MATH0094:Market Risk Measures and Portfolio Theory

Solutions to LN Exercises

2021-2022

1 Fundamentals of market theory

Exercise 1.1. The following table summarises all the questions.

	Predictable	Adapted	Markovian	Martingale
1	N	Y	Y	N
2	N	Y	Y	N
3	N	Y	Y	Y
4	N	Y	Y	N
5	N	Y	Y	N
6	Y	Y	N	N
7	N	N	N	N

Table 1: Summary of answers for this question

Let us see the details for some of these:

1. The process is clearly adapted but not predictable (for the latter it suffices to see that $\hat{X}_1$ is not deterministic).

   Since $X_t \perp \mathcal{F}_{t-1}$, $X_t \perp \hat{X}_t$, so

   $$\mathbb{E}[\hat{X}_t|\mathcal{F}_{t-1}] = \frac{t-1}{t}\hat{X}_{t-1} + \frac{1}{t}\mathbb{E}[X_t] = \mathbb{E}[\hat{X}_t|\hat{X}_{t-1}]. \tag{1}$$

   A simple induction together with the chain rule then shows that the process is Markovian.

   Equation (1) also shows that the process is not a Martingale in general, since

   $$\hat{X}_{t-1} \neq \mathbb{E}[\hat{X}_t|\mathcal{F}_{t-1}].$$

2. If Y is the process in 1., this process is just $\exp(Y)$, so it is also not predictable, but it is adapted and Markovian, since

   $$\mathbb{E}[\hat{X}_t|\mathcal{F}_{t-1}] = (\hat{X}_{t-1})^{\frac{t-1}{t}}\mathbb{E}[\exp(\frac{X_t}{t})] = \mathbb{E}[\hat{X}_t|\hat{X}_{t-1}]$$

   where we used independence to separate the two terms in the conditional expectation.

   Also, the process is not in general a martingale.

3. This is a special modification of 2: if we call the process solution of 2 Z, we get

$$\hat{X}_t = \exp(-t\mathbb{E}[X_1] - \frac{1}{2}t\mathrm{var}[\mathrm{X}_1])(\mathrm{Z}_\mathrm{t})^\mathrm{t}.$$

Note that the first term is deterministic. Thus, it inherits adaptability, non-predictability and Markovianity. Hence, we just need to verify if this is a martingale. Since for a random variable $N \sim \mathcal{N}(m, \sigma^2)$ we get that

$$\mathbb{E}[\exp(N)] = \exp(m + \frac{1}{2}\sigma^2),$$

it follows that

$$\mathbb{E}[\hat{X}_t|\mathcal{F}_{t-1}] = \hat{X}_{t-1}\exp(\mathbb{E}[X_1] + \frac{1}{2}\mathrm{var}(X_1))\exp(-\mathbb{E}[X_1] - \frac{1}{2}\mathrm{var}[\mathrm{X}_1]) = \hat{\mathrm{X}}_{\mathrm{t}-1}$$

Hence, it is a martingale.

4. It follows easily that the process is not predictable but is adapted. Moreover,

$$\begin{aligned}\mathbb{E}[\xi_t|\mathcal{F}_{t-1}] &= \begin{pmatrix}\mathbb{E}[\hat{X}_t|\mathcal{F}_{t-1}]\\ \mathbb{E}[\sigma_t|\mathcal{F}_{t-1}]\end{pmatrix} = \begin{pmatrix}\mathbb{E}[\sigma_t X_t|\mathcal{F}_{t-1}]\\ \alpha_0 + \alpha_1\hat{X}_{t-1}^2 + \beta_t\sigma_{t-1}^2\end{pmatrix}\\ &= \begin{pmatrix}(\alpha_0 + \alpha_1\hat{X}_{t-1}^2 + \beta_t\sigma_{t-1}^2)^{1/2}\mathbb{E}[X_1]\\ \alpha_0 + \alpha_1\hat{X}_{t-1}^2 + \beta_t\sigma_{t-1}^2\end{pmatrix} = \mathbb{E}[\xi_t|\xi_{t-1}]\end{aligned}$$

where we used once more independence and the fact that the expression before the last one only depends on the values of ξ_{t-1}. Also, the same expression shows that in general it will not be a martingale.

5. The process $\hat{X}$ is in this case adapted but not predictable, since the event $\{X_t > \max_{0\le s<t} X_s\}$ is $\mathcal{F}_t$ but not $\mathcal{F}_{t-1}$ measurable.

   Since

$$\hat{X}_t = \max\{\hat{X}_{t-1}, X_t\},$$

it is clear that the process is Markovian. Note also that

$$\mathbb{E}[\hat{X}_t|\mathcal{F}_{t-1}] = \mathbb{E}[\max\{\hat{X}_{t-1}, X_t\}|\hat{X}_{t-1}]$$

and so in general it will not be a martingale.

6. We can write

$$\hat{X}_t = \begin{cases}\hat{X}_{t-1} & \text{if } X_{\hat{X}_{t-1}} > 1\\ t & \text{otherwise}\end{cases},$$

which is $\mathcal{F}_{t-1}$ measurable. Hence, it is predictable and adapted, and in particular $\mathbb{E}[\hat{X}_t|\mathcal{F}_{t-1}] = \hat{X}_t$. Now, note that the value of $\hat{X}_t$ is not completely known by conditioning by $\hat{X}_{t-1}$. For instance,

$$\mathbb{E}[\hat{X}_t|\hat{X}_{t-1} = t-1] = \begin{cases}t-1 & \text{if } X_{t-1} > 1\\ t & \text{otherwise.}\end{cases}.$$

Hence, it follows that this is not a Markovian process.

7. The process $\hat{X}$ in this case is not even adapted as it uses information for $\mathcal{F}_s$ for $s > t$.

Exercise 1.2. 1. π is a valid strategy if it denotes the actions of investors, and it is a predictable process.

π is deterministic, so that it is trivially predictable. Moreover, both π^+ and π^- are predictable processes since $\pi_1^{+,-}$ are both deterministic while $\pi_2^{+,-}$ are $\mathcal{F}_1$ measurable.

2. Calling w_0 the initial investment, we have that:

$$\begin{aligned}
S_1^\pi &= S_1^{\pi^+} = S_1^{\pi^-} = \frac{w_0}{2}(1+R_1^1)\\
S_2^\pi &= \frac{w_0}{4}(1+R_1^1)(1+R_2^1)\\
S_2^{\pi^+} &= \frac{w_0}{4}(1+R_1^1)\left[1+R_2^1+2\delta(R_2^1-1)\mathrm{sign}(R_1^1-\frac{u+d}{2})\right]\\
S_2^{\pi^-} &= \frac{w_0}{4}(1+R_1^1)\left[1+R_2^1-2\delta(R_2^1-1)\mathrm{sign}(R_1^1-\frac{u+d}{2})\right],
\end{aligned}$$

where the function $\mathrm{sign}(x) = \begin{cases} |x|x^{-1} & \text{if } x \neq 0 \\ 0 & \text{otherwise} \end{cases}$.

Table 2 contains the values of the risky asset under different outcomes. Replacing the values, we obtain Table 3 (note the sense of inequalities marked in yellow, which come from the assumptions).

	$(0,0)$	$(1,1)$	$(0,1)$	$(1,0)$
S_2^1	$S_0^1 d^2$	$S_0^1 u^2$	$S_0^1 du$	$S_0^1 du$

Table 2: Values of risky asset under different outcomes.

	$S_2^{\pi^-}$	S_2^π	$S_2^{\pi^+}$
$(0,0)$	$\frac{w_0}{4}(1+d)[1+d-2\delta(1-d)]$	$\frac{w_0}{4}(1+d)^2$	$\frac{w_0}{4}(1+d)[1+d+2\delta(1-d)]$
$(1,1)$	$\frac{w_0}{4}(1+u)[1+u-2\delta(u-1)]$	$\frac{w_0}{4}(1+u)^2$	$\frac{w_0}{4}(1+u)[1+u+2\delta(u-1)]$
$(0,1)$	$\frac{w_0}{4}(1+d)[1+u+2\delta(u-1)]$	$\frac{w_0}{4}(1+d)(1+u)$	$\frac{w_0}{4}(1+d)[1+u-2\delta(u-1)]$
$(1,0)$	$\frac{w_0}{4}(1+u)[1+d+2\delta(1-d)]$	$\frac{w_0}{4}(1+d)(1+u)$	$\frac{w_0}{4}(1+u)[1+d-2\delta(1-d)]$

Table 3: Values of portfolios under different outcomes: in the cases $\omega \in \{(0,0),(1,1)\}$ the values increase from left to right; in the cases $\omega \in \{(0,1),(1,0)\}$ the values decrease from left to right.

Now, if we want $S_2^{\pi^+} > S_2^{\pi^-}$, we can take both $\mathbb{P}[S_1^1 = u], \mathbb{P}[S_1^1 = u] > 0$; while setting

$$\mathbb{P}[S_2^1 = u | S_1^1 = u] = \mathbb{P}[S_2^1 = d | S_1^1 = d] = 1.$$

We can easily verify the inequality by observing the table 3, since we do not have the cases $(0,1)$ or $(1,0)$.

3. Similar to previous point but for $S_2^{\pi^+} < S_2^{\pi^-}$ we choose $\mathbb{P}[S_2^1 = u | S_1^1 = d] = \mathbb{P}[S_2^1 = d | S_1^1 = u] = 1$.

4. By using again the table, we can see that for $S_2^\pi > S_2^{\pi^-}$ we would need

$$\mathbb{P}[S_2^1 = u|S_1^1 = d] = \mathbb{P}[S_2^1 = d|S_1^1 = u] = 0,$$

but to get $S_2^\pi > S_2^{\pi^+}$ we require

$$\mathbb{P}[S_2^1 = d|S_1^1 = d] = \mathbb{P}[S_2^1 = u|S_1^1 = u] = 0.$$

All these conditions cannot be imposed simultaneously, so it is impossible.

Exercise 1.3. We show that we can deduce the risk premia from the SDF. From the properties of covariance, conditional expectation and (1.14) in the Lecture Notes, we have

$$\mathrm{Cov}_t\left(\frac{M_{t+1}}{M_t}, R_{t+1}^i\right) = \mathbb{E}_t\left[\frac{M_{t+1}R_{t+1}^i}{M_t}\right] - \mathbb{E}_t\left[\frac{M_{t+1}}{M_t}\right]\mathbb{E}_t[R_{t+1}^i] = 1 - \frac{\mathbb{E}_t[M_{t+1}]\mathbb{E}_t[R_{t+1}^i]}{M_t}.$$

On the other hand, turning our attention to the money market account, we deduce from (1.14) in the LN and the predictable property of R^0 that

$$M_t = \mathbb{E}_t[M_{t+1}R_{t+1}^0] = R_{t+1}^0\mathbb{E}_t[M_{t+1}].$$

Thus,

$$\mathrm{Cov}_t\left(\frac{M_{t+1}}{M_t}, R_{t+1}^i\right) = 1 - \frac{\mathbb{E}_t[R_{t+1}^i]}{R_{t+1}^0}$$

which after reordering concludes the proof.

Exercise 1.4. (i) Is there any risk-less asset in this market?

Yes, note that the first entry of $\boldsymbol{S}_1(\omega_1)$, $\boldsymbol{S}_1(\omega_2)$, $\boldsymbol{S}_1(\omega_3)$ are all equal to 1. Hence, it is deterministic and thus risk-less. Its rate is $R_1^0 = 1/1 = 1$.

(ii) The return of the risky assets:

$$R^1 = [\boldsymbol{S}_1^1(\omega_1)/\boldsymbol{p}^1, \boldsymbol{S}_1^1(\omega_2)/\boldsymbol{p}^2, \boldsymbol{S}_1^1(\omega_3)/\boldsymbol{p}^3] \tag{2}$$
$$= [\frac{3}{2}, \frac{1}{2}, \frac{5}{2}] \tag{3}$$

i.e., $R^1(\omega_1) = \frac{3}{2}$, $R^1(\omega_2) = \frac{1}{2}$, $R^1(\omega_3) = \frac{5}{2}$. And $R^2 = [\frac{9}{7}, \frac{5}{7}, \frac{10}{7}]$.

If probability is uniform, we have that the risk premia are:

$$\mathbb{E}[R^1] - R^0 = \frac{3}{2}\mathbb{P}[\{\omega_1\}] + \frac{1}{2}\mathbb{P}[\{\omega_2\}] + \frac{5}{2}\mathbb{P}[\{\omega_3\}] - 1 \tag{4}$$
$$= \frac{1}{3}[\frac{3}{2} + \frac{1}{2} + \frac{5}{2}] - 1 = \frac{3}{2} - 1 = \frac{1}{2} \tag{5}$$
$$\mathbb{E}[R^2] - R^0 = \frac{9}{7}\mathbb{P}[\{\omega_1\}] + \frac{5}{7}\mathbb{P}[\{\omega_2\}] + \frac{10}{7}\mathbb{P}[\{\omega_3\}] - 1 \tag{6}$$
$$= \frac{1}{3}[\frac{9}{7} + \frac{5}{7} + \frac{10}{7}] - 1 = \frac{8}{7} - 1 = \frac{1}{7} \tag{7}$$

(iii) We now check if the market model is complete and arbitrage free. This kind of problem can be solved in several ways using the matrix properties we reviewed, or exposing explicitly arbitrage opportunities or non-replicable profiles.

We follow a method based on linear algebra. Since the matrix is square, we can use the determinant to check if the matrix $\mathcal{M}_{S_1}$ is invertible. Remember that if this is the case, then there is a unique solution to any linear system associated to it.

We obtain that

$$\det(\mathcal{M}_{S^1}) = \det(\mathcal{M}_{S^1}^{\intercal}) = (1)(10-25) - (1)(30-45) + (1)(15-9) = -15 + 15 + 6 = 6.$$

Hence, any linear system has a unique solution, so $\mathcal{M}_{S^1}$ has full range. The market is therefore complete.

It can also be deduced that there is at most one set of AD prices. We need to check that they are all positive. By solving the linear system

$$\boldsymbol{S}_0 = \mathcal{M}_{S_1}\boldsymbol{p}^{AD}$$

We get

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 3 & 1 & 5 & 2 \\ 9 & 5 & 10 & 7 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & 2 & -1 \\ 0 & -4 & 1 & -2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1/2 \\ 0 & 1 & -1 & 1/2 \\ 0 & 0 & -3 & 0 \end{array}\right)$$

Note that from the last equation, we have that $p_3^{AD} = 0$. As this is the only solution, we get that there is no strictly positive AD prices (or SDF or equivalent risk neutral probability).

This means that the market has an arbitrage.

Exercise 1.5. (a) We know that absence of arbitrage is equivalent to the existence of a positive SDF, positive AD prices, or an equivalent risk-neutral probability. Using the latter, there is no-arbitrage if and only if there exists $\mathbb{Q}$ equivalent to the original probability such that $\mathbb{E}^{\mathbb{Q}}[R] = R^0$.

Define $\mathbb{Q}\{\omega_1\} = q_1$, $\mathbb{Q}\{\omega_2\} = q_2$. We solve the system:

$$R^0 = q_1 R^1(\omega_1) + q_2 R^1(\omega_2) = q_1 R^u + q_2 R^d$$

and

$$q_1 + q_2 = 1 \text{(from probability properties)}$$

Moreover, from no arbitrage we require, $q_1 > 0$, $q_2 > 0$. Solving, we get:

$$q_2 = \frac{R^u - R^0}{R^u - R^d} \quad q_1 = \frac{R^0 - R^d}{R^u - R^d} \tag{8}$$

But since $R^u > R^d$, then $q_2 > 0$ and $q_1 > 0$ if and only if $R^u > R^0$ and $R^0 > R^d$. Therefore, the condition is $R^u > R^0 > R^d$.

(b) We obtained the risk neutral probabilities as a by product of the previous exercise, as given in (8)

(c) There are two Arrow-Debreu securities, one for each $\{\omega_1, \omega_2\}$.

$$p_1^{AD} = \mathbb{E}^{\mathbb{Q}}[\frac{\mathbb{1}_{\omega_1}}{R^0}] = q_1 \frac{\mathbb{1}_{\{\omega_1\}}(\omega_1)}{R^0} + q_2 \frac{\mathbb{1}_{\{\omega_1\}}(\omega_2)}{R^0} = \frac{q_1}{R^0} = \frac{R^0 - R^d}{(R^u - R^d)R^0} \tag{9}$$

$$p_2^{AD} = \mathbb{E}^{\mathbb{Q}}[\frac{\mathbb{1}_{\omega_2}}{R^0}] = \frac{q_2}{R^0} = \frac{R^u - R^0}{(R^u - R^d)R^0} \tag{10}$$

(d) If a new asset paying $(S_1 - K)^+$ is introduced, we can use the risk neutral probabilities to find p_{opt}, the price of this asset.

$$p_{opt} = \mathbb{E}^{\mathbb{Q}}[\frac{(S_1 - K)^+}{R^0}] = \frac{q_1(R^u p - K)^+}{R^0} + \frac{q_2(R^d p - K)^+}{R^0} \tag{11}$$

$$= \frac{1}{R^0(R^u - R^d)}[(R^u p - K)^+(R^0 - R^d) + (R^d p - K)^+(R^u - R^0)] \tag{12}$$

but since $R^u > R^0 > R^d$, and $K = R^0 p$

$$p_{opt} = \frac{p}{R^0(R^u - R^d)}[(R^u - R^0)(R^0 - R^d)]$$

Exercise 1.6. (a) Arbitrage-free $\Rightarrow$ law of one price. Take $\boldsymbol{\theta}$,$\boldsymbol{\xi}$ such that $\boldsymbol{\theta} \cdot \boldsymbol{S}_1 = \boldsymbol{\xi} \cdot \boldsymbol{S}_1$. If there is no-arbitrage, there exists a positive SDF. Hence, $\mathbb{E}[MS_1^i] = p^i$.

But then, by linearity:

$$\boldsymbol{\theta} \cdot \boldsymbol{p} = \boldsymbol{\theta} \cdot \mathbb{E}[M\boldsymbol{S}_1] = \mathbb{E}[M(\boldsymbol{\theta} \cdot \boldsymbol{S}_1)] \tag{13}$$

$$= \mathbb{E}[M(\boldsymbol{\xi} \cdot \boldsymbol{S}_1)] = \boldsymbol{\xi} \cdot \mathbb{E}[M\boldsymbol{S}_1] = \boldsymbol{\xi} \cdot \boldsymbol{p} \tag{14}$$

(b) Example of market with arbitrage and such that the law of one price is satisfied. Take the market in question 2.1. We showed it admitted an arbitrage. Let $\boldsymbol{\theta}$, $\boldsymbol{\xi}$ such that $\boldsymbol{\theta} \cdot \boldsymbol{S}_1 = \boldsymbol{\xi} \cdot \boldsymbol{S}_1$. This means that

$$[\theta^0, \theta^1, \theta^2] \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 5 \\ 9 & 5 & 10 \end{bmatrix}}_{M_{S_1}} = [\xi^1, \xi^2, \xi^3] \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 5 \\ 9 & 5 & 10 \end{bmatrix}. \tag{15}$$

Since we showed M_{S_1} is invertible, because $\det(\mathcal{M}_{S_1}) = 6$. Therefore, we conclude that $\boldsymbol{\theta} = \boldsymbol{\xi}$, so trivially the law of one price holds.

Exercise 1.7. Recall that M is an SDF if for each market instrument, $i = 0, ..., n$, we have $S_0^i = \mathbb{E}[S_1^i M]$. Let $M_1^\alpha = \alpha M_1 + (1 - \alpha)\tilde{M}_1$ for two SDFs M_1, $\tilde{M}_1$. Then,

$$\mathbb{E}[(\alpha M_1 + (1 - \alpha)\tilde{M}_1)S_1^i] = \alpha \mathbb{E}[M_1 S_1^i] + (1 - \alpha)\mathbb{E}[\tilde{M}_1 S_1^i] \tag{16}$$

$$= \alpha S_0^i + (1 - \alpha)S_0^i = S_0^i \tag{17}$$

Moreover, if $M_1 > 0$ and $\tilde{M}_1 > 0$, we have that for $0 \leq \alpha \leq 1$, $\alpha M_1 > 0$ and $(1 - \alpha)\tilde{M}_1 > 0$ and thus $M_1^\alpha > 0$.

Exercise 1.8. We show the two applications: Assume that the price for the asset satisfies $p^{new} = \mathbb{E}[S_1^{new} M]$ for some strictly positive SDF M. Then, M is a strictly positive SDF for the market with $n + 2$ assets, and by the FTAP-1, there is no arbitrage.

On the other hand, if $p^{new} \neq \mathbb{E}[D^{new} M]$ for every M strictly positive SDF on the market with $n + 1$ assets, then we will not be able to find an SDF on the market with $n + 2$ assets. By the FTAP-1, this implies there exists an arbitrage opportunity.

Exercise 1.9. By Exercise 1.8, we have that $\tilde{p} = \mathbb{E}[D^{new}\tilde{M}_1]$ and $\hat{p} = \mathbb{E}[D^{new}\hat{M}_1]$ for two strictly positive SDFs $\tilde{M}_1$, $\hat{M}_1$. But since

$$\alpha\tilde{p} + (1 - \alpha)\hat{p} = \mathbb{E}[D^{new}\{\alpha\tilde{M}_1 + (1 - \alpha)\hat{M}_1\}] \tag{18}$$

$$= \mathbb{E}[D^{new} M_1^\alpha], \tag{19}$$

and M_1^α is a strictly positive SDF by Exercise 1.7. Hence, it is an arbitrage free price.

2 Utility functions

Exercise 2.1.

1. The investor chooses from the two alternatives the one with larger expected utility. Hence, we need to compare the expected utility of a final wealth $W_1 = w_0 = 200$ and a final wealth

$$W_2 = \begin{cases} w_0 + 100 = 300 & \text{with probability } \frac{1}{2} \\ w_0 - 100 = 100 & \text{with probability } \frac{1}{2} \end{cases}.$$

   Now, note that $\mathbb{E}[W_2] = \frac{1}{2}300 + \frac{1}{2}100 = 200 = w_0 = W_1$, and also that $u_A(x) = \frac{x^\gamma}{\gamma}$ is concave (in fact it is a CRRA utility function). Thus, by Jensen's inequality we have that

$$\mathbb{E}[u_A(W_2)] \leq u_A[\mathbb{E}[W_2]] = u_A(W_1) = \mathbb{E}[u_A(W_1)]$$

   Therefore, the investor prefers W_1.

   The argument is the same for $u_B(x) = -\exp(-x)$, since it is also a concave function.

2. In this case we have that

$$W_1 = \begin{cases} 11 = & \text{with probability } \frac{1}{3} \\ 31 = & \text{with probability } \frac{2}{3} \end{cases}; \qquad W_2 = \begin{cases} 21 & \text{with probability } 0.9 \\ 1 & \text{with probability } 0.1 \end{cases}.$$

   We can numerically make the calculation to get, $\mathbb{E}[\log(W_1)] = \frac{1}{3}\log(11) + \frac{2}{3}\log(31) \approx= 3.09$, while $\mathbb{E}[\log(W_2)] = 0.93\log(21) + 0.1\log(1) \approx= 2.74$. Clearly, the investor will choose the first investment.

Exercise 2.2. Since the functions to be considered are twice differentiable, it suffices to look at the sign of the second derivative: we need to verify that the second derivative is negative on the domain of definition.

1. $f_1(x) = \log(x)$ for $x > 0$: We get

$$f'(x) = \frac{1}{x}; \qquad \text{and} \qquad f''(x) = \frac{-1}{x} < 0.$$

2. $f_2(x) = a - bx^2$ for $b > 0$: We get

$$f'(x) = -2bx; \qquad \text{and} \qquad f''(x) = -2b < 0.$$

Hence, both functions are concave. Recall though that the quadratic function needs a restricted domain of definition to be considered a utility function.

Exercise 2.3. To evaluate the expected utility, we can either use integral methods or Monte Carlo methods.

```
# Import modules
import numpy as np
import scipy.stats as st
```

```python
# We use two approaches, either using the 'stats' module
# and the 'expect' method, or using Monte Carlo with a large
# number of simulations

# We initialise the function
myf = lambda x: np.log(x)
mc_number = 1000000

# Pareto with shape alpha=2 and scale (or mode) xm = 0.5

# Method 1
u = st.pareto.expect(myf, args=(2,), scale=0.5)
print('Pareto, method 1:', u)
# Method 2
sample_p = (np.random.pareto(2, mc_number) + 1) * 0.5
u = myf(sample_p).mean()
print('Pareto, method 2:', u)
```

```
Pareto, method 1: -0.19314718055994637
Pareto, method 2: -0.1932406330316995
```

```python
# Exponential with lambda = 1
# Method 1
u = st.expon.expect(myf, scale=1)
print('Exponential, method 1:', u)
# Method 2
sample_e = np.random.exponential(size=(mc_number,))
u = myf(sample_e).mean()
print('Exponential, method 2:', u)
```

```
Exponential, method 1: -0.5772156649008394
Exponential, method 2: -0.5804589164442893
```

```python
# Log normal (standard)
# Method 1
u = st.lognorm.expect(myf, args=(1,))
print('Exponential, method 1:', u)
# Method 2
sample_ln = np.random.lognormal(size=(mc_number,))
u = myf(sample_ln).mean()
print('Exponential, method 2:', u)
```

```
Exponential, method 1: 4.844603335998414e-17
Exponential, method 2: -0.00023813580546780955
```

Note that there is a good agreement between the two methods.

Exercise 2.4. An affine function on $\mathbb{R}$ is any function $g : \mathbb{R} \to \mathbb{R}$ of the type $g(x) = ax + b$ for $a, b \in \mathbb{R}$. Clearly, if $a > 0$, we have that

$$x < y \Rightarrow ax < ay \Rightarrow ax + b < ay + b \Rightarrow g(x) < g(y)$$

so that an affine function with $a > 0$ preserves the order. To show the converse, note that if $a = 0$, for all $x < y$ we have $g(x) = g(y)$. And similarly, if $a < 0$, we get

$$x < y \Rightarrow ax > ay \Rightarrow ax + b > ay + b \Rightarrow g(x) > g(y),$$

so that the order is reversed. Hence, the only affine functions that preserve the order are the ones with $a > 0$.

Exercise 2.8. (i) We take the case $u(x) = \log(x)$. Take w initial wealth and wX additional gamble.

$$\begin{aligned} \mathbb{E}[u(w(1+X))] &= \mathbb{E}[\log(w(1+X))] && (20)\\ &= \mathbb{E}[\log(w) + \log(1+X)] && (21)\\ &= \log(w) + \mathbb{E}[\log(1+X)] && (22) \end{aligned}$$

On the other hand,

$$\begin{aligned} u(\mathbb{E}[w(1+X)] - w\eta(X)) &= \log(w(\mathbb{E}[1+X] - \eta(X))) && (23)\\ &= \log(w) + \log(\mathbb{E}[1+X] - \eta(X)) && (24) \end{aligned}$$

$$\begin{aligned} &\Rightarrow \eta(X) \text{ solves: } \log(\mathbb{E}[1+X] - \eta(X)) = \mathbb{E}[\log(1+X)] && (25)\\ &\Rightarrow \eta(X) = \mathbb{E}[1+X] + \exp(\mathbb{E}[\log(1+X)]), && (26) \end{aligned}$$

which is independent of ω. Similar development for the case $u(x) = \frac{1}{1-\gamma}x^{1-\gamma}$ with $\gamma \in (0,1) \cup (1,\infty)$.

(ii) $\mathbb{E}[R] = \mathbb{E}[\exp(Z)]$ with $Z \sim \mathcal{N}(-\frac{\sigma^2}{2}, \sigma)$.

$$\mathbb{E}[R] = \exp(\mathbb{E}[Z] + \frac{1}{2}\text{var}(Z)) = \exp(-\frac{\sigma^2}{2} + \frac{\sigma^2}{2}) = 1 \tag{27}$$

Since we have that u is CRRA, we can use the results of the previous exercise, and solve the problem for initial wealth 1 If the initial wealth is $w_0 \neq 1$ we simply multiply by w_0. Let us first focus on the case $u(x) = \log(x)$. We get,

$$\log(\mathbb{E}[R] - \eta(R)) = \mathbb{E}[\log(R)] = \mathbb{E}[Z] = -\frac{\sigma^2}{2}, \tag{28}$$

$$\log(1 - \eta(R)) = -\frac{\sigma^2}{2} \Rightarrow \eta(R) = 1 - \exp(-\frac{\sigma^2}{2}) > 0. \tag{29}$$

Hence, $\eta(w_0 R) = w_0(1 - \exp(-\frac{\sigma^2}{2}))$.

In the case of $u(x) = \frac{1}{1-\gamma}x^{1-\gamma}$,

$$\frac{1}{1-\gamma}(1-\eta(R))^{1-\gamma} = \mathbb{E}[\frac{1}{1-\gamma}[\exp(Z)]^{1-\gamma}] \tag{30}$$
$$= \mathbb{E}[\frac{1}{1-\gamma}\exp((1-\gamma)Z)] \tag{31}$$
$$= \frac{1}{1-\gamma}\exp((1-\gamma)(-\frac{\sigma^2}{2}) + \frac{\sigma^2}{2}(1-\gamma)^2) \tag{32}$$
$$= \frac{1}{1-\gamma}\exp(-\frac{\sigma^2}{2}(1-\gamma)\gamma) \tag{33}$$
$$= \frac{1}{1-\gamma}[\exp(-\frac{\sigma^2}{2}\gamma)]^{1-\gamma} \tag{34}$$
$$\Leftrightarrow (1-\eta(R)) = \exp(-\frac{\sigma^2}{2}\gamma) \tag{35}$$
$$\eta(R) = 1-\exp(-\frac{\sigma^2}{2}\gamma) \tag{36}$$
$$\eta(w_0 R) = w_0(1-\exp(-\frac{\sigma^2}{2}\gamma)) \tag{37}$$

We see that the solution is the same independently of the relative risk aversion coefficient.

Exercise 2.9. (i) Since $W \sim \mathcal{N}(w, \sigma_w^2)$, we calculate

$$\mathbb{E}[u(W)] = \mathbb{E}[-\exp(-\alpha W)] = -\exp(-\alpha w + \frac{1}{2}\alpha^2\sigma_w^2). \tag{38}$$

On the other hand,

$$\mathbb{E}[u(W + D - p_{BID})] = \mathbb{E}[-\exp(-\alpha(W + D - p_{BID}))] \tag{39}$$
$$= -\exp(-\alpha(w + m - p_{BID}) + \frac{1}{2}\alpha^2(\sigma_w^2 + \sigma^2 + 2\sigma\sigma_w\rho)) \tag{40}$$
$$\text{Hence, } -\alpha w + \frac{1}{2}\alpha^2\sigma_w^2 = -\alpha(w + m - p_{BID}) + \frac{1}{2}\alpha^2(\sigma_w^2 + \sigma^2 + 2\sigma\sigma_w\rho) \tag{41}$$
$$\Rightarrow p_{BID} = m - \frac{1}{2}\alpha(\sigma^2 + 2\sigma\sigma_w\rho) \tag{42}$$

where in (40) we use the expression for the mean of the exponential of a Gaussian distributed random variable and the fact that if the joint vector $(A, B)$is Gaussian,

$$A + B \sim \mathcal{N}(\mathbb{E}[A+B], \operatorname{var}(A+B)) \tag{43}$$
$$= \mathcal{N}(\mathbb{E}[A] + \mathbb{E}[B], \operatorname{var}(A) + \operatorname{var}(B) + 2\operatorname{cov}(A,B)) \tag{44}$$

(ii) By similar arguments as before,

$$\mathbb{E}[u(W - D + p_{ask})] = -\exp(-\alpha(w - m + p_{ask}) + \frac{1}{2}\alpha^2(\sigma_w^2 + \sigma^2 - 2\sigma\sigma_w\rho)) \tag{45}$$
$$\text{Hence, } -\alpha w + \frac{1}{2}\alpha^2\sigma_w^2 = -\alpha(w - m + p_{ask}) + \frac{1}{2}\alpha^2(\sigma_w^2 + \sigma^2 - 2\sigma\sigma_w\rho) \tag{46}$$
$$\Leftrightarrow p_{ask} = m + \frac{1}{2}\alpha(\sigma^2 - 2\sigma\sigma_w\rho) \tag{47}$$

Note that this simple model proposes an explanation to the fact that there is a bid-ask spread around the average value of the asset m. This simple model captures the fact that an investor perceives an asset to be worth more than its average price if they want to sell, and to be worth less if they want to buy.

Exercise 2.11.

i. Following the hint, we take W such that

$$\mathbb{P}[W = 2^i] = 2^{-i}; \text{ for all } i = 1, 2, \dots.$$

We easily verify

$$\mathbb{E}[W] = \infty.$$

ii. Let us now modify the above to get the required condition. Take for example $\tilde{W} = \exp(-W)$, i.e.,

$$\mathbb{P}[W = e^{1/2^i}] = 2^{-i}; \text{ for all } i = 1, 2, \dots.$$

Clearly, $\tilde{W}$ is always positive. Moreover,

$$\mathbb{E}[\log(\tilde{W})] = \mathbb{E}[-W] = -\mathbb{E}[W] = -\infty.$$

3 Portfolio choice

One period case

Exercise 3.4. In this setting, we have only two possible assets. Let ϕ be the amount to be invested in the risky asset. Note that if we set $w_0 - \phi$ the amount to be invested in the risk-free asset, the budget constraint on the initial time, is satisfied, since $w_0 = \phi + (w_0 - \phi)$. Hence, we can write the optimisation problem as

$$\begin{aligned} \max_{\phi \in \mathbb{R}} \quad & \psi(\theta) := \mathbb{E}[u(W(\phi))] \\ \text{s.t.} \quad & W(\phi) = (w_0 - \phi)R^0 + \phi R^1 \end{aligned} \tag{48}$$

In this case, we take $u(x) = \log(x)$, and we have,

$$\mathbb{E}[u(W)] = \frac{1}{2}u((w_0 - \phi)R^0 + \phi u) + \frac{1}{2}u((w_0 - \phi)R^0 + \phi d) \tag{49}$$

Using the first order conditions, we have that

$$\frac{\mathrm{d}}{\mathrm{d}\phi}\mathbb{E}[u(W(\phi))]\Big|_{\phi=\phi^*} = 0 \tag{50}$$

$$\Rightarrow u'((w_0 - \phi^*)R^0 + \phi^* u)\frac{1}{2}(u - R^0) + u'((w_0 + \phi^*)R^0 + \phi^* d)\frac{1}{2}(d - R^0) = 0 \tag{51}$$

$$\Rightarrow \frac{u - R^0}{(w_0 - \phi^*)R^0 + \phi^* u} = \frac{R^0 - d}{(w_0 - \phi^*)R^0 + \phi^* d} \tag{52}$$

$$\Rightarrow \phi^*(u - R^0)(d - R^0) + (u - R^0)w_0R^0 = \phi^*(R^0 - d)(u - R^0) + (R^0 - d)(w_0R^0) \tag{53}$$

$$\phi^* = \frac{w_0R^0(u - 2R^0 + d)}{2(R^0 - d)(u - R^0)} \tag{54}$$

Compare with $\bar{\phi} = \frac{w_0R^0(K-1)}{u-R^0+K(R^0-d)}$ (Eq. 5.15 from lecture notes). Given that $\rho = 1$, we have that

$$K = \frac{u - R^0}{R^0 - d} \Rightarrow \bar{\phi} = \frac{w_0R^0(u - R^0 - R^0 - d)}{(u - R^0)(R^0 - d) + (R^0 - d)(u - R^0)}. \tag{55}$$

Thus, it follows that the expression is unchanged.

Exercise 3.5. The main difference between this problem and Exercise (3.4) is the additional constraint on having no short positions. In terms of the variable ϕ (the amount invested in the risky asset), this means that

$$0 \leq \phi \leq w_0. \tag{56}$$

Indeed, ϕ cannot be negative nor be greater than the total (otherwise we would need to short the risk-free asset). Hence, the optimisation problem now reads,

$$\begin{aligned} \max_{\phi \in [0, w_0]} \quad & \psi(\theta) := \mathbb{E}[u(W(\phi))] \\ \text{s.t.} \quad & W(\phi) = (w_0 - \phi)R^0 + \phi R^1 \end{aligned} \tag{57}$$

For the rest of the solution of this exercise, let us call $\bar{\phi}$ to *the solution to the problem* (48) (i.e., the solution with allowed short positions), and let ϕ^* be *the solution to* (57), i.e., with no short positions. Let us remark that since $[0, w_0] \subset \mathbb{R}$, we have that

$$\psi(\bar{\phi}) \geq \psi(\bar{\phi}^*).$$

Clearly, from this inequality we deduce that if $\bar{\phi} \in [0, w_0]$, we have equality and $\bar{\phi} = \phi^*$. Let us explicitly write this condition in terms of the data of the problem. From (55) we get:

$$0 \leq \bar{\phi} \leq w_0 \Leftrightarrow 0 \leq \frac{w_0 R^0 (u - 2R^0 + d)}{2(R^0 - d)(u - R^0)} \leq w_0 \Leftrightarrow 0 \leq R^0(u - 2R^0 + d) \leq 2(R^0 - d)(u - R^0) \tag{58}$$

$$\Leftrightarrow 0 \leq R^0((u - R^0) - (R^0 - d)) \leq 2(u - R^0)(R^0 - d) \tag{59}$$

Recall that we are working under the assumption $0 < d < R_0 < u$. We now consider the cases where (59) does not hold (i.e. if either the left or the right inequality fail).

- If $(u - R^0) < (R^0 - d)$: we would have as optimal $\phi^* = 0$. To verify this, let us check the directional derivative version of the first order conditions. We obtain

$$\begin{aligned}\partial_\phi \mathbb{E}[u(W(\phi))]|_{\phi=0} &= u^1(w_0R^0)\frac{(u - R^0)}{2} + u^1(w_0R^0)\frac{(d - R^0)}{2} \\ &= \frac{u^1(w_0R^0)}{2}((u - R^0) - (R^0 - d)) < 0\end{aligned}$$

  Thus, $\partial_\phi \mathbb{E}[u(W(\phi))]|_{\phi=0}(\phi - 0) \leq 0 \ \forall \phi \in [0, w_0]$. This means that 0 is a local (and thanks to concavity) global minimum of our problem. Note also that in this case, $\bar{\phi} < \phi^*$, so as a result of not being able to short we invest more on the risky asset (or more appropriately, we do not short it).

- If $R^0((u - R^0) - (R^0 - d)) \geq 2(u - R^0)(R^0 - d)$, we claim that $\phi^* = w_0$. The verification follows the same lines as above. In this case, $\bar{\phi}^* < \bar{\phi}$, so we invest less on the risky asset (in order not to short the risk-free one).

Exercise 3.6. The optimisation problem reads

$$\begin{aligned} \max_{\boldsymbol{\phi} \in \mathbb{R}^n} \quad & \mathbb{E}[u(W(\phi))] \\ \text{s.t.} \quad & w_0 = \boldsymbol{\phi} \cdot \mathbf{1}, \\ & W(\phi) = \boldsymbol{\phi} \cdot \boldsymbol{R}_1 \end{aligned} \tag{60}$$

where $u(x) \coloneqq -\exp(-\alpha x)$. We can replace directly the second constraint on the utility function. Moreover, recalling that since $\boldsymbol{R}_1$ is a Gaussian vector, $(\boldsymbol{\phi} \cdot \boldsymbol{R}_1$ is a Gaussian random variable with mean $\boldsymbol{\phi} \cdot \boldsymbol{\mu}$ and variance $\boldsymbol{\phi}^\top \bar{\bar{\Sigma}} \boldsymbol{\phi}$, it follows that

$$\mathbb{E}[u(\boldsymbol{\phi} \cdot \boldsymbol{R}_1))] = \mathbb{E}[-\exp(-\alpha \boldsymbol{\phi} \cdot \boldsymbol{R}_1)] = -\exp(-\alpha \boldsymbol{\phi} \cdot \boldsymbol{\mu} + \frac{1}{2}\alpha^2 \boldsymbol{\phi}^\top \bar{\bar{\Sigma}} \boldsymbol{\phi}). \tag{61}$$

The problem (60) then reads

$$\begin{aligned} \max_{\boldsymbol{\phi} \in \mathbb{R}^n} \quad & -\exp\left(-\alpha \boldsymbol{\phi} \cdot \boldsymbol{\mu} + \frac{1}{2}\alpha^2 \boldsymbol{\phi}^\top \bar{\bar{\Sigma}} \boldsymbol{\phi}\right) \\ \text{s.t.} \quad & w_0 = \boldsymbol{\phi} \cdot \mathbf{1} \end{aligned} \tag{62}$$

The main difference of (62) with respect to section 5.5. is that the control variable is an element of $\mathbb{R}^n$, which corresponds to the fact that there is no risk-free asset. However, as before, observing that $-\exp(-x)$ is increasing, we get that $\boldsymbol{\phi}^*$ solves (62) if and only if it also solves

$$\begin{aligned} \max_{\boldsymbol{\phi} \in \mathbb{R}^n} \quad & \alpha \boldsymbol{\phi} \cdot \boldsymbol{\mu} - \frac{1}{2} \alpha^2 \boldsymbol{\phi}^\top \bar{\bar{\Sigma}} \boldsymbol{\phi} \\ \text{s.t.} \quad & w_0 = \boldsymbol{\phi} \cdot \mathbf{1} \end{aligned} \tag{63}$$

We solve (63) by using the Lagrange multipliers technique[1], i.e. we set

$$L(\boldsymbol{\phi}, \lambda) := \alpha \boldsymbol{\phi} \cdot \boldsymbol{\mu} - \frac{1}{2} \alpha^2 \boldsymbol{\phi}^\top \bar{\bar{\Sigma}} \boldsymbol{\phi} - \lambda (\boldsymbol{\phi} \cdot \mathbf{1} - w_0) \tag{64}$$

and applying the first order conditions, to get

$$\nabla_{\boldsymbol{\phi}} L(\boldsymbol{\phi}^*, \lambda^*) = -\alpha \boldsymbol{\mu} + \alpha^2 \bar{\bar{\Sigma}} \boldsymbol{\phi}^* - \lambda^* \mathbf{1} = \mathbf{0} \tag{65}$$

and also,

$$\partial_\lambda L(\boldsymbol{\phi}^*, \lambda^*) = \boldsymbol{\phi}^* \cdot \mathbf{1} - w_0 = 0 \tag{66}$$

Solving in (65) for $\boldsymbol{\phi}^*$ gives

$$\boldsymbol{\phi}^* = \frac{1}{\alpha^2} \bar{\bar{\Sigma}}^{-1} (\lambda^* \mathbf{1} + \alpha \boldsymbol{\mu}) \tag{67}$$

On the other hand, multiplying (inner product) by $\mathbf{1}$ and using (66) we can write

$$\begin{aligned} w_0 = \boldsymbol{\phi}^* \cdot \mathbf{1} &= (\frac{1}{\alpha^2} \bar{\bar{\Sigma}}^{-1} (\lambda^* \mathbf{1} + \alpha \boldsymbol{\mu})) \cdot \mathbf{1} & (68) \\ &= \frac{\lambda^*}{\alpha^2} \mathbf{1}^\top \bar{\bar{\Sigma}}^{-1} \mathbf{1} + \frac{1}{\alpha} \mathbf{1}^\top \bar{\bar{\Sigma}}^{-1} \boldsymbol{\mu} & (69) \\ \Rightarrow \lambda^* &= \frac{\alpha^2 (w_0 - \frac{1}{\alpha} \mathbf{1}^\top \bar{\bar{\Sigma}}^{-1} \boldsymbol{\mu})}{\mathbf{1}^\top \bar{\bar{\Sigma}}^{-1} \mathbf{1}} = \frac{\alpha^2 w_0 - \alpha \mathbf{1}^\top \bar{\bar{\Sigma}}^{-1} \boldsymbol{\mu}}{\mathbf{1}^\top \bar{\bar{\Sigma}}^{-1} \mathbf{1}} & (70) \end{aligned}$$

which gives us the value of λ^*. Replacing in (67) we conclude that

$$\boldsymbol{\phi}^* = \frac{1}{\alpha} \bar{\bar{\Sigma}}^{-1} \boldsymbol{\mu} + \left(\frac{\alpha w_0 - \mathbf{1}^\top \bar{\bar{\Sigma}}^{-1} \boldsymbol{\mu}}{\alpha \mathbf{1}^\top \bar{\bar{\Sigma}}^{-1} \mathbf{1}} \right) \bar{\bar{\Sigma}}^{-1} \mathbf{1} \tag{71}$$

as claimed. It is interesting to compare this equation with the one in the case where a risk-free asset is available. Note first that the optimal portfolio now *does* depend on the initial wealth! The point is that because there is no risk-free investment and all the initial wealth has to be invested, all alternatives are risky. So the small perturbations around risk-free alternatives are not possible.

To have more insight, let us call $\boldsymbol{\varphi}$ the optimal portfolio when there is a risk-free asset. From Section 5.5 we have that $\boldsymbol{\varphi} = \alpha^{-1} \bar{\bar{\Sigma}}^{-1} \boldsymbol{\mu}$. By replacing and reordering terms, we get

$$\boldsymbol{\phi}^* = \boldsymbol{\varphi} + (w_0 - \mathbf{1} \cdot \boldsymbol{\varphi}) \frac{\bar{\bar{\Sigma}}^{-1} \mathbf{1}}{\mathbf{1}^\top \bar{\bar{\Sigma}}^{-1} \mathbf{1}}$$

[1] It is possible to use the same technique as in the section 5.5., but this time is slightly more involved to write this time

In other words, we are just dividing the amount that would be assigned to the risk-free asset among the remaining ones. This is done proportionally to the sum of all covariances of each asset.

Exercise 3.7. In this case, we find that the optimisation problem can be written

$$\begin{aligned} \max_{c_0 \in \mathbb{R}, \boldsymbol{\phi} \in \mathbb{R}} \quad & \mathbb{E}[u(c_0) + u(C_1)] \\ \text{s.t.} \quad & w_0 = c_0 + \boldsymbol{\phi} \cdot \mathbf{1}, \\ & C_1 = \boldsymbol{\phi} \cdot \boldsymbol{R} \end{aligned} \tag{72}$$

where $\boldsymbol{R} = \begin{bmatrix} R^0 \\ \hat{\boldsymbol{R}} \end{bmatrix}$. Note that using the constraints, we have:

$$\mathbb{E}[u(c_0) + u(C_1)] = u(c_0) + \mathbb{E}[u(C_1)] = u(c_0) + \mathbb{E}[u(\phi^0 R^0 + \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{R}})] \tag{73}$$

$$= u(c_0) + \mathbb{E}[u((w_0 - c_0 - \hat{\boldsymbol{\phi}} \cdot \mathbf{1})R^0 + \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{R}})] \tag{74}$$

Once again, we see that $C_1 = (w_0 - c_0 - \hat{\boldsymbol{\phi}} \cdot \mathbf{1})R^0 + \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{R}}$ is Gaussian with mean $\mathbb{E}[C_1] = (w_0 - c_0 - \hat{\boldsymbol{\phi}} \cdot \mathbf{1})R^0 + \hat{\boldsymbol{\phi}} \cdot \boldsymbol{\mu}$ and variance $\text{var}[C_1] = \hat{\boldsymbol{\phi}}^\top \bar{\bar{\Sigma}} \hat{\boldsymbol{\phi}}$. Hence, we have that

$$\mathbb{E}[u(c_0) + u(C_1)] = -\exp(-\alpha c_0) - \exp(-\alpha \mathbb{E}[C_1] + \frac{1}{2}\alpha^2 \text{var}(C_1)) \tag{75}$$

$$= -\exp(-\alpha c_0) - \exp\underbrace{(-\alpha[(w_0 - c_0 - \hat{\boldsymbol{\phi}} \cdot \mathbf{1})R^0 + \hat{\boldsymbol{\phi}} \cdot \mu] + \frac{1}{2}\alpha^2 \hat{\boldsymbol{\phi}}^\top \bar{\bar{\Sigma}} \hat{\boldsymbol{\phi}})}_{\varphi} \tag{76}$$

Let us apply the first order conditions. We get

$$\partial_{c_0} \mathbb{E}[u(c_0^*) + u(c_1^*)] = \alpha \exp(-\alpha c_0^*) - \alpha R^0 \exp(\varphi) = 0 \tag{77}$$

$$\nabla_{\hat{\boldsymbol{\phi}}} \mathbb{E}[u(c_0^*) + u(c_1^*)] = \alpha \exp(-\alpha \varphi)(\boldsymbol{\mu} - \mathbf{1}R^0 - \alpha \bar{\bar{\Sigma}} \hat{\boldsymbol{\phi}}^*) = 0. \tag{78}$$

Since the exponential function is always positive, from (78), we obtain

$$\hat{\boldsymbol{\phi}}^* = \frac{1}{\alpha} \bar{\bar{\Sigma}}^{-1} (\boldsymbol{\mu} - \mathbf{1}R^0) \tag{79}$$

Let us highlight that this is exactly as in the case of the pure investment problem! The intuition is that riskiness comes only from investment (as the consumption at initial time is deterministic), so that this investor prioritises his decision on how to invest on the risky assets. Also, note that if we express this problem in terms of the number of shares, we have

$$\theta^{i,*} = \frac{\phi^{i,*}}{S_0^i} = \frac{1}{\alpha S_0^i} \bar{\bar{\Sigma}}^{-1} (\boldsymbol{\mu} - \mathbf{1}R^0)$$

It remains to find c_0^* and the amount to be invested in the risk-free asset. Using (77), we have by reordering, dividing by α and taking log that

$$-\alpha c_0^* = -\alpha[(w_0 - c_0^* - \hat{\boldsymbol{\phi}}^* \cdot \mathbf{1})R^0 + \hat{\boldsymbol{\phi}}^* \cdot \boldsymbol{\mu}] + \frac{1}{2}\alpha^2 \hat{\boldsymbol{\phi}}^{*\top} \bar{\bar{\Sigma}} \hat{\boldsymbol{\phi}}^*] + \log(R^0) \tag{80}$$

$$\Rightarrow c_0^* = \frac{1}{1 + R^0}[(w_0 - \hat{\boldsymbol{\phi}}^* \cdot \mathbf{1})R^0 + \hat{\boldsymbol{\phi}}^* \cdot \boldsymbol{\mu} - \frac{1}{2}\alpha \hat{\boldsymbol{\phi}}^{*\top} \bar{\bar{\Sigma}} \hat{\boldsymbol{\phi}}^* - \frac{1}{\alpha}\log(R^0)] \tag{81}$$

Noting that $\hat{\boldsymbol{\phi}}^* \cdot \mathbf{1} = \frac{1}{\alpha}\mathbf{1}^\top \bar{\bar{\Sigma}}^{-1}(\boldsymbol{\mu} - \mathbf{1}R^0)$; $\hat{\boldsymbol{\phi}}^* \cdot \boldsymbol{\mu} = \frac{1}{\alpha}\boldsymbol{\mu}^\top \bar{\bar{\Sigma}}^{-1}(\boldsymbol{\mu} - \mathbf{1}R^0)$ and

$$\hat{\boldsymbol{\phi}}^{*\top}\bar{\bar{\Sigma}}\boldsymbol{\phi} = \frac{1}{\alpha^2}(\boldsymbol{\mu} - \mathbf{1}R^0)^\top \bar{\bar{\Sigma}}^{-1}(\boldsymbol{\mu} - \mathbf{1}R^0)$$

we finally get

$$c_0^* = \frac{1}{1+R^0}[w_0 R^0 + \frac{1}{2\alpha}(\boldsymbol{\mu} - \mathbf{1}R^0)^\top \bar{\bar{\Sigma}}^{-1}(\boldsymbol{\mu} - \mathbf{1}R^0) - \frac{1}{\alpha}\log(R^0)].$$

Using that $\phi^{0,*} = w_0 - c_0^* - \hat{\boldsymbol{\phi}}^*$, we also get the initial amount invested in the risky-free asset.

Exercise 3.8. The problem can be written

$$\max_{\theta,\eta}\left[u\left(\eta\omega_0\right) + \delta\mathbb{E}\left[u\left(W^{w_0(1-\eta),\phi}\right)\right] = \max_{\eta}\left[u(\eta w_0) + \delta\max_{\phi}\mathbb{E}\left[u\left(W^{w_0(1-\eta),\phi}\right)\right]\right]\right.$$

By Example 3.1.3 we have that

$$\phi^* = \frac{w_0(1-\eta)R^0(x-1)}{u - R^0 + K\left(R^0 - d\right)}$$

solves the max problem with initial wealth $w_0(1-\eta)$.

We now find the value of expected utility at this maximum. We look first at $W^{w_0(1-eta),\phi^*}$.

We have that if $R^1 = u$,

$$W^{w_0(1-\eta),\Phi^*}(\omega = 1) = w_0(1-\eta)R^0\left(1 + \frac{(K-1)\left(u - R^0\right)}{u - R^0 + K\left(R^0 - d\right)}\right)$$

but since $K^\rho = \frac{u-R^0}{R^0-d}$, we have

$$W^{w_0(1-\eta),\Phi^*} = w_0(1-\eta)R^0\left(1 + \frac{K-1}{1+K^{1-\rho}}\right).$$

By a similar procedure, we get that if $R^1 = d$

$$W^{w_0(1-\eta),\Phi^*}(\omega = 0) = w_0(1-\eta)R^0\left(1 + \frac{1-K}{K^\rho + K}\right).$$

Hence,

$$\mathbb{E}\left[u\left(W^{w_0(u-v)_1\phi^2}\right)\right] = \frac{\left(w_0R^0(1-\eta)\right)^{1-\rho}}{2(1-\rho)}\underbrace{\left[\left(\frac{K+K^{1-\rho}}{1+K^{1-\rho}}\right)^{1-\rho} + \left(\frac{K^\rho+1}{K^\rho+K}\right)\right]}_{=C}$$

Finally, we solve

$$\max_{\eta}\left[\frac{\left(w_0\eta\right)^{1-\rho}}{1-\rho} + \frac{\left(w_0R^0(1-\eta)\right)^{1-\rho}}{2(1-\rho)}C\delta\right]$$

which by a first order condition gives

$$
\begin{aligned}
&(w_0\eta)^{-\rho} w_0 - \tfrac{\left(w_0 R^0(1-\eta)\right)^{-\rho}}{2}(C\delta)\left(w_0 R^0\right) = 0 \\
&\Rightarrow \eta = \tfrac{R^0(1-\eta)2^{1/\rho}}{(C\delta)^{1/\rho}(R^0)^{1/\rho}} \\
&\Rightarrow \eta = \tfrac{(R^0)^{\frac{\rho-1}{\rho}} 2^{1/\rho}}{(C\delta)^{1/\rho}+(R^0)^{(\rho-1)/\rho}2^{1/\rho}}
\end{aligned}
$$

Some checks are in order: note that $\eta \in (0,1)$ as expected. Moreover, when $\delta \to 0$, $\eta \to 1$.

Exercise 3.9. Since we have only one riskless asset, we have that the only uncertainty on this case comes from the random endowment. The decision variables are c_0 and ϕ (the amount to be invested in the risk-free asset), and the problem reads

$$
\begin{aligned}
\max_{c_0,\,\phi\,\in\,\mathbb{R}} \quad & \mathbb{E}[u(c_0) + \delta u(C_1)] \\
\text{s.t.} \quad & w_0 = c_0 + \phi, \\
& C_1 = \phi R_0 + Y
\end{aligned}
\tag{82}
$$

We can introduce both restrictions on the utility as follows: from the first restriction, we have $\phi = (w_0 - c_0)$ and replacing on the end of period constraint, we get that $C_1 = (w_0 - c_0)R^0 + Y$. Thus, we can solve the unconstrained problem

$$
\max_{c_0\,\in\,\mathbb{R}} \quad u(c_0) + \delta\mathbb{E}[u((\omega_0 - c_0)R^0 + Y)] \tag{83}
$$

which solves the first question. For the second claim, because $u'' < 0$, we know that u is strictly concave. Using first order conditions, we get

$$
u'(c_0^*) - \delta\mathbb{E}[u'((\omega_0 - c_0^*)R^0 + Y)]R^0 = 0. \tag{84}
$$

On the other hand, using $u''' > 0$, we get by Jensen's inequality that

$$
\mathbb{E}[u'((\omega_0 - c_0^*)R^0 + Y)] \geq u'(\mathbb{E}[(\omega_0 - c_0^*)R^0 + Y]) = u'((\omega_0 - c_0^*)R^0) \tag{85}
$$

where we used the fact that $\mathbb{E}[Y] = 0$. Equations (84) and (85) imply that

$$
u'(c_0^*) \geq \delta u'((\omega_0 - c_0^*)R^0)R^0. \tag{86}
$$

Assuming now that $Y = 0$ (denoting $\hat{c}_0$ the optimal in this case), we have

$$
u'(\hat{c}_0) = \delta u'((\omega_0 - \hat{c}_0)R^0)R^0. \tag{87}
$$

We claim now that (86) and (87) deduce that $\hat{c}_0 \geq c_0^*$. Assume by contradiction that $\hat{c}_0 < c_0^*$, and observe that the left hand of (86) is monotonously decreasing as function of c_0^* (due to concavity) while the right-hand side is increasing with c_0^* (also due to concavity). Hence, we would get from (87) and $\hat{c}_0 < c_0^*$ that $u'(c_0^*) < \delta u'((\omega_0 - c_0^*)R^0)$, which would contradict (86). It follows that $\hat{c}_0 \geq c_0^*$ as desired.

Multi-period case

Exercise 3.10. By definition 1.29, we have that a measure $\mathcal{Q}$ is risk neutral if and only if the Radon-Nikodym density with respect to $\mathbb{P}$ can be written

$$
\frac{\mathrm{d}\mathcal{Q}}{\mathrm{d}\mathbb{P}} := M_T S_T^0
$$

for some SDF M. Hence, it follows that for any $s = 0, \ldots, T$

$$\mathbb{E}^{\mathcal{Q}}\left[\frac{C_s}{S_s^0}\right] = \mathbb{E}\left[M_T S_T^0 \frac{C_s}{S_s^0}\right] = \mathbb{E}\left[\mathbb{E}_s[M_T S_T^0]\frac{C_s}{S_s^0}\right]$$

where we used the definition of the RN density and of conditional expectation, given that both C_s, S_s^0 are $\mathcal{F}_s$ measurable. But by the definition of SDF, $\mathbb{E}_s[M_T S_T^0] = M_s S_s^0$. Hence,

$$\mathbb{E}^{\mathcal{Q}}\left[\frac{C_s}{S_s^0}\right] = \mathbb{E}\left[M_s C_s\right].$$

Exactly the same arguments show that

$$\mathbb{E}^{\mathcal{Q}}\left[\frac{I_s}{S_s^0}\right] = \mathbb{E}\left[M_s I_s\right].$$

Hence, the claim follows by Lemma 3.17.

Exercise 3.12. As per the question, let us call $\pi^* \in \mathbb{R}^{n+1}$ the portfolio achieving the maximum in the log-utility maximisation for one period.

Let $\pi \in \mathbb{R}^{n+1}$ be other non-optimal strategy (that we keep fixed), and we need to compare $S_t(\pi)$ and $S_t(\pi^*)$ for t large enough. Let

$$\Delta := \mathbb{E}[\log(\pi^* R_1)] - \mathbb{E}[\log(\pi R_1)].$$

By assumption, we have that $0 < \Delta < \infty$.

Let us now recall that for a dividend-reinvested portfolio without consumption, we have that

$$S_0(\pi) = w_0; \quad S_t(\pi) = w_0 \prod_{s=1}^{t} (\pi^\top R_s),$$

Clearly the strict monotonicity of log implies that

$$S_t(\pi) < S_t(\pi^*) \text{ if and only if } \frac{1}{t}\log(S_t(\pi)) < \frac{1}{t}\log(S_t(\pi^*)).$$

So we focus in showing a statement on the right-hand side. Indeed, note that

$$\frac{1}{t}\log(S_t(\pi)) = \frac{\log(w_0)}{t} + \frac{1}{t}\sum_{s=1}^{t}\log(\pi^\top R_s);$$

and, similarly for π^*. Thus,

$$\frac{1}{t}\log(S_t(\pi^*)) - \frac{1}{t}\log(S_t(\pi)) = \frac{1}{t}\sum_{s=1}^{t}\{\log((\pi^*)^\top R_s) - \log(\pi^\top R_s)\}. \tag{88}$$

Now, the law of large numbers implies that $\frac{1}{t}\log(S_t(\pi^*)) \to \mathbb{E}[\log(S_t(\pi^*))]$ almost surely (and similarly for π). Their difference converges almost surely then to Δ. Hence, there exists $T > 0$ such that for all $t > T$,

$$\mathbb{P}\left(|\frac{1}{t}\log(S_t(\pi^*)) - \frac{1}{t}\log(S_t(\pi)) - \Delta| \le \frac{\Delta}{2}\right) = 1$$

From where we deduce that

$$\mathbb{P}\left(\frac{1}{t}\log(S_t(\pi^*)) - \frac{1}{t}\log(S_t(\pi)) \ge \frac{\Delta}{2} > 0\right) = 1$$

for all $t > T$, as wanted.

4 Risk Measures

Note: In the following, Φ is the c.d.f of the standard Gaussian and ϕ is the p.d.f of the standard Gaussian.

Exercise 4.1. We consider first the case $X \sim \mathcal{N}(0,1)$. Since $-X \sim \mathcal{N}(0,1)$ and Φ is invertible, we have by definition that

$$\mathrm{V@R}^{\alpha}(\mathrm{X}) = \mathrm{q}_{-\mathrm{X}}(\alpha) = \Phi^{-1}(\alpha).$$

Now, in the general case if $X \sim \mathcal{N}(\mu, \sigma^2)$ then

$$Z := \frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1).$$

Hence, using the cash invariant and positive homogeneity properties of V@R we have

$$\mathrm{V@R}^{\alpha}(\mathrm{X}) = \mathrm{V@R}^{\alpha}(\sigma\mathrm{Z}+\mu) = \mathrm{V@R}^{\alpha}(\sigma\mathrm{Z}) - \mu = \sigma\mathrm{V@R}^{\alpha}(\mathrm{Z}) - \mu = \sigma\Phi^{-1}(\alpha) - \mu$$

Exercise 4.2. Expected shortfall:

$$\begin{aligned}
\mathrm{ES}^{\alpha}(\mathrm{X}) &= \frac{1}{1-\alpha}\int_{\alpha}^{1} \mathrm{V@R}^{\mathrm{u}}(\mathrm{X})\mathrm{du} && (89)\\
&= \frac{1}{1-\alpha}\int_{\alpha}^{1} (\sigma\Phi^{-1}(u) - \mu)du && (90)\\
&= \sigma\left[\frac{1}{1-\alpha}\int_{\alpha}^{1}\Phi^{-1}(u)du\right] - \mu && (91)
\end{aligned}$$

Since $u = \Phi(x)$, we have $du = \phi(x)dx$. If we set $v = -\frac{x^2}{2}$, we have $dv = -xdx$. Now,

$$\begin{aligned}
\int_{\alpha}^{1}\Phi^{-1}(u)du = \int_{\Phi^{-1}(\alpha)}^{\infty} x\phi(x)dx = \frac{1}{\sqrt{2\pi}}\int_{\Phi^{-1}(\alpha)}^{\infty} xe^{-\frac{x^2}{2}}dx && (92)\\
= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\frac{(\Phi^{-1}(\alpha))^2}{2}} e^{v}dv = \frac{1}{\sqrt{2\pi}}e^{-\frac{(\Phi^{-1}(\alpha))^2}{2}} = \phi(\Phi^{-1}(\alpha)). && (93)
\end{aligned}$$

Hence, $\mathrm{ES}^{\alpha}(\mathrm{X}) = \sigma\frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha} - \mu$.

Exercise 4.3. (a) In the uniform case, if $X \sim \mathcal{U}[0,1]$, we have $L = -X \sim \mathcal{U}[-1,0]$.

$$\mathrm{V@R}^{\alpha}(\mathrm{X}) = \inf_{\mathrm{x}}\{\mathrm{F_L}(\mathrm{x}) \geq \alpha\} = \alpha - 1 \qquad (94)$$

$$\begin{aligned}
\mathrm{ES}^{\alpha}(\mathrm{X}) &= \frac{1}{1-\alpha}\int_{\alpha}^{1}\mathrm{V@R}^{\mathrm{u}}(\mathrm{X})\mathrm{du} = \frac{1}{1-\alpha}\int_{\alpha}^{1}(\mathrm{u}-1)\mathrm{du} && (95)\\
&= \frac{1}{1-\alpha}\left[\frac{(1-\alpha)^2}{2} - (1-\alpha)\right] = \frac{(1-\alpha)}{2} - 1 = -\frac{(1+\alpha)}{2} && (96)
\end{aligned}$$

(b) In the log-normal case, $X = e^{\sigma Z + m}$ with $Z \sim \mathcal{N}(0,1)$.

$$\begin{aligned}
\mathrm{V@R}^\alpha(\mathrm{X}) &= \inf\{x : \mathbb{P}(-X \le x) \ge \alpha\} && (97)\\
&= \inf\{x : \mathbb{P}(-e^{\sigma Z + m} \le x) \ge \alpha\} && (98)\\
&= \inf\{x : \mathbb{P}(e^{\sigma Z + m} \ge -x) \ge \alpha\} && (99)\\
&= \inf\{x : \mathbb{P}(e^{\sigma Z + m} < -x) \le 1-\alpha\}, && (100)
\end{aligned}$$

since $\mathbb{P}(e^{\sigma Z+m} < -x) + \mathbb{P}(e^{\sigma Z+m} \ge -x) = 1$. Remark that if $x \ge 0$, $\mathbb{P}(e^{\sigma Z+m} < -x) = 0$, so that we restrict our search to $x < 0$. Then, by taking logarithm

$$\begin{aligned}
&= \inf\{x : \mathbb{P}(\sigma Z + m < \log(-x)) \le 1-\alpha\} && (101)\\
&= \inf\{x : \mathbb{P}(Z + \frac{m - \log(-x)}{\sigma} \le 0) \le 1-\alpha\} && (102)
\end{aligned}$$

where we use continuity to turn $<$ to $\le$. Note that $\frac{m-\log(-x)}{\sigma}$ is an increasing function, $\frac{m-\log(-x)}{\sigma} = \Phi^{-1}(\alpha)$. Hence,

$$\mathrm{V@R}^\alpha(\mathrm{X}) = -\exp(\mathrm{m} - \Phi^{-1}(\alpha)\sigma). \tag{103}$$

Expected shortfall: Note that

$$\{-X \ge \mathrm{V@R}^\alpha(\mathrm{X})\} \Leftrightarrow \{-\exp(\mathrm{m} + \sigma\mathrm{Z}) \ge -\exp(\mathrm{m} - \Phi^{-1}(\alpha)\sigma)\} \Leftrightarrow \{\mathrm{Z} \le -\Phi^{-1}(\alpha)\}$$

Hence,

$$\begin{aligned}
\mathrm{ES}^\alpha(\mathrm{X}) &= -\mathbb{E}(X | -X \ge \mathrm{V@R}^\alpha(\mathrm{X})) && (104)\\
&= -\mathbb{E}(X | Z \le -\Phi^{-1}(\alpha)) && (105)\\
&= -\frac{1}{1-\alpha}\int_{-\infty}^{-\Phi^{-1}(\alpha)} e^{\sigma y + m}\frac{1}{\sqrt{2\pi}}e^{\frac{-y^2}{2}}dy && (106)\\
&= -\frac{1}{1-\alpha}\int_{-\infty}^{-\Phi^{-1}(\alpha)} \frac{1}{\sqrt{2\pi}}e^{\frac{-y^2}{2}+\sigma y - \frac{\sigma^2}{2}}e^{\frac{\sigma^2}{2}+m}dy && (107)\\
&= -\frac{e^{\frac{\sigma^2}{2}+m}}{1-\alpha}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\Phi^{-1}(\alpha)} e^{-\frac{(y-\sigma)^2}{2}}dy && (108)\\
&= -\frac{e^{\frac{\sigma^2}{2}+m}}{1-\alpha}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\Phi^{-1}(\alpha)-\sigma} e^{-\frac{\hat{y}^2}{2}}d\hat{y} \text{ where } \hat{y} = y + \sigma, && (109)\\
&= -\frac{e^{\frac{\sigma^2}{2}+m}}{1-\alpha}\Phi(-\Phi^{-1}(\alpha) - \sigma) = -\frac{e^{\frac{\sigma^2}{2}+m}}{1-\alpha}[1 - \Phi(\Phi^{-1}(\alpha) + \sigma)] && (110)
\end{aligned}$$

Exercise 4.5. We have $d = 100$ defaultable bonds and $P = 100$ price. The total P&L X is

$$X = \begin{cases} 5 & \text{with probability}: \ 98\% \\ -100 & \text{with probability}: \ 2\% \end{cases} \tag{111}$$

We can easily see that in the case of portfolio A, $X_A = 100X$, and hence

$$\mathrm{V@R}^{0.95}(\mathrm{X_A}) = \mathrm{V@R}^{0.95}(100\mathrm{X}) = 100\mathrm{V@R}^{0.95}(\mathrm{X}) = 100\mathrm{q_{-X}}(0.95) = -500.$$

On the other hand, we have that for portfolio B, the P& L can be seen in terms of 100 rescaled Bernoulli trials. Indeed, $X = 105B - 100$ where B is Bernoulli with probability $p = 0.98$. Hence, the total P&L is

$$X_B = \sum_{i=1}^{100} X_i = \sum_{i=1}^{100} 105B_i - 100 = 105Y - 10000$$

where Y is Binomial with $p = 0.98$ and 100 trials. Now, we can use the computer to make the calculation of the quantile. Here instead we use the De Moivre-Laplace theorem (or the central limit theorem) to argue that we can approximate the cumulative distribution of Y by $F_Y \approx \mathcal{N}(100p, 100p(1-p)) = \mathcal{N}(98, 1.96)$, and quantiles should also be closed together. Hence, using Exercise 4.1 and the properties of value at risk, we get

$$\begin{aligned} \mathrm{V@R^{0.95}(X_B)} &\approx \mathrm{V@R^{0.95}(105Y - 1000)} = \mathrm{105V@R^{0.95}(Y) + 10000} \\ &= 105(\sqrt{1.96}\Phi^{-1}(0.95) - 98) + 1000 \approx -48.2. \end{aligned}$$

Clearly $\mathrm{V@R^{0.95}(X_B)} > \mathrm{V@R^{0.95}(X_A)}$. This is a counterexample for subadditivity of value at risk: indeed, if subadditivity held we would have

$$\mathrm{V@R^{0.95}(X_B) = V@R^{0.95}(\sum_{i=1}^{100} X_i) \leq 100V@R^{0.95}(X) = V@R^{0.95}(100X) = V@R^{0.95}(X_A)}.$$

which is a contradiction.

Exercise 4.7. Show that $\rho(X) = \mathbb{E}[-X]$ is a coherent risk measure. Let us verify our properties:

(i) Cash invariance:

$$\rho(X + a) = \mathbb{E}[-X - a] = \mathbb{E}[-X] - a = \rho(X) - a \tag{112}$$

(ii) Monotonicity: Assume that $X_1 \leq X_2$, then $-X_1 \geq -X_2$.

$$\mathbb{E}[-X_1] \geq \mathbb{E}[-X_2] \Rightarrow \rho(X_1) \geq \rho(X_2) \tag{113}$$

(iii) Subadditivity:

$$\rho(X_1 + X_2) = \mathbb{E}[-X_1 - X_2] = \mathbb{E}[-X_1] + \mathbb{E}[-X_2] = \rho(X_1) + \rho(X_2) \tag{114}$$

(iv) Positive homogeneity: Assume that $\lambda > 0$,

$$\rho(\lambda X) = \mathbb{E}[-\lambda X] = \lambda\mathbb{E}[-X] = \lambda\rho(X) \tag{115}$$

Exercise 4.8. Let us show that $\rho(X) = \sup_\omega - X(\omega)$ is a coherent risk measure.

(i) Cash invariance:

$$\begin{aligned} \rho(X + a) &= \sup\nolimits_\omega(-X(\omega) - a) && (116) \\ &= \sup\nolimits_\omega(-X(\omega)) - a && (117) \\ &= \rho(X) - a && (118) \end{aligned}$$

(ii) Monotonicity: Assume $X(\omega) \leq Y(\omega)$ for all $\omega \in \Omega$. Then, $-X(\omega) \geq -Y(\omega)$.

$$\begin{aligned} \sup\nolimits_\omega - X(\omega) &\geq \sup\nolimits_\omega - Y(\omega) && (119) \\ \rho(X) &\geq \rho(Y) && (120) \end{aligned}$$

(iii) Subadditivity:

$$\rho(X_1 + X_2) = \sup_{\omega\in\Omega}(-X_1(\omega) - X_2(\omega)) \le \sup_{\omega\in\Omega}(-X_1(\omega)) + \sup_{\omega\in\Omega}(-X_2(\omega)) \tag{121}$$

$$\le \rho(X_1) + \rho(X_2) \tag{122}$$

(iv) Positive homogeneity: Assume that $\lambda > 0$,

$$\rho(\lambda X) = \sup_{\omega\in\Omega}(-\lambda X(\omega)) = \lambda \cdot \sup_{\omega\in\Omega}(-X(\omega)) = \lambda\rho(X) \tag{123}$$

Exercise 4.9. Assume that ρ is a convex risk measure, and $\rho(0) = 0$.

(a) Show that $\rho(\lambda X) \le \lambda\rho(X)$ if $\lambda \in [0, 1]$. By convexity, $\rho(\lambda X+(1-\lambda)0) \le \lambda\rho(X)+(1-\lambda)\rho(0)$ from where the claim follows.

(b) Show that $\rho(\lambda X) \ge \lambda\rho(X)$ if $\lambda \in [1, \infty)$. Assume that $\lambda > 1$. Then, take $\hat{\lambda} = \frac{1}{\lambda}$ and $\hat{X} = \lambda X$. Since $\hat{\lambda} \in [0, 1]$, we have from the first part,

$$\rho(\hat{\lambda}\hat{X}) \le \hat{\lambda}\rho(\hat{X}) \Leftrightarrow \rho\left(\frac{1}{\lambda} \cdot \lambda X\right) \le \frac{1}{\lambda}\rho(\lambda X) \tag{124}$$

$$\Leftrightarrow \lambda\rho(X) \le \rho(\lambda X). \tag{125}$$

Exercise 4.10. Recall that the amount of capital to cover risks measured with a risk measure ρ is $(\rho(X))^+$. Thus, we need to calculate $\mathrm{V@R}^\alpha(\mathrm{X})$ and $\mathrm{ES}^\alpha(\mathrm{X})$, where $X = S_1 - S_0$.

By the cash invariant and positive homogeneity properties, we have

$$\mathrm{V@R}^\alpha(\mathrm{X}) = \mathrm{V@R}^\alpha(\mathrm{S_1 - S_0}) = \mathrm{S_0 V@R}^\alpha(\mathrm{S_1/S_0}) + \mathrm{S_0},$$

and by using the result of Exercise 4.3, we have

$$\mathrm{V@R}^\alpha(\mathrm{X}) = \mathrm{S_0}(1 - \exp(\mathrm{m} - \Phi^{-1}(\alpha)\sigma)).$$

In a similar way,

$$\mathrm{ES}^\alpha(\mathrm{X}) = \mathrm{S_0}\left\{1 - \frac{\mathrm{e}^{\frac{\sigma^2}{2}+\mathrm{m}}}{1-\alpha}[1 - \Phi(\Phi^{-1}(\alpha) + \sigma)]\right\}.$$

The required capital is the positive part of both measures above.

(*) Exercise 4.12. Clearly, if $\rho(X) = \mathbb{E}_{\mathbb{Q}}[-X]$, we have that

$$\rho(X_1 + X_2) = \mathbb{E}_{\mathbb{Q}}[-(X_1 + X_2)] = \mathbb{E}_{\mathbb{Q}}[-X_1] + \mathbb{E}_{\mathbb{Q}}[-X_2] = \rho(X_1) + \rho(X_2). \tag{126}$$

Hence, if the class $\mathcal{Q}$ reduces to a unique probability measure $\mathbb{Q}$, we have that ρ is additive. To prove the reciprocal, assume that the class $\mathcal{Q}$ contains at least two different $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Without loss of generality, we also assume normality, i.e., that $\rho(0) = 0$.

Now, since $\mathcal{Q}_1 \neq \mathcal{Q}_2$, there exists a set $A \in \mathcal{A}$ such that $\mathcal{Q}_1(A) \neq \mathcal{Q}_2(A)$. Without loss of generality, assume that $\mathcal{Q}_1(A) < \mathcal{Q}_2(A)$. Define $X_1 = \mathbb{1}_A$; $X_2 = \mathbb{1}_{A^c}$. It follows that $\rho(X_1 + X_2) = \rho(1) = -1$ thanks to cash invariance and normality. But then

$$\rho(X_1) + \rho(X_2) = \sup_{Q\in\mathcal{Q}}(-\mathbb{E}[X_1]) + \sup_{Q\in\mathcal{Q}}(-\mathbb{E}[X_2]) \tag{127}$$

$$\ge \mathbb{E}^{Q_1}[-X_1] + \mathbb{E}^{Q_2}[-X_2] \tag{128}$$

$$= -\mathcal{Q}_1(A) - (1 - \mathcal{Q}_2(A)) \tag{129}$$

$$= -1 + (\mathcal{Q}_2(A) - \mathcal{Q}_1(A)) > -1. \tag{130}$$

Hence, ρ is not subadditive.

5 Equilibrium and CAPM

Exercise 5.1. To deduce an expression like CAPM in Theorem 5.5, we study the equilibrium market problem when the utility function of each one of the participants is

$$U_k[X] = \xi_k \mathbb{E}[X] - \frac{\text{var}[X]}{2}.$$

We keep the same notation and assumptions as in the theorem.

Recall that a market is in equilibrium if two conditions are satisfied: i. The market clears; and ii. All investors are optimally allocated.

Participants are optimally invested. Let us start by finding the optimal investment for the k−th participant. We call $\hat{\boldsymbol{\phi}}_k$ the vector of the **amount** of wealth the k−th participant invests on the risky assets. We get that the final wealth of the investor is

$$W_{1,k} = \hat{\boldsymbol{\phi}}_k \cdot \hat{\boldsymbol{R}} + (w_{0,k} - \hat{\boldsymbol{\phi}}_k \cdot \hat{\mathbf{1}})R^0$$

and so,

- $\mathbb{E}[W_{1,k}] = \hat{\boldsymbol{\phi}}_k \cdot \boldsymbol{\mu} + (w_{0,k} - \hat{\boldsymbol{\phi}}_k \cdot \hat{\mathbf{1}})R^0 = \hat{\boldsymbol{\phi}}_k \cdot (\boldsymbol{\mu} - R^0 \hat{\mathbf{1}}) + w_{0,k} R^0$
- $\text{var}[W_{1,k}] = \hat{\boldsymbol{\phi}}_k^\top \bar{\bar{\Sigma}} \hat{\boldsymbol{\phi}}_k$.

Note that we have encoded the constraints of the optimal investment problem, so we only need to find

$$\max_{\hat{\boldsymbol{\phi}}_k \in \mathbb{R}^n} U_k(W_{1,k}).$$

We find the first order condition for optimal investment by taking the gradient with respect to ϕ_k of the reward function and making it equal to zero. We get

$$\nabla_{\hat{\boldsymbol{\phi}}_k} U_k[W_{1,k}] = \xi_k(\boldsymbol{\mu} - R^0 \hat{\mathbf{1}}) + \bar{\bar{\Sigma}} \hat{\boldsymbol{\phi}}_k^* = \hat{\mathbf{0}},$$

and solving for $\hat{\boldsymbol{\phi}}_k$,

$$\hat{\boldsymbol{\phi}}_k^* = \xi_k \bar{\bar{\Sigma}}^{-1}(\boldsymbol{\mu} - R^0 \hat{\mathbf{1}}).$$

Market clears. We look now at the condition that the market clears. As in Theorem 7.1., we assume that assets are on unit net supply, so that the total amount invested on the risky assets by all the M market participants should coincide with the price of the assets, i.e.

$$\hat{\boldsymbol{p}} = \sum_{k=1}^{M} \hat{\boldsymbol{\phi}}_k^* = \sum_{k=1}^{M} \xi_k \bar{\bar{\Sigma}}^{-1}(\boldsymbol{\mu} - R^0 \hat{\mathbf{1}}).$$

Also, we assume that the risk-free asset is in zero-net supply, and thus,

$$\sum_{k=1}^{M}(w_{0,k} - \hat{\boldsymbol{\phi}}_k^* \cdot \hat{\mathbf{1}}) = 0 \Leftrightarrow \sum_{k=1}^{M} w_{0,k} = \sum_{k=1}^{M} \hat{\boldsymbol{\phi}}_k^* \cdot \hat{\mathbf{1}} = (\sum_{k=1}^{M} \hat{\boldsymbol{\phi}}_k^*) \cdot \hat{\mathbf{1}} = \hat{\boldsymbol{p}} \cdot \hat{\mathbf{1}} =: p_m.$$

Hence,

$$\hat{\boldsymbol{\pi}}_m = \frac{1}{p_m}\hat{\boldsymbol{p}} = \underbrace{\left(\frac{1}{p_m}\sum_{k=1}^{M} \xi_k\right)}_{\bar{\Gamma}} \bar{\bar{\Sigma}}^{-1}(\boldsymbol{\mu} - R^0 \hat{\mathbf{1}}) \quad \text{and} \quad \hat{\boldsymbol{\pi}}_m \cdot \hat{\mathbf{1}} = 1$$

So that exactly as in the Theorem, we have for $R_m := \hat{\boldsymbol{\pi}}_m \cdot R$

$$\operatorname{cov}(R, R_m) = \operatorname{cov}(R, \hat{\boldsymbol{\pi}}_m \cdot R) = \bar{\bar{\bar{\Sigma}}} \hat{\boldsymbol{\pi}}_m = \bar{\Gamma}(\mu - R^0 \hat{\mathbf{1}})$$

and

$$\operatorname{var}(R_m) = \hat{\boldsymbol{\pi}}_m^T \bar{\bar{\bar{\Sigma}}} \hat{\boldsymbol{\pi}}_m = \bar{\Gamma} \hat{\boldsymbol{\pi}}_m^T (\boldsymbol{\mu} - R^0 \hat{\mathbf{1}}) = \Gamma(\mu_m - R^0)$$

So that

$$\beta_i = \frac{\operatorname{cov}(R, R_m)}{\operatorname{var}(R_m)} = \frac{\mu^i - R^0}{\mu_m - R^0}$$

as desired.

6 Efficient frontiers

Exercise 6.1. We need to prove that for any two different portfolios $\bar{\boldsymbol{\pi}}_a$ and $\bar{\boldsymbol{\pi}}_b$ with $\bar{\boldsymbol{\pi}}_a \neq \bar{\boldsymbol{\pi}}_b$ express any other different portfolio (say $\bar{\boldsymbol{\pi}}_c$) by

$$\bar{\boldsymbol{\pi}}_c = \lambda\bar{\boldsymbol{\pi}}_a + (1-\lambda)\bar{\boldsymbol{\pi}}_b \text{ for some } \lambda \in \mathbb{R} \tag{131}$$

Now, we know from the two fund portfolios that

$$\bar{\boldsymbol{\pi}}_a = \alpha_a\bar{\boldsymbol{\pi}}_\mu + (1-\alpha_a)\bar{\boldsymbol{\pi}}_1 \tag{132}$$
$$\bar{\boldsymbol{\pi}}_b = \alpha_b\bar{\boldsymbol{\pi}}_\mu + (1-\alpha_b)\bar{\boldsymbol{\pi}}_1 \tag{133}$$

We can solve these equations to obtain $\bar{\boldsymbol{\pi}}_\mu$ and $\bar{\boldsymbol{\pi}}_1$ in terms of $\bar{\boldsymbol{\pi}}_a$ and $\bar{\boldsymbol{\pi}}_b$:

$$\underbrace{\frac{\alpha_b}{\alpha_b-\alpha_a}}_{\lambda_1}\bar{\boldsymbol{\pi}}_a + (1-\frac{\alpha_b}{\alpha_b-\alpha_a})\bar{\boldsymbol{\pi}}_b = \bar{\boldsymbol{\pi}}_1 \tag{134}$$
$$\underbrace{\frac{1-\alpha_b}{\alpha_a-\alpha_b}}_{\lambda_\mu}\bar{\boldsymbol{\pi}}_a + (1-\frac{1-\alpha_b}{\alpha_a-\alpha_b})\bar{\boldsymbol{\pi}}_b = \bar{\boldsymbol{\pi}}_\mu \tag{135}$$

So we can express $\bar{\boldsymbol{\pi}}_1$ and $\bar{\boldsymbol{\pi}}_\mu$ as a portfolio on the two portfolios $\bar{\boldsymbol{\pi}}_a$ and $\bar{\boldsymbol{\pi}}_b$. Moreover,

$$\begin{aligned}
\bar{\boldsymbol{\pi}}_c &= \alpha_c\bar{\boldsymbol{\pi}}_\mu + (1-\alpha_c)\bar{\boldsymbol{\pi}}_1 && (136)\\
&= \alpha_c(\lambda_1\bar{\boldsymbol{\pi}}_a + (1-\lambda_1)\bar{\boldsymbol{\pi}}_b) + (1-\alpha_c)(\lambda_\mu\bar{\boldsymbol{\pi}}_a + (1-\lambda_\mu)\bar{\boldsymbol{\pi}}_b) && (137)\\
&= (\alpha_c\lambda_1 + \lambda_\mu - \lambda_\mu\alpha_c)\bar{\boldsymbol{\pi}}_a + (\alpha_c - \lambda_1\alpha_c + 1 - \alpha_c - \lambda_\mu + \lambda_\mu\alpha_c)\bar{\boldsymbol{\pi}}_b && (138)\\
&= \lambda_c\bar{\boldsymbol{\pi}}_a + (1-\lambda_c)\bar{\boldsymbol{\pi}}_b && (139)
\end{aligned}$$

with $\lambda_c = \alpha_c\lambda_1 + \lambda_\mu - \lambda_\mu\alpha_c$, so that our claim is shown.

Conclusion: any two distinct portfolios in the mean-variance frontier generate the whole frontier.

Exercise 6.2. Since we are looking for the optimal point and not the optimal value, we can solve for convenience $\min_{\bar{\boldsymbol{\pi}}\in\mathbb{R}^d} \frac{1}{2}\bar{\boldsymbol{\pi}}^\top\bar{\bar{\Sigma}}\bar{\boldsymbol{\pi}}$ s.t. $\boldsymbol{\pi}\cdot\mathbf{1} = 1$: indeed, the constant '1/2' factor does not change the optimal.

Following our procedure, we transform the problem in an unconstrained problem via the Lagrangian

$$\min_{\bar{\boldsymbol{\pi}}\in\mathbb{R}^d,\sigma\in\mathbb{R}} L(\bar{\boldsymbol{\pi}},\sigma) \text{ where } L(\bar{\boldsymbol{\pi}},\sigma) = \bar{\boldsymbol{\pi}}^\top\bar{\bar{\Sigma}}\bar{\boldsymbol{\pi}} + \sigma(\bar{\boldsymbol{\pi}}\cdot\mathbf{1}-1) \tag{140}$$

The first order conditions imply:

$$\nabla_{\bar{\boldsymbol{\pi}}} L(\bar{\boldsymbol{\pi}}^*,\sigma^*) = \bar{\bar{\Sigma}}\bar{\boldsymbol{\pi}}^* + \sigma^*\mathbf{1} = \mathbf{0}; \tag{141}$$
$$\partial_\sigma L(\bar{\boldsymbol{\pi}},\sigma^*) = (\bar{\boldsymbol{\pi}}^*\cdot\mathbf{1}-1) = 0 \tag{142}$$

Hence, from (141), we get

$$\bar{\boldsymbol{\pi}}^* = -\sigma^*\bar{\bar{\Sigma}}^{-1}\mathbf{1} \tag{143}$$

and replacing in (142),

$$-\sigma^* \mathbf{1}^\top \bar{\bar{\Sigma}}^{-1} \mathbf{1} = 1 \tag{144}$$

$$\Rightarrow -\sigma^* = \frac{1}{\mathbf{1}^\top \bar{\bar{\Sigma}}^{-1} \mathbf{1}} = \frac{1}{C} \text{ (with the notation the section)} \tag{145}$$

And from (143),

$$\bar{\boldsymbol{\pi}}^* = \frac{1}{C} \bar{\bar{\Sigma}}^{-1} \mathbf{1} = \bar{\boldsymbol{\pi}}_1$$

as wanted

Exercise 6.3. By assumption, we have that

$$R^0 = \bar{\boldsymbol{\pi}}_1^\top \cdot \boldsymbol{\mu} = \frac{\boldsymbol{\mu}^\top \bar{\bar{\Sigma}}^{-1} \mathbf{1}}{\mathbf{1}^\top \bar{\bar{\Sigma}}^{-1} \mathbf{1}} \tag{146}$$

So that

$$\mathbf{1}^\top \bar{\bar{\Sigma}}^{-1} (\boldsymbol{\mu} - R^0 \mathbf{1}) = \mathbf{1}^\top \bar{\bar{\Sigma}}^{-1} \boldsymbol{\mu} - R^0 \mathbf{1}^\top \bar{\bar{\Sigma}}^{-1} \mathbf{1} = \mathbf{1}^\top \bar{\bar{\Sigma}}^{-1} \boldsymbol{\mu} - \mathbf{1}^\top \bar{\bar{\Sigma}}^{-1} \mathbf{1} \frac{\boldsymbol{\mu}^\top \bar{\bar{\Sigma}}^{-1} \mathbf{1}}{\mathbf{1}^\top \bar{\bar{\Sigma}}^{-1} \mathbf{1}} = 0 \tag{147}$$

Looking at the expression for the tangency portfolio, this means that $\boldsymbol{\mu}_{tan}$ and $\bar{\boldsymbol{\pi}}_{tan}$ are not well-defined. This can be explained as follows: all the efficient portfolio cases are given by

$$\bar{\boldsymbol{\pi}}_p^* = \sigma_i^* \bar{\bar{\Sigma}}^{-1} (\boldsymbol{\mu} - R^0 \mathbf{1}). \tag{148}$$

Now, if we pre-multiply by $\mathbf{1}^\top$ and use (147), we find

$$\mathbf{1}^\top \bar{\boldsymbol{\pi}}_p^* = \sigma_i^* \mathbf{1}^\top \bar{\bar{\Sigma}}^{-1} (\boldsymbol{\mu} - R^0 \mathbf{1}) = 0, \tag{149}$$

so that any efficient portfolio has zero net position on the risky assets. In particular this is incompatible with a fully inverted portfolio in risky assets. It is the only case in which there is no tangency portfolio.

Exercise 6.5. Recall that the Euler capital allocation assigns the capital $C_i = \frac{\partial}{\partial \pi_i} \rho(R(\boldsymbol{\pi}))$ to the investment/unit i.

1. Here, ρ is the standard deviation. Let $\bar{\bar{\Sigma}}$ be the variance-covariance matrix. Then,

$$C_i(\boldsymbol{\pi}) = \frac{\partial}{\partial \pi_i} [\boldsymbol{\pi}^\top \bar{\bar{\Sigma}} \boldsymbol{\pi}]^{\frac{1}{2}} \tag{150}$$

$$\Rightarrow [C_1, ..., C_n]^\top = \nabla_{\boldsymbol{\pi}} [\boldsymbol{\pi}^\top \bar{\bar{\Sigma}} \boldsymbol{\pi}]^{\frac{1}{2}} = \frac{2 \bar{\bar{\Sigma}} \boldsymbol{\pi}}{2 [\boldsymbol{\pi}^\top \bar{\bar{\Sigma}} \boldsymbol{\pi}]^{\frac{1}{2}}} \tag{151}$$

$$\Rightarrow C_i(\boldsymbol{\pi}) = \frac{\sum_{j=1}^n cov(R_i, R_j) \pi_j}{sd(R(\boldsymbol{\pi}))} \quad (*) \tag{152}$$

But, by linearity

$$\sum_{j=1}^n cov(R_i, R_j) \pi_j = cov(R_i, \sum_{j=1}^n R_j \pi_j) = cov(R_i, R(\boldsymbol{\pi})) \tag{153}$$

Thus, we get,

$$C_i(\boldsymbol{\pi}) = \frac{cov(R_i, R(\boldsymbol{\pi}))}{sd(R(\boldsymbol{\pi}))}. \tag{154}$$

In the case $\boldsymbol{\pi} = \mathbf{1}$,

$$C_i = \frac{cov(R_1, R)}{sd(R)}. \tag{155}$$

2. Indeed,

$$\sum_{i=1}^{n} \pi_i C_i = \frac{\sum_{i=1}^{n} \pi_i cov(R_1, R)}{sd(R)} = \frac{cov(R, R)}{sd(R)} = \frac{var(R)}{sd(R)} = \frac{(sd(R))^2}{sd(R)} = sd(R). \tag{156}$$

3.

$$RORAC(R(\boldsymbol{\pi})) = \frac{\mathbb{E}[R(\boldsymbol{\pi}) - 1]}{sd(R(\boldsymbol{\pi}))} = \frac{\mathbb{E}[\sum_{i=1}^{n}(R_i - 1)\pi_i}{\sqrt{\boldsymbol{\pi}^\top \bar{\bar{\Sigma}} \boldsymbol{\pi}}} \tag{157}$$

and therefore, from the chain rule and $(*)$ we get

$$\frac{\partial}{\partial \pi_i} RORAC(R(\boldsymbol{\pi})) = \frac{\mathbb{E}[R_i - 1] sd(R(\boldsymbol{\pi})) - \frac{\partial}{\partial \pi_i} sd(R(\boldsymbol{\pi}))\mathbb{E}[R(\boldsymbol{\pi}) - 1]}{var(R(\boldsymbol{\pi}))} \tag{158}$$

$$= \frac{\mathbb{E}[R_i - 1] sd(R(\boldsymbol{\pi})) - C_i \mathbb{E}[R(\boldsymbol{\pi}) - 1]}{var(R(\boldsymbol{\pi}))} \tag{159}$$

Note that, if each original allocated capital is positive, then,

$$\frac{\partial}{\partial \pi_i} RORAC(R(\boldsymbol{\pi})) \geq 0 \Leftrightarrow \frac{\mathbb{E}[R_i - 1]}{C_i} > \frac{\mathbb{E}[R(\boldsymbol{\pi}) - 1]}{sd(R(\boldsymbol{\pi}))} \tag{160}$$

$$\Leftrightarrow RORAC(R_i) > RORAC(R). \tag{161}$$

Exercise 6.6.

1. Since $F^\lambda(x) = P[R(\lambda) \leq x]$, and using the independence of R_1, R_2 we have:

$$\begin{aligned}
\mathbb{P}[R(\lambda) \leq x] =& \mathbb{P}[\lambda_1 R_1 + \lambda_2 R_2 \leq x] = \mathbb{E}[\mathbb{1}_{\{\lambda_1 R_1 + \lambda_2 R_2 \leq x\}}] \\
=& \mathbb{E}\left[\mathbb{1}_{\{R_2 \leq \frac{1}{\lambda_2}[x - \lambda_1 R_1]\}}\right] = \mathbb{E}\left[P\left(R_2 \leq \frac{1}{\lambda_2}[x - \lambda_1 R_1\right)\right] \\
=& \mathbb{E}\left[F_2\left(\frac{1}{\lambda_2}[x - \lambda_1 R_1]\right)\right]
\end{aligned}$$

   The other result follows by symmetry.

2. From the definition, we have that

$$\mathbb{E}[R_1 | R(\lambda) = x] = \int_{-\infty}^{\infty} y_1 f_{R_1|R(\lambda)}(y_1|x) dy_1 \tag{162}$$

To find $f_{R_1|R(\lambda)}$, we first study the joint law

$$F_{R_1,R(\lambda)}(x_1,x) = \mathbb{P}[R_1 \leq x_1 \text{ and } R(\lambda) \leq x].$$

From part 1., we get

$$F_{R_1,R(\lambda)}(x_1,x) = \int_{-\infty}^{x_1}\int_{-\infty}^{\frac{1}{\lambda_2}(x-\lambda_1 y_1)} f_2(y_2)f_1(y_1)\mathrm{d}y_1\mathrm{d}y_2,$$

we get by differentiating partially twice,

$$f_{R_1,R(\lambda)}(x_1,x) = \frac{\partial^2}{\partial x_1 \partial x}F_{R_1,R(\lambda)}(x_1,x) = \frac{1}{\lambda_2}f_2\left(\frac{1}{\lambda_2}(x-\lambda_1 x_1)\right)f_1(x_1)$$

and we deduce using the definition of the exercise that

$$f_{R_1|R(\lambda)}(x_1|x) = \frac{f_{R_1,R(\lambda)}(x_1,x)}{f^\lambda(x)} = \frac{\frac{1}{\lambda_2}f_2\left(\frac{1}{\lambda_2}(x-\lambda_1 x_1)\right)f_1(x_1)}{f^\lambda(x)}$$

and hence, replacing in (162),

$$\begin{aligned}\mathbb{E}[R_1|R(\lambda)=x] &= \int_{-\infty}^{\infty} y_1 \frac{\frac{1}{\lambda_2}f_2\left(\frac{1}{\lambda_2}(x-\lambda_1 y_1)\right)f_1(y_1)}{f^\lambda(x)}dy_1\\ &= \frac{1}{\lambda_2 f^\lambda(x)}\mathbb{E}\left[R_1 f_2\left(\frac{1}{\lambda_2}[x-\lambda_2 R_1]\right)\right]\end{aligned}$$

3. Set $Z = -\mathrm{V@R}^\alpha(R(\lambda))$. Using our assumption of continuity we know that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda_1}F^\lambda(Z) = \frac{\mathrm{d}}{\mathrm{d}\lambda_1}F^\lambda(-\mathrm{V@R}^\alpha(R(\lambda))) = \frac{\mathrm{d}}{\mathrm{d}\lambda_1}\alpha = 0.$$

On the other hand, from the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda_1}F^\lambda(-\mathrm{V@R}^\alpha[R(\lambda)]) = \frac{\partial}{\partial\lambda_1}F^\lambda(Z) + \left(\frac{\partial}{\partial z}F^\lambda(z)\Big|_{z=Z}\right)\left(-\frac{\partial}{\partial\lambda_1}\mathrm{V@R}^\alpha[R(\lambda)]\right) = 0.$$

Hence, provided that $\frac{\partial}{\partial z}F^\lambda(Z) \neq 0$, (i.e. there is no plateau of the probability at the point $-\mathrm{V@R}^\alpha[R(\lambda)]$), we have

$$\frac{\partial}{\partial\lambda_1}\mathrm{V@R}^\alpha[R(\lambda)] = \frac{\frac{\partial}{\partial\lambda_1}F^\lambda(Z)}{\frac{\partial}{\partial z}F^\lambda(Z)}$$

Replacing the results of 1. (using once more uniform integrability), and then replacing the result of 2., we get

$$\begin{aligned}\frac{\partial}{\partial\lambda_1}\mathrm{V@R}^\alpha[R(\lambda)] &= \frac{\frac{\partial}{\partial\lambda_1}F^\lambda(Z)}{\frac{\partial}{\partial x}F^\lambda(Z)} = \frac{\frac{\partial}{\partial\lambda_1}E\left[F_2\left(\frac{1}{\lambda_2}[Z-\lambda_1 R_1]\right)\right]}{f^\lambda(Z)} = -\frac{E\left[\frac{1}{\lambda_2}R_1 f_2\left(\frac{1}{\lambda_2}[Z-\lambda_1 R_1]\right)\right]}{f^\lambda(Z)}\\ &= \mathbb{E}[-R_1|R(\lambda)=Z]\end{aligned}$$

9 Factor models

Exercise 9.2. We need to show that

$$\delta = \frac{\mathrm{var}(R_*)}{\mathbb{E}[R_*] - R^0}.$$

Now, since $R_* = R^{\boldsymbol{\pi}}$ for a portfolio $\boldsymbol{\pi}$ in the mean variance frontier, note that from (6.20) in the lecture notes, we have that

$$\delta = \frac{\mathbb{E}[R_*] - R^0}{D} \tag{163}$$

where $D = (\boldsymbol{\mu} - \mathbf{1}R^0)^\top \bar{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{1}R^0)$. On the other hand, again since it belongs to the mean variance frontier, we have that its Sharpe ratio is maximal and equal to (the conclusion after Remark 6.2)

$$S(R_*) = \frac{\mathbb{E}[R_*] - R^0}{\mathrm{var}(R_*)^{1/2}} = D^{1/2} \tag{164}$$

Thus, replacing (164) in (163), we get

$$\delta = \frac{\mathbb{E}[R_*] - R^0}{S^2} = \frac{\mathrm{var}(R_*)}{\mathbb{E}[R_*] - R^0}$$

as claimed.