Exercise 2

Exercise

Suppose the data is modeled as i.i.d. $\text{Exp}(\theta)$, and the prior is

$$p(\theta) = \text{Gamma}(\theta|a, b) = \frac{b^a}{\Gamma(a)}\theta^{a-1}e^{-b\theta}\mathbb{1}(\theta > 0).$$

We know that the posterior is

$$p(\theta|x_{1:n}) = \text{Gamma}(\theta|\alpha, \beta)$$

where $\alpha = a + n$ and $\beta = b + \sum_{i=1}^{n} x_i$.

What is the posterior predictive density $p(x_{n+1}|x_{1:n})$? Give your answer as a closed-form expression (not an integral). Next, find the marginal likelihood $p(x_{1:n})$.

Solution

Denoting $x' = x_{n+1}$ for short, the posterior predictive is

$$
\begin{aligned}
p(x'|x_{1:n}) &= \int p(x'|\theta)p(\theta|x_{1:n})d\theta \\
&= \int_0^\infty \theta e^{-\theta x'} \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta} d\theta \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \theta^{(\alpha+1)-1} e^{-(\beta+x')\theta} d\theta \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{(\beta+x')^{\alpha+1}} \int \mathrm{Gamma}(\theta \mid \alpha+1,\, \beta+x') d\theta \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{(\beta+x')^{\alpha+1}}.
\end{aligned}
$$

The marginal likelihood is

$$
\begin{aligned}
p(x_{1:n}) &= \int p(x_{1:n}|\theta)p(\theta)d\theta \\
&= \int_0^\infty \theta^n e^{-\theta \sum x_i} \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta} d\theta \\
&= \frac{b^a}{\Gamma(a)} \int_0^\infty \theta^{a+n-1} \exp\big(-(b+\textstyle\sum x_i)\theta\big) d\theta \\
&= \frac{b^a}{\Gamma(a)} \frac{\Gamma(a+n)}{(b+\sum x_i)^{a+n}} \int \mathrm{Gamma}\big(\theta \mid a+n,\, b+\textstyle\sum x_i\big) d\theta \\
&= \frac{b^a}{\Gamma(a)} \frac{\Gamma(a+n)}{(b+\sum x_i)^{a+n}} = \frac{b^a}{\Gamma(a)} \frac{\Gamma(\alpha)}{\beta^\alpha}.
\end{aligned}
$$

The marginal likelihood can also be found by using Bayes' theorem: for any θ,

$$
p(x_{1:n}) = \frac{p(x_{1:n}|\theta)p(\theta)}{p(\theta|x_{1:n})} = \frac{\frac{b^a}{\Gamma(a)} \theta^{\alpha-1} e^{-\beta\theta}}{\mathrm{Gamma}(\theta|\alpha,\beta)} = \frac{\frac{b^a}{\Gamma(a)}}{\frac{\beta^\alpha}{\Gamma(\alpha)}}.
$$