Department of Computer Science
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Market risk, measures and portfolio theory

MSc Examination
2017

TIME ALLOWED: 2.5 HOURS

Full marks will be awarded for complete answers to FOUR questions. Only the best FOUR questions will count towards the total mark. Each question is worth 25 marks.

Calculators are **not** permitted. Clear and readable symbolic calculations are acceptable replacements for numerical results.

Problem 1.

(a) [8 points] Consider the function

$$u(x) = -\exp(-ax).$$

- Giving mathematical arguments, explain if an investor having u as utility function is risk-seeking or risk-averse.
- Find the coefficient of absolute risk aversion for this investor, and show it is constant.
- Show that another investor with utility function $u'(x) = mu(x) + b$ for $b, m \in \mathbb{R}$ and $m > 0$ has the same preferences.
- Calculate the expected utility $E[u(W)]$ if W is normal with mean μ and variance σ^2.

(b) [10 points] Let $a > 0$. Consider the function

$$v(x) = -\log\left(1 + \exp\left(-ax\right)\right)$$

- Show that an investor with utility function v is risk-averse and insatiable.
- Find the *certainty equivalent* x for this investor when its wealth is given by a random variable W, in terms of the expectation $\hat{v} = \mathbb{E}[v(W)]$. What is the financial interpretation of the certainty equivalent?
- Which wealth profile among W_1, W_2 defined below will an investor with utility function v choose?

$$W_1 = \begin{cases} 3/4 & \text{probability } 1/2 \\ 0 & \text{probability } 1/2 \end{cases}; \quad W_2 = \begin{cases} 1 & \text{probability } 1/4 \\ 1/2 & \text{probability } 1/4 \\ 0 & \text{probability } 1/2 \end{cases}.$$

(c) [7 points] Write a python code that approximates, using a Monte Carlo simulation with 10^6 samples, the expected value of $v(W)$ assuming that W follows a uniform distribution in the interval $[-10, 10]$, and $a = 1$. The following lines have already been included:

```
import numpy as np
from numpy.random import rand
from numpy import exp, log
```

– SOLUTION –

(a) An investor has a risk-averse utility function if she prefers to avoid fair bets. That is, for any random variable W such that $\mathbb{E}[u(W)] < \infty$, we have $u(\mathbb{E}[W]) > \mathbb{E}[u(W)]$.

Now, by Jensen's inequality, it suffices to verify that the function is concave on the whole domain, which follows from the fact that

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}u(x) = -a^2 \exp(-ax) \leq 0.$$

To calculate the absolute risk aversion coefficient, we recall and apply the definition to find

$$\alpha(x) = -\frac{\frac{\mathrm{d}^2}{\mathrm{d}x^2}u(x)}{\frac{\mathrm{d}}{\mathrm{d}x}u(x)} = \frac{a^2 \exp(-ax)}{a \exp(-ax)} = a$$

(the result still holds with different arguments for a=0), which is constant.

The equivalence between investors after a monotonous afine transformation can be proved in general. It suffices to note that if W_1, W_2 are two random variables, we have from the linearity of expectation and the properties of the order relation in the real numbers that

$$\mathbb{E}[u(W_1)] \leq E[u(W_2)] \Leftrightarrow \mathbb{E}[mu(W_1) + b] \leq \mathbb{E}[mu(W_2) + b].$$

Finally, by recalling that for a random variable $W \sim \mathcal{N}(\mu, \sigma^2)$ we have $\mathbb{E}[\exp(W)] = \exp(\mu + \frac{1}{2}\sigma^2)$, and recalling the rescaling properties of a Gaussian random variable, we deduce

$$\mathbb{E}[u(W)] = -\mathbb{E}[\exp(-aW)] = -\exp(-a\mu + \frac{1}{2}a^2\sigma^2).$$

(b) Since the function v is smooth, we can verify the characteristics of an investor associated to it by checking on its derivative and second derivative:

$$\frac{\mathrm{d}}{\mathrm{d}x}v(x) = \frac{a \exp(-ax)}{1 + \exp(-ax)} = \frac{a}{1 + \exp(ax)} > 0;$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}v(x) = \frac{-a^2 \exp(ax)}{(1 + \exp(ax))^2} < 0.$$

Hence the investor is insatiable (strictly prefers to consume more, i.e. monotonocity) and risk-averse (avoid fair bets).

The certainty equivalent is a sure quantity x such that it is indifferent for the investor to have x or the random wealth W. Hence,

$$v(x) = \mathbb{E}[v(W)] \Rightarrow x = v^{-1}(\hat{v}),$$

where the existence of the inverse of v is guaranteed by its monotonicity. It is easily seen that

$$x = -\frac{1}{a}\log[\exp(-\hat{v}) - 1].$$

Finally, in the choice problem between W_1 and W_2, the investor chooses W_1. The argument[1] uses only strict-concavity, which in this case is deduced from the second derivative calculation above. Indeed,

$$\begin{aligned}\mathbb{E}[v(W_1)] &= \frac{1}{2}v(0) + \frac{1}{2}v(\frac{3}{4}) = \frac{1}{2}v(0) + \frac{1}{2}v(\frac{1}{2}\{\frac{1}{2}\} + \frac{1}{2}\{1\}) \\ &> \frac{1}{2}v(0) + \frac{1}{4}v(\frac{1}{2}) + \frac{1}{4}v(1) = \mathbb{E}[v(W_2)].\end{aligned}$$

(c) A possible implementation is

```
import numpy as np
from numpy.random import rand
from numpy import exp,log

mc_it = 1e6
a = 1
sample = (rand(mc_it)-0.5)*20.
v = -log(1+exp(-a*sample))
v_mean = v.mean()
print(v_mean)
```

[1] Any alternative correct argument that leads to a strict preference is accepted.

Problem 2. In a one period framework, consider a risky asset with price $S_0 > 0$ at the beginning of the period. Assume its price at the end of the period "S_1" is log-normally distributed with

$$S_1 = S_0 \exp(\xi); \quad \text{for} \quad \xi \sim \mathcal{N}(m, \sigma^2).$$

Both m, σ^2 are assumed to be known. Let us measure the risk for a buyer of this asset in one period, that is $L = S_0 - S_1$.

(a) [10 points] Give the definition and financial interpretation of Value at Risk, and show that

$$\mathrm{V@R}^{\alpha}(L) = S_0 \left(1 - \exp\left[m + \sigma \Phi^{-1}(1-\alpha)\right]\right),$$

where Φ is the cumulative distribution function of a standard Gaussian.

(b) [8 points] We are now interested in measuring, in absence of any hedging, the risk of **selling** a European call option on the asset described in (a). Consider a call with maturity one-period and strike K. Set C to be the (known) price of this call. Let L' be the random variable representing losses for the **seller** of this call option in absence of any hedging, that is $L' = (S_1 - K)^+ - C$.

Show that for $\alpha \in (0,1)$, $\mathrm{V@R}^{\alpha}(\mathrm{L}') = \max\{S_0 - K - C - \mathrm{V@R}^{1-\alpha}(L), -C\}$

(c) [7 points] Assume that the model in (a) describes well the behaviour of a given asset in several periods, so that $S_{n+1} = S_n \exp(\xi_{n+1})$, where all the $\xi_1, \xi_2 \ldots$, are i.i.d. with $\mathcal{N}(m, \sigma^2)$. Set $L_i = S_{i-1} - S_i$

- Show that the random variables $I_i = \mathbb{1}_{\{L_i \geq \mathrm{V@R}^{\alpha}(L_i)\}}$, for $i = 1, \ldots$ are i.i.d. and give their distribution. Recall that $\mathbb{1}_A$ denotes the indicator function of the set A.
- In a set of 400 periods, the observed losses are seen to exceed the estimated value at risk at 99% confidence on 10 occasions. Perform a backtest to establish the covering property of a reserve made with this value at risk calculation. Fix the confidence of the test at 97.5%. You may use that $\Phi^{-1}(0.975) \approx 1.96$.

– Solution –

(a) Value at risk at level α is a risk measure that quantifies the minimal value that would cover losses with a probability α. Because L represents losses, mathematically, this can be written as

$$\mathrm{V@R}^{\alpha}(L) = \inf\{x : P[L \leq x] \geq \alpha\}.$$

For continuous random variables as in this case, value at risk is simply the quantile at a given level. To deduce an expression for value at risk, we will express those quantiles in terms of the quantiles of a standard normal variable.
Indeed, using properties of inequalities and the monotonicity of the log function, we have that for any $x \in (-\infty, S_0)$,

$$\begin{aligned}
\mathbb{P}[S_0 - S_1 \leq x] =& \mathbb{P}[S_0\{1-\exp(\xi)\} \leq x] = \mathbb{P}\left[\log(1-\frac{x}{s_0}) \leq \xi\right] \\
&= 1 - \mathbb{P}\left[\xi < \log(1-\frac{x}{s_0})\right] \\
&= 1 - \mathbb{P}\left[\sigma X + m < \log(1-\frac{x}{s_0})\right]; \qquad \text{for } X \sim \mathcal{N}(0,1) \\
&= 1 - \Phi(\theta); \qquad \text{with } \theta = \frac{1}{\sigma}[\log(1-\frac{x}{s_0}) - m].
\end{aligned}$$

The result then follows by choosing $\theta = \Phi^{-1}(1-\alpha)$ and solving for x.

(b) We follow a similar approach as in the previous question: we try to express the probability of losses being bounded above in terms of the already known quantiles for the underlying. Now, for any $x \in \mathbb{R}$, we have

$$\begin{aligned}
\mathbb{P}[(S_1-K)^+ - C \leq x] &= \mathbb{P}[\{S_1 - K - C \leq x\} \cap \{S_1 \geq K\}] + \mathbb{P}[S_1 < K]\mathbb{1}_{\{-c \leq x\}} \\
&= \mathbb{1}_{\{-c\leq x\}} P[S_1 \leq x + C + K] \\
&= \mathbb{1}_{\{-c\leq x\}} (1 - P[L \leq S_0 - x - C - K]).
\end{aligned}$$

We must now be careful since there is an atom in this distribution (which corresponds to all the cases where the option is not called). We apply directly the definition of value at risk to conclude that

$$\begin{aligned}
\text{V@R}^\alpha(L') &= \inf\{x : P[L' \leq x] \geq \alpha\} \\
&= \inf\{x : \mathbb{1}_{\{-c\leq x\}}(1 - P[L \leq S_0 - x - C - K]) \geq \alpha\} \\
&= \inf\{x : x \geq -c, P[L \leq S_0 - x - C - K]) \leq 1 - \alpha\} \\
&= \max\{-C, S_0 - C - K - \text{V@R}^{1-\alpha}(L)\}
\end{aligned}$$

(c) The key point is that we can study the i.i.d. property of the defined random variables, by using the corresponding property for the sequence of $\xi_1, \xi_2, \ldots$. Indeed, we use the expression found in (a) to deduce that

$$\{L_i \geq \text{V@R}^\alpha(L_i)\} \Leftrightarrow \{\xi_i \leq m + \sigma\Phi^{-1}(1-\alpha)\}$$

Hence, as expected, the i.i.d. property of the set $(I_i)_{i\in\mathcal{N}_+}$ follows from the same property for $(\xi_i)_{i\in\mathcal{N}_+}$. Moreover by the definition of value at risk, we find that I_i follows a Bernoulli distribution with parameter $p = 1 - \alpha$.

We can now perform a hypothesis testing. Taking as our null assumption that the reserves are sufficient, i.e. that VaR of the asset is less than the one given by the formula, we can define as estimator

$$Z = \frac{\sqrt{N}(\frac{1}{N}\sum_{i=1}^{N} I_i - (1-\alpha))}{\sqrt{\alpha(1-\alpha)}} = \frac{200(\frac{10}{400} - 0.01)}{\sqrt{0.01(0.99)}} \approx \frac{20(0.025 - 0.01)}{0.1} = 3 > 1.96.$$

Therefore, using the i.i.d. property for $(I_i)_{i\in\mathcal{N}_+}$ and invoking the central limit theorem, we conclude that the probability that the number of VaR violations was due to randomness is less than 2.5%. We reject the assumption that the reserves are sufficient.

Problem 3. Assume an agent has taken a position in d different investing possibilities whose losses are represented by $(L_1, \ldots, L_d)$. Her total loss profile is given by

$$L = \sum_{i=1}^{d} L_i.$$

Let Σ denote the covariance matrix of the vector $(L_1, \ldots, L_d)^\top$, and $-\bar{\mu}$ its mean vector (i.e. $\bar{\mu}$ is the mean of profits). Assume all the components of $\bar{\mu}$ are positive. Set $L(\bar{\lambda}) := \sum_{i=1}^{d} \lambda_i L_i$.

(a) [9 points] We consider first the standard deviation risk measure $\rho_{sd} : X \to \sqrt{\mathrm{var}(X)}$. Explain the financial meaning of a capital allocation and compute the optimal capital allocation for L, according to the Euler principle if capital is calculated with ρ_{sd}.

(b) [9 points] Deduce, using the previous capital allocation, an expression in terms of the known quantities for the RORAC of L and each one of the individual positions L_i.

What condition should satisfy the ratio $\frac{\mu_i}{\sum_{j=1}^{d} \mu_j}$ to conclude that the asset i is not underperforming according to RORAC ?

(c) [7 points] We will now consider another risk measure. Let ρ be a convex risk measure. Define

$$\nu(L) := \inf_{z>0} \left\{ \frac{1}{z} [\rho(zL)] \right\}.$$

Show that ν is also a convex risk measure.

Hint. Remember that if $g_1(z) \le g_2(z)$ for all $z \in A$, then $\inf_{z \in A} g_1(z) \le \inf_{z \in A} g_2(z)$

– SOLUTION –

(a) Suppose that an investor has an aggregated position, that is, a position resulting from the sum of individual investments. Then, a capital allocation is a rule to divide the capital allocated for the total position among the subcomponents.
If a risk measure is positive homogeneous, we can use the Euler capital allocation: that is, we assign to each position the derivative with respect to λ of the aggregated capital of $L(\bar{\lambda})$. In the case of standard deviation we have:

$$\pi_i^{Euler}(\bar{\lambda}) = \frac{\partial}{\partial \lambda_i}\sqrt{\mathrm{var}(L(\bar{\lambda}))} = \frac{1}{2}\frac{2\sum_{j=1}^{d}(\Sigma_{i,j}\lambda_j)}{\sqrt{\mathrm{var}(L(\lambda))}} = \frac{\mathrm{cov}(L(\bar{\lambda}), L_i)}{\sqrt{\mathrm{var}(L(\bar{\lambda}))}}.$$

We obtain the capital allocation in our case by simply fixing $\bar{\lambda} = \bar{1}$ (vector of ones in $\mathbb{R}^d$).

(b) The RORAC is the ratio between the expected income over the required capital. Let us denote the RORAC by $\mathcal{R}$.
We have that

$$\mathcal{R}(L) = \frac{\mathbb{E}[-L]}{\sqrt{\mathrm{var}[L]}} = \frac{\sum_{j=1}^{d}\mu_j}{\sqrt{\mathrm{var}[L]}}$$

while

$$\mathcal{R}_i^{sd}(L) = \frac{\mu_i\sqrt{\mathrm{var}[L]}}{\mathrm{cov}(L, L_i)}$$

An asset is under-performing according to RORAC if $\mathcal{R}_i^{sd}(L) < \mathcal{R}(L)$. Hence, we require,

$$\frac{\sum_{j=1}^{d}\mu_j}{\sqrt{\mathrm{var}[L]}} \leq \frac{\mu_i\sqrt{\mathrm{var}[L]}}{\mathrm{cov}(L, L_i)}$$

i.e.

$$\frac{\mu_i}{\sum_{j=1}^{d}\mu_j} \geq \frac{\mathrm{cov}(L, L_i)}{\mathrm{var}[L]} = \frac{\sum_{j=1}^{d}(\Sigma_{i,j})}{\sum_{j=1}^{d}\sum_{k=1}^{d}(\Sigma_{j,k})},$$

which is the correlation coefficient. Informally, this tells us that the ratio between the mean profit of an individual position and the aggregate must be at least as big as the contribution of the position to the overall variance.

(c) To prove that ν_β is a convex risk measure, we verify each one of the convex measure assumptions. In the following X, Y are bounded random variables.

- Monotonicity: Assume $X \geq Y$ almost surely. Then, by the monotonicity property of ρ, we have (a.s.) that for each $z > 0$,

$$zX \geq zY \Rightarrow \rho(zX) \geq \rho(zY) \Rightarrow \frac{1}{z}[\rho(zX)] \geq \frac{1}{z}[\rho(zY)].$$

  We conclude by taking the infimum over z on both sides of the inequality.

- Cash (translation) invariance: By the cash invariance property of ρ,

$$\nu(X + a) = \inf_{z>0}\left\{\frac{1}{z}[\rho(z(X + a))]\right\} = \inf_{z>0}\left\{\frac{1}{z}[\rho(zX) + az]\right\} = \nu(X) + a.$$

- Convexity: Let $\lambda \in (0,1)$, then for every $z > 0$,

$$\frac{1}{z}\rho(z(\lambda X + (1-\lambda)Y)) \leq \lambda \left(\frac{1}{z}\rho(zX)\right) + (1-\lambda)\left(\frac{1}{z}\rho(zY)\right).$$

  By taking the infimum over z on both sides of the inequality,

$$\begin{aligned}\nu(\lambda X + (1-\lambda)Y) &\leq \inf_{z>0}\left\{\lambda\left(\frac{1}{z}\rho(zX)\right) + (1-\lambda)\left(\frac{1}{z}\rho(zY)\right)\right\} \\ &\leq \lambda\nu(X) + (1-\lambda)\nu(Y).\end{aligned}$$

Problem 4. Let us consider a one-period market model. Suppose it is composed by a risk-free asset with gross return $R_f > 0$ and one risky asset with gross return R given by

$$R = \begin{cases} u & \text{with probability } \alpha \\ d & \text{with probability} (1-\alpha) \end{cases}$$

for some $0 < \alpha < 1$ and $d < u$.

(a) [10 points] What is an arbitrage opportunity? Show that if $R_f \leq d$ or $R_f \geq u$, there is an arbitrage opportunity. Explain what is a Stochastic Discount Factor (SDF). Find a SDF assuming that $d < R_f < u$.

(b) [10 points] Assume that an investor has CRRA utility function

$$v(x) = (1-\rho)^{-1} x^{1-\rho}$$

with $\rho > 1$. Solve the portfolio choice problem, in terms of the amount invested in the risky asset (ϕ), for an investor with initial wealth w_0, no initial consumption and no endowments.

(c) [5 points] Now, we also allow consumption at the beginning of the period. Thus, the (representative) investor has a concave utility function $U(c_0, c_1)$ with two variables. Suppose U has positive and bounded first order derivatives. Assuming optimal investment, derive a formula for the stochastic discount factor in terms of the utility function U.

– Solution –

(a) An arbitrage opportunity in one period is a strategy with non-positive initial cost such that almost surely it does not produce any loss at the end of the period a positive profit with non-trivial probability.
In the present framework it can be represented as a vector $\theta \in \mathbb{R}^2$ such that

- $\theta_1 + \theta_2 \leq 0$,
- $\mathbb{P}[\theta_1 R + \theta_2 R_f \geq 0] = 1$
- $\mathbb{P}[\theta_1 R + \theta_2 R_f > 0] > 0$.

Note that since $d < u$, either $R_f > u$ or $R_f < d$. If $R_f < d$, take any $x > 0$. Fix $\theta_1 = -\theta_2 = x$. Then, $\theta_1 + \theta_2 = x - x = 0$, and since $\theta_1 R + \theta_2 R_f = xR - xR_f > xd - xR_f > 0$ we have a strategy that creates profit almost surely, and is therefore an arbitrage. If $R_f > u$, the same argument works with $\theta_1 = -\theta_2 = -x$.
When $d \geq R_f \geq u$, we can find a stochastic discount factor. It is a positive random variable M such that[2]

$$E[M\hat{R}] = 1$$

where $\hat{R}$ is any return of a traded asset. In this framework, there are only two traded assets and hence we get

$$R_f E[M] = 1 = E[MR] = uE[M\mathbb{1}_{R=u}] + dE[M\mathbb{1}_{R=d}]$$

If we assume that M is constant on each one of the events $\mathbb{1}_{R=u}$, and $\mathbb{1}_{R=d}$ (that we denote in the following by M_u and M_d), we get

$$\alpha R_f M_u + (1-\alpha) R_f M_d = 1; \qquad \alpha u M_u + (1-\alpha) d M_d = 1$$

i.e. a system with two equations and two unknowns. By solving it we get

$$M_u = \frac{R_f - d}{\alpha R_f (u-d)}; \quad M_d = \frac{u - R_f}{(1-\alpha) R_f (u-d)}$$

(b) The portfolio choice problem can be stated as follows:

$$\max_{\phi \in \mathbb{R}} \{\mathbb{E}[v(W)] : W = \phi R + (w_0 - \phi) R_f)\}$$

where in this representation, ϕ is the wealth invested on the risky asset. It is an optimisation problem in dimension one with constraints. We can plug the constraints to get

$$\mathbb{E}[v(W)] = \frac{1}{(1-\rho)} \{\alpha[\phi(u - R_f) + w_0 R_f]^{1-\rho} + (1-\alpha)[\phi(d - R_f) + w_0 R_f]^{1-\rho}\}$$

As we showed in the lecture, the problem is concave and the restriction is linear. There is a unique optimal solution that we can find by considering the first order condition $\frac{\mathrm{d}}{\mathrm{d}\phi}\mathbb{E}[u(W_*)] = 0$, which implies

$$0 = \alpha(u - R_f)[\phi_*(u - R_f) + w_0 R_f]^{-\rho} + (1-\alpha)(d - R_f)[\phi_*(d - R_f) + w_0 R_f]^{-\rho}$$

Rearranging we obtain

$$\alpha(u - R_f)[\phi_*(u - R_f) + w_0 R_f]^{-\rho} = (1-\alpha)(R_f - d)[\phi_*(d - R_f) + w_0 R_f]^{-\rho}$$

[2] It can be alternatively defined in terms of prices and payoffs.

and then using the fact that $d < R_f < u$

$$\left(\frac{\phi_*(u - R_f) + w_0 R_f}{\phi_*(d - R_f) + w_0 R_f}\right)^{\rho} = \frac{\alpha(u - R_f)}{(1-\alpha)(R_f - d)}.$$

Let us define

$$K := \left(\frac{\alpha(u - R_f)}{(1-\alpha)(R_f - d)}\right)^{\frac{1}{\rho}}$$

Then, we get

$$\phi_*(u - R_f) + w_0 R_f = K(\phi_*(d - R_f) + w_0 R_f)$$

and by rearranging,

$$\phi_* = \frac{w_0 R_f (K - 1)}{u - R_f + K(R_f - d)}.$$

(c) Since we allow for consumption, the problem can be written now:

$$\max_{\phi, \phi_f \in \mathbb{R}} \{\mathbb{E}[U(C_0, C_1)] : w_0 = C_0 + \phi + \phi_f, C_1 = \phi R + \phi_f R_f)\}$$

As the problem is concave, we can use the first order conditions to identify the maximum, that is

$$\partial_\phi \mathbb{E}[U(C_0^*, C_1^*)] = 0; \qquad (1)$$
$$\partial_{\phi_f} \mathbb{E}[U(C_0^*, C_1^*)] = 0. \qquad (2)$$

Enforcing the constraints and using the fact that the derivatives of U exist and are continuous and bounded (which allow us to exchange the integration order), we get from (1) and (2) respectively

$$\mathbb{E}[-\partial_{C_0} U(w_0 - \phi - \phi_f, \phi R + \phi_f R_f) + \partial_{C_1} U(w_0 - \phi - \phi_f, \phi R + \phi_f R_f)R] = 0$$
$$\mathbb{E}[-\partial_{C_0} U(w_0 - \phi - \phi_f, \phi R + \phi_f R_f) + \partial_{C_1} U(w_0 - \phi - \phi_f, \phi R + \phi_f R_f)R_f] = 0,$$

from where we deduce that

$$\mathbb{E}\left[\frac{\partial_{C_1} U(C_0, C_1)}{\mathbb{E}[\partial_{C_0} U(C_0, C_1)]} R\right] = 1; \qquad \mathbb{E}\left[\frac{\partial_{C_1} U(C_0, C_1)}{\mathbb{E}[\partial_{C_0} U(C_0, C_1)]} R_f\right] = 1$$

Which in this market with only two asset gives us an expression for the stochastic discount factor:

$$M = \frac{\partial_{C_1} U(C_0, C_1)}{\mathbb{E}[\partial_{C_0} U(C_0, C_1)]}.$$

Problem 5. Fix a probability space $(\Omega, \mathbb{P}, \mathcal{F}, \{\mathcal{F}_t\}_{t=1,\dots,T})$. In this question we consider the investment consumption problem in a discrete setting with finite horizon T. We focus on the dynamic programming principle applied to the case of an investor with additive discounted utility

$$U(\bar{C}) = \sum_{i=0}^{T} \delta^i v(C_i)$$

where v is a concave function. Consider a market consisting of one risky asset and a money market account. Assume that the reinvested money market account satisfies $B_{t+1} = B_t R_f$ and $B_1 = 1$ for a deterministic value $R_f > 0$.
Assume also that the gross returns $(R_i)_{i=1,\dots T}$ of the risky asset are independent and identically distributed.

(a) [7 points] State the investment consumption problem for an investor with initial wealth w_0 and no endowments, in terms of the consumption at each time and the investment strategy.

(b) [8 points] Give the definition of conditional expectation at time $t, (\mathbb{E}_t[.])$. Show that for all X, Y $\mathcal{F}$-measurable random variables with finite expectation,

- $\mathbb{E}_t[aX + bY] = a\mathbb{E}_t[X] + b\mathbb{E}_t[Y]$, for all $a, b \in \mathbb{R}, 0 \leq t \leq T$.
- $\mathbb{E}_s[\mathbb{E}_t[X]] = \mathbb{E}_s[X]$, for all $0 \leq s \leq t \leq T$.

(c) [10 points] Define

$$J(t, w) = \sup_{\substack{C_i \in \mathcal{A}_i, \pi_i \in \mathbb{R} \\ i=t,\dots,T}} \left\{ \mathbb{E}_t \left[\sum_{i=t}^{T} \delta^{i-t} v(C_i) \right] : \text{ budget constraints from } t \text{ to } T \right\},$$

where $\mathcal{A}_i$ are the $\mathcal{F}_i$ measurable random variables with finite expectation and the budget constraints are

$$W_t = w; \qquad W_T = C_T$$
$$W_{i+1} = (W_i - C_i)(\pi(R_{i+1} - R_f) + R_f) \text{ for } i = t+1, \dots, T-1.$$

Explain the financial interpretation of this function. Then, use it to sketch how we can apply the dynamic programming principle approach to solve the optimal investment consumption problem.

– SOLUTION –

(a) In the investment consumption problem, the agent looks to maximise the expected utility she gets from her consumption at all periods. Her decision variables are the consumption at each period and her investment strategy. Hence, the problem can be written as

$$\sup_{\substack{C_i \in \mathcal{A}_i, \pi_i \in \mathbb{R} \\ i=0,\dots,T}} \mathbb{E}\left(\sum_{i=0}^{T} \delta^i v(C_i)\right)$$

subject to the budget constraints

$$W_{i+1} = (W_i - C_i)(\pi(R_{i+1} - R_f) + R_f) \text{ for } i = 0, \dots, T-1; \quad W_T = C_T; \quad W_0 = w_0.$$

In this representation, π_i is the proportion of the wealth invested on the risky asset at time i. Alternative equivalent representations can be obtained in terms of invested amounts or the number of shares.

(b) The conditional expectation at time t in this setting can be defined for each $F-$ measurable random variable with finite variance X by

$$\mathbb{E}_t[X] := \arg\inf\{E[(X - Z)^2] : Z \text{ is } F_t \text{ measurable and } E[Z^2] < \infty\},$$

and then extended by approximation to every integrable random variable.
Alternatively, $\mathbb{E}_t[X]$ for an integrable $\mathcal{F}$ measurable random variable X can be characterised as the (a.s.)-unique integrable F_t random variable Z such that $\mathbb{E}[XY] = \mathbb{E}[ZY]$ for all F_t measurable Y such that $\mathbb{E}[|XY|] < \infty$.
The second definition is probably more convenient to show the two properties. Indeed,

- Define $Z = a\mathbb{E}_t[X] + b\mathbb{E}_t[Y]$, and note that it is $\mathcal{F}_t-$ measurable. Moreover, we have that for any ξ F_t-measurable, we deduce from the linearity of expectation and the definition of conditional expectation that:

$$\begin{aligned}\mathbb{E}[(Z\xi)] &= \mathbb{E}[(a\mathbb{E}_t[X] + b\mathbb{E}_t[Y])\xi] = a\mathbb{E}[\mathbb{E}_t[X]\xi] + b\mathbb{E}[\mathbb{E}_t[Y]\xi] \\ &= a\mathbb{E}[X\xi] + b\mathbb{E}[Y\xi] = \mathbb{E}[(aX + bY)\xi].\end{aligned}$$

  and we conclude that $Z = \mathbb{E}_t[aX + bY]$.

- For the second claim, take $Z = \mathbb{E}_s[\mathbb{E}_t[X]$. By definition of conditional expectation, it is $\mathcal{F}_s-$ measurable. Let ξ be F_s-measurable. Using the fact that ξ is also F_t- measurable (as $\mathcal{F}_s \subset \mathcal{F}_t$) and twice the property of conditional expectation, we get

$$\mathbb{E}[(Z\xi)] = \mathbb{E}[\mathbb{E}_t[\mathbb{E}_s[X]]\xi] = \mathbb{E}[\mathbb{E}_t[X]\xi] = \mathbb{E}[X\xi].$$

  Hence, $Z = \mathbb{E}_s[X]$

(c) The question asks for a sketch: well reasoned arguments that points on the same line of what follows are accepted: The function $J(t, w)$ can be interpreted as the solution of the investment consumption problem when considered at time t, for the period $[t, T]$, given that at time t the available wealth is w.
Let us use this function to discuss the dynamic principle approach: the main goal is to be able to turn a complex optimisation problem into a sequence of simpler optimisation problems.
Note that from the definition of $J(t, w)$, we can see that $J(0, w_0)$ coincides with the investment consumption problem. So, it suffices to find this value to solve the problem. Moreover, we can rewrite the definition of $J(t, w)$ to get a backward recursive definition. The idea is to rewrite the condition as a contribution of the utility at time t and the optimal utility from there on, using the shown properties for conditional expectations as follows:

$$J(t, w) = \sup_{\substack{C_i \in \mathcal{A}_i, \pi_i \in \mathbb{R} \\ i=t,\ldots,T}} \left\{ \mathbb{E}_t[v(C_t)] + \delta \mathbb{E}_t \left[\sum_{i=t+1}^{T} \delta^{i-t-1} v(C_i) \right] : \text{ b.c. from } t \text{ to } T \right\}$$

$$= \sup_{\substack{C_i \in \mathcal{A}_i, \pi_i \in \mathbb{R} \\ i=t,\ldots,T}} \left\{ \mathbb{E}_t[v(C_t)] + \delta \mathbb{E}_t \left(\mathbb{E}_{t+1} \left[\sum_{i=t+1}^{T} \delta^{i-t-1} v(C_i) \right] \right) : \text{ b.c. from } t \text{ to } T \right\}$$

where we used the conditional expectation properties proved in (b). We can now identify the term

$$\mathbb{E}_{t+1} \left[\sum_{i=t+1}^{T} \delta^{i-t-1} v(C_i) \right]$$

as an utility function considered at time $t + 1$, with the initial budget $\{w - C_t\}\{\pi(R - R_f) + R_f\}$ (due to the budget restriction), that must satisfy the budget restrictions from $t + 1$ to T. Hence, our interpretation of $J(t + 1, .)$, allows us to conclude that

$$J(t, w) = \sup_{C_t \in \mathcal{A}_t, \pi_t \in \mathbb{R}} \{ v(C_t) + \delta \mathbb{E}_t[J(t + 1, \{w - C_t\}\{\pi(R - R_f) + R_f\})].\} .$$

This is a one-period optimisation problem. Hence, to solve the consumption investment problem, we can solve a succession of simpler one period optimisation problems.