Introduction to Laboratory 11

Laboratory 11 concerns the implementation of a **Monte Carlo integration method** to calculate the integral $\int_a^b f(x)dx$. The idea of the method is to calculate the average value $\langle f \rangle$ of the function $f(x)$ in the interval $[a, b]$. Then, the integral can be easily obtained as (see Figure 1)

$$\int_a^b f(x)dx = (b-a)\langle f \rangle$$

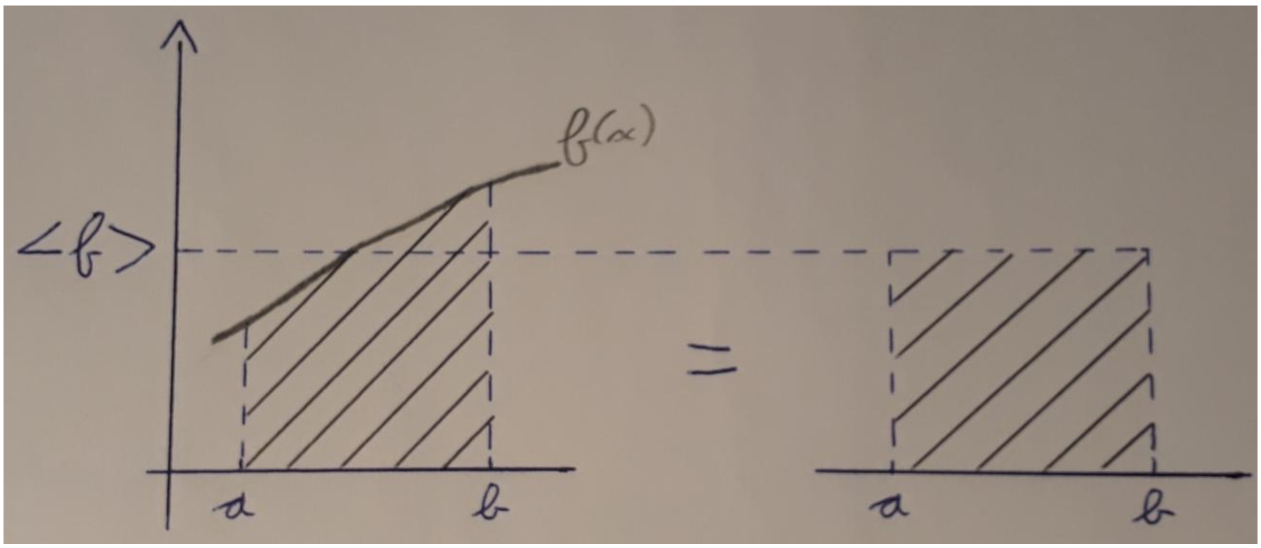

Figure 1: (left) the integral $\int_a^b f(x)dx$ is equal to the surface under the curve of $f(x)$ (black hatches), (right) the integral is equal to $(b-a)\langle f \rangle$.

In the **Monte Carlo integration method** the average of the function is estimated using a list of N random numbers uniformly distributed between a and b. For a given list of N numbers, the average is

$$\langle f \rangle_N = \frac{1}{N}\sum_{i=1}^{N} f(x_i)$$

and the integral can be estimated as

$$\int_a^b f(x)dx \approx (b-a)\langle f \rangle_N$$

Additionally, the standard deviation σ_N can be calculated (see Laboratory 11). This means that the value of the integral is between $(b-a)(\langle f\rangle_N - \sigma_N)$ and $(b-a)(\langle f\rangle_N + \sigma_N)$ with 68.3% confidence.

In Laboratory 11 you have to calculate the integral $\int_0^1 e^x dx$ and the standard deviation σ_N for different values of N.

Then, you have to calculate the same integral using the **Importance sampling method**. The idea of this method is to re-write the integral in a way that the integrand becomes nearly constant in the integration interval. In that case, the calculation of the average $\langle f\rangle$ will require smaller values of N (i.e., if the integrand would be exactly constant, one random number would be enough to calculate $\langle f\rangle$ exactly).

In our example, the integrand e^x can be approximated by $1+x$ (first-order Taylor series at $x=0$) in the interval $[0,1]$. The integral can be written as

$$\int_0^1 e^x dx = \int_0^1 \frac{e^x}{1+x}(1+x)dx$$

The idea is that $e^x/1+x$ is nearly equal to 1 in the interval $[0,1]$. Making the change of variable $y = x + \frac{x^2}{2}$ we have that $\frac{dy}{dx} = 1+x$ and that $x = -1 + \sqrt{1+2y}$ for $0 \le x \le 1$. Then, we can write the integral as

$$\int_0^1 e^x dx = \int_0^{\frac{3}{2}} \frac{e^{-1+\sqrt{1+2y}}}{\sqrt{1+2y}} dy$$

The evaluation of the integral on the right-hand side with the **Monte Carlo integration method** will necessitate smaller values of N, making the calculation more efficient.

Finally, you should calculate the distribution of integral values. This distribution takes a Gaussian shape with a standard deviation σ_N.