# *Facial Image Characterization and Reconstruction Using Singular Value Decomposition*


**An Assignment Submitted by:-**

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**TO THE**

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**J.K LAKSHMIPAT UNIVERSITY JAIPUR**

Dataset-

Figure 1: Dataset Image

Notation:

Let M denote the total number of images in the dataset (M=3000). Additionally, let n and m represent the row and column dimension of a given image matrix, respectively. N denote the total number of elements in a given image matrix. Note: N=62500 (250×250).

Let, Image be denoted by $\varphi \in \mathrm{R}^{n \times m}$ where $\varphi = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ x_{31} & x_{32} & \cdots & x_{3m} \\ \vdots & \vdots & \ddots & \vdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$

where $x_{ij}$ is the grayscale intensity of pixel at row i and column j.

1. **The Average Image**: This is the mean image of all the images in the dataset. Denoted by $\bar{\varphi}$. Constructed using the formula: - $\bar{\varphi} = \frac{1}{M}\sum_{i=1}^{M} \varphi_i$.
   - Every pixel in the average image is Average grayscale intensity of that pixel across all faces
   - Represents the "common structure" of all faces in dataset.

Figure 2: Average Image

●Note: For easy computation we have flattened the matrix from m × n matrix to an N × 1 column vector.

2. **The Perturbation Matrix**: From the average image, we are constructing the perturbation matrix. This matrix will provide a value for each individual images deviation from this database average as we are subtracting the average matrix from each image there are many ways to perturbate a matrix but we have chosen to subtract the average of them so that we get how much each image has deviated from average face. Denoted by P, the perturbation matrix is $P = [\phi_1, \phi_2, \phi_3, \ldots, \phi_M] \in R^{N\times M}$ where $\phi_i = \varphi_i - \bar{\varphi}$.

●Note: Dimension of the P (Perturbation matrix) is much larger than that of the individual images.
Reason: It is denoting characterization of all images in dataset.

3. **The Covariance Matrix:** The next important step is to compute the eigen vectors of this covariance matrix for constructing the image.
   First let's define what a Covariance matrix is: -
   A covariance matrix is a square matrix that shows how each pair of variables in a dataset varies together.

Suppose you have a dataset with: n observations (rows) and d features (columns)

Let matrix-

$$X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1d} \\ x_{21} & x_{22} & \cdots & x_{2d} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nd} \end{bmatrix}$$

- First compute mean-centred matrix**:** $\bar{X} = X - \mu$
- Then the covariance matrix is:$\Sigma = \frac{1}{n-1} \widetilde{X^T} \tilde{X}$

For 3 variables it looks like:

$$\Sigma = \begin{bmatrix} \mathrm{Var}(X_1) & \mathrm{Cov}(X_1, X_2) & \mathrm{Cov}(X_1, X_3) \\ \mathrm{Cov}(X_2, X_1) & \mathrm{Var}(X_2) & \mathrm{Cov}(X_2, X_3) \\ \mathrm{Cov}(X_3, X_1) & \mathrm{Cov}(X_3, X_2) & \mathrm{Var}(X_3) \end{bmatrix}$$

- Diagonal elements = variances
- Off-diagonal elements = covariance between variable pairs
- Matrix is symmetric**:** Cov(i,j)=Cov(j,i)

- ❖ Meaning of variance –
- ❖ **Positive covariance -** variables increase together
- ❖ **Negative covariance** -one increases, other decreases
- ❖ **Zero covariance** -variables are independent / unrelated

**Properties of Covariance matrix-**

1. $A\ covariance\ matrix\ is\ always\ positive\ semidefinite\ (PSD).$

Take any non-zero vector $v$: $v^T C v = \frac{1}{n-1} v^T(\widetilde{X^T}\tilde{X})v$

$v^T C v = \frac{1}{n-1} ||\widetilde{X}v||^2$

Since a squared norm is always $\geq 0$: $||\tilde{X}v||^2 \geq 0$

Therefore: $v^T C v \geq 0$ Thus, the covariance matrix $C$ is PSD.

2. It's Symmetric matrix. Cov(i,j)=Cov(j,i)

3. Eigenvalues of a covariance matrix real**,** non-negative and orthogonal.

4. Sum of Eigen values = Total variance $\sum_i \lambda_i = \mathrm{Trace}(\Sigma) = \text{Total variance}$

$C = \frac{1}{M-1}\sum_{i=1}^{M} P\,P^T = \frac{1}{M-1}\sum_{i=1}^{M} \phi_i\phi_i^T$ But due to very high memory requirements we cannot save this covariance matrix and since we need only eigen vectors of covariance matrix let us use SVD.

● Singular Value Decomposition: - is the representation of any n× m matrix by three distinct matrices representing a three-step transformation.

$A = U\,\Sigma\,V^T$ where U is called left singular vector, Σ is a diagonal matrix with its diagonal entries as Singular values of matrix A in decreasing order and V is right singular vector.

U and V are orthonormal matrices. $UU^T = I$ and $VV^T = I$

$$U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1m} \\ u_{21} & u_{22} & \cdots & u_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \in \mathbb{R}^{nxm},\ V = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1m} \\ v_{21} & v_{22} & \cdots & v_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \in \mathbb{R}^{mxm},\ \Sigma = \begin{bmatrix} \sigma_{11} & 0 & 0 & \cdots \\ 0 & \sigma_{22} & 0 & \cdots \\ 0 & 0 & \sigma_{33} & \vdots \\ \cdots & \cdots & \cdots & \ddots \end{bmatrix} \in \mathbb{R}^{mxm}$$

● Geometric Interpreatation of SVD

Let A be a 2×2 matrix with arbitrary elements. Let S be a unit circle in $R^2$. The transformation of A to the unit circle S results in a hyper-ellipse, which will be denoted AS.

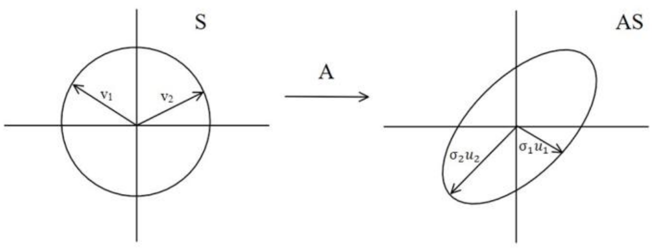



Figure 3: Unit circle S transformed by 2x2 matrix A.

$Therefore, Av_i = \sigma_i u_i$

Post multiplying both side by $v_i^T$ we have (since $v_i v_i^T = I$ ( *as V is an orthogonal matrix*))

$$Av_i v_i^T = \sigma_i u_i v_i^T = A = \sum_{i=1}^{m} \sigma_i \; u_i v_i^T$$

$$A = U \Sigma V^T$$

- Each matrix, U, Σ and V, and their respective vectors have significance in relation to the geometric transformation done.
- They also each hold important properties and details about the matrix A which they represent.

- The **V matrix** is the collection of unit vectors which create the axes of the unit circle transformed by A, classified as the **right singular vectors** of A.
- The **U matrix** is the collection of directional unit vectors which represent the orientation of the hyper-ellipse as a result of the A transformation, classified as the **left singular vectors** of A.
- The **Σ matrix** is a diagonal matrix holding the values which attach to the left singular vectors as magnitudes and represent the stretching of the hyper-ellipse.

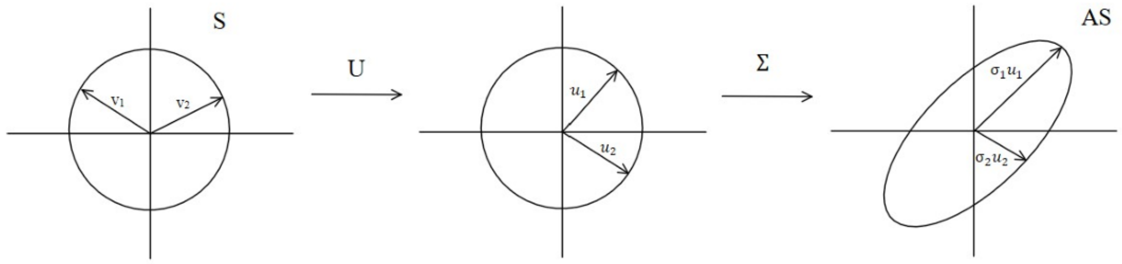



Figure 4: Unit circle S transformed by 2 x 2 matrix A

An interesting note about some of the functionality of SVD is that the values obtained from the singular value decomposition of a given matrix actually provide useful information about patterns and structures present in the original matrix. The left singular vectors, as they form an orthonormal basis for the matrix A (on which SVD is performed), represent some of the principal components of that matrix. SVD is useful in this way because it relies on the fact that the data being analysed is not truly random; instead, there are natural patterns and recurring structures embedded within the data.

The next Figure shows exactly the same properties of SVD plotted by taking Singular values

1,10,20,50,80,250 and it is clearly showing this in images.

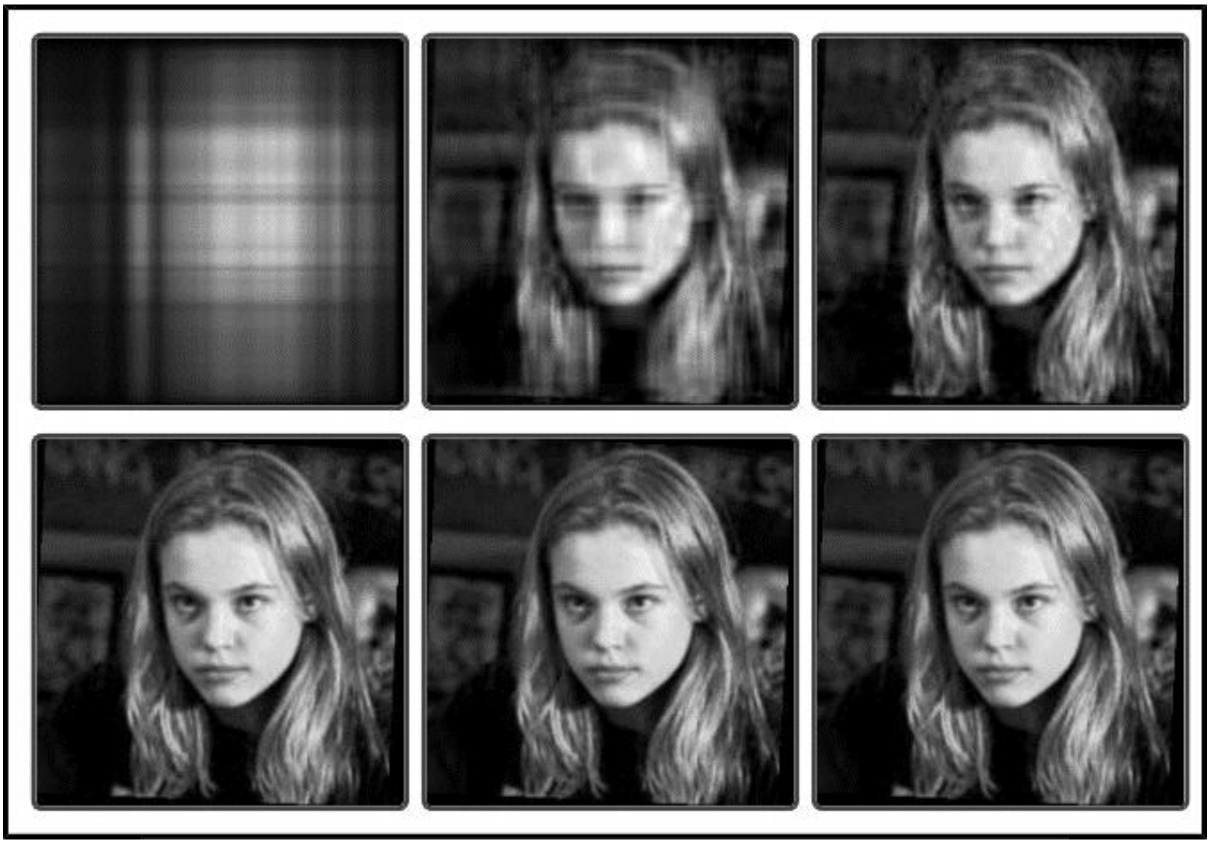


Figure 5: Left singular vectors displayed as images

Some Properties of SVD-

Let A be an n ×m matrix. Then the left singular vectors of A, within its singular value decomposition, are the.

Proof: $AA^T = (U\Sigma V^T)(U\Sigma V^T)^T = (U\Sigma V^T)(V\Sigma^T U^T) = U(\Sigma\Sigma^T)U^T$

Therefore, $AA^T = U\,\Sigma\Sigma^T U^T$ Post Multiplying by U we have,

$AA^T U = U\,\Sigma\Sigma^T$ Considering A$A^T$ multiplied by a single vector of U gives: $AA^T u_i = \sigma_i^2\, u_i$

$\Sigma\Sigma^T$ *diagonal entires are multiplied in such a way that they result in* $\sigma_i^2$ *term.*

From $AA^T u_i = \sigma_i{}^2\, u_i$ as we know $A\,x \; = \; \lambda x$

Hence, we can say eigenvectors of $AA^T$ *is left singular vector.*

Note: Definition: Let the Frobenius norm of an n x m matrix be defined by-

$||A||_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2\right)^{\frac{1}{2}}$

● Using Rank k approximation-

$A_k = \sum_{j=1}^{k} \sigma_j u_j v_j^T.$

$||A - A_k||_F^2 = \sigma_{v+1}^2 + \sigma_{v+2}^2 + \cdots + \sigma_m^2.$

This theorem shows that a given matrix, when represented by a sum of k rank-one singular value decompositions, has an error determined by the remaining m-k singular values.

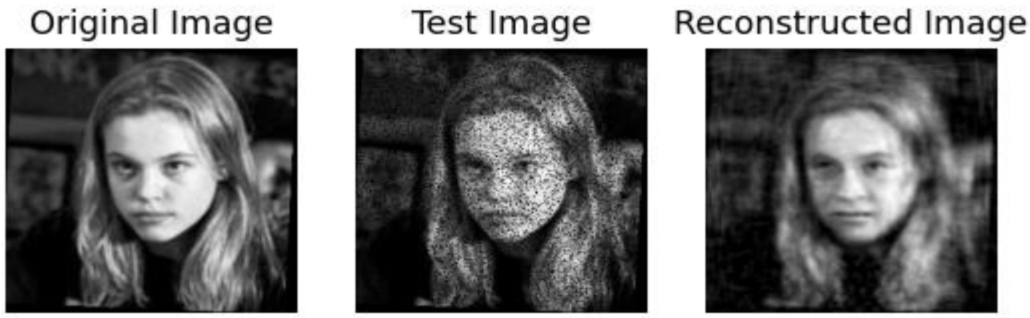
Now after finding the Eigen vector of that now let's reconstruct the image and lets the reconstructed image be denoted by $\hat{\varphi}$.

$\widehat{\varphi} = \overline{\varphi} + \sum_{i=1}^{M} a_i u_i$ where the coefficients $a_i$ are evaluated a $a_i = (u_i, \varphi_i - \overline{\varphi})$.

Note: $a_n$ is the Euclidian product of the corresponding two vectors.

Now for different singular values constructing the images and here are the results-





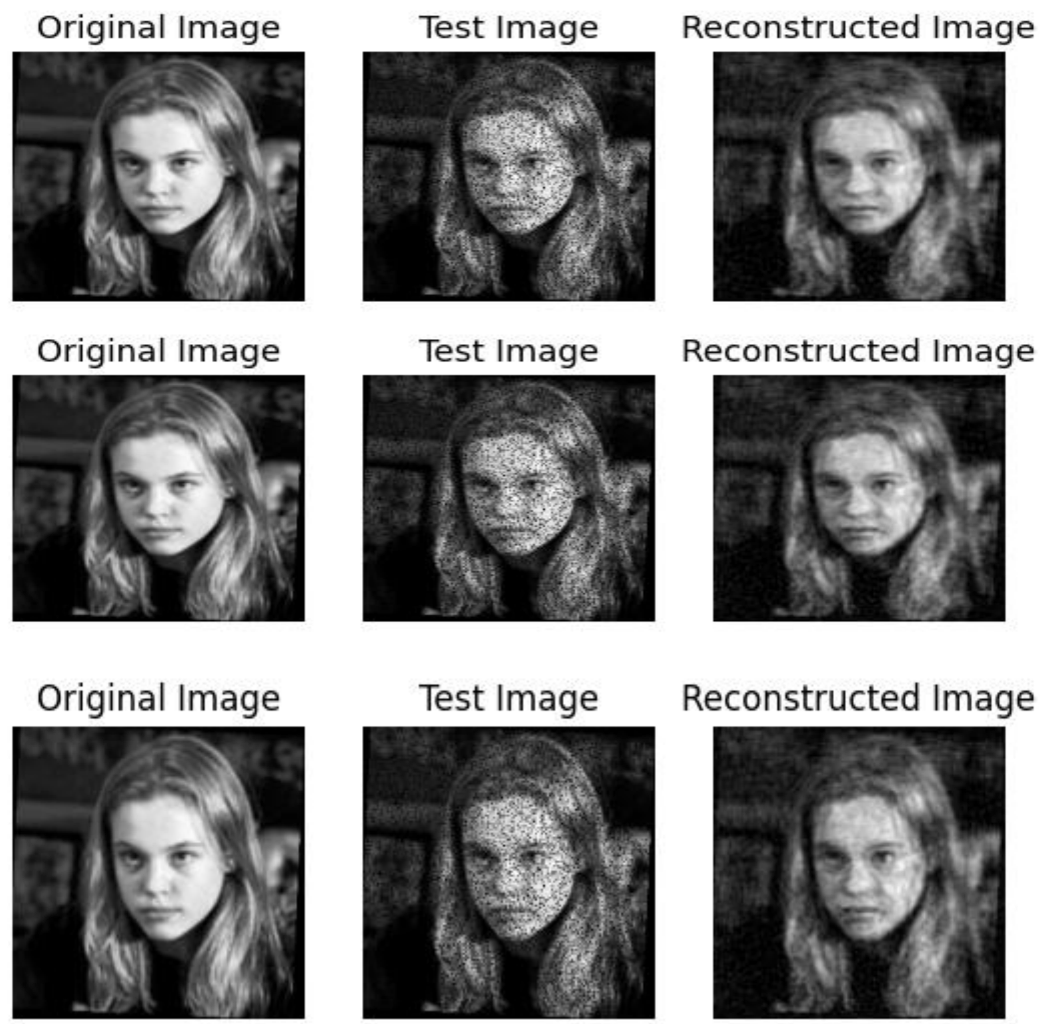




Figure 6: Output Images

Taking the dataset images at a time differently 0, 500 to 3000 as the values increases the image reconstructed improves.

Using Frobenius norm calculating error- $\left|\left|\varphi - \widehat{\varphi_i}\right|\right|_F = \sqrt{\sigma_{i+1}^2 + \cdots + \sigma_m^2}$

By this, if a given image is reconstructed by means of i database images, the error of this image in comparison to the original (due to the use of singular value decomposition) is given by the square rooted sum of the singular values of the images in the range from i +1 to m.

Here a Graph Plotted between error values and left singular vector used in reconstruction.

Showing that as the Dataset size or left singular values increase the error decreases.

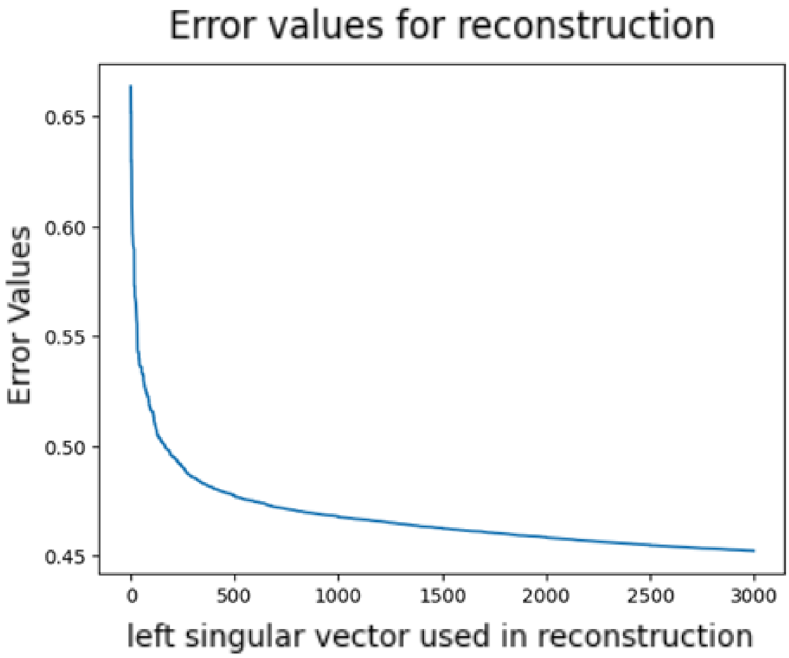



Figure 6:Error Values for Reconstructing Graph

Hence, we can conclude that as the size increase of left singular vectors used in image reconstruction the images improve hence error reduces.