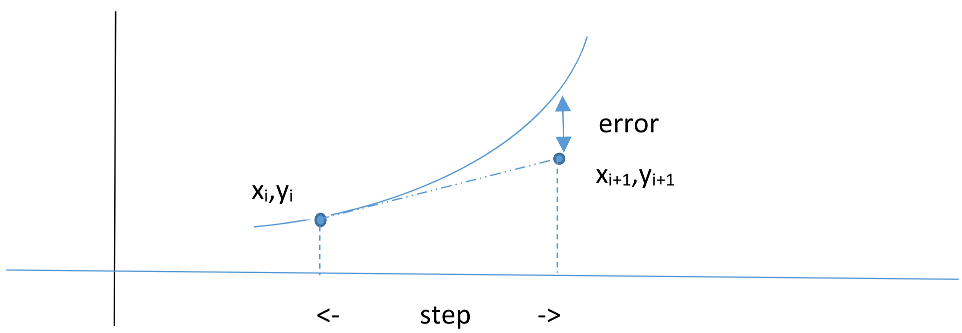
Heun's Method

Heun's method is an improvement of Euler's method for solving Ordinary Differential Equations. It is a "predictor-corrector" method.

The problem with Euler's method is that it uses the slope $\frac{dy}{dx})_i$ = $g(x_i,y_i)$ at the start of the step to calculate the y_{i+1} value projection at the end of the step h:

$y_{i+1} = y_i + g(x_i,y_i)*h$

For a derivative that increases with x and y the derivative at the start is less than the derivative anywhere else in the step interval and so gives a value for the end projection y_{i+1} that is too low. For a derivative that decreases with x and y the derivative at the start is greater than the derivative anywhere else in the step interval and so gives a value for the end projection y_{i+1} that is too high. Here is the picture for the derivative that increases with x and y:



A better result would be obtained with a derivative that represents the average slope over the step interval – that's what is used in Heun's method.

Heun Step 1 - predictor

Heun's method follows Euler's method to begin with, but recognises that the predicted value y_{i+1} is incorrect and calls it y'_{i+1} – the *predictor*. However it is an *estimate* of y at the end of the step:

$y'_{i+1} = y_i + g(x_i,y_i)*h$

Heun Step 2 - corrector

Now use the predictor from step 1 to calculate the derivative at the step end:

$g'(x_{i+1}, y'_{i+1})$

Then calculate the average of $g(x_i,y_i)$ and $g'(x_{i+1}, y'_{i+1})$ and use that to recalculate the final better y_{i+1} at the end of the step:

$y_{i+1} = y_i + ½(g(x_i,y_i) + g'(x_{i+1},y'_{i+1}))*h$

Example: Heun's Method for dy/dx = x + y

This is the example in the book so we will be able to see how much better it is.
Solve the differential equation g = dy/dx = x + y, from (0,1.0) to x = 0.5 in steps of Δx = 0.1

start
$x_0 = 0.0$, $y_0 = 1.0$.

<u>step 1 to $x_1 = 0.1$</u>
$g(x_0,y_0) = x_0 + y_0 = 0 + 1 = 1.0;$
$y'_1 = 1.0 + 1.0*0.1 = 1.1;$
$g' = 0.1 + 1.1 = 1.2;$
$y_1 = 1.0 + (1.0+1.2)/2 * 0.1 = 1.11$

<u>step 2 to x = 0.2</u>
$g(x_1,y_1) = x_1 + y_1 = 0.1 + 1.11 = 1.21;$
$y'_2 = 1.11 + 1.21*0.1 = 1.11 + 0.121 = 1.231;$
$g' = 0.2 + 1.231 = 1.431;$
$y_2 = 1.11 + (1.21+1.431)/2 * 0.1 = 1.11 + 1.3205 = 1.24205$

<u>step 3 to x = 0.3</u>
$g(x_2,y_2) = x_2 + y_2 = 0.2 + 1.24205 = 1.44205;$
$y'2 = 1.24205 + 1.44205*0.1 = 1.24205 + 0.144205 = 1.386255;$
$g' = 0.3 + 1.386255 = 1.686255;$
$y_3 = 1.24205+ (1.44205+1.686255)/2 * 0.1 = 1.24205 + 0.15641525 = 1.39846525$

Here is a comparison of the three results from the book example: dy/dx = x + y

x	Euler	Euler %error	Heun	Heun %error	Euler/Heun error	Exact
0.0	1.0	0	1.0	0	-	1.0
0.1	1.1	0.92%	1.11	0.027%	34	1.1103
0.2	1.22	1.83%	1.24205	0.06%	30.5	1.2428
0.3	1.362	2.69%	1.39846525	0.088%	30.56	1.3997
0.4	1.5282	***You do this one***	***You do this one***	***You do this one***	***You do this one***	1.5836
0.5	1.7210	***You do this one***	***You do this one***	***You do this one***	***You do this one***	1.7974

Heun is consistently 30 times more accurate than Euler.