Portfolio Choice

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Market risk and portfolio theory

Portfolio choice fundamentals

If we assume that investors are rational, they will choose portfolios to achieve a goal.

- Maximise their utility as a function of wealth or consumption
- Maximise performance
- Minimise the risk associated to a given investment/position

They are thus frequently drawn to choose portfolios via solving optimisation problems.

Optimisation problem

Optimisation problem (P)

$$\max_{\boldsymbol{x} \in K \subseteq \mathbb{R}^n} F(x)$$

- K is the set of admissible values:
  - If $K = \mathbb{R}^n$, the problem is *unconstrained*
  - Otherwise it is *constrained*. A common case:

$$K = \{x \in \mathbb{R}^n : G(x) \leqslant 0\} \cap \{x \in \mathbb{R}^n : H(x) = 0\}.$$

- $F : K \to \mathbb{R}$ is the *criterion* or *objective function*.

Minimisation is covered by considering $-F(x)$ instead.

Optimisation problem

Optimisation problem (P)

$$\max_{\boldsymbol{x} \in K \subseteq \mathbb{R}^n} F(x)$$

- x^* is a (global) optimal if $x^* \in K$ and

$$F(x^*) \geqslant F(y) \text{ for all } y \in K$$

- x^* is a (local) optimal if $x^* \in K$ and there is $V(x^*) \ni x^*$ open, such that

$$F(x^*) \geqslant F(y) \text{ for all } y \in K \cap V(x^*).$$

Some key results in optimisation

General existence result

- If F is continuous (even u.s.c.) and K is non-empty and compact then a maximal exists.

Some key results in optimisation

The convex case with concave objective function

Assume that K is convex and F is concave.

- Any local solution to the optimisation problem is a global solution
- If F strictly concave any optimal is unique
- Assume F is differentiable at x^*. Then,

$$x^* \text{ is optimal if and only if } \nabla F(x^*) \cdot (y - x^*) \leqslant 0 \text{ for all } y \in K.$$

If in addition $x^* \in \mathrm{Int}(K)$ then

$$\nabla F(x^*) = \mathbf{0}.$$

Some key results in optimisation

The case $K = \mathbb{R}^n$ with concave goal function

- There is a solution if F is concave and $F(x) > -\infty$ for some $x \in \mathbb{R}$ and
$$\lim_{|a|\uparrow\infty} F(ax) = -\infty, \forall x \neq 0$$
(F is coercive)
- If F is strictly concave the above condition is also necessary.

The problem with convex constraints

Assume F is concave, and

$$K = \{x \in \mathbb{R}^n : G(x) \leqslant 0\} \cap \{x \in \mathbb{R}^n : H(x) = 0\}$$

with G is convex, H is linear, and there exists such that $G(x) < 0$ and $H(x) = 0$.

A point x^* is maximal if and only if there exist (γ^*, λ^*) so that the (KT) conditions are satisfied:

- Stationarity $\nabla F(x^*) - \nabla G(x^*)^\top \gamma - \nabla H(x^*)^\top \lambda = 0$
- Feasibility $H(x^*) = 0$, $G(x^*) \leqslant 0$
- Complementary slackness $\gamma \cdot G(x^*) = 0$ and $\gamma \geqslant 0$

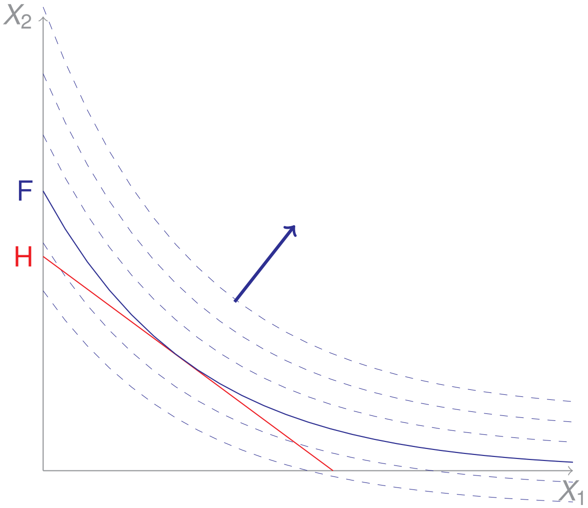
This can be remembered as a F.O.C. on the Lagrangian

$$\max_{x \in \mathbb{R}^n} \min_{\gamma \in \mathbb{R}^m_+, \lambda \in \mathbb{R}^\ell} \mathcal{L}(x, \gamma, \lambda) := F(x) - G(x) \cdot \gamma - H(x) \cdot \lambda.$$

plus the complementary slackness.

KT/Lagrange multipliers

$$\max_{x_1, x_2} F(x_1, x_2); \quad \text{subject to: } H(x_1, x_2) = c.$$



KKT/Lagrange multipliers

- Recall: gradients are orthogonal to level sets
- This means that at a tangency point, there is a value λ such that

$$\nabla F = \lambda \nabla H \quad \rightarrow \quad \nabla F - \lambda \nabla H = 0$$

- The maximum of the concave problem with linear equality constraints must therefore satisfy the previous condition and be in the feasible set.

Setup of the investor's choice problem in one period

Assumptions on the investor:

- Goal: maximise the expected utility of wealth at the end of the period
- Initial wealth w_0
- At the end of the period they receive a random endowment I (e.g. income due to labour)

Recall outstanding assumptions:

- Positions can be short or long at will (including fractions)
- No transaction costs
- Prices at time 0 and random variables at time 1 known
- Market modelled in a given probability space by adapted stochastic processes.

Mathematical formulation

Problem (P):

$$\max_{\boldsymbol{\theta} \in \mathbb{R}^{n+1}} \mathbb{E}[u(W_1)] \qquad \text{(goal, objective)}$$

subject to the (budget) constraints

$$\begin{aligned} t = 0; &\quad w_0 = \boldsymbol{\theta} \cdot \boldsymbol{S}_0; \\ t = 1; &\quad W_1 = I_1 + \boldsymbol{\theta} \cdot \boldsymbol{S}_1. \end{aligned}$$

If the domain of $E \subsetneq \mathbb{R}$ (we only treat in addition the case $[a, \infty)$) we add the constraint

$$W_1 \geqslant a \quad \text{(technical constraint)}$$

We say that $\boldsymbol{\theta}$ is *feasible* if it satisfies the constraints.

Alternative formulations

Problem (P'):

$$\max_{\boldsymbol{\pi} \in \mathbb{R}^{n+1}} \mathbb{E}[u(W_1)]$$

Subject to

$$1 = \sum_{i=0}^{n} \pi^i = \boldsymbol{\pi} \cdot \mathbf{1};$$
$$W_1 = I_1 + w_0(\boldsymbol{\pi} \cdot \boldsymbol{R}_1),$$

(plus technical constraint). Here, π^i is the percentage of wealth allocated to the asset i. Similarly, we can rephrase in terms of rate of return.

Existence and uniqueness

Theorem

Assume u is a strictly concave, and either that

- $E = \mathbb{R}$ *and u is bounded from above; or*
- $E = (a, \infty]$*, and*

$$\mathbb{E}|u(I_1 + \boldsymbol{S}_1^{\boldsymbol{\theta}})| < \infty$$

  for all feasible $\boldsymbol{\theta}$*.*

Then:

- *There exists a solution to the optimal investment problem if and only if the market is arbitrage-free;*
- *This solution is unique if and only if the market has no redundancies.*

Idea of the proof

It is clear that:

- If redundancies in the market, the solution cannot be unique. If no redundancies, uniqueness is deduced from strict concavity.
- If there is an arbitrage on the market, there is no maximum.

In the case $E = [a, \infty)$

- Show that the space of feasible portfolios is compact and the function $F(\boldsymbol{\theta}) := \mathbb{E}[u(I_1 + \boldsymbol{S}_1^{\boldsymbol{\theta}})]$ is upper semicontinuous.

In the case $E = \mathbb{R}$,

- Show that F is upper semicontinuous using Fatou's lemma, and that F tends to $-\infty$ at infinity using the no-arbitrage assumption.

Solving the problem

We have two main possibilities:

- Replace inequality constraints: From the first,

$$w_0 = \sum_{i=0} \theta^i S_0^i \Rightarrow \theta_0 = w_0 - \sum_{i=1}^{n} \theta^i S_0^i$$

  replacing on the second

$$\begin{aligned} W_1 &= I_1 + \sum_{i=0}^{n} \theta^i S_1^i = I_1 + \sum_{i=1}^{n} \theta^i S_1^i + S_1^0 \left(w_0 - \sum_{i=1}^{n} \theta^i S_0^i \right) \\ &= I_1 + S_1^0 \left(w_0 + \sum_{i=1}^{n} \theta^i \left(\frac{S_1^i}{S_1^0} - S_0^i\right) \right) = I_1 + S_1^0 \left(w_0 + \boldsymbol{\theta}^\top \bar{\boldsymbol{Y}} \right) \end{aligned}$$

  So, it is equivalent to solving (plus technical assumption)

$$\max_{\boldsymbol{\theta} \in \mathbb{R}^{n+1}} \mathbb{E}[u(I_1 + S_1^0(w_0 + \boldsymbol{\theta}^\top \boldsymbol{Y}))]$$

- Use F.O.C.

First Order Conditions - $E = \mathbb{R}$

The Lagrangian becomes

$$L(\boldsymbol{\theta}, \lambda) := \mathbb{E}\left[u\left(I_1 + \boldsymbol{\theta} \cdot \boldsymbol{S}_1\right)\right] - \lambda\left(\boldsymbol{\theta} \cdot \boldsymbol{S}_0 - w_0\right),$$

and the associated first order condition is

$$\partial_{\theta^i} L(\boldsymbol{\theta}^*, \lambda^*) = 0 \text{ for all } i = 0, \ldots n; \quad \text{and} \quad \partial_\lambda L(\boldsymbol{\theta}^*, \lambda^*) = 0.$$

Interlude: connection with SDFs - $E = \mathbb{R}$

If we assume in addition for all feasible $\boldsymbol{\theta}$ that

$$\mathbb{E}|u'(I_1 + \boldsymbol{\theta} \cdot \boldsymbol{S}_1)\boldsymbol{S}_1 \cdot \mathbf{1}| < \infty,$$

we can apply F.O.C., exchange derivatives and expectations to get

$$\mathbb{E}[u'(W^*)S_1^i] - \lambda^* S_0^i = 0, \text{ for all } i \text{ and } \boldsymbol{\theta} \cdot \boldsymbol{S}_0 = w_0$$

- $\mathbb{E}[u'(W^*)S_1^i] - \lambda^* S_0^i = 0,$ for all i and $\boldsymbol{\theta} \cdot \boldsymbol{S}_0 = w_0$;
- $\mathbb{E}[u'(W^*)(R_1^i - R_1^j)] = 0$;

where $W^* = I_1 + \boldsymbol{\theta}^* \cdot \boldsymbol{S}_1$ is optimal end of period wealth.

We conclude we can define an SDF

$$M_1 = \frac{u'(W^*)}{\lambda^*}$$

if the market is complete, this is the unique SDF.

First Order Conditions - $E = [a, \infty)$

The technical constraint cannot be always treated using Lagrange multipliers (because u might not be defined).

We use instead the more general first order optimality condition

$$\nabla L(\boldsymbol{\theta}^*, \lambda^*)^\top (\boldsymbol{\theta} - \boldsymbol{\theta}^*) \leqslant 0 \text{ for all feasible } \boldsymbol{\theta}$$

Note that for elements in the interior of the feasible set, both conditions are equivalent.

Examples

1. Optimal investment for CRRA under Binomial model
2. Optimal investment for CARA under Gaussian returns.

Multiperiod setting

Multiperiod investor-consumer problems

- Consumer-investor problem in finite period: maximise utility from consumption from given initial wealth and endowment process.

Notation

- $(W_t)_{0 \leqslant t \leqslant T}$ wealth
- $(C_t)_{0 \leqslant t \leqslant T}$ consumption
- $(I_t)_{1 \leqslant t \leqslant T}$: random endowment
- $(\boldsymbol{\theta}_t)_{1 \leqslant t \leqslant T}$: strategy as number of shares (or $\boldsymbol{\pi}_t^i := \frac{\theta_t^i S_{t-1}^i}{S_{t-1}^{\boldsymbol{\theta}}}$ when expressed as percentage)

Mathematical statement of the problem

Given W_0 and a time horizon T, find

$$\max_{(C,\boldsymbol{\theta})\in\mathcal{A}} \mathbb{E}[u(C_0,\ldots,C_T)]$$

s.t.

- Invest what is not consumed
  $\boldsymbol{\theta}_{t+1}\boldsymbol{S}_t = S_t^{\boldsymbol{\theta}} = (W_t - C_t)$ for $0 \leqslant t < T$;
- Wealth only from endowment and investment
  $W_{t+1} = I_{t+1} + \boldsymbol{\theta}_{t+1}\boldsymbol{S}_{t+1}$ for $0 \leqslant t < T$;
- Closure constraint
  $W_T = C_T$.
- (plus eventual technical constraints)

where $\mathcal{A}$ includes all processes with values in $\mathbb{R}\times\mathbb{R}^{n+1}$ that are integrable, adapted and predictable in their second component.

A conservation lemma

Lemma

Let M be an SDF, and suppose that $(C, \boldsymbol{\theta})$ is a strategy that satisfies all budget constraints. Then

$$\sum_{s=0}^{T} \mathbb{E}\left[M_s C_s\right] = \sum_{s=1}^{T} \mathbb{E}\left[M_s I_s\right] + W_0. \tag{1}$$

Similarly, if $\mathbb{Q}$ is a martingale measure,

$$\sum_{s=0}^{T} \mathbb{E}^{\mathbb{Q}}\left[\frac{C_s}{S_s^0}\right] = \sum_{s=1}^{T} \mathbb{E}^{\mathbb{Q}}\left[\frac{I_s}{S_s^0}\right] + W_0.$$

Proof of lemma

$$\sum_{s=0}^{T} \mathbb{E}\left[M_s C_s\right] = \sum_{s=1}^{T} \mathbb{E}\left[M_s I_s\right] + W_0.$$

First Order optimality condition (necessity)

Theorem

Assume that $(C^*, \boldsymbol{\theta}^*)$ is an optimal solution for the consumer-investor problem with $W_0 = w_0$ and T periods. Assume that $u \in C^1(\mathbb{R}^{T+1}, \mathbb{R})$ and such that ∇u is bounded. Then,

$$\mathbb{E}_t[\partial_{c_t} u(C^*)] S_t^i = \mathbb{E}_t[\partial_{c_{t+1}} u(C^*) S_{t+1}^i] \text{ for all } 0 \leqslant t < T; i = 1, \dots n \quad (2)$$

First Order optimality condition (necessity)

Proof

Some comments

$$\mathbb{E}_t[\partial_{c_t} u(C^*)] S^i_t = \mathbb{E}_t[\partial_{c_{t+1}} u(C^*) S^i_{t+1}] \text{ for all } 0 \leqslant t < T; i = 1, \dots n$$

- For time-additive case $u(C) := \sum_{i=0}^{T} \delta^i \tilde{u}(C_i)$:

$$\tilde{u}'(C^*_t) S^i_t = \mathbb{E}_t[\tilde{u}'(C_{t+1}) S^i_{t+1}]$$

- Assumption on bounded derivatives can be relaxed: only used to exchange derivative and integral
- Cases where technical conditions are present can be also included but are more delicate.

SDFs from optimal utility

Lemma

Assume that $\mathbb{E}[|\partial_{C_T} u(C^*)\boldsymbol{S}_T|] < \infty$ *and that*

$$\mathbb{E}_t[\partial_{c_t} u(C^*)]S_t^i = \mathbb{E}_t[\partial_{c_{t+1}} u(C^*)S_{t+1}^i] \text{ for all } 0 \leqslant t < T; i = 1, \ldots n$$

then we can construct an SDF M by

$$M_t = \frac{\mathbb{E}_t[\partial_{C_t} u(C^*)]}{\mathbb{E}[\partial_{C_0} u(C^*)]}$$

SDFs from optimal utility

Lemma

Assume that $\mathbb{E}[|\partial_{C_T} u(C^*)\boldsymbol{S}_T|] < \infty$ *and that*

$$\mathbb{E}_t[\partial_{c_t} u(C^*)]S_t^i = \mathbb{E}_t[\partial_{c_{t+1}} u(C^*)S_{t+1}^i] \text{ for all } 0 \leqslant t < T; i = 1, \ldots n$$

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Corollary

Under the assumptions of the theorem for necessity of first order constraints, we can define an SDF

First Order optimality condition (sufficiency)

Theorem

Consider the optimal solution for the consumer-investor problem with $W_0 = w_0$ and T periods. Assume that $u \in C^1(\mathbb{R}^{T+1}, \mathbb{R})$ and ∇u is bounded. Assume further that u is concave.
Let (C, θ) be such that it satisfies all the budget constraints and

$$\mathbb{E}_t[\partial_{c_t} u(C)] S_t^i = \mathbb{E}_t[\partial_{c_{t+1}} u(C) S_{t+1}^i] \text{ for all } 0 \leqslant t < T; i = 1, \ldots n \tag{3}$$

Then, (C, θ) is an optimal solution for the problem.

First Order optimality condition (sufficiency)

Proof

Dynamic programming principle

Key idea: transform the optimisation problem on processes into a sequence of simpler one-period optimisation problems.

Example: Consider a time-additive utility and a Markovian model for both the market and the endowments.

We can write the problem[1] as

$$J^* := \max_{C,\boldsymbol{\pi}} \mathbb{E}\left[\sum_{t=0}^{T} \delta^t \tilde{u}(C_t)\right]$$

s.t.

$$W_{t+1} = (W_t - C_t)\pi_t^\top R_{t+1};$$

$$W_T = C_T; \qquad \pi_t^\top \bar{1} = 1.$$

[1] for simplicity take no endowments

Dynamic programming principle

Define **value function** $J : \{0, \ldots, T\} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$: For $t < T$

$$J(t, w, R_t) := \max_{c, \boldsymbol{\pi} \cdot \mathbf{1} = 1} \left\{ \tilde{u}(c) + \delta \mathbb{E}[J(t+1, (w-c)(\boldsymbol{\pi} \boldsymbol{R}_{t+1}), R_{t+1}) | R_t] \right\},$$

and $J(T, w, r) := \tilde{u}(w)$.

Dynamic programming principle

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and $J(T, w, r) := \tilde{u}(w)$.

- J is defined iteratively backwards in time if all one-period problems are well-defined.
- We say the problem satisfies a DPP if $J^* = J(0, w_0, \mathbf{1})$.
- In this case, setting C_t^*, π^* the optimal couple at stage t gives us an optimal solution.

Example: CRRA for i.i.d returns and finite horizon

Assumptions:

- No endowments $I_t = 0$
- R_t are i.i.d and positive
- u is time-additive and $\tilde{u}$ is CRRA with $\rho > 1$

Example

In this case

$$J(t,w) := \max_{c\in\mathbb{R},\lambda\in\mathbb{R}^{n+1};\lambda^\top\bar{1}=1}\left[u(c)+\delta\mathbb{E}[J(t+1,(w-c)(\lambda^\top R_{t+1})]\right]$$

Example

In this case

$$J(t, w) := \max_{c \in \mathbb{R}, \lambda \in \mathbb{R}^{n+1}; \lambda^\top \bar{1} = 1} \left[u(c) + \delta \mathbb{E}[J(t+1, (w-c)(\lambda^\top R_{t+1})] \right]$$

We show by induction:

- The optimal consumption is $C_t = \xi_t W_t$ with

$$\frac{1}{\xi_t} = \sum_{s=0}^{T-t} (\delta B^{1-\rho})^{\frac{s}{\rho}} \tag{4}$$

  for B the certainty equivalent of the best investment.
- The optimal portfolio is the same as in the one-period case.
- The optimal utility satisfies for all $w \geqslant 0$,

$$J(t, w) = \frac{w^{1-\rho} \xi_t^{-\rho}}{1-\rho} \tag{5}$$

Example

We show $C_t = \xi_t W_t$; $(\xi_t)^{-1} = \sum_{s=0}^{T-t} (\delta B^{1-\rho})^{\frac{s}{\rho}}$ and $J(t, w) = \frac{w^{1-\rho} \xi_t^{-\rho}}{1-\rho}$.

Other connected financial problems

- Pricing of contingent claims
- (Partial) hedging of risky positions
- Optial investment with transaction costs

Recall: Dynamic programming principle does not always hold, but in many simple cases it does.

The pricing and hedging problem

The most common approach to find the price of a contingent payoff X to be paid at time T is to find its 'minimal super hedging price'.

It is the minimal price that allows to form a super-hedging self-financing strategy.

The pricing and hedging problem

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$$\begin{aligned} \min_{\boldsymbol{\theta}} \quad & w_0 \\ \text{s.t.} \quad & S_0^{\boldsymbol{\theta}} = w_0 \\ & S_T^{\boldsymbol{\theta}} \geqslant X \\ & \boldsymbol{\theta}_t \boldsymbol{S}_t = \boldsymbol{\theta}_{t+1} \boldsymbol{S}_t \text{ for all } t = 1, \ldots, T-1 \end{aligned}$$

The pricing and hedging problem

When the market is **complete**, the problem can be reduced to

$$
\begin{aligned}
\min_{\boldsymbol{\theta}} \quad & w_0 \\
\text{s.t.} \quad & S_0^{\boldsymbol{\theta}} = w_0 \\
& S_T^{\boldsymbol{\theta}} = X \\
& \boldsymbol{\theta}_t \boldsymbol{S}_t = \boldsymbol{\theta}_{t+1} \boldsymbol{S}_t \text{ for all } t = 1, \ldots, T-1
\end{aligned}
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& \boldsymbol{\theta}_t \boldsymbol{S}_t = \boldsymbol{\theta}_{t+1} \boldsymbol{S}_t \text{ for all } t = 1, \dots, T-1
\end{aligned}
$$

Note that the constraints are connected to the constraints of the optimal consumption-investment problem if we interpret $W_T := X$, $I_t = 0$ for all $t = 1, \dots, T$, and $C_t = 0$ for all $t = 0, \dots, T-1$.
(The goal is different, though).

The pricing and hedging problem

Let us assume that there is a known SDF M. From the conservation lemma, we get that

$$\mathbb{E}[M_T X] = w_0.$$

In this case, we only need to find a self-financing strategy $\boldsymbol{\theta}$ that, starting from $S_0^{\boldsymbol{\theta}}$ has terminal wealth X. To find it we can solve

$$\begin{aligned} \min_{\boldsymbol{\theta}} \quad & \mathbb{E}[(S_T^{\boldsymbol{\theta}} - X)^2] \\ \text{s.t.} \quad & S_0^{\boldsymbol{\theta}} = \mathbb{E}[M_T X] \\ & \boldsymbol{\theta}_t \boldsymbol{S}_t = \boldsymbol{\theta}_{t+1} \boldsymbol{S}_t \text{ for all } t = 1, \dots, T-1 \end{aligned}$$

The pricing and hedging problem

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In this case, we only need to find a self-financing strategy $\boldsymbol{\theta}$ that, starting from $S_0^{\boldsymbol{\theta}}$ has terminal wealth X. To find it we can solve

$$\begin{aligned}
\max_{\boldsymbol{\theta}} \quad & -\mathbb{E}[(S_T^{\boldsymbol{\theta}} - X)^2] \\
\text{s.t.} \quad & S_0^{\boldsymbol{\theta}} = \mathbb{E}[M_T X] \\
& \boldsymbol{\theta}_t \boldsymbol{S}_t = \boldsymbol{\theta}_{t+1} \boldsymbol{S}_t \text{ for all } t = 1, \ldots, T-1
\end{aligned}$$

which can be seen as an instance of the consumption-investment problem with utility given by $u(C) = -(C_t - X)$. Tools like the DPP are then readily available.