COURSE #5

Thursday, November 2, 2023 6:00 PM

THE SIMPLEX ALGORITHM

$$LP \begin{cases} \min(c^T x) \\ Ax = b \\ x \geq 0 \end{cases}$$

Start with basic feasible soln B_0

Two-phase simplex

→ incompatible

(while $\exists\, i \in N$ s.t. $\tilde{c}_i < 0$)

REDUCED COST of variable x_i

choose $i \in N$ s.t. $\tilde{c}_i < 0$ ← EXPL Take i s.t. $\tilde{c}_i < 0$ and $|c_i|$ maximal.

choose $j \in B$ ← $\boxed{\dfrac{\bar{a}_{ij}}{\bar{b}_j} = \max \left\{ \dfrac{\bar{a}_{ik}}{\bar{b}_k} ;\ \bar{a}_{ik} > 0 \right\}}$

pivot (x_i, x_j)

what if r.h.s is empty?

$\boxed{OPT = -\infty}$

current soln OPTIMAL

$\bar{a}_{ij} = 0$

STALLING / CYCLING

BLAND'S RULE

if $\bar{a}_{ij} = 0 \Rightarrow$ pivot (x_i, x_j) does not improve objective value.

SCENARIO TO AVOID

$$\underbrace{B_i \xrightarrow{\Delta OBJ = 0} B_{i+1} \xrightarrow{\Delta OBJ = 0} \dots B_{i+k} \dots \longrightarrow B_i}_{\text{STALLED}}$$

CYCLING!

RULE FOR CHOOSING VARIABLES THAT GUARANTEES EVENTUAL PROGRESS

What if multiple variables in N are "the same"?
What if multiple candidates in B "the same"?

BLAND'S RULE

1. Chose a fixed var. ordering.
2. If several $i \in N$ candidates choose smallest i
3. If for that i there are multiple j's that are candidates to exit the basis

multiple j's that are candidates
to exit the basis
choose smallest j

USING BLAND'S RULE GUARANTEES LACK OF CYCLING

IN PRACTICE

PROBLEM: USING BLAND'S RULE SLOWS DOWN SIMPLEX

PRACTICAL COMPROMISE

Choose # of steps k (e.g. $k = 250$)
if you have STALLED for k steps start using BLAND'S RULE UNTIL YOU MAKE PROGRESS

REMAINING PROBLEM

How DO WE ACTUALLY IMPLEMENT ALL THIS ?

TABLEAU FORM OF THE SIMPLEX ALGORITHM

$$\begin{cases} \min(z = c^T x) \\ Ax = b \\ x \geq 0 \end{cases}$$

$$\begin{array}{c|cc|cc|c} & \overbrace{x_1 \cdots x_n}^{N} & & \overbrace{x_{n+1} \cdots x_{n+m}}^{B} & & \\ \hline x_{n+1} & & & 1 & 0 & \bar b_1 \\ \vdots & \bar A & & & \ddots & \vdots \\ x_{n+m} & & & 0 & 1 & \bar b_m \\ \hline & -\tilde c_1 & -\tilde c_n & 0 \cdots & 0 & -z \end{array}$$

$$\begin{cases} x_{n+1} = \bar b_1 - \sum_{j=1}^{n} a_{n+j,1} x_j \\ x_{n+m} = \bar b_m - \sum_{j=1}^{m} a_{n+j,m} x_j \end{cases}$$

minus the red costs of vars in N

minus the obj value at current BFS

write the obj value in terms of vars in N only

CAUTION: Exact form of the tableau varies from author to author.

EXAMPLE

EXAMPLE

$$\begin{cases} \max\ (x_1 + 2x_2) \\ x_1 \le 2 \quad \mid x_3 \\ x_2 \le 2 \quad \mid x_4 \\ x_1 + x_2 \le 3 \quad \mid x_5 \\ x_1, x_2 \geq 0 \end{cases} \qquad \begin{cases} \min\ (z = -x_1 - 2x_2) \\ x_1 + x_3 = 2 \\ x_2 + x_4 = 2 \\ x_1 + x_2 + x_5 = 3 \\ x_1, x_2, x_3, x_4, x_5 \geq 0 \end{cases}$$

$$\boxed{\text{OPT} \begin{cases} x_1 = 1 \\ x_2 = 2 \\ \text{val} = 5 \end{cases}}$$

$\{x_3, x_4, x_5\}$ basic feasible solution

$$\begin{cases} \min(-x_1 - 2x_2) \\ x_3 = 2 - x_1 \\ x_4 = 2 - x_2 \qquad 2 = 2/1 \\ x_5 = 3 - x_1 - x_2 \qquad 3 = 3/1 \\ x_i \geq 0 \end{cases}$$

	N: x_1	x_2	B: x_3	x_4	x_5		
x_3	1	0	1	0	0	2	0/2
$\leftarrow x_4$	0	1	0	1	0	2	1/2
x_5	1	1	0	0	1	3	1/3
	1	2	0	0	0	0	

largest

BFS
$$\begin{bmatrix} x_1 = 0 \\ x_2 = 0 \\ x_3 = 2 \\ x_4 = 2 \\ x_5 = 3 \end{bmatrix}$$

(both x_1, x_2 candidates for pivoting)

2,

$$\boxed{\text{pivot}(x_2, x_4)}$$

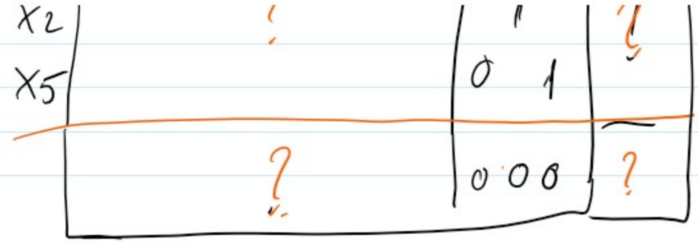
$$\boxed{\text{HOW DO I DO PIVOTING IN TABLEAU FORM?}}$$

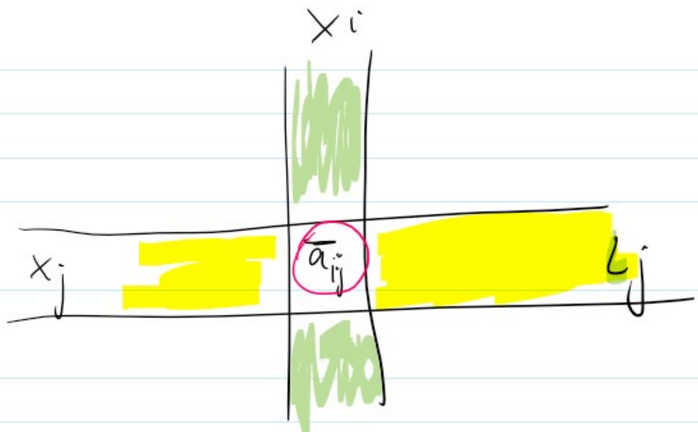
$$\begin{cases} \min(z = -x_1 - 2x_2) \\ x_3 = 2 - x_1 \\ x_2 = 2 - x_4 \\ x_5 = 3 - x_1 - x_2 \end{cases} \Longrightarrow \begin{cases} \min(z = -x_1 - 2(2 - x_4)) = -4 - x_1 + 2x_4 \\ x_3 = 2 - x_1 \\ x_2 = 2 - x_4 \\ x_5 = 3 - x_1 - (2 - x_4) = 1 - x_1 + x_4 \end{cases}$$

$$\begin{cases} \min(z = -4 - x_1 + 2x_4) \\ x_3 = 2 - x_1 \\ x_2 = 2 - x_4 \\ x_5 = 1 - x_1 + x_4 \end{cases}$$

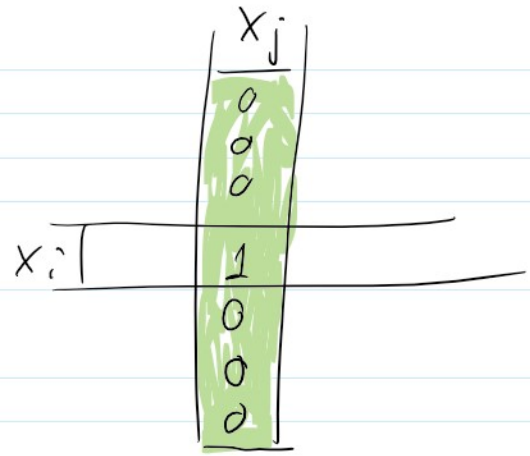
	x_1	x_4	$x_3\ x_2\ x_5$	
x_3			1 0	
x_2	?		1	?
x_5			0 1	

$$x_5 = 1 - x_1 + x_4$$





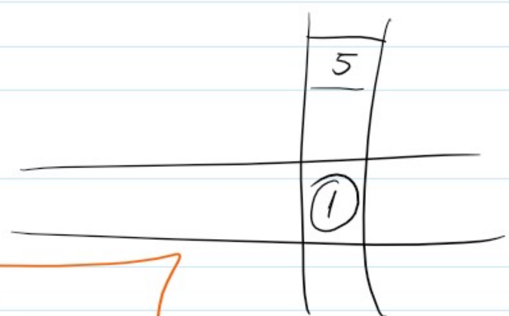
$\Rightarrow$



① $$\boxed{L_j' = \frac{L_j}{\bar{a}_{ij}}}$$

divide all the elements of the line of the pivot by the pivot.

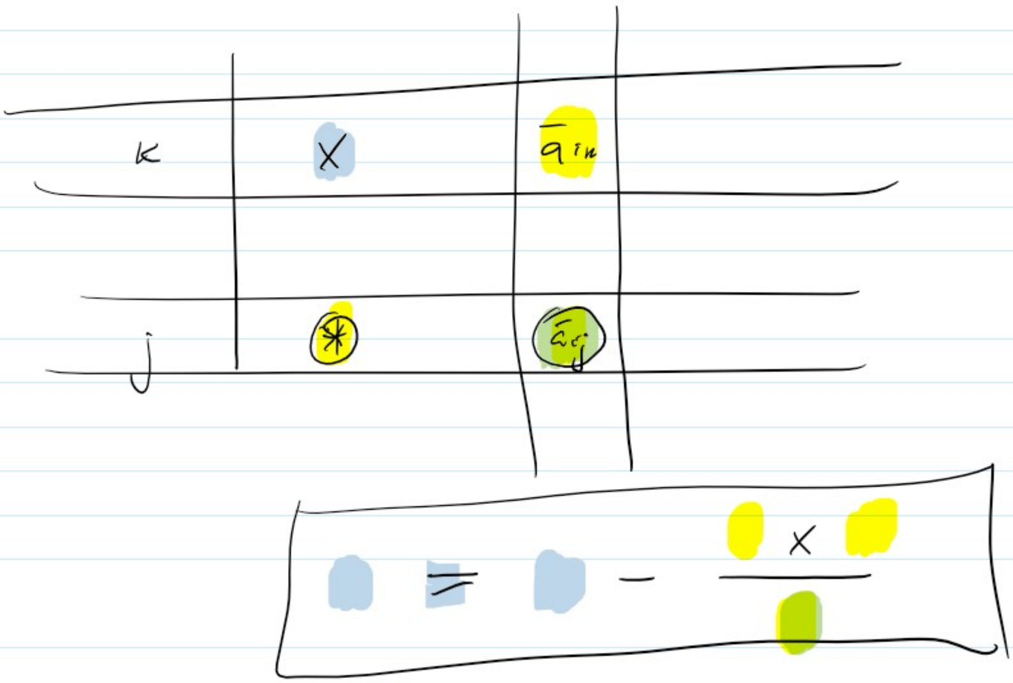
② $k \neq j$ $$\boxed{L_k' = L_k - \bar{a}_{ik} L_j'}$$



(1) $\wedge$ (2) $$\boxed{L_k' = L_k - \frac{\bar{a}_{ik}}{\bar{a}_{ij}} L_j}$$

new values

old values



$$\boxed{\text{WHAT HAPPENS TO FREE TERMS/ REDUCED COSTS ?}}$$

Ans Apply the same rule to them!

Example

	x_1	x_2	x_3	x_4	x_5	
x_3	1	0	1	0	0	2
x_4	0	1	0	1	0	2
x_5	1	1	0	0	1	3
	1	2	0	0	0	0

$\Downarrow$ pivot (x_2, x_4)

	x_1	x_4	x_3	x_2	x_5	

	x_1	x_4	x_3	x_2	x_5		
x_3	1	0	1	0	0	2	1/2
x_2	0	1	0	1	0	2	
x_5	1	0	0	-1	1	1	1/1
	1	0	0	-2	0	-4	

c'

$\Downarrow$ pivot (x_1, x_5)

	x_5	x_4	x_3	x_2	x_1	
x_3	0	0	1	1	-1	1
x_2	0	1	0	1	0	2
x_1	1	0	0	-1	1	1
	0	0	0	-1	-1	-5

no more positive values

OPTIMAL

$x_4 = x_5 = 0$

$x_3 = 1$

$\boxed{x_2 = 2,\quad x_1 = 1,\quad OPT = 5}$

$\boxed{\text{gi65t3j}}$