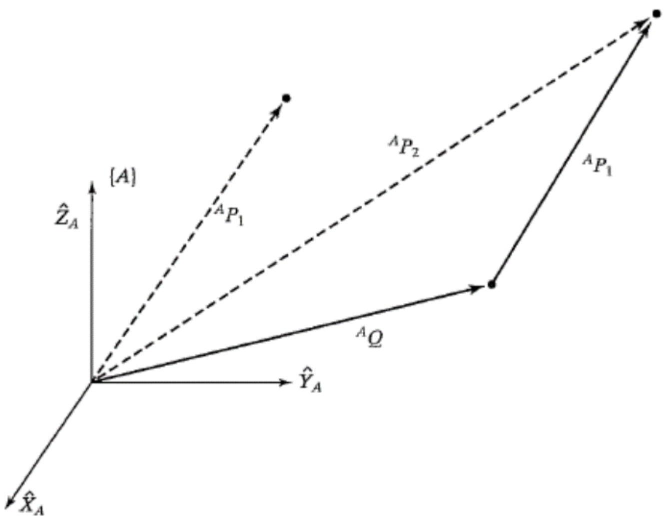
The process of mapping points between frames can be generalized and also applied to translate points, rotate vectors, or do both.

- Translational operators:
  A translation moves a point in space a finite distance along a given vector direction. Translating a point in space is accomplished with the same mathematics as mapping the point to a second frame. The Figure below shows the translation of vector ${}^{A}P_1$ by vector ${}^{A}Q$.



  The consequence of the translation is ${}^{A}P_2$ given by

$${}^{A}P_2 = {}^{A}P_1 + {}^{A}Q$$

  Which is similar to the equation used to map translated frames with no rotation. To write this translation in a compact form, a homogeneous transform matrix operator $D_Q(q)$ is defined as shown below.

$${}^{A}P_2^* = D_Q(q)\ {}^{A}P_1^*$$

  Here, q is the signed magnitude of the translation along the vector direction $\widehat{Q}$.

$$D_Q(q) = \begin{bmatrix} 1 & 0 & 0 & q_x \\ 0 & 1 & 0 & q_y \\ 0 & 0 & 1 & q_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Rotational operators:
  Likewise, rotating a vector ${}^{A}P_1$ to a new vector ${}^{A}P_2$ is accomplished in exactly the same way that mapping involving rotated frames and no translation is carried out. For the sake of consistency, a rotational operator that clearly indicates which axis is being rotated about is defined below:

$${}^{A}P_2 = R_k(\theta)\ {}^{A}P_1$$

  Here, $R_k(\theta)$ is a homogeneous transform matrix operator that performs a rotation about the axis direction $\widehat{K}$ by θ degrees and is given below. Here, $R(\theta)$ is the rotation matrix for rotation in direction $\widehat{K}$ by θ degrees.

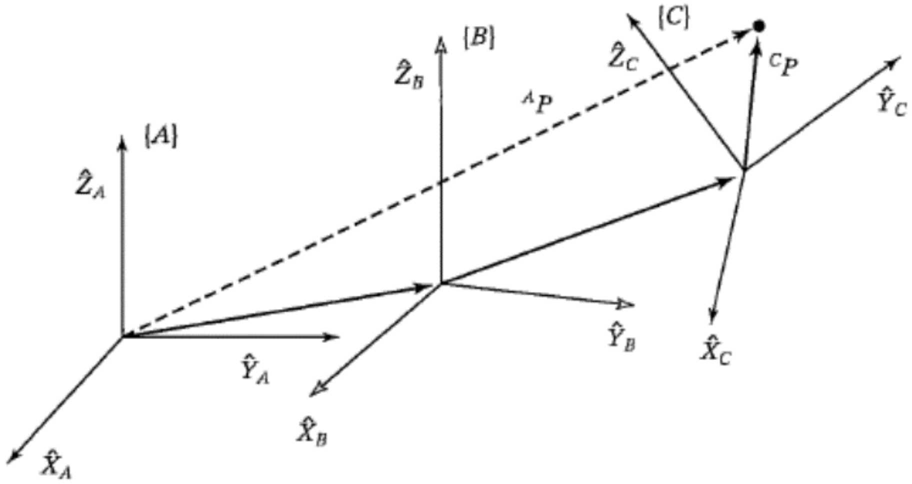
$$R_k(\theta) = \begin{bmatrix} & R(\theta) & & 0 \\ & & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Transformation operators:
  As explained in the previous lecture, mapping involving both translated and rotated frames can be expressed in compact form using a homogenous transform matrix ${}^{A}_{B}T$. However, for the general case where only one coordinate system is involved, symbol T is used without subscripts or superscripts. The operator T rotates and translates a vector ${}^{A}P_1$ to compute a new vector ${}^{A}P_2$ as shown below
  $${}^{A}P_2 = T\ {}^{A}P_1$$
  NOTE: Homogeneous transforms uses 4×4 matrices that contain orientation and position information. The term transformation is a generalization of translation and rotation. Hence, the term transform is often used even when speaking of a pure rotation or translation.

TRANSFORMATION ARITHMETIC

Multiplication of transforms and inversion of transforms are two important elementary operations.

- Compound transformations
  In the figure below, we are given ${}^{C}P$ and are required to find ${}^{A}P$.



  Frame {C} is known relative to frame {B}, and frame {B} is known relative to frame {A}. Hence, ${}^{C}P$ can be transformed into ${}^{B}P$ as below:
  $${}^{B}P = {}^{B}_{C}T\ {}^{C}P$$
  Likewise, ${}^{B}P$ can be transformed into ${}^{A}P$ as below:
  $${}^{A}P = {}^{A}_{B}T\ {}^{B}P$$
  Thus combining the equations, we get
  $${}^{A}P = {}^{A}_{B}T\ {}^{B}_{C}T\ {}^{C}P$$
  Resulting in the following equation
  $${}^{A}_{C}T = {}^{A}_{B}T\ {}^{B}_{C}T$$
  Equivalently, the homogenous 4×4 transform matrix is obtained as
  $${}^{A}_{C}T = \begin{bmatrix} {}^{A}_{B}R\ {}^{B}_{C}R & {}^{A}_{B}R\ {}^{B}P_{CORG} + {}^{A}P_{BORG} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}$$

- Inverting a transform

  Consider a frame {B} that is known with respect to a frame {A}. To get a description of {A} relative to {B}, we need to invert the transform. Here, we know ${}^{A}_{B}T$ and need to find ${}^{B}_{A}T$.

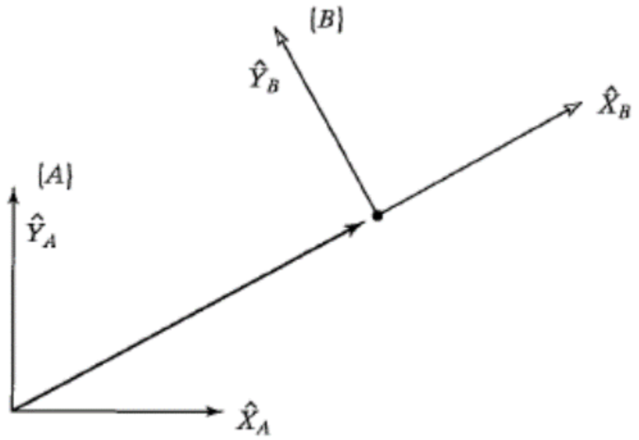
$$ {}^{B}_{A}T = {}^{A}_{B}T^{-1} $$

  The following 4×4 homogenous transform matrix can be used to compute ${}^{B}_{A}T$.

$$ {}^{B}_{A}T = \begin{bmatrix} & {}^{A}_{B}R^{T} & & -{}^{A}_{B}R^{T}\ {}^{A}P_{BORG} \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

EXAMPLE 2.5

The figure below shows a frame {B} that is rotated relative to frame {A} about $\hat{Z}$ by 30 degrees and translated four units in $\hat{X}_A$ and three units in $\hat{Y}_A$. Thus, we have a description of ${}^{A}_{B}T$. Find ${}^{B}_{A}T$.



The frame defining {B} is

$$ {}^{A}_{B}T = \begin{bmatrix} 0.866 & -0.500 & 0.000 & 4.000 \\ 0.500 & 0.866 & 0.000 & 3.000 \\ 0.000 & 0.000 & 1.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 1.000 \end{bmatrix} $$

Therefore

$$ {}^{B}_{A}T = \begin{bmatrix} 0.866 & 0.500 & 0.000 & -4.964 \\ -0.500 & 0.866 & 0.000 & -0.598 \\ 0.000 & 0.000 & 1.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 1.000 \end{bmatrix} $$