COURSE #7
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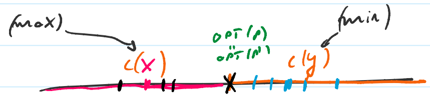
LAST TIME DUALITY IN LINEAR PROGRAMMING

(P) $\longleftrightarrow$ (P')

dual of pb. P.

STRONG DUALITY THM.



If both P and P' have finite OPT

$\Downarrow$

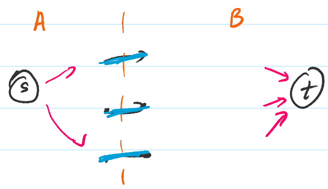
$\boxed{OPT(P) = OPT(P')}$

WEAK DUALITY

If x soln for (max)
y sol for (min) $\Big\}\Rightarrow c(x) \le c(y)$

$\max(\dots) = \min(\dots)$

MAX-FLOW MIN-CUT THM



$cut(A;B) =$

$= \sum_{e:A\to B} c(e)$

$\boxed{\max(\text{flow } s\to t) = \min_{(A,B)}(\text{cut})}$

CUT: $V = A \cup B$

$s\in A$ $A\cap B = \phi$
$t\in B$

There is a LP formulation P for the max-flow problem,
s.t Dual D gives min-cut.

exp many variables! n = # of vertices / edges

Let P be a simple path from s to t.
Let $\mathcal{P}$ = the set of all such paths.

For $P\in\mathcal{P}$ f_P = flow that travels from s to t along path P

(primal) $\begin{cases} \max\left(\sum_{P\in\mathcal{P}} f_P\right) \\ \sum_{P\ni e} 1\cdot f_P \le c(e), \quad \forall e\in E \end{cases}$

Dual

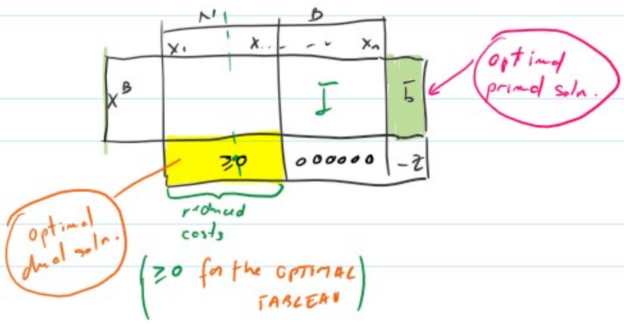
y_e $\begin{cases} \min\left(\sum_{e\in E} c(e) y_e\right) \\ \sum_{P\ni e} y_e \ge 1 \quad (P) \end{cases}$

(primal) $\begin{cases} \sum_{p \ni e} 1 \cdot f_p \le c(e), \quad \forall e \in E \\ f_p \ge 0 \end{cases}$ $\Big|_{y_e}$ $\begin{cases} \sum_{p \ni e} y_e \ge 1 \ (p) \\ y_e \ge 0 \end{cases}$

Conclusion: Can solve Max Flow Min-Cut using simplex.

if i can solve exponentially large problems

OPTIMAL SIMPLEX TABLEAU FOR PRIMAL Pd contains an opt. soln for the dual



OPT BFS $\quad x_N = 0$, $\quad x_B \to$

reduced costs

compute the costs only in vars in N

$Ax = b \;\xrightarrow{(A_B)^{-1}}\; A_B x^B + A_N x^N = b$

$\Updownarrow$

$x^B + A_B^{-1} A_N x^N = \underbrace{(A_B)^{-1} b}_{= \bar{b}}$

$$\boxed{x^B = \bar{b} - (A_B)^{-1} A_N x^N} \quad (*)$$

$$\text{cost} = c^T x = c_B^T x^B + c_N^T x_N$$

$\downarrow (*)$

$$= \underbrace{\left(c_B^T \bar{b}\right)}_{OPT} - c_B^T (A_B)^{-1} A_N x^N + c_N^T x^N$$

$$= OPT + \sum_N \big(\underbrace{c_N^T - c_B^T A_B^{-1} A_N}_{\text{reduced cost}}\big) x^N$$

$$= OPT + \sum_{n \in N} \tilde{c}_n x_n$$

$$\tilde{c}_N = c_N^T - c_B^T A_B^{-1} A_N$$

At optimality $\quad \tilde{c}_N \ge 0 \quad \tilde{c}_N = c_N - c_B A_B^{-1} A_N$

At optimality $\quad \tilde{c}_N \geq 0 \quad \tilde{c}_N = c_N - c_B A_B^{-1} A_N$

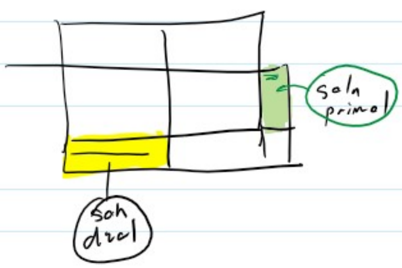
$$\tilde{c}_B = 0 = c_B - c_B A_B^{-1} A_B = c_B - c_B = 0$$

Dual $\begin{cases} \max (b^T y) \\ A^T y \leq c \\ y \geq 0 \end{cases} \quad \tilde{c} = \begin{pmatrix} \tilde{c}_N \\ \tilde{c}_B \end{pmatrix} \geq 0$

$$\tilde{c} = \boxed{c - (\underbrace{c \cdot A_B^{-1}}_{\pi}) A} \geq 0$$

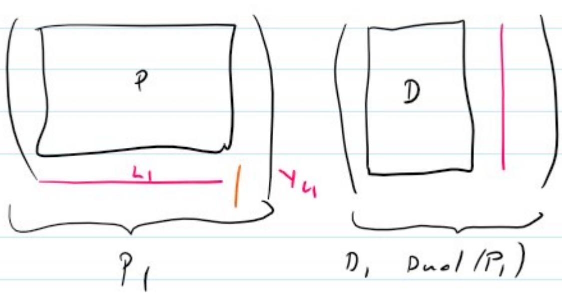
$\boxed{\pi \text{ is soln for the dual problem}}$

$\boxtimes$



Application

Suppose I have solved a pb (P) to optimality but want to add another inequality L_1



Adding L_1 to $P \iff$ adding y_{L_1} to D

$OPT(P) = OPT(D)$ but I want $OPT(P_1)$

$\downarrow$

$\tilde{y}$

$(\tilde{y}, y_{L_1} = 0)$ feasible soln for (D_1)

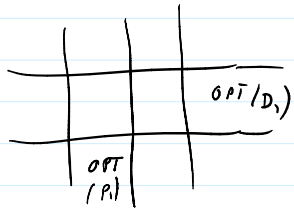
Can solve $P + L_1$ starting from the BFS for D_1

$\Downarrow$

solve the dual instead of P_1

$\Downarrow$

OPT (D_1)



Dual Simplex Alg $\longrightarrow$ can read about it in the TEXTBOOK

has tableau version with slightly different rules

can be seen as applying TABLEAU SIMPLEX on the dual problem

APPLICATIONS OF L.P. TO ALGORITHMS

(TOTAL UNIMODULARITY)

$$(LP) \begin{cases} \min (c^T x) \\ Ax = b \\ x \geq 0 \end{cases} \qquad (ILP) \begin{cases} \min (c^T x) \\ Ax \\ x \geq 0 \\ x \in \mathbb{Z} \end{cases}$$

integer

LP: has efficient algorithms

ILP: NP-complete $\longrightarrow$ (−) all algorithms have bad cases (complexity is exponential)

$\searrow$ (+) can model many pbs.

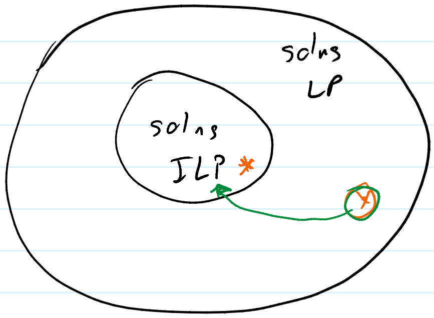
$A, b, c \in \mathbb{Z}$ ...

(*) When can we solve an ILP by solving the corresponding LP and by chance the OPTIMAL soln. for the LP has integral values?

$\downarrow$

(Total Unimodularity)

If $\text{OPT}(LP) \in \mathbb{Z} \Rightarrow \boxed{\text{OPT}(ILP) = \text{OPT}(LP)}$



soln space

in general

$\boxed{OPT(LP) \leq OPT(ILP)}$

can solve ILP using simplex

In general $OPT(ILP) \neq OPT(LP)$

$$(ILP) \begin{cases} \min(x) \\ 2x \geq 1 \\ x \geq 0 \\ x \in \mathbb{Z} \end{cases} \Longrightarrow (LP) \begin{cases} \min(x) \\ 2x \geq 1 \\ x \geq 0 \end{cases}$$

$\boxed{x = 1}$ $\boxed{x = 1/2}$

$\boxed{OPT(ILP) = 1}$ $\boxed{OPT(LP) = 0.5}$

Condition on A s.t. $OPT(ILP) = OPT(LP)$

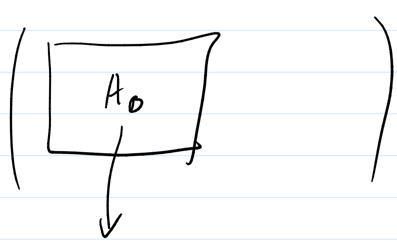
↓

total unimodularity

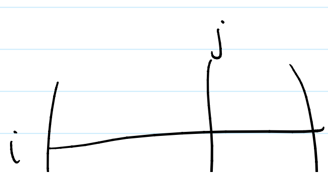
SIMPLEX

optimal BFS $\begin{cases} x_N = 0 \\ x_B = \underbrace{(A_B)^{-1} b} \end{cases}$

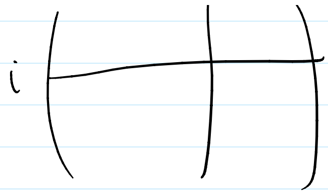
$Ax = b$



$\det(A_B) \neq 0$ $\begin{cases} A_B x^B = b \\ x^N = 0 \end{cases}$



$\boxed{(-1)^{i+j} \Delta_{ij}}$



$$\boxed{\frac{(-1)^{i+j}\,\Delta_{ij}}{\Delta}}$$

$$\Delta = \det(A_B)$$

If $A, b \in \mathbb{Z}$ $\quad \boxed{x_B \in \mathbb{Q}}$ (not necessarily integral)

<u>Obs</u> x_B is guaranteed to be integral when $\Delta \in \{\pm 1\}$.

<u>Problem</u> Don't know B without running simplex.

<u>Def</u> A matrix A is called <u>totally unimodular</u> iff every minor Δ of A <u>has determinant</u> $\{\pm 1, 0\}$

<u>Obs</u> A is T.U. $\Rightarrow$ a_{ij} (1×1 minor) $\in \{\pm 1, 0\}$

(T) If A is T.U.

$\Downarrow$

opt soln $\begin{cases} \min(c^T x) \\ Ax = b \\ x \geq 0 \end{cases}$ <u>has integral values</u>

Exp $\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$ T.U $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

Obs Same result is true

$$(2)\ \begin{cases} \min (c^T x) \\ Ax \geq b \ \big|\ y \text{ slack variables} \\ x \geq 0 \end{cases}$$

$$(2) \iff \begin{cases} \min (c^T x + 0^T \cdot y) \\ \underline{\underline{A}}x - y = b \\ x \geq 0 \\ y \geq 0 \end{cases}$$

matrix $(A, -I)$

$$\tilde{A} = \begin{array}{cc} x & y \\ \left(\boxed{\ A\ } \right. & \left. \boxed{\begin{matrix} -1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{matrix}} \right) \end{array}$$

Every minor of $\tilde{A}$ corresponds to columns of A and colns of $-I$

Δ $\qquad \begin{pmatrix} & \vdots \\ & 0 \\ & -1 \\ & 0 \\ & 0 \\ & \vdots \end{pmatrix}$

$\Delta \to \Delta'$ minor of A

If A is T.U $\Rightarrow$ $\bar{A}$ is T.U.

Exp (a) $A = \begin{bmatrix} 1 & -1 & -1 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ is T.U.

(b) $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ not T.U.
($\det(A) = 2$)

Properties of T.U matrices

① If A is T.U $\Rightarrow$ $\{A; I\}$, $\{A, -I\}$, $\begin{Bmatrix} A \\ I \end{Bmatrix}$,

$\begin{Bmatrix} A \\ -I \end{Bmatrix}$ $\quad \begin{bmatrix} A & 0 \\ B & I \end{bmatrix}$ T.U.

$\forall B$

Pf A minor of $\begin{bmatrix} A & 0 \\ B & I \end{bmatrix}$ has det equal to a minor of A

$$\left(\begin{array}{c|c} A & 0 \\ \hline B & I \end{array}\right)$$

(2) If A is T.U $\implies$ A^T is also T.U.

reason $\det(\Delta^T) = \det(\Delta)$