Second-Order Statistics of One-Sided CPM Signals

Donatella Darsena, *Senior Member, IEEE*, Giacinto Gelli, Ivan Iudice, and Francesco Verde, *Senior Member, IEEE*

***Abstract*—This letter deals with second-order statistics (SOS) of continuous-phase modulated (CPM) signals. To overcome some mathematical inconsistencies emerging from the idealized assumption that the CPM signal evolves from $t = -\infty$, we consider a one-sided model for the signal, which starts from $t = 0$, noting also that such a model emerges naturally when building practical SOS estimators. On the basis of such a model, we first evaluate the SOS of the pseudosymbols, which arise when expressing a CPM signal in terms of its Laurent representation, as well as closed-form expressions of the cyclic autocorrelation and conjugate correlation functions of one-sided CPM signals.**

***Index Terms*—Continuous-phase modulated (CPM) signals, Laurent representation, second-order statistics (SOS).**

I. Introduction

KNOWLEDGE of *second-order statistics* (SOS) of the received signal is required in the synthesis of receiving structures based on quadratic cost measures, like the minimum mean square error, minimum output energy, or maximum signal-to-noise-ratio criteria. When a bandpass signal is represented in terms of its complex envelope $x(t)$, SOS characterization requires [1], [2] evaluation of both its statistical *autocorrelation function* (ACF) $R_{xx}(t,\tau) \triangleq \mathbb{E}\,[x(t)\,x^*(t-\tau)]$ and its *conjugate correlation function* (CCF) $R_{xx^*}(t,\tau) \triangleq \mathbb{E}\,[x(t)\,x(t-\tau)]$. Since many man-made signals exhibit SOS that are periodic or almost periodic functions of time, i.e., the signals obey a *cyclostationary* or *almost cyclostationary* [3] model, their ACF and CCF can be expanded in a Fourier series with respect to the variable t, whose coefficients are the *cyclic* ACF/CCF [3].

Continuous-phase modulated (CPM) *signals* [4], [5] are widely employed in wireless communication systems, due to their favorable spectral and constant-modulus properties, as well as their good error-probability performance. Modeling and evaluation of the SOS of CPM signals is complicated by the memory and nonlinearity of the modulation process. To this aim, a useful tool is the *Laurent representation* [6], wherein a CPM signal is expanded as a linear superposition of pulse-amplitude modulated (PAM) signals. Based on this representation, an expression for the ACF and power spectrum of the complex envelope of the CPM signal has been derived in [6]: however, no discussion about the CCF and cyclostationarity properties was provided.

In [7], evaluation of cyclic SOS and higher-order statistics (HOS) of CPM signals (including ACF and CCF) has been carried out in the nonstochastic or *fraction-of-time* (FOT) probability framework [8]; however, a problem of convergence of infinite products has been solved by introducing an "undetermined constant" that can assume values ± 1. A careful analysis of the derivations in [7] reveals that such a constant stems from the assumption that the CPM signal starts from $t = -\infty$: however, in practice, CPM signals evolve starting from a finite time-epoch. Moreover, practical SOS estimators are built by evaluating suitable time averages of sampled signal data, taken starting from a particular time epoch, e.g., $t = 0$.

We show in this letter that the aforementioned problem of convergence, as well as the practical issues of SOS estimation, can be dealt with by modeling the CPM signal as a *one-sided* random process [9], i.e., as a process that starts from $t = 0$. In particular, on the basis of such model and exploiting the linearity of the Laurent representation, we evaluate closed-form expressions for the cyclic ACF and CCF of the CPM signal, which depend in their turn on the SOS of the *pseudosymbols* [6] of the Laurent representation.

II. One-Sided CPM Signal Model

The complex envelope $x(t)$ of a continuous-time CPM signal with baud-rate $1/T$ defined for $t \geq 0$ (one-sided model) can be obtained by straightforward modifications of the classical two-sided model (see, e.g., in [4] and [5]) as follows

$$x(t) = \exp\left[j2\pi h \sum_{n=0}^{+\infty} a_n\, g(t-nT)\right] \tag{1}$$

where h is the *modulation index*, the symbol sequence $\{a_n\}_{n\geq 0}$ assumes values in the M-ary alphabet $A \triangleq \{\pm 1, \pm 3, \ldots, \pm(M-1)\}$, $g(t) \triangleq \int_0^t f(u)\,du$ is the *phase response*, and $f(t)$ is the *frequency response* satisfying the three conditions: $f(t) \equiv 0$ for each $t \notin [0, LT]$; $f(t) = f(LT-t)$; and $\int_0^{LT} f(u)\,du = g(LT) = 1/2$, with $L \in \mathbb{N}$.

Assuming that h is not an integer and $M = 2$ (binary alphabet), by straightforward modifications of the Laurent representation proposed in [6], it can be proven that $x(t)$ for $t \geq 0$ is a linear superposition of $Q \triangleq 2^{L-1}$ PAM waveforms[1]

$$x(t) = \sum_{q=0}^{Q-1}\sum_{n=0}^{+\infty} s_{q,n}\, c_q(t-nT) \tag{2}$$

[1]Equations (1) and (2) should be slightly modified when $0 \leq t < (L-1)T$, to account for the finite-length of the Laurent pulses: however, such a transient phenomenon is not relevant when evaluating infinite-time averages.

Manuscript received May 31, 2017; revised August 2, 2017; accepted August 7, 2017. Date of publication August 17, 2017; date of current version September 5, 2017. The work of I. Iudice was supported in part by the Italian National Project TECVOL-II. The associate editor coordinating the review of this manuscript and approving it for publication was Dr. Seung-Jun Kim. *(Corresponding author: Francesco Verde.)*

D. Darsena is with the Department of Engineering, Parthenope University, Naples 80147, Italy (e-mail: darsena@uniparthenope.it).

G. Gelli and F. Verde are with the Department of Electrical Engineering and Information Technology, University Federico II, Naples 80125, Italy (e-mail: gelli@unina.it; f.verde@unina.it).

I. Iudice is with Italian Aerospace Research Centre, Capua 81043, Italy (e-mail: i.iudice@cira.it).

Color versions of one or more of the figures in this letter are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/LSP.2017.2740964

where the following *nonlinear* functions of $\{a_n\}_{n\geq 0}$

$$s_{q,n} = \exp\left[j\pi h \left(\sum_{\ell=0}^{n} a_\ell - \sum_{\ell=0}^{\min(n,L-1)} a_{n-\ell}\beta_{q,\ell}\right)\right] \quad (3)$$

are the *pseudosymbols*, with $n \geq 0$, $\beta_{q,\ell} \in \{0,1\}$ is the ℓth bit of the radix-2 representation of $q \in \{0,1,\ldots,Q-1\}$, i.e., $q = \sum_{\ell=1}^{L-1} 2^{\ell-1}\beta_{q,\ell}$ (with $\beta_{q,0}=0$), for $\ell \in \{1,2,\ldots,L-1\}$, and $c_q(t)$ is a real-valued pulse (see [6] for its expression). The Laurent representation can be extended to multilevel CPM signaling [10] and integer modulation indexes [11].

III. SOS of One-Sided CPM Signals

The time-averaged ACF of a two-sided CPM signal has been evaluated in [6, eq. (28)] in terms of its Laurent representation. The corresponding statistical ACF for the one-sided model can be written, for $t \geq (\tau)^+$, with $(\tau)^+ \triangleq \max(\tau, 0)$, as

$$R_{xx}(t,\tau) = \sum_{q_1,q_2=0}^{Q-1}\sum_{n=0}^{+\infty}\sum_{m=-\infty}^{n} R_{s_{q_1}s_{q_2}}(n,m) \times c_{q_1}(t-nT)\,c_{q_2}(t-nT+mT-\tau) \quad (4)$$

where $R_{s_{q_1}s_{q_2}}(n,m) \triangleq \mathbb{E}\left[s_{q_1,n}\,s^*_{q_2,n-m}\right]$, for $n \geq (m)^+$, with $(m)^+ \triangleq \max(m,0)$, is the cross-correlation function of the pseudosymbols. Evaluation of the CCF for the two-sided CPM signal has not been carried out in [6]. It can be shown that, for the one-sided model, one has

$$R_{xx^*}(t,\tau) = \sum_{q_1,q_2=0}^{Q-1}\sum_{n=0}^{+\infty}\sum_{m=-\infty}^{n} R_{s_{q_1}s^*_{q_2}}(n,m) \times c_{q_1}(t-nT)\,c_{q_2}(t-nT+mT-\tau) \quad (5)$$

where $R_{s_{q_1}s^*_{q_2}}(n,m) \triangleq \mathbb{E}\left[s_{q_1,n}\,s_{q_2,n-m}\right]$, for $n \geq (m)^+$, is the conjugate cross-correlation function of $\{s_{q,n}\}_{n\geq 0}$.

Both (4) and (5) depend on the SOS $R_{s_{q_1}s_{q_2}}(n,m)$ and $R_{s_{q_1}s^*_{q_2}}(n,m)$ of the pseudosymbols, which are evaluated in the following.[2] Starting from (3), it can be proven that, for $n \geq (m)^+$, one has

$$R_{s_{q_1}s_{q_2}}(n,m) = [\cos(\pi h)]^{\Delta^+_{q_1q_2}(n,m)} \quad (6)$$

where, for any $m \in \mathbb{Z}$, $\Delta^+_{q_1q_2}(n,m) \triangleq \Delta_{q_1q_2}(m) + \widetilde{\Delta}_{q_1q_2}(n,m)$, where $\Delta_{q_1q_2}(m)$ is an integer, whose explicit expression is given in [6, eq. (26)], whereas

$$\widetilde{\Delta}_{q_1q_2}(n,m) = \sum_{\ell=n+1}^{\min(L+m-1,L-1)} 2\beta_{q_2,\ell-m}\,\beta_{q_1,\ell} - \sum_{\ell=n+1}^{L-1}\beta_{q_1,\ell} - \sum_{\ell=n-m+1}^{L-1}\beta_{q_2,\ell} \quad (7)$$

is a correction term, which vanishes for $n \geq L-1+(m)^+$; in the latter case, the cross-correlation function for the one-sided model turns out to be the same of that for the two-sided model after a small transient and, moreover, it does not depend on n, i.e., one has, for $n \geq L-1+(m)^+$

$$R_{s_{q_1}s_{q_2}}(n,m) = R_{s_{q_1}s_{q_2}}(m) = [\cos(\pi h)]^{\Delta_{q_1q_2}(m)}. \quad (8)$$

Moreover, starting again from (3), it can be inferred that, for $n \geq (m)^+$, the conjugate cross-correlation function of pseudosymbols assumes the form [3] given in (9) shown at the bottom of this page. Note that, for $n \geq L-1+(m)^+$, (9) can be factorized as

$$R_{s_{q_1}s^*_{q_2}}(n,m) = R^+_{s_{q_1}s^*_{q_2}}(m)\,[\cos(2\pi h)]^n \quad (10)$$

where $R^+_{s_{q_1}s^*_{q_2}}(m)$ is given in (11) shown at the bottom of this page.

It is seen from (10) that for $h \neq \frac{1}{2}+k$, with $k \in \mathbb{Z}$, i.e., for $|\cos(2\pi h)| < 1$, the conjugate cross-correlation function of the pseudosymbols vanishes as n increases; in this case, we will prove later that the one-sided CPM signal is asymptotically *circular* or *proper* (see [1] and [2]). Instead, when $h = \frac{1}{2}+k$, with $k \in \mathbb{Z}$, i.e., $\cos(2\pi h) = -1$, the conjugate cross-correlation function of the pseudosymbols does not vanish as n increases; in this case, the CPM signal exhibits asymptotically nonvanishing *noncircular* or *improper* [1], [2] features.

IV. Cyclic SOS of One-Sided CPM Signals

With reference to the one-sided CPM signal model, we observe [see (4) and (5)] that the signal exhibits in general time-varying SOS. Such time-varying features cannot be estimated

[2] Cyclic SOS and HOS of CPM signals have been calculated in [7] in the nonstochastic FOT framework, without however giving explicit expressions for the SOS of the pseudosymbols.

[3] When the lower-limit of a sum [product] is larger than its upper-limit, the sum [product] is conventionally equal to zero [one].

$$R_{s_{q_1}s^*_{q_2}}(n,m) = \prod_{\ell=0}^{m-1}\cos[\pi h(1-\beta_{q_1,\ell})]\prod_{\ell=0}^{-m-1}\cos[\pi h(1-\beta_{q_2,\ell})]\prod_{\ell=(m)^+}^{\min\left[n,L+m-1-(m)^+\right]}\cos\left[\pi h\left(2-\beta_{q_1,\ell}-\beta_{q_2,\ell-m}\right)\right]$$
$$\times\prod_{\ell=L-m}^{\min(n-m,L-1)}\cos\left[\pi h\left(2-\beta_{q_2,\ell}\right)\right]\prod_{\ell=L+m}^{\min(n,L-1)}\cos\left[\pi h\left(2-\beta_{q_1,\ell}\right)\right][\cos(2\pi h)]^{\max\left[0,n-L+1-(m)^+\right]} \quad (9)$$

$$R^+_{s_{q_1}s^*_{q_2}}(m) = \prod_{\ell=0}^{m-1}\cos[\pi h(1-\beta_{q_1,\ell})]\prod_{\ell=0}^{-m-1}\cos[\pi h(1-\beta_{q_2,\ell})]\prod_{\ell=(m)^+}^{L+m-1-(m)^+}\cos\left[\pi h\left(2-\beta_{q_1,\ell}-\beta_{q_2,\ell-m}\right)\right]$$
$$\times\prod_{\ell=L-m}^{L-1}\cos\left[\pi h\left(2-\beta_{q_2,\ell}\right)\right]\prod_{\ell=L+m}^{L-1}\cos\left[\pi h\left(2-\beta_{q_1,\ell}\right)\right][\cos(2\pi h)]^{-L+1-(m)^+} \quad (11)$$

in practice unless a structured model for time variations is assumed. When time variations in SOS are described by a periodic or almost periodic model for $t \geq 0$, they can be conveniently measured and estimated by defining the cyclic ACF at the cycle frequency $\alpha \in \mathbb{R}$ as

$$R_{xx}^{\alpha}(\tau) = \langle R_{xx}(t,\tau)\, e^{-j2\pi\alpha t} \rangle_{+} \tag{12}$$

where $\langle f(t,\tau) \rangle_{+} \triangleq \lim_{Z \to +\infty} \frac{1}{Z} \int_{(\tau)^{+}}^{Z} f(t,\tau)\, dt$ denotes the one-sided time average operator. An estimator of (12) is the finite time-average of $x(t)\, x^{*}(t-\tau)\, e^{-j2\pi\alpha t}$

$$\widehat{R}_{xx}^{\alpha}(\tau) = \frac{1}{Z} \int_{(\tau)^{+}}^{Z} x(t) x^{*}(t-\tau)\, e^{-j2\pi\alpha t}\, dt\,. \tag{13}$$

It is clear that $\widehat{R}_{xx}^{\alpha}(\tau)$ is an asymptotically (for $Z \to +\infty$) unbiased estimator of $R_{xx}^{\alpha}(\tau)$; under mild conditions (see [12] and [13]), it can be proven that it is also a consistent estimator.

To obtain the theoretical expression of $R_{xx}^{\alpha}(\tau)$, (4) must be substituted in (12). It is convenient first to rewrite (4) as

$$\begin{aligned} R_{xx}(t,\tau) = & \sum_{q_1,q_2=0}^{Q-1} \sum_{m=-\infty}^{+\infty} \sum_{n=(m)^{+}}^{+\infty} R_{s_{q_1} s_{q_2}}(n,m) \\ & \times p_{q_1 q_2}(t-nT, \tau-mT) \end{aligned} \tag{14}$$

where $p_{q_1 q_2}(t,\tau) \triangleq c_{q_1}(t) c_{q_2}(t-\tau)$. Thus, one has

$$\begin{aligned} R_{xx}^{\alpha}(\tau) = & \sum_{q_1,q_2=0}^{Q-1} \sum_{m=-\infty}^{+\infty} \Bigg\langle \sum_{n=(m)^{+}}^{+\infty} R_{s_{q_1} s_{q_2}}(n,m) \\ & \times p_{q_1 q_2}(t-nT, \tau-mT)\, e^{-j2\pi\alpha t} \Bigg\rangle_{+}. \end{aligned} \tag{15}$$

Recalling the transient analysis of $R_{s_{q_1} s_{q_2}}(n,m)$ discussed with reference to (6)–(8), the time average in (15) can be decomposed as shown in (16) shown at the bottom of this page. The first time average in (16) is zero due to the finite duration of the signal involved. With reference to the second term, the two-sided version of the sum over n is clearly periodic in t of period T, thus it can be expanded as

$$\sum_{n=-\infty}^{+\infty} p_{q_1 q_2}(t-nT, \tau-mT) = \sum_{k=-\infty}^{+\infty} X_k\, e^{j2\pi \frac{k}{T} t} \tag{17}$$

where $\{X_k\}_{k \in \mathbb{Z}}$ are the Fourier series coefficients, given by $X_k \triangleq \frac{1}{T} P_{q_1 q_2}(f, \tau-mT)\big|_{f=k/T}$, where

$$\begin{aligned} P_{q_1 q_2}(f,\tau) &\triangleq \int_{-\infty}^{+\infty} p_{q_1 q_2}(t,\tau)\, e^{-j2\pi f t} dt \\ &= \int_{-\infty}^{+\infty} C_{q_1}(\lambda)\, C_{q_2}^{*}(\lambda - f)\, e^{j2\pi(\lambda - f)\tau} d\lambda \end{aligned} \tag{18}$$

is the Fourier transform of $p_{q_1 q_2}(t,\tau)$ with respect to t, and the second expression arises by virtue of Parseval identity, with $C_q(f)$ denoting the Fourier transform of $c_q(t)$. Since (17) holds for any t, it holds *a fortiori* for the values of t involved in the one-sided sum over n in (16); by substituting (17) in (16), one has

$$R_{xx}^{\alpha}(\tau) = \sum_{q_1,q_2=0}^{Q-1} \sum_{m=-\infty}^{+\infty} R_{s_{q_1} s_{q_2}}(m) \sum_{k=-\infty}^{+\infty} X_k \left\langle e^{j2\pi(\frac{k}{T}-\alpha)t} \right\rangle_{+}. \tag{19}$$

Observing that $\langle e^{j2\pi(\frac{k}{T}-\alpha)t} \rangle_{+} = \delta_{\alpha - k/T}$, with δ_k denoting the Kronecker delta, (19) shows that the CPM signal exhibits, in general, *wide-sense cyclostationarity* [14] with *cycle frequencies* $\alpha = \frac{k}{T}$, with $k \in \mathbb{Z}$, thus one has

$$R_{xx}^{\frac{k}{T}}(\tau) = \frac{1}{T} \sum_{q_1,q_2=0}^{Q-1} \sum_{m=-\infty}^{+\infty} R_{s_{q_1} s_{q_2}}(m)\, P_{q_1 q_2}\left(\frac{k}{T}, \tau - mT\right) \tag{20}$$

with $R_{s_{q_1} s_{q_2}}(m)$ and $P_{q_1 q_2}(f,\tau)$ given by (8) and (18), respectively.

Let us consider now the CCF given by (5), which can be conveniently rewritten as

$$\begin{aligned} R_{xx^{*}}(t,\tau) = & \sum_{q_1,q_2=0}^{Q-1} \sum_{m=-\infty}^{+\infty} \sum_{n=(m)^{+}}^{+\infty} R_{s_{q_1} s_{q_2}^{*}}(n,m) \\ & \times p_{q_1 q_2}(t-nT, \tau-mT). \end{aligned} \tag{21}$$

Let us define the cyclic CCF $R_{xx^{*}}^{\alpha}(\tau) \triangleq \langle R_{xx^{*}}(t,\tau)\, e^{-j2\pi\alpha t} \rangle$, on the basis of (9) and (10), one obtains (22) shown at the bottom of this page. Similarly to (16), it can be proven that the first time average in (22) goes to zero due to the finite duration of the signal involved. With reference to the second term, two distinct cases must be discussed, according to the value of h.

$$\begin{aligned} R_{xx}^{\alpha}(\tau) = & \sum_{q_1,q_2=0}^{Q-1} \sum_{m=-\infty}^{+\infty} \left\langle \sum_{n=(m)^{+}}^{L-2+(m)^{+}} R_{s_{q_1} s_{q_2}}(n,m)\, p_{q_1 q_2}(t-nT, \tau-mT)\, e^{-j2\pi\alpha t} \right\rangle_{+} \\ & + \sum_{q_1,q_2=0}^{Q-1} \sum_{m=-\infty}^{+\infty} R_{s_{q_1} s_{q_2}}(m) \left\langle \sum_{n=L-1+(m)^{+}}^{+\infty} p_{q_1 q_2}(t-nT, \tau-mT)\, e^{-j2\pi\alpha t} \right\rangle_{+} \end{aligned} \tag{16}$$

$$\begin{aligned} R_{xx^{*}}^{\alpha}(\tau) = & \sum_{q_1,q_2=0}^{Q-1} \sum_{m=-\infty}^{+\infty} \left\langle \sum_{n=(m)^{+}}^{L-2+(m)^{+}} R_{s_{q_1} s_{q_2}^{*}}(n,m)\, p_{q_1 q_2}(t-nT, \tau-mT)\, e^{-j2\pi\alpha t} \right\rangle_{+} \\ & + \sum_{q_1,q_2=0}^{Q-1} \sum_{m=-\infty}^{+\infty} R_{s_{q_1} s_{q_2}^{*}}^{+}(m) \left\langle \sum_{n=L-1+(m)^{+}}^{+\infty} [\cos(2\pi h)]^{n}\, p_{q_1 q_2}(t-nT, \tau-mT)\, e^{-j2\pi\alpha t} \right\rangle_{+} \end{aligned} \tag{22}$$

Let us first consider the case where $h \neq \frac{1}{2} + k$, which implies that $|\cos(2\pi h)| < 1$; one has

$$\begin{aligned}
&\left\langle \sum_{n=L-1+(m)^+}^{+\infty} [\cos(2\pi h)]^n \, p_{q_1 q_2}(t-nT, \tau-mT)\, e^{-j2\pi\alpha t} \right\rangle_+ \\
&= \lim_{Z\to+\infty} \frac{1}{Z} \sum_{n=L-1+(m)^+}^{+\infty} [\cos(2\pi h)]^n \\
&\times \int_{-\infty}^{+\infty} 1_{[(\tau)^+, Z]}(t)\, p_{q_1 q_2}(t-nT, \tau-mT)\, e^{-j2\pi\alpha t} dt
\end{aligned} \quad (23)$$

where $1_{[a,b]}(t) = 1$ for $t \in (a,b)$ and zero otherwise. The last integral in (23) represents a scalar product in $L^2(\mathbb{R})$, thus, resorting to Parseval identity and applying straightforward properties of the Fourier transform, it can be expressed as

$$\begin{aligned}
&\int_{-\infty}^{+\infty} 1_{[(\tau)^+, Z]}(t)\, [p_{q_1 q_2}(t-nT, \tau-mT)\, e^{j2\pi\alpha t}]^* dt \\
&= [Z-(\tau)^+] \int_{-\infty}^{+\infty} \operatorname{sinc}\{[Z-(\tau)^+] f\} \\
&\quad \times e^{-j2\pi f z(\tau)} P^*_{q_1 q_2}(f-\alpha, \tau-mT)\, e^{j2\pi(f-\alpha)nT} df
\end{aligned} \quad (24)$$

where $z(\tau) \triangleq \frac{Z+(\tau)^+}{2}$. As a consequence, the following inequality

$$\begin{aligned}
&\left| \frac{1}{Z} \sum_{n=L-1+(m)^+}^{+\infty} [\cos(2\pi h)]^n \, [Z-(\tau)^+] \int_{-\infty}^{+\infty} \operatorname{sinc}\{[Z-(\tau)^+] f\} \right. \\
&\qquad \left. \times e^{-j2\pi f z(\tau)} P^*_{q_1 q_2}(f-\alpha, \tau-mT)\, e^{j2\pi(f-\alpha)nT} df \right| \\
&\qquad\quad \leq \frac{1}{1-\cos(2\pi h)} \int_{-\infty}^{+\infty} |\operatorname{sinc}\{[Z-(\tau)^+] f\}| \\
&\qquad\qquad \times |P_{q_1 q_2}(f-\alpha, \tau-mT)|\, df
\end{aligned} \quad (25)$$

can be simply derived. The last integral in (25) goes to zero when $Z \to +\infty$ due to the behavior of the sinc function. Thus, when $h \neq \frac{1}{2} + k$, a CPM signal exhibits a zero CCF, hence it obeys a circular signal model.

Instead, let us consider the case where $h = \frac{1}{2} + k$, with $k \in \mathbb{Z}$, in which case $\cos(2\pi h) = -1$. It turns out that

$$\begin{aligned}
R^{\alpha}_{xx^*}(\tau) &= \sum_{q_1, q_2 = 0}^{Q-1} \sum_{m=-\infty}^{+\infty} R^{+}_{s_{q_1} s^*_{q_2}}(m) \\
&\times \left\langle \sum_{n=L-1+(m)^+}^{+\infty} (-1)^n p_{q_1 q_2}(t-nT, \tau-mT)\, e^{-j2\pi\alpha t} \right\rangle_+ .
\end{aligned} \quad (26)$$

If we consider the two-sided version of the sum over n, we observe that it is periodic in t of period $2T$, thus it can be expanded as

$$\sum_{n=-\infty}^{+\infty} (-1)^n p_{q_1 q_2}(t-nT, \tau-mT) = \sum_{k=-\infty}^{+\infty} Y_k \, e^{j2\pi \frac{k}{2T} t} \quad (27)$$

where $\{Y_k\}_{k\in\mathbb{Z}}$ are the Fourier series coefficients, given by $Y_k \triangleq \frac{1}{2T} G_{q_1 q_2}(f, \tau-mT)\big|_{f=k/2T}$, where

$$G_{q_1 q_2}(f, \tau) \triangleq P_{q_1 q_1}(f, \tau) \left[1 - e^{-j2\pi f T}\right] . \quad (28)$$

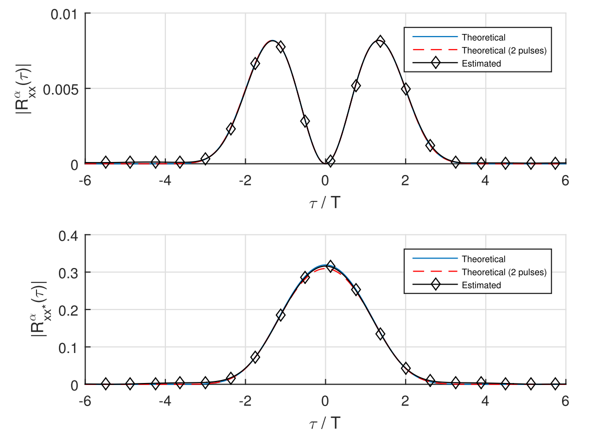

Fig. 1. Magnitude of the cyclic ACF (upper) at $\alpha = 1/T$ and cyclic CCF (lower) at $\alpha = 1/2T$ for a GMSK signal with $h = 0.5$, $L = 4$, and $BT = 0.25$.

In this case, reasoning similarly to (19), it can be proven that the CPM signal exhibits, in general, *conjugate wide-sense cyclostationarity* [14] with *cycle frequencies* $\alpha = \frac{k}{2T}$, with $k \in \mathbb{Z}$, thus one has

$$\begin{aligned}
R^{\frac{k}{2T}}_{xx^*}(\tau) &= \frac{1}{2T} \sum_{q_1, q_2 = 0}^{Q-1} \sum_{m=-\infty}^{+\infty} R^{+}_{s_{q_1} s^*_{q_2}}(m) \\
&\times G_{q_1 q_2}\left(\frac{k}{2T}, \tau-mT\right) .
\end{aligned} \quad (29)$$

However, taking into account (28), it turns out that

$$G_{q_1 q_2}\left(\frac{k}{2T}, \tau-mT\right) = \begin{cases} 2\, P_{q_1 q_2}\left(\frac{k}{2T}, \tau-mT\right), & k \text{ odd;} \\ 0, & k \text{ even} \end{cases} \quad (30)$$

hence, the cyclic CCF is nonzero only for $\alpha = \frac{1}{2T} + \frac{k}{T}$, with $k \in \mathbb{Z}$, and one has

$$\begin{aligned}
R^{\frac{1}{2T}+\frac{k}{T}}_{xx^*}(\tau) &= \frac{1}{T} \sum_{q_1, q_2 = 0}^{Q-1} \sum_{m=-\infty}^{+\infty} R^{+}_{s_{q_1} s^*_{q_2}}(m) \\
&\times P_{q_1 q_2}\left(\frac{k}{T} + \frac{1}{2T}, \tau-mT\right)
\end{aligned} \quad (31)$$

for all $k \in \mathbb{Z}$, with $R^{+}_{s_{q_1} s^*_{q_2}}(m)$ and $P_{q_1 q_2}(f, \tau)$ given by (11) and (18), respectively.

To validate our findings, we plot in Fig. 1 the cyclic ACF given by (20) and CCF given by (31) for a Gaussian minimum-shift keying (GMSK) signal with $h = 0.5$, $L = 4$, and $BT = 0.25$, together with their estimates obtained over 2^{15} symbols, with 2^5 samples/symbol. We also report the approximate expressions of the cyclic ACF and CCF obtained by considering only the first two Laurent pulses, which contain a large portion of the signal energy in many cases of interest [15]. All the curves show a very good agreement between simulation and analytical results.

V. Conclusion

In this letter, by adopting a one-sided model for a CPM signal, we derived closed-form expressions for its cyclic SOS, in terms of the SOS of the pseudosymbols of its Laurent representation. The obtained expressions can be useful to design receiving structures for CPM signals based on optimization of quadratic cost-functions.

REFERENCES

[1] F. D. Neeser and J. L. Massey, "Proper complex random processes with applications to information theory," *IEEE Trans. Inf. Theory*, vol. 39, no. 4, pp. 1293–1302, Jul. 1993.

[2] B. Picinbono, "On circularity," *IEEE Trans. Veh. Technol.*, vol. 42, no. 12, pp. 3473–3482, Dec. 1994.

[3] W. A. Gardner, Ed., *Cyclostationarity in Communications and Signal Processing*. Piscataway, NJ, USA: IEEE Press, 1994.

[4] J. B. Anderson, T. Aulin, and C. E. Sundberg, *Digital Phase Modulation*. New York, NY, USA: Springer, 1986.

[5] J. G. Proakis and M. Salehi, *Digital Communications*, 5th ed. New York, NY, USA: McGraw-Hill, 2007.

[6] P. Laurent, "Exact and approximate construction of digital phase modulations by superposition of amplitude modulated pulses (AMP)," *IEEE Trans. Commun.*, vol. 34, no. 2, pp. 150–160, Feb. 1986.

[7] A. Napolitano and C. M. Spooner, "Cyclic spectral analysis of continuous-phase modulated signals," *IEEE Trans. Signal Process.*, vol. 49, no. 1, pp. 30–44, Jan. 2001.

[8] W. A. Gardner and W. A. Brown, "Fraction-of-time probability for time-series that exhibit cyclostationarity," *Signal Process.*, vol. 23, no. 3, pp. 273–292, 1991.

[9] R. M. Gray, *Probability, Random Processes, and Ergodic Properties*. New York, NY, USA: Springer-Verlag, 2009.

[10] U. Mengali and M. Morelli, "Decomposition of M-ary CPM signals into PAM waveforms," *IEEE Trans. Inf. Theory*, vol. 41, no. 5, pp. 1265–1275, Sep. 1995.

[11] X. Huang and Y. Li, "The PAM decomposition of CPM signals with integer modulation index," *IEEE Trans. Commun.*, vol. 51, no. 4, pp. 543–546, Apr. 2003.

[12] H. L. Hurd, "Nonparametric time series analysis for periodically correlated processes," *IEEE Trans. Inf. Theory*, vol. 35, no. 2, pp. 350–359, Mar. 1989.

[13] A. V. Dandawate and G. B. Giannakis, "Nonparametric polyspectral estimators for kth-order (almost) cyclostationary processes," *IEEE Trans. Inf. Theory*, vol. 40, no. 1, pp. 67–84, Jan. 1994.

[14] W. A. Gardner, "Cyclic Wiener filtering: Theory and method," *IEEE Trans. Commun.*, vol. 41, no. 1, pp. 151–163, Jan. 1993.

[15] G. K. Kaleh, "Simple coherent receivers for partial response continuous phase modulation," *IEEE J. Sel. Areas Commun.*, vol. 7, no. 9, pp. 1427–1436, Dec. 1989.