Market model - Fundamentals of Market Theory

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Market risk and portfolio theory

“All models are wrong, but some are useful”

George Box

(Normative) market model - preliminaries

Goal: Be able to price assets, produce meaningful strategies and measure risks.

What are the minimal elements we would need to have a meaningful model?

(Normative) market model - preliminaries

Goal: Be able to price assets, produce meaningful strategies and measure risks.

Minimal meaningful model: Focus on the evolution of asset prices themselves but taking into account uncertainty:

- We can observe the price that same stock has had in the past...
- but we cannot predict accurately its value in the future.
- For simplicity, we limit ourselves to deal with prices at periodic intervals.

A finite time discrete market model

We make our analysis during a fixed period and in discrete time.

Such a model also assumes that

- A single agent cannot impact market values.
- Investors can buy or short as many instances of an asset as they want, in fractional quantities if they so desire.
- Market values contain all available information.

A finite time discrete market model (cont.)

We set a *filtered probability space in finite discrete time* $(\Omega, \mathbb{P}, \mathcal{F}, \{\mathcal{F}_t\}_{t=0,\dots,T})$.

- Assets *characterised by adapted random processes* $\boldsymbol{S}^j = (S^j_0, \dots, S^j_T)$ that represent the *total market value of the asset* j for $j \in \{0, 1, \dots, n\}$.
- Values expressed in terms of a common *numéraire* or currency.
- The asset 0 will be reserved for a *bank account* which we will assume has some special properties.

For convenience, we introduce also the following matrix-like notation: we write $\boldsymbol{S}_t = (S^0_t, \dots, S^n_t)^\top$, that is, a column vector with the random values at time t of each asset.

Probability background

Probability space, sigma algebra, filtration, measurability, random variable, stochastic process, adapted and predictable processes, expectation

Generated filtration

Generated filtration

A filtration generated by X ($(\mathcal{F}^X_s)_{s\in\mathbb{N}}$) is the minimal filtration such that X_s is adapted

Conditional expectation

Definition (**Conditional Expectation**)

An operator from $\mathcal{F}$-measurable random variables X with finite variance to $\mathcal{F}_t$- measurable random variables given by,

$$\mathbb{E}[X|\mathcal{F}_t] = \arg \inf_{Z \in L^2(\Omega, \mathcal{F}_t)} \mathbb{E}[(Z - X)^2].$$

$\mathbb{E}_t$ can be completed to cover random variables with finite mean.

When filtration is clear we use $\mathbb{E}_t[X]$. *Alternatively*, it is the only (up to measure zero) $\mathcal{F}_t$-measurable, integrable random variable such that

$$\mathbb{E}[XY] = \mathbb{E}[\mathbb{E}_t[X]Y] \text{ for all } \mathcal{F}_t - \text{measurable } Y$$

Some properties of conditional expectation

Let X be $\mathcal{F}-$measurable.

1. (Invariance 1): If X is $\mathcal{F}_t$-measurable, $\mathbb{E}_t[X] = X$
2. (Invariance 2): If Y is $\mathcal{F}_t$-measurable, $\mathbb{E}_t[XY] = Y\mathbb{E}_t[X]$
3. (Orthogonality): If X is independent of $\mathcal{F}_t$, $\mathbb{E}_t[X] = \mathbb{E}[X]$
4. (Tower property): If $s \leqslant t$, $\mathbb{E}_s[X] = \mathbb{E}_s[\mathbb{E}_t[X]]$.

Similar to expectation:

1. (Linearity): We have $\mathbb{E}_t[a(X + X')] = a(\mathbb{E}_t[X] + E_t[X'])$ for all $a \in \mathbb{R}$
2. (Positivity): if $X \geqslant 0$, $\mathbb{E}_t[X] \geqslant 0$
3. Behaviour with limits: monotone convergence, dominated convergence...
4. Jensen's inequality: If $f\colon \mathbb{R} \to \mathbb{R}$ is convex, then $f(\mathbb{E}_t[X]) \leqslant \mathbb{E}_t[f(X)]$

Example of properties

Example:
Set $(X_s)_{s\in\mathbb{N}}$ i.i.d, with each entry being integrable, and assume we are working under the filtration generated by X.

Assume also that $Y_s = \sum_{i=0}^{s} a_i X_i$. For all $t < s$. What is $\mathbb{E}_t[Y_s]$?

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$$
\begin{aligned}
\mathbb{E}_t[Y_s] = \mathbb{E}_t[\sum_{i=0}^{s} a_i X_i] &= \sum_{i=0}^{t} a_i \mathbb{E}_t[X_i] + \sum_{i=t+1}^{s} a_i \mathbb{E}_t[X_i] \\
&= \sum_{i=0}^{t} a_i X_i + \sum_{i=t+1}^{s} a_i \mathbb{E}[X_1] \\
&= Y_t + \mathbb{E}[X_1] \sum_{i=t+1}^{s} a_i
\end{aligned}
$$

A finite time discrete market model (cont.)

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Bank account

As announced the asset 0 will play a special role. A very convenient assumption is that it is a locally risk-free asset.

Definition

We call *money market*, *bank account* or *(locally) risk-free* asset to an asset whose price is **strictly positive** and **predictable** with $S_0^0 = 1$.

We call any asset whose price process is not predictable *risky*

Some useful transformations

- Excess prices: Prices in excess of the money market account

$$E_t^i = S_t^i - S_t^0.$$

- Discounted prices: Prices in the units of the money market account

$$Y_t^i = \frac{S_t^i}{S_t^0}.$$

- (Gross) Returns: End-of-period price over start-of-period price

$$R_t^i = \frac{S_t^i}{S_{t-1}^i};\ \text{for } t = 1, \ldots, T$$

- Rate of return (net return): Gains in period over start-of-period price

$$r_t^i = \frac{S_t^i - S_{t-1}^i}{S_{t-1}^i};\ \text{for } t = 1, \ldots, T$$

- Discounted net gain:

$$\bar{Y}_t^i = \frac{S_t^i}{S_t^0} - \frac{S_{t-1}^i}{S_{t-1}^0}$$

Illustration of transformations

An investor decides to invest an amount φ_0 in asset *i* at time 0.

- At the end of the period they have

$$\varphi_0 R_1^i = \varphi_0 \frac{S_1^i}{S_0^i} = \theta_0^i S_1^i = \varphi_1$$

  where θ_0^i represents the number of shares that can be bought with an amount φ_0.

- The investor can evaluate their net gain or loss to be

$$\varphi_0 r_1^i = \varphi_0 (R_1^i - 1) = \varphi_1 - \varphi_0.$$

Note: since it is possible to buy any fraction of an asset the market can be equally modelled by any of the transformations as main variables, as long that denominators are not zero.

Examples of models

Independent and identically distributed returns We model the returns as i.i.d., and take the filtration generated by the return process. Hence, for all $t < \ell$ and all $i = 0, \ldots, n$

$$R^i_\ell \perp \mathcal{F}_t; \text{ and } R^i_\ell \sim R^i_t;$$

and,

$$S^i_t = S^i_0 \prod_{\ell=1}^{t} R^i_\ell.$$

A popular version is the log-normal model where

$$\boldsymbol{R}_t = \exp(\boldsymbol{Z}_t); \boldsymbol{Z}_t \sim \mathcal{N}(\mu, \Sigma).$$

In this case

$$\boldsymbol{S}_t = \boldsymbol{S}_0 \odot \exp(\sum_{s \leqslant t} \boldsymbol{Z}_s)$$

(where the $\odot$ denotes entry-wise product) which is also log-normal.
[menti]

Examples of models

Another, instance of the i.i.d. model uses a discrete distribution. For example $R_t^1 = (u-d)Z_t + d$ where Z_t are Bernoulli random variables with probability α.

More examples

- Some models might propose a dependence of prices or returns at a given time on more than the information on the period immediately before (non Markovian)

- Some models make hypothesis as the ones presented, but instead of assuming an explicit distribution they assume an observed distribution.

Strategies

Definition (Strategy)

Actions that an investor can decide to perform on the market.

Mathematically: It is a predictable process $\boldsymbol{\theta}$ with values in $\mathbb{R}^{n+1}$ so that θ_t^j denotes the number of shares of asset j to be held during the period $[t-1, t]$.

In a one-period model, choosing a strategy means simply choosing a portfolio composition, i.e. a vector $\boldsymbol{\theta} \in \mathbb{R}^{n+1}$.

Examples

- Time-series momentum strategy

$$\boldsymbol{\theta}_{t+1} = \psi(\boldsymbol{R}_t - \boldsymbol{R}_{t-1})$$

  for ψ entrywise and increasing.

- Discrete delta-hedging of derivative (on asset i).

$$\varphi^i_{t+1} = \varphi^i_t + N\{\partial_S u(t, S^i_t)\}(S^i_t - S^i_{t-1})$$

$$\varphi^0_{t+1} = \varphi^0_t - N\{\partial_S u(t, S^i_t)\}(S^i_t - S^i_{t-1})$$

Self-financing strategies

We denote by $S^{\boldsymbol{\theta}}$ the value of a portfolio that follows the strategy $\boldsymbol{\theta}$.

We let $S^{\boldsymbol{\theta}}_{t-}$ be the value of the strategy just before time t and $S^{\boldsymbol{\theta}}_{t}$ the value at time t.

Definition (Self-financing strategies)

Strategies that do not require additional resources apart from the initial one.

Mathematically: A strategy is self-financing if for all $t = 1, \ldots, T-1$,

$$S^{\boldsymbol{\theta}}_{t-} = \boldsymbol{\theta}_t \boldsymbol{S}_t = \boldsymbol{\theta}_{t+1} \boldsymbol{S}_t = S^{\boldsymbol{\theta}}_{t}.$$

Arbitrage opportunities

Definition (Arbitrage opportunity)

A costless self-financing strategy that produces some returns but never losses by the end of the modelling time.

Mathematically: a self-financing strategy $\boldsymbol{\theta}$ such that

1. $S_0^{\boldsymbol{\theta}} \leqslant 0$
2. $\mathbb{P}[S_T^{\boldsymbol{\theta}} \geqslant 0] = 1$
3. $\mathbb{P}[S_T^{\boldsymbol{\theta}} > 0] > 0$.

We call a market model where arbitrages are not possible *arbitrage-free*.

Replication

Definition (Replication)

A wealth profile W (at time T) can be *replicated* if there exists a self-financing strategy $\boldsymbol{\theta}$ such that

$$S_T^{\boldsymbol{\theta}} = W \text{ almost surely.}$$

In this case we say that $\boldsymbol{\theta}$ replicates W. We call $S_0^{\boldsymbol{\theta}}$ the replication price.

Completeness

A market is *complete* if every $\mathcal{F}_T$-measurable wealth W with $W \geqslant 0$ and $\mathbb{P}(W < \infty) = 1$ can be replicated.

Obstacles to completeness in practice:

- regulation
- high operational risk
- sophistication aversion

Example: The case of a finite probability space - one

- Finite probability space: $\Omega = \{\omega_1, \ldots, \omega_k\}$
- $\mathbb{P}(\{\omega_i\}) > 0$ for all $i = 1, \ldots, k$.

The market is characterised by $\boldsymbol{S}_0$ and

$$\mathcal{M}_{S_1} := \begin{bmatrix} S_1^0(\omega_1) & \ldots & S_1^0(\omega_k) \\ \vdots & \ddots & \vdots \\ S_1^n(\omega_1) & \ldots & S_1^n(\omega_k) \end{bmatrix}$$

Example: Take n = 2, k= 3 , $\boldsymbol{S}_0 = [1, 3]^\top$ and

$$\mathcal{M}_{S_1} := \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

- It admits arbitrage: for instance: $\boldsymbol{\theta} = [3/2, -1/2]^\top$

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- It can replicate some payoffs: ex: $X = [1, 0, 0]$ can be replicated with $\boldsymbol{\theta} = [-1/2, 1/2]^\top$
- But it is incomplete: $X = [0, 1, 0]$ cannot be replicated.

Example: The case of a finite probability space

- Finite probability space: $\Omega = \{\omega_1, \dots, \omega_k\}$
- $\mathbb{P}(\{\omega_i\}) > 0$ for all $i = 1, \dots, k$.

The market is also characterised by

$$\mathcal{M}_{ext} := [-\boldsymbol{S}_0 \; \mathcal{M}_{S_1}] = \begin{bmatrix} -S_0^0 & S_1^0(\omega_1) & \dots & S_1^0(\omega_k) \\ \vdots & \vdots & \ddots & \vdots \\ -S_0^n & S_1^n(\omega_1) & \dots & S_1^n(\omega_k) \end{bmatrix}$$

Theorem

There is no-arbitrage if and only if for all $\boldsymbol{\theta}^\top \in \mathbb{R}^n$, if $\boldsymbol{\theta}^\top \mathcal{M}_{ext} \geqslant \mathbf{0}$ then $\boldsymbol{\theta}^\top \mathcal{M}_{ext} = \mathbf{0}$.

There is completeness if and only if the range of $\mathcal{M}_{S_1}^\top$ is $\mathbb{R}^k$. In particular, if $\mathcal{M}_{S_1}$ is squared, it is complete if $\mathcal{M}_{S_1}$ is invertible if and only if its determinant is not zero.

The pricing problem on arbitrage-free markets

Definition (Arbitrage-free price)

p^{new} is an *arbitrage-free price* of a new contingent claim if the market extended with the new asset at the new price is still arbitrage-free.

Martingales

Definition (Martingale)

An **integrable** and adapted process is a martingale if

$$\mathbb{E}_s[M_t] = M_s \text{ for all } s \leqslant t,$$

i.e., if the “best” $\mathcal{F}_s$ -measurable estimation of the process at every time bigger that s is its value at s.

Stochastic Discount Factor (SDF)

Definition

An adapted process M in $\mathbb{R}$ is an SDF, if $M_0 = 1$, and for each $i = 0, \dots, n$

$$\mathbb{E}[|M_T S^i_T|] < \infty; \tag{1}$$

$$M_t S^i_t = \mathbb{E}_t[M_{t+1} S^i_{t+1}]. \tag{2}$$

i.e. the process $(M_t S^i_t)_{t=0,\dots,n}$ is a martingale for each i.

Properties SDFs

- The tower property deduces claims like

$$\mathbb{E}[M_t S_t^i] = S_0^i$$

- For non-negative asset values, property (2) in the definition is equivalent to

$$\mathbb{E}_t[M_{t+1} R_{t+1}^i] = M_t$$

  And we can also deduce

$$\mathbb{E}[M_t(R_t^i - R_t^j)] = 0; \qquad i \neq j$$

- If there is a risk-free asset,

$$\mathbb{E}_t[M_{t+1}] = \frac{M_t}{R_{t+1}^0}.$$

Example

Take a market modelled in one-period $\Omega = \{0, 1\}$ with $\mathcal{F} = \mathcal{P}(\Omega)$, $\mathbb{P}(\omega = 0) = \alpha$, $\mathbb{P}(\omega = 1) = 1 - \alpha$,

$$R_1^1 = \begin{cases} u & \text{if } \omega_1 = 0 \\ d & \text{if } \omega_1 = 1 \end{cases}; \quad \text{and} \quad R_1^0 = R^0 \text{ is constant.}$$

Where $0 < d < R^0 < u$. Let us find an SDF M (if it exists).

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Where $0 < d < R^0 < u$. Let us find an SDF M (if it exists).

- We know already we should have $M_0 = 1$
- The random variable M_1 solves the system of equations

$$1 = M_0 = \mathbb{E}[M_1 R_1^0] = \alpha R^0 M_1(0) + (1 - \alpha) R^0 M_1(1)$$
$$1 = M_0 = \mathbb{E}[M_1 R_1^1] = \alpha u M_1(0) + (1 - \alpha) d M_1(1)$$

The unique solution is

$$M_1 = \begin{cases} \frac{1}{\alpha R^0} \frac{R^0 - d}{u - d} & \text{if } \omega_1 = 0 \\ \frac{1}{(1-\alpha) R^0} \frac{u - R^0}{u - d} & \text{if } \omega_1 = 1 \end{cases}.$$

Fundamental Theorems of Asset Pricing

For a general market model SDFs can exist or not, be unique or not.

These properties of SDFs are remarkably connected with structural properties of the market. In our setting:

Theorem (First fundamental theorem asset pricing)

A market has no arbitrage opportunities if and only if there exists a strictly positive SDF.

Theorem (Second fundamental theorem asset pricing)

An arbitrage-free market model is complete if and only if there is a unique SDF.

Application

Question: Take the binomial market we saw before. We want to introduce a call option on the risky asset with strike K and maturity the end of the period.

Is there a price that avoids generating arbitrage opportunities?

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Question: Take the binomial market we saw before. We want to introduce a call option on the risky asset with strike K and maturity the end of the period.

Is there a price that avoids generating arbitrage opportunities?

Answer: Recall there is only one SDF on this market and it is strictly positive, namely

$$M_1 = \begin{cases} \frac{1}{\alpha R^0} \frac{R^0 - d}{u - d} & \text{if } \omega_1 = 0 \\ \frac{1}{(1-\alpha) R^0} \frac{u - R^0}{u - d} & \text{if } \omega_1 = 1 \end{cases}.$$

Hence, there is only one price P that works, and is given by

$$P = \mathbb{E}[M_1 (S_1^1 - K)^+] = \frac{1}{R^0} \frac{R^0 - d}{u - d} (S_0^1 u - K)^+ + \frac{1}{R^0} \frac{u - R^0}{u - d} (S_0^1 d - K)^+.$$

Pricing of assets

A price p^{new} for a new European option with unique payoff H at time τ is arbitrage-free if and only if $p^{new} = \mathbb{E}[M_\tau H]$ for some strictly positive SDF (M).

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Let $A(H, \tau)$ be the set of all arbitrage-free prices.

- Non-empty if market is arbitrage-free and a singleton if in addition market is complete
- There exists a super-hedging strategy θ of a European claim with payoff H if and only if S_0^θ is an upper bound for $A(H, \tau)$. A super-hedging is a self-funded strategy $\boldsymbol{\theta}$ such that

$$S_\tau^\theta \geqslant H.$$

Equivalent measures

Consider different probability measures on a measurable space $(\Omega, \mathcal{F})$.

Definition

Q_1 is absolutely continuous with respect to Q_2 (denoted $Q_1 \ll Q_2$), if for all $A \in \mathcal{F}$ with $Q_2(A) = 0 \Rightarrow Q_1(A) = 0$.

If both $Q_1 \ll Q_2$ and $Q_2 \ll Q_1$, we say the two measures are equivalent (denoted $Q_1 \sim Q_2$).

Densities

The Radon-Nikodym theorem shows that two equivalent measures are connected by (an almost surely unique) random variable called **a density** so that for any random variable X measurable on $(\Omega, \mathcal{F})$ if $Q_1 \sim Q_2$,

$$\mathbb{E}^{Q_2}[X] = \int_\Omega X(\omega)\mathrm{d}Q_2(\omega) = \int_\Omega X(\omega)\frac{\mathrm{d}Q_2}{\mathrm{d}Q_1}(\omega)\mathrm{d}Q_1(\omega) = \mathbb{E}^{Q_1}\left[X\frac{\mathrm{d}Q_2}{\mathrm{d}Q_1}\right].$$

Similarly, if Z is a strictly positive r.v. with $\mathbb{E}^{Q_1}[Z] = 1$, we can define a probability $Q_2 \sim Q_1$ by

$$Q_2(A) := \int_A Z(\omega)\mathrm{d}Q_1(\omega); \quad \textit{i.e.}, \frac{\mathrm{d}Q_2}{\mathrm{d}Q_1} = Z$$

Note also that for any r.v. in the space,

$$\mathbb{E}_t^{Q_2}[X] = \frac{\mathbb{E}_t^{Q_1}[XZ]}{\mathbb{E}_t^{Q_1}[Z]}.$$

Risk-neutral measure

A *risk-neutral* or *martingale* measure ($\mathbb{Q}$) can be defined in terms of a strictly positive SDF as an equivalent measure to $\mathbb{P}$ with density

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} := M_T S_T^0.$$

Note that $\mathbb{E}[M_T S_T^0] = M_0 S_0^0 = 1$, so that the density is well-defined.

$\mathbb{Q}$ is characterised by being an equivalent measure to $\mathbb{P}$ such that discounted prices $Y_t^i = \frac{S_t^i}{S_t^0}$ are martingales.

Risk-neutral measure properties

- Equivalence to strictly positive SDFs (if we have one such measure we can retrieve an SDF using the density and conditional expectation.)

- Fundamental theorems of asset pricing hold: no-arbitrage is equivalent to existence and completeness to uniqueness.

- There is no mean excess return under this measure:

$$\mathbb{E}_t^{\mathbb{Q}}[R_{t+1}^i - R_{t+1}^0] = 0.$$