# Risk measures

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Market risk and portfolio theory

# Risk measures

How to measure the “risk” encoded in a financial position $X$ ?

## Definition

A (unidimensional) **risk measure** is a function $\rho : \mathcal{X} \to \mathbb{R}$ that assigns to a given random variable $X$ representing a financial position a real number representing its “riskiness”.

Our convention is $X$ is positive for profits/wealth.

# Risk measures

- R.M. generalise utility functions: they put more emphasis on profit and losses than in the perception from the investor
- They allow defining risk management strategies and regulatory limits (e.g. minimal capital requirements, risk taking limits, etc.)
- They can be used in conjunction with profit analysis to evaluate performance

Some examples:

Variance: $\mathrm{var}(\mathrm{X}) = \mathbb{E}[\mathrm{X}^2] - \mathbb{E}[\mathrm{X}]^2$

Standard deviation: $\mathrm{sd}(\mathrm{X}) = (\mathrm{var}(\mathrm{X}))^{1/2}$

# Properties of risk measures

- Monotonicity: For all $X_1, X_2 \in \mathcal{X}$ such that $X_1 \leqslant X_2$ almost surely we have $\rho(X_1) \geqslant \rho(X_2)$ (Smaller risk for larger profits)
- Translation (cash) invariance: For all $X \in \mathcal{X}$ and for every $a \in \mathbb{R}$, we have $\rho(X + a) = \rho(X) - a$ (Risk reduces with additional sure amounts)
- Normalisation: $\rho(0) = 0$.

Risk measures that satisfy the above are called monetary risk measures: they can be used for **capital allocation**.

# Properties of risk measures

We assume from now on that $\mathcal{X}$ is a convex subset of a vector space.

- Convexity: For all $X_1, X_2 \in \mathcal{X}$ and $\lambda \in [0, 1]$ then

$$\rho[\lambda X_1 + (1-\lambda)X_2] \leqslant \lambda\rho(X_1) + (1-\lambda)\rho(X_2).$$

A monetary risk measure with the convexity property is called a *convex* risk measure. They favour diversification.

# Properties of risk measures

If we have a convex risk measure, and we assume in addition that the space $\mathcal{X}$ is a *convex cone* and that the risk measure satisfies

- Positive homogeneity: For all $X \in \mathcal{X}$ and every $\lambda > 0$ we have

$$\rho(\lambda X) = \lambda\rho(X) \text{ (Scaling)}$$

The measure is called *coherent* risk measure.

Alternatively we can assume

- Subadditivity: For all $X_1, X_2 \in \mathcal{X}$, we have

$$\rho(X_1 + X_2) \leqslant \rho(X_1) + \rho(X_2)$$

In fact, any two properties out of positive homogeneity, convexity and subadditivity imply the third one.

# Other properties

Comonotonic additivity (No diversification for total dependence): Let $f_1, f_2$ be two increasing functions. Then

$$\rho(f_1(X) + f_2(X)) = \rho(f_1(X)) + \rho(f_2(X))$$

Law invariance (Context independence): If $X_1 \sim X_2$, then

$$\rho(X_1) = \rho(X_2)$$

Ellicitability (Can be comparatively tested): Let $\mathcal{P}$ be a class of probability measures in $\Omega$. $\rho$ is said to be *ellicitable* relative to the class $\mathcal{P}$ if there is a scoring function $s : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that

$$\rho(X) = \arg\min_{x \in \mathbb{R}} \mathbb{E}[s(x, X)]$$

# Utility based risk measures

Several risk measures can be defined from a given a concave utility function $u$:

- Simple loss: Satisfies convexity (not monetary in general)

$$\rho_u^{sl}(X) := \mathbb{E}[-u(X)]$$

- Certainty equivalent: Monetary risk measure (for CARA, also convex)

$$\rho_u^{ce}(X) = -u^{-1}(\mathbb{E}[u(X)])$$

- Shortfall risk: Convex risk measure

$$\rho_u^{SR}(X) := \inf\{z : \mathbb{E}[u(X+z)] \geqslant \varphi\}.$$

# Example: Entropic risk measure

For some $\theta > 0$, the *entropic risk measure* with parameter $\theta$ (that we denote $(\rho_\theta^{\exp})$) is defined by

$$\rho_\theta^{\exp}(X) := \frac{1}{\theta} \log \left( \mathbb{E}[e^{-\theta X}] \right)$$

It is in general **not** coherent, since for $\lambda > 1$

$$\rho_\theta^{\exp}(\lambda X) = \frac{1}{\theta} \log \mathbb{E}[e^{-\theta \lambda X}] \geqslant \frac{1}{\theta} \log \left( \mathbb{E}[e^{-X\theta}]^\lambda \right) = \lambda \rho_\theta^{\exp}(X)$$

with a strict inequality for most distributions (take for example a standard Gaussian and $\lambda = 2$).

# Tail risk measures and quantile function

We can define measures that focus more on the results than the investor, by using the tail of the distributions.

## Definition (Quantile function)

The quantile function of $X$, $q_X : (0,1) \to \mathbb{R}$, is

$$q_X(\lambda) = \inf\{x : \lambda \leqslant F_X(x)\} = \inf\{x : \lambda \leqslant P(X \leqslant x)\}$$

Equivalently the quantile is characterised as the **only left continuous with right limits** function defined from $(0,1)$ to $\mathbb{R}$ such that

$$q_X(\lambda) \leqslant x \Leftrightarrow \lambda \leqslant F_X(x) \text{ for all } \lambda \in (0,1).$$

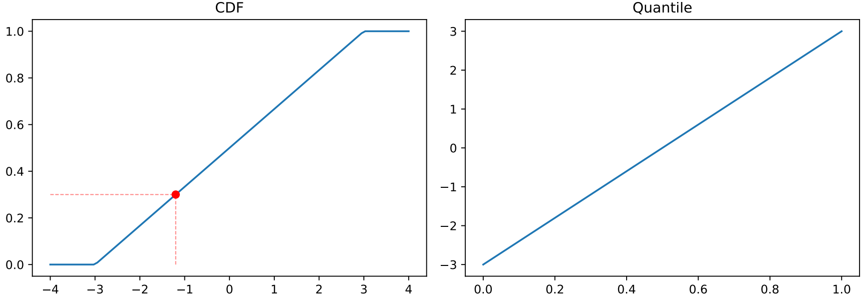
# Example of quantile

Let $X \sim U[-3, 3]$, uniformly distributed. Then we have,

$$F_X(x) = \frac{1}{6}(x+3)\mathbb{1}_{\{-3<x\leqslant 3\}} + \mathbb{1}_{\{x>3\}}$$

and we find for $\lambda \in (0, 1)$

$$q_X(\lambda) = 6\lambda - 3,$$



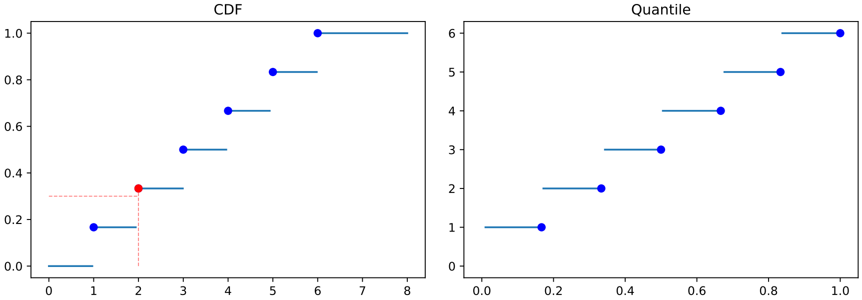
# Example of quantile

Let $X$ represent the outcome of one fair dice throw. We have

$$F_X(x) = \min\left\{\max\left\{0, \frac{\lfloor x \rfloor}{6}\right\}, 1\right\}$$

Then, we can find that the quantile function is

$$q_X(\lambda) = \lceil 6\lambda \rceil,$$



# Value at risk

Value at risk ($\mathrm{V@R}$) at level $\alpha$ minimal amount sufficient to assure no losses with probability $\alpha$.

$$\mathrm{V@R}^{\alpha}(X) := q_{-X}(\alpha) = \inf\{z \in \mathbb{R} : \mathbb{P}(-X \leqslant z) \geqslant \alpha\}$$

Mind the convention. Some references take $\alpha$ as the probability of having losses, which is equivalent to taking $1 - \alpha$ here.

With our convention, typical values used in market risk management practice are: $\alpha = 95\%$ or $\alpha = 99\%$.

# Example of Value at Risk

An investment produces profit of 100 with probability 0.75, or a loss of 150 with probability 0.25.

What is the value at risk at level $\alpha \in (0, 1)$ of the P& L of this investment?

$$F_{-X}(x) = 0.75\ \mathbb{1}_{\{x \geqslant -100\}} + 0.25\ \mathbb{1}_{\{x \geqslant 150\}}$$

Hence,

$$\mathrm{V@R}^{\alpha}(X) = q_{-X}(\lambda) = -100\ \mathbb{1}_{\{0 < \lambda \leqslant 0.75\}} + 150\ \mathbb{1}_{\{0.75 < \lambda < 1\}}$$

Note: $\mathrm{V@R}^{0.99}(X) = 150$, but $\mathrm{V@R}^{0.75}(X) = -100$.

# Expected shortfall

Expected shortfall $(\mathrm{ES})$ at level $\alpha$ is the average of values at risk above level $\alpha$.

$$\mathrm{ES}^{\alpha}(X) = \frac{1}{1-\alpha} \int_{\alpha}^{1} \mathrm{V@R}^{u}(X) du$$

It has several alternative representations:

$$\mathrm{ES}^{\alpha}(X) = \inf \left\{ z + \frac{1}{1-\alpha} \mathbb{E}[(X+z)^{-}] : z \in \mathbb{R} \right\}$$

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and, with $\tilde{\alpha} := F_{-X}(\mathrm{V@R}^{\alpha}(X))$,

$$\mathrm{ES}^{\alpha}(X) = \frac{1}{1-\alpha}\left\{ \mathbb{E}[-X \cdot \mathbb{1}_{\{-X > \mathrm{V@R}^{\alpha}(X)\}}] + (\tilde{\alpha} - \alpha)\mathrm{V@R}^{\alpha}(X) \right\}.$$

Mind the convention.

# Expectiles

For $\tau \in (0, 1)$, the $\tau$-expectile of a random variable $X$ with finite mean is the only solution to the equation

$$\tau \mathbb{E}[(X - \mathrm{e}_\tau)^+] - (1 - \tau)\mathbb{E}[(X - \mathrm{e}_\tau)^-] = 0,$$

- Generalisation of the mean: weighted mean squares minimisation
- Usual mean when $\tau = 1/2$.

# Summary of properties

| | var | sd | $\rho_\theta^{\mathsf{exp}}$ | $\mathrm{V@R}^\alpha$ | $\mathrm{ES}^\alpha$ | $\mathrm{e}_\tau$ |
|---|---|---|---|---|---|---|
| Monotonicity | $\star$ | $\star$ | ✓ | ✓ | ✓ | ✓ |
| Translation invariance | | | ✓ | ✓ | ✓ | ✓ |
| Subadditivity | | ✓ | ✓ | † | ✓ | ✓ |
| Positive homogeneity | | ✓ | | ✓ | ✓ | ✓ |
| Convexity | | ✓ | ✓ | † | ✓ | ✓ |
| Normalisation | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| Comonotonic additivity | | | | ✓ | ✓ | |
| Law invariance | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| Ellicitability | | | ✓ | ✓ | | ✓ |

$\star$ Variance and standard deviation are monotonous when restricting to losses with the same mean.

† $\mathrm{V@R}$ is subadditive (and hence convex) when considering linear combinations of a multidimensional elliptic function.

# Conditional and dynamic risk measures

## Definition

Let $t \in 0, \dots, T$. A mapping $\rho_t : L^\infty(\Omega, \mathcal{F}, \mathbb{R}) \to L^\infty(\Omega, \mathcal{F}_t, \mathbb{R})$ is called a conditional risk measure.

Examples:

- $\mathrm{V@R}_t^\alpha(X) := \operatorname{ess\,inf}\{Z \in L_t^\infty : \mathbb{E}_t[\mathbb{1}_{-X \leqslant Z}] \geqslant \alpha\}$.
- $\mathrm{ES}_t^\alpha(X) := \operatorname{ess\,inf}\{Z + \frac{1}{1-\alpha}\mathbb{E}_t[(X+Z)^-] : Z \in L_t^\infty\}$.

A sequence of conditional risk measures $\{\rho_t\}_{t=0,\dots,T}$ is denoted a *dynamic risk measure*.

# Concepts in risk management

Risk management is the action of identifying, evaluating and prioritizing risks.

## Risk management system

It comprises: policies, organisational structure, quantitative models and indicators

Its goal is:

- understanding the risk sources and exposures of a company
- deciding and monitoring when they are within what is acceptable
- taking action in cases when they are not acceptable: mitigate (collateral, hedging, diversification), avoid.

# Risk management

In the face of risk, there are three main strategies:

- Accept
- Mitigate
- Avoid

# Acceptable risks

A helpful tool in risk management is to determine which risks are acceptable

Mathematically, we define this set $\mathcal{A} \subset \mathcal{W}$, where $\mathcal{W}$ is the set of all possible wealth values that a market participants can have

## Definition

The acceptability set associated to a risk measure $\rho$ is defined by

$$\mathcal{A}_\rho = \{X : X \in \mathcal{X}; \rho(X) \leqslant 0\}.$$

A position $X$ is **acceptable** if $X \in \mathcal{A}_\rho$.

Risks that cannot be accepted can be mitigated to render them acceptable, or avoided if mitigation is unfeasible or too expensive.

# Collateral addition

**Collateral**: financial deposit, provided to mitigate economic losses associated to a default. Ex: mortgages and **capital**

*The deposit is not used unless the risk is materialised*.

The amount of collateral required depends on the *numeraire* on which it is provided:

- If collateral in a risk-less numéraire and $\rho$ is **monetary** $\kappa = (\rho(X))^+$
- If collateral in a defaultable numéraire $N$,

$$\kappa(X) = \inf_{r \in \mathbb{R}^+} \{r : \rho(X + rN) \leqslant 0\}.$$

If $\rho$ is strictly convex , there is at most a unique solution. If the solution is infinity (or infinity in practice) another mitigation tool must be used.

# Capital

Capital can be seen as a collateral that the owners of a financial company provide to make their business acceptable.

Financial regulation[1] imposes some maximal acceptability sets: forbids some operations, and making minimal capital requirements.It also defines the types of instruments acceptable for capital provision, and their maximum proportions.

**Capital determination:**

standard Based on the linearisation-normal approximation. Sensitivities times exposures times regulated coefficients.

internal model Each institution proposes a model for capital calculation adjusted to its business. Model subject to tests and approval by regulator.

[1] To learn more, visit each regulator's sites: BoE, BIS, FED,...

# Hedging

To *hedge* a risk means to make a sequence of operations in the market to obtain payoffs that offset the initial risks.

In our framework: find a strategy $\theta$ with minimal initial cost such that $\rho(\boldsymbol{S}_1^{\theta} - X) \leqslant 0$.

**Types of hedging**

Partial hedging Risk is mitigated but the probability of having losses is not zero.

Replication The risk of the operation is exactly matched: no losses and no profits.

Super hedging No losses and some profits.

# Risk sharing / Diversification

Risk sharing: Transfer part of your risk to another institution.

Diversification: Avoid depending on a few factors.

# Some final comments

- Risk measures have *blind spots*.
- Mitigation strategies might introduce new risks that need to be considered
- There are many additional interesting properties that risk measures can have.

# (**) Robust representations: coherent measures

Assume:

$$L_n \uparrow L \Rightarrow \rho(L_n) \uparrow \rho(L) \quad \text{(continuity from below; a.k.a. Fatou property)}$$

**Examples**: $\mathrm{V@R}$, $\mathrm{ES}$, expectiles.

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Let $\mathcal{M}$ be set of probability measures s.t. $\forall Q \in \mathcal{M}$, $\mathbb{E}^Q[X]$ is well-defined for all $X \in \mathcal{X}$.

## Theorem (Robust representation for coherent risk measures)

*There exists a set* $\mathcal{Q} \subset \mathcal{M}$ *such that*

$$\rho(X) = \sup_{Q \in \mathcal{Q}} \left\{ \mathbb{E}^Q[-X] \right\}$$

Intuition: A coherent risk measure is the worst average loss that can be obtained amongst all the possible distributions in $\mathcal{Q}$.

# (**)Robust representations: convex measures

## Theorem (Robust representation for convex risk measures)

*Let* $\rho$ *be a convex risk measure with Fatou property. Then there exist a set* $\mathcal{Q} \subset \mathcal{M}$ *and a function* $\alpha : \mathcal{M} \to \mathbb{R}$ *such that*

$$\rho(X) = \sup_{Q \in \mathcal{Q}} \left\{ \mathbb{E}^Q[X] - \alpha(Q) \right\}.$$

*Furthermore, defining* $\mathcal{A}_\rho = \{X : X \in \mathcal{X}; \rho(X) \leqslant 0\}$

$$\alpha(Q) = \sup_{X \in A_\rho} \left\{ \mathbb{E}^Q[X] \right\}$$

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Loosely speaking, we have penalised (through the term $\alpha$) probability distributions that are considered "unrealistic".

The converse of both robust representation theorems also holds.