Intro to Bayesian Methods

Rebecca C. Steorts
Predictive Modeling and Data Mining: STA 521

October 2015

Topics

- Why Bayes'
- Bayes' Theorem
- Hierarchical models
- Conjugacy
- Examples
- Lab: applied examples

- Why should we learn about Bayesian concepts?
- Natural if thinking about unknown parameters as random.
- They naturally give a full distribution when we perform an update.
- We automatically get uncertainty quantification.
- Drawbacks: They can be slow and inconsistent.

Suppose we have some noisy data. How can we recover the underlying structure of the data?

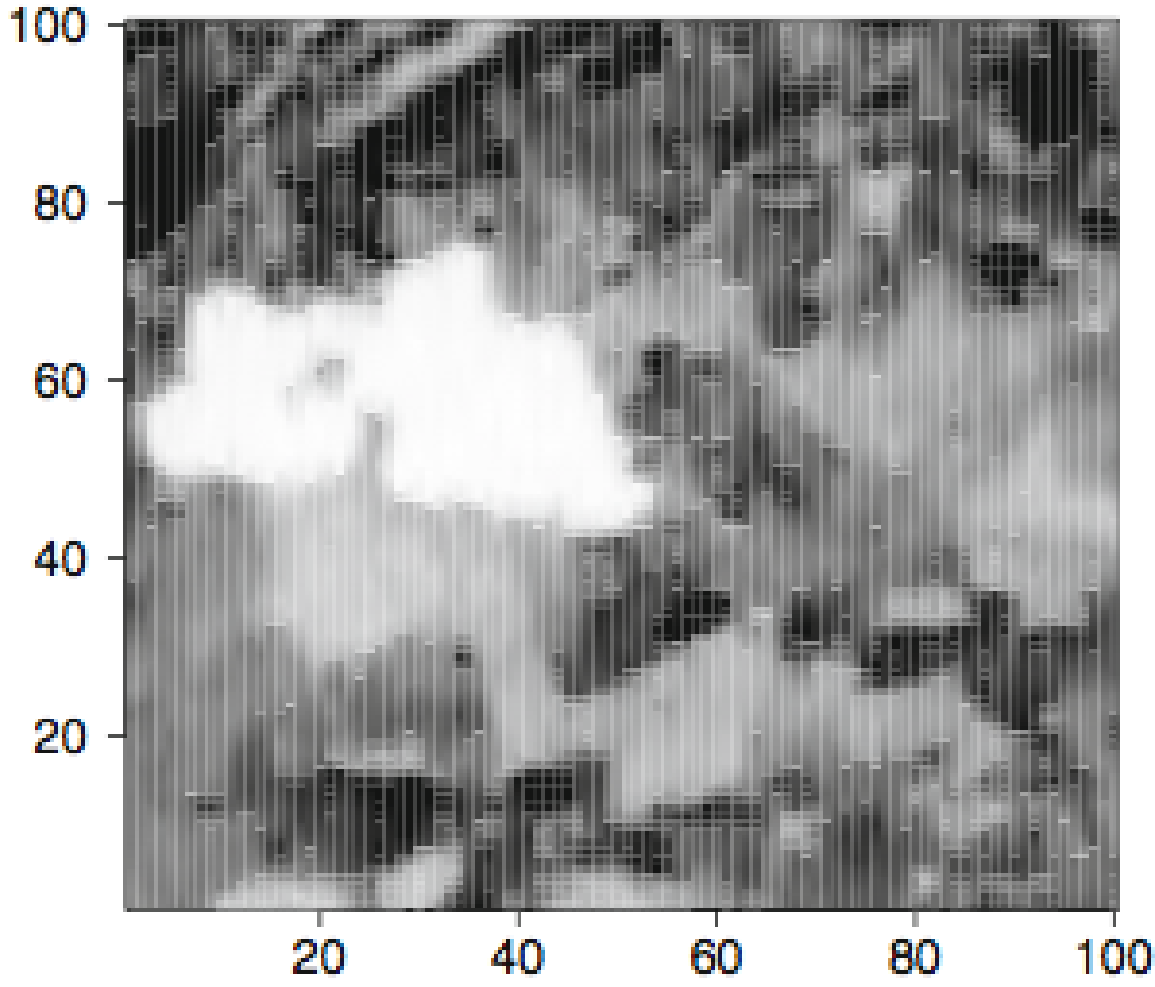

Figure 1: Satellite image of the lake of Menteith, Scotland

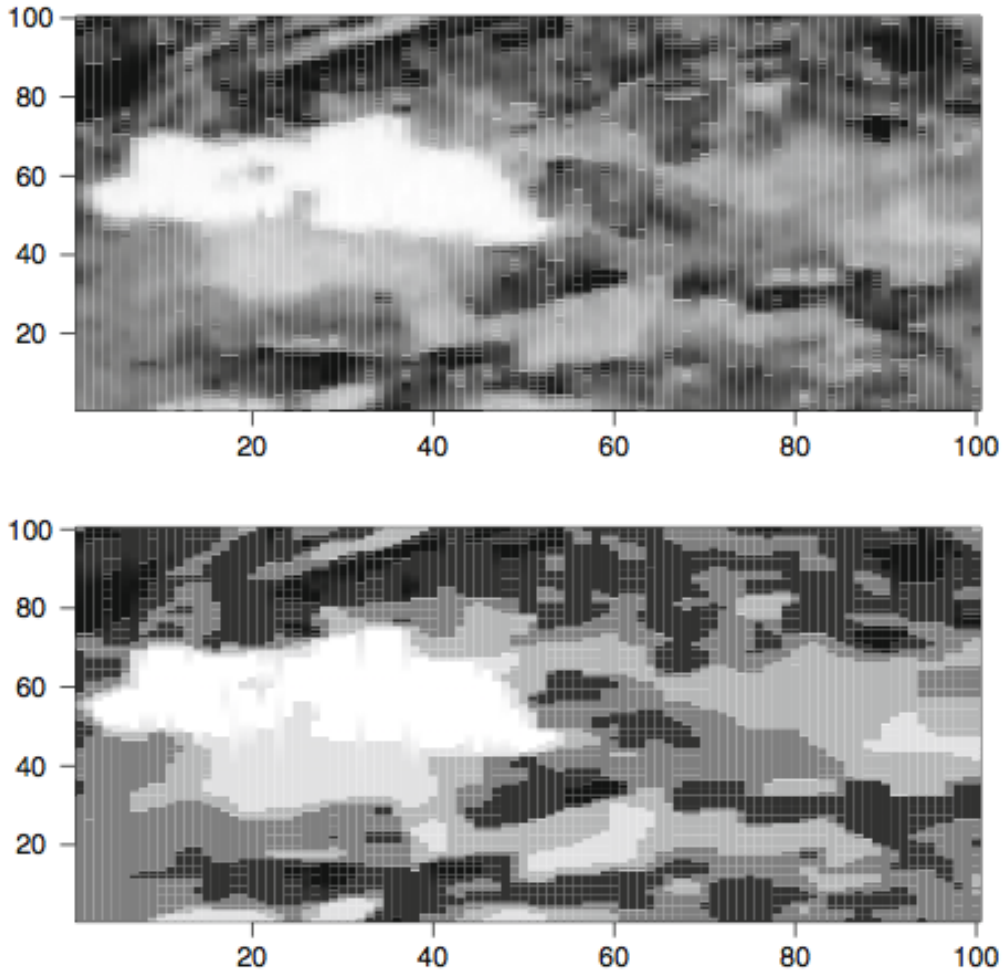

Figure 2: Dataset Menteith: (top) the observed image and (bottom) Segmented image

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$p(\theta)$	prior

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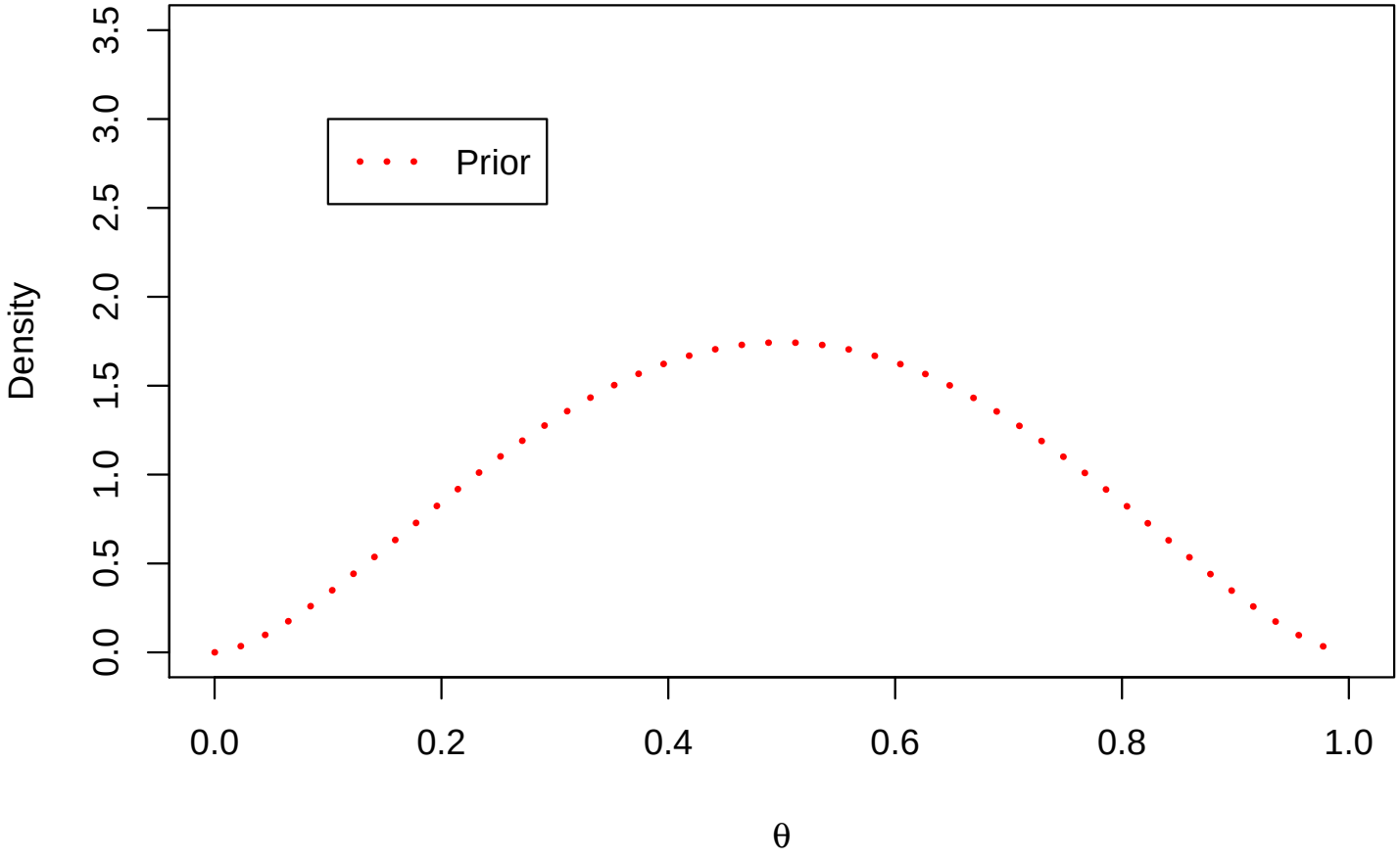
- We can plot the prior and likelihood distributions in `R` and then see how the two mix to form the posterior distribution.


Prior
Density
0.0
0.5
1.0
1.5
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2.5
3.0
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0.0
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0.4
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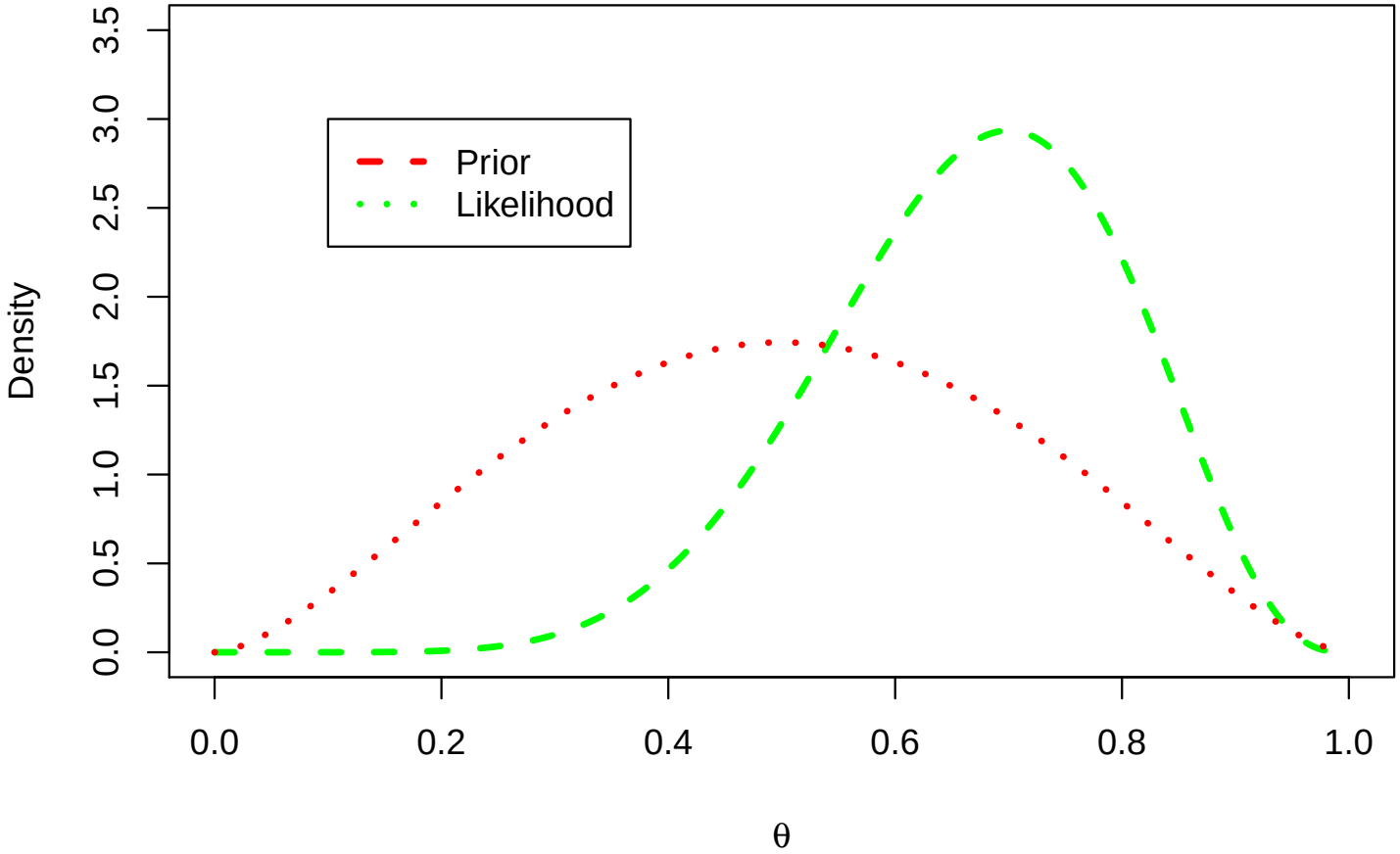
0.8
1.0
θ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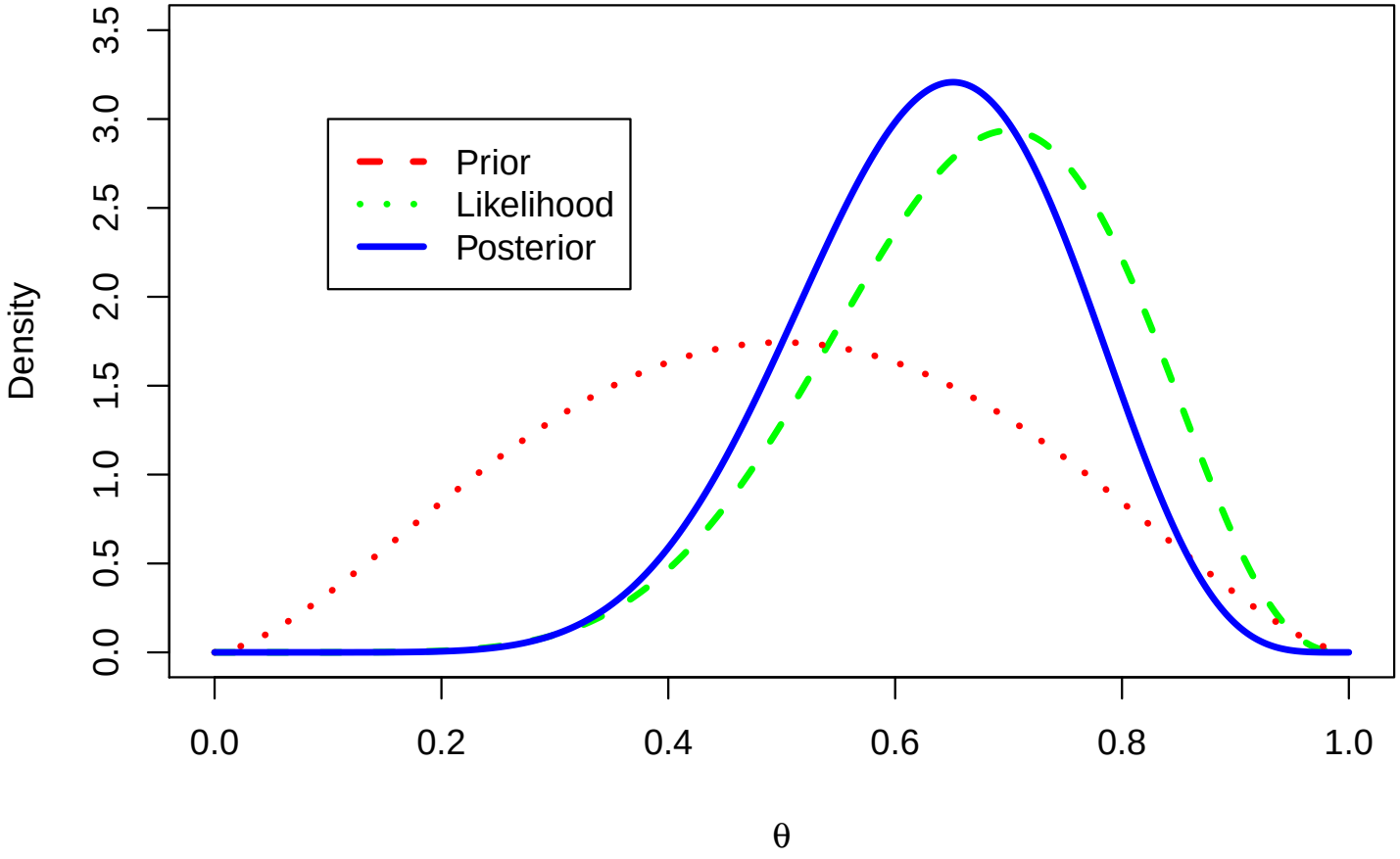
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- Bayesians treat an unknown parameter θ as *random*.
- Frequentists treat θ as unknown but *fixed*.

Stopping Rule

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- The proper way to do this depends on exactly which of the following two experiments was actually performed:

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So we fail to reject H_0 at $\alpha = 0.05$.

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&= 1 - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}\right) = \frac{1}{32} = 0.03125 < 0.05.
\end{aligned}$$

So we reject H_0 at $\alpha = 0.05$.

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- The result our hypothesis test depends on whether we would have stopped flipping if we had gotten a tails sooner.
- The tests are dependent on what we call the *stopping rule*.

- The likelihood for the actual value of x that was observed is the same for both experiments (up to a constant):

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- A Bayesian approach would take the data into account only through this likelihood.
- This would provide the same answers regardless of which experiment was being performed.

The Bayesian analysis is independent of the stopping rule since it only depends on the likelihood (show this at home!).

Hierarchical Bayesian Models

In a hierarchical Bayesian model, rather than specifying the prior distribution as a single function, we specify it as a hierarchy.

Hierarchical Bayesian Models

$$
\begin{aligned}
X|\theta &\sim f(x|\theta) \\
\Theta|\gamma &\sim \pi(\theta|\gamma) \\
\Gamma &\sim \phi(\gamma),
\end{aligned}
$$

where we assume that $\phi(\gamma)$ is known and not dependent on any other unknown *hyperparameters*.

Conjugate Distributions

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- Then let P denote the class of prior distributions on θ.
- Then P is said to be conjugate to F if for every $p(\theta) \in P$ and $p(y|\theta) \in F$, $p(y|\theta) \in P$.

Simple definition: A family of priors such that, upon being multiplied by the likelihood, yields a posterior in the same family.

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\end{aligned}$$

$$\theta|x \sim \text{Beta}(x+a, n-x+b).$$

How Much Do You Sleep

We are interested in a population of American college students and the proportion of the population that sleep at least eight hours a night, which we denote by θ.

Prior Data

- *The Gamecock*, at the USC printed an internet article "College Students Don't Get Enough Sleep" (2004).
  - Most students spend six hours sleeping each night.
- 2003: University of Notre Dame's paper, *Fresh Writing*.
  - The article reported took random sample of 100 students:
  - "approximately 70% reported to receiving only five to six hours of sleep on the weekdays,
  - 28% receiving seven to eight,
  - and only 2% receiving the healthy nine hours for teenagers."

- Have a random sample of 27 students is taken from UF.
- 11 students record that they sleep at least eight hours each night.
- Based on this information, we are interested in estimating θ.

- From USC and UND, believe it's probably true that most college students get less than eight hours of sleep.
- Want our prior to assign most of the probability to values of $\theta < 0.5$.
- From the information given, we decide that our best guess for θ is 0.3, although we think it is very possible that θ could be any value in $[0, 0.5]$.

- Given this information, we believe that the median of θ is 0.3 and the 90th percentile is 0.5.
- Knowing this allows us to estimate the unknown values of a and b.
- After some calculations we find that $a = 3.3$ and $b = 7.2$. How did we actually calculate a and b?

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We would need to solve the following equations:

$$\int_0^{0.3} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \theta^{a-1}(1-\theta)^{b-1}\, d\theta = 0.5$$

$$\int_0^{0.5} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \theta^{a-1}(1-\theta)^{b-1}\, d\theta = 0.9$$

- In non-calculus language, this means the 0.5 quantile (50th percentile) $= 0.3$. The 0.9 quantile (90th percentile) $= 0.5$.
- We can easily solve this numerically in `R` using a numerical solver `BBsolve`.

Since you won't have used this command before, you'll need to install the package `BB` and load the library.

Here is the code in R to find a and b.

```
## install the BBsolve package
install.packages("BB", repos="http://cran.r-project.org")
library(BB)
fn = function(x){qbeta(c(0.5,0.9),x[1],x[2])-c(0.3,0.5)}
BBsolve(c(1,1),fn)

## alternative way
myfn <- function(shape){
test <- pbeta(q = c(0.3, 0.5), shape1 = shape[1],
 shape2 = shape[2]) - c(0.5, 0.9)
return(test)
}
BBsolve(c(1,1), myfn)
```

Using our calculations from the Beta-Binomial our model is

$$
\begin{aligned}
X \mid \theta &\sim \text{Binomial}(27, \theta) \\
\theta &\sim \text{Beta}(3.3, 7.2) \\
\theta \mid x &\sim \text{Beta}(x + 3.3, 27 - x + 7.2) \\
\theta \mid 11 &\sim \text{Beta}(14.3, 23.2)
\end{aligned}
$$

```
th = seq(0,1,length=500)
a = 3.3
b = 7.2
n = 27
x = 11
prior = dbeta(th,a,b)
like = dbeta(th,x+1,n-x+1)
post = dbeta(th,x+a,n-x+b)
pdf("sleep.pdf",width=7,height=5)
plot(th,post,type="l",ylab="Density",lty=2,lwd=3,
xlab = expression(theta))
lines(th,like,lty=1,lwd=3)
lines(th,prior,lty=3,lwd=3)
legend(0.7,4,c("Prior","Likelihood","Posterior"),
lty=c(3,1,2),lwd=c(3,3,3))
dev.off()
```

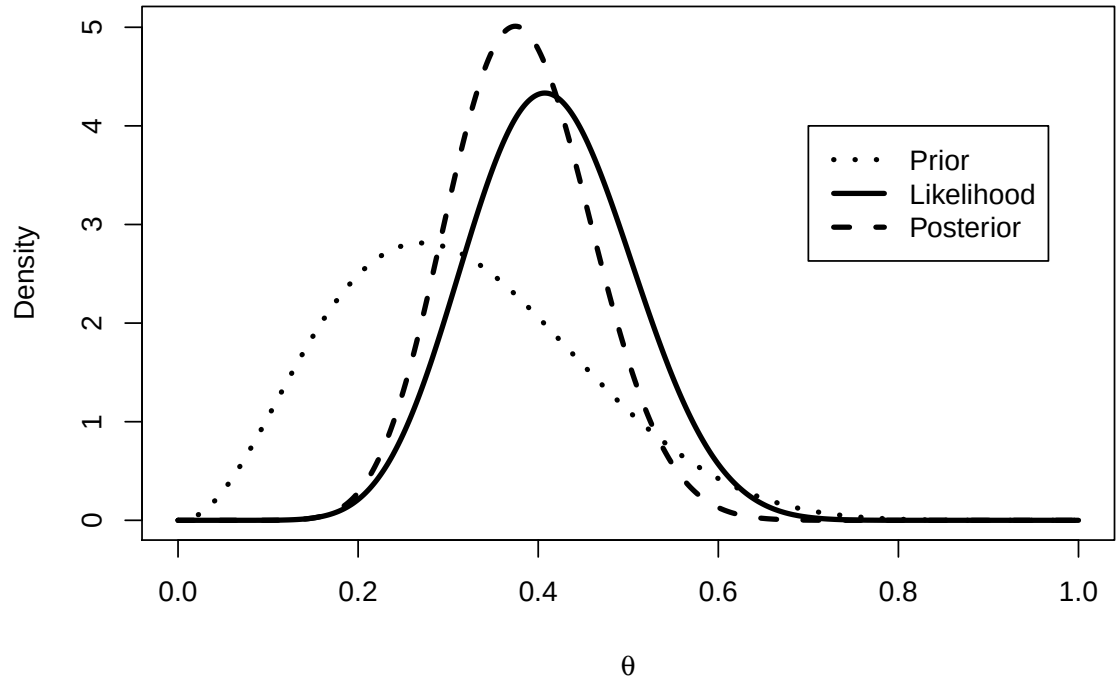

Figure 3: Likelihood $p(X|\theta)$, Prior $p(\theta)$, and Posterior Distribution $p(\theta|X)$