Problem 1.

(a) Consider a T-period market model with $n+1$ assets (including a bank account) defined on a probability space with our usual assumptions. Let \left{\boldsymbol{R}{t}^{i}\right}{i=0, \ldots, n ; t=1, \ldots, T} be the one-period gross returns (all well-defined). Show that if Z is an adapted process such that $Z_0 = 1$ and

$$\mathbb{E}t\left[Zt+1R_{t+1}^{i}\right]=1; \quad \text{for all } i=0,\ldots,n; t=0,\ldots,T-1,$$

then $M_t = \prod_{\ell=0}^{t} Z_\ell$ is an SDF. [9 marks]

Consider a probability space with $\Omega = 1, 2^2$, the power sigma algebra $\mathcal{F} = \mathcal{P}(\Omega)$ and uniform probability. On this space, we define a market model in one period composed of two assets: a riskfree asset with return $R_1^0 = R_2^0 = 3/2$, and a risky asset with $S_0^1 = 8$ and
$$
R_{1}^{1}\left(\left(\omega_{1}, \omega_{2}\right)\right)=\left{

$$1/2 \quad \text{if } \omega_1 = 1\ 2 \quad \text{if } \omega_1 = 2$$

; \quad R_{2}^{1}\left(\left(\omega_{1}, \omega_{2}\right)\right)=

$$\{1/2 \quad \text{if } \omega_2 = 1\ 2 \quad \text{if } \omega_2 = 2$$

\right.
$$

(b) Find all SDFs (if any) that can be defined on this market model. Conclude on whether the market is arbitrage-free and/or complete. [9 marks]
(c) A butterfly option on the risky asset with maturity $T=2$ wants to be introduced into the market: this is the option with payoff $\phi\left(S_2^1\right)$ with

$$\phi(x) := \left(1 - \frac{1}{4}|x-8|\right)^{+}.$$

Find the set of arbitrage-free prices for this option. [7 marks]

Problem 2.

Q2 (25 points)

A pure investor has utility function

$$u(x) = \{ x^{1/2} \quad \text{if } x \geq 0 \; - |x|^{1/2} \quad \text{if } x < 0$$

(a) Find the coefficients of relative risk aversion of this investor as a function of their wealth. How does the investor behave facing risk? [8 marks]

(b) Let W be a continuous random variable with even density f (i.e. $f(x) = f(-x)$ for all $x \in \mathbb{R}$), and so that $\mathbb{E}[|u(W)|] < \infty$. Find the certainty equivalent of a wealth W for this investor. [8 marks]

(c) Suppose that our investor has initial wealth w_0 . They will act on a one-period market with two assets: one risk-free with return R^0 and one risky with a binomial return

$$R^1 = \{ h \quad \text{with probability } p \; \ell \quad \text{with probability } 1 - p$$

with $\ell < R^0 < h$. Find the optimal investment strategy with no short positions for this agent, at time 0 , in terms of the amount to invest on each asset. [9 marks]

Problem 3.

Q3 (25 points)

Let $g : [0, 1] \to \mathbb{R}+$be a positive function such that

$$\int 0^1 g(u)\mathrm{d}u = 1$$

For $X \in L^1$, define the risk measure

$$\rho_g(X) := \int_0^1 g(u) q_{-X}(u)\mathrm{d}u$$

where q_{-X} is the quantile function of $-X$.

(a) Find an example of a function g satisfying the assumptions above and such that the measure ρ_g is coherent. [8 marks]

(b) Show that ρ_g is a monetary risk measure.

Remark: You can use properties of risk measures studied in class. [8 marks]

(c) Take a sample of size T of results of one-trading periods. Consider the excess indicators $(I_{t,u})\, t = 1, \ldots, T; u \in (0, 1)$ where Assume that $It, u\textbf{\Perp}I_{s,v}$ for all $t \neq s$ and that $\mathbb{P}[I_{t,u} = 1] = 1 - u$ for all $t = 1, \ldots, T; u \in (0, 1)$. Let

$$Y_g := \frac{1}{T} \sum_{t=1}^{T} \int_0^1 g(u) I_{t,u}\, \mathrm{d}u$$

Show that

$$\mathbb{E}[Y_g] = \int_0^1 g(u)(1 - u)\mathrm{d}u.$$

Then, using in addition the (given) fact that

$$\operatorname{var}(Y_g) = \frac{2}{T} \int_0^1 \int_v^1 g(u)g(v)(1 - u)\mathrm{d}u\, \mathrm{d}v - \mathbb{E}[Y_g]^2 < \infty$$

propose a coverage backtest for the measure ρ_g. [9 marks]

Problem 4.

Q4 (25 points)

A one-period market model contains a risk-free asset with return R^0 and n risky assets. The mean-variance frontier (excluding the risk-free asset) of returns in this market contains two portfolios, π_1, π_2 such that:

- $\mathbb{E}\left(R^{\pi_1}\right) = R^0 + a$ for $a > 0, \mathbb{E}\left(R^{\pi_2}\right) = R^0$;
- $\mathrm{sd}\left(\mathrm{R}^{\pi_1}\right) = \mathrm{sd}\left(\mathrm{R}^{\pi_2}\right) = \sigma > 0$;
- $\mathrm{corr}\left(R^{\pi_1}, R^{\pi 2}\right) = 1/2$.

  (a) Find an expression that describes all portfolios in the mean-variance frontier (without risk-free asset), and give their corresponding mean and variance. [9 marks]

  (b) Find the tangency portfolio and the maximal Sharpe ratio $\left(S_{\max}\right)$ in this market. [9 marks]

  (c) Assume that an investor would like to choose a portfolio with maximal Sharpe ratio, mean return larger than the risk-free rate, and no short position on the risk-free asset. Describe the set of portfolios that satisfy these properties. [7 marks]