Performance measurement, efficient frontiers and capital allocation

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Market risk and portfolio theory

Efficiency frontiers

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Definition

Efficiency: set of allocations for which it is impossible to reallocate to improve one feature without making another worse off.

This depends, usually, on the risk measure that we choose.

The mean variance case

Why work with mean and variance?

- Simplicity
- It coincides with optimal for CARA and quadratic under Gaussian returns
- For distributions like Gaussian this is equivalent to most risk measures
- Is valid for short perturbation asymptotics

The mean variance case

Definition (Mean-variance dominance)

A portfolio with return R_1 dominates another with return R_2 if either

- $\mathbb{E}[R_1] > \mathbb{E}[R_2]$ and $\mathrm{var}(R_1) \leqslant \mathrm{var}(R_2)$; or
- $\mathbb{E}[R_1] \geqslant \mathbb{E}[R_2]$ and $\mathrm{var}(R_1) < \mathrm{var}(R_2)$.

Our goal is to find all the portfolios not dominated by any other: this is the efficient mean-variance frontier.

Finding efficient portfolios: no risk-free asset

We start by *excluding the risk-free asset from our initial calculation*.

We are given the mean vector of all risky returns $\hat{\boldsymbol{\mu}}$ and its associated variance covariance $\bar{\Sigma}$.

For a given average return μ_p, we pose the problem

$$\min_{\hat{\boldsymbol{\pi}}} \frac{1}{2}\hat{\boldsymbol{\pi}}^\top \bar{\bar{\Sigma}} \hat{\boldsymbol{\pi}}$$

s.t.

$$\boldsymbol{\mu}^\top \hat{\boldsymbol{\pi}} = \mu_p;$$
$$\mathbf{1}^\top \hat{\boldsymbol{\pi}} = 1.$$

Mean-variance frontier: obtained by taking all possible $\mu_p \in \mathbb{R}$.
Efficient frontier: subset taking largest return for equal standard deviation.

Solution of optimisation problem - no risk-free asset

We first compute the Lagrangian

$$\mathcal{L}(\hat{\boldsymbol{\pi}}, \delta, \gamma) = \frac{1}{2}\hat{\boldsymbol{\pi}}^\top \bar{\bar{\Sigma}} \hat{\boldsymbol{\pi}} - \delta(\boldsymbol{\mu}^\top \hat{\boldsymbol{\pi}} - \mu_p) - \gamma(\mathbf{1}^\top \hat{\boldsymbol{\pi}} - 1)$$

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This yields the first-order conditions

$$0 = \nabla_{\hat{\boldsymbol{\pi}}} \mathcal{L}(\hat{\boldsymbol{\pi}}^*, \delta^*, \gamma^*) = \bar{\bar{\bar{\Sigma}}} \hat{\boldsymbol{\pi}}^* - \delta^* \boldsymbol{\mu} - \gamma^* \mathbf{1}.$$

$$0 = \boldsymbol{\mu}^\top \hat{\boldsymbol{\pi}}^* - \mu_p; \qquad 0 = \mathbf{1}^\top \hat{\boldsymbol{\pi}}^* - 1$$

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From where we get $\boxed{\hat{\boldsymbol{\pi}}^* = \delta^*\overline{\overline{\overline{\Sigma}}}^{-1}\boldsymbol{\mu} + \gamma^*\overline{\overline{\overline{\Sigma}}}^{-1}\mathbf{1}}$.

$$\mu_p = \delta^* \overbrace{\boldsymbol{\mu}^\top \overline{\overline{\overline{\Sigma}}}^{-1}\boldsymbol{\mu}}^{A} + \gamma^* \overbrace{\boldsymbol{\mu}^\top \overline{\overline{\overline{\Sigma}}}^{-1}\mathbf{1}}^{B};$$

$$1 = \delta^* \underbrace{\mathbf{1}^\top \overline{\overline{\overline{\Sigma}}}^{-1}\boldsymbol{\mu}}_{B} + \gamma^* \underbrace{\mathbf{1}^T \overline{\overline{\overline{\Sigma}}}^{-1}\mathbf{1}}_{C}.$$

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Solving,

$$\delta^* = \frac{\mu_p C - B}{AC - B^2}, \quad \gamma^* = \frac{A - \mu_p B}{AC - B^2}$$

Two fund spanning

Note

$$B\delta^* + C\gamma^* = 1$$

We then rewrite the F.O.C. in terms of two fully-invested portfolios

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where $\hat{\boldsymbol{\pi}}_\mu \cdot \mathbf{1} = 1$ and $\hat{\boldsymbol{\pi}}_1 \cdot \mathbf{1} = 1$, $\alpha = B\delta^* = \frac{B(\mu_p C - B)}{AC - B^2}$.

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$$\hat{\boldsymbol{\pi}}_\mu = \frac{1}{B} \bar{\Sigma}^{-1} \boldsymbol{\mu}; \qquad \hat{\boldsymbol{\pi}}_1 = \frac{1}{C} \bar{\Sigma}^{-1} \mathbf{1};$$

$$\mu_\mu = \hat{\boldsymbol{\pi}}_\mu \cdot \boldsymbol{\mu} = \frac{A}{B} \qquad \mu_1 = \hat{\boldsymbol{\pi}}_1 \cdot \boldsymbol{\mu} = \frac{B}{C}$$

$$\sigma_\mu^2 = \hat{\boldsymbol{\pi}}_\mu^\top \bar{\Sigma} \hat{\boldsymbol{\pi}}_\mu = \frac{A}{B^2} \qquad \sigma_1^2 = \hat{\boldsymbol{\pi}}_1^\top \bar{\Sigma} \hat{\boldsymbol{\pi}}_1 = \frac{1}{C}$$

Variance, Standard Deviation and Graphical

It follows that the optimal variance for a portfolio with mean μ_p is

$$\sigma_p^2 = \alpha^2\sigma_\mu^2 + (1-\alpha)^2\sigma_1^2 + 2\alpha(1-\alpha)\frac{1}{C} = \frac{A - 2B\mu_p + C\mu_p^2}{AC - B^2}$$

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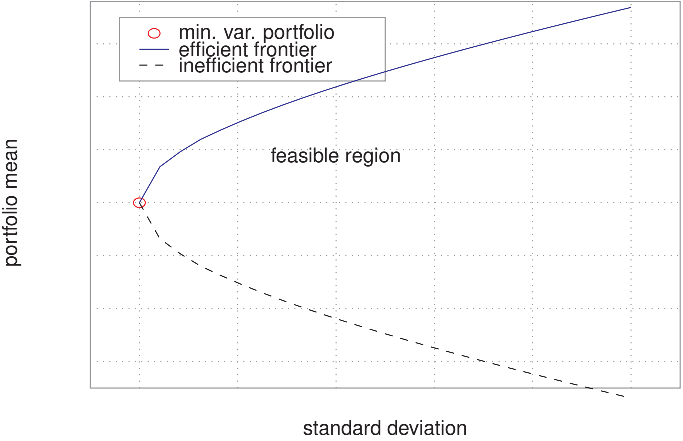

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We find the global minimal variance portfolio with only risky-assets:

- We can solve a new optimisation problem **without** the mean constraint.
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$$\sigma_p^2 = \frac{A - 2B\mu_p + C\mu_p^2}{AC - B^2} = \frac{(C^{1/2}\mu_p - BC^{-1/2})^2 + A - B^2C^{-1}}{AC - B^2}$$

So that the minimal variance is $\sigma_{min}^2 = \frac{1}{C}$ and occurs when $\mu_p = \frac{B}{C}$.

This is exactly the mean and variance of $\hat{\boldsymbol{\mu}}_1$. For this reason, it is the global minimal variance portfolio of risky assets.

The case with risk-free asset

We add again the risk free asset to the picture: note that

$$\boldsymbol{\pi} = (\pi^0, \hat{\boldsymbol{\pi}}) = (\pi^0, (1 - \pi^0)\bar{\boldsymbol{\pi}})$$

where $\bar{\boldsymbol{\pi}} = \frac{1}{1-\pi^0}\hat{\boldsymbol{\pi}}$ satisfies $\bar{\boldsymbol{\pi}}^\top \mathbf{1} = 1$.

Mean-variance optimisation with risk-free asset

The mathematical formulation in this case is

$$\min_{\hat{\boldsymbol{\pi}}} \frac{1}{2} \hat{\boldsymbol{\pi}}^\top \bar{\Sigma} \hat{\boldsymbol{\pi}}$$

subject to

$$\underbrace{\boldsymbol{\mu}^T \hat{\boldsymbol{\pi}}}_{\text{mean return risky assets}} \quad + \quad \underbrace{(1 - \mathbf{1}^\top \hat{\boldsymbol{\pi}}) R^0}_{\text{return risk-free}} \quad = \quad \mu_p;$$

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This constraint can also be written

$$\underbrace{(\boldsymbol{\mu} - R^0\mathbf{1})^\top\hat{\boldsymbol{\pi}}}_{\text{Excess of return portfolio}} \quad = \quad \underbrace{\mu_p - R^0}_{\text{Desired excess of return}} \quad ;$$

As before, we can instead solve the unconstrained problem

$$\min_{\hat{\boldsymbol{\pi}}\in\mathbb{R}^d, \delta\in\mathbb{R}} L(\hat{\boldsymbol{\pi}}, \delta); \quad \text{with } L(\hat{\boldsymbol{\pi}}, \delta) = \frac{1}{2}\hat{\boldsymbol{\pi}}^\top\bar{\Sigma}\hat{\boldsymbol{\pi}} - \delta((\boldsymbol{\mu} - R^0\mathbf{1})^\top\hat{\boldsymbol{\pi}} - (\mu_p - R_0))$$

Solution: the case with risk-free asset

$$L(\hat{\boldsymbol{\pi}}, \delta) = \frac{1}{2}\hat{\boldsymbol{\pi}}^\top \bar{\Sigma}\hat{\boldsymbol{\pi}} - \delta((\boldsymbol{\mu} - R^0\mathbf{1})^\top \hat{\boldsymbol{\pi}} - (\mu_p - R_0))$$

Applying first order conditions, we get the couple of equations

$$\bar{\Sigma}\hat{\boldsymbol{\pi}}^* - \delta^*(\boldsymbol{\mu} - R^0\mathbf{1}) = 0 \quad (*); \qquad (\boldsymbol{\mu} - R^0\mathbf{1})^\top \hat{\boldsymbol{\pi}}^* = \mu_p - R^0 \quad (**)$$

From $(*)$, we get

$$\hat{\boldsymbol{\pi}}^* = \delta^* \bar{\Sigma}^{-1}(\boldsymbol{\mu} - R^0\mathbf{1})$$

and replacing in $(**)$ we then obtain

$$\delta^* = \frac{\mu_p - R^0}{(\boldsymbol{\mu} - R^0\mathbf{1})^\top \bar{\Sigma}^{-1}(\boldsymbol{\mu} - R^0\mathbf{1})}$$

Hence,

$$\hat{\boldsymbol{\pi}}^* = \frac{(\mu_p - R^0)}{(\boldsymbol{\mu} - R^0\mathbf{1})^\top \bar{\Sigma}^{-1}(\boldsymbol{\mu} - R^0\mathbf{1})} \bar{\Sigma}^{-1}(\boldsymbol{\mu} - R^0\mathbf{1})$$

Standard Deviation

$$
\begin{aligned}
\sqrt{\hat{\boldsymbol{\pi}}^{*}\bar{\Sigma}\hat{\boldsymbol{\pi}}^{*}} &= \sqrt{(\mu_p - R^0)^2 \frac{(\boldsymbol{\mu} - R^0\mathbf{1})^\top \bar{\Sigma}^{-1} (\boldsymbol{\mu} - R^0\mathbf{1})}{\left[(\boldsymbol{\mu} - R^0\mathbf{1})^\top \bar{\Sigma}^{-1} (\boldsymbol{\mu} - R^0\mathbf{1})\right]^2}} \\
&= \frac{|\mu_p - R^0|}{\sqrt{(\boldsymbol{\mu} - R^0\mathbf{1})^\top \bar{\Sigma}^{-1} (\boldsymbol{\mu} - R^0\mathbf{1})}}
\end{aligned}
$$

Sharpe ratio

Sharpe ratio: Ratio between risk premium and standard deviation:

$$\mathcal{S}_p := \frac{\mu_p - R^0}{\sqrt{\hat{\boldsymbol{\pi}}_p^\top \bar{\bar{\Sigma}} \hat{\boldsymbol{\pi}}_p}}$$

The Sharpe ratio is a (risk-adjusted) **performance** measurement: the larger it is, the better relation between return and risk.

The efficient mean-variance portfolio (case with risky asset) has constant maximal Sharpe ratio

$$\mathcal{S}^* = \sqrt{(\boldsymbol{\mu} - R^0\mathbf{1})^\top \bar{\bar{\Sigma}}^{-1} (\boldsymbol{\mu} - R^0\mathbf{1})}$$

Tangency portfolio

Recall that efficient (with riskless asset) portfolios have the form

$$\hat{\boldsymbol{\pi}}^* = (\mu_p - R^0)\frac{\bar{\Sigma}^{-1}(\boldsymbol{\mu} - R^0\mathbf{1})}{(\boldsymbol{\mu} - R^0\mathbf{1})^\top\bar{\Sigma}^{-1}(\boldsymbol{\mu} - R^0\mathbf{1})}$$

We need to choose (at least one) mean μ_{tan} such that the associated portfolio satisfies $\mathbf{1}^\top\boldsymbol{\pi}_{tan} = 1$ (only risky assets).

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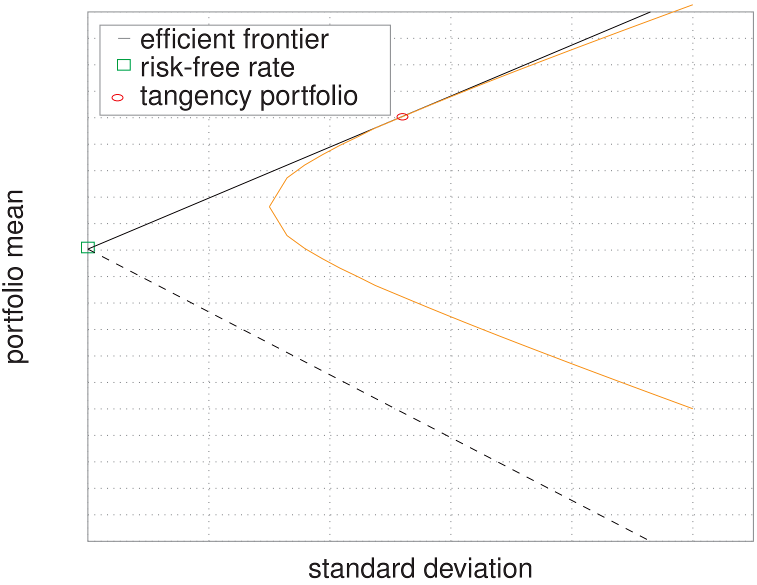
We need to choose (at least one) mean μ_{tan} such that the associated portfolio satisfies $\mathbf{1}^\top \boldsymbol{\pi}_{tan} = 1$ (only risky assets). Assume $B \neq R^0 C$, and set

$$\mu_p = \mu_{tan} = \frac{\boldsymbol{\mu}^\top \bar{\Sigma}^{-1}(\boldsymbol{\mu} - R^0\mathbf{1})}{\mathbf{1}^T \bar{\Sigma}^{-1}(\boldsymbol{\mu} - R^0\mathbf{1})}$$

Then, we verify the constraint and find

$$\hat{\boldsymbol{\pi}}_{tan} = \frac{1}{\mathbf{1}^T \bar{\Sigma}^{-1}(\boldsymbol{\mu} - R^0\mathbf{1})} \bar{\Sigma}^{-1}(\boldsymbol{\mu} - R^0\mathbf{1}). \tag{1}$$

Graphical representation



Connection with CAPM

If assumptions of CAPM in the CARA - Normal case are true: the market portfolio has shape

$$\hat{\boldsymbol{\pi}}_m^{CAPM} = \Gamma\bar{\Sigma}^{-1}(\boldsymbol{\mu} - R^0\mathbf{1})$$

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In practice the estimations of these quantities **do not coincide**.

Drawbacks

- The model relies on estimations of mean and variance that might be difficult to obtain
- Tangency portfolio is not very stable
- Assuming we canshort as much as desired is not realistic
- Not a unique candidate for proxy of risk-free rate
- Measuring risk with standard deviation/variance is sometimes too simplistic.

Extension to other risk measures

We can repeat a similar analysis with other risk measures, and find, for example, the mean-expected shortfall frontier.

If the risk measure is convex, then the plot of the set of feasible portfolios is also convex (the frontier is not always a quadratic function, except possibly on elliptic function case).

RORAC

Sharpe ratio expresses the excess return per 'risk unit' as measured by std. dev. We extend this idea.

Let $\boldsymbol{\phi} \in \mathbb{R}_{+}^{n}$ be the vector of amounts invested in risky positions, and ρ a monetary risk measure.

Definition (RORAC (returns of risk-adjusted capital))

$$\mathcal{R}^{\rho}(\boldsymbol{\phi}) = \frac{\text{Expected profit of portfolio } \boldsymbol{\phi}}{\text{Risk capital of portfolio } \boldsymbol{\phi}} = \frac{\mathbb{E}[\boldsymbol{\phi}(\boldsymbol{R} - \mathbf{1})]}{\rho[\boldsymbol{\phi}(R - \mathbf{1})]},$$

That is, the ratio of expected net-gains over their risk.

It is a good measure to compare performance between assets with different risks.

Risk allocation

Motivation:

- Financial investment: Establish what is the riskiness of individual investments considered *as part* of a given portfolio.
- Loan pricing: Understand how much capital should be **allocated** to each loan within a portfolio of n loans.

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A **risk allocation** is a set of risk values $\kappa^1, \dots, \kappa^n$ such that κ^i can be understood as the risk contribution of the asset i, and such that

$$\kappa = \sum_{i=1}^{n} \kappa^i.$$

where $\kappa = \rho(\boldsymbol{\phi}(\boldsymbol{R} - \mathbf{1}))$ is the total risk of the portfolio.

Euler per-unit capital allocation

Let ρ be monetary, positive homogeneous and differentiable at $\mathbb{R}_+$.

Define the risk measure function $r^\rho : \boldsymbol{x} \in \mathbb{R}_+^n \to \rho(\boldsymbol{x}(\boldsymbol{R} - 1))$. It follows from Euler's theorem that

$$\kappa = \rho(\boldsymbol{\phi}(\boldsymbol{R} - 1)) = r^\rho(\boldsymbol{\phi}) = \sum_{i=1}^{n} \phi^i \partial_{\phi_i} r^\rho(\boldsymbol{\phi})$$

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This defines the Euler per-unit capital allocation rule: $\kappa_i := \phi^i \partial_{\phi_i} r^\rho(\boldsymbol{\phi})$.

Examples

Some examples of Euler capital allocation:

$$\text{sd: } \kappa_i^{\mathrm{sd}}(\boldsymbol{\phi}) = \phi_i \frac{\mathrm{cov}(R_i, R(\boldsymbol{\pi}))}{\sqrt{\mathrm{var}(R(\boldsymbol{\pi}))}}$$

$$\mathrm{V@R}^{\alpha}\text{: } \kappa_i^{\mathrm{V@R}^{\alpha}}(\boldsymbol{\phi}) = \phi_i \mathbb{E}[-R_i | \boldsymbol{\phi}(\boldsymbol{R} - 1) = \mathrm{V@R}^{\alpha}(\boldsymbol{\phi}(\boldsymbol{R} - 1)))]$$

$$\mathrm{ES}^{\alpha}\text{: } \kappa_i^{\mathrm{ES}^{\alpha}}(\boldsymbol{\phi}) = \phi_i \mathbb{E}[-R_i | \boldsymbol{\phi}(\boldsymbol{R} - 1) \geqslant \mathrm{V@R}^{\alpha}(\boldsymbol{\phi}(\boldsymbol{R} - 1)))]$$

Euler is suitable for performance comparison

The Euler capital allocation is *suitable for performance measurement*: i.e.

$$\frac{\partial \mathcal{R}^{r_\rho}(\boldsymbol{\phi})}{\partial \phi_i} \begin{cases} > 0 & \text{if } \frac{\mathbb{E}[\phi^i(R_i - 1)]}{\kappa_i^\rho(\boldsymbol{\phi})} > \mathcal{R}^\rho(\boldsymbol{\phi}) \\ < 0 & \text{if } \frac{\mathbb{E}[\phi^i(R_i - 1)]}{\kappa_i^\rho(\boldsymbol{\phi})} < \mathcal{R}^\rho(\boldsymbol{\phi}) \end{cases}$$

In words: *increasing a position in assets with better RORAC using as capital the allocated one increases the overall RORAC*.

In fact, (Tasche (1999)) shows this is the only per-unit capital allocation principle suitable for performance measurement.