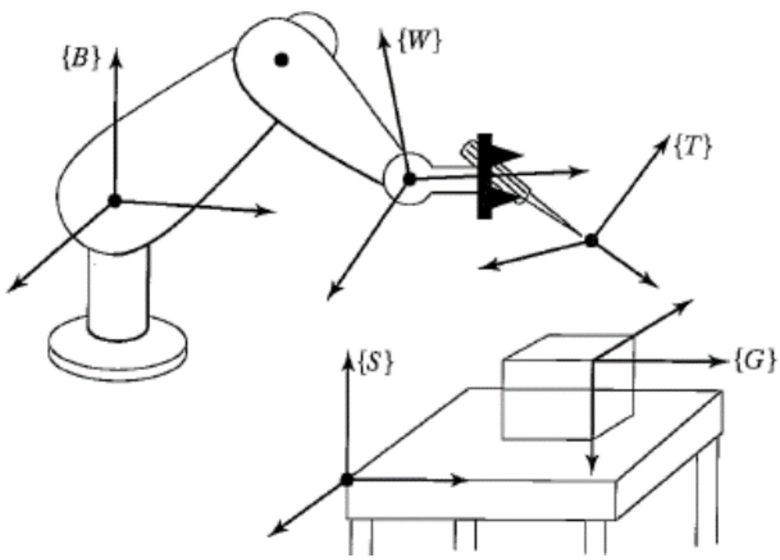
STANDARD FRAMES

Generally, to facilitate robot programming and control, the following five frames that relate to robot motions are defined:

- Base Frame $\{B\}$: This frame is located at the base of the manipulator. It is simply another name for frame $\{0\}$.
- Station Frame $\{S\}$: This frame serves as the origin of the task-relevant location and all actions of the robot are performed relative to it. This frame is sometimes called the task frame, the world frame, or the universe frame. The station frame is always specified with respect to the base frame. I.e. ${}^{B}_{S}T$.
- Wrist Frame $\{W\}$: This frame is attached to the last link of the manipulator. It is another name for frame $\{n\}$. Usually, $\{W\}$ has its origin fixed at a point called the wrist of the manipulator, and $\{W\}$ moves with the last link of the manipulator. It is defined relative to the base frame. I.e. $\{W\} = {}^{B}_{W}T = {}^{0}_{n}T$.
- Tool Frame $\{T\}$: This frame is attached to the end of any tool/device that the robot happens to be holding. If the robot isn't holding any device, $\{T\}$ is usually located at the origin of the end effector. The tool frame is always specified with respect to the wrist frame.
- Goal Frame $\{G\}$: This frame describes the location to which the robot is to move the tool. Specifically this means that, at the end of the motion, the tool frame should be brought to the location of the goal frame. $\{G\}$ is always specified relative to the station frame.



A critical capability of a robot is its ability to calculate the location (position and orientation) of any device/tool it is holding (or of its empty hand) with respect to a convenient coordinate system. That is, we wish to calculate the value of the tool frame $\{T\}$ relative to the station frame $\{S\}$. Once ${}^{B}_{W}T$ has been computed, we can use the following transforms to calculate $\{T\}$ relative to $\{S\}$.

$${}^{S}_{T}T = {}^{B}_{S}T^{-1}\ {}^{B}_{W}T\ {}^{W}_{T}T$$

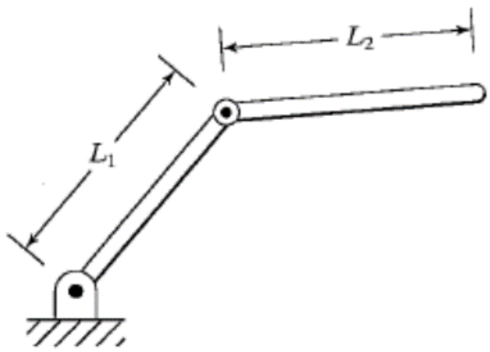
This equation implements what is called the WHERE function in some robot systems. It computes where the arm is.

MANIPULATOR'S WORKSPACE

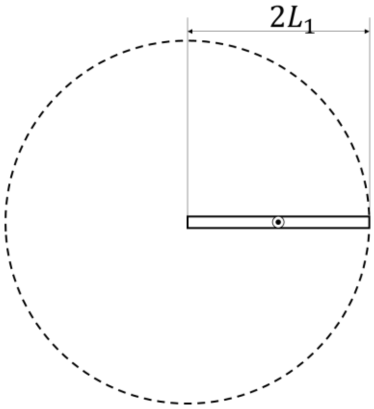
A manipulator's workspace is the volume of space that the end-effector of the manipulator can reach. Generally, two definitions of workspace are frequently used:

1. Dextrous workspace: This is the volume of space that the robot's end-effector can reach with all orientations. That is, at each point in the dextrous workspace, the end-effector can be arbitrarily oriented.
2. Reachable workspace: This is the volume of space that the robot can reach in at least one orientation.

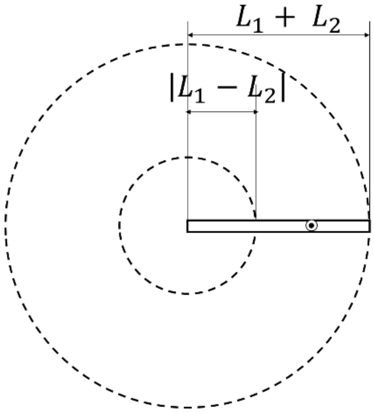
The dextrous workspace is a subset of the reachable workspace. Consider the workspace of the two-link manipulator below.



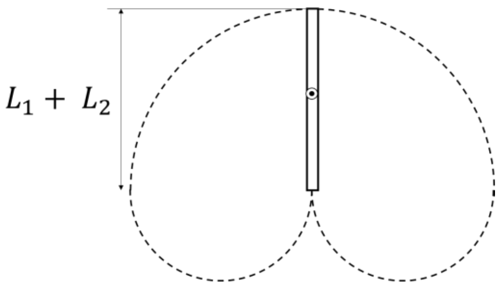
If $L_1 = L_2$, then the reachable workspace consists of a disc of radius $2L_1$. Here, the dextrous workspace consists of only a single point, the origin.



If $L_1 \neq L_2$, then the reachable workspace becomes a ring of outer radius $L_1 + L_2$ and inner radius $|L_1 - L_2|$. Here, there is no dextrous workspace as no point can be reached with all orientation.



It should be noted that the workspaces considered above assumed that all the joints can rotate 360 degrees. In practice, this is rarely true due to structural restrictions and as a result the actual workspace is reduced. For instance, for the two link manipulator considered, where $L_1 = L_2$, if θ_2 has 360° motion and $0^\circ \leq \theta_1 \leq 180^\circ$, then the resulting reachable workspace is as shown below.



The foregoing discussions on workspace have assumed the robot isn't holding any device, hence $\{T\}$ is located at the origin of the end effector (equivalent to no tool frame). However, the workspace also depends on the tool frame transformation because it is usually the tool tip that is discussed when reachable points in space is strictly considered.

SUMMARY OF FORWARD (DIRECT) KINEMATIC

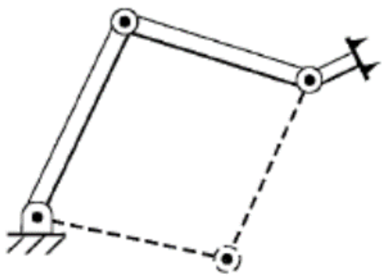
For revolute-jointed mechanical manipulators, the forward kinematics simply seeks to determine the location (orientation and position) of the end effector given the joint angles and link lengths of the robot arm. Here the link lengths are constant while the joint angles vary. Thus given any joint angle set, the first step is to determine $\{W\} = {}^{B}_{W}T = {}^{0}_{n}T$, which shows the location of the end-effector relative to the base. However, for many operations, the tool frame $\{T\}$ relative to the station frame $\{S\}$ information is needed and can then be found using ${}^{S}_{T}T = {}^{B}_{S}T^{-1}\ {}^{B}_{W}T\ {}^{W}_{T}T$.

Inverse Kinematics

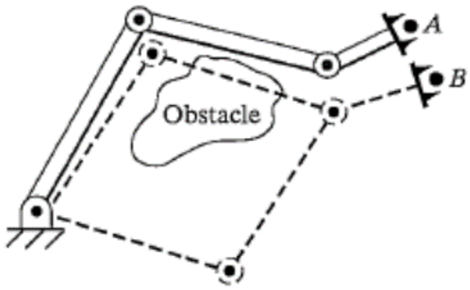
The inverse kinematics problem for a revolute-jointed mechanical manipulators is to find the values of the joint angles given the location (position and orientation) of the end-effector relative to the base and the values of all of the geometric link parameters. Here, the notion of workspace is particularly important since we have to specify a location. Thus we cannot specify locations (goal frames) outside the reachable workspace or else the manipulator cannot achieve the goal. Another possible problem encountered in inverse kinematics is that of multiple solutions. A planar arm with three or more revolute joints has a large dextrous workspace in the plane because more locations in the interior of its workspace can be reached with any orientation. Thus the solution of an inverse kinematics problem is not always unique as the same end effector location can be reached with several configurations.

NOTE: The dextrous workspace increases as the number of links of a manipulator increases

The figure below shows a three-link planar arm with its end-effector at a certain location. The dashed lines in the figure indicate a second possible configuration by which the same end-effector location can be achieved.



The fact that a manipulator has multiple solutions can cause problems, because the system has to be able to choose one. The criteria upon which to base a decision vary, but a very reasonable choice would be the closest solution. For example, in the figure below, if the manipulator is at point A and the goal is to move it to point B, a good choice would be the solution that minimizes the amount of rotation for each joint. Hence, the upper dashed configuration would be the obvious solution. However, the presence of the obstacle clearly makes this solution not feasible. Consequently, the solution is the lower dashed configuration.



The number of possible solutions depends upon the number of joints in the manipulator, the link parameters (a_i, α_i and d_i for a rotary joint manipulator), and the allowable ranges of motion of the joints. It should be noted that the more nonzero link parameters there are, the more ways there will be to reach a certain goal. As an example, for a 6R manipulator, the table below shows how the maximum number of solutions is related to how many of the link length parameters (a_i) that are zero. The more that are nonzero, the bigger the maximum number of solutions. For a completely general rotary-jointed manipulator with six degrees of freedom, there are up to sixteen possible solutions.

$\boldsymbol{a_i}$	**Number of Solutions**
$a_1 = a_3 = a_5 = 0$	≤ 4
$a_3 = a_5 = 0$	≤ 8
$a_3 = 0$	≤ 16
All $a_i \neq 0$	≤ 16

Generally, an algorithm is used to find the solution of an inverse kinematics problem of a manipulator. The manipulator problem is considered solvable only if all the sets of joint variables associated with a given location can be determined by the algorithm. It should be noted that according to the definition of solvability, all systems with revolute and prismatic joints having a total of six degrees of freedom in a single series chain are numerically solvable. For robots with six degrees of freedom to be solved analytically (closed-form), the robots are characterized either by having several intersecting joint axes or by having many α_i equal to 0 or $\pm 90^{\circ}$. Hence, nowadays virtually all industrial manipulators are designed sufficiently simply that a closed-form solution can be developed. A sufficient condition that a manipulator with six revolute joints have a closed form solution is that three neighbouring joint axes intersect at a point. Almost every manipulator with six degrees of freedom built today has three axes intersecting. For example, axes 4, 5, and 6 of the PUMA 560 intersect.

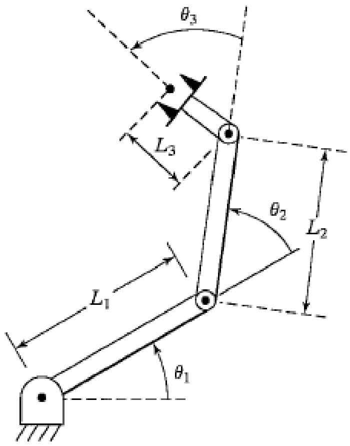
To facilitate the discussion on inverse kinematics, let's first give a proper means to specify the subspace of a manipulator.

SUBSPACE: The manipulator subspace is the smallest p-dimensional space that contains the manipulator reachable workspace.

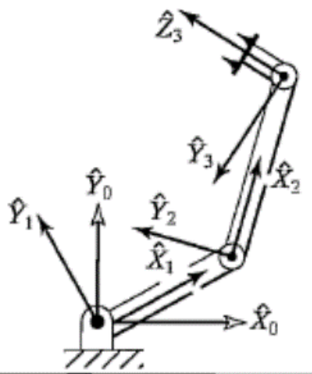
One common way to specify the subspace is to give an expression for a manipulator's wrist or tool frame as a function of n variables that locate it.

EXAMPLE 4.1

Give a description of the subspace of ${}^{B}_{W}T$ for the three-link manipulator below



The link parameters are shown below



i	$\alpha_i - 1$	$\alpha_i - 1$	d_i	θ_i
1	0	0	0	θ_1
2	0	L_1	0	θ_2
3	0	L_2	0	θ_3

As previously been established,

$$
{}^{B}_{W}T = {}^{0}_{3}T = \begin{bmatrix} \cos(\theta_1+\theta_2+\theta_3) & -\sin(\theta_1+\theta_2+\theta_3) & 0 & L_1\cos(\theta_1)+L_2\cos(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2+\theta_3) & \cos(\theta_1+\theta_2+\theta_3) & 0 & L_1\sin(\theta_1)+L_2\sin(\theta_1+\theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Rewriting the expression above as a function of the three (3) variables that locate it, results in the equivalent form below:

$$
{}^{B}_{W}T = \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) & 0 & x \\ \sin(\varphi) & \cos(\varphi) & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Here x and y give the position of the wrist and φ describes the orientation of the terminal link. As x, y and φ are allowed to take on arbitrary values, the subspace is generated. Any wrist frame that does not have the structure above lies outside the subspace (and therefore lies outside the workspace) of this manipulator. Link lengths and joint limits restrict the workspace of the manipulator to be a subset of this subspace.

In this course, we will focus on the following closed-form solution methods:

1. Algebraic: An algebraic approach to solving kinematic equations is basically one of manipulating the given equations into a form for which a solution is known.
2. Geometric: A geometric approach aims to decompose the spatial geometry of the arm into several plane-geometry problems.

Algebraic Approach

As an example, given the three link planar manipulator above, whose goal has been expressed in terms of the wrist frame relative to the base frame ${}^{B}_{W}T$, we can express the goal given by x, y and φ by using the equations below:

$$\varphi = \theta_1 + \theta_2 + \theta_3$$

$$x = L_1\cos(\theta_1) + L_2\cos(\theta_1+\theta_2)$$

$$y = L_1\sin(\theta_1) + L_2\sin(\theta_1+\theta_2)$$

By algebraic manipulation, these equations can be combined to yield

$$\cos(\theta_2) = \frac{x^2 + y^2 - L_1^2 - L_2^2}{2L_1L_2}$$

In order for a solution to exist, the right-hand side of this equation must have a value between -1 and 1. Physically, if this constraint is not satisfied, then the goal point is too far away for the manipulator to reach. If the goal is in the workspace, then

$$\sin(\theta_2) = \pm\sqrt{1 - \cos(\theta_2)^2}$$

Here, the choice of signs corresponds to the multiple solution in which we can choose the "elbow-up" or the "elbow-down" solution. Hence, to ensure that all solutions are obtained and that the solved angle is in the proper quadrant, $\theta_2$can be computed from

$$\theta_2 = \text{atan2}(\sin(\theta_2), \cos(\theta_2))$$

To find θ_1, we rewrite $x = L_1 \cos(\theta_1) + L_2 \cos(\theta_1 + \theta_2)$ and $y = L_1 \sin(\theta_1) + L_2 \sin(\theta_1 + \theta_2)$ using intermediate variables $k_1$and k_2, which are given by

$$k_1 = L_1 + L_2 \cos(\theta_2)$$

$$k_2 = L_2 \sin(\theta_2)$$

This yields

$$x = k_1 \cos(\theta_1) - k_2 \sin(\theta_1)$$

$$y = k_1 \sin(\theta_1) + k_2 \cos(\theta_1)$$

Thus $\theta_1$can be computed from

$$\theta_1 = \text{atan2}(y, \text{x}) - \text{atan2}(k_2, k_1)$$

Note that, when a choice of sign is made in the solution of θ_2, it will cause a sign change in k_2, thus affecting θ_1. Finally, the equation below can be used to solve for θ_3.

$$\theta_1 + \theta_2 + \theta_3 = \text{atan2}(\sin(\varphi), \cos(\varphi)) = \varphi$$

NOTE: $\text{atan2}(y, x)$ computes $\tan^{-1}\left({}^{y}/_{x}\right)$ but uses the signs of both x and y to identify the quadrant in which the resulting angle lies.

Geometric Approach