EXERCISES 31.4

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the resulting differential equations.

1. In Example 1, change the right side to $3dx$.
2. In Example 3, change the right side to $2dx$.

In Exercises 3–28, solve the given differential equations.

3. $dy + y\,dx = e^{-x}dx$
4. $dy + 3y\,dx = e^{-3x}dx$
5. $dy + 2y\,dx = e^{-4x}dx$
6. $di + i\,dt = e^{-t}\cos t\,dt$
7. $\dfrac{dy}{dx} - 2y = 4$
8. $2\dfrac{dy}{dx} = 5 - 6y$
9. $dy = 3x^2(2 - y)dx$
10. $x\,dy + 3y\,dx = dx$
11. $2x\,dy + y\,dx = 8x^3dx$
12. $3x\,dy - y\,dx = 9x\,dx$
13. $dr + r\cot\theta\,d\theta = d\theta$
14. $y' = x^2y + 3x^2$
15. $\sin x\dfrac{dy}{dx} = 1 - y\cos x$
16. $\dfrac{dv}{dt} - \dfrac{v}{t} = \ln t$
17. $y' + y = x + e^x$
18. $y' + 2y = \sin x$
19. $ds = (te^{4t} + 4s)dt$
20. $y' - 2y = 2e^{2x}$
21. $y' = x^3(1 - 4y)$
22. $y' + y\tan x = -\sin x$
23. $x\dfrac{dy}{dx} = y + (x^2 - 1)^2$
24. $dy = dt - \dfrac{y\,dt}{(1 + t^2)\tan^{-1}t}$
25. $\sqrt{1 + x^2}dy + x(1 + y)dx = 0$
26. $(1 + x^2)dy + xy\,dx = x\,dx$
27. $\tan\theta\dfrac{dr}{d\theta} - r = \tan^2\theta$
28. $y' + y = y^2$ (Solve by letting $y = 1/u$ and solving the resulting linear equation for u.)

(W) *In Exercises 29 and 30, solve the given differential equations. Explain how each can be solved using either of two different methods.*

29. $y' = 2(1 - y)$
30. $x\,dy = (2x - y)dx$

In Exercises 31–36, find the indicated particular solutions of the given differential equations.

31. $\dfrac{dy}{dx} + 2y = e^{-x}$; $x = 0$ when $y = 1$
32. $dq - 4q\,du = 2\,du$; $q = 2$ when $u = 0$
33. $y' + 2y\cot x = 4\cos x$; $x = \pi/2$ when $y = 1/3$
34. $y'\sqrt{x} + \frac{1}{2}y = e^{\sqrt{x}}$; $x = 1$ when $y = 3$
35. $(\sin x)y' + y = \tan x$; $x = \pi/4$ when $y = 0$
36. $f(x)dy + 2yf'(x)dx = f(x)f'(x)dx$; $f(x) = -1$ when $y = 3$

In Exercises 37–40, solve the given problems.

37. The differential equation $y' + P(x)y = Q(x)y^2$ is not linear. Show that the substitution $u = y^{-1}$ will transform it into a linear equation.
38. An equation used in the analysis of rocket motion is $m\,dv + kv\,dt = 0$, where m and k are positive constants. Solve this equation for v as a function of t in two ways.
39. The electric current i in a circuit with a voltage V, resistance R, and inductance L, is a function of the time t given by $\dfrac{di}{dt} + \dfrac{R}{L}i = \dfrac{V}{L}$. Solve for i as a function of t if the initial current is zero.
40. If A dollars are placed in an account that pays 5% interest, compounded *continuously*, and A dollars are added to the account each year, the number of dollars n in the account after t years is given by $dn/dt = A + 0.05n$. Solve for n as a function of t.

Answer to Practice Exercise

1. $y = x^4 + cx^2$

31.5 Numerical Solutions of First-order Equations

Euler's Method • Runge–Kutta Method

Many differential equations do not have exact solutions. Therefore, in this section, we show one basic method and one more advanced method of solving such equations numerically.

EULER'S METHOD

■ Named for the Swiss mathematician Leonhard Euler (1707–1783).

To find an approximate solution to a differential equation of the form $dy/dx = f(x, y)$, that passes through a known point (x_0, y_0), we write the equation as $dy = f(x, y)dx$ and then approximate dy as $y_1 - y_0$, and replace dx with Δx. From Section 24.8, we recall that Δy closely approximates dy for a small dx and that $dx = \Delta x$. This gives us

$$y_1 = y_0 + f(x_0, y_0)\Delta x \qquad \text{and} \qquad x_1 = x_0 + \Delta x$$

Therefore, we now know another point (x_1, y_1) that is on (or very nearly on) the curve of the solution. We can now repeat this process using (x_1, y_1) as a known point to obtain a next point (x_2, y_2). Continuing this process, we can get a series of points that are approximately on the solution curve. The method is called ***Euler's method.***

EXAMPLE 1 Euler's method

For the differential equation $dy/dx = x + y$, use Euler's method to find the y-values of the solution for $x = 0$ to $x = 0.5$ with $\Delta x = 0.1$, if the curve of the solution passes through $(0, 1)$.

■ See Appendix C for a graphing calculator program EULRMETH. It gives numerical values of the solution of a differential equation using Euler's method.

Using the method outlined above, we have $x_0 = 0$, $y_0 = 1$, and

$$y_1 = 1 + (0 + 1)(0.1) = 1.1 \quad \text{and} \quad x_1 = 0 + 0.1 = 0.1$$

This tells us that the curve passes (or nearly passes) through the point $(0.1, 1.1)$. Assuming this point is correct, we use it to find the next point on the curve.

$$y_2 = 1.1 + (0.1 + 1.1)(0.1) = 1.22 \quad \text{and} \quad x_2 = 0.1 + 0.1 = 0.2$$

Therefore, the next approximate point is $(0.2, 1.22)$. Continuing this process, we find a set of points that would approximately satisfy the function that is the solution of the differential equation. Tabulating results, we have the following table.

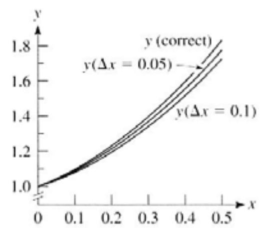

Fig. 31.3

x	y	Correct value of y
0.0	1.0000	1.0000
0.1	1.1000	1.1103
0.2	1.2200	1.2428
0.3	1.3620	1.3997
0.4	1.5282	1.5836
0.5	1.7210	1.7974

the values shown have been rounded off, although more digits were carried in the calculations

In this case, we are able to find the correct values since the equation can be written as $dy/dx - y = x$, and the solution is $y = 2e^x - x - 1$. Although numerical methods are generally used with equations that cannot be solved exactly, we chose this equation so that we could compare values obtained with known values.

We can see that as x increases, the error in y increases. More accurate values can be found by using smaller values of Δx. In Fig. 31.3 the solution curves using $\Delta x = 0.1$ and $\Delta x = 0.05$ are shown along with the correct values of y.

Euler's method is easy to use and understand, but it is less accurate than other methods. We will show one of the more accurate methods in the next example. ■

RUNGE–KUTTA METHOD

■ Named for the German mathematicians Carl Runge (1856–1927) and Martin Kutta (1867–1944).

For more accurate numerical solutions of a differential equation, the ***Runge–Kutta method*** is often used. Starting at a first point (x_0, y_0), the coordinates of the second point (x_1, y_1) are found by using a weighted average of the slopes calculated at the points where $x = x_0$, $x = x_0 + \frac{1}{2}\Delta x$, and $x = x_0 + \Delta x$. The formulas for y_1 and x_1 are

$$y_1 = y_0 + \frac{1}{6}H(J + 2K + 2L + M) \quad \text{and} \quad x_1 = x_0 + H \quad (\text{for convenience, } H = \Delta x)$$

where

$$\begin{aligned} J &= f(x_0, y_0) \\ K &= f(x_0 + 0.5H, y_0 + 0.5HJ) \\ L &= f(x_0 + 0.5H, y_0 + 0.5HK) \\ M &= f(x_0 + H, y_0 + HL) \end{aligned}$$

We have used uppercase letters to correspond to calculator use. Traditional sources normally use h for H and a lowercase letter (such as k) with subscripts for J, K, L, and M, and express 0.5 as 1/2.

■ See Appendix C for a graphing calculator program RUNGKUTT. It gives numerical values of the solution of a differential equation using the Runge–Kutta method.

As with Euler's method, once (x_1, y_1) is determined, we use the formulas again to find (x_2, y_2) by replacing (x_0, y_0) with (x_1, y_1). The following example illustrates the use of the Runge–Kutta method.

EXAMPLE 2 Runge–Kutta method

x	y
0.0	0.0
0.1	0.0050125208
0.2	0.0202013395
0.3	0.0460278455
0.4	0.0832868181
0.5	0.1331460062

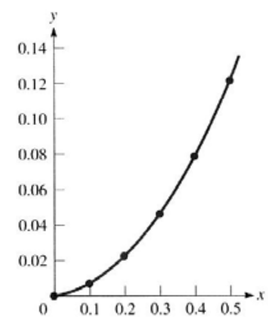

Fig. 31.4

For the differential equation $dy/dx = x + \sin xy$, use the Runge–Kutta method to find y-values of the solution for $x = 0$ to $x = 0.5$ with $\Delta x = 0.1$, if the curve of the solution passes through $(0, 0)$.

Using the formulas and method outlined above, we have the following solution, with calculator notes to the right of the equations. Also, calculator symbols are used on the right sides of the equations to indicate the way in which they should be entered.

$x_0 = 0$ store as X

$y_0 = 0$ store as Y

$H = 0.1$

$J = X + \sin XY = 0$ store as J

$K = X + 0.5H + \sin[(X + 0.5H)(Y + 0.5HJ)] = 0.05$ store as K

$L = X + 0.5H + \sin[(X + 0.5H)(Y + 0.5HK)] = 0.050\,125$ store as L

$M = X + H + \sin[(X + H)(Y + HL)] = 0.100\,501\,25$ store as M

$y_1 = Y + (H/6)(J + 2K + 2L + M) = 0.005\,012\,520\,8$ store as Y

$x_1 = X + H = 0.1$ store as X

We now use (x_1, y_1) as we just used (x_0, y_0) to get the next point (x_2, y_2), which is then used to find (x_3, y_3), and so on. A table showing calculator values and a graph of these values is shown in Fig. 31.4. ■

EXERCISES 31.5

In Exercises 1–8, use Euler's method to find y-values of the solution for the given values of x and Δx, if the curve of the solution passes through the given point. Check the results against known values by solving the differential equations exactly. Plot the graphs of the solutions in Exercises 1–4.

1. $\dfrac{dy}{dx} = x + 1$; $x = 0$ to $x = 1$; $\Delta x = 0.2$; $(0, 1)$
2. $\dfrac{dy}{dx} = \sqrt{2x + 1}$; $x = 0$ to $x = 1.2$; $\Delta x = 0.3$; $(0, 2)$
3. $\dfrac{dy}{dx} = y(0.4x + 1)$; $x = -0.2$ to $x = 0.3$; $\Delta x = 0.1$; $(-0.2, 2)$
4. $\dfrac{dy}{dx} = y + e^x$; $x = 0$ to $x = 0.5$; $\Delta x = 0.1$; $(0, 0)$
5. The differential equation of Exercise 1 with $\Delta x = 0.1$
6. The differential equation of Exercise 2 with $\Delta x = 0.1$
7. The differential equation of Exercise 3 with $\Delta x = 0.05$
8. The differential equation of Exercise 4 with $\Delta x = 0.05$

In Exercises 9–14, use the Runge–Kutta method to find y-values of the solution for the given values of x and Δx, if the curve of the solution passes through the given point.

9. $\dfrac{dy}{dx} = xy + 1$; $x = 0$ to $x = 0.4$; $\Delta x = 0.1$; $(0, 0)$
10. $\dfrac{dy}{dx} = x^2 + y^2$; $x = 0$ to $x = 0.4$; $\Delta x = 0.1$; $(0, 1)$
11. $\dfrac{dy}{dx} = e^{xy}$; $x = 0$ to $x = 1$; $\Delta x = 0.2$; $(0, 0)$
12. $\dfrac{dy}{dx} = \sqrt{1 + xy}$; $x = 0$ to $x = 0.2$; $\Delta x = 0.05$; $(0, 1)$
13. $\dfrac{dy}{dx} = \cos(x + y)$; $x = 0$ to $x = 0.6$; $\Delta x = 0.1$; $(0, \pi/2)$
14. $\dfrac{dy}{dx} = y + \sin x$; $x = 0.5$ to $x = 1.0$; $\Delta x = 0.1$; $(0.5, 0)$

In Exercises 15–18, solve the given problems.

(W) 15. In Example 1, use Euler's method to find the y-values from $x = 0$ to $x = 3$ with $\Delta x = 1$. Compare with the value found using the exact solution. Comment on the use of Euler's method in finding the value of y for $x = 3$.

16. For the differential equation $dy/dx = x + 1$, if the curve of the solution passes through $(0, 0)$, calculate the y-value for $x = 0.04$ with $\Delta x = 0.01$. Find the exact solution, and compare the result using three terms of the Maclaurin series that represents the solution.

17. An electric circuit contains a 1-H inductor, a 2-Ω resistor, and a voltage source of $\sin t$. The resulting differential equation relating the current i and the time t is $di/dt + 2i = \sin t$. Find i after 0.5 s by Euler's method with $\Delta t = 0.1$ s if the initial current is zero. Solve the equation exactly and compare the values.

18. An object is being heated such that the rate of change of the temperature T (in °C) with respect to time t (in min) is $dT/dt = \sqrt[3]{1 + t^3}$. Find T for $t = 5$ min by using the Runge–Kutta method with $\Delta t = 1$ min, if the initial temperature is 0°C.