Department of Mathematics
University College London

MATH0094 Market Risk and Portfolio Theory

Mock Exam

TIME: 2.5 HOURS

Each question is worth 25 marks.

Question 1. For a constant $c \in \mathbb{R}$, let $u : \mathbb{R} \to \mathbb{R}$ be given by

$$u(x) = \begin{cases} \sqrt{x-c} & x \geq c \\ -\sqrt{c-x} & x < c \end{cases},$$

where $\sqrt{x}$ is the non-negative square root for $x \geq 0$.

(a) Calculate the coefficient of absolute risk aversion and the coefficient of relative risk aversion as a function of wealth for an investor with this utility function.

[8 marks]

(b) Two investment opportunities are available to an investor with the above utility function and initial wealth c: i) An investment with (gross) return R distributed like $U[1/2, 3/2]$; or ii) An investment with return $\tilde{R}$ distributed like $U[2/3, 4/3]$.

According to utility theory, which of these investments will the investor prefer? Generalise your answer to the comparison of two investments with distribution $U[1-p, 1+p]$.

[9 marks]

(c) Consider the risk measure ρ_u defined by

$$\rho(X) := -u^{-1}(\mathbb{E}[u(X)]), \quad \text{for } X \text{ such that } \mathbb{E}|u(X)| < \infty.$$

Show that u *is not* a monetary risk measure.

[8 marks]

[Total: 25 marks]

Question 2.
Set a one-period market model with a risk-free asset with rate of return r_1^0 and n risky assets with normal rate of return vector $\boldsymbol{r}_1 \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$, where $\boldsymbol{\mu} \in \mathbb{R}^n$ and $\Sigma \in \mathbb{R}^{n \times n}$ is definite positive and invertible. Recall that

$$r_1^i := \frac{S_1^i - S_0^i}{S_0^i}; i = 0, \ldots, n.$$

Assume that $0 < r_1^0 < \mu^i$ for $i = 1, \ldots n$.

(a) Show that if M is an SDF in this market, then M is negatively correlated with each rate of return r_1^i (that is, show that $\mathrm{corr(M, r_1^i)} < 0$).

[7 marks]

(b) Find the optimal investment in this market for a pure investor with CARA utility and absolute risk aversion α. Assume no endowments and initial wealth w_0.

[10 marks]

Consider now a two-period version of the above market, assuming that the rate of returns in time are i.i.d. so that $\boldsymbol{r}_2 \sim \boldsymbol{r}_1$, $\boldsymbol{r}_2 \perp\!\!\!\perp \boldsymbol{r}_1$ and $r_2^0 = r_1^0$.

(c) The same investor as in (b) wants now to maximise

$$\mathbb{E}[u(W_2)]$$

where W_2 is their wealth at time 2. Assume they have initial wealth w_0, no consumption and no endowments. Find the optimal investment strategy for this investor.
Hint: Take advantage of the independence assumption and use dynamic programming.

[8 marks]

[Total: 25 marks]

Question 3.

(a) Show that for $\alpha \in (0,1)$ and any two random variables X, Y such that (X, Y) is jointly Gaussian we have that $\mathrm{V@R}^\alpha(X+Y) \leq \mathrm{V@R}^\alpha(X) + \mathrm{V@R}^\alpha(Y)$.

[9 marks]

(b) Show via a counterexample that Value at Risk is, however, not subadditive in general. Verify on the same example the subadditivity of expected shortfall.

[6 marks]

(c) Assume that you are calculating capital on the daily returns of a portfolio using as risk measure value at risk at 97.5%.

Use a Z-test (Gaussian test) to find the minimal number of excess losses in a trading year (252 days) that would put in doubt the coverage property of your calculation, if the accepted I-type error is 5%.

[10 marks]

[Total: 25 marks]

Question 4. Consider a financial market composed by one risk-free asset with return R^0, and n risky assets with returns $\hat{\boldsymbol{R}} = (R^1, \dots, R^n)^\top$. We define $\boldsymbol{\mu} = (\mu^1, \dots, \mu^n)^\top$ where $\mu^i = E[R^i]$, and the matrix Σ where $\Sigma_{ij} = \mathrm{cov}[R^i, R^j]$, for $i, j = 1, \dots, n$. Assume that Σ is invertible and that

$$R^0 \neq \frac{\boldsymbol{\mu}^\top \Sigma^{-1} \mathbf{1}}{\mathbf{1}^\top \Sigma^{-1} \mathbf{1}},$$

Furthermore, let $\boldsymbol{\pi} = (\pi^0, \dots, \pi^n)^\top$ where π^i denotes the proportion of investment in the i-th risky asset, and $\hat{\boldsymbol{\pi}}$ is the sub-vector corresponding to the risky assets.

(a) We showed that all portfolios in the mean-variance frontier can be written as

$$\hat{\boldsymbol{\pi}}^* = \delta_p \Sigma^{-1}(\boldsymbol{\mu} - R^0\mathbf{1}); \quad \pi^{*0} = 1 - \mathbf{1}^\top \hat{\boldsymbol{\pi}}^*$$

Show that the value of the constant δ_p needed to obtain a portfolio in the frontier with mean μ_p is

$$\delta_p = \frac{\mu_p - R^0}{(\boldsymbol{\mu} - R^0\mathbf{1})^\top \Sigma^{-1} (\boldsymbol{\mu} - R^0\mathbf{1})}.$$

[7 marks]

(b) For a portfolio $\boldsymbol{\pi}$ let R_π, μ_π and σ_π be respectively its return, expected return and standard deviation of its return. Recall that the Sharpe ratio is given by

$$\mathcal{S}(\boldsymbol{\pi}) = \frac{\mu_\pi - R^0}{\sigma_\pi}.$$

Show that

$$|\mathcal{S}(\boldsymbol{\pi})| \leq \sqrt{(\boldsymbol{\mu} - R^0\mathbf{1})^\top \Sigma^{-1} (\boldsymbol{\mu} - R^0\mathbf{1})}$$

with equality if and only if $\boldsymbol{\pi}$ is in the mean-variance frontier.

[9 marks]

(c) Show that if there is a beta pricing model having as factor the return of a market portfolio $\boldsymbol{\pi}^*$, i.e., if for any market portfolio π we have that

$$\mu_\pi - R^0 = (\mu_{\pi^*} - R^0) \frac{\mathrm{cov}(R_\pi, R_{\pi^*})}{\sigma^2_{\pi^*}},$$

then π^* is a portfolio in the mean-variance frontier. *Hint: Use (b).*

[9 marks]

[Total: 25 marks]