1.
(a) When is a model arbitrage free? complete?

A model is arbtrage free if it does not admits arbitrage.
An arbitrage is a portfolio consists of trading assets so that $p(0) \leq 0$ and $p(T) \geq 0$ are with $p(T) > 0$ having positive probability. Model is complete if any contingent claim can be replicated.

(b). $S_0 = [10, 15, 30]^T$ $S_1 = [12, 12, 36]^T$, $S_2 = [12, 24, 24]^T$, $S_3 = [12, 24, 48]^T$

Show model is arbitrage free and complete.

proof. Choosing first asset as unit, we then have

$$\tilde{S}_0 = [1, 1.5, 3]^T, \hat{S}_2 = [1, 1, 3]^T, \hat{S}_2 = [1, 2, 2]^T, \hat{S}_3 = [1, 2, 4]^T$$

We seek p_1, p_2, p_3 so that $p_1 + p_2 + p_3 = 1$, $0 \leq p_i \leq 1$

$$p_1 \hat{S}_1 + p_2 \hat{S}_2 + p_3 \hat{S}_3 = \hat{S}_0$$

$$\Rightarrow \quad p_1 = 1/2, \quad p_2 = 1/4, \quad p_3 = 1/4$$

Hence we constructed a unique measure (Martingale measure) therefore market is arbitrage free and complete.

(c) Adding call option $P^c = 3$, strike $= 36$.

Adding call to the assets

$$\tilde{S}_0 = [1, 1.5, 3, 0.3]$$

$$\hat{S}_1 = [1, 1, 3, 0]$$

$$\tilde{S}_2 = [1, 2, 2, 0]$$

$$\tilde{S}_3 = [1, 2, 4, 1]$$

But $p_1 \hat{S}_1 + p_2 \hat{S}_2 + p_3 \tilde{S}_3 = \hat{S}_0$ has no solution

Because if it has solution it must be $p_1=1/2,\ p_2=1/4,\ p_3=1/4$

But $p_1\cdot 0+p_2\cdot 0+p_3\cdot 1 \neq 0.3$.

By the fundamental theorem of asset pricing,

it is not arbitrage-free anymore.

Q3. (a) $S_1 = S_0 \exp Y$. $Y \sim U(-m,m)$.

Let $VaR(\Delta S) = VaR(S_1-S_0) = VaR(S_0(\exp Y - 1)) = X$

We should have the $P(S_1-S_0 \leq -X) = 1-\alpha$

However $S_1-S_0 \leq -X \Leftrightarrow S_0(e^Y-1) \leq -X$

$\Leftrightarrow e^Y - 1 \leq -X/S_0 \Leftrightarrow Y \leq \ln(1-X/S_0)$

Hence $P(\Delta S \leq -X) = P(Y \leq \ln(1-X/S_0))$

$$= \frac{\ln(1-X/S_0)+m}{2m} = 1-\alpha$$

$$\Rightarrow \ln(1-X/S_0) = m - 2\alpha m$$

$$\Rightarrow X = S_0(1-e^{m-2\alpha m}).$$

(b) RORAC by definition is $E(S_0e^Y - S_0)/VaR(\Delta S_0)$

We calculate $E(e^Y) = \frac{1}{2m}\int_{-m}^{m} e^u\,du = e^m - e^{-m}/2m$

$$\Rightarrow E(S_0e^Y - S_0) = \frac{S_0}{2m}(e^m - e^{-m} - 2m)$$

$$V(\Delta S_0) = S_0(1-e^{m-2\alpha m})$$

$$\Rightarrow RoRAC = \frac{e^m - e^{-m} - 2m}{1-e^{m-2\alpha m}}$$

(c). The program calcute 1% expected loss

Let $Y \sim U(-1,1)$. We calculate

$$E(Y \cdot I_{Y \le -0.8}) = \int_{-0.8}^{1} (e^{y}-1)\,dy = e^{1} - e^{-0.8} - 1.8$$

Q4.

(a) proof.

$W = [w_1, \cdots, w_n]^T$, total allocation to risky assets is $W^T l$. $l = (1,1,\cdots,1)^T$

and we use $1 - W^T l$ to risk free asset

Hence return is $W^T \mu + (1 - W^T l) R_0$

We formulate the following optimitization

$$\begin{cases} \min & \frac{1}{2} W^T \Sigma W \\ \text{subject:} & W^T \mu + (1 - W^T l) R_0 = \mu_p \end{cases} \quad (*)$$

let $f(w) = \frac{1}{2} W^T \Sigma W - \lambda \left(W^T \mu + R_0 - \mu_p - W^T R_0 l \right)$

$$\nabla f(w) = \Sigma W - \lambda (\mu - R_0 l) = 0$$

$$\Rightarrow \quad W = \lambda \Sigma^{-1} (\mu - R_0 l)$$

plug into $(*)$ $\Rightarrow \lambda = \dfrac{\mu_p - R_0}{(\mu - R_0 l)^T \Sigma^{-1} (\mu - R_0 l)}$

(b) For CARA utility $u(x) = -a e^{-\alpha x} + b$

when $x \sim N(\mu, \sigma^2)$ we have

$$E(-a e^{-\alpha x} + b) = -a e^{-\alpha E(x) + \frac{1}{2}\alpha V(x)} + b$$

$$= -a e^{-\alpha (E(x) - \frac{1}{2} V(x))} + b.$$

$$\max E(u(x)) \Leftrightarrow \max \left(E(x) - \tfrac{1}{2} V(x)\right)$$

Now when invest to Gaussian return assets as in (a)

our $\quad x = w^T r + (1 - w^T 1) R_0 \quad$ where r is the risky

return and R_0 is the risk free return.

$$E(x) = w^T \mu + (1 - w^T 1) R_0$$

$$V(x) = w^T \Sigma w$$

$$E(x) - \frac{1}{2} V(x) = w^T \mu + (1 - w^T 1) R_0 - \frac{1}{2} w^T \Sigma w = f(w)$$

$$\nabla f(w) = (\mu - R_0 1) - \Sigma w = 0$$

$$\Rightarrow w = \Sigma^{-1} (\mu - R_0 1)$$

This is equivalent to $\lambda = 1 \Leftrightarrow \mu_p = R + (\mu - R_0 1)^T \Sigma^{-1} (\mu - R_0 1)$

in the mean-variance optimal portfolio.

Q5.

(a). $f(x) = \dfrac{a+bx}{c+dx} \Rightarrow f'(x) = \dfrac{ac-bd}{(c+dx)^2}$

$$V_t = \frac{a+bx}{c+dx} \Rightarrow cV_t + dx V_t = a + bx$$

$$\Rightarrow x (dV_t - b) = a - cV_t$$

$$\Rightarrow f^{-1}(V_t) = \frac{a - cV_t}{dV_t - b}$$

Hence $\quad V_t = f(\exp \xi_t) \Rightarrow x = e^{\xi_t}$

$$\frac{dV}{d\xi} = f'(x) \cdot x = \frac{bd - ac}{(c+dx)^2} \frac{a - cV_t}{dV_t - b}$$

$$= \frac{(cV_t - a)(dV_t - b)}{ad - bc}$$

(b). $\Delta V_{t+1} \approx \dfrac{dV}{d\xi} \cdot \Delta\xi = \dfrac{(cV-a)(dV-b)}{ad-bc} \cdot \Delta\xi$

But $\Delta\xi \sim N(0, 2\sigma^2)$,

$$\Rightarrow P(\Delta V_{t+1} \leq -x) = P\left(\frac{(cV-a)(dV-b)}{\quad} \Delta\xi \leq -x\right)$$

$$\Rightarrow P(\Delta V_{t+1} \leq -x) = P\left(\frac{(cV-a)(dV-b)}{ad-bc}\Delta\xi \leq -x\right)$$

$$\Leftrightarrow P\left(\Delta\xi \leq \frac{-x(ad-bc)}{(cV-a)(dV-b)}\right)$$

$$= \phi\left(\frac{-x(ad-bc)}{(cV-a)(dV-b)}\right) = 1\%.$$

$$\Rightarrow \frac{x(ad-bc)}{(cV-a)(dV-b)} = 2.33$$

$$\Rightarrow x = 2.33 \cdot \frac{(cV-a)(dV-b)}{ad-bc}.$$

(c). Assuming one year 250 samples and Gauss Returns Basel Committee requires that

	number of excesses
Green	≤ 4
Yellow	$5 \sim 9$
Red	≥ 10

Q2. (a). $u(x) = \frac{2x+1}{x+1} = 2 - \frac{1}{x+1}$

Hence $u(x)$ increasing and concave. Investor is risk averse and insatiable.

$$u'(x) = \frac{1}{(x+1)^2} \qquad u'' = -\frac{2}{(x+1)^3}$$

$\alpha(x) = -\frac{u''(x)}{u'(x)} = \frac{2}{x+1}$ is the absolute risk aversion.

(b) $E(W) = \frac{1}{6}u(0) + \frac{1}{3}u(1) + \frac{1}{2}u(2)$

$$= \frac{1}{6} + \frac{1}{3}\frac{3}{2} + \frac{1}{2}\frac{5}{3}$$

$$= \frac{2}{3} + \frac{5}{6} = \frac{3}{2}$$

certainty equivalent is $u^{-1}(\frac{3}{2}) = 1$.

Meaning at this level the utility is at average.

(c). Assuming cost of asset is x

with D. we have payoff

$$W+D = \begin{cases} 1 & W = \omega_1 \\ 4 & \omega = \omega_2 \\ 1 & \omega = \omega_3 \end{cases}$$

$$\Rightarrow \quad E(u(W+D)) - x = \frac{1}{6}u(1) + \frac{1}{3}u(4) + \frac{1}{2}u(1) - x$$

And the maximum cost x satisfies

$$E(u(W+D)) - x = E(u(W))$$

$$\Rightarrow \quad \frac{1}{6}u(1) + \frac{1}{3}u(4) + \frac{1}{2}u(1) - x = u(1)$$

$$\Rightarrow \quad x = \frac{1}{3}(u(4) - u(1)) = \frac{1}{3}\left(\frac{9}{5} - \frac{3}{2}\right) = \frac{1}{10}.$$