Gödel's incompleteness theorems

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Godel's incompleteness theorems deals with the limit of provability in formal axiomatic theories. It states that any set of axiom could not completely prove or justify any possible foundation of mathematics i.e. it will always be inevitably incomplete. It also state that no candidate set of axioms can ever prove its own consistency. So, it implies that there can be no universal mathematical theory, no unification of what is provable and true. What mathematicians can prove is determined by their beginning assumptions, rather than by any underlying ground truth from which all solutions emerge.

The incompleteness theorems are applicable to formal systems of sufficient complexity to describe consistent, comprehensive, and effectively axiomatized natural number arithmetic. If the set of a formal system is recurringly listed, it is said to be axiomatized efficiently. A set of axioms is complete if the axioms prove that statement or its negation for any statement in the axioms' language. If there is no statement for which the axioms can establish both the statement and its negation, the set of axioms is consistent; otherwise, it is inconsistent.

The incompleteness theorems consists of two theorems which are:

1. First incompleteness theorem
"Any consistent formal system F within which a certain amount of elementary arithmetic can be carried out is incomplete; i.e., there are statements of the language of F which can neither be proved nor disproved in F."

2. Second incompleteness theorem"
Assume F is a consistent formalized system which contains elementary arithmetic. Then $F \nvdash Cons(F)$"

The incompleteness theorem is closely connected with several outcomes in recursion theory on undecidable sets. Many theorem in recursion theory such as Matiyasevich's solution to Hilbert's 10th problem, Kleene's halting problem, Chaitin's incompleteness theorem helped to prove Gödel's incompleteness theorem.