Department of Mathematics
University College London

MATH0094 Market Risk and Portfolio Theory

MSc Examination

2022-2023

TIME ALLOWED: 3 HOURS

The exam contains FOUR questions. All questions should be attempted. Each question is worth 25 marks.

Calculators **are** permitted.

Question 1. Consider a T-periods *arbitrage-free* market with n risky assets and a risk-free asset S^0. As usual, we assume that $S_0^0 = 1$.

(a) Let $\boldsymbol{\theta}$ and $\boldsymbol{\xi}$ be self-funding strategies such that $S_t^{\boldsymbol{\theta}} = S_t^{\boldsymbol{\xi}}$ for some $t \in \{1, \ldots, T\}$. Show that $S_0^{\boldsymbol{\theta}} = S_0^{\boldsymbol{\xi}}$, and explain the financial meaning of this property.

[7 marks]

Let $K \in \mathbb{R}_+$ and $\tau \in \{1, \ldots, T\}$ be arbitrary but fixed values. A *call option* on the asset $i \in \{1, \ldots, n\}$ with strike K and maturity τ is an asset with payoff $(S_\tau^i - K)^+$ where $(.)^+$ denotes positive part. Likewise, a *put option* on the asset i with strike K and maturity τ is an asset with payoff $(K - S_\tau^i)^+$.

In what follows, let p_c^i be an arbitrage-free price of a contingent call option on the asset i with strike K and maturity τ (respectively let p_p^i be the arbitrage-free price of the analogous put option). Further, assume that the risk-free asset is constant and equal to 1, i.e. $S_t^0 = 1, \forall t = 0, \ldots, T$.

(b) Show that

$$p_c^{i,K} - p_p^{i,K} = S_0^i - K.$$

Hint: Recall that $a = (a)^+ - (-a)^+$.

[9 marks]

(c) Suppose that $\{\boldsymbol{\theta}_t\}_{t=0,\ldots,\tau}$ is a self-funding strategy (in terms of units) replicating a call option on the asset i with strike K and maturity τ. Find a self-funding strategy replicating the corresponding put option (i.e. same asset, strike, and maturity).

[9 marks]

[Total: 25 marks]

Question 2.
Consider a pure investor with CRRA utility function v with constant relative risk aversion $\rho = 1$. The investor acts in a one-period market model with finite probability, with n-risky assets, a risk-free asset, and m outcomes in the probability space. The market is characterised by the random returns vector $\boldsymbol{R}$.

(a) Let $\mathcal{R} \in \mathbb{R}^{(n+1)\times m}$ be the matrix of returns

$$\mathcal{R} = \{R_1^i(\omega_j)\}_{i=0,\dots,n;j=1,\dots m}.$$

Find conditions in terms of $\mathcal{R}$ guaranteeing existence and uniqueness of the solution to the utility maximisation problem.

[8 marks]

Consider the investor-consumer problem

$$\begin{aligned}\max_{C_0,\pi} & \ \mathbb{E}[v(C_0) + \delta v(C_1)] \\ \text{s.t.} & \\ & C_1 = (w_0 - C_0)((\hat{\boldsymbol{R}}_1 - R_1^0\mathbf{1}) \cdot \hat{\boldsymbol{\pi}} + R_1^0);\end{aligned}$$

for $\delta \in (0, 1)$.

(b) Suppose that the conditions in (a) hold. i) Find the optimal consumption strategy; and ii) show that the optimal investment strategy satisfies

$$\frac{1}{R_1^0}\mathbb{E}\left[\frac{R_1^i}{(\hat{\boldsymbol{R}}_1 - R_1^0\mathbf{1}) \cdot \hat{\boldsymbol{\pi}}^* + R_1^0}\right] = \mathbb{E}\left[\frac{1}{(\hat{\boldsymbol{R}}_1 - R_1^0\mathbf{1}) \cdot \hat{\boldsymbol{\pi}}^* + R_1^0}\right],$$

for any $i = 0, \dots, n$.

[11 marks]

(c) Find an SDF in this market in terms of the optimal investment strategy.

[6 marks]

[Total: 25 marks]

Question 3. Let X be a continuous random variable with probability distribution function f_x and cumulative density function F_X. Assume further that F_X is invertible. Recall that expected shortfall is defined by

$$\mathrm{ES}^{\alpha}(X) = \frac{1}{1-\alpha}\int_{\alpha}^{1} \mathrm{V@R}^{u}(X)du.$$

(a) Show that

$$\mathrm{ES}^{\alpha}(X) = \mathrm{V@R}^{\alpha}(X) + \frac{1}{1-\alpha}\mathbb{E}[(-X - \mathrm{V@R}^{\alpha}(X))^{+}]$$

where $(a)^{-}$ is the negative part operator, i.e. $(a)^{-} = (-a)^{+}$.

[7 marks]

(b) Prove that

$$\mathrm{ES}^{\alpha}(X) = \inf_{z\in\mathbb{R}}\{z + \frac{1}{1-\alpha}\mathbb{E}[(-X - z)^{+}]\}$$

Hint: Apply first order conditions to the optimization problem on the right-hand side, and use the fact that -X has a density.

[9 marks]

(c) Use (b) to show that ES^{α} satisfies convexity, and give a financial interpretation for this property.

[9 marks]

[Total: 25 marks]

Question 4. Consider a market with three risky assets and a risk-free asset. The risk-free asset has unit return, while the risky assets have mean return μ and variance-covariance Σ given by

$$\mu = \begin{pmatrix} 3 \\ 7 \\ 9 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 5 & 5 \\ 1 & 5 & 13 \end{pmatrix}. \qquad \text{Note that } \Sigma^{-1} = \frac{1}{8}\begin{pmatrix} 10 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

(a) Find an expression that characterises all portfolios in the mean-variance frontier *excluding the risk-free asset* in terms of the above data.

[8 marks]

(b) Find in this market:

i. the tangency portfolio; and

ii. the maximal Sharpe ratio ($\mathcal{S}_{\max}$) of the market *including* the risk-free asset.

[9 marks]

(c) An investor evaluates risk using standard deviation. They would like to fix a target average return of 3. Find their optimal portfolio in the market (*including* risk-free asset).

[8 marks]

[Total: 25 marks]