COURSE # 4

Thursday, October 26, 2023 6:07 PM

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23rd of NOVEMBER → NO CLASS, RESCHEDULED

LAB → <u>OK</u>

$\boxed{\text{LAST TIME}}$

<u>SIMPLEX ALGORITHM</u>

LOOKING FOR BASIC FEASIBLE SOLUTION

WE <u>PIVOT</u> FROM BFS TO BFS

$$\begin{cases} \min (c^T x) \\ \underline{Ax = b} \\ x \geq 0 \end{cases} \qquad B_0 \to \underbrace{B_1 \to B_2}_{\text{pivoting}} \to \dots \to B_n$$

$c^T \cancel{X}_2 \leq c^T X_1$

<u>OPTIMAL</u>

<u>PROBLEMS TO ANSWER</u>

(1) How do we find <u>our first BFS</u> Bo?

(2) LP → finite OPT, infinite OPT, incompatible } how do we figure out <u>which</u> <u>case we are</u> <u>in?</u>

(3) How do we recognize the fact that B_n is OPTIMAL?

(4) How do we choose which variables to pivot on?

(5) How do we prevent <u>cycling?</u>

$$B_0 \to B_1 \to \quad B_i \to B_{i+1} \dots \to B_j$$

(6) How do we <u>efficiently implement</u> all this?

BASIC FEASIBLE SOLUTION

$$A = \Big(\boxed{A_B} \ \boxed{A_N} \Big)$$

columns $\equiv$ variables

$\det(A_B) \neq 0$

$\operatorname{rank}(A) = \operatorname{rank}(A_B)$

When we partition columns (variables) into B and N

we implicitly create a solution (not necessarily feasible)

Basic feasible solution, when $\boxed{X \geq 0}$

vector components are all ≥ 0

B_0

BFS

$\underbrace{B_i \longrightarrow B_{i+1}}_{\text{pivoting}} \Rightarrow B_{i+1}$ be a BFS

$$\left\{ \begin{pmatrix} \\ \\ \end{pmatrix}_B = \begin{pmatrix} \\ \\ \end{pmatrix}_{v} - \begin{pmatrix} & \\ & \end{pmatrix}\begin{pmatrix} \\ \\ \end{pmatrix}_N \right.$$

What is the solution corresponding to B?

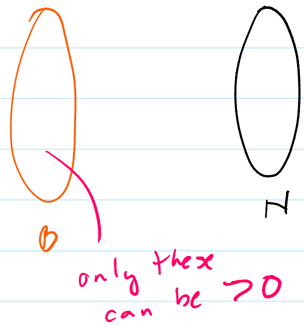
$$\begin{cases} X_N = 0 \\ X_B = \bar{b} \end{cases}$$

$$A_B^{-1} \Big| \quad A_B X^B + A_N X^N = b$$

$$X^B + A_B^{-1} A_N X^N = \underbrace{A_B^{-1} b}_{\bar{b}}$$

$$\boxed{X^B = \bar{b} - A_B^{-1} A_N X^N} \Longrightarrow \begin{cases} X^B = \bar{b} = A_B^{-1} b \geq 0 \\ X^N = 0 \end{cases}$$

BFS



if I want to increase some variable $x_i \in N$ to make it > 0

$\Downarrow$

x_i has to be in B

swap/pivot x_i for some x_j in B

$$\begin{cases} x_3 = 1 - 3x_1 - 2x_2 \\ x_4 = 1 - 2x_1 \\ x_5 = 1 - x_1 \end{cases}$$

$\underline{BFS}_3$ $\begin{cases} x_3 = 1 \\ x_4 = 1 \\ x_5 = 1 \end{cases}$ $\quad x_1 = 0$, $x_2 = 0$

Why $x_1 = 1/3$ and not $x_1 = 1$?

$x_1 = 1/3 \Rightarrow$ $x_3 = 0$

$x_4 = 1 - 2/3 = 1/3 \geq 0$

$x_5 = 1 - 1/3 = 2/3 \geq 0$

If we swap x_1 and $x_3 \Rightarrow$ $\underline{BFS}$ ✓ $\begin{cases} x_1 = 1/3 \\ x_2 = 0 \\ x_3 = 0 \\ x_4 = 1/3 \\ x_5 = 1/3 \end{cases}$

if we swapped instead x_1 and x_5

$$\begin{cases} x_1 = 1 \\ x_3 = -2 \\ x_5 = 0 \\ x_2 = 0 \\ x_4 = 1 \end{cases}$$

<u>not</u> a BFS !

<u>Rule</u> if $B_i \xrightarrow[\text{pivoting}]{} B_{i+1}$

B_i BFS then <u>B_{i+1} has to be a BFS</u>

$\min \left(z = x_1 + x_2 + x_3 + x_4 + x_5 \right)$

BFS

$$\min \left(z = x_1 + x_2 + x_3 + x_4 + x_5\right)$$

$$\begin{aligned} 3x_1 + 2x_2 + x_3 \quad\quad\quad &= 1 \\ 5x_1 + x_2 \quad + x_4 \quad &= 1 \\ 2x_1 + 5x_2 \quad\quad + x_5 &= 1 \end{aligned}$$

$$\begin{cases} x_3 = 1 - 3x_1 - 2x_2 & x_1 \le 1/3 \\ x_4 = 1 - 5x_1 - x_2 & x_1 \le 1/5 \\ x_5 = 1 - 2x_1 - 5x_2 & x_1 \le 1/2 \end{cases}$$

BFS
$x_1 = x_2 = 0$
$x_3 = x_4 = x_5 = 1$

We would like to increase both x_1 and x_2 because they have positive coeffs in the objective.

$$\boxed{\text{pivot}(x_1, x_4)}$$

Solution to PROBLEMS (1) and (2)

(1): first BFS
(2): STATUS

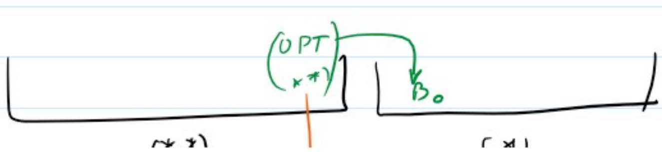
TWO PHASE SIMPLEX

Instead of solving (1) and (2) for $(*)\begin{cases} \min(c^T x) \\ Ax = b \\ x \ge 0 \end{cases}$

I am going to first solve another problem (**) for which (1) is easy

$(**)\begin{cases} \min(1^T y) \\ Ax + y = b \\ x, y \ge 0 \end{cases}$

OPT(**) $\longrightarrow$ STATUS of Pb (*)
$\downarrow$
first BFS for Pb (*)



$(**)$ STOP B_0 $(*)$

in words We add to every equation of $(*)$ a new variable y_i

Exp

$$(*)\begin{cases} \min(x_1+x_2+x_3+x_4+x_5) \\ 3x_1+2x_2+x_3 = 1 \\ 5x_1+x_2+x_4 = 1 \\ 2x_1+5x_2+x_5 = 1 \\ x_i \geq 0 \end{cases} \Rightarrow (**)\begin{cases} \min(y_1+y_2+y_3) \\ 3x_1+2x_2+x_3+y_1 = 1 \\ 5x_1+x_2+x_4+y_2 = 1 \\ 2x_1+5x_2+x_5+y_3 = 1 \\ x_i, y_i \geq 0 \end{cases}$$

if $(*)$ is compatible then Pb $(**)$ has a soln with value 0

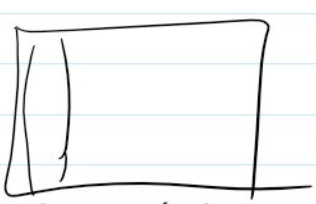
$$\underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ soln for } (*) \quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}_{\text{soln for } (**) \text{ with obj value } = 0}$$

Conclusion if $\underbrace{OPT(**)}_{\sum y_i} > 0 \Rightarrow (*)$ incompatible STOP!

if $OPT(**) = 0$

$$\Downarrow$$

$$\left.\begin{cases} \tilde{y}_i = 0 \\ \tilde{x}_i \end{cases} \Longrightarrow \begin{array}{l} \boxed{\tilde{x} \geq 0} \\ A\tilde{x} = b \end{array}\right\} \text{BFS for pb } (*)$$



$\longrightarrow \{B_0\}$

B OPT $(**)$

soln to (1) + 1/2 soln to (2)   infinite optimum ?

D OPT (+ +)

$\boxed{\text{soln to (1) + 1/2 soln to (2)}}$ infinite OPTIMUM ?

$$z = (x_1 + x_2 + x_3 + x_4 + x_5)$$

$$\begin{cases} x_3 = 1 - 3x_1 - 2x_2 \\ x_4 = 1 - 5x_1 - x_2 \\ x_5 = 1 - 2x_1 - 5x_2 \end{cases}$$

BFS $\begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = x_4 = x_5 = 1 \end{cases}$

$\boxed{OBJ = 3}$

To figure out which r.h.s variables are candidates for swapping (pivoting) we have to get rid of l.h.s. vars in the objective

$$z = (x_1 + x_2 + 1 - 3x_1 - 2x_2 + 1 - 5x_1 - x_2 + 1 - 2x_1 - 5x_2)$$

reduced cost of x_1

$$z = 3 - 5x_1 - 7x_2$$

negative coefficients

Rule variables with negative reduced cost (≤ 0) are candidates to enter the basis

reduced cost $\nearrow$ < 0 pivoting improves objective
$= 0$ obj doesn't change.

EXAMPLE OF RULE

Pick up variable in N with <u>smallest</u> reduced cost

QUESTION

What if all reduced costs are ≥ 0 ?

Current soln OPTIMAL !

STOP

1 ✓
2 ✓
3 ✓

WHICH VAR EXITS THE BASIS?

$$\begin{cases} x_3 = 1 - 3x_1 - 2x_2 \\ x_4 = 1 - 5x_1 - x_2 \\ x_5 = 1 - 2x_1 - 5x_2 \end{cases}$$

1/3
1/5 $\leftarrow$
1/2

What if x_1 appeared positively in all equations?

Can increase x_1 as much as I want

without creating neg values

$\downarrow$

$\boxed{OPT = -\infty}$ $\longrightarrow$ STOP

$\boxed{\text{Choose var to exit the basis only from } \underline{\text{positive terms}}}$

What if free term is zero?

$$x^B = \begin{pmatrix} 0 \end{pmatrix} - \left(\quad \right) \left(\quad \right)$$

$\boxed{\text{swapping that variable does not improve obj value.}}$

$$\boxed{x_i = \bar{b}_i - \sum_k \bar{a}_{ik} x_k}$$

B $\qquad\qquad$ N

Assume x_{k_0} enters the basis

choose $x_j \in B$ that minimizes

$$\frac{\bar{b}_j}{\bar{a}_{ij}} = \min \left\{ \frac{\bar{b}_{k_0}}{\bar{a}_{ik_0}} \;\middle|\; \frac{\bar{b}_k}{\bar{a}_{ik}} > 0 \right\}$$

$$\boxed{\text{pivot}\,(x_{k_0}, x_j)}$$

STILL TO COME

(1) How do I avoid cycling?

(2) How do I implement everything?