From previous lectures, it has been established that orientation can be represented using a 3×3 rotation matrix. Rotation matrices are special in that all columns are mutually orthogonal and have unit magnitude. In addition, the determinant of a rotation matrix is always equal to $+1$. Thus rotation matrices are PROPER ORTHONORMAL (ORTHOGONAL) MATRICES due to the fact that the determinant is $+1$. Cayley's formula for orthonormal matrices states that for any proper orthonormal matrix R, there exists a skew-symmetric matrix S such that

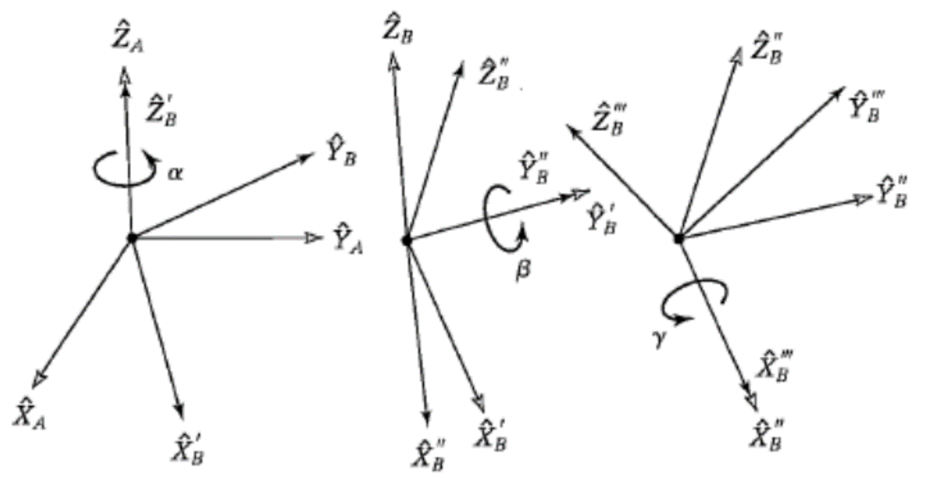
$$R = (I_3 - S)^{-1}(I_3 + S)$$

Here, I_3 is the 3×3 unit/identity matrix and the skew-symmetric matrix S of dimension 3 is specified by three parameters (S_x, S_y, S_z)as shown below:

$$S = \begin{bmatrix} 0 & -S_z & S_y \\ S_z & 0 & -S_x \\ -S_y & S_x & 0 \end{bmatrix}$$

NOTE: Since S is skew-symmetric, then $S = -S^T$. Also, since R is orthonormal, then $R^{-1} = R^T$.

Hence, any 3×3 rotation matrix can be specified by just three parameters. There exits several three-parameter representations of rotation matrices. A common method of describing the orientation of a frame {B} is to describe the orientation in terms of three rotations that take place about an axis in a fixed reference frame {A}. This convention utilizes the so-called roll, pitch, and yaw angles. However, for this class, emphasis will be on using the $Z - Y - X$ $(\alpha - \beta - \gamma)$ Euler angle convention. In this representation, each rotation is performed about an axis of the moving system {B} rather than about an axis of the fixed reference {A} as shown in the following figure.



This description can be stated as:

> Start with the fame coincident with a known frame {A}. Rotate {B} first about $\hat{Z}_B$ by an angle α, then about $\hat{Y}_B$ by an angle β, and finally about $\hat{X}_B$ by an angle γ.

Such sets of three rotations are called Euler angles.

NOTE: Each rotation takes place about an axis whose location depends upon preceding rotations.

A rotation matrix which is parameterized by $Z-Y-X$ Euler angles will be indicated by the notation ${}^{A}_{B}R_{Z'Y'X'}(\alpha,\beta,\gamma)$ as shown below to indicate that the rotation is described by Euler angles.

$${}^{A}_{B}R_{Z'Y'X'}(\alpha,\beta,\gamma) = R_Z(\alpha)R_Y(\beta)R_X(\gamma)$$

$$= \begin{bmatrix} \cos\beta\cos\alpha & \sin\gamma\sin\beta\cos\alpha - \cos\gamma\sin\alpha & \cos\gamma\sin\beta\cos\alpha + \sin\gamma\sin\alpha \\ \cos\beta\sin\alpha & \sin\gamma\sin\beta\sin\alpha + \cos\gamma\cos\alpha & \cos\gamma\sin\beta\sin\alpha - \sin\gamma\cos\alpha \\ -\sin\beta & \sin\gamma\cos\beta & \cos\gamma\cos\beta \end{bmatrix}$$

NOTE: This result is exactly the same as that previously obtained for the same three rotations taken in the opposite order about fixed axes. In general, three rotations taken about fixed axes yield the same final orientation as the same three rotations taken in opposite order about the axes of the moving frame.

Thus given a rotation matrix

$${}^{A}_{B}R_{Z'Y'X'}(\alpha,\beta,\gamma) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

The Euler angles can be computed using:

$$\beta = \operatorname{atan2}\left(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\right)$$

$$\alpha = \operatorname{atan2}\left({}^{r_{21}}/_{\cos\beta}, {}^{r_{11}}/_{\cos\beta}\right)$$
$$\gamma = \operatorname{atan2}\left({}^{r_{32}}/_{\cos\beta}, {}^{r_{33}}/_{\cos\beta}\right)$$

NOTE: $\operatorname{atan2}(y,x)$ computes $\tan^{-1}({}^{y}/_{x})$ but uses the signs of both x and y to identify the quadrant in which the resulting angle lies.

The preceding discussions have shown that the Euler angles can be computed from the rotation matrix. Conversely, the rotation matrix can be computed from the Euler angles. However, the specific mathematics to prove this, is beyond the scope of this class.

Example:

Given the following rotation matrix, compute the Euler angles

$${}^{A}_{B}R_{Z'Y'X'}(\alpha,\beta,\gamma) = \begin{bmatrix} 0.866 & -0.433 & 0.250 \\ 0.500 & 0.750 & -0.433 \\ 0.000 & 0.500 & 0.866 \end{bmatrix}$$

$$\alpha = 30°, \beta = 0°, \gamma = 30°$$

Manipulator kinematics

Kinematics is the science of motion without studying the forces that cause it. Here the study of position and its derivatives such as velocity and acceleration is the focus. Therefore, the study of the kinematics of manipulators refers to all the geometrical and time-based properties of the manipulator's motion.

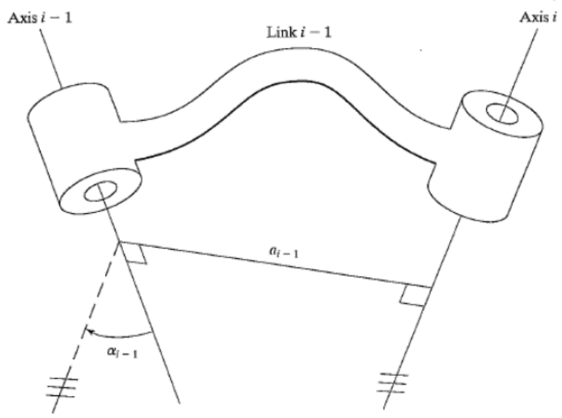
We start by considering the position and orientation of the manipulator linkages in static situations. As stated in previous lectures, we attach frames (coordinate systems) to the various parts of the manipulator and then describe the relationships between these frames. At the moment, the study is aimed towards deriving a method to compute the position and orientation of the manipulator's end-effector relative to the base of the manipulator as a function of the joint variables. This is also known as forward kinematics.

A manipulator is basically a set of bodies connected in a chain by joints. These bodies are called links. Joints form a connection between a neighbouring pair of links. Most manipulators are constructed from revolute joints that exhibit just one degree of freedom. Therefore, we will focus only on these type of manipulators. The links are numbered starting from the base (generally immobile), which is typically 0. The first moving body is link 1 and so on, out to the free end of the manipulator (tool), which is link n. Generally, to adequately specify the location of an end-effector in 3-D space, a minimum of six joints is required because the description of an object in space requires six parameters: three for position (the three axes specification) and three for orientation (In our case, the Euler angles).

For the purposes of obtaining the kinematic equations of the manipulator, a link is considered only as a rigid body that defines the relationship between two neighbouring joint axes of a manipulator. JOINT AXES are defined by lines in space. Joint axis i is defined by a line in space, or a vector direction, about which link i rotates relative to link $i-1$. It turns out that, for kinematic purposes, a link can be specified with two numbers, which define the relative location of the two axes in space.

The first parameter is the LINK LENGTH. For any two axes in 3-D space, the distance between them is measured along a line that is mutually perpendicular to both axes. The figure below shows a link and the mutually perpendicular line along which the link length a_{i-1} is measured.

The second parameter needed to define the relative location of the two axes is called the LINK TWIST. First we project the two axes onto the plane whose normal is the mutually perpendicular line to both axes. The angle measured from axis $i-1$ to axis i in the right hand sense about a_{i-1} denotes the link twist as indicated by α_{i-1} in the figure.



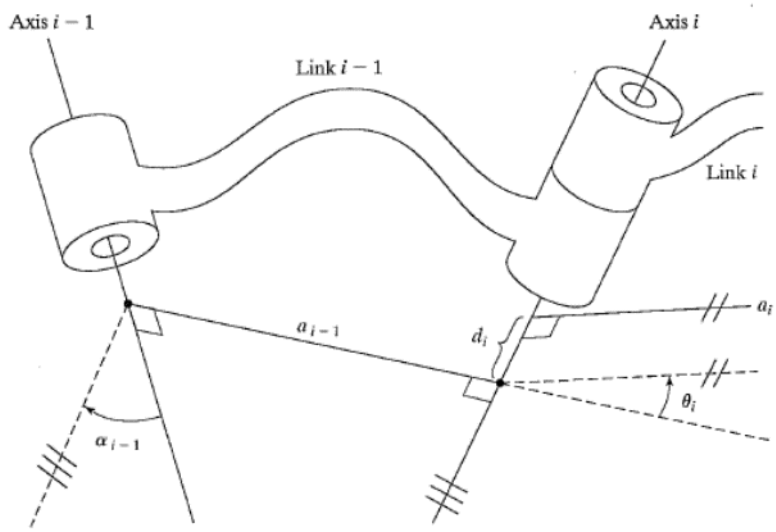
In kinematics, the following two quantities are used to describe the nature of the connection between neighbouring links and completely specify the way in which links are connected together.

1. Link offset (d): This parameter describes the distance along the common axis from one link to the next. The offset at joint i is denoted by d_i.
2. Joint angle (θ): This parameter describes the amount of rotation about the common axis between one link and its neighbour. The joint angle at joint i is denoted by θ_i.

NOTE: Neighbouring links have a common joint axis between them.

Given link lengths a_{i-1} and a_i, which corresponds to link $i-1$and i, respectively. The link offset d_i of joint i is the signed distance measured along the axis of joint i from the point where a_{i-1}intersects the axis to the point where a_i intersects the axis as shown in the figure below. The joint angle θ_i of joint i is the angle made between extensions of a_{i-1} and a_i measured about the axis of joint i.

NOTE: Generally for revolute joints, the link offset is fixed while the link angle is variable.



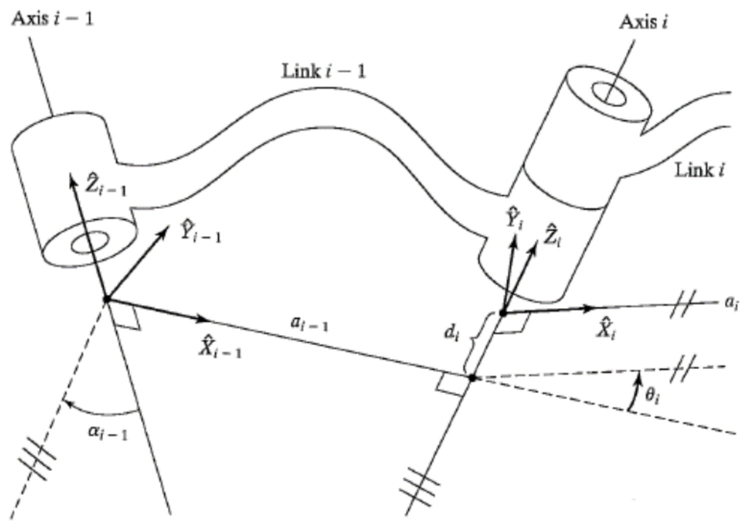
NOTE: Link length, a_i and link twist, α_i depend on joint axes i and $i+1$.

These four parameters, taken together, are generally used to kinematically describe robots. This description is known as the Denavit—Hartenberg notation. Generally, for revolute joints, only the joint angle is variable. All other parameters are fixed and are known as FIXED LINK PARAMETERS. The joint angle is known as the JOINT VARIABLE.

For our study, the following assumptions/conventions will be used:

1. The base and the end effector are end of the link chain and are generally considered immobile. The links are numbered from 0 to n, where the base is numbered 0 and the end effector numbered n.
2. The link length and link twist for both the base and end effector is assigned 0. I.e., $a_0 = a_n = 0$ and $\alpha_0 = \alpha_n = 0$. This is due to the fact that these parameters for the base and end effector play no part in the location of the end effector.
3. Since our focus is on revolute joints with one degree of freedom for each joint, the lint offset for the base, link 1 and the end effector are assigned 0. I.e.,$d_0 = d_1 = d_n = 0$.
4. The link frames are named by number according to the link to which they are attached. That is, frame $\{i\}$ is attached rigidly to link i.
5. The Z-axis of frame $\{i\}$, referred to as $\hat{Z}_i$, is coincident with the joint axis i.
6. The origin of frame $\{i\}$ is located where the a_i perpendicular intersects the joint i axis.
7. The X-axis, $\hat{X}_i$ points along a_i in the direction from joint i to joint $i+1$.
8. The Y-axis, $\hat{Y}_i$ is formed by the right-hand rule to complete the i^{th} frame.
9. Frame$\{0\}$ is attached to the base of the robot and is assumed stationary for our study. It can be considered the reference frame with which the locations of other frames may be described. Since Frame$\{0\}$ is generally arbitrarily chosen, it is common convention to choose $\hat{Z}_0$ along axis 1 and to place frame$\{0\}$ so that it coincides with Frame$\{1\}$ when joint variable 1 (θ_1) is 0.
10. Frame$\{n\}$ is attached to the end effector of the robot. It is common convention to choose $\hat{X}_n$ so that it aligns with $\hat{X}_{n-1}$ when joint variable n (θ_n) is 0.
11. The origin of frame$\{n\}$ is chosen so that $d_n = 0$.

The figure below shows the location of frames $\{i-1\}$ and $\{i\}$ for a general manipulator.



Using the conventions listed, the following relationships can be used to determine the link parameters in terms of the link frames:

- a_i = distance from $\hat{Z}_i$ to $\hat{Z}_{i+1}$ measured along $\hat{X}_i$
- α_i = angle from $\hat{Z}_i$ to $\hat{Z}_{i+1}$ measured along $\hat{X}_i$
- d_i = distance from $\hat{X}_{i-1}$ to $\hat{X}_i$ measured along $\hat{Z}_i$
- θ_i = angle from $\hat{X}_{i-1}$ to $\hat{X}_i$ measured along $\hat{Z}_i$

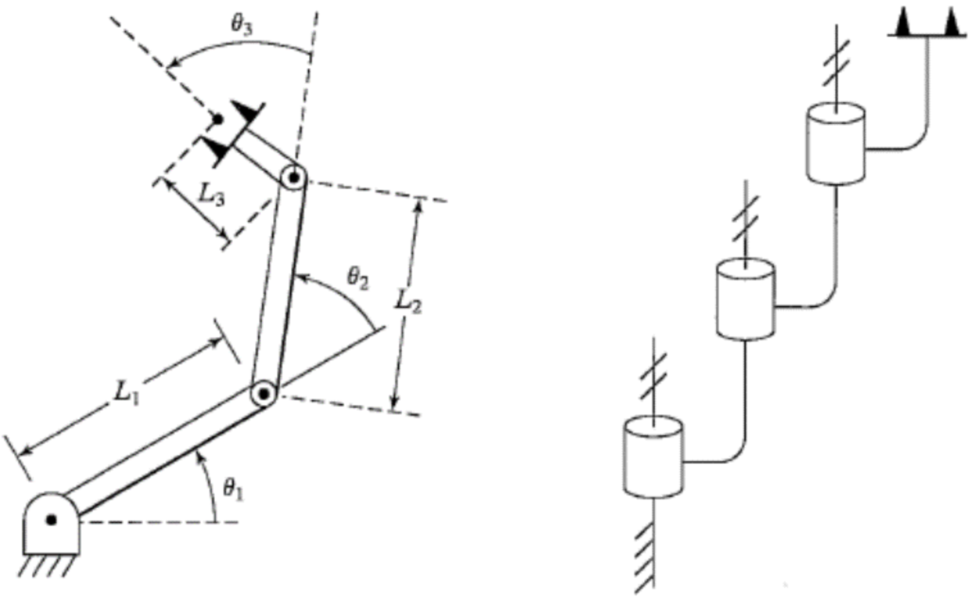
In summary, the following procedure can be used for link-frame attachment:

1. Identify the joint axes.
2. Identify the common perpendicular between the joint axes (point of intersection).
3. Assign the link-frame origin of the i^{th} joint at the point where the common perpendicular meets the i^{th} axis.
4. Assign the $\hat{Z}_i$ axis pointing along the i^{th} joint axis.
5. Assign the $\hat{X}_i$ axis pointing along the common perpendicular.
6. Assign the $\hat{Y}_i$ axis to complete a right-hand coordinate system.
7. Assign frame $\{0\}$ to match frame $\{1\}$ when the first joint variable is zero.
8. For frame $\{n\}$, choose an origin location and freely assign $\hat{X}_n$ direction so as to ensure that as many linkage parameters as possible equal zero.

EXAMPLE:

The figure to the left below shows a three-link planar arm also known as an RRR (or 3R) mechanism because all three joints are revolute. The goal is to assign link frames to the mechanism and give the Denavit-Hartenberg parameters.

NOTE: The figure to the right below shows a simple schematic notation for representing the same mechanism, where hash marks on the axes indicate that they are mutually parallel.



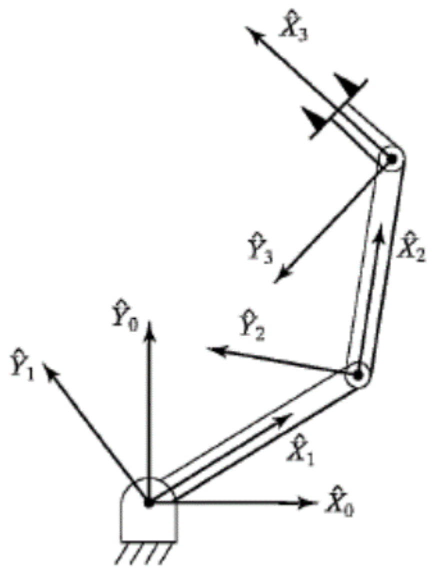
SOLUTION:

1. For this arm, all joint axes are oriented perpendicular to the plane of the arm axes and point out of the paper (normal to the page).
2. Because the arm lies in a plane, all $\hat{Z}$ axes are parallel.
3. Since all $\hat{Z}$ axes are parallel and lie in the same plane, there are no link offsets and no link twist. That is, all $d_i = \alpha_i = 0$.
4. All joints are revolute (rotational), so when they are at zero degrees, all $\hat{X}$ axes must align.
5. Assign frame $\{0\}$ to match frame $\{1\}$ when $\theta_1 = 0$ as shown in the figure below. Here $\hat{Z}_0$is aligned with joint 1 axis.

Using points 1 to 5 above, the Denavit-Hartenberg parameters of the three-link planar manipulator is shown in the table below.

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	L_1	0	θ_2
3	0	L_2	0	θ_3

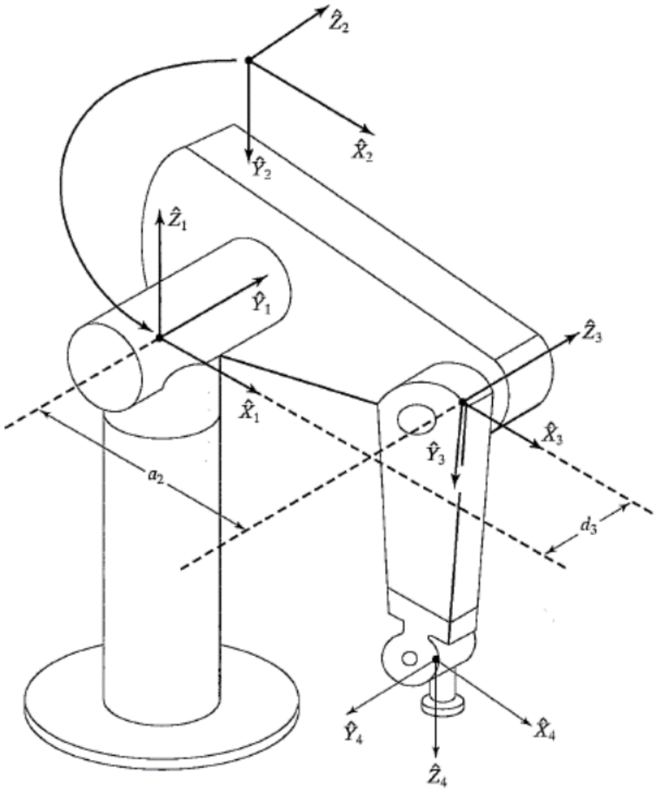
The figure below shows the frame assignment.



NOTE: The kinematic analysis always ends at a frame whose origin lies on the last joint axis, therefore, L_3 does not appear in the link parameters.

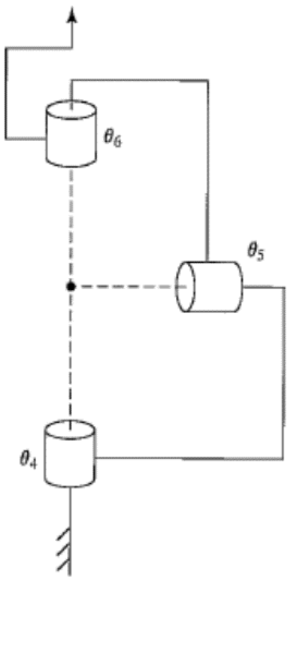
EXAMPLE: KINEMATICS OF UNIMATION PUMA 560 INDUSTRIAL ROBOT

The Unimation PUMA 560 is a rotary-joint manipulator with six degrees of freedom. I.e., it is a 6R mechanism. The figure below shows some kinematic parameters and frame assignments for the PUMA 560 manipulator.



NOTE:

- Frame $\{0\}$ is coincident with frame $\{1\}$ when $\theta_1 = 0$.
- Joints 2 and 3 are always parallel.
- This robot utilizes a wrist mechanism that is typical of many industrial robots as shown in the figure below. Joint axes 5 and 6 are located on the end effector (the gripper). The joint axes $4, 5,$ and 6 are mutually orthogonal and intersect at a common point, which coincides with the origin of frames $\{4\}, \{5\},$ and $\{6\}$.



The table below shows the link parameters of the PUMA 560.

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	$-90°$	0	0	θ_2
3	0	a_2	d_3	θ_3
4	$-90°$	a_3	d_4	θ_4
5	$90°$	0	0	θ_5
6	$-90°$	0	0	θ_6