resource allocation for disease prediction

Suppose public health officials in a small city need to decide how much resources to devote toward prevention and treatment of a certain disease, but the fraction θ of infected individuals in the city is unknown.

Resource allocation for disease prediction (continued)

Suppose they allocate enough resources to accomodate a fraction c of the population.

- ▶ If c is too large, there will be wasted resources, while if it is too small, preventable cases may occur and some individuals may go untreated.
- ▶ After deliberation, they tentatively adopt the following loss function:

$$\ell(\theta, c) = \begin{cases} |\theta - c| & \text{if } c \geq \theta \\ 10|\theta - c| & \text{if } c < \theta. \end{cases}$$

Resource allocation for disease prediction (continued)

- ▶ By considering data from other similar cities, they determine a prior $p(\theta)$. For simplicity, suppose $\theta \sim \text{Beta}(a, b)$ (i.e., $p(\theta) = \text{Beta}(\theta|a, b)$), with $a = 0.05$ and $b = 1$.
- ▶ They conduct a survey assessing the disease status of $n = 30$ individuals, x_1, \dots, x_n .

This is modeled as $X_1, \dots, X_n \stackrel{iid}{\sim} \text{Bernoulli}(\theta)$, which is reasonable if the individuals are uniformly sampled and the population is large. Suppose all but one are disease-free, i.e., $\sum_{i=1}^n x_i = 1$.

The Bayes procedure

The **Bayes procedure** is to minimize the posterior expected loss

$$\rho(c, x) = \mathbb{E}(\ell(\theta, c)|x) = \int \ell(\theta, c)p(\theta|x)d\theta$$

where $x = x_{1:n}$.

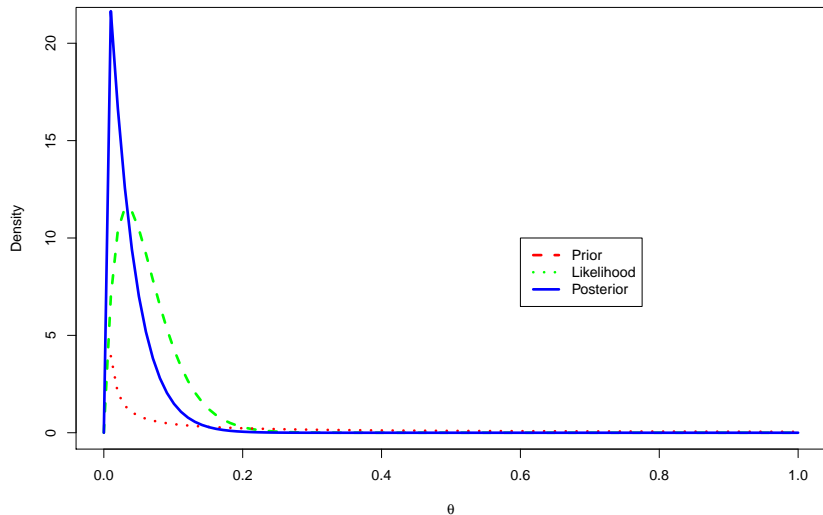
1. We know $p(\theta|x)$ as an updated Beta, so we can numerically compute this integral for each c .
2. Figure 1 shows $\rho(c, x)$ for our example.
3. The minimum occurs at $c \approx 0.08$, so under the assumptions above, this is the optimal amount of resources to allocate.
4. How would one perform a sensitivity analysis of the prior assumptions?

Resource allocation for disease prediction in R

```
# set seed
set.seed(123)

# data
sum_x = 1
n = 30
# prior parameters
a = 0.05; b = 1
# posterior parameters
an = a + sum_x
bn = b + n - sum_x
th = seq(0,1,length.out = 100)
like = dbeta(th, sum_x+1,n-sum_x+1)
prior = dbeta(th,a,b)
post = dbeta(th,sum_x+a,n-sum_x+b)
```

Likelihood, Prior, and Posterior



The loss function

```
# compute the loss given theta and c
loss_function = function(theta, c){
  if (c < theta){
    return(10*abs(theta - c))
  } else{
    return(1 = abs(theta - c))
  }
}
```

Posterior risk

```
# compute the posterior risk given c  
# s is the number of random draws  
# compute the posterior risk given c  
# s is the number of random draws  
posterior_risk = function(c, s = 30000){  
  # random draws from beta distribution  
  theta = rbeta(s, an, bn)  
  
  loss <- apply(as.matrix(theta), 1, loss_function, c)  
  # average values from the loss function  
  risk = mean(loss)  
}
```

Posterior Risk (continued)

```
# a sequence of c in [0, 0.5]  
c = seq(0, 0.5, by = 0.01)  
post_risk <- apply(as.matrix(c), 1, posterior_risk)  
head(post_risk)
```

```
## [1] 0.33917940 0.25367603 0.18868962 0.14489894 0.116931
```


Posterior expected loss/posterior risk for disease prevalence

```
# plot posterior risk against c
```

```
pdf(file="posterior-risk.pdf")  
plot(c, post_risk, type = 'l', col='blue',  
      lwd = 3, ylab = 'p(c, x)' )  
dev.off()
```

```
## pdf
```

```
## 2
```

```
# minimum of posterior risk occurs at c = 0.08
```

```
(c[which.min(post_risk)])
```

```
## [1] 0.08
```

Posterior expected loss/posterior risk for disease prevalence

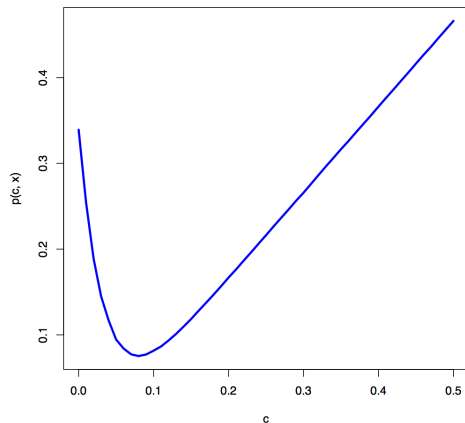


Figure 1:

Frequentist and Integrated Risk

1. Consider a decision problem in which $S = \theta$.
2. The **risk** (or **frequentist risk**) associated with a decision procedure δ is

$$R(\theta, \delta) = \mathbb{E}(\ell(\theta, \delta(X)) \mid \theta = \theta),$$

where X has distribution $p(x|\theta)$. In other words,

$$R(\theta, \delta) = \int \ell(\theta, \delta(x)) p(x|\theta) dx$$

if X is continuous, while the integral is replaced with a sum if X is discrete.

3. The **integrated risk** associated with δ is

$$r(\delta) = \mathbb{E}(\ell(\theta, \delta(X))) = \int R(\theta, \delta) p(\theta) d\theta.$$

Example: Resource allocation, revisited

1. The frequentist risk provides a useful way to compare decision procedures in a prior-free way.
2. In addition to the Bayes procedure above, consider two other possibilities: choosing $c = \bar{x}$ (sample mean) or $c = 0.1$ (constant).

Example: Resource allocation, revisited

3. Figure 2 shows each procedure as a function of $\sum x_i$, the observed number of diseased cases. For the prior we have chosen, the Bayes procedure always picks c to be a little bigger than \bar{x} .

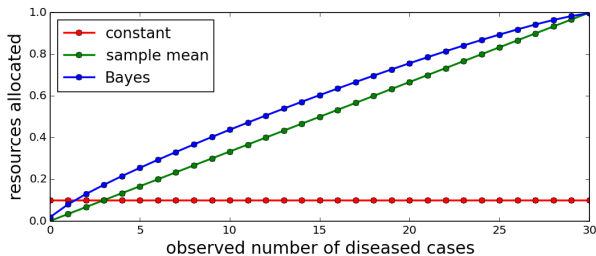


Figure 2: Resources allocated c , as a function of the number of diseased individuals observed, $\sum x_i$, for the three different procedures.

Example: Resource allocation, revisited

4. Figure 3 shows the risk $R(\theta, \delta)$ as a function of θ for each procedure. Smaller risk is better. (Recall that for each θ , the risk is the expected loss, averaging over all possible data sets. The observed data doesn't factor into it at all.)

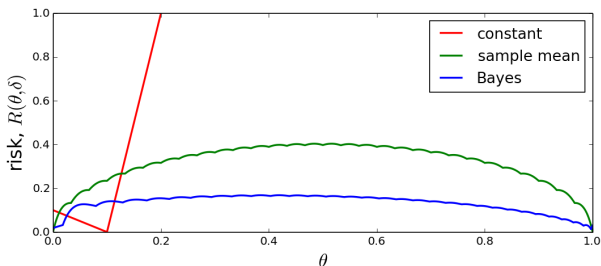


Figure 3: Risk functions for the three different procedures.

Example: Resource allocation, revisited

5. The constant procedure is fantastic when θ is near 0.1, but gets very bad very quickly for larger θ . The Bayes procedure is better than the sample mean for nearly all θ 's. These curves reflect the usual situation—some procedures will work better for certain θ 's and some will work better for others.
6. A decision procedure which is **inadmissible** is one that is dominated everywhere. That is, δ is **inadmissible** if there is no δ' such that

$$R(\theta, \delta') \leq R(\theta, \delta)$$

for all θ and $R(\theta, \delta') < R(\theta, \delta)$ for at least one θ . (A decision procedure that is not **inadmissible** is said to be **admissible**).

7. Bayes procedures are admissible under very general conditions.
8. Admissibility is nice to have, but it doesn't mean a procedure is necessarily good. Silly procedures can still be admissible—e.g., in this example, the constant procedure $c = 0.1$ is admissible too!

Exercise

Consider $X_1, \dots, X_n \mid \theta \stackrel{iid}{\sim} N(\theta, \sigma^2)$ Suppose that we assume very weak prior information on θ . That is, suppose that $p(\theta) \propto 1$.

- ▶ What does the likelihood and prior distribution look like (what is your intuition)? Now let's verify this in markdown.
- ▶ What is the posterior distribution for θ ?

Exercise (continued)

$$p(x_{1:n} \mid \theta) = \left(\frac{1}{\sqrt{2\pi\sigma^2}} \right)^n \exp \left(- \frac{1}{2\sigma^2} \sum_i (x_i - \theta)^2 \right) \quad (1)$$

$$\left(\frac{1}{\sqrt{2\pi\sigma^2}} \right)^n \exp \left(- \frac{1}{2\sigma^2} \sum_i (x_i + \bar{x} - \bar{x} - \theta)^2 \right) \quad (2)$$

$$\left(\frac{1}{\sqrt{2\pi\sigma^2}} \right)^n \exp \left(- \frac{n}{2\sigma^2} (\theta - \bar{x})^2 \right) \quad (3)$$