Hybrid 2 Maclaurin (Taylor) Series

Exercise 1

Using the basic definition of a Maclaurin series find the first three nonzero terms of the following functions.

#1 f(x) = e^x	answer: $1 + x + \frac{1}{2}x^2 + \ldots$
#2. f(x) = cos(x)	answer: $1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \ldots$
#3 $f(x) = e^{-2x}$	answer: $1 - 2x + 2x^2 - \ldots$
#4 $f(x) = \cos 4\pi x$	answer: $1 - 8\pi^2 x^2 + \frac{32}{3}\pi^4 x^4 - \ldots$

Exercise 2

In the following, find the first four nonzero terms of the Maclaurin expansions of the given functions by using the following standard results:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

#1 $f(x) = e^{3x}$	answer: $1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \ldots$
#2 $f(x) = \sin\frac{1}{2}x$	answer: $\frac{x}{2} - \frac{x^3}{2^3 3!} + \frac{x^5}{2^5 5!} - \frac{x^7}{2^7 7!} + \ldots$
#3 $f(x) = x\cos 4x$	answer: $x - 8x^3 + \frac{32}{3}x^5 - \frac{256}{45}x^7 + \ldots$

Exercise 3.

Calculate the value of each of the given functions. Use the indicated number of terms of the appropriate Maclaurin series. Compare with the value found directly on a calculator.

#1 $e^{0.2}$ (3)	answer: 1.22, 1.221 402 8
#2 sin 0.1 (2)	answer: 0.099 833 3, 0.099 833 4
#3 e (7)	answer: 2.718 055 6, 2.718 281 8
#4 $\cos \pi^{\circ}$ (2)	answer: 0.998 496 77, 0.998 497 15

Exercise 4.

Calculate the value of the following function using a Taylor series, taking a = 1.0, using all the terms up to and including x^3

#1 $e^{1.2}$	answer: 3.32

Exercise 5.

#1a. Derive the Maclaurin series expansion for the function $f(x) = (e^x + e^{-x})/2$ for the first three non-zero terms	*Answer.* $e^x = 1 + x + x^2/2 + x^3/6 + x^4/24 + \ldots.$ $e^{-x} = 1 - x + x^2/2 - x^3/6 + x^4/24 + \ldots.$ So $(e^x + e^{-x})/2 = 1 + x^2/2 + x^4/24 +$
#1b. Write a numerical expression for the estimated % fractional (relative) error in your series from 1**a** at x = 1.0 when only two terms are used.	*Answer.* At x = 1 with two terms f(x) = 1 + 1/2! = 3/2 error is ~first truncated term = 1/4! = 1/24 so the % fractional error = 100 x 1/24 x 2/3 = 2.8%
#2a. Derive the Maclaurin series expansion for the function $f(x) = (e^x - e^{-x})/2$ for the first three non-zero terms.	*Answer.* $e^x = 1 + x + x^2/2 + x^3/6 + x^4/24 + x^5/120 + \ldots.$ $e^{-x} = 1 - x + x^2/2 - x^3/6 + x^4/24 - x^5/120 + \ldots.$ So $(e^x - e^{-x})/2 = x + x^3/6 + x^5/120 + \ldots$
#2b. Write a numerical expression for the estimated % fractional (relative) error in your series from **2a** at x = 1.0 when only two terms are used.	*Answer.* At x = 1, f(x) = 1 + 1/6 = 7/6 error is ~first truncated term = 1/120 so the % fractional error = 100 x 1/120 x 6/7 = 100/140 = 0.71%

Exercise 6

#1a. Derive the Maclaurin series expansion for the function f(x) = xcos(4x) for the first three nonzero terms	*Answer.* *This is an example from exercise 2:* $f(x) = x(1 - (4x)^2/2! + (4x)^4/4!) = x - 4^2x^3/2! + 4^4x^5/4! = x - 8x^3 + 32x^5/3$
#1b. Write down a numerical expression for an estimate of the % relative series error in your series from 1a at x = 0.1 when the series is truncated after the second term (the third term and higher are omitted from the series)	*Answer.* *At x = 0.1 for two terms* $f(x) = 0.1 - 8*(0.1)^3 = 0.1 - 0.008 = 0.092$ *%relative error from 3rd term* $= 100*(32*10^{-5}/3)/0.092 = 0.116\%$

Exercise 7

#1a. Derive the Maclaurin series expansion for the function `f(x) = e`^{x}`cos(x)` up to and including the term in x^4.	*Answer.* $e^x \cos x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)$ By multiplying the series on the right, we have the following result, considering through the x^4 terms in the product. $1\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right)$ $x\left(1 - \frac{x^2}{2!}\right)$ $\frac{x^2}{2!}\left(1 - \frac{x^2}{2!}\right)$ $\left(\frac{x^3}{3!} + \frac{x^4}{4!}\right)(1)$ $e^x \cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + x - \frac{x^3}{2} + \frac{x^2}{2} - \frac{x^4}{4} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$ $= 1 + x - \frac{1}{3}x^3 - \frac{1}{6}x^4 + \cdots$
#1b. Write down a numerical expression for the estimated % fractional error in your series from 1a at x = 1 when the series contains only up to and including the term in x^3	*Answer.* At x = 1, f(x) = 1 + 1 - 1/3 = 5/3 The error is approximated as the first neglected term = $-x^4/6 = -1/6$ so the % fractional error = -(1/6)/(5/3)*100 = (-1/10)*100 = -10%