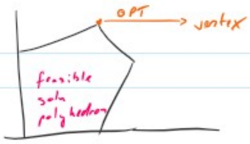
COURSE #3

Thursday, October 19, 2023 5:58 PM

NOVEMBER 23^{rd} → ALTERNATE DATE ?

LAST TIME GEOMETRIC INTERPRETATION OF 2VAR LP.

LP →



Normal forms for LP problem.

LP

$$\max(2x+3y)$$
$$x+y \le 3 \quad | \; z$$
$$2x+y \ge 4 \quad | \; t$$
$$x, y \ge 0$$

"slack variables"

$$z = 3-(x+y)$$
$$t = 2x+y-4$$

Normal (equational) form

- always a minimization pb
- all constraints equalities
- all variables are ≥ 0

⇓

$$\begin{cases} \min(-2x-3y) \\ x+y+z=3 \\ 2x+y-t=4 \\ x,y,z,t \ge 0 \end{cases}$$

- max → min : multiply by -1
- add slack variables to inequations

s.v. = bigger side − smaller side

- convert unrestricted variables to vars ≥ 0

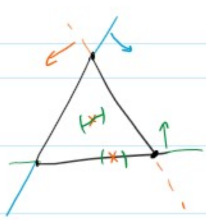
$w \in \mathbb{R}$ $\qquad w = w_1 - w_2$, $w_1, w_2 \ge 0$

$$\begin{cases} \min(-2x-3y+0\cdot z+0\cdot t) \\ x+y+z=3 \\ 2x+y+t=4 \\ x,y,z,t \ge 0 \end{cases} \Rightarrow \begin{cases} \min(c^T\cdot X) \\ AX=b \\ X \ge 0 \end{cases}$$

$$c = \begin{pmatrix} -2 \\ -3 \\ 0 \\ 0 \end{pmatrix} \qquad X = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \qquad b = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}$$

$c^T = (\quad)$ $\qquad c^T\cdot X = -2x-3y$

- feasible sols polyhedron in $\mathbb{R}^n$
- optimum reached at one of the vertices

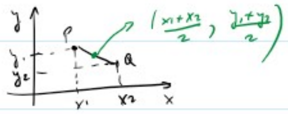


convex polyhedron: intersection of half-spaces

in $\mathbb{R}^n$ half-space inequality

in a half-space inequality

Point P in a polyhedron **vertex** iff P is not inside any segment $[QR]$, Q, R points of the polyhedron



$R = (x, y)$

$R \in [PQ]$
$$\begin{cases} x = \lambda x_1 + (1-\lambda) x_2 \\ y = \lambda y_1 + (1-\lambda) y_2 \end{cases}$$
$0 \le \lambda \le 1$ convex combination

$P = (x_1 \dots x_n)$
$Q = (y_1 \dots y_n)$
$\Rightarrow [PQ] = (z_1 \dots z_n)$
$$\begin{cases} z_1 = \lambda x_1 + (1-\lambda) y_1 \\ \vdots \\ z_n = \lambda x_n + (1-\lambda) y_n \end{cases}$$

W convex $\Leftrightarrow \forall P, Q \in W \quad [PQ] \subseteq W$

(T) set of solns of $\begin{cases} Ax = b \\ x \ge 0 \end{cases}$ convex set

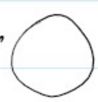
Pf $x_1, x_2 \in W \quad \lambda \in [0,1]$

$x = \lambda x_1 + (1-\lambda) x_2$

$Ax = b \qquad Ax = A(\lambda x_1 + (1-\lambda) x_2)$
$= \lambda A x_1 + (1-\lambda) A x_2$
$= \lambda b + (1-\lambda) b = b$

$\left. \begin{matrix} x_1 \ge 0 \\ x_2 \ge 0 \end{matrix} \right\} \Rightarrow x \ge 0$

$Ax = b$
$x \ge 0$

- convex (bounded) polyhedron 
  - finite optimum
- non-empty but extends to ∞ $(-\infty)$
  - opt. infinite
- empty.

(T) There exists a vertex of the soln polyhedron which is opt soln.

Pf
$$\begin{cases} \min (c^T x) \\ Ax = b \\ x \ge 0 \end{cases}$$
Assume $P_1 \dots P_k$ all vertices of soln polyhedron

Suppose $\exists\, x$ feasible s.t.
$$\boxed{\begin{matrix} c^T x < c^T x_1 \\ \vdots \\ c^T x < c^T x_k \end{matrix}}$$
optimum none of the vertices.

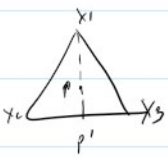
Claim For every x in a polyhedron there exist

$\lambda_1, \dots \lambda_k \geq 0 \qquad \lambda_1+\lambda_2+\dots+\lambda_k=1$

s.t.

$$\boxed{x = \lambda_1 x_1 + \lambda_2 x_n + \dots + \lambda_n x_n}$$

$$x = \lambda_1 x_1 + \underbrace{(1-\lambda_1)}_{\lambda_2} x_2$$



E.g. center of gravity

$\lambda_1 = \lambda_2 = \lambda_3 = 1/3$

$G = \frac{1}{3} x_1 + \frac{1}{3} x_2 + \frac{1}{3} x_3$

$$c^T \cdot x = \lambda_1 \cdot c^T x_1 + \lambda_2 \cdot c^T x_2 + \dots \lambda_n \cdot c^T x_n$$

$\lambda_1 c^T x < c^T x_1 \cdot \lambda_1$

$\lambda_2 c^T x < c^T x_2 \cdot \lambda_2$

$\vdots$

$\lambda_n c^T x < c^T x_n \cdot \lambda_n$

$$\underbrace{(\lambda_1 + \dots + \lambda_n)}_{1} \cdot c^T x < c^T (\underbrace{\lambda_1 x_1 + \dots \quad \lambda_n \cdot x_n}_{x})$$

$\boxed{c^T x < c^T x}$ contradiction ☒

vertex $\begin{cases} Ax = b \\ x \geq 0 \end{cases}$

$\updownarrow$

basic feasible solution

$$\begin{cases} x + y + z = 3 \\ 2x + y + t = 4 \end{cases} \qquad A = \begin{array}{c} \text{aux. variables} \\ \begin{array}{cccc} x & y & z & t \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

det $\neq 0$

$$\begin{cases} z = 3 - x - y \\ t = 4 - x - y \end{cases}$$

for every $x, y \Rightarrow \begin{pmatrix} z \\ t \end{pmatrix}$

If we set $x = y = 0$ $\begin{cases} z = 3 \geq 0 \\ t = 4 \geq 0 \end{cases}$

$\begin{pmatrix} 0 \\ 0 \\ 3 \\ 4 \end{pmatrix}$ soln to $\begin{cases} Ax = b \\ x \geq 0 \end{cases}$

$\begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}$ soln to $\begin{cases} Ax = b \\ x \geq 0 \end{cases}$

basic feasible soln.

feasible solns

Exp $\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \neq 0$ $\quad \begin{cases} x + y = 3 - z \\ 2x + y = 4 - t \end{cases}$

$$z = t = 0$$

$$x = 1, \; y = 2 \geq 0$$

$\begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}$ is a basic feasible soln

---

in general $\quad A = (A_B ; A_N)$ $\quad$ Exp $\quad A_B = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$ $\quad A_N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$X^B = \begin{pmatrix} x \\ y \end{pmatrix} \qquad X^N = \begin{pmatrix} z \\ t \end{pmatrix}$$

$$\boxed{A \cdot X = A_B \cdot X^B + A_N X^N}$$

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} x + y + z \\ 2x + y + t \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z \\ t \end{pmatrix} = \begin{pmatrix} x + y \\ 2x + y \end{pmatrix} + \begin{pmatrix} z \\ t \end{pmatrix} = \begin{pmatrix} x + y + z \\ 2x + y + t \end{pmatrix}$$

$$Ax = b \iff A_B X^B + A_N X^N = b$$

$\det(A_B) \neq 0$

$$(A_B)^{-1} \,|\, A_B X^B + A_N X^N = b$$

$$X^B + (A_B)^{-1} A_N X^N = \underbrace{(A_B)^{-1} b}_{\bar{b}}$$

$\Delta \begin{cases} X^N = 0 \\ X^B = (A_B)^{-1} b \end{cases}$ Canonical Soln. associated to basis B

Basic feasible soln if $X^B = (A_B)^{-1} b \geq 0$

Def $\quad \Delta$ is BFS iff $X^B \geq 0$

T $\quad$ Vertices of Soln polyhedron $\begin{matrix} Ax = b \\ x \geq 0 \end{matrix}$ $\iff$ Basic feasible solns

(T) Vertices of soln polyhedron $Ax=b$, $x \geq 0$ $\iff$ Basic feasible solns

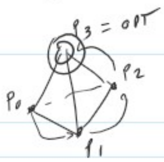
$\boxed{\text{SIMPLEX ALGORITHM}}$ (DANTZIG 1947)

1. Start with a BFS B_0
2. While (I can improve current BFS)

IDEA

$$B_i \rightarrow B_{i+1}$$

3. Declare basis B_i as optimal



$$\max (x_1 + x_2)$$

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 1 \\ 2x_1 \qquad + x_4 &= 1 \\ x_1 \qquad + x_5 &= 1 \end{aligned}$$

first hit x_3

$$\begin{cases} x_3 = 1 - 3x_1 - 2x_2 & x_1 \leq 1/3 \\ x_4 = 1 - 2x_1 & x_1 \leq 1/2 \\ x_5 = 1 - x_1 & x_1 \leq 1 \end{cases}$$

$$B_0 = \begin{pmatrix} x_1 = 0 \\ x_2 = 0 \\ x_3 = 1 \\ x_4 = 1 \\ x_5 = 1 \end{pmatrix}$$

$$\boxed{OBJ(B_0) = 0}$$

We'd like to increase x_1 (x_2).

ONLY basic variables are >0 in the BFS.

swap x_3 for x_1 on the l.h.s.

$$3x_1 = 1 - 2x_2 - x_3$$

$$\Updownarrow$$

$$x_1 = 1/3 - 2/3\, x_2 - \frac{1}{3} x_3$$

$$x_4 = 1 - 2x_1 = 1 - 2(1/3 - 2/3\, x_2 - 1/3\, x_3)$$

$$x_5 = 1 - x_1 = 1 - (1/3 - \tfrac{2}{3} x_2 - \tfrac{1}{3} x_3)$$

$$\begin{cases} \boxed{x_1 = 1/3 - 2/3\, x_2 - 1/3\, x_3} \\ x_4 = 1/3 + 4/3\, x_2 - 2/3\, x_3 \\ x_5 = 2/3 + 2/3\, x_2 + 1/3\, x_3 \end{cases}$$

$$\rightarrow \begin{pmatrix} x_1 = 1/3 \\ x_2 = 0 \end{pmatrix}$$

$$\beta_1 \quad \begin{pmatrix} x_1 = 1/3 \\ x_2 = 0 \\ x_3 = 0 \\ x_4 = 1/3 \\ x_5 = 2/3 \end{pmatrix}$$

pivot (x_1, x_3)

$$\boxed{OBJ(\beta_1) = 1/3}$$

pivot (x_1, x_2)

$$\frac{}{2} \cdot \frac{}{3} - \frac{}{2} \cdot x_1 - \frac{3}{2} \cdot \frac{1}{3} x_3$$