Seminar Nr. 5, Continuous Random Variables and Continuous Random Vectors

Theory Review

$X : S \to \mathbb{R}$ continuous random variable with pdf $f : \mathbb{R} \to \mathbb{R}$ and cdf $F : \mathbb{R} \to \mathbb{R}$. Properties:

1. $F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)dt$

2. $f(x) \ge 0, \forall x \in \mathbb{R}, \int_{\mathbb{R}} f(x) = 1$

3. $P(X = x) = 0, \forall x \in \mathbb{R}, P(a < X < b) = P(a \le X \le b) = \int_{a}^{b} f(t)dt$

4. $F(-\infty) = 0, F(\infty) = 1$

$(X, Y) : S \to \mathbb{R}^2$ continuous random vector with pdf $f = f_{(X,Y)} : \mathbb{R}^2 \to \mathbb{R}$ and

cdf $F = F_{(X,Y)} : \mathbb{R}^2 \to \mathbb{R},\ F(x, y) = P(X \le x, Y \le y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u, v)\, dv\, du,\ \forall (x, y) \in \mathbb{R}^2$. Properties:

1. $P(a_1 < X \le b_1, a_2 < Y \le b_2) = F(b_1, b_2) - F(a_1, b_2) - F(b_1, a_2) + F(a_1, a_2)$
2. $F(\infty, \infty) = 1, F(-\infty, y) = F(x, -\infty) = 0,\ \forall x, y \in \mathbb{R}$
3. $F_X(x) = F(x, \infty),\ F_Y(y) = F(\infty, y),\ \forall x, y \in \mathbb{R}$ (marginal cdf's)
4. $P((X, Y) \in D) = \iint_D f(x, y)\, dy\, dx$
5. $f_X(x) = \int_{\mathbb{R}} f(x, y)dy,\ \forall x \in \mathbb{R},\ f_Y(y) = \int_{\mathbb{R}} f(x, y)dx,\ \forall y \in \mathbb{R}$ (marginal densities)
6. X and Y are independent $<=>$ $f_{(X,Y)}(x, y) = f_X(x)f_Y(y),\ \forall (x, y) \in \mathbb{R}^2$.

Function $Y = g(X)$: X r.v., $g : \mathbb{R} \to \mathbb{R}$ differentiable with $g' \ne 0$, strictly monotone

$f_Y(y) = \dfrac{f_X\left(g^{-1}(y)\right)}{|g'\left(g^{-1}(y)\right)|},\ y \in g(\mathbb{R})$

Uniform distribution $U(a, b),\ -\infty < a < b < \infty$: pdf $f(x) = \dfrac{1}{b-a}, x \in [a, b]$.

Normal distribution $N(\mu, \sigma), \mu \in \mathbb{R}, \sigma > 0$: pdf $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, x \in \mathbb{R}$.

Gamma distribution $Gamma(a, b),\ a, b > 0$: pdf $f(x) = \dfrac{1}{\Gamma(a)b^a} x^{a-1} e^{-\frac{x}{b}}\ ,\ x > 0$.

Exponential distribution $Exp(\lambda) = Gamma(1, 1/\lambda),\ \lambda > 0$: pdf $f(x) = \lambda e^{-\lambda x}, x > 0$.
- Exponential distribution models *time*: waiting time, interarrival time, failure time, time between rare events, etc; the parameter λ represents the frequency of rare events, measured in time^{-1}.
- Gamma distribution models the *total* time of a multistage scheme.
- For $\alpha \in \mathbb{N}$, a $Gamma(\alpha, 1/\lambda)$ variable is the sum of α independent $Exp(\lambda)$ variables.

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1. The lifetime, in years, of some electronic component is a random variable with density

$$f(x) = \begin{cases} \dfrac{k}{x^4}, & \text{for} \quad x \ge 1 \\ 0, & \text{for} \quad x < 1. \end{cases}$$

Find
a) the constant k;

b) the corresponding cdf F;
c) the probability for the lifetime of the component to exceed 2 years.

2. (The Uniform property) Let $X \in U(a, b)$. For any $h > 0$ and $t, s \in [a, b-h]$,

$$P(s < X < s+h) = P(t < X < t+h).$$

The probability is only determined by the length of the interval, but not by its location.
Example: A certain flight can arrive at any time between 4:50 and 5:10 pm. Let X denote the arrival time of the flight.
a) What distribution does X have?
b) When is the flight more likely to arrive: between 4:50 and 4:55 or between 5 and 5:05; before 4:55 or after 5:05?

3. On the average, a computer experiences breakdowns every 5 months. The time until the first breakdown and the times between any two consecutive breakdowns are independent Exponential random variables. After the third breakdown, a computer requires a special maintenance.
a) Find the probability that a special maintenance is required within the next 9 months;
b) Given that a special maintenance was not required during the first 12 months, what is the probability that it will not be required within the next 4 months?

4. The joint density for (X, Y) is $f_{(X,Y)}(x, y) = \dfrac{1}{16}x^3y^3$, $x, y \in [0, 2]$.
a) Find the marginal densities f_X, f_Y.
b) Are X and Y independent?
c) Find $P(X \leq 1)$.

5. Let X be a random variable with density $f_X(x) = \dfrac{1}{4}xe^{-\frac{x}{2}}$, $x \geq 0$ and let $Y = \dfrac{1}{2}X + 2$. Find f_Y.

6. Let $X \in N(0, 1)$. Find the probability density function of $Y = |X|$.

Bonus Problems:

7. An internet service provider has two connection lines for its customers. Eighty percent of customers are connected through Line I, the rest through Line II. Line I has a $Gamma(3, 1/2)$ connection time (in minutes), while Line II has a $U(20, 50)$ connection time (in seconds). Find the probability that it takes a randomly selected customer more than 30 seconds to connect to the internet.

8. Let $X, Y \in N(0, 1)$ be independent random variables. Let D_r be the disk centered at the origin with radius r. Find r such that $P((X, Y) \in D_r) = 0.3$.