Utility functions

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Market risk and portfolio theory

A chance game

Consider the following chance game:

- The initial prize is fixed to £1
- A fair coin is thrown
- If the coin falls in HEADS, the player receives the prize
- If the coin falls in TAILS, the prize is duplicated and the game is played again

For example, an instance of the game can be given by: TAILS, TAILS, HEADS. In this case the player would receive as prize 4 at the end of the game.

How much are you willing to pay to play the game?

St. Petersburg Paradox

Let us calculate the *expected* prize. Let X be the number of throws, and P the prize received:

- The game consists on exactly i times if and only if $i-1$ TAILS are followed by a HEADS
- Using independence, this means $\mathbb{P}[X=i] = (\frac{1}{2}) \prod_{k=1}^{i-1} \frac{1}{2} = \frac{1}{2^i}$
- On the other hand, the prize is a function of $X=i$, namely $P = 2^{X-1}$.

Hence, the expected prize is

$$\mathbb{E}[P] = \mathbb{E}[2^{X-1}] \sum_{i=1}^{\infty} 2^{i-1} \mathbb{P}[X=i] = \sum_{i=1}^{\infty} \frac{1}{2} = \infty.$$

On the other hand, most people are willing to pay only a few pounds to play this game. This is the so-called St. Petersburg Paradox.

Utility functions

Key idea: investors ('econs') are sensitive to their expected wellbeing rather than pure return.

Definition

A (one-dimensional) utility function is a continuous and increasing mapping $u : E \to \mathbb{R}$, with E an interval in $\mathbb{R}$.

- Introduced by D. Bernoulli to solve "St. Petersburg Paradox"

Utility functions

If an investor has a choice between W_1 and W_2:

The investor chooses W_1 over W_2 if and only if $\mathbb{E}[u(W_1)] > \mathbb{E}[u(W_2)]$

If both have the same expected utility they are said to be equivalent for the investor.

Examples of utility function

$u(x) := -\exp(-x)$ (exponential, defined over $\mathbb{R}$) , $u(x) := \log(x)$ (logarithmic, defined over $\mathbb{R}_+$).

Risk attitude

Definition (Risk aversion)

"*weakly*" risk averse $u(\mathbb{E}(W)) \geqslant \mathbb{E}[u(W)]$ for all W

"*weakly*" risk seeking $u(\mathbb{E}(W)) \leqslant \mathbb{E}[u(W)]$ for all W

From Jensen's inequality, if u is a *concave* function, it is risk averse.

Certainty equivalent

Definition

Let W be a wealth gamble. x_W is a *certainty equivalent* of W if

$$x_W := u^{-1}(\mathbb{E}[u(W)])$$

Thus an investor is risk-averse if and only if $x_W \leqslant \mathbb{E}[W]$ for all W.

If a utility u is strictly increasing, it is invertible and is unique.

Insurance premium

The difference between the certainty equivalent and the expected wealth can be understood as a premium to avoid uncertainty.

Definition

Let W be a wealth gamble. The *insurance premium*[1] of W is given by

$$\eta(W) = \mathbb{E}[W] - x_W$$

The insurance premium can be understood as a measure of risk aversion.

[1] Also known as risk premium in books like Back 2010

(Arrow-Pratt) coefficients of risk aversion

Coefficient of absolute risk aversion

Definition (Coefficient of "*absolute*" risk aversion)

At wealth level w is given by

$$\alpha(w) = -\frac{u''(w)}{u'(w)}.$$

Relative risk aversion

A similar development but in relative terms ($W = w + w\hat{\xi}$ and $\eta = w\hat{\eta}$) motivates the following definition.

Definition (Coefficient of "*relative*" risk aversion)

At wealth level w is given by

$$\rho(w) = w\alpha(w) = -w\frac{u''(w)}{u'(w)}.$$

Monotone affine transforms

Two investors, one with utility function $u(w)$ and a second one with utility function $v(w) := au(w) + b$, with $a > 0$, will

- Make the same choices;
- share the same certainty equivalent;
- pay the same insurance premium; and
- have the same coefficients of risk aversion (when defined).

From a modelling point of view they cannot be distinguished.

Constant absolute risk aversion (CARA)

If absolute risk aversion is the same at **every** level of wealth, that is, $\alpha(w) \equiv \alpha \in \mathbb{R}$, then an investor has CARA utility.

They are characterised by

$$u(w) = -e^{-\alpha w} \text{ (if } \alpha \neq 0\text{); and} \quad u(w) = w\text{(if } \alpha = 0\text{).} \tag{1}$$

In fact, every CARA utility is a monotone affine transform of (1).

Constant relative risk aversion (CRRA)

If relative risk aversion is the same at **every** level of wealth, that is, $\rho(w) \equiv \rho \in \mathbb{R}$, then an investor has CRRA utility.

(i) $u(x) = x$ if $\rho = 0$;

(ii) $u(x) = \log(x)$ if $\rho = 1$;

(iii) $u(x) = \frac{1}{1-\rho} x^{1-\rho}$ otherwise.

❦ $E = \mathbb{R}_+$, i.e. only well-defined for positive values.

More examples of utility functions

Some final examples of utility functions:

Linear $u(x) := ax + b$ with $a > 0$, with $E = \mathbb{R}$

Quadratic $u_Q(x) = x - bx^2/2$, with $b > 0$, and $E = (-\infty, 1/b)$

HARA For $\gamma \neq 0$, $\frac{x}{1-\gamma} - \hat{x} > 0$,

$$u(x) = \frac{1-\gamma}{\gamma}\left(\frac{x}{1-\gamma} - \hat{x}\right)^{\gamma}.$$

Utility functions in multi-period

Definition

A (multi-dimensional) utility function is a component-wise increasing and continuous mapping $u : E \to \mathbb{R}$, with $E \subset \mathbb{R}^{T+1}$ a convex set.

We say the utility function is *time-discounting* if there exists $\epsilon > 0$ such that for all $r \in (0, \epsilon)$

$$u(\mathbf{x} + r\mathbf{e}_t) \leqslant u(\mathbf{x} + r\mathbf{e}_\ell) \quad \forall 0 \leqslant \ell \leqslant t \leqslant T.$$

Time-additive utility

A time-additive utility is based on a one-period utility $\hat{u}$

$$u(x) = \sum_{t=0}^{T} \delta^t \hat{u}(x_t),$$

where $0 \leqslant \delta \leqslant 1$ is a discount factor that models preference for immediate consumption.

Some final remarks

Utility maximisation theory is very convenient, but it does not work in many situations.

Main alternative theory is based on *behavioural finance*: based on the works of D. Kahneman and A. Tversky (see for example Tversky and Kahneman 1986 and Tversky and Kahneman 1992), but also the subject of study of Richard Thaler and Robert Shiller.

However, these developments are just modifications of the framework: usually the model only changes the type of function being maximised and the reference on which it acts.