UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE	:	**COMPG004**
ASSESSMENT PATTERN	:	**COMPG004A**
MODULE NAME	:	**Market Risk, Measures and Portfolio Theory**
DATE	:	**Thursday 26 April 2018**
TIME	:	**14:30**
TIME ALLOWED	:	**2 hrs 30 mins**

This paper is suitable for candidates who attended classes for this module in the following academic year(s):

Year

2017/18

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TURN OVER

Department of Computer Science
University College London

Market risk, measures and portfolio theory

MSc Examination
2018

TIME ALLOWED: 2.5 HOURS

Market risk, measures and portfolio theory, COMPG004, 2018
Full marks will be awarded for answers to FOUR out of FIVE questions. Only the best FOUR questions will count towards the total mark. Each question is worth 25 marks.
Marks for each part of each question are indicated in square brackets.
Standard calculators are permitted. Clear and readable symbolic calculations are acceptable replacements for numerical results.

Question 1. Consider a one-period market model with three assets. Assume it is defined in a finite probability space with $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and uniform probability. The assets are characterised by their price at time 0, given by the vector

$$\boldsymbol{S}_0 = [10, 15, 30]^\top,$$

and their corresponding value at the end of the period under each scenario, given by

$$\boldsymbol{S}_1(\omega_1) = \begin{bmatrix} 12 \\ 12 \\ 36 \end{bmatrix}; \quad \boldsymbol{S}_1(\omega_2) = \begin{bmatrix} 12 \\ 24 \\ 24 \end{bmatrix}; \quad \boldsymbol{S}_1(\omega_3) = \begin{bmatrix} 12 \\ 24 \\ 48 \end{bmatrix}.$$

(a) When is a market model said to be arbitrage-free? When is it complete?

[5 marks]

(b) Show that this market model is both arbitrage-free and complete.
Hint: Recall the Fundamental Theorem of Asset Pricing.

[12 marks]

(c) Assume that a call option on the third asset with strike 36 (i.e., an asset with $S_1^c = (S_1^3 - 36)^+$) is introduced to the market at a price $p^c = 3$. Would the market still be arbitrage-free? Justify your answer.

[8 marks]

[Total: 25 marks]

Question 2. Consider a model with three possible probability outcomes $\Omega = \{\omega_1, \omega_2, \omega_3\}$, with $\mathbb{P}\{\omega_1\} = 1/6, \mathbb{P}\{\omega_2\} = 1/3, \mathbb{P}\{\omega_3\} = 1/2$; and an investor who has utility function

$$u(x) = \frac{1+2x}{1+x}, \text{ for } x > -1.$$

(a) Show that this investor is risk averse and insatiable. Then, calculate her associated coefficient of absolute risk aversion.

[8 marks]

(b) The investor's wealth at the end of the period, if nothing is done, is estimated to be

$$W(\omega) = \begin{cases} 0 & \omega = \omega_1 \\ 1 & \omega = \omega_2 \\ 2 & \omega = \omega_3 \end{cases}$$

What is the certainty equivalent of this wealth for the given utility function? What is the meaning of this quantity?

[10 marks]

(c) This investor can buy an asset with the following payoff at the end of the period

$$D(\omega) = \begin{cases} 1 & \omega = \omega_1 \\ 3 & \omega = \omega_2 \\ -1 & \omega = \omega_3 \end{cases}$$

Find the maximum value the investor is willing to pay for this asset.

Hint: Compare the certainty equivalent of the wealth profile with and without the new asset.

[7 marks]

[Total: 25 marks]

Question 3. In a one period framework, consider a risky asset with initial price $S_0 > 0$. Its value at the end of the period is "S_1". Assume that

$$S_1 = S_0 \exp(Y), \text{ where } Y \sim U[-m, m],$$

i.e. S_1 is "log-uniform". Set $\Delta S_1 = S_1 - S_0$, the one-period profit of the investment.

(a) Show that

$$\mathrm{V@R}^{\alpha}(\Delta S_1) = S_0(1 - \exp(m - 2m\alpha)).$$

What is the financial interpretation of this value?

[9 marks]

(b) Calculate the RORAC of this investment, assuming that the amount of capital that needs to be allocated on this position is determined by $\mathrm{V@R}^{\alpha}(\Delta S_1)$. Why is RORAC useful in finance?

[7 marks]

(c) The following Python code approximates the calculation of a risk measure ρ on the profit ΔS_1 for some given values of the defined parameters:

```python
import numpy as np
num1 = 10
num2 = 10000
num3 = (num2)**(-.5)*num1
uni_sample = np.random.rand(num2)*2-1
profit_sample = num1*( np.exp(uni_sample)-1 )
profit_sample.sort()    # increasing order
aux = int( num2 * num3 )
rho_approx = -1*( profit_sample[:aux+1].mean() )
```

Identify the risk measure and the parameters for the risk measure and the distribution used in the simulation. Then, give a close form expression for the exact value of the risk measure ρ.

[9 marks]

[Total: 25 marks]

Question 4. Consider a financial market composed by one risk-free asset with return R^0, and n risky assets with returns $\hat{\boldsymbol{R}} = (R^1, \ldots, R^n)^\top$. We define $\boldsymbol{\mu} = (\mu^1, \ldots, \mu^n)^\top$ where $\mu^i = E[R^i]$, and the matrix Σ where $\Sigma_{ij} = \mathrm{cov}[R^i, R^j]$, for $i, j = 1, \ldots, n$. Assume that Σ is invertible and that

$$R^0 \neq \frac{\boldsymbol{\mu}^\top \Sigma^{-1} \mathbf{1}}{\mathbf{1}^\top \Sigma^{-1} \mathbf{1}},$$

Furthermore, let $\boldsymbol{\pi} = (\pi^0, \ldots, \pi^n)^\top$ where π^i denotes the proportion of investment in the i-th risky asset, and $\hat{\boldsymbol{\pi}}$ is the sub-vector corresponding to the risky assets.

(a) Show that the portfolios in the mean-variance frontier can be written as a function of their mean return μ_p by

$$\hat{\boldsymbol{\pi}}^*(\mu_p) = \delta_p \Sigma^{-1}(\boldsymbol{\mu} - R^0 \mathbf{1}),$$

where

$$\delta_p = \frac{\mu_p - R^0}{(\boldsymbol{\mu} - R^0\mathbf{1})^\top \Sigma^{-1} (\boldsymbol{\mu} - R^0 \mathbf{1})}.$$

Hint: Find the mean-variance frontier portfolios as the solution of an optimisation problem with constrained mean.

[12 marks]

(b) Consider the optimal investment problem for an investor with CARA utility with constant risk aversion α and initial wealth w_0. Assume further that the risky returns are Gaussian. Show that the optimal portfolio is on the mean-variance frontier and give its expected return.

[13 marks]

[Total: 25 marks]

Question 5. Suppose the price of an asset follows the "rational log-normal" model

$$V_t = f(\exp(\xi_t)); \quad \text{for} \quad f(x) = \frac{a + bx}{c + dx},$$

where $a, b, c, d > 0$, $bc - ad \neq 0$ and $(\xi_{t+1})_{t \in \mathbb{N}}$ are stationary with $\xi_1 \sim \mathcal{N}(m, \sigma^2)$.

Note: For later reference, here are some quantiles for the Gaussian distribution: $\Phi^{-1}(0.95) \approx 1.65$, $\Phi^{-1}(0.99) \approx 2.33$, $\Phi^{-1}(0.999) \approx 3.09$.

(a) Find $\frac{\mathrm{d}}{\mathrm{d}x} f(x)$ and $f^{-1}(V_t)$. Then, show that the sensitivity of V_t with respect to ξ_t can be written

$$\delta_t = \frac{(cV_t - a)(dV_t - b)}{ad - bc}$$

[10 marks]

(b) Use the above sensitivity to approximate the Value at Risk at level 99%, conditional at time t, for ΔV_{t+1}.

[5 marks]

(c) Now, we want to evaluate the coverage property of our estimated one- day $\mathrm{V@R}^{0.99}$. Recall that we say there is an 'excess' if the realised losses exceed the estimation.

Suppose that we perform the backtest during one year (250 business days), and assume that 'excesses' occur independently and with the same distribution. Use a Gaussian coverage backtest, to approximate the ranges on the number of exceptions that fall into each zone of the Basel supervisory framework for covering backtesting:

Zone	Backtest (confidence for Type I error)
Green	Passed with confidence 95%
Yellow	Passed with confidence 99.9% but failed with confidence 95%
Red	Failed with confidence 99.9%

[10 marks]

[Total: 25 marks]