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Technical Report on

Uniform Hypergraphic Degree Sequences

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Abstract

Given a graph/hypergraph G with n vertices and e edges, degree of v is the no of edges in G containing v. The set of all degrees in a graph/hypergraph is called degree sequence when arranged in non-increasing order.The question "Which sequence of integers are degree sequences?" is discussed in this report. This report is based on the theorems of Erdős and Gallai,Havel and Hakimi, and A. K. Dewdney. Sufficiency condition on k-uniform hypergraphic degree sequences is also dicussed briefly.

Contents

1 Introduction

Graph is one of the most ubiquitous models of data structures. They can be used to model many types of relations and process dynamics in computer science, physical, biological and social systems. Many problems of practical interest can be represented by graphs. In general graph theory has a wide range of applications in diverse fields. We explored the degree sequence realization problem during the internship period.

If we are given a graph, it is easy to find the degree of the vertices. But what if we are given the degrees of the vertices and asked to construct a graph which satisfies it. It is not as simple a that.

Any sequence of non-negative integers can not form a degree sequence. So first, we need to check if the sequence is valid. Now, how do we know whether the given sequence is degree sequence or not? What are the conditions to be satisfied by a sequence to be degree sequence? What are the characteristics of a degree sequence? If we can draw at least one graph which satisfies the degree sequence, then we can say that the sequence is realizable and the sequence is degree sequence. There can be several graphs that realize a single degree sequence. So, is it possible to find all of them? If yes, how? These are the topics that we discussed during this period under the guidance of the mentor.

Constructive proof by Havel and Hakimi for degree sequence realization, Characterization of degree sequence by Erdős and Gallai and the constructive proof for sufficiency, degree sequence of k-uniform hypergraph by Dewdney, etc. are the papers that we studied. The ideas about the degree sequence of graph/hypergraph that I studied and understood from this internship is briefly summarized in this report.

2 Known Results on Characterization of Realizability

2.1 Erdős-Gallai Characterization

Theorem 2.1. *[3][4]A non-increasing sequence $D = [d_i]_1^n$ of non-negative numbers is graphic if and only if the sum of the D is even and the inequality*

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{j=k+1}^{n} min(d_j, k)$$

is satisfied for every integer k, $1 \leq k \leq n$

Proof.
Necessity: Let U be the graph with vertices v_1 to v_k. U may be complete, which will contribute maximum to the sum. i.e $k(k-1)/2$ edges, each contributing 2 to the sum. And there may be edges between U and U^c. Each of those edges will contribute only 1 to the sum. For every vertex v_j in set U^c , if $d_j > k$ it can have at most k edges with U. Otherwise d_j edges. So maximum no of edges that can be drawn from vertex v_j to U is min(d_j,k).
Sufficiency: Sufficiency of the theorem can be proved either by induction method or by construction method.
Proof by Induction: Induction is on the sum s. $s = 2$ is the basic step. $D =$ (1,1,0,0,. . . ,0) is the only possible degree sequence with sum 2. It is clearly graphic. The induction hypothesis is, any sequence D' with sum less than or equal to $s - 2$ which satisfies the inequality is graphic. Using this hypothesis we will prove that, any sequence of sum s that satisfies the inequalities are graphic. If D is a sequence which satisfies the inequality then we define D' as follows.

$$D' = \begin{cases} d_i & 1 \leq i \leq t-1 \\ d_t - 1 & i = t \\ d_i & t+1 \leq i \leq n-1 \\ d_n - 1 & i = n \end{cases}$$

where t is the smallest integer, such that $d_t > d_{t+1}$. Now we should show that D' satisfies the inequalities. Since the sum of D' is $s - 2$ we can claim that D' is graphic based on our hypothesis. Just by adding an edge between the vertex v_t and vertex v_n in D' we can realize D.
Constructive Proof: Constructing a realization through successive sub-realizations is the idea. Initial sub-realization is n vertices and no edges. Critical index r can be defined as the

largest index such that d(v_i)=d_i for $1 \leq i < r$. While $r \leq n$, we obtain new sub-realization with smaller deficiency at vertex v_r without changing the degree of any vertex v_i,where $i < r$. We will maintain the set $S = \{ v_{r+1}, \ldots, v_n \}$ independent. At any stage, if no more sub-realization is possible then it means that sum exceeded the limit. i.e the sequence is not satisfying the inequality, but it contradicts our assumption that the sequence satisfies the inequality. Hence the theorem is sufficient. □

2.2 Havel-Hakimi Theorem

Theorem 2.2 ([1][2])**.** *A non-negative integer sequence D=$[d_i]_1^n$ is degree sequence if and only if the sequence D' obtained by laying off the maximum term from D is graphic.*

Laying off is the procrss of deleting an element say d_k from the sequence and reduce 1 from the next d_k largest elements of the sequence.

Proof.
Sufficiency: if D' is graphic then we can add a new vertex v_1 and add d_1 edges to those vertices which are degree deficient with respect to D.
Necessity: D=$[d_i]_{i=1}^n$ is the non-increasing degree sequence,

$$d_1 \geq d_2 \geq \ldots \geq d_n$$

We need to show that the sequence $(d_2 - 1,\ d_3 - 1, \ldots\ldots, d_{d_1+1} - 1,\ d_{d_1+2}, \ldots, d_n)$ is graphic. Let the vertices v_{d_1+2} to v_n forms the set S. We claim that in the graph realizing D, v_1 is adjacent to vertices v_2 to v_{d_1+1}. If not, then there exist a vertex v_i in S which is adjacent v_1 and a vertex v_j in $V - \{v_1\} - S$ which is not adjacent to v_1. So there exists a vertex v_h which is adjacent to v_j but not to v_i(since $d_j > d_i$). We can replace the edges v_1v_i and v_jv_h with v_1v_j and v_iv_h. We can repeat this process until we get the required graph.

The Havel, Hakimi theorem gives only one graph even if a lot of other realizations are possible. We should find other realizations by elementary degree preserving transformations(EDT). We can also use Wang-Kleitman algorithm to generate more graphs by laying off any random vertex in each step. Even though we can generate more graphs, Wang-Kleitman algorithm cannot generate all the realizations. □

2.3 Degree Sequence of Uniform Hypergraphs

A hypergraph is a generalization of graph in which an edge can connect any number of vertices. A hyperedge is an arbitrary set of nodes and can therefore contain an arbitrary number of nodes. k-uniform hypergraph is a hypergraph such that all its hyperedges has size k.

Theorem 2.3 ([5])**.** *The non-increasing sequence $\pi = (d_1, d_2, \ldots, d_n)$ is k-graphic if and only if there is a non-increasing sequence π'=$(d_2', d_3', \ldots, d_n')$ for which the following conditions hold.*

1. *π' is $(k-1)-$ graphic.*
2. *$\sum_{i=2}^{n} d'_i = (k-1)d_1$.*
3. *$\pi" = (d_2 - d'_2, d_3 - d'_3, \ldots, d_n - d'_n)$ is k-graphic.*

Proof.
Sufficiency: $\pi"$ is k-graphic realized by graph $K"$ and π' is $(k-1)$-graphic realized by K'. Include vertex v_1 to all the edges in K' and take union of the resulting k-graph with $K"$.i.e $K = (v_1 * K')UK"$. Since the sum of π' is $(k-1)d_1$, there are d_1 edges and hence the degree of v_1 will be d_1.The degree of any vertex other than v_1 will be $d_i - d'_i + d'_i$=d_i for $2 \leq i \leq n$. Therefore, the resulting graph will be a k-uniform realization of degree sequence π.So the theorem is sufficient.

Necessity: Here it is given that K is k-graphic. We can define K' and $K"$ as follows. Take all the edges of K in which v_1 is included and remove v_1 from each of these edges. Now we will get a $(k-1)$-uniform graph. That is our K'.
$K"$ is graph obtained after the removal of vertex v_1 from K. It is clearly k-graphic.

K' will have d_1 edges with size $k-1$, so the sum of the sequence π' will be equal to $(k-1)d_1$. Sometimes the sequence π' may not be non-increasing. In such cases we can rearrange the edges in such a way that the sequence π remains same and π' becomes non-increasing. □

2.4 Sufficiency condition on the Degree Sequence of Uniform Hypergraphs

Theorem 2.4. *[6]Let π be a non-increasing sequence with maximum entry Δ and t entries that are at least $\Delta - 1$. If k divides $\sigma(\pi)$ and*

$$\binom{t-1}{k-1} \geq \Delta$$

then π is k-graphic.

Proof.
We will show that π is k-graphic by constructing a $(k-1)$ graphic link sequence and k-graphic residual sequence. We will prove the theorem for $\Delta > 1$ and $k \geq 3$.
Consider the least k for which the theorem does not hold. Among all such non-increasing sequence that do not satisfy the theorem for this k, consider the sequence with smallest maximum entry and minimum multiplicity for the maximum term. Let $\pi = (d_{0,1}, \ldots, d_{n-1})$ be that sequence.

$$c = max\{i \in Z : \sum_{j=1}^{n-1} max\{0, d_j - i\} \geq (k-1)\Delta\}$$

Let L' be the sequence $(l'_1, l'_2, \dots, l'_{n-1})$ where $l'_i = max\{0, d_i - c\}$
Let $s = \sigma(L') - (k-1)\Delta$
Now we will create the link sequence $L = (l_1, l_2, \dots, l_{n-1})$ by subtracting 1 from each of the first s entries of L'.

$$L = \{ \begin{array}{cc} l'_i - 1 & 1 \leq i \leq s \\ l'_i & i > s \end{array}$$

The residual sequence $R = (r_1, r_2, \dots, r_{n-1})$ can be created by subtracting L from π. Now we just need to show that L is $(k-1)$ graphic and R is k-graphic. □

3 Progress Made So Far

We also discussed the following points

- Any sequence of the form x^n is realizable if it satisfies the following basic conditions
  1. $x.n$ is even
  2. $x \leq n-1$
- Any degree set with element 1 is tree realizable.
- Any set of positive integers can be realized by a connected graph and the minimum order of such a graph is $M+1$ where M is the maximum integer in the set.

3.1 Attempt to find the necessary condition for degree sequence that is realizable by Hypergraph.

Any sequence $[d_i]_1^n$ of positive integers is realizable. But the question is whether it is realizable by a simple hypergraph(i.e no repeated edges). We did an attempt to generalize Erdős-Gallai characterization of 2-uniform graph to hypergraphs. The necessary condition on the sequence to be hypergraphic is

$$\sum_{i=1}^{k} d_i \leq k2^{k-1} + Y$$

for every k, $1 \leq k \leq n$

The first term $k2^{k-1}$ gives the maximum possible contribution to the sum of degrees by the edges formed by the vertices of the set $S = \{v_1, v_2, \dots, v_k\}$

Number of edges with size 1 = $\binom{k}{1}$, each contributing 1 to the sum
Number of edges with size 2 = $\binom{k}{2}$, each contributing 2 to the sum
.
.
.
Number of edges with size k = $\binom{k}{k}$, each contributing k to the sum

So the maximum possible contribution to the sum by the edges formed by the vertices of set $S = 1.\binom{k}{1} + 2.\binom{k}{2} + 3.\binom{k}{3} + \cdots + k.\binom{k}{k} = k.2^{k-1}$
The term Y gives the maximum contribution to the sum by the edges formed between S and S^c.

$$\begin{array}{cc}
 & \begin{array}{ccccc} 1 & 2 & 3 & \quad & n\text{-}k \end{array} \\
\begin{array}{c} k \\ k\text{-}1 \\ k\text{-}2 \\ \\ 1 \end{array} &
\begin{bmatrix}
k & k/2 & k/3 & \cdots\cdots & k/n\text{-}k \\
k\text{-}1 & (k\text{-}1)/2 & (k\text{-}1)/3 & \cdots\cdots & (k\text{-}1)/(n\text{-}k) \\
k\text{-}2 & (k\text{-}2)/2 & (k\text{-}2)/3 & \cdots\cdots & (k\text{-}2)/(n\text{-}k) \\
\vdots & \vdots & \vdots & & \\
1 & 1/2 & 1/3 & \cdots\cdots & 1/(n\text{-}k)
\end{bmatrix}
\end{array}$$

We can construct a k x $(n-k)$ matrix where row number represent the no. of vertices taken from set S and column number represent the no. of vertices taken from set S^c.
In the matrix, each term represent the contribution to degree sum per vertex.The contribution of edge formed by one vertex from set S^c and all the k vertices from set S will contribute k to the sum(since the one vertex alone contribute k to the sum), while an edge formed by 2 vertices of S^c and k vertices from S will contribute only $k/2$ per vertex(since 2 vertices together contribute k). Also the sum will be bounded by the degree sequence $R = (v_{k+1}, v_{k+2}, \ldots, v_n)$. We will choose edges and add its contribution to the sum until the sequence R is satisfied. We will consider the edges in the order in which it contributes. The edge that contributes the most will be selected first (maximum no of such edges will be taken), then the next one and so on.
For example consider the sequence (7,3,2,2) which is not realizable by a simple hypergraph. First $k = 1$, $\sum_{i=1}^{1} d_i$=7. Now find the upper bound on sum. $k.2^{k-1}$=1.2^0=1. The remaining sequence R=(3,2,2). We will consider edges v_1*$\{\{v_2\}$,$\{v_3\}$,$\{v_4\}\}$ first. Each contribute 1 to the sum. Now the degree to be satisfied became (2,1,1). Now consider edges v_1*$\{\{v_2 v_3\}, \{v_2 v_4\}\}$, each contributing 1. Now R is satisfied. So the upper bound will be 3.1 + 2.1 = 5. The sum is greater than the upper bound. Hence the given sequence is not realizable.

4 Conclusion

The matrix changes as the value of k and n changes. Also, the edges and no. of edges that we can choose depends on the remaining sequence. Hence we couldn't generalize it more concisely. The general hypergraph is an area in which we do not have any results. Hence, it is still a question, whether we can optimize?
Can there be a more efficient characterization of the general/k-uniform degree sequence? What could be the minimum no of edges that could realize a hypergraph? How many realizations are possible for a given degree sequence? There are so many unanswered questions in this area that we can work on. So clearly, it is a fruitful area for research enthusiasts and igniting minds.

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