Q 1

(a) Since the market is arbitrage-free, there is a strictly positive SDF M,

Take θ, ξ s.t. $\theta \cdot S_{t_0} = \xi \cdot S_{t_0}$, so $E(M \cdot S_{t_0}) > 0$

$S_0^\theta = \theta \cdot S_0 = \theta \cdot E(M \cdot S_{t_0}) = E(M \cdot (\theta \cdot S_{t_0})) = E(M(\xi \cdot S_{t_0})) = \xi \cdot E(M S_{t_0}) = \xi \cdot S_0 = S_0^\xi$

i.e. $S_0^\theta = S_0^\xi$

(b) $\forall i$, let $p_{new}^{i,k}$ be the arbitrage-free price for a new asset that can have unique European option payoff H at time τ.

By proposition, $p_{new}^{i,k} = E(M_\tau \cdot H)$ for some strictly positive SDF.

Similarly, by definition of $p_c^{i,k}, p_p^{i,k}$, $p_c^{i,k} = E(M_\tau (S_\tau^i - K)^+)$, $p_p^{i,k} = E(M_\tau (K - S_\tau^i)^+)$

So $p_c^{i,k} - p_p^{i,k} = E(M_\tau((S_\tau^i - K)^+ - (K - S_\tau^i)^+))$

$= E(M_\tau(S_\tau^i - K))$

$= E(M_\tau \cdot S_\tau^i) - M_0 \cdot K$

$= S_0^i - K$

(c) Since θ is a self-funding replication of call option on asset i, strike K, maturity τ,

So $S_\tau^\theta = (S_\tau^i - K)^+$, $S_t^\theta = E_t(M_\tau (S_\tau^i - K)^+)$, $t = 0, 1, \cdots, \tau$

we want to find ξ to replicate put option, which means ξ should satisfy

$S_\tau^\xi = (K - S_\tau^i)^+$, $S_t^\xi = E_t(M_\tau (K - S_\tau^i)^+)$, $t = 0, 1, \cdots, \tau$

so $S_\tau^\theta - S_\tau^\xi = S_0^i - K$, $S_t^\theta - S_t^\xi = E_t(M_\tau (S_\tau^i - K))$, $t = 0, 1, \cdots, \tau$

Let $t = 0$, $S_0^\theta - S_0^\xi = E_0(M_\tau(S_\tau^i - K)) = S_0^i - K$

So $S_0^\xi = S_0^\theta - (S_0^i - K)$

So $\xi = \theta - (S_0^i - K, 0, 0, \cdots, 0)$ satisfy above condition

This means ξ has $(S_0^i - K)$ unit of S^0 less than θ, and same on the other assets at $t = 0$

Q2

(a) Existance : arbitrage-free (Because if have infinite utility in limit fund, one can arbitrage).

$\Leftrightarrow$ $R \cdot p_1^{AD} = S_0$ has strict positive solution, where p_1^{AD} is Arrow-Debreu security.

Uniqueness : no redundancies

$\Leftrightarrow$ $\forall i$, return R_1^i cannot be replicated by remaining R_1^i

$\Leftrightarrow$ R has full column rank m, $\text{rank}(R) = m$. (Obviously $n+1 \geq m$)

(b) $\rho = 1$, $u(x) = \log(x)$, $f(c_0, \vec{\pi}) = E(u(x) + \delta u(c_1)) = E(\log(c_0) + \delta \log((W_0 - c_0)((\vec{R}_1 - R_1^0 \vec{1})^T \cdot \vec{\pi} + R_1^0)))$

(i) FOC : $\frac{\partial f}{\partial c_0} = \frac{1}{c_0^*} - \delta \frac{1}{c_1^*} \cdot E((\vec{R}_1 - R_1^0 \vec{1})^T \vec{\pi}^* + R_1^0)) = 0$

So $\frac{1}{c_0^*} = \frac{\delta}{W_0 - c_0^*}$, so $c_0^* = \frac{\delta}{1+\delta} \cdot W_0$

(ii) $\frac{\partial f}{\partial \vec{\pi}} = E(\delta \frac{(W_0 - c_0)}{c_1} (\vec{R}_1 - R_1^0 \vec{1})) = \vec{0}$

So $E\left(\frac{R_1^i - R_1^0}{(\vec{R}_1 - R_1^0 \vec{1})^T \vec{\pi}^* + R_1^0}\right) = 0$, $i = 0, \cdots, n$

$\frac{1}{R_1^0} E\left(\frac{R_1^i}{(\vec{R}_1 - R_1^0 \vec{1})^T \vec{\pi}^* + R_1^0}\right) = E\left(\frac{1}{(\vec{R}_1 - R_1^0 \vec{1})^T \vec{\pi}^* + R_1^0}\right)$

(c) Let $M = E\left(\frac{1}{(\vec{R}_1 - R_1^0 \vec{1})^T \vec{\pi}^* + R_1^0}\right)^{-1} \cdot \frac{1}{(\vec{R}_1 - R_1^0 \vec{1})^T \cdot \vec{\pi}^* + R_1^0} \cdot \frac{1}{R_1^0}$

This will satisfy $E(M \cdot R_1^i) = 1 \quad \forall i = 0, \cdots, n$

from question (b).

Q3 Notation: 'X' is upper case of letter 'x', not 'χ'. Upper case 'z' is 'Z'.

(a)
$$ES^\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 V@R^u(X)\,du$$

$$= V@R^\alpha(X) + \frac{1}{1-\alpha}\int_\alpha^1 \left(V@R^u(X) - V@R^\alpha(X)\right)du$$

By definition, $V@R^\alpha(X) = q_{1-\alpha}(X) = \inf\{z \in \mathbb{R} : P(X+z \geq 0) \geq \alpha\}$

Notice that $V@R^\alpha(X)$ is monotonically increasing for α,

it means $\forall\, \alpha \leq u$ $\quad V@R^\alpha(X) \leq V@R^u(X)$, so $V@R^u(X) - V@R^\alpha(X) \geq 0$

so $(V@R^u(X) - V@R^\alpha(X))^+ = (V@R^u(X) - V@R^\alpha(X))_{1\{X \geq \alpha\}}$

So
$$ES^\alpha(X) = V@R^\alpha(X) + \frac{1}{1-\alpha}\int_\alpha^1 (q_{1-u}(X) - V@R^\alpha(X))_{1\{X \geq \alpha\}}$$

$$= V@R^\alpha(X) + \frac{1}{1-\alpha} E\left((-X - V@R^\alpha(X)) \mid X \geq V@R^\alpha(X)\right)$$

$$= V@R^\alpha(X) + \frac{1}{1-\alpha} E\left((-X - V@R^\alpha(X))^+\right)$$

(b) By definition, $V@R^\alpha(X) = q_{1-\alpha}(X) = \inf\{z \in \mathbb{R} : P(X+z \geq 0) \geq \alpha\}$,

$$ES^\alpha(X) = V@R^\alpha(X) + \frac{1}{1-\alpha} E\left((-X - V@R^\alpha(X)) \mid X \geq V@R^\alpha(X)\right\}$$

$$= \inf\{z \in \mathbb{R} : P(X+z \geq 0) \geq \alpha\} + \frac{1}{1-\alpha} E\left(-X - \inf\{z \in \mathbb{R} : P(X+z \geq 0) \geq \alpha \mid X \geq q_{1-\alpha}(X)\}\right)$$

$$= \inf\{z \in \mathbb{R} : z + \frac{1}{1-\alpha} E(-X - z \mid X \geq q_{1-\alpha}(X))$$

$$= \inf\{z \in \mathbb{R} : z + \frac{1}{1-\alpha} E((-X - z)^+)\}$$

(c) $\forall\, X, Y$ are r.v., $\lambda \in [0,1]$, Let $Z = \lambda X + (1-\lambda)Y$, so

$$\lambda ES^\alpha(X) + (1-\lambda)ES^\alpha(Y) - ES^\alpha(\lambda X + (1-\lambda)Y)$$

$$= \lambda \inf\{z \in \mathbb{R} : \frac{1}{1-\alpha} E((-X - z) \mid X \geq z)\} + (1-\lambda)\inf\{z \in \mathbb{R} : \frac{1}{1-\alpha} E((-Y - z) \mid Y \geq z)\}$$

$$- \inf\{z \in \mathbb{R} : \frac{1}{1-\alpha} E((-\lambda X - (1-\lambda)Y - z) \mid \lambda X + (1-\lambda)Y \geq z\}$$

$$\geq \lambda\left(\inf\{z \in \mathbb{R} : \frac{1}{1-\alpha} E((-X) \mid X \geq z)\} - \inf\{z \in \mathbb{R} : \frac{1}{1-\alpha} E((-X) \mid \lambda X + (1-\lambda)Y \geq z\}\right)$$

$$+ (1-\lambda)\left(\inf\{z \in \mathbb{R} : \frac{1}{1-\alpha} E((-Y) \mid X \geq z)\} - \inf\{z \in \mathbb{R} : \frac{1}{1-\alpha} E((-Y) \mid \lambda X + (1-\lambda)Y \geq z\}\right)$$

≥ 0 $\quad$ (The $\inf_z$ of former is higher than latter one)

Interpretation:

Expected shortfall is a measure of risk, the convexity of ES encourage us to diversify our portfolio, because this can reduce the risk of the portfolio lower than the linear combination of the original portfolio

Q4

(a) Using Two spam theory, any mean-variance frontier portfolio can be characterized by

$$\vec{\pi} = \lambda \vec{\pi}_{\mu} + (1-\lambda)\vec{\pi}_1 \quad , \lambda \in \mathbb{R}$$

where $A = \vec{\mu}^T \Sigma^{-1} \vec{\mu}$, $B = \vec{\mu}^T \Sigma^{-1} \vec{1}$, $C = \vec{1}^T \Sigma^{-1} \vec{1}$, $\vec{\pi}_{\mu} = \frac{1}{B} \Sigma^{-1} \vec{\mu}$, $\vec{\pi}_1 = \frac{1}{C} \Sigma^{-1} \vec{1}$.

Calculate them, we have

$$\begin{cases} A = 13.5 \\ B = 3 \\ C = 1 \end{cases} \qquad \begin{cases} \vec{\pi}_{\mu} = \frac{1}{3}(2, 0.75, 0.25)^T \\ \vec{\pi}_1 = (1, 0, 0)^T \end{cases}$$

Also, $\lambda = \dfrac{\mu_p - \vec{\mu}^T \vec{\pi}_1}{\vec{\mu}^T \vec{\pi}_{\mu} - \vec{\mu}^T \vec{\pi}_1} = \frac{2}{3}\mu_p - 3$

(b)

i) Tangency has max SR., where

$$SR = \frac{\mu_p}{\sigma_p} \quad , \sigma_p = \sqrt{\frac{A - 2B\mu_p + C\mu_p^2}{AC - B^2}}$$

$$f(x) := \log(x) - \frac{1}{2}\log\left(\frac{13.5 - 6x + x^2}{4.5}\right)$$

$$f'(x) = \frac{1}{x} - \frac{1}{2} \cdot \frac{(-6+2x)}{4.5} \cdot \frac{4.5}{13.5 - 6x + x^2} = 0 \quad \Rightarrow x = 4.5 = \mu_p, \ \sigma_p^2 = 1.5$$

ii)

(c) $\vec{\pi} = \dfrac{(\mu - R^0)^T \Sigma^{-1} (\mu - R^0\vec{1})}{(\mu - R^0\vec{1})^T \Sigma^{-1} (\mu - R^0\vec{1})}$, where $\mu_p = 3$

$$= (3-1)\frac{\Sigma^{-1}(\mu - R^0\vec{1})}{(\mu - R^0\vec{1})^T \Sigma^{-1}(\mu - R^0\vec{1})}$$

$$= 2 \cdot \frac{(1, 0.75, 0.25)^T}{8.5}$$

$$= (0.24, 0.18, 0.06)^T$$

So the part is $1 - \vec{1}^T\vec{\pi} = 0.47$