The aim is to obtain the transform that defines a frame $\{i\}$ relative to frame $\{i-1\}$. In general, this transformation, ${}^{i-1}_{i}T$ will be a function of the four link parameters. For our manipulator, this transformation will be a function of the joint variable (joint angle) since the other three parameters are fixed by mechanical design. By defining a frame for each link, the kinematics problem is broken into a n sub problems. As previously explained in preceding lectures, the transformation used to map vectors defined with reference to $\{i\}$ to their description with reference to $\{i-1\}$ is given by:

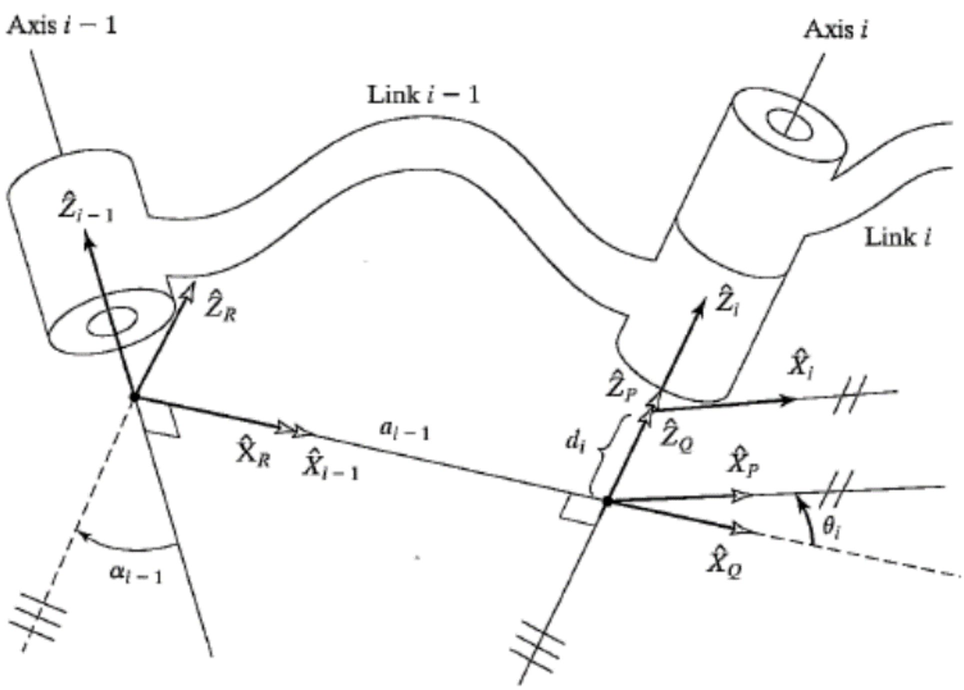
$${}^{i-1}P = {}^{i-1}_{i}T \; {}^{i}P$$

Generally, the transformation ${}^{i-1}_{i}T$ is a homogenous transform that can be expressed as:

$${}^{i-1}_{i}T = \begin{bmatrix} & {}^{i-1}_{i}R & & {}^{i-1}P_{iORG} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

However, for the general case as can be observed in the figure below, the vector ${}^{i-1}P_{iORG}$ which links the origin of frame $\{i\}$ to the origin of frame $\{i-1\}$ isn't well defined. This is because a number of sub transformations exist between the two frames. Concretely, to get the frame $\{i\}$ from frame $\{i-1\}$, the following operations are done.

1. First, rotate frame $\{i-1\}$ by α_{i-1}
2. Then, translate the resulting frame by a_{i-1}
3. Next, Rotate the resulting frame by θ_i
4. Finally, translate the resulting frame by d_i



Therefore, the transformation ${}^{i-1}_{i}T$ could be viewed as a product of four sub transformations, which result in the general form for ${}^{i-1}_{i}T$ given below:

$$
{}^{i-1}_{i}T = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 & a_{i-1} \\ \sin\theta_i \cos\alpha_{i-1} & \cos\theta_i \cos\alpha_{i-1} & -\sin\alpha_{i-1} & -\sin\alpha_{i-1} d_i \\ \sin\theta_i \sin\alpha_{i-1} & \cos\theta_i \sin\alpha_{i-1} & \cos\alpha_{i-1} & \cos\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

EXAMPLE

Using the link parameters of the three-link planar arm, compute the individual transformations for each link.

The Denavit-Hartenberg parameters of the three-link planar manipulator is shown in the table below:

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	L_1	0	θ_2
3	0	L_2	0	θ_3

$$
{}^{0}_{1}T = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{1}_{2}T = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & L_1 \\ \sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{2}_{3}T = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & L_2 \\ \sin\theta_3 & \cos\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

FORWARD KINEMATICS

The goal of forward kinematics is to obtain the location of the end effector relative to the base. This is achieved by using the homogenous transform matrix ${}^{0}_{n}T$, which relates the end effector frame to the base frame.

Once the link frames have been established and the corresponding link parameters obtained, the individual link-transformation matrices ${}^{i-1}_{i}T$ can be computed. Multiplying the link transformation matrices yields the single homogenous transform matrix ${}^{0}_{n}T$ that relates the end effector frame n to the base frame 0 as shown below:

$$ {}^{0}_{n}T = {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T \cdots {}^{n-1}_{n}T $$

For the three-link planar manipulator that we have considered, the homogenous transform matrix is given by:

$$ {}^{0}_{3}T = {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T $$

$$ = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & L_1 \\ \sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & L_2 \\ \sin\theta_3 & \cos\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Using the following trigonometric identities,

$$ \begin{aligned} \cos(\theta_1 + \theta_2) &= c_{12} = c_1c_2 - s_1s_2 \\ \sin(\theta_1 + \theta_2) &= s_{12} = c_1s_2 + s_1c_2 \\ \cos(\theta_1 - \theta_2) &= c_1c_2 + s_1s_2, \\ \sin(\theta_1 - \theta_2) &= s_1c_2 - c_1s_2. \end{aligned} $$

It can be shown that

$$ {}^{0}_{3}T = \begin{bmatrix} \cos(\theta_1+\theta_2+\theta_3) & -\sin(\theta_1+\theta_2+\theta_3) & 0 & L_1\cos(\theta_1) + L_2\cos(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2+\theta_3) & \cos(\theta_1+\theta_2+\theta_3) & 0 & L_1\sin(\theta_1) + L_2\sin(\theta_1+\theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

For the puma560, the individual transformations for each link is as follows:

$$ {}^{0}_{1}T = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{1}_{2}T = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s\theta_2 & -c\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{2}_{3}T = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & a_2 \\ s\theta_3 & c\theta_3 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{3}_{4}T = \begin{bmatrix} c\theta_4 & -s\theta_4 & 0 & a_3 \\ 0 & 0 & 1 & d_4 \\ -s\theta_4 & -c\theta_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{4}_{5}T = \begin{bmatrix} c\theta_5 & -s\theta_5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s\theta_5 & c\theta_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{5}_{6}T = \begin{bmatrix} c\theta_6 & -s\theta_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s\theta_6 & -c\theta_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

From which the homogenous transform matrix that relates frame $\{6\}$ to frame $\{0\}$ is given by:

$$ {}^{0}_{6}T = {}^{0}_{1}T\,{}^{1}_{6}T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Here,

$$
\begin{aligned}
r_{11} &= c_1[c_{23}(c_4c_5c_6 - s_4s_5) - s_{23}s_5c_5] + s_1(s_4c_5c_6 + c_4s_6), \\
r_{21} &= s_1[c_{23}(c_4c_5c_6 - s_4s_6) - s_{23}s_5c_6 - c_1(s_4c_5c_6 + c_4s_6), \\
r_{31} &= -s_{23}(c_4c_5c_6 - s_4s_6) - c_{23}s_5c_6, \\
\\
r_{12} &= c_1[c_{23}(-c_4c_5s_6 - s_4c_6) + s_{23}s_5s_6] + s_1(c_4c_6 - s_4c_5s_6), \\
r_{22} &= s_1[c_{23}(-c_4c_5s_6 - s_4c_6) + s_{23}s_5s_6] - c_1(c_4c_6 - s_4c_5s_6), \\
r_{32} &= -s_{23}(-c_4c_5s_6 - s_4c_6) + c_{23}s_5s_6, \\
\\
r_{13} &= -c_1(c_{23}c_4s_5 + s_{23}c_5) - s_1s_4s_5, \\
r_{23} &= -s_1(c_{23}c_4s_5 + s_{23}c_5) + c_1s_4s_5, \\
r_{33} &= s_{23}c_4s_5 - c_{23}c_5, \\
\\
p_x &= c_1[a_2c_2 + a_3c_{23} - d_4s_{23}] - d_3s_1, \\
p_y &= s_1[a_2c_2 + a_3c_{23} - d_4s_{23}] + d_3c_1, \\
p_z &= -a_3s_{23} - a_2s_2 - d_4c_{23}.
\end{aligned}
$$

The position of all the links of a manipulator of n degrees of freedom can be specified with a set of n joint variables. This set of variables is often referred to as the $n \times 1$ joint vector. The space of all such joint vectors is referred to as joint space.

NOTE: Although it has been implicitly assumed that each kinematic joint is actuated directly by a separate actuator, it should be noted that in the case of many industrial robots, this is not so. Sometimes two actuators work together in a pair to move a single joint, or sometimes a linear actuator is used to rotate a revolute joint, through the use of linkages, and son on.