Q1 (25 points)

Consider a probability space with $\Omega = \{0,1\}^T$, the power sigma algebra $\mathcal{F} = \mathcal{P}(\Omega)$ and uniform probability. On this space, we define a market model composed of two assets: a risk-free asset with constant per-period gross return $R_t^0 = R^0$, and a risky asset with

$$R_t^1 = \begin{cases} \lambda_\ell & \text{if } \omega_t = 0 \\ \lambda_h & \text{if } \omega_t = 1 \end{cases};$$

for $t = 1, \dots, T$ and $\lambda_h > \lambda_\ell$.

(a) *(8 marks)* State a condition on λ_h, λ_ℓ, and R^0 implying there are no arbitrage opportunities on this market. Justify your answer.

Hint: Consider first the one-period case $T = 1$.

(b) *(9 marks)* Assuming that the condition in the previous question is satisfied, find the set of risk neutral measures that can be defined on this market model.

(c) *(8 marks)* Set $T = 2$, $R^0 = 1$, $\lambda_u = 1.1$, $\lambda_\ell = 0.9$. Assume further that $S_0^1 = 100$. A butterfly option wants to be introduced into the market, with maturity T. Its payoff is $\phi(S_2^1)$ with

$$\phi(x) := \left(1 - \frac{1}{2}|x - 100|\right)^+.$$

Find the set of arbitrage-free prices for this option.

Q2 (25 points)

(a) *(8 marks)* Give an example of a utility function (as defined in the course) that is not risk-averse nor risk-seeking. Justify your answer.

Consider a consumer-investor with time-additive and constant absolute risk aversion (CARA) coefficient $\alpha > 0$ and time discounting $\delta \in (0,1)$. This agent aims to maximise expected utility in a one-period arbitrage-free and complete market model with an initial wealth w_0, i.e., they aim to solve

$$\max_{\boldsymbol{\pi}\in\mathbb{R}^{n+1};c_0\in\mathbb{R}} \quad u(c_0)+\delta\mathbb{E}[u(C_1)] \qquad \text{s. t.} \qquad C_1=(w_0-c_0)\boldsymbol{\pi}\cdot\boldsymbol{R}; \qquad \boldsymbol{\pi}\cdot\boldsymbol{1}=1;$$

where u is the CARA utility, c_0 and C_1 represent consumption at times 0 and 1 respectively, $\boldsymbol{\pi}$ is the investment strategy in percentage and $\boldsymbol{R}$ is the vector of gross returns of market instruments.

(b) *(9 marks)* Show that if (c_0^*, C_1^*) with $c_0^* < w_0$ is the optimal consumption vector associated with the problem above, then

$$M=\delta e^{\alpha(c_0^*-C_1^*)}$$

is a stochastic discount factor (SDF).

(c) *(8 marks)* Find an expression for c_0^* exclusively as a function of the data of the problem and M.

Remark: note that this reduces the problem to finding an SDF in the given setting.

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Q3 (25 points)

For $\alpha,\beta\in(0,1)$, let $\rho_{\alpha,\beta}: L^1(\mathbb{R})\to\mathbb{R}$ be a risk measure defined by

$$\rho_{\alpha,\beta}(X):=\max\{\mathrm{V@R}^{\alpha}(X),\mathrm{ES}^{\beta}(X)\},$$

with the convention for $\mathrm{V@R}^{\alpha}$ and ES^{β} used in class, that is, assuming that α (respectively β) is related to coverage.

(a) *(8 marks)* Verify, by giving a proof or a counter-example, whether $\rho_{\alpha,\beta}$ is monetary;

(b) *(8 marks)* Show that

$$\rho_{\alpha,\beta}(\lambda X+(1-\lambda)Y)\le\lambda\rho_{\alpha,\beta}(X)+(1-\lambda)\rho_{\alpha,\beta}(Y)$$

if either i) $\beta\ge\alpha$; or ii) (X,Y) is jointly Gaussian.

(c) *(9 marks)* Suppose that $\alpha=99\%$ and $\beta=95\%$, and X is your estimation of P&L on an investment.

Assume that in a sample of size 256 of results of one trading days (supposed to be drawn i.i.d.), the one-period losses exceed your $\mathrm{V@R}^{0.99}(X_t)$ estimation on 8 days. Using a backtest, find whether your value at risk calculation is correct with a confidence of 95%. Argue whether your conclusion has any consequence on the estimation of the measure $\rho_{0.99,0.95}(X)$.

Hint: You can use the following small table for the c.d.f. of a standard Gaussian distribution.

x	-3	-2	-1	0	1	2	3
$\Phi(x)$	0.0013	0.0228	0.1587	0.5	0.8413	0.9772	0.9987

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Q4 (25 points)

A one-period market model contains a risk-free asset with return R^0 and n risky assets. Let $\boldsymbol{R} \in L^2(\mathbb{R}^{n+1})$ be the vector of returns of all market instruments.

Let M be a (square integrable) stochastic discount factor (SDF) in this market model, and set $\tilde{\phi}$ to be the solution of the ordinary linear regression problem on the space of returns, that is, $\tilde{\phi}$ solves

$$\min_{\phi \in \mathbb{R}^{n+1}} \mathbb{E}|M - \phi \cdot \boldsymbol{R}|^2.$$

Set $\tilde{M} := \tilde{\phi} \cdot \boldsymbol{R}$.

(a) *(9 marks)* Show that $\tilde{M}$ is an SDF (attainable via investment). Moreover, show that

$$\mathbb{E}[\tilde{M}] = \frac{1}{R_0}; \qquad \mathbb{E}[\tilde{M}^2] = 1 \cdot \tilde{\phi}$$

Hint: Use the first order condition on the optimisation problem.

(b) *(9 marks)* Let $\mathcal{S}(\pi)$ be the Sharpe ratio of a (weights) portfolio π. Show that

$$|\mathcal{S}(\pi)| \leq \frac{\mathrm{sd}(\tilde{M})}{|\mathbb{E}[\tilde{M}]|}.$$

Hint: recall that $|\mathrm{corr}(R^{\pi}, \tilde{M})| \leq 1$.

(c) *(7 marks)* The mean-variance frontier (excluding the risk-free asset) of returns in this market contains two portfolios, π_1, π_2 such that:

- $\mathbb{E}(R^{\pi_1}) = R^0 + a$ for $a > 0$, $\mathbb{E}(R^{\pi_2}) = R^0$;
- $\mathrm{sd}(\mathrm{R}^{\pi_1}) = \mathrm{sd}(\mathrm{R}^{\pi_2}) = \sigma > 0$;
- $\mathrm{corr}(R^{\pi_1}, R^{\pi_2}) = 1/2$.

Find an expression that describes all portfolios in the mean-variance frontier (without risk-free asset), and give their corresponding Sharpe ratios.