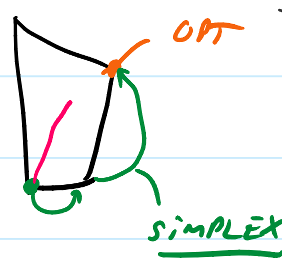
COURSE #6

Thursday, November 9, 2023 5:54 PM

LAST SIMPLEX ALG IN TABLEAU FORM

COMPLEXITY OF LINEAR PROGRAMMING

LP →
- simplex
- interior point methods → gradient ascent
  {not going to cover in this course}
- ellipsoid method (Khachiyan)
  - poly-time complexity
  - construct sequence of ellipsoids guaranteed to contain OPT (cheating a bit)



ellipsoid NOT competitive in practice

simplex (+ int. point) EXPONENTIAL COMPLEXITY good "in practice"

EXP. BAD INSTANCES FOR SIMPLEX

$$\max\ (10^{n-1} x_1 + 10^{n-2} x_2 + \cdots + 10 x_{n-1} + x_n)$$

$$x_1 \le 1$$

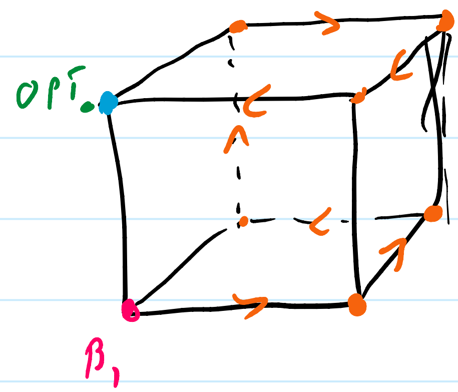
$$20 x_1 + x_2 \le 100$$

$$200 x_1 + 20 x_2 + x_3 \le 100^2$$

$$\vdots$$

$$2\left[\sum_{j=1}^{i-1} 10^{i-j} x_j\right] + x_i \le 100^{i-1}$$

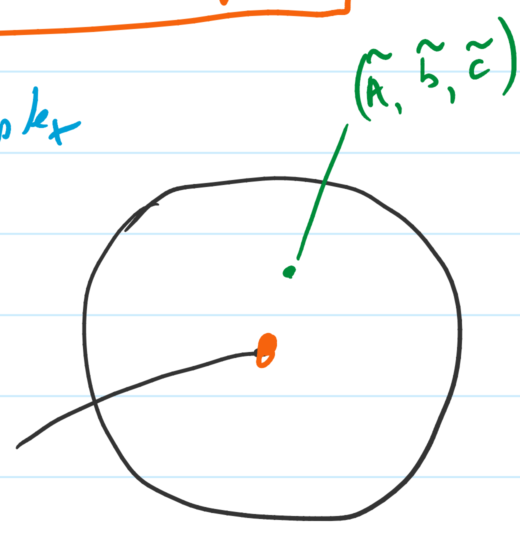
$$x_1, \ldots, x_n \ge 0$$



Theoretical explanation why simplex is good

"smoothed complexity" of simplex

random perturbation $(\tilde{A}, \tilde{b}, \tilde{c})$ of (A, b, c)



average complexity of simplex on perturbed instances

for every (A,b,c)

easy instances dominate

dense around any "hard" instance

for every (a,b,c)
average complexity polynomial.

DUALITY IN LINEAR PROGRAMMING

Motivating example

$$(1)\quad \begin{cases} \min(x+y) \\ 2x+y \geq 3 \\ x+2y \geq 4 \\ x,y \geq 0 \end{cases}$$

Want LB on OPT
without actually computing OPT

CLAIM For every soln of (1) $\boxed{x+y \geq 7/3}$

Added
$$\begin{array}{r} 2x+y \geq 3 \\ x+2y \geq 4 \\ \hline 3x+3y \geq 7 \end{array} \Rightarrow \boxed{x+y \geq 7/3}$$

$\boxed{\text{"smarter" method}}$

$$\begin{array}{l|l} 2x+y \geq 3 & a \geq 0 \\ x+2y \geq 4 & b \geq 0 \end{array} \qquad \begin{array}{r} 2ax+ay \geq 3a \\ bx+2by \geq 4b \\ \hline x\underbrace{(2a+b)}_{\leq 1}+y\underbrace{(a+2b)}_{\leq 1} \geq 3a+4b \end{array}$$

$x+y \geq$

$$\boxed{OPT \geq 3a+4b}$$

How do I find the best pair (a,b)?

to answer have to solve LP (2)

$$(2)\begin{cases} \max(3a+4b) \\ 2a+b \leq 1 & \mid u \\ a+2b \leq 1 & \mid v \\ a,b \geq 0 \end{cases}$$

$$OPT(2) \leq \quad \underset{(1)}{\overset{(3)}{\parallel}} \begin{cases} \min(u+v) \\ 2u+v \geq 3 \\ u+2v \geq 4 \\ u,v \geq 0 \end{cases}$$

$$2au + bu \leq u$$
$$av + 2bv \leq v$$

$$3a+4b \leq a(\underbrace{2u+v}_{\geq 3}) + b(\underbrace{u+2v}_{\geq 4}) \leq u+v$$

$$(1) \rightleftarrows (2)$$

Conclusion

Problems come "in pairs"

(2) the dual of pb (1)

primal   dual

RULES FOR COMPUTING THE DUAL PROBLEM

$$(1)\begin{cases} \min(1\cdot X + 1\cdot Y) \\ 2X+Y \geq 3 & \mid a \end{cases}$$

$$(2)\begin{cases} \max(3a+4b) \\ 2a+3b \leq 1 & \mid X \end{cases}$$

$$(1)\begin{cases} 2x + y \geq 3 & \mid a \\ 3x + 2y \geq 4 & \mid b \\ x, y \geq 0 \end{cases}$$

$$(2)\begin{cases} 2a + 3b \leq 1 & \mid x \\ a + 2b \leq 1 & \mid y \\ a, b \geq 0 \end{cases}$$

$A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$

$b = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ $c = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

transposed

$\begin{pmatrix} 2 \times 3 \end{pmatrix}$ $\begin{pmatrix} 3 \times 2 \end{pmatrix}$

- if primal MIN/MAX then dual MAX/MIN
- every line in the primal ⇒ variable in the dual
- system matrix of dual = TRANSPOSED of SYSTEM MATRIX primal

vars of dual ≡ Constraints primal
Constraints of dual ≡ vars of primal

non-negative variables primal → inequalities in the dual
Inequalities primal → non-negative vars in the dual

equalities primal → unconstrained var. in the dual
unconstrained var → equalities in the

(unconstrained var primal $\longrightarrow$ equalities in the dual.)

$\min(\quad)$

$\leq \Rightarrow$ multiply by -1 $[\quad \geq \quad]$

bad for min

Example

$$\max(x+z)$$

$$\begin{cases} 2x - y - z \leq 5 & \mid a \\ 1 \cdot x + 3y + 2z = 7 & \mid b \\ x \geq 0 \\ y \text{ unconstrained} \\ z \geq 0 \end{cases}$$

$$\min(5a + 7b)$$

$$\begin{cases} 2a + 1 \cdot b \geq 1 & \mid x \\ -a + 3b = 0 & \mid y \\ -a + 2b \geq 1 & \mid z \\ a \geq 0 \\ b \text{ unconstrained} \end{cases}$$

$$\begin{pmatrix} 2 & -1 & -1 \\ 1 & 3 & 2 \end{pmatrix}$$

Primal

Dual

How does OPT (Primal) compare to the OPT (Dual)

assuming both optima exist!

Answer {STRONG DUALITY THM.}

$\boxed{\text{OPT (primal)} = \text{OPT (dual)}!}$

Not going to prove this

Ⓣ {WEAK DUALITY THM.}

Let

(P) $\begin{cases} \min(c^T x) \\ Ax \geq b \\ x \geq 0 \end{cases}$ (D) $\begin{cases} \max(b^T y) \\ yA^T \leq c \\ y \geq 0 \end{cases}$

be dual problems. Let x^* and y^* be feasible solutions to (P), (D)

Then $\boxed{c^T x^* \geq b^T y^*}$

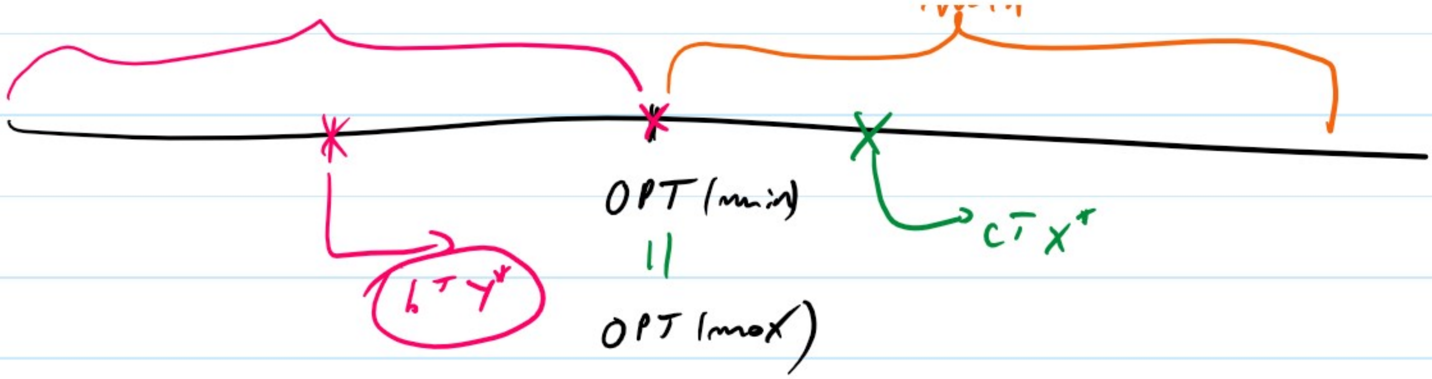
val min

val maximization

$\boxed{\text{OPT(min)} \geq \text{OPT(max)}}$

max

min



PROOF OF WEAK DUALITY THM.

$$c^T x^* - b^T y^* =$$ (want to prove ≥ 0)

$$= c^T x^* - (y^*)^T A x^*$$

$$+ (y^*)^T A x^* - \underbrace{b^T y^*}_{\substack{\text{real } \# \\ || \\ (y^*)^T b}}$$

real number
scalar product of two vectors of same size

$$y^T = (\longrightarrow) \qquad Ax^*$$

$$= \left(c^T - (y^*)^T A\right) x^* + (y^*)^T (A x^* - b)$$

$$||$$

$$\left(\underbrace{c - A^T y^*}_{\geq 0}\right)^T \underbrace{x^*}_{\geq 0} + \underbrace{(y^*)^T}_{\geq 0} \underbrace{(A x^* - b)}_{\geq 0} \geq 0$$

non neg vector

$\boxtimes$

STRONG DUALITY $\boxed{\max(P) = \min(D)}$

$\boxed{\text{MAX-FLOW MIN-CUT THM}}$ example of duality