Practice Problems, Part I

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1. Let $X_1, \ldots, X_n$ be iid Poisson(θ) variables, where $\theta \in (0, \infty)$. Let $L(\theta, \delta) = (\theta - \delta)^2/\theta$. Assuming the prior,
$$g(\theta) = \frac{\exp\{-\theta\alpha\}\alpha^{\beta}\theta^{\beta-1}}{\Gamma(\beta)} I_{[\theta>0]},$$
where $\alpha > 0$ and $\beta > 0$ are given. Show that the Bayes estimator of θ is given by
$$h(\boldsymbol{X}) = \begin{cases} \dfrac{\sum_i X_i + \beta - 1}{n + \alpha} & \text{if } \sum_i X_i + \beta - 1 > 0 \\ 0 & \text{if } \textit{otherwise.} \end{cases}$$

2. Suppose $X \mid p \sim \text{Bin}(n, p)$ and that $p \sim \text{Beta}(a, b)$.
   (a) Show that the marginal distribution of X is the beta-binomial distribution with mass function
   $$m(x) = \binom{n}{x} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(x+a)\Gamma(n+b-x)}{\Gamma(n+a+b)}.$$
   (b) Show that the mean and variance of the beta-binomial is given by $EX = \dfrac{na}{a+b}$ and $\text{V}X = n\left(\dfrac{a}{a+b}\right)\left(\dfrac{b}{a+b}\right)\left(\dfrac{a+b+n}{a+b+1}\right)$.
   Hint: For part(b): Use the formulas for iterated expectation and iterated variance.

3. DasGupta (1994) presents an identity relating the Bayes risk to bias, which illustrates that a small bias can help achieve a small Bayes risk. Let $X \sim f(x|\theta)$ and $\theta \sim \pi(\theta)$. The Bayes estimator under squared error loss is $\hat{\delta} = E(\theta|X)$. Show that the Bayes risk of $\hat{\delta}$ can be written
$$r(\pi, \hat{\delta}) = \int_{\Theta}\int_{\mathcal{X}} [\theta - \hat{\delta}(X)]^2 f(x|\theta)\pi(\theta) dx\, d\theta = -\int_{\Theta} \theta\, b(\theta)\pi(\theta)\, d\theta$$
where $b(\theta) = E[\hat{\delta}|\theta] - \theta$ is the bias of $\hat{\delta}$.

4. Suppose that
$$\begin{aligned} X|\theta &\sim f(x|\theta) \\ \theta|\lambda &\sim \pi(\theta|\lambda) \\ \lambda &\sim \pi(\lambda). \end{aligned}$$
Using the HB model above,

(a) prove that $E[\theta|x] = E[\ E[\theta|x,\lambda]\ ]$.

(b) prove that $V[\theta|x] = E[\ V[\theta|x,\lambda]\ ] + V[\ E[\theta|x,\lambda]\ ]$.

Remark: when proving (a) and (b) above, you may show this two ways, either by integrals in which say what you are integrating over or you may simply just use expectations (and in this case specifying what you are taking an expectation over).

5. Albert and Gupta (1985) investigate theory and application of the hierarchical model

$$\begin{aligned} X_i|\theta_i &\overset{ind}{\sim} \text{Bin}(n,\theta_i),\ i = 1,\dots,p \\ \theta_i|\eta &\sim \text{Beta}(k\eta, k(1-\eta)),\ k \text{ known} \\ \eta &\sim \text{Uniform}(0,1). \end{aligned}$$

(a) Show that

$$E(\theta_i|x) = \frac{n}{n+k}\frac{x_i}{n} + \frac{k}{n+k}E(\eta|x)$$

and

$$V(\theta_i|x) = \frac{E[\theta_i|x](1-E[\theta_i|x])}{n+k+1} + \frac{k^2V(\eta|x)}{(n+k)(n+k+1)}.$$

Hint: You should show along the way that

$$V(\theta_i|x) = \frac{x_i(n+k-x_i) + E(\eta|x)k(n+k-2x_i) - k^2E(\eta^2|x)}{(n+k)^2(n+k+1)} + \frac{k^2V(\eta|x)}{(n+k)^2}.$$

General Remark: Note that $E(\eta|x)$ and $V(\eta|x)$ are not expressible in a simple form and hence you can leave them as such.

(b) Unconditionally on η, the θ_i's have conditional covariance

$$\text{Cov}(\theta_i,\theta_j|x) = \left(\frac{k}{n+k}\right)^2 V(\eta|x) \text{ for } i \neq j.$$

Show this.

(c) Ignoring the prior on η, show how to construct an EB estimator of θ_i. Again, this is not expressible in a simple form. That is, simply derive the marginal distribution and then explain using software how you would find an estimator for η. Then give a *simple* construction for the EB estimator.