2017 Q2

a). $\xi \sim N(m, \sigma^2)$, $S_1 = S_0 \cdot e^{\xi}$, $X = S_1 - S_0$

$V@R_X(\alpha) = q_{-X}(\alpha) = \inf\{z : P(-X \le z) \ge \alpha\}$

Notice that $\log(\frac{X}{S_0}+1) \sim N(m, \sigma^2)$, $\frac{1}{\sigma}(\log(\frac{X}{S_0}+1)-m) \sim N(0,1)$, Φ is cdf of stdnd

$$P(-X \le V@R_X(\alpha)) = P(X \ge -V@R_X(\alpha)) \ge P(\tfrac{1}{\sigma}((\log(\tfrac{X}{S_0}+1)-m) \ge \tfrac{1}{\sigma}((\log(\tfrac{-V@R_X(\alpha)}{S_0}+1)-m))$$

$$\frac{1}{\sigma}(\log(\frac{-V@R_X(\alpha)}{S_0}+1)-m) = \Phi^{-1}(\alpha) \Rightarrow V@R_X(\alpha) = -S_0(e^{\sigma\Phi^{-1}(\alpha)+m}-1)$$

b) $X = S_1 - S_0$, $L = (S_1-K)^+ - C$, $P = -L$, show $V@R^{\alpha}(P) = \max\{S_0 - K - C - V@R_{P}(1-\alpha), -C\}$

$$\begin{aligned}
V@R^{\alpha}(P) &= \inf\{z : P(P+z \ge 0) \ge \alpha\} \\
&= \inf\{z : P((S_1-K)^+ - C \le z) \ge \alpha\} \\
&= \inf\{z : P(\max\{S_1-K, 0\} - C \le z) \ge \alpha\} \\
&= \inf\{z : P(\max\{S_1-K-C, -C\} \le z) \ge \alpha\} \\
&= \inf\{z : P(\{S_1-K-C \le z\} \cap \{-C \le z\}) \ge \alpha\} \\
&= \inf\{z : P(S_0+X-K-C \le z) \ge \alpha,\ z \ge -C\} \\
&= \max\{-C, \inf\{z : P(S_0+X-K-C \le z) \ge \alpha\}\} \\
&= \max\{-C, \inf\{z : P(z - X - S_0+K+C \ge 0) \ge \alpha\}\} \\
&= \max\{-C, V@R^{\alpha}(-X-S_0+K+C)\} \\
&= \max\{-C, S_0-K-C-V@R^{1-\alpha}(X)\}
\end{aligned}$$

If $z < -C$, then $P(\cdots) = 0$, impossible

By Lemma 1, cash invariance

Lemma 1: When X is continuous r.v., $V@R^{\alpha}(-X) = -V@R^{1-\alpha}(X)$

Proof:
$$\begin{aligned}
V@R^{\alpha}(-X) &= \inf\{z : P(-X+z \ge 0) \ge \alpha\} \\
&= \inf\{z : P(X-z > 0) \ge 1-\alpha\} \\
&= \inf\{z : P(X-z \ge 0) \ge 1-\alpha\} \\
&= -V@R^{1-\alpha}(X)
\end{aligned}$$

continuity

c) $L_i \ge V@R^{\alpha}(L_i) \iff \xi_i \le m + \sigma\Phi^{-1}(1-\alpha)$, ξ_i iid, $\Rightarrow I_i$ iid binomial

$$I_i = \begin{cases} 1, & L_i \ge V@R^{\alpha}(L_i), \quad p = 1-\alpha \\ 0, & \text{others}, \quad p = \alpha \end{cases}$$

Let $S = \frac{1}{N}\sum_{i=1}^{n} I_i$, $N = 400$

$E(I_i) = 1-\alpha$, $V(I_i) = \alpha(1-\alpha)$,

$E(S) = 1-\alpha$, $V(S) = \frac{1}{N}\alpha(1-\alpha)$

H_0: L_i iid and $P(L_i \ge V@R^{\alpha}(L_i)) \ge \alpha$

By CLT, $Z = \dfrac{\hat{S} - E(S)}{\sqrt{V(S)}}$ is std Gaussian, $\hat{S} = \frac{10}{400}$ observed, $\alpha = 0.99$

$Z_{test} = \dfrac{20 \times (\frac{10}{400} - 0.01)}{\sqrt{0.01 \times 0.99}} \approx 3 > 1.96$ so reject H_0

2018 Q1

(a) ...

(b)
$$S_0 = \begin{bmatrix} 10 \\ 15 \\ 30 \end{bmatrix}, \quad M_{S_1} = \begin{bmatrix} 12 & 12 & 12 \\ 12 & 24 & 24 \\ 36 & 24 & 48 \end{bmatrix},$$

$p_1^{AD} \cdot M_{S_1} = S_0$ has strictly positive solution $\Rightarrow$ arbitrage free

$\det(M_{S_1}) \neq 0 \Rightarrow$ complete

(c).
$$S_0^c = \begin{bmatrix} 10 \\ 15 \\ 30 \\ 5 \end{bmatrix} \quad M_{S_1^c} = \begin{bmatrix} 12 & 12 & 12 \\ 12 & 24 & 24 \\ 36 & 24 & 48 \\ 0 & 0 & 12 \end{bmatrix}$$

Judge $p_1^{AD} \cdot M_{S_1} = S_0^c$ has strictly positive solution :

$$\left[\begin{array}{ccc|c} 12 & 12 & 12 & 10 \\ 12 & 24 & 24 & 15 \\ 36 & 24 & 48 & 30 \\ 0 & 0 & 12 & 5 \end{array}\right] = \frac{5}{12}\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 3 & 2 & 4 & 6 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$= \frac{5}{12}\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] = \frac{5}{12}\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$= \frac{5}{12}\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & \frac{1}{2} \end{array}\right]$$ no sol !

In fact, arbitrage : $\theta = [0, 0, 5, -5]^T$

2018 Q2

(a) $u(x) = \frac{1+2x}{1+x} = 2 - \frac{1}{1+x}$ $\quad u'(x) = \frac{1}{(x+1)^2}$ $\quad u''(x) = -2 \cdot \frac{1}{(x+1)^3} < 0$

$ara = \frac{-u''(x)}{u'(x)} = \frac{2}{x+1}$

(b)

	p	$\overline{w}$	$u(\overline{w})$
w_1	$\frac{1}{6}$	0	1
w_2	$\frac{1}{3}$	1	$\frac{3}{2}$
w_3	$\frac{1}{2}$	2	$\frac{5}{3}$

$u(x) = \frac{1+2x}{1+x} = y$

$1+2x = y + yx$

$(y-2)x = 1-y \Rightarrow u^{-1}(x) = \frac{1-x}{x-2}$

$u(E(\overline{w}) - \eta) = E(u(\overline{w}))$

$E(\overline{w}) = \frac{1}{6}\cdot 0 + \frac{1}{3}\cdot 1 + \frac{1}{2}\cdot 2 = \frac{4}{3}$

$E(u(\overline{w})) = \frac{1}{6}\cdot 1 + \frac{1}{3}\cdot\frac{3}{2} + \frac{1}{2}\cdot\frac{5}{3} = \frac{1+3+5}{6} = \frac{3}{2}$

$u(\frac{4}{3} - \eta) = \frac{3}{2} \Rightarrow \frac{4}{3} - \eta = u^{-1}(\frac{3}{2}) = \frac{1-\frac{3}{2}}{\frac{3}{2}-2} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1 \Rightarrow \eta = \frac{1}{3}$

meaning : show how risk averse she is.

c)

her payoff without new asset : $\overline{W}_1 \begin{cases} 0 & w_1 \\ 1 & w_2 \\ 2 & w_3 \end{cases}$ and $E(u(\overline{W}_1)) = \frac{3}{2}$

her payoff with new asset : $\overline{W}_2 \begin{cases} 0+1 = 1 & w_1 \\ 1+3 = 4 & w_2 \\ 2-1 = 1 & w_3 \end{cases}$ $\quad E(\overline{W}_2) = \frac{1}{6}\times 1 + \frac{1}{3}\times 4 + \frac{1}{2}\times 1 = \frac{1+8+3}{6} = 2$

her willing to pay η s.t. $u(E(\overline{W}_2) - \eta) \geq E(u(\overline{W}_1))$ I'm not sure about the formula.

$u(2-\eta) \geq \frac{3}{2} \Rightarrow 2 - \eta = u^{-1}(\frac{3}{2}) = 1 \Rightarrow \eta = 1$

2018 Q3

(a)

2018 Q4

(a) notes P110

(b) CARA : $u = -ae^{-\alpha x} + b$

When $X \sim N(\mu, \sigma^2)$, $E(-ae^{-\alpha x} + b) = -ae^{-\alpha\mu + \frac{1}{2}\alpha^2\sigma^2} + b$

since $-e^{-\alpha f(x)} + b$ has same increasing with $f(x)$, let $f(x) = \mu + \frac{1}{2}\alpha\sigma^2$

so $\max E(u(X)) \Longleftrightarrow \max\{\mu + \frac{1}{2}\alpha\sigma^2\}$

$W_1 = (W_0 - \vec{\phi}^T \vec{1}) R^0 + \vec{\phi}^T \vec{R} = \vec{\phi}^T(\vec{R} - R^0\vec{1}) + W_0 R^0$

$\mu = E(W_1) = \vec{\phi}^T(\vec{\mu} - R^0\vec{1}) + W_0 R^0$

$\sigma^2 = E((W_1 - \mu)^2) = \vec{\phi}^T \Sigma \vec{\phi}$

$L(\vec{\phi}) = \mu + \frac{1}{2}\alpha\sigma^2$

$\frac{\partial}{\partial \vec{\phi}} L(\hat{\vec{\phi}}) = \alpha \Sigma \hat{\vec{\phi}} + \vec{\mu} - R^0 \vec{1} = 0 \Rightarrow \vec{\phi} = \frac{1}{\alpha}\Sigma^{-1}(\vec{\mu} - R^0\vec{1})$

It's on the frontier because we can let $\delta_p = \frac{1}{\alpha}$,

so $\frac{\mu_p - R^0}{(\vec{\mu} - R^0\vec{1})\Sigma^{-1}(\vec{\mu} - R^0\vec{1})} = \frac{1}{\alpha} \Rightarrow \mu_p = R^0 + \frac{1}{\alpha}(\vec{\mu} - R^0\vec{1})\Sigma^{-1}(\vec{\mu} - R^0\vec{1})$

2021 m Q4

(a) $\pi = \lambda \pi_1 + (1-\lambda)\pi_2$, $\lambda \in \mathbb{R}$

$E(R(\pi)) = \lambda(R^0 + a) + (1-\lambda)R^0 = R^0 + \lambda \cdot a$

$V(R(\pi)) = V(\lambda\pi_1 + (1-\lambda)\pi_2) = \lambda^2\sigma^2 + (1-\lambda)^2\sigma^2 + 2\cdot\frac{1}{2}\cdot\lambda\cdot(1-\lambda)\sigma^2$

$= (\lambda^2 + \lambda^2 - 2\lambda + 1 + \lambda - \lambda^2)\sigma^2 = (\lambda^2 - \lambda + 1)\sigma^2$

(b) $\pi_* = \lambda_*\pi_1 + (1-\lambda_*)\pi_2$

$E(R(\pi_*)) = R^0 + \lambda_* \cdot a$

$V(R(\pi_*)) = (\lambda_*^2 - \lambda_* + 1)\sigma^2$

$SR(\pi_*) = \dfrac{E(R(\pi_*)) - R^0}{sd(R(\pi_*))} = \dfrac{\lambda_* \cdot a}{(\lambda_*^2 - \lambda_* + 1)\sigma}$

Since tangency portfolio has max SR, $\lambda_* = 1$

$\Rightarrow$ tangency portfolio $\pi_* = \pi_2$, $SR_{max} = \frac{a}{\sigma}$

(c) Let π_0 stand for risk-free asset portfolio

$A = \{\pi : \pi = \lambda\pi_* + (1-\lambda)\pi_0 , \ 0 < \lambda \leq 1\}$,

$\forall \hat{\pi} \in A$:

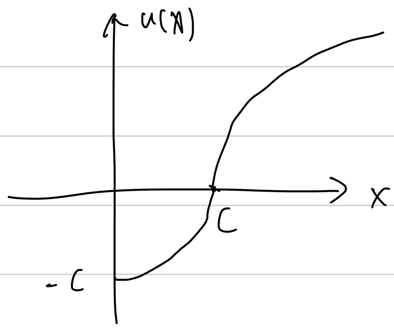
$SR(\hat{\pi}) = SR(\lambda\pi_*) = SR_{max}$ $\Rightarrow SR_{max}$

$E(R(\hat{\pi})) = \lambda(R^0 + a) + (1-\lambda)R^0 = R^0 + \lambda a > R^0$ $\Rightarrow$ return $> R^0$

$1 - \lambda > 0$ $\Rightarrow$ no short position.

2023 Q1

(a)



$ara = -\frac{u''(x)}{u'(x)}$, $rra = \frac{-u''(x)}{u'(x)} \cdot x$

(b) $u = \begin{cases} \sqrt{x-c}, & x \geq c \\ -\sqrt{c-x}, & x < c \end{cases}$, $x = cr$, $r \sim U(1-p, 1+p)$

$\left(E(R) = 1, \quad V(R) = E((R-1)^2) = \int_{1-p}^{1+p} (R-1)^2 dR = \int_{-p}^{p} R^2 dR = \frac{2}{3} p^3. \right)$

$E(u(w)) = \int_{1-p}^{1} -\sqrt{c-cr} \cdot \frac{1}{2p} dr + \int_{1}^{1+p} \sqrt{cr-c} \cdot \frac{1}{2p} dr = \frac{\sqrt{c}}{p} \int_0^p \sqrt{r}\, dr = \frac{\sqrt{c}}{p} \frac{2}{3} \cdot p^{\frac{3}{2}} = \frac{2}{3}\sqrt{cp}$

is monotinicity increasing. so prefer lower p.

(c) monetary: monotonicity: $u(X) \uparrow$, $E(u(X)) \uparrow$, $-u^{-1}(x) \uparrow \Rightarrow -u^{-1}(E(u(X))) \uparrow$ ✓

Proof by contradiction: If has cash invariance: If $c \neq 0$,

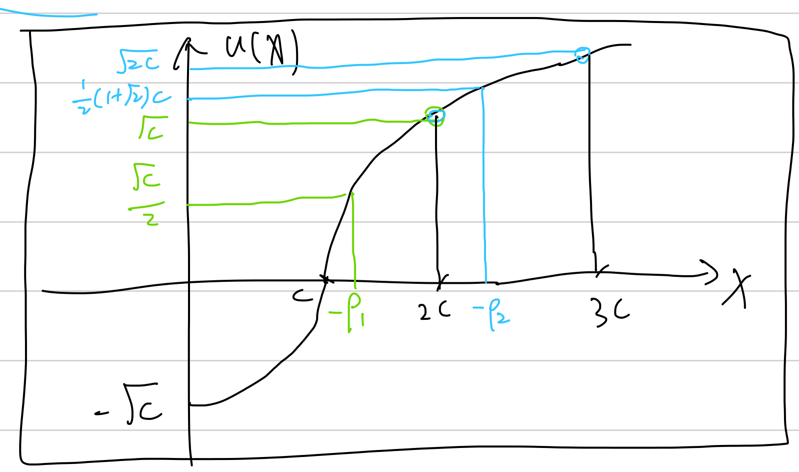
Let $X_1 = \begin{cases} c, & w = w_1, \ p = \frac{1}{2} \\ 2c, & w = w_2, \ p = \frac{1}{2} \end{cases}$ $\quad X_2 = X_1 + c$

$\rho(X) = -u^{-1}(E(u(X)))$

$u^{-1}(x) = \begin{cases} c + x^2, & x \geq 0, \\ c - x^2, & x < 0 \end{cases}$

$E(u(X_1)) = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot \sqrt{c} = \frac{1}{2}\sqrt{c}$

$\rho(X_1) = -u^{-1}\left(\frac{\sqrt{c}}{2}\right) = -c + \frac{1}{4}c = -\frac{5}{4}c$



$E(u(X_2)) = \frac{1}{2} \cdot \sqrt{c} + \frac{1}{2}\sqrt{2c} = \frac{1}{2}(1+\sqrt{2})\sqrt{c}$

$\rho(X_2) = -u^{-1}\left(\frac{1}{2}(1+\sqrt{2})\sqrt{c}\right) = -c - \frac{1}{4}(1+\sqrt{2})^2 c = -\left(1 + \frac{\sqrt{2}}{2}\right)c$ ②

By ①, ②:

$\rho(X_2) = \rho(X_1 + c) = \rho(X_1) - c \Rightarrow -\left(1+\frac{\sqrt{2}}{2}\right)c = -\frac{1}{4}c - c$

which is contradict to $c \neq 0$, hence not monetary

If $c = 0$, change from c to 1 in previous proof. The rest are same.

2023 Q2

(a) Stronger conclusion (Exercise 1.3): $E_t^P(R_{t+1}^i) - R_{t+1}^0 = -R_{t+1}^0 \, cov_t\left(\frac{M_{t+1}}{M_t}, R_{t+1}^i\right)$

Proof:

$$E_t(R_{t+1}^i) - R_{t+1}^0 = -R_{t+1}^0 \, cov_t\left(\frac{M_{t+1}}{M_t}, R_{t+1}^i\right)$$

$$cov_t\left(\frac{M_{t+1}}{M_t}, R_{t+1}^i\right) = E_t\left(\frac{M_{t+1}R_{t+1}^i}{M_t}\right) - E_t\left(\frac{M_{t+1}}{M_t}\right) \cdot E_t(R_{t+1}^i)$$

$$cov_t\left(\frac{M_{t+1}}{M_t}, R_{t+1}^i\right) = 1 - \frac{1}{R_{t+1}^0} \cdot E_t(R_{t+1}^i)$$

$$-R_{t+1}^0 \, cov_t(\cdots) = -R_{t+1}^0 + E_t(R_{t+1}^i)$$

(b) cara: $u = -e^{-\alpha x}$ (ignore a, b in $-ae^{-\alpha x} + b$)

The problem reads: $\max_{\vec{\phi}} E(u(w_1(\vec{\phi})))$, $w_1(\vec{\phi}) = (W_0 - \vec{\phi}^T \cdot 1)(1+r_1^0) + \vec{\phi}^T \cdot (\vec{1} + \vec{r}_1)$

$$u(w_1(\vec{\phi})) = -e^{-\alpha((W_0 - \vec{\phi}^T \cdot 1)(1+r_1^0) + \vec{\phi}^T \cdot (\vec{1} + \vec{r}_1))}$$

$$E(-\alpha W_1(\vec{\phi})) = -\alpha(W_0 - \vec{\phi}^T \cdot \vec{1})(1+r_1^0) + \vec{\phi}^T(\vec{1} + \vec{\mu})$$

$$V(-\alpha w_1(\vec{\phi})) = \alpha^2 \vec{\phi}^T \bar{\bar{\Sigma}} \vec{\phi}$$

$E(u(w_1(\vec{\phi}))) = -e^{-\alpha(W_0 - \vec{\phi}^T \cdot \vec{1})(1+r_1^0) + \frac{1}{2}\alpha^2 \vec{\phi}^T \bar{\bar{\Sigma}} \vec{\phi}}$ is a quadratic function of $\vec{\phi}$

(or using FOC) $\vec{\phi}^* = \frac{1}{\alpha} \bar{\bar{\Sigma}}^{-1} \cdot \vec{1}(1+r_1^0)$

(c) DPP: $W_0 = 0$, $\vec{\phi}_1$, $\vec{\phi}_2$ s.t.

$w_1 = (W_0 - \vec{\phi}_1^T \cdot \vec{1}) \cdot (1+r_1^0) + \vec{\phi}_1^T \cdot (1+\vec{r}_1)$

$W_2 = (W_1 - \vec{\phi}_2^T \cdot \vec{1}) \cdot (1+r_2^0) + \vec{\phi}_2^T(1+\vec{r}_2)$, and $r_2^0 = r_1^0$, $\vec{r}_2 \sim \vec{r}_1$, $\vec{r}_2 \perp \vec{r}_1$

The problem reads $\max_{\vec{\phi}_1, \vec{\phi}_2} E(u(W_2(\vec{\phi}_1, \vec{\phi}_2)))$

First, $\vec{\phi}_2^* = \arg\max_{\vec{\phi}_2} E(u(W_2(\vec{\phi}_1, \vec{\phi}_2))) = \arg\max_{\vec{\phi}_2} e^{\alpha E(W_2) + \frac{1}{2}\alpha^2 V(W_2)}$

$$\vec{\phi}_2^* = \frac{1}{\alpha} \bar{\bar{\Sigma}}^{-1} \vec{1}(1+r_2^0) = \frac{1}{\alpha} \bar{\bar{\Sigma}}^{-1} \vec{1}(1+r_1^0) \quad (\text{since } r_1^0 = r_2^0)$$

It means $\vec{\phi}_2^*$ is independent with W_1, also, it's independent with W_0.

Then, $\vec{\phi}_1^* = \arg\max_{\vec{\phi}_1} E\left(u\left(\left(\underbrace{(W_0 - \vec{\phi}_1^T \cdot \vec{1})(1+r_1^0) + \vec{\phi}_1^T \cdot (\vec{1}+\vec{r}_1)}_{=W_1}\right) - \vec{\phi}_2^{*T} \cdot \vec{1}\right)(1+r_2^0) + \vec{\phi}_2^{*T}(\vec{1}+\vec{r}_2)\right)\right)$

$$E(-\alpha W(\vec{\phi}_1, \vec{\phi}_2^*)) = -\alpha\left(\left(\underbrace{(W_0 - \vec{\phi}_1^T \cdot \vec{1})(1+r_1^0) + \vec{\phi}_1^T \cdot (\vec{1}+\vec{\mu})}_{=W_1}\right) - \vec{\phi}_2^{*T} \cdot \vec{1}\right)(1+r_2^0) + \vec{\phi}_2^{*T}(\vec{1}+\vec{\mu})\right)$$

$$V(-\alpha W(\vec{\phi}_1, \vec{\phi}_2^*)) = \alpha^2\left((1+r_2^0)^2 \vec{\phi}_1^T \bar{\bar{\Sigma}} \vec{\phi}_1 + \vec{\phi}_2^* \bar{\bar{\Sigma}} \vec{\phi}_2^*\right)$$

So $\underset{\vec{\phi}_1}{\operatorname{argmax}} = e^{E(-\alpha w(\vec{\phi}_1, \vec{\phi}_2^*)) + \frac{1}{2} V(-\alpha w(\vec{\phi}_1, \vec{\phi}_2^*))}$

FOC, $\frac{\partial}{\partial \vec{\phi}_1}\left(E(-\alpha w(\vec{\phi}_1, \vec{\phi}_2^*)) + \frac{1}{2} V(-\alpha w(\vec{\phi}_1, \vec{\phi}_2^*))\right) = 0$

$$-\alpha\left(\left(-\vec{1}\cdot(1+r_1^0) + (\vec{1}+\vec{\mu})\right)(1+r_2^0) + \vec{\phi}_2^{*T}(\vec{1}+\vec{\mu})\right) + \alpha^2(1+r_2^0)^2 \bar{\bar{\Sigma}} \vec{\phi}_1^* = 0$$

$$\vec{\phi}_1^* = \frac{1}{\alpha(1+r_2^0)^2} \bar{\bar{\Sigma}}^{-1}\left(\left(-\vec{1}\cdot(1+r_1^0) + (\vec{1}+\vec{\mu})\right)(1+r_2^0) + \vec{\phi}_2^{*T}(\vec{1}+\vec{\mu})\right)$$

2023 Q3

(a) $X, Y \sim N(\vec{\mu}, \Sigma)$

$VaR_{X+Y}(\alpha) \leq VaR_X(\alpha) + VaR_Y(\alpha)$?

Proof: $\Phi(t)$ is cdf. of $t \sim N(0,1)$

$F_X(x) = \Phi(\frac{x-\mu}{\sigma})$

$F_X(x) = P(X \leq x) = P(\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}) = \Phi(\frac{x-\mu}{\sigma})$

$F_{-X}(x) = P(-X \leq x) = P(X \geq -x) = 1 - P(X < -x) = 1 - \Phi(\frac{-x-\mu}{\sigma}) = \Phi(\frac{\mu+x}{\sigma})$

$F_{-X}(x) = \Phi(\frac{\mu_1+x}{\sigma_1})$, now find $F_{-X}^{-1}(x)$

Let $y = \Phi(\frac{\mu_1+x}{\sigma_1})$, $\Phi^{-1}(y) = \frac{\mu_1+x}{\sigma_1}$, $x = \sigma_1 \cdot \Phi^{-1}(y) - \mu_1$

So $F_{-X}^{-1}(x) = -\mu_1 + \sigma_1 \cdot \Phi^{-1}(x)$

$VaR_X(\alpha) = q_{-X}(\alpha) = F_{-X}^{-1}(\alpha) = -\mu_1 + \sigma_1 \cdot \Phi^{-1}(\alpha)$ ①

$VaR_Y(\alpha) = q_{-Y}(\alpha) = F_{-Y}^{-1}(\alpha) = -\mu_2 + \sigma_2 \cdot \Phi^{-1}(\alpha)$ ②

$VaR_{X+Y}(\alpha) = q_{-X-Y}(\alpha) = F_{-X-Y}^{-1}(\alpha) = -\mu_1 - \mu_2 + \sqrt{2\rho\sigma_1\sigma_2}\,\Phi^{-1}(\alpha)$ ③

① + ② − ③ :

$VaR_X(\alpha) + VaR_Y(\alpha) - VaR_{X+Y}(\alpha) = (\sigma_1 + \sigma_2 - \sqrt{2\rho\sigma_1\sigma_2})\,\Phi^{-1}(\alpha) \geq 0$

(b)

	X	Y	$X+Y$	$\mathbb{P}$
w_1	-1	0	-1	$\frac{1}{4}$
w_2	0	-1	-1	$\frac{1}{4}$
w_3	1	1	2	$\frac{1}{4}$
w_4	2	2	4	$\frac{1}{4}$

$VaR^{\frac{3}{4}}(X) = VaR^{\frac{3}{4}}(Y) = 0$

$VaR^{\frac{3}{4}}(X+Y) = 1$ contradict!

$ES^{\frac{3}{4}}(X) = 4 \cdot \int_{\frac{3}{4}}^{1} VaR^{u}(X)\,du = 4 \cdot \int_{\frac{3}{4}}^{1} 1 \cdot du = 1 = ES^{\frac{3}{4}}(Y)$

$ES^{\frac{3}{4}}(X+Y) = 4\int_{\frac{3}{4}}^{1} VaR^{u}(X+Y)\,du = 4 \cdot \int_{\frac{3}{4}}^{1} 1 \cdot du = 1$

(c) ?

2023 Q4

(a) √ (notes P110)

(b) $\sigma_\pi = \sqrt{\pi^T \Sigma \pi}$

$$|S(\pi)| = \frac{|\mu_\pi - R^0|}{\sqrt{\pi^T \Sigma \pi}}$$

$\arg\max_\pi$ of $|S(\pi)| = \arg\min_\pi$ of $\frac{1}{2}\pi^T \Sigma \pi$

$$|S(\pi)| \leq |S(\hat{\pi})| = \sqrt{(\boldsymbol{\mu} - R^0\mathbf{1})^T \bar{\bar{\Sigma}}^{-1}(\boldsymbol{\mu} - R^0\mathbf{1})}.$$

This yields the first-order condition

$$0 = \nabla_{\boldsymbol{\pi}} \mathcal{L}(\hat{\boldsymbol{\pi}}, \delta) = \bar{\bar{\Sigma}}\hat{\boldsymbol{\pi}} - \delta(\boldsymbol{\mu} - R^0\mathbf{1}). \tag{6.19}$$

Exercise. Show that we obtain the frontier portfolios

$$\hat{\boldsymbol{\pi}} = \frac{(\mu_p - R^0)}{(\boldsymbol{\mu} - R^0\mathbf{1})^T \bar{\bar{\Sigma}}^{-1}(\boldsymbol{\mu} - R^0\mathbf{1})} \bar{\bar{\Sigma}}^{-1}(\boldsymbol{\mu} - R^0\mathbf{1}). \tag{6.20}$$

Moreover, check that we have the following.

$$\sigma_\pi = \sqrt{\hat{\boldsymbol{\pi}}^T \bar{\bar{\Sigma}}\hat{\boldsymbol{\pi}}} = \frac{|\mu_p - R^0|}{\sqrt{(\boldsymbol{\mu} - R^0\mathbf{1})^T \bar{\bar{\Sigma}}^{-1}(\boldsymbol{\mu} - R^0\mathbf{1})}}.$$

Definition 6.3. The **Sharpe ratio** is the ratio of risk premium to standard deviation:

$$\frac{\mu_p - R^0}{\sqrt{\hat{\boldsymbol{\pi}}^T \bar{\bar{\Sigma}}\hat{\boldsymbol{\pi}}}}$$

Remark 6.2. Sharpe ratios are not always suitable to compare different investment opportunities. For example, one opportunity has a (deterministic return) of 100%, implying a Sharpe ration of infinity. Another investment opportunity might have a return of 100% or 200%, each with probability 1/2. The second opportunity is clearly much better than the first one, but has a finite Sharpe ratio.

Any frontier portfolio has Sharpe ratio

$$\pm\sqrt{(\boldsymbol{\mu} - R^0\mathbf{1})^T \bar{\bar{\Sigma}}^{-1}(\boldsymbol{\mu} - R^0\mathbf{1})}.$$

(c)