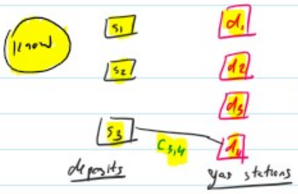
COURSE #2
Thursday, October 12, 2023 5:52 PM

LINEAR PROGRAMMING



$c_{3,4}$ = cost of moving 1 ton of gas from $s_3 \to d_4$

$X_{3,4}$ = amount of gas we are actually sending $s_3 \to d_4$

$X = (x_{11}, \dots X_{34})$ optimal allocation

(LP)
$$\begin{cases} \min \left(\sum_{i=1}^{3} \sum_{j=1}^{4} c_{ij} x_{ij} \right) & \text{total cost} \\ \sum_k x_{ik} \le s_i & \forall i = \overline{1,3} \quad \text{("we have enough gas in } s_i \text{ for the allocation")} \\ \sum_r x_{rj} \ge d_j & \forall j = \overline{1,4} \quad \text{("demand } d_j \text{ is met } \forall j\text{")} \\ x_{ij} \ge 0 & \text{(real numbers)} \quad \text{("physical constraints")} \end{cases}$$

linear constraints

constants

linear function

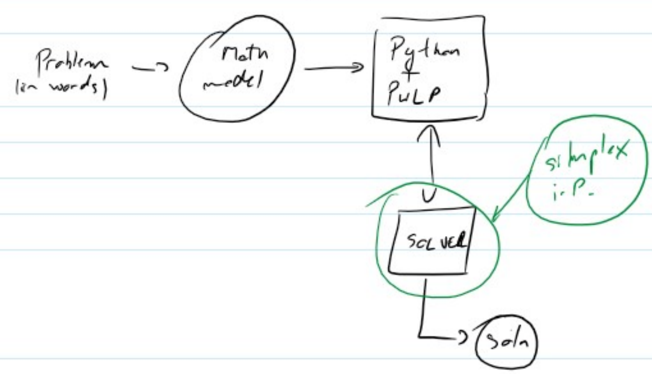
$\le, \ge, =$ ✓

$<, >, \ne$ X

Exp.
$$\begin{cases} \max (2x + 3y) \\ x + y \le 3 \\ 2x + y \le 4 \\ x, y \ge 0 \end{cases}$$

GOOD NEWS LP has efficient (in theory / in practice) algorithms

ALGORITHMS Simplex alg.
INTERIOR POINT METHODS



BAD NEWS FEW(ER) PROBLEMS CAN BE MODELED AS L.P.

ILP — LP with the restriction that we only care about sol'ns where all variables take integral values

different algorithms

BAD NEWS — ILP harder
- theory: ILP is NP-complete problem
- practice: order of magnitude smaller than LP

GOOD NEWS — Way more problems can be modeled as ILP (ILP is NP-complete)

$\boxed{\text{ILP being NP-complete?}}$

(COMPUTATIONAL COMPLEXITY)

P≠NP problem

ALGORITHMS — $O(n \log(n))$ $O(n^2)$, $O(n^3)$

"good algorithm" $O(n^k)$ for some $k>0$

DECISION PROBLEMS — INPUT x; ANSWER YES/NO.

P = class of decision problems that have algs with complexity $O(n^k)$

"easy (decision) problems"

$LP \in P$ (KHACHIAN "ellipsoid method") — New York Times

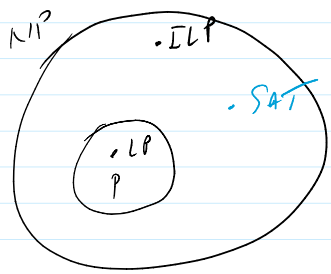
good in theory, not so in practice

SIMPLEX ALG. (DANTZIG '47) — bad in theory but good in practice

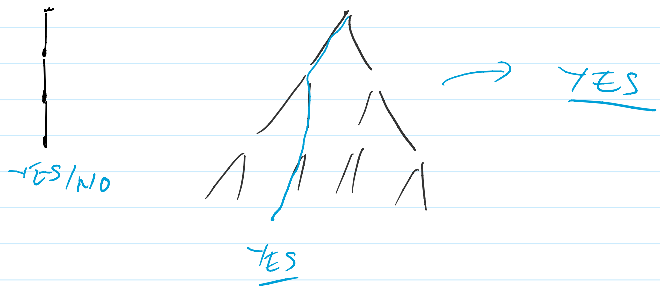
(exponential)

$\boxed{\text{problem LP}}$ → change LP a little bit

likely easy
pb for SIMPLEX



NP nondeterministic polynomial time.

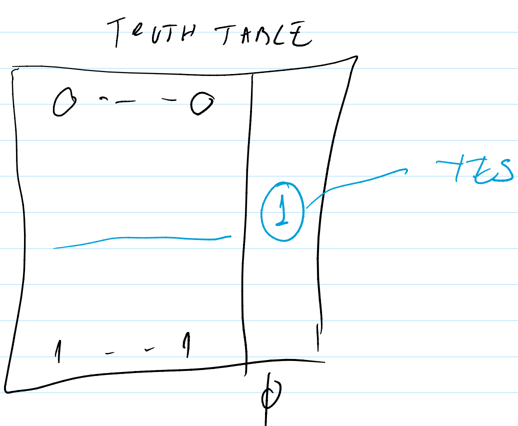


guess soln → easy to verify that it is soln indeed

EXAMPLE SAT (satisfiability problem)

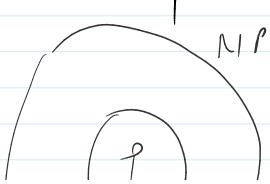
INPUT Logical formula $\phi(x_1, \ldots x_n)$

TO DECIDE Can I assign values TRUE / FALSE to $x_1 \ldots x_n$ to satisfy ϕ ?

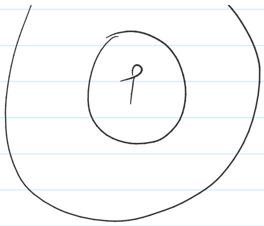


$\boxed{P \neq NP ?}$

1.000.000 $
(reward)
(CLAY INSTITUTE)



Probably SAT does not



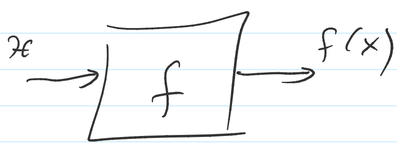
Probably SAT does **not** have poly time algorithms

{Probably ILP does **not** have poly time algs}

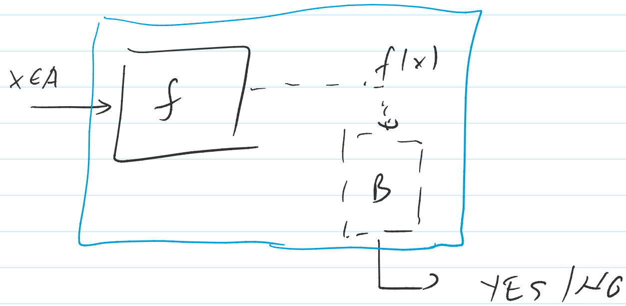
A, B decision problems

$A \leq B$ (A reduces to B)

$\exists$ poly time alg f

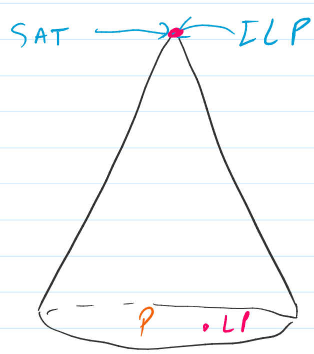


$x \in A \iff f(x) \in B$



A reduces to B

$\boxed{\text{B is "harder" / "more expressive" than A}}$



$A \leq A$

$A \leq B,\ B \leq C \implies A \leq C$

$A \leq B$ and $B \in P$ then $A \in P$

(T) (Cook-Karp)

LP

(T) (Cook - Karp)

SAT / ILP NP-complete

$\boxed{\begin{array}{l} SAT \in NP \\ \forall A \in NP \quad A \leq SAT \end{array}}$

I can solve every problem in NP using a (SAT solver) (ILP solver)

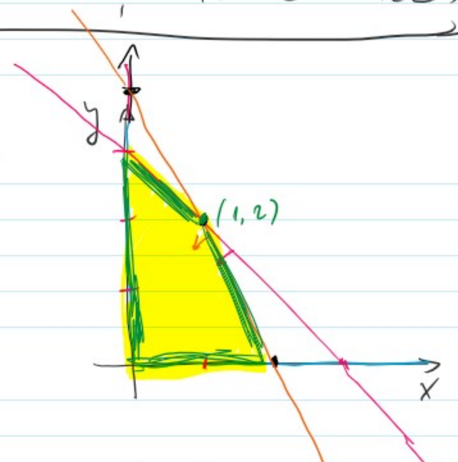
LINEAR PROGRAMMING, GEOMETRICALLY

$\max(2x + 3y)$

$x + y \leq 3$

$2x + y \leq 4$

$x, y \geq 0$



feasible soln — satisfies all constraints, not nec. OPTIMAL

P = feasible solns → polygon

vertices (0,0)

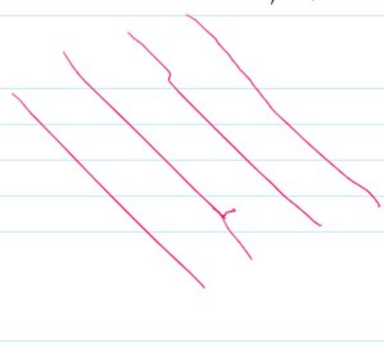
(2,0)

(0,3)

(1,2)

$2x + 3y = \lambda$ line

When λ changes



Conclusion max is reached at some vertex of the polygon.

	$2x + 3y$
(0,0)	0

OPTIMAL

(0,0)	0	
(2,0)	4	
(0,3)	9	
(1,2)	8	

OPTIMAL SOLN

X = 0
y = 3 OPT Soln

WHAT CAN GO WRONG?

(1) Problem is INCOMPATIBLE (INFEASIBLE)

Exp Gas STATION Pb.

TOTAL GAS 1 ton

Total demand 2 tons

(2) INFINITELY MANY OPTIMAL SOLNS.

(3) INFINITE OPTIMUM

$$\begin{cases} \max(x) \\ x \geq \cancel{1} \\ x \geq 0 \end{cases}$$

(4) n variables

GOOD NEWS Set of feasible sol'ns in Polyhedron
$\mathbb{R}^n$

BAD NEWS exponentially many (in n)
vertices.