UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE	:	**MATH0094**
ASSESSMENT PATTERN	:	**MATH0094A7PC**
MODULE NAME	:	**MATH0094 - Market Risk and Portfolio Theory**
LEVEL:	:	**Postgraduate**
DATE	:	**27/04/2021**
TIME	:	**10:00**

This paper is suitable for candidates who attended classes for this module in the following academic year(s):

Year
2020/21

Additional material	
Special instructions	
Exam paper word count	

TURN OVER

Department of Mathematics
University College London

MATH0094 Market Risk and Portfolio Theory

MSc Examination

2020-2021

The exam contains FOUR questions. Each question is worth 25 marks.

Question 1. Consider a probability space with $\Omega = \{1, 2, 3\}$, the power sigma algebra $\mathcal{F} = \mathcal{P}(\Omega)$ and uniform probability. On this space, we define a market model in one period conformed by two assets: a risk-free asset with return $R_1^0 = 6/5$, and a risky asset with

$$S_0^1 = 5 \qquad\qquad S_1^1(\omega) = \begin{cases} 3 & \text{if } \omega = 1 \\ 6 & \text{if } \omega = 2 \\ 9 & \text{if } \omega = 3 \end{cases}.$$

(a) Obtain all martingale measures that can be defined on this market. State if the market is arbitrage-free and/or complete.

[10 marks]

(b) Consider a put option on the risky asset with strike 7 expiring at the end of the period (recall that the payoff is in this case $(7 - S_1^1)^+$). State what an arbitrage-free price is and find all possible arbitrage-free prices for this option.

[8 marks]

(c) Can we find a strategy with initial value 5/3 that super-hedges the option defined in the previous item? Provide it explicitly or argue why it cannot exist.

[7 marks]

[Total: 25 marks]

Question 2. Consider a random variable with an exponential distribution $X \sim \text{Exp}[\lambda]$ (i.e. it has pdf $f_X(x) = \lambda e^{-\lambda x} \mathbb{1}_{x>0}$).

(a) For $c \in \mathbb{R}$, calculate $\mathbb{E}[e^{cX}]$

[4 marks]

Consider a pure investor with CARA utility function with risk-aversion coefficient $\alpha > 0$ in a one-period market model with two assets: a risk-free asset with return R_1^0 and a risky asset with return $R_1^1 \sim \text{Exp}[\lambda]$.

(b) Explain why, a priori, there exists a unique optimal strategy for this investor to maximize its expected utility, assuming its initial wealth is $w_0 > 0$. Then, find the optimal strategy .

[12 marks]

Consider now the problem of the same investor in a two-period version of the market model, assuming that $R_2^0 = R_1^0$ is known and the returns in time are i.i.d. $\text{Exp}[\lambda]$ random variables.

(c) Still under the conditions in (a), find the optimal investment strategy for this investor in the two-period market model: that is, maximize

$$\mathbb{E}[u(W_2)]$$

where W_2 is their wealth at time 2. Assume they have initial wealth w_0, no consumption and no endowments.

[9 marks]

[Total: 25 marks]

Question 3.
Let X be a continuous random variable with probability density function

$$f_X(x) = \begin{cases} \frac{b_1}{2} e^{b_1 x} & \text{if } x < 0 \\ \frac{b_2}{2} e^{-b_2 x} & \text{if } x \geq 0 \end{cases}$$

for $b_1, b_2 > 0$.

(a) Calculate $\mathrm{V@R}^{\alpha}(X)$ for $\alpha \in (0, 1)$.

[9 marks]

Consider a stock price whose daily returns satisfy

$$R_{k+1} = \frac{S_{k+1}}{S_k} = e^{\xi_{k+1} + \mu_k}$$

with $(\mu_k)_{k=1,2,\ldots}$ positive and adapted, and where the sequence $(\xi_k)_{k=1,2,\ldots}$ is i.i.d. and adapted with $\xi_k \sim X$ as defined above.

(b) Find the value at risk at level α of the one-period P&L, $\Delta_{k+1} := (S_{k+1} - S_k)$, conditional on $\mathcal{F}_{t_k}$.

[8 marks]

(c) Assume that the risk capital on P&L is calculated using a one-day value at risk at level $\alpha = 0.99$.

Find the maximum number of exceptions in a five-year trading period ($\sim$ 1250 days) that, with a confidence of 95%, do not lead you to reject the assumption that the calculated risk is covered. State explicitly the type of test and statistic that you use.

[8 marks]

[Total: 25 marks]

Question 4.
A market has n risky assets with one-period excess returns $\tilde{\boldsymbol{R}} = (\tilde{R}^1, \ldots, \tilde{R}^n)$ with mean $\mathbb{E}[\tilde{\boldsymbol{R}}] = \boldsymbol{\mu}$ and variance-covariance matrix $\operatorname{cov}(\tilde{\boldsymbol{R}}, \tilde{\boldsymbol{R}}) = \Sigma$ for some positive definite matrix $\Sigma \in \mathbb{R}^{n \times n}$.

Consider an investor taking long positions in the risky assets of the market. The investment is characterized by a vector $\boldsymbol{\varphi} \in (0, \infty)^n$ denoting the amount invested in each asset. Assume the investor chooses to measure the risk of their investment by using standard deviation. Also, assume they measure the performance using Sharpe ratio.

(a) Calculate the Sharpe ratio of the investor's portfolio returns.

[5 marks]

(b) Find the allocation of the risk of the overall portfolio to each component using a per capita Euler allocation measure. Compare with the risk of each asset considered independently.

[11 marks]

(c) The investor is interested in improving the performance of their portfolio by rebalancing *slightly* its composition. Explain how the investor should proceed to be sure to increase its performance. Then, write a Python function that receives $\boldsymbol{\mu}, \Sigma, \boldsymbol{\varphi}$ and returns three lists 'increase','decrease' and 'maintain' that contain the indices of assets for which the position should be respectively increased, decreased or maintained.

[9 marks]

[Total: 25 marks]