Module 4: Introduction to the Normal Gamma Model

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Agenda

- NormalGamma distribution
- posterior derivation
- application to IQ scores

NormalGamma distribution

Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \mathcal{N}(\mu, \lambda^{-1})$ and assume **both**

- the mean μ and
- the precision $\lambda = 1/\sigma^2$ are **unknown**.

The NormalGamma(m, c, a, b) distribution, with $m \in \mathbb{R}$ and $c, a, b > 0$, is a joint distribution on (μ, λ) obtained by letting

$$\begin{aligned} \boldsymbol{\mu}|\lambda &\sim \mathcal{N}(m, (c\lambda)^{-1}) \\ \boldsymbol{\lambda} &\sim \text{Gamma}(a, b) \end{aligned}$$

In other words, the joint p.d.f. is

$$p(\mu, \lambda) = p(\mu|\lambda)p(\lambda) = \mathcal{N}(\mu \mid m, (c\lambda)^{-1})\,\text{Gamma}(\lambda \mid a, b)$$

which we will denote by NormalGamma$(\mu, \lambda \mid m, c, a, b)$.

NormalGamma distribution (continued)

It turns out that this provides a **conjugate prior** for (μ, λ).

One can show the posterior is

$$\boldsymbol{\mu}, \boldsymbol{\lambda} | x_{1:n} \sim \text{NormalGamma}(M, C, A, B) \tag{1}$$

i.e., $p(\mu, \lambda | x_{1:n}) = \text{NormalGamma}(\mu, \lambda \mid M, C, A, B)$, where

$$\begin{aligned}
M &= \frac{cm + \sum_{i=1}^n x_i}{c + n} \\
C &= c + n \\
A &= a + n/2 \\
B &= b + \tfrac{1}{2}\big(cm^2 - CM^2 + \textstyle\sum_{i=1}^n x_i^2\big).
\end{aligned}$$

NormalGamma distribution (continued)

For interpretation, B can also be written (by rearranging terms) as

$$B = b + \frac{1}{2}\sum_{i=1}^{n}(x_i - \bar{x})^2 + \frac{1}{2}\frac{cn}{c+n}(\bar{x} - m)^2. \tag{2}$$

NormalGamma distribution (continued)

- M: Posterior mean for μ. It is a weighted average (convex combination) of the prior mean and the sample mean:

$$M = \frac{c}{c+n} m + \frac{n}{c+n} \bar{x}.$$

- C: "Sample size" for estimating μ. (The standard deviation of $\mu|\lambda$ is $\lambda^{-1/2}/\sqrt{C}$.)
- A: Shape for posterior on λ. Grows linearly with sample size.
- B: Rate (1/scale) for posterior on λ. Equation 2 decomposes B into the prior variation, observed variation (sample variance), and variation between the prior mean and sample mean:

$$\begin{aligned} B &= (\text{prior variation}) + \tfrac{1}{2} n (\text{observed variation}) \qquad (3) \\ &\quad + \tfrac{1}{2} \tfrac{cn}{c+n} (\text{variation bw means}). \qquad (4) \end{aligned}$$

Posterior derivation

First, consider the NormalGamma density. Dropping constants of proportionality, multiplying out $(\mu - m)^2 = \mu^2 - 2\mu m + m^2$, and collecting terms, we have

$$
\begin{aligned}
&\mathrm{NormalGamma}(\mu, \lambda \mid m, c, a, b) && (5)\\
&= \mathcal{N}(\mu \mid m, (c\lambda)^{-1})\, \mathrm{Gamma}(\lambda \mid a, b) \\
&= \sqrt{\frac{c\lambda}{2\pi}} \exp\Big(-\tfrac{1}{2} c\lambda(\mu - m)^2\Big) \frac{b^a}{\Gamma(a)} \lambda^{a-1} \exp(-b\lambda) \\
&\underset{\mu,\lambda}{\propto} \lambda^{a-1/2} \exp\Big(-\tfrac{1}{2}\lambda(c\mu^2 - 2cm\mu + cm^2 + 2b)\Big). && (6)
\end{aligned}
$$

Posterior derivation (continued)

Similarly, for any x,

$$\mathcal{N}(x \mid \mu, \lambda^{-1}) = \sqrt{\frac{\lambda}{2\pi}} \exp\Big(-\tfrac{1}{2}\lambda(x-\mu)^2\Big)$$
$$\underset{\mu,\lambda}{\propto} \lambda^{1/2} \exp\Big(-\tfrac{1}{2}\lambda(\mu^2 - 2x\mu + x^2)\Big). \qquad (7)$$

Posterior derivation (continued)

$$
\begin{aligned}
& p(\mu, \lambda | x_{1:n}) \\
& \underset{\mu,\lambda}{\propto} p(\mu, \lambda) p(x_{1:n} | \mu, \lambda) \\
& \underset{\mu,\lambda}{\propto} \lambda^{a-1/2} \exp\Big(-\tfrac{1}{2}\lambda(c\mu^2 - 2cm\mu + cm^2 + 2b)\Big) \\
& \qquad \times \lambda^{n/2} \exp\Big(-\tfrac{1}{2}\lambda(n\mu^2 - 2(\textstyle\sum x_i)\mu + \sum x_i^2)\Big) \\
& = \lambda^{a+n/2-1/2} \exp\Big(-\tfrac{1}{2}\lambda\big((c+n)\mu^2 - 2(cm + \textstyle\sum x_i)\mu + cm^2 + 2b + \sum x_i^2\big)\Big) \\
& \overset{(a)}{=} \lambda^{A-1/2} \exp\Big(-\tfrac{1}{2}\lambda\big(C\mu^2 - 2CM\mu + CM^2 + 2B\big)\Big) \\
& \overset{(b)}{\propto} \mathrm{NormalGamma}(\mu, \lambda \mid M, C, A, B)
\end{aligned}
$$

where step (b) is by Equation 6, and step (a) holds if $A = a + n/2$, $C = c + n$, $CM = (cm + \sum x_i)$, and

$$CM^2 + 2B = cm^2 + 2b + \sum x_i^2.$$

Posterior derivation (continued)

This choice of A and C match the claimed form of the posterior, and solving for M and B, we get $M = (cm + \sum x_i)/(c + n)$ and

$$B = b + \tfrac{1}{2}(cm^2 - CM^2 + \sum x_i^2),$$

as claimed.

Alternative derivation: complete the square all way (longer and much more tedious).

Do a teacher's expectations influence student achievement?

Do a teacher's expectations influence student achievement? In a famous study, Rosenthal and Jacobson (1968) performed an experiment in a California elementary school to try to answer this question. At the beginning of the year, all students were given an IQ test. For each class, the researchers randomly selected around 20% of the students, and told the teacher that these students were "spurters" that could be expected to perform particularly well that year. (This was not based on the test—the spurters were randomly chosen.) At the end of the year, all students were given another IQ test. The change in IQ score for the first-grade students was:[1]

[1] The original data is not available. This data is from the `ex1321` dataset of the `R` package `Sleuth3`, which was constructed to match the summary statistics and conclusions of the original study.

Do a teacher's expectations influence student achievement?

```
## Warning: package 'dplyr' was built under R version 3.4.

##
## Attaching package: 'dplyr'

## The following objects are masked from 'package:plyr':
##
##     arrange, count, desc, failwith, id, mutate, rename,
##     summarize

## The following objects are masked from 'package:stats':
##
##     filter, lag

## The following objects are masked from 'package:base':
##
##     intersect, setdiff, setequal, union
```

Spurters/Control Data

```
#spurters
x <- c(18, 40, 15, 17, 20, 44, 38)
#control
y <- c(-4, 0, -19, 24, 19, 10, 5, 10,
      29, 13, -9, -8, 20, -1, 12, 21,
      -7, 14, 13, 20, 11, 16, 15, 27,
      23, 36, -33, 34, 13, 11, -19, 21,
      6, 25, 30,22, -28, 15, 26, -1, -2,
      43, 23, 22, 25, 16, 10, 29)
iqData <- data.frame(Treatment =
      c(rep("Spurters", length(x)),
      rep("Controls", length(y))),
      Gain = c(x, y))
```

An initial exploratory analysis

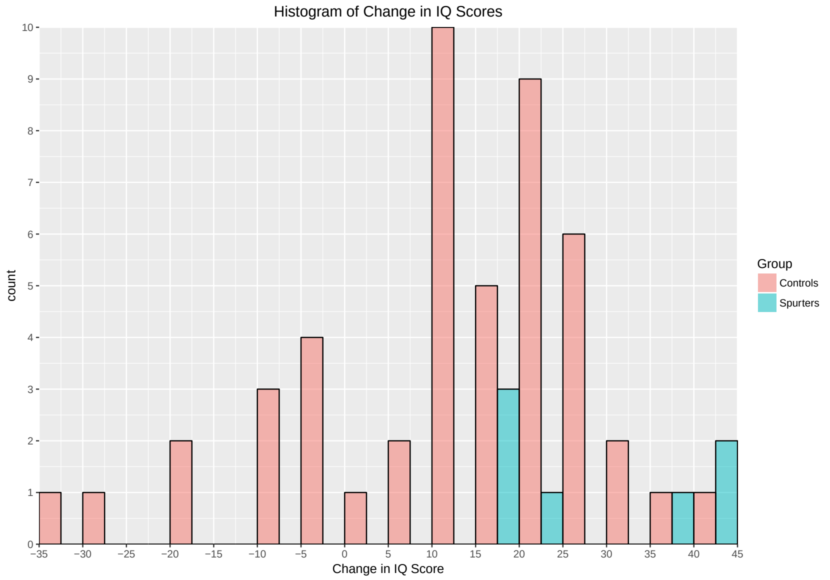
Plot the **number of students** versus the **change in IQ score** for the two groups. How strongly does this data support the hypothesis that the teachers' expectations caused the spurters to perform better than their classmates?

Histogram of Change in IQ Scores

```
xLimits = seq(min(iqData$Gain) - (min(iqData$Gain) %% 5),
              max(iqData$Gain) + (max(iqData$Gain) %% 5),
              by = 5)

ggplot(data = iqData, aes(x = Gain,
                          fill = Treatment,
                          colour = I("black"))) +
  geom_histogram(position = "dodge", alpha = 0.5,
                 breaks = xLimits, closed = "left")+
  scale_x_continuous(breaks = xLimits,
                     expand = c(0,0))+
  scale_y_continuous(expand = c(0,0),
                     breaks = seq(0, 10, by = 1))+
  ggtitle("Histogram of Change in IQ Scores") +
  labs(x = "Change in IQ Score", fill = "Group") +
  theme(plot.title = element_text(hjust = 0.5))
```

Histogram of Change in IQ Scores



IQ Tests and Modeling

IQ tests are purposefully calibrated to make the scores normally distributed, so it makes sense to use a normal model here:

$$\text{spurters: } X_1, \ldots, X_{n_S} \overset{\text{iid}}{\sim} \mathcal{N}(\mu_S, \lambda_S^{-1})$$
$$\text{controls: } Y_1, \ldots, Y_{n_C} \overset{\text{iid}}{\sim} \mathcal{N}(\mu_C, \lambda_C^{-1}).$$

- We are interested in the difference between the means—in particular, is $\mu_S > \mu_C$?
- We don't know the standard deviations $\sigma_S = \lambda_S^{-1/2}$ and $\sigma_C = \lambda_C^{-1/2}$, and the sample seems too small to estimate them very well.

IQ Tests and Modeling

On the other hand, it is easy using a Bayesian approach: we just need to compute the posterior probability that $\mu_S > \mu_C$:

$$\mathbb{P}(\boldsymbol{\mu}_S > \boldsymbol{\mu}_C \mid x_{1:n_S}, y_{1:n_C}).$$

Let's use independent NormalGamma priors:

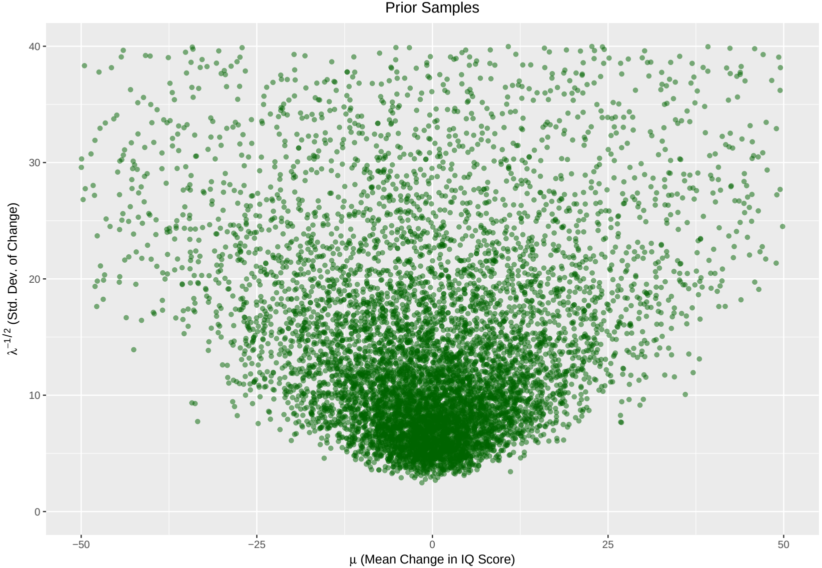
$$\text{spurters: } (\boldsymbol{\mu}_S, \boldsymbol{\lambda}_S) \sim \text{NormalGamma}(m, c, a, b)$$
$$\text{controls: } (\boldsymbol{\mu}_C, \boldsymbol{\lambda}_C) \sim \text{NormalGamma}(m, c, a, b)$$

Hyperparameter settings

- $m = 0$ (Don't know whether students will improve or not, on average.)
- $c = 1$ (Unsure about how big the mean change will be—prior certainty in our choice of m assessed to be equivalent to one datapoint.)
- $a = 1/2$ (Unsure about how big the standard deviation of the changes will be.)
- $b = 10^2 a$ (Standard deviation of the changes expected to be around $10 = \sqrt{b/a} = \mathbb{E}(\lambda)^{-1/2}$.)

Prior Samples

```
## Warning: Removed 2148 rows containing missing values (ge
```



Original question

"What is the posterior probability that $\mu_S > \mu_C$?"

- The easiest way to do this is to take a bunch of samples from each of the posteriors, and see what fraction of times we have $\mu_S > \mu_C$.
- This is an example of a Monte Carlo approximation (much more to come on this in the future).
- To do this, we draw $N = 10^6$ samples from each posterior:

$$(\mu_S^{(1)}, \lambda_S^{(1)}), \ldots, (\mu_S^{(N)}, \lambda_S^{(N)}) \sim \text{NormalGamma}(A_1, B_1, C_1, D_1)$$
$$(\mu_C^{(1)}, \lambda_C^{(1)}), \ldots, (\mu_C^{(N)}, \lambda_C^{(N)}) \sim \text{NormalGamma}(A_2, B_2, C_2, D_2)$$

Original question (continued)

What are the updated posterior values?

and obtain the approximation

$$\mathbb{P}(\boldsymbol{\mu}_S > \boldsymbol{\mu}_C \mid x_{1:n_S}, y_{1:n_C}) \approx \frac{1}{N} \sum_{i=1}^{N} I(\mu_S^{(i)} > \mu_C^{(i)}) = ??.$$

In lab this week, you will explore this data set more and analyze this in your homework assignment.