Department of Computer Science
University College London

Market risk, measures and portfolio theory

MSc Examination
2017

TIME ALLOWED: 2.5 HOURS

Full marks will be awarded for complete answers to FOUR questions. Only the best FOUR questions will count towards the total mark. Each question is worth 25 marks.

Calculators are **not** permitted. Clear and readable symbolic calculations are acceptable replacements for numerical results.

Problem 1.

(a) [8 points] Consider the function

$$u(x) = -\exp(-ax).$$

- Giving mathematical arguments, explain if an investor having u as utility function is risk-seeking or risk-averse.
- Find the coefficient of absolute risk aversion for this investor, and show it is constant.
- Show that another investor with utility function $u'(x) = mu(x) + b$ for $b, m \in \mathbb{R}$ and $m > 0$ has the same preferences.
- Calculate the expected utility $E[u(W)]$ if W is normal with mean μ and variance σ^2.

(b) [10 points] Let $a > 0$. Consider the function

$$v(x) = -\log\left(1 + \exp\left(-ax\right)\right)$$

- Show that an investor with utility function v is risk-averse and insatiable.
- Find the *certainty equivalent* x for this investor when its wealth is given by a random variable W, in terms of the expectation $\hat{v} = \mathbb{E}[v(W)]$. What is the financial interpretation of the certainty equivalent?
- Which wealth profile among W_1, W_2 defined below will an investor with utility function v choose?

$$W_1 = \begin{cases} 3/4 & \text{probability } 1/2 \\ 0 & \text{probability } 1/2 \end{cases}; \quad W_2 = \begin{cases} 1 & \text{probability } 1/4 \\ 1/2 & \text{probability } 1/4 \\ 0 & \text{probability } 1/2 \end{cases}.$$

(c) [7 points] Write a python code that approximates, using a Monte Carlo simulation with 10^6 samples, the expected value of $v(W)$ assuming that W follows a uniform distribution in the interval $[-10, 10]$, and $a = 1$. The following lines have already been included:

```
import numpy as np
from numpy.random import rand
from numpy import exp, log
```

Problem 2. In a one period framework, consider a risky asset with price $S_0 > 0$ at the beginning of the period. Assume its price at the end of the period "S_1" is log-normally distributed with

$$S_1 = S_0 \exp(\xi); \quad \text{for} \quad \xi \sim \mathcal{N}(m, \sigma^2).$$

Both m, σ^2 are assumed to be known. Let us measure the risk for a buyer of this asset in one period, that is $L = S_0 - S_1$.

(a) [10 points] Give the definition and financial interpretation of Value at Risk, and show that

$$\mathrm{V@R}^\alpha(L) = S_0 \left(1 - \exp\left[m + \sigma\Phi^{-1}(1-\alpha)\right]\right),$$

where Φ is the cumulative distribution function of a standard Gaussian.

(b) [8 points] We are now interested in measuring, in absence of any hedging, the risk of **selling** a European call option on the asset described in (a). Consider a call with maturity one-period and strike K. Set C to be the (known) price of this call. Let L' be the random variable representing losses for the **seller** of this call option in absence of any hedging, that is $L' = (S_1 - K)^+ - C$.

Show that for $\alpha \in (0,1)$, $\mathrm{V@R}^\alpha(\mathrm{L}') = \max\{S_0 - K - C - \mathrm{V@R}^{1-\alpha}(L), -C\}$

(c) [7 points] Assume that the model in (a) describes well the behaviour of a given asset in several periods, so that $S_{n+1} = S_n \exp(\xi_{n+1})$, where all the $\xi_1, \xi_2 \ldots$, are i.i.d. with $\mathcal{N}(m, \sigma^2)$. Set $L_i = S_{i-1} - S_i$

- Show that the random variables $I_i = \mathbb{1}_{\{L_i \geq \mathrm{V@R}^\alpha(L_i)\}}$, for $i = 1, \ldots$ are i.i.d. and give their distribution. Recall that $\mathbb{1}_A$ denotes the indicator function of the set A.
- In a set of 400 periods, the observed losses are seen to exceed the estimated value at risk at 99% confidence on 10 occasions. Perform a backtest to establish the covering property of a reserve made with this value at risk calculation. Fix the confidence of the test at 97.5%. You may use that $\Phi^{-1}(0.975) \approx 1.96$.

Problem 3. Assume an agent has taken a position in d different investing possibilities whose losses are represented by $(L_1, \ldots, L_d)$. Her total loss profile is given by

$$L = \sum_{i=1}^{d} L_i.$$

Let Σ denote the covariance matrix of the vector $(L_1, \ldots, L_d)^\top$, and $-\bar{\mu}$ its mean vector (i.e. $\bar{\mu}$ is the mean of profits). Assume all the components of $\bar{\mu}$ are positive. Set $L(\bar{\lambda}) := \sum_{i=1}^{d} \lambda_i L_i$.

(a) [9 points] We consider first the standard deviation risk measure $\rho_{sd} : X \to \sqrt{\text{var}(X)}$. Explain the financial meaning of a capital allocation and compute the optimal capital allocation for L, according to the Euler principle if capital is calculated with ρ_{sd}.

(b) [9 points] Deduce, using the previous capital allocation, an expression in terms of the known quantities for the RORAC of L and each one of the individual positions L_i.

What condition should satisfy the ratio $\frac{\mu_i}{\sum_{j=1}^{d} \mu_j}$ to conclude that the asset i is not underperforming according to RORAC ?

(c) [7 points] We will now consider another risk measure. Let ρ be a convex risk measure. Define

$$\nu(L) := \inf_{z>0} \left\{ \frac{1}{z} [\rho(zL)] \right\}.$$

Show that ν is also a convex risk measure.

Hint. Remember that if $g_1(z) \leq g_2(z)$ for all $z \in A$, then $\inf_{z \in A} g_1(z) \leq \inf_{z \in A} g_2(z)$

Problem 4. Let us consider a one-period market model. Suppose it is composed by a risk-free asset with gross return $R_f > 0$ and one risky asset with gross return R given by

$$R = \begin{cases} u & \text{with probability } \alpha \\ d & \text{with probability} (1-\alpha) \end{cases}$$

for some $0 < \alpha < 1$ and $d < u$.

(a) [10 points] What is an arbitrage opportunity? Show that if $R_f \leq d$ or $R_f \geq u$, there is an arbitrage opportunity. Explain what is a Stochastic Discount Factor (SDF). Find a SDF assuming that $d < R_f < u$.

(b) [10 points] Assume that an investor has CRRA utility function

$$v(x) = (1-\rho)^{-1} x^{1-\rho}$$

with $\rho > 1$. Solve the portfolio choice problem, in terms of the amount invested in the risky asset (ϕ), for an investor with initial wealth w_0, no initial consumption and no endowments.

(c) [5 points] Now, we also allow consumption at the beginning of the period. Thus, the (representative) investor has a concave utility function $U(c_0, c_1)$ with two variables. Suppose U has positive and bounded first order derivatives. Assuming optimal investment, derive a formula for the stochastic discount factor in terms of the utility function U.

Problem 5. Fix a probability space $(\Omega, \mathbb{P}, \mathcal{F}, \{\mathcal{F}_t\}_{t=1,\ldots,T})$. In this question we consider the investment consumption problem in a discrete setting with finite horizon T. We focus on the dynamic programming principle applied to the case of an investor with additive discounted utility

$$U(\bar{C}) = \sum_{i=0}^{T} \delta^i v(C_i)$$

where v is a concave function. Consider a market consisting of one risky asset and a money market account. Assume that the reinvested money market account satisfies $B_{t+1} = B_t R_f$ and $B_1 = 1$ for a deterministic value $R_f > 0$.
Assume also that the gross returns $(R_i)_{i=1,\ldots T}$ of the risky asset are independent and identically distributed.

(a) [7 points] State the investment consumption problem for an investor with initial wealth w_0 and no endowments, in terms of the consumption at each time and the investment strategy.

(b) [8 points] Give the definition of conditional expectation at time $t, (\mathbb{E}_t[.])$. Show that for all X, Y $\mathcal{F}$-measurable random variables with finite expectation,

- $\mathbb{E}_t[aX + bY] = a\mathbb{E}_t[X] + b\mathbb{E}_t[Y]$, for all $a, b \in \mathbb{R}, 0 \leq t \leq T$.
- $\mathbb{E}_s[\mathbb{E}_t[X]] = \mathbb{E}_s[X]$, for all $0 \leq s \leq t \leq T$.

(c) [10 points] Define

$$J(t, w) = \sup_{\substack{C_i \in \mathcal{A}_i, \pi_i \in \mathbb{R} \\ i=t,\ldots,T}} \left\{ \mathbb{E}_t \left[\sum_{i=t}^{T} \delta^{i-t} v(C_i) \right] : \text{ budget constraints from } t \text{ to } T \right\},$$

where $\mathcal{A}_i$ are the $\mathcal{F}_i$ measurable random variables with finite expectation and the budget constraints are

$$W_t = w; \qquad W_T = C_T$$
$$W_{i+1} = (W_i - C_i)(\pi(R_{i+1} - R_f) + R_f) \text{ for } i = t+1, \ldots, T-1.$$

Explain the financial interpretation of this function. Then, use it to sketch how we can apply the dynamic programming principle approach to solve the optimal investment consumption problem.