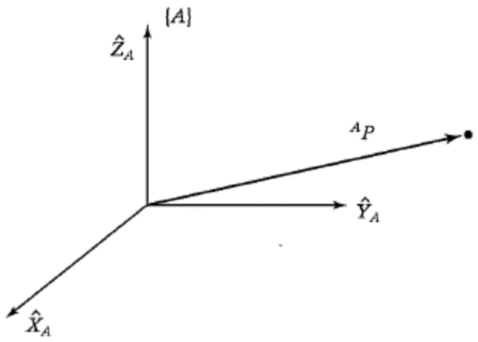
In the previous class, it was established that in order to describe the position and orientation of a body in space, a coordinate system is generally attached rigidly to the body. Once a coordinate system is established, the position of all points can be described using a 3×1 position vector in reference to the established coordinate system. As an example, given an established coordinate system, {A}, the position of a point p with reference to {A} is given by the vector below:

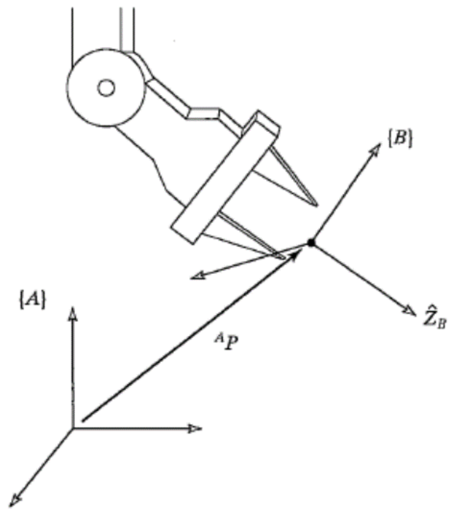
$$^{A}P = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

Where $p_x, p_y,$ and p_z are numerical values that indicate distances along the axes of {A}. Each of these distances can be thought of as the result of projecting the vector onto the corresponding axis.



In order to describe the orientation of a body, the coordinate system that is attached to the body is described relative to an established reference coordinate system.

In the figure below, coordinate system {B} is attached to the body. A description of {B} relative to {A} describes the orientation of the body.



One way to do this is by expressing the unit vectors of the three axes of {B} with reference to {A}. If we denote the unit vectors of {B} by $\hat{X}_B, \hat{Y}_B$ and $\hat{Z}_B$ then when written with reference to {A}, we have $^{A}\hat{X}_B$, $^{A}\hat{Y}_B$ and $^{A}\hat{Z}_B$. These represent the projection of each axis of {B} onto the axes of {A}. Generally, $^{A}\hat{X}_B$, $^{A}\hat{Y}_B$ and $^{A}\hat{Z}_B$ are stacked together resulting in the 3×3 matrix below, which is known as the rotation matrix.

$$
{}^{A}_{B}R = \left[\ {}^{A}\hat{X}_B \ \ {}^{A}\hat{Y}_B \ \ {}^{A}\hat{Z}_B\right] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}
$$

NOTE: This implies that the columns of a rotation matrix all have unit magnitude.

If we denote the unit vectors of {A} by $\hat{X}_A$, $\hat{Y}_A$ and $\hat{Z}_A$, then the rotation matrix can be expressed in terms of the dot product of the unit vectors of {A} and {B} as shown below:

$$
{}^{A}_{B}R = \left[\ {}^{A}\hat{X}_B \ \ {}^{A}\hat{Y}_B \ \ {}^{A}\hat{Z}_B\right] = \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix}
$$

NOTE: For two equal length column vectors X and Y, the dot product is given by:

$$
X^T Y = Y^T X = \|X\| \, \|Y\| \, \cos\theta
$$

Here, θ is the angle between the vectors. Given that all vectors concerned are the unit vector in each axis direction, it implies that the dot product simply yields the cosine of the angle between the vectors. Hence, the components of rotation matrices are often referred to as direction cosines. Concretely, the rotation matrices about the three principal axes are given below:

- Rotation of θ_x about the x-axis is defined as

$$
R_X(\theta_x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{bmatrix}
$$

- Rotation of θ_y about the y-axis is defined as

$$
R_Y(\theta_y) = \begin{bmatrix} \cos\theta_y & 0 & \sin\theta_y \\ 0 & 1 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y \end{bmatrix}
$$

- Rotation of θ_z about the z-axis is defined as

$$
R_Z(\theta_z) = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

Rotation about the three principal axes can be viewed as a sequence of three rotations that can be represented by the matrix product below:

$$
R_{ZYX} = R_Z(\theta_z)\, R_Y(\theta_y)\, R_X(\theta_x)
$$

$$
= \begin{bmatrix} \cos\theta_y \cos\theta_z & \sin\theta_x \sin\theta_y \cos\theta_z - \cos\theta_x \sin\theta_z & \cos\theta_x \sin\theta_y \cos\theta_z + \sin\theta_x \sin\theta_z \\ \cos\theta_y \sin\theta_z & \sin\theta_x \sin\theta_y \sin\theta_z + \cos\theta_x \cos\theta_z & \cos\theta_x \sin\theta_y \sin\theta_z - \sin\theta_x \cos\theta_z \\ -\sin\theta_y & \sin\theta_x \cos\theta_y & \cos\theta_x \cos\theta_y \end{bmatrix}
$$

However, since matrix multiplication does not commute, the order of the axes is significant. For the matrix product above, it is assumed that rotation about the x-axis is done first, then rotation about the y-axis and finally rotation about the z-axis. It should be observed that

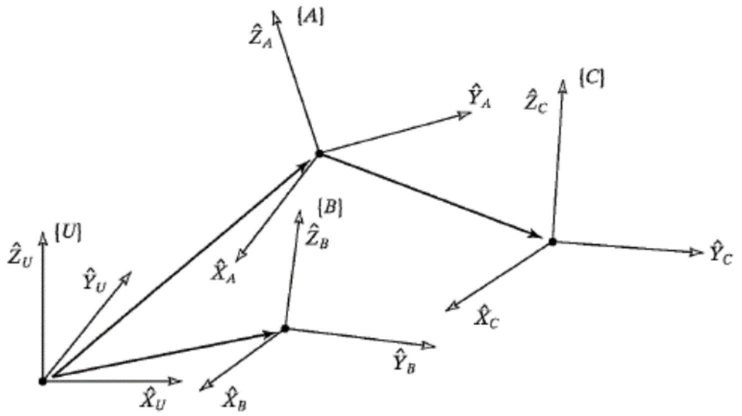
$$
{}^{A}_{B}R = \left[\ {}^{A}\hat{X}_B \ \ {}^{A}\hat{Y}_B \ \ {}^{A}\hat{Z}_B\right] = \begin{bmatrix} {}^{B}\hat{X}_A^T \\ {}^{B}\hat{Y}_A^T \\ {}^{B}\hat{Z}_A^T \end{bmatrix} = {}^{B}_{A}R^T
$$

This also means that the inverse of a rotation matrix is equal to its transpose. (Use $A^{-1}A = I$ to prove)

From the last lecture, we learned that both the position and the orientation are needed to completely specify the location of a body. Generally, the origin of the body is the point that we use and to which the coordinate system is attached. In robotics, a FRAME is a set of four vectors ($\equiv$ position vector combined with rotation matrix) used to specify the position and orientation of a body. As an example, frame {B} in the previous discussions is described by both ${}^{A}P_{BORG}$ and ${}^{A}_{B}R$. Here, ${}^{A}P_{BORG}$ is the position vector that locates the origin of the {B} with reference to {A}. Therefore frame {B} can be written as $\{B\} = \{{}^{A}_{B}R\ ,\ \ {}^{A}P_{BORG}\}$. A further example is shown in the figure below, where there are three frames given along with the universal (reference) coordinate system. As can be observed from the diagram, Frames {A} and {B} are known relative to the universal coordinate system, and frame {C} is known relative to frame {A}.

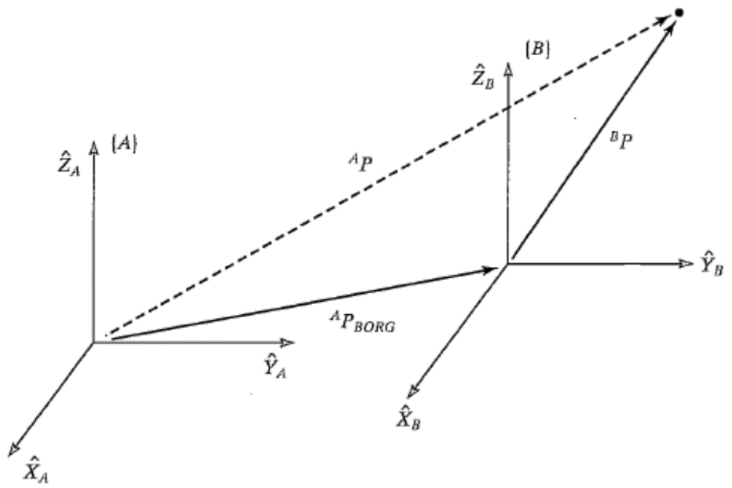
NOTE: A frame is generally used as a description of one coordinate system relative to another.

Alternatively, Positions could be represented by a frame whose rotation matrix part is the identity matrix and whose position vector part locates the point of interest. Likewise, orientations could be represented by a frame whose position vector part is the zero vector.



In robotics, mapping simply means expressing the same quantity in terms of various reference coordinate systems. In other words, changing descriptions from one frame to another frame.

1. Mapping involving translated frames and no rotation
   The figure below shows a point expressed in the terms of {B} as defined by position vector ${}^{B}P$. By mapping, we wish to express this point in space in terms of {A}. In this case, {B} differs from {A} only by a translation, which is given by ${}^{A}P_{BORG}$, a vector that locates the origin of {B} relative to {A}.

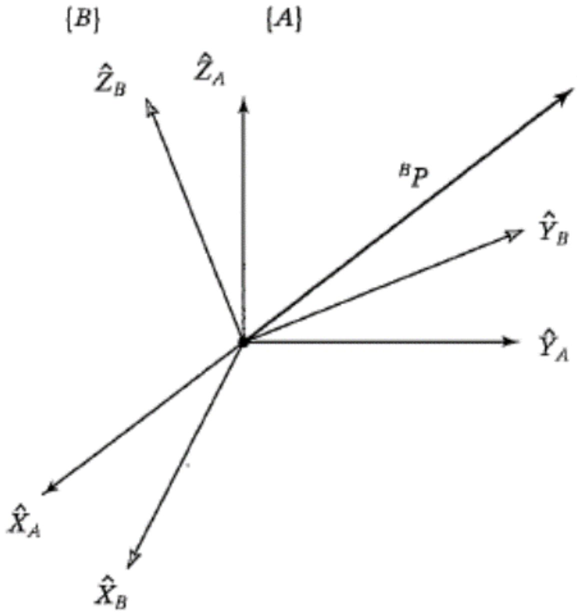


Because both ^{A}P and ^{B}P are defined relative to frames with the same orientation, we calculate the description of point p relative to {A} by the vector addition below:

$$^{A}P = {}^{B}P + {}^{A}P_{BORG}$$

NOTE: To add or subtract two vectors, simply add or subtract the corresponding components. We say that the vector $^{A}P_{BORG}$ defines this mapping because all the information needed to perform the change in description is contained in $^{A}P_{BORG}$ (along with the knowledge that the frames have the same orientation).

2. Mapping involving rotated frames and no translation
   The figure below shows a point expressed in the terms of {B} as defined by position vector ^{B}P. Here, we would like to know its definition with respect to {A}.

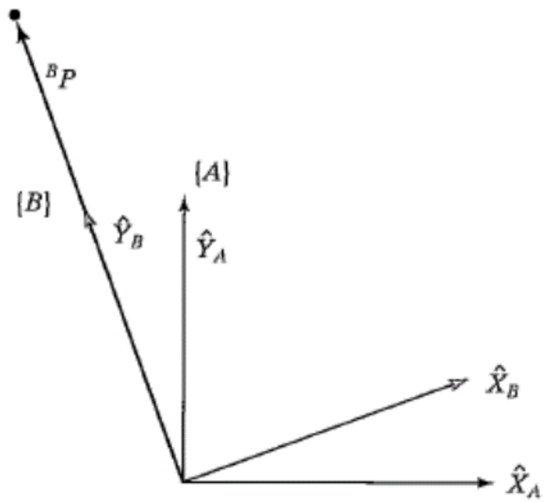


As can be observed in the figure, the origins of the two frames are coincident. The description of point p relative to {A} as defined by ${}^{A}P$ is given by the matrix multiplication below:

$$ {}^{A}P = {}^{A}_{B}R\,{}^{B}P $$

EXAMPLE 2.1

The figure below shows a frame {B} that is rotated relative to frame {A} about the z-axis by 30 degrees. Given ${}^{B}P = \begin{bmatrix} 0.0 \\ 2.0 \\ 0.0 \end{bmatrix}$, compute ${}^{A}P$.



Using

$$ R_Z(\theta_z) = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

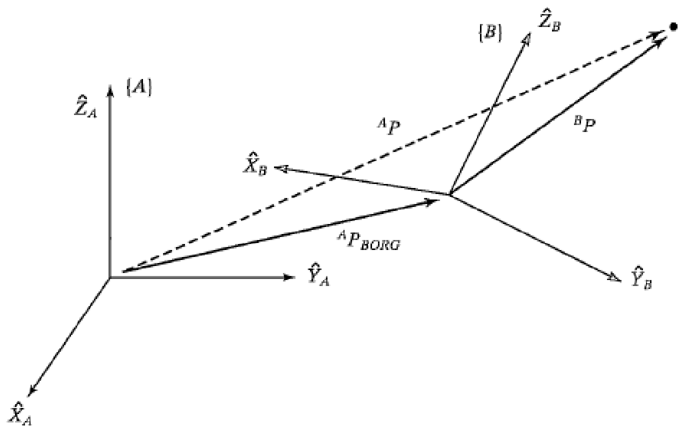
The rotation matrix can be computed as

$$ {}^{A}_{B}R = \begin{bmatrix} 0.866 & -0.500 & 0.000 \\ 0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix} $$

$$ \therefore\ {}^{A}P = {}^{A}_{B}R\,{}^{B}P = \begin{bmatrix} 0.866 & -0.500 & 0.000 \\ 0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix} \begin{bmatrix} 0.0 \\ 2.0 \\ 0.0 \end{bmatrix} = \begin{bmatrix} -1.000 \\ 1.732 \\ 0.000 \end{bmatrix} $$

3. Mapping involving both translated and rotated frames

   The figure below shows a point expressed in the terms of {B} as defined by position vector ${}^{B}P$. Here, we wish to express this point in space in terms of {A}. In this case, {B} differs from {A} both by a translation, which is given by ${}^{A}P_{BORG}$, a vector that locates the origin of {B} relative to {A} and orientation as the origins of the two frames are not coincident. Here, {B} is rotated with respect to {A} as described by ${}^{A}_{B}R$.



The description of point p relative to {A} as defined by ^AP is given by the equation below:

$$^AP = {}^A_BR\, {}^BP + {}^AP_{BORG}$$

In order to get a more compact form of the equation above, a homogeneous transform matrix ${}^A_BT$ is defined as shown below:

$$^AP^* = {}^A_BT\ {}^BP^*$$

Here, ${}^A_BT$ is a 4×4 matrix, while $^AP^*$ and $^BP^*$are 4×1 position vectors as shown below:

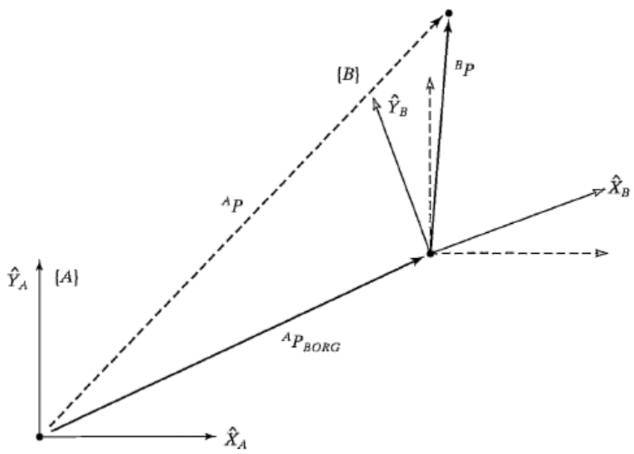
$$\begin{bmatrix} ^AP \\ 1 \end{bmatrix} = \begin{bmatrix} & {}^A_BR & & {}^AP_{BORG} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ^BP \\ 1 \end{bmatrix}$$

Where

1. A “1” has been added as the last element of position vectors ^AP and ^BP resulting in 4×1 position vectors $^AP^*$ and $^BP^*$.
2. A row "0 0 0 1" has been added as the last row of the 4×4 matrix created by stacking ${}^A_BR$ and $^AP_{BORG}$.

EXAMPLE 2.2

The figure below shows a frame {B}, which is rotated relative to frame {A} about the z-axis by 30 degrees and translated 10 units in the x-axis with reference to {A} and translated 5 units in the y-axis with reference to {A}. Given $^BP = \begin{bmatrix} 3.0 \\ 7.0 \\ 0.0 \end{bmatrix}$, compute ^AP.



Using

$$R_Z(\theta_z) = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The rotation matrix can be computed as

$$^A_BR = \begin{bmatrix} 0.866 & -0.500 & 0.000 \\ 0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix}$$

Also, from the question $^AP_{BORG} = \begin{bmatrix} 10.0 \\ 5.0 \\ 0.0 \end{bmatrix}$

This results in $^A_BT = \begin{bmatrix} 0.866 & -0.500 & 0.000 & 10.000 \\ 0.500 & 0.866 & 0.000 & 5.000 \\ 0.000 & 0.000 & 1.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 1.000 \end{bmatrix}$

Using $^AP^* = \, ^A_BT \; ^BP^*$, ^AP can be computed. ($^AP = \begin{bmatrix} 9.098 \\ 12.562 \\ 0.000 \end{bmatrix}$)