3D Transformations

CMPT 361

Introduction to Computer Graphics

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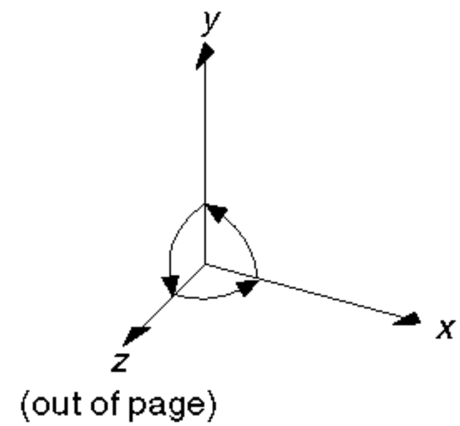
Schedule

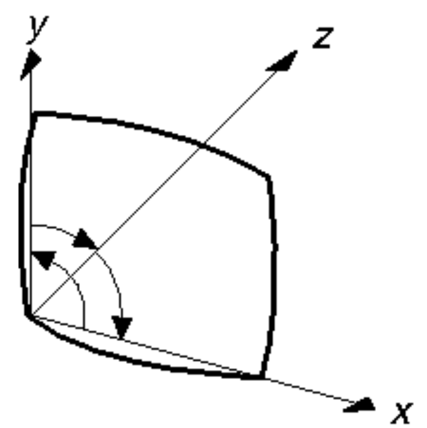
- *Geometry basics*
- *Affine transformations*
- *Use of homogeneous coordinates*
- *Concatenation of transformations*
- 3D transformations
- Transformation of coordinate systems
- Transform the transforms
- Transformations in OpenGL

Transformations in 3D

- Add a z-axis to (x, y) plane
  - right-handed system:
    - positive z pointing towards us
    - positive rotation counter-clockwise
    - Standard math convention (used in our presentation and OpenGL)



  - left-handed system:
    - positive z pointing away from us
    - positive rotation clockwise
    - Used in some graphics systems (z-axis as depth), e.g., POV-Ray, and Renderman



Translation in 3D

- Again, we use homogeneous coordinates

$$T(t_x, t_y, t_z) = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Scaling in 3D

$$S(s_x, s_y, s_z) = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation in 3D

- Around z-axis

$$R_z(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Around x-axis

$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Around y-axis

$$R_y(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Properties of rotation matrix

- Property 1: columns and rows are mutually orthogonal unit vectors, i.e, orthonormal
- Property 2: determinant of M = 1

$$M = \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- product of any pair of orthonormal matrices is also orthonormal
- orthonormality: inverse = transpose ($P^T = P^{-1}$)

Another nice property

- row vectors: unit vectors which rotate into principal axes, i.e., $[1\ 0\ 0]^T$, $[0\ 1\ 0]^T$, and $[0\ 0\ 1]^T$
- column vectors: unit vectors into which principle axes rotate (obviously)

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} r_{11} \\ r_{12} \\ r_{13} \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} r_{21} \\ r_{22} \\ r_{23} \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} r_{31} \\ r_{32} \\ r_{33} \end{bmatrix}$$

Shearing in 3D

- In (y, z) w.r.t. x value $SH_{yz} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ sh_y & 1 & 0 & 0 \\ sh_z & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

- In (z, x) w.r.t. y value $SH_{xz} = \begin{bmatrix} 1 & sh_x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & sh_z & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

- In (x, y) w.r.t. z value $SH_{xy} = \begin{bmatrix} 1 & 0 & sh_x & 0 \\ 0 & 1 & sh_y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Inverse Transforms

- Translation: negate t_x, t_y, t_z
- Scaling: change s_x to $1/s_x$, etc.
- Rotation: negate the angle
- Shearing: negate sh_y, sh_z, etc.

General 3D transformations

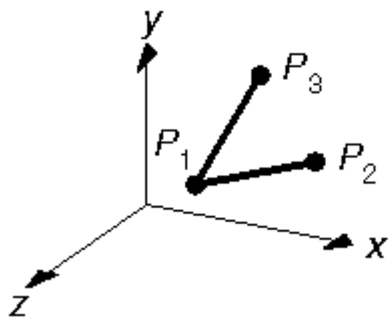
- Any arbitrary sequence of rotation, translation scaling, and shear can be represented as:

$$M = \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

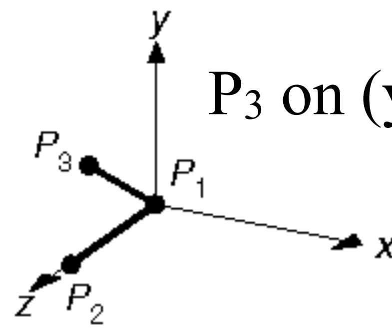
- where upper left 3×3 is the combined scaling, rotation, and shearing; $[t_x \ \ t_y \ \ t_z]^T$ for translation

Compound transforms

- Just like in 2D, however …
- Rotation is about more than one axis



(a) Initial position



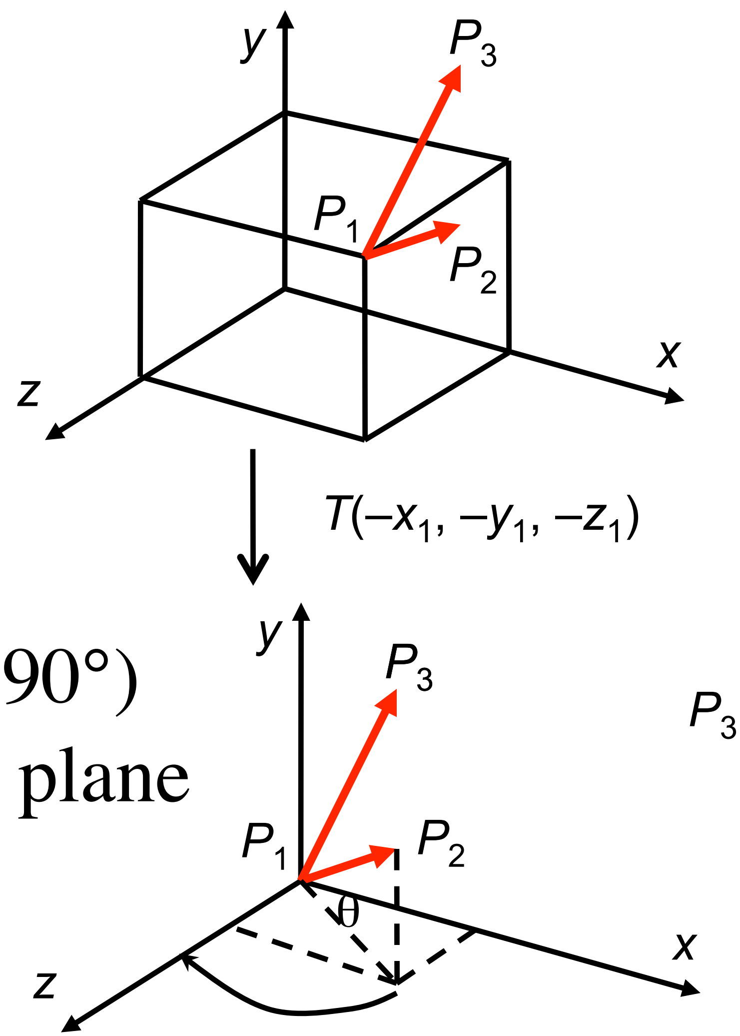
(b) Final position

P_3 on (y, z) plane

- How should we do this?

Compute compound transform

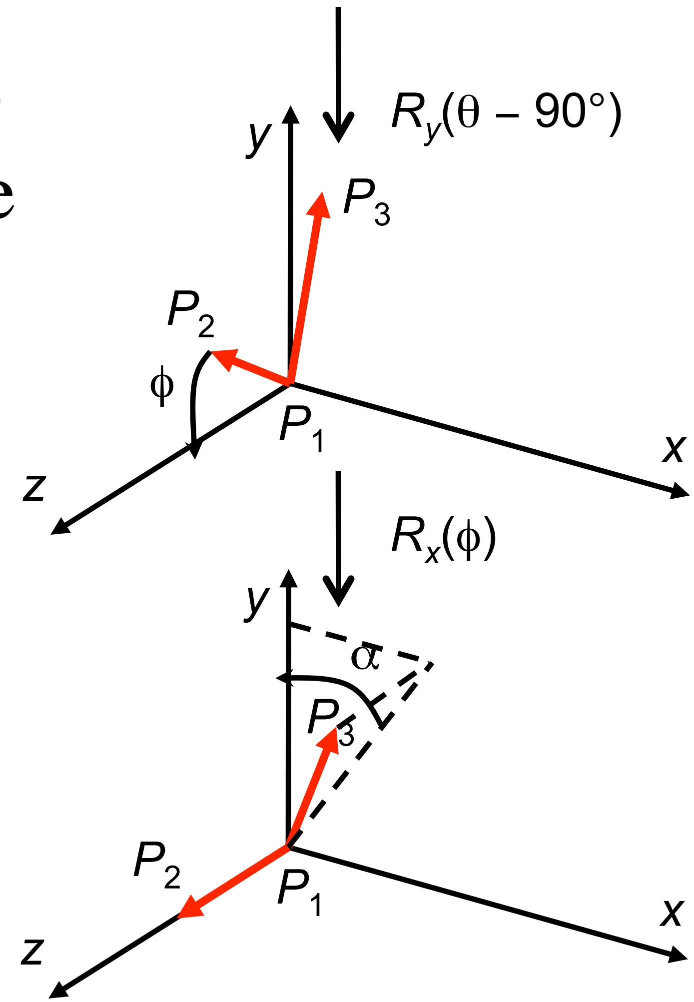
- Use of right-hand CS
- Translation by P_1 so that P_1 is at origin
- Rotation about y by $(\theta - 90°)$ to get P_1P_2 onto the (y, z) plane



Compute compound transform

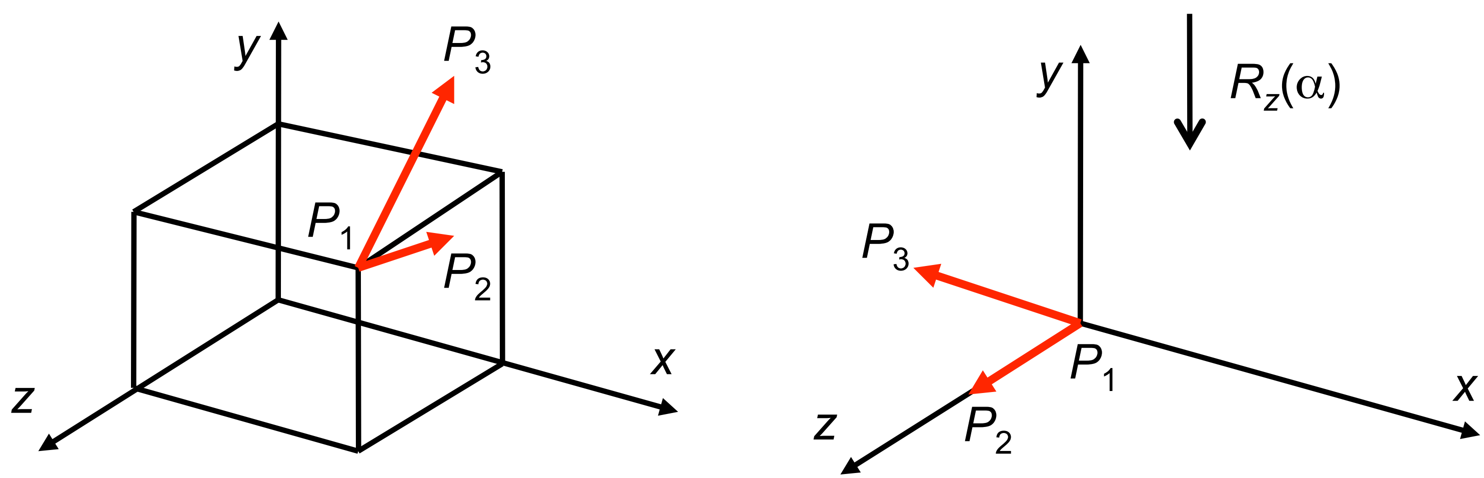
- Rotation about x by ϕ to get P_1P_2 to align with the positive z-axis

- Rotation about z by α to get P_1P_3 onto the (y, z) plane



Compute compound transform

- Combined transformation:



$$R_z(\alpha) \times R_x(\phi) \times R_y(\theta - 90^\circ) \times T(-P_1)$$

Alternative composition

- Recall the "nice" properties of the rotation matrices:

$$R = \begin{bmatrix} r_{1x} & r_{2x} & r_{3x} \\ r_{1y} & r_{2y} & r_{3y} \\ r_{1z} & r_{2z} & r_{3z} \end{bmatrix} = \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix}$$

- R_i's are the unit row vectors which rotate into principal coordinate axes, e.g., $RR_x^T = [1\ 0\ 0]^T$
- Let us try to construct these directly, assuming the translation $T(-P_1)$ is already done.

Alternative composition

- the unit vector to move to lie on the positive z axis is:

$$R_z = \frac{\overline{P_1P_2}}{\left|\overline{P_1P_2}\right|}$$

- the unit vector that rotates into x is normal to the plane $P_1P_2P_3$.

$$R_x = \frac{\overline{P_1P_3} \times \overline{P_1P_2}}{\left|\overline{P_1P_3} \times \overline{P_1P_2}\right|}$$

(a) Initial position (b) Final position

Alternative composition

- By definition, $R_z \times R_x$ must rotate into the remaining y-axis and:

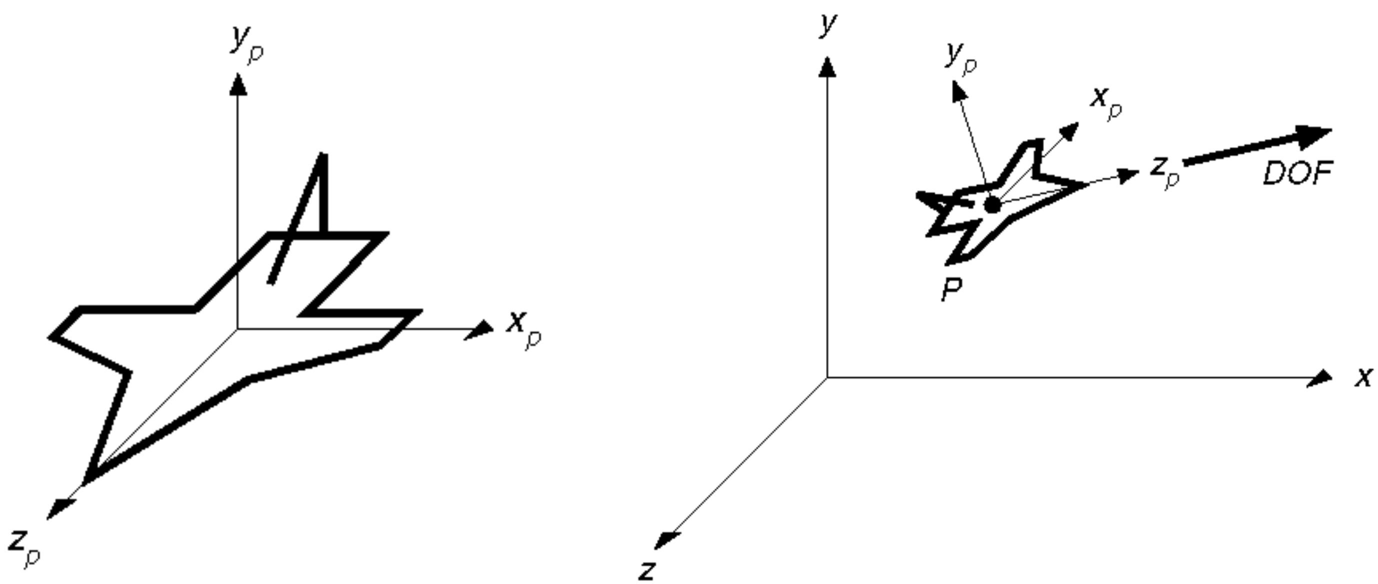
$$R_y = R_z \times R_x$$

- We are done:

$$M = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} \times T(-P_1), \text{where } R = \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix}$$

Exercise

- How to get the jet into the desired direction of flight (DOF)?



Special transformations

- Points: we have been doing this so far
- Lines: just transform the endpoint of a line
- Planes: trickier
  - if defined by 3 points, can transform points,
  - but ... more often defined by a plane equation

$$Ax + By + Cz + D = 0$$

Plane transform

- By homogeneous coordinates we can write:

$$N = \begin{bmatrix} A & B & C & D \end{bmatrix}^T$$

- with P = $[x\ y\ z\ 1]^T$:

$$N^T P = 0$$

- Now, suppose we want to transform our space by matrix M
- To maintain $N^T P = 0$, we must also transform N. Let this transform be Q.

Plane transform: derivation

- After the transform we have:

$$N_n = QN \qquad P_n = MP$$

- and we would like to have:

$$N_n^T P_n = 0$$

- now some algebra:

$$\begin{aligned} N_n^T P_n &= (QN)^T (MP) \\ &= N^T (Q^T M) P \\ &= 0 \end{aligned}$$

Plane transform: result

- This will hold when:

$$Q^T M = kI$$

- hence:

$$Q = M^{-T}$$

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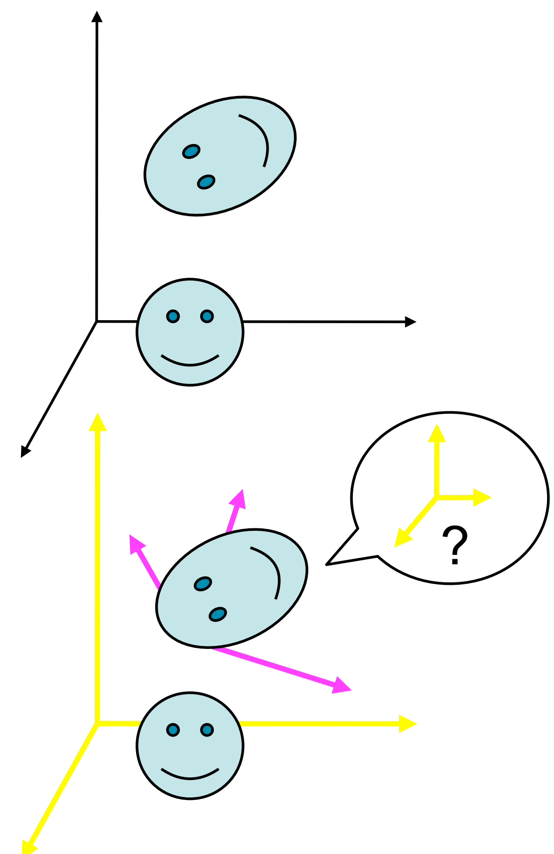
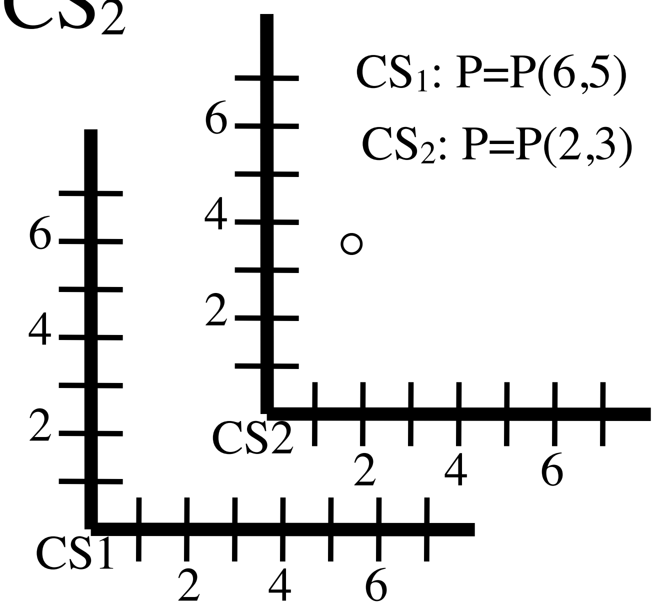
Transformation of CS

- So far: transform points on one object with respect to the **same** coordinate system (CS)
- Sometimes need to change CS
- e.g. we may have many objects, each in its own CS, and we want to express all of them in some GLOBAL CS



Transformation of CS

- $P^{(i)}$ = point in coordinate system i
- $M_{2\leftarrow 1}$ converts representation of point in CS_1 to representation of point in CS_2
- Alternate interpretation: $M_{2\leftarrow 1}$ transforms axes of CS_2 into axes of CS_1



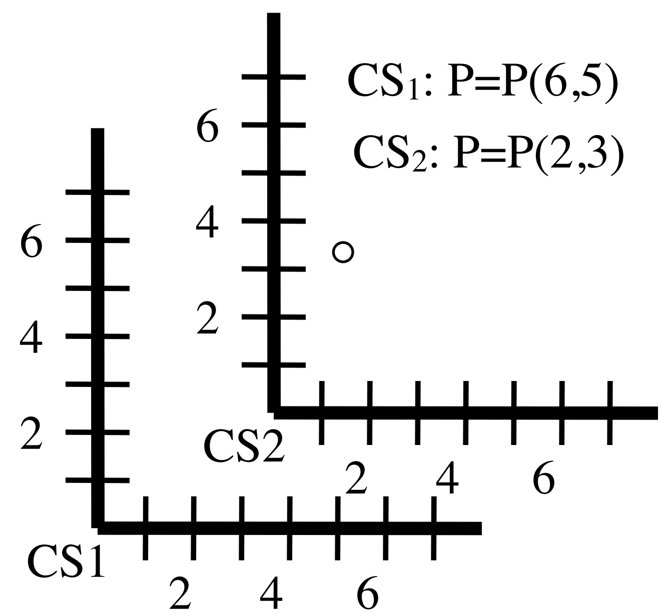
Derivation

- By definition: $P^{(2)} = M_{2\leftarrow 1} P^{(1)}$
- Hence: $CS_2 P^{(2)} = CS_1 P^{(1)}$
- Therefore: $CS_2 M_{2\leftarrow 1} = CS_1$

- with other words,
  $M_{2\leftarrow 1}$ transforms CS_2 into CS_1

Transform of CS: example

- Example: $M_{2\leftarrow 1} = T(-4, -2)$, this is seen by inspection
- $(2,3)^T = T(-4, -2)(6,5)^T$
- $CS_1 = CS_2\ T(-4, -2)$

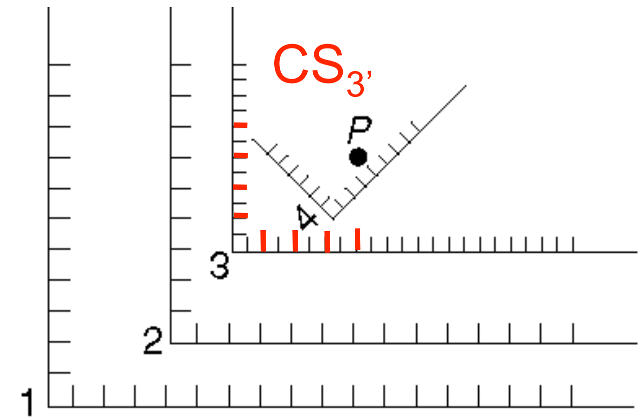


Transform of CS: transitivity

- Observe transitivity of this operator:
  - Given $$P^{(j)} = M_{j \leftarrow i} P^{(i)}$$ $$P^{(k)} = M_{k \leftarrow j} P^{(j)}$$
  - then $$P^{(k)} = M_{k \leftarrow j} M_{j \leftarrow i} P^{(i)}$$
  - so that $$M_{k \leftarrow i} = M_{k \leftarrow j} M_{j \leftarrow i}$$
- this is/was our basic concatenation of transformations

Transformation of CS

- Example: what is $M_{2\leftarrow 3}$?
  - $M_{2\leftarrow 3} = M_{2\leftarrow 3'} M_{3'\leftarrow 3}$ with $CS_{3'}$ aligned with CS_3 but having the same scale as CS_2
  - $M_{3'\leftarrow 3}$ transforms $CS_{3'}$ to CS_3: $S(0.5, 0.5)$
  - $M_{2\leftarrow 3'}$ transforms CS_2 to $CS_{3'}$: $T(2, 3)$
  - $M_{2\leftarrow 3} = T(2, 3)S(0.5, 0.5)$
  - Verify:
  - Alternative: $M_{2\leftarrow 3} = S(0.5, 0.5)\ T(4, 6)$



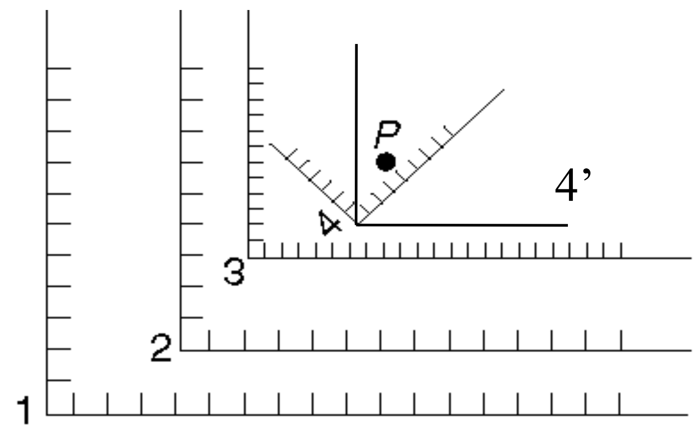
$$P^{(1)} = (10,8) \quad P^{(2)} = (6,6)$$

$$P^{(3)} = (8,6) \quad P^{(4)} = (4,2)$$

$$\begin{bmatrix} 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 8 \\ 6 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Transform of CS: example

- What is $M_{3 \leftarrow 4}$?
  - $M_{3\leftarrow 4} = M_{3\leftarrow 4'} M_{4'\leftarrow 4}$ with $CS_{4'}$ as shown
  - $M_{4'\leftarrow 4}$ transforms $CS_{4'}$ to CS_4: R(+45)
  - $M_{3\leftarrow 4'}$ transforms CS_3 to $CS_{4'}$: T(6.7, 1.8)
  - $M_{3\leftarrow 4}$ = T(6.7, 1.8)R(+45)
  - Verify:



$$P^{(1)} = (10,8) \quad P^{(2)} = (6,6)$$

$$P^{(3)} = (8,6) \quad P^{(4)} = (4,2)$$

$$\begin{bmatrix} 8 \\ 6 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 6.7 \\ 1.8 \end{bmatrix}$$

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Transforming the transforms

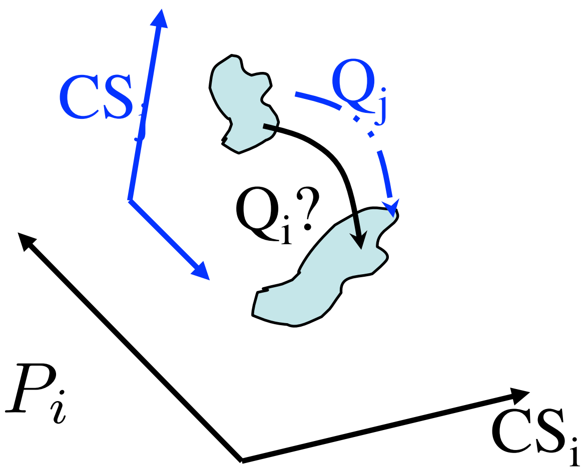
- Suppose Q_j is a transformation in CS_j
- Need Q_i that acts on points, with respect to CS_i, just like Q_j would on the same points
- Assume that we know $M_{i \leftarrow j}$

$$P_i = M_{i\leftarrow j}P_j \quad P_i' = M_{i\leftarrow j}P_j'$$

$$P_i' = M_{i\leftarrow j}P_j' \qquad = M_{i\leftarrow j}Q_jP_j$$

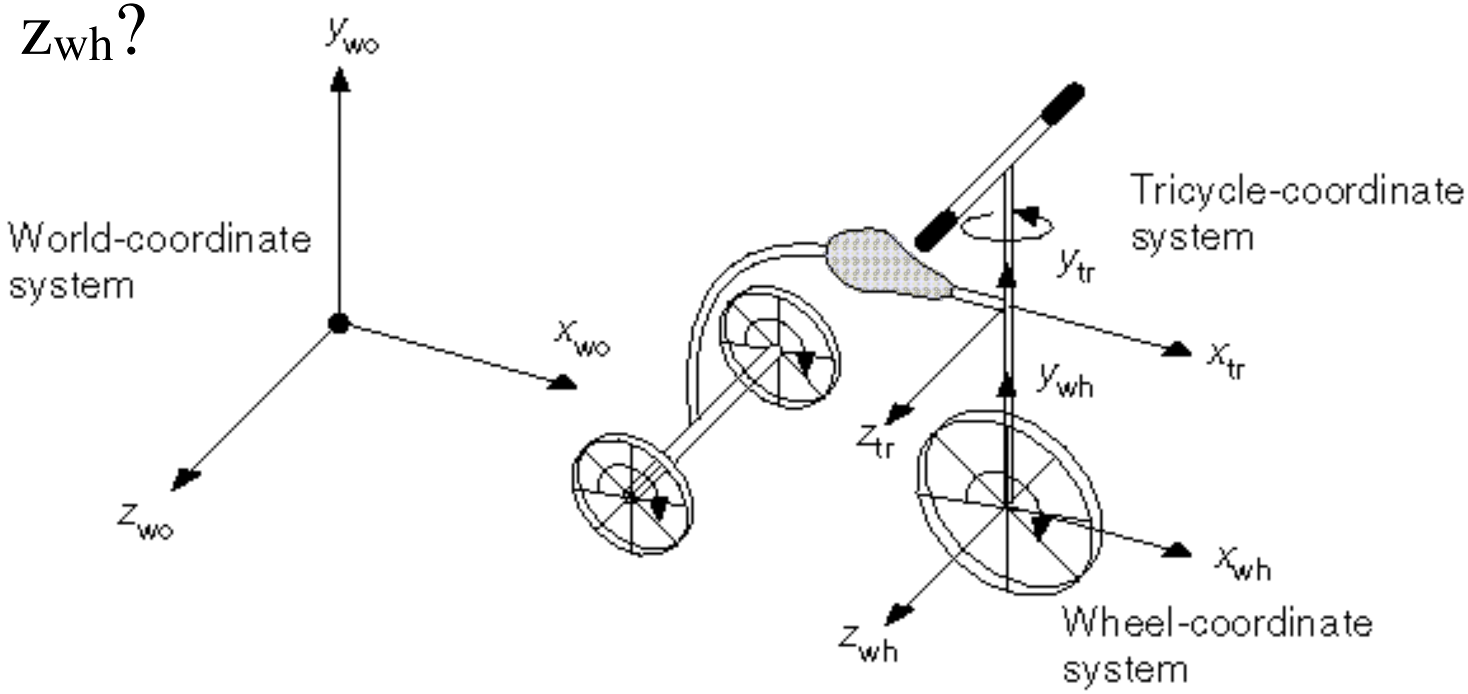
$$P_j' = Q_jP_j \qquad = [M_{i\leftarrow j}Q_jM_{i\leftarrow j}^{-1}]P_i$$

$$Q_i \quad = M_{i\leftarrow j}Q_jM_{i\leftarrow j}^{-1}$$



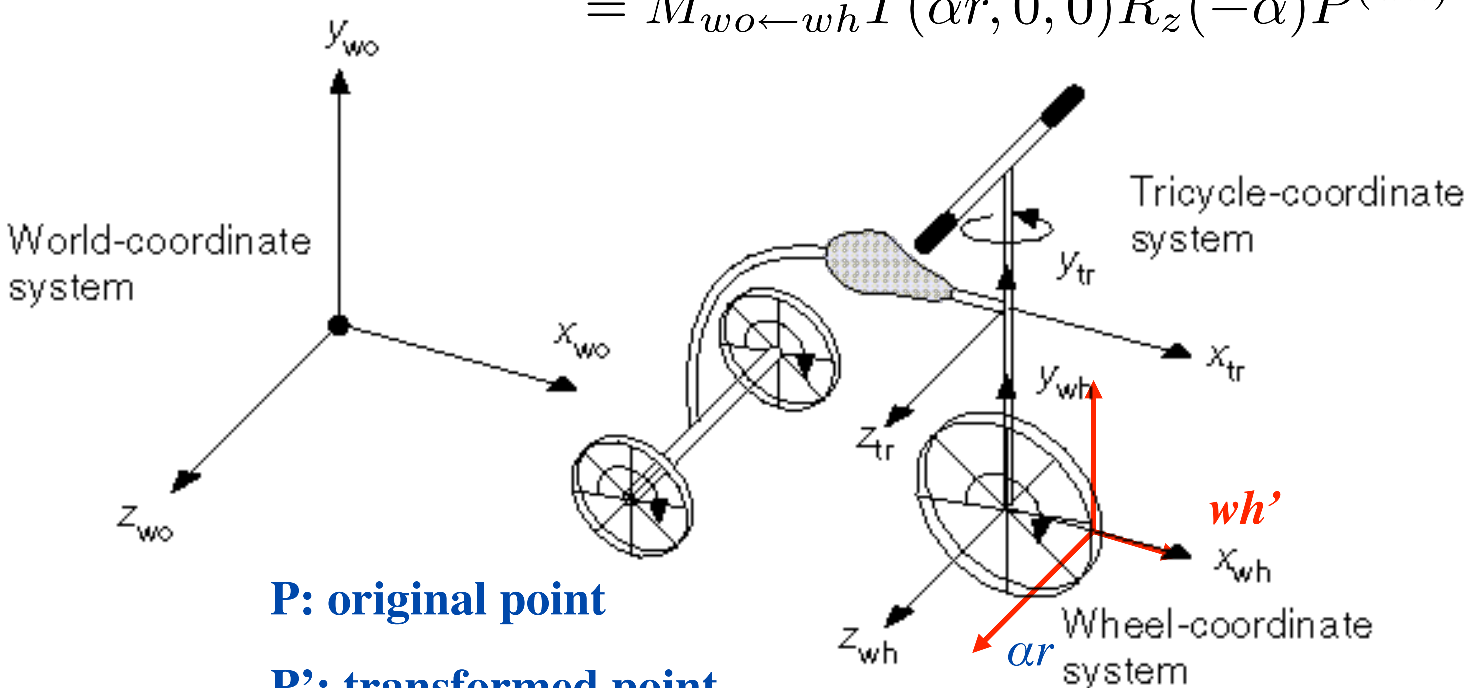
Transforming the transforms

Example: How does a point P on the front tricycle wheel move in the world CS (wo) when the wheel rotates forward by an angle of α about its own z_{wh}?



Transforming the transforms

$$
\begin{aligned}
P'^{(wo)} &= M_{wo \leftarrow wh} P'^{(wh)} \\
&= M_{wo \leftarrow wh} M_{wh \leftarrow wh'} P'^{(wh')} \\
&= M_{wo \leftarrow wh} T(\alpha r, 0, 0) P'^{(wh')} \\
&= M_{wo \leftarrow wh} T(\alpha r, 0, 0) R_z(-\alpha) P^{(wh)}
\end{aligned}
$$



P: original point

P': transformed point

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Coordinate Systems

- The units in **`points`** are determined by the application and are called
  - *object* (or *model*) coordinates
  - *world* coordinates

model view transform

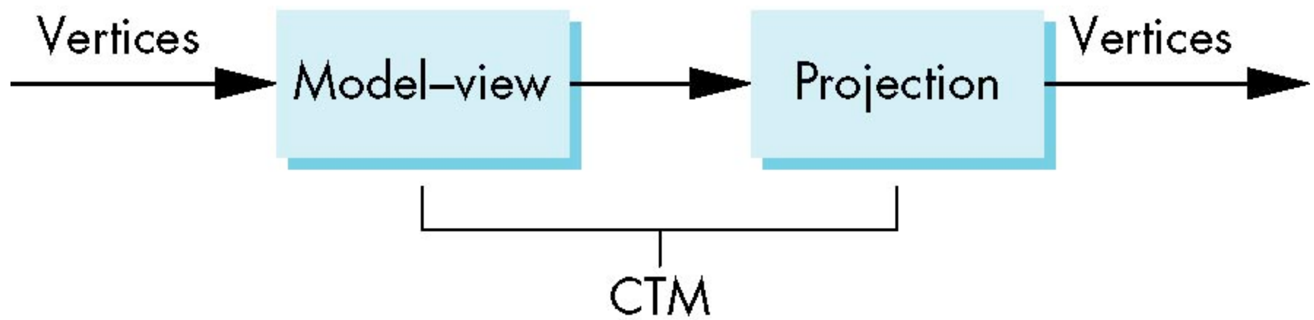
- Viewing specifications usually are also in object coordinates
- transformed through
  - *eye* (or *camera*) coordinates
  - *clip* coordinates
  - normalized *device* coordinates
  - *window* (or *screen*) coordinates

projection transform

- OpenGL also uses some internal representations that usually are not visible to the application but are important in the shaders

CTM in OpenGL

- OpenGL had a model-view and a projection matrix in the pipeline which were concatenated together to form the CTM
- Angel emulates this process



Rotation, Translation, Scaling

Create an identity matrix:

```
mat4 m = Identity();
```

Multiply on right by rotation matrix of **`theta`** in degrees where (**`vx, vy, vz`**) define axis of rotation

```
mat4 r = Rotate(theta, vx, vy, vz)
m = m*r;
```

Do same with translation and scaling:

```
mat4 s = Scale( sx, sy, sz)
mat4 t = Transalate(dx, dy, dz);
m = m*s*t;
```

Rotation, Translation, Scaling

- Create an identity matrix:

```
mat4 m = Identity();
```

- Multiply on right by rotation matrix of **`theta`** in degrees where (**`vx`**, **`vy`**, **`vz`**) define axis of rotation

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mat4 s = Scale( sx, sy, sz)
mat4 t = Transalate(dx, dy, dz);
m = m*s*t;
```

Example

- Rotation about z axis by 30 degrees with a fixed point of (1.0, 2.0, 3.0)

```
mat4 m = Identity();
m = Translate(1.0, 2.0, 3.0)*
  Rotate(30.0, 0.0, 0.0, 1.0)*
  Translate(-1.0, -2.0, -3.0);
```

- Remember that last matrix specified in the program is the first applied

Arbitrary Matrices

- Can load and multiply by matrices defined in the application program
- Matrices are stored as one dimensional array of 16 elements which are the components of the desired 4 x 4 matrix stored by columns
- OpenGL functions that have matrices as parameters allow the application to send the matrix or its transpose