30.2 Maclaurin Series

Power-Series Expansion • Interval of Convergence • Maclaurin Series Expansion

In this section, we develop a very important basic polynomial form of a function. Before developing the method using calculus, we will review how this can be done for some functions algebraically.

EXAMPLE 1 Algebraic function represented by series

$$\begin{array}{r} 1 + \frac{x}{2} \\ 2 - x\overline{)2} \\ \underline{2 - x} \\ x \\ \underline{x - \frac{x^2}{2}} \\ \frac{x^2}{2} \end{array}$$

By using long division (as started at the left), we have

$$\frac{2}{2-x} = 1 + \frac{1}{2}x + \frac{1}{4}x^2 + \cdots + \left(\frac{1}{2}x\right)^{n-1} + \cdots \tag{1}$$

where n is the number of the term of the expression on the right. Since x represents a number, the right-hand side of Eq. (1) becomes a geometric series.

From Eq. (30.3), we know that the sum of a geometric series with first term a_1 and common ratio r is

$$S = \frac{a_1}{1-r}$$

where $|r| < 1$ and the series converges.

If $x = 1$, the right-hand side of Eq. (1) is

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^{n-1} + \cdots$$

For this series, $r = \frac{1}{2}$ and $a_1 = 1$, which means that the series converges and $S = 2$. If $x = 3$, the right-hand side of Eq. (1) is

$$1 + \frac{3}{2} + \frac{9}{4} + \cdots + \left(\frac{3}{2}\right)^{n-1} + \cdots$$

which diverges since $r > 1$. Referring to the left side of Eq. (1), we see that it also equals 2 when $x = 1$. Thus, we see that the two sides agree for $x = 1$, but that the series diverges for $x = 3$. In fact, as long as $|x| < 2$, the series will converge to the value of the function on the left. From this we conclude that the series on the right properly represents the function on the left, as long as $|x| < 2$. ■

From Example 1, we see that an algebraic function may be properly represented by a function of the form

Power Series

$$\boxed{f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots} \tag{30.4}$$

Equation (30.4) is known as a **power-series expansion** *of the function* $f(x)$. The problem now arises as to whether or not functions in general may be represented in this form. If such a representation were possible, it would provide a means of evaluating the transcendental functions for the purpose of making tables of values. Also, since a power-series expansion is in the form of a polynomial, it makes algebraic operations much simpler due to the properties of polynomials. A further study of calculus shows many other uses of power series.

In Example 1, we saw that the function could be represented by a power series as long as $|x| < 2$. That is, if we substitute any value of x in this interval into the series and also into the function, the series will converge to the value of the function. *This interval of values for which the series converges is called the* **interval of convergence.**

EXAMPLE 2 Interval of convergence

In Example 1, the interval of convergence for the series

$$1 + \frac{1}{2}x + \frac{1}{4}x^2 + \cdots + \left(\frac{1}{2}x\right)^{n-1} + \cdots$$

is $|x| < 2$. We saw that the series converges for $x = 1$, with $S = 2$, and that the value of the function is 2 for $x = 1$. This verifies that $x = 1$ is in the interval of convergence.

Also, we saw that the series diverges for $x = 3$, which verifies that $x = 3$ is not in the interval of convergence. ■

At this point, we will assume that unless otherwise noted, the functions with which we will be dealing may be properly represented by a power-series expansion (it takes more advanced methods to prove that this is generally possible), for appropriate intervals of convergence. We will find that the methods of calculus are very useful in developing the method of general representation. Thus, writing a general power series, along with the first few derivatives, we have

$$\begin{aligned}
f(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots + a_nx^n + \cdots \\
f'(x) &= a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \cdots + na_nx^{n-1} + \cdots \\
f''(x) &= 2a_2 + 2(3)a_3x + 3(4)a_4x^2 + 4(5)a_5x^3 + \cdots + (n-1)na_nx^{n-2} + \cdots \\
f'''(x) &= 2(3)a_3 + 2(3)(4)a_4x + 3(4)(5)a_5x^2 + \cdots + (n-2)(n-1)na_nx^{n-3} + \cdots \\
f^{\text{iv}}(x) &= 2(3)(4)a_4 + 2(3)(4)(5)a_5x + \cdots + (n-3)(n-2)(n-1)na_nx^{n-4} + \cdots
\end{aligned}$$

NOTE ➧ Regardless of the values of the constants a_n for any power series, ***if $x = 0$, the left and right sides must be equal,*** and all the terms on the right are zero except the first. Thus, setting $x = 0$ in each of the above equations, we have

$$f(0) = a_0 \qquad f'(0) = a_1 \qquad f''(0) = 2a_2$$
$$f'''(0) = 2(3)a_3 \qquad f^{\text{iv}}(0) = 2(3)(4)a_4$$

Solving each of these for the constants a_n, we have

$$a_0 = f(0) \qquad a_1 = f'(0) \qquad a_2 = \frac{f''(0)}{2!} \qquad a_3 = \frac{f'''(0)}{3!} \qquad a_4 = \frac{f^{\text{iv}}(0)}{4!}$$

Substituting these into the expression for $f(x)$, we have

Maclaurin Series

$$\boxed{f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots + \frac{f^{n}(0)x^n}{n!} + \cdots} \qquad \textbf{(30.5)}$$

■ Named for the Scottish mathematician Colin Maclaurin (1698–1746).

Equation (30.5) is known as the **Maclaurin series expansion** *of a function.* For a function to be represented by a Maclaurin expansion, the function and all of its derivatives must exist at $x = 0$. Also, we note that the factorial notation introduced in Section 19.4 is used in writing the Maclaurin series expansion.

As we mentioned earlier, one of the uses we will make of series expansions is that of determining the values of functions for particular values of x. If x is sufficiently small, successive terms become smaller and smaller and the series will converge rapidly. This is considered in the sections that follow.

The following examples illustrate Maclaurin expansions for algebraic, exponential, and trigonometric functions.

EXAMPLE 3 Maclaurin series for algebraic function

Find the first four terms of the Maclaurin series expansion of $f(x) = \dfrac{2}{2-x}$.

$$f(x) = \frac{2}{2-x} \quad f(0) = 1 \quad f''(x) = \frac{4}{(2-x)^3} \quad f''(0) = \frac{1}{2}$$

$$f'(x) = \frac{2}{(2-x)^2} \quad f'(0) = \frac{1}{2} \quad f'''(x) = \frac{12}{(2-x)^4} \quad f'''(0) = \frac{3}{4}$$

find derivatives and evaluate each at $x = 0$

$$f(x) = 1 + \frac{1}{2}x + \frac{1}{2}\left(\frac{x^2}{2!}\right) + \frac{3}{4}\left(\frac{x^3}{3!}\right) + \cdots$$

using Eq. (30.5)

$$\frac{2}{2-x} = 1 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \cdots$$

■

Practice Exercise

1. Find the first four terms of the Maclaurin series expansion for $f(x) = \dfrac{1}{1+x}$.

■ Compare with Example 1.

EXAMPLE 4 Maclaurin series for exponential function

Find the first four terms of the Maclaurin series expansion of $f(x) = e^{-x}$.

$$f(x) = e^{-x} \quad f(0) = 1 \quad f''(x) = e^{-x} \quad f''(0) = 1$$

$$f'(x) = -e^{-x} \quad f'(0) = -1 \quad f'''(x) = -e^{-x} \quad f'''(0) = -1$$

find derivatives and evaluate each at $x = 0$

$$f(x) = 1 + (-1)x + 1\left(\frac{x^2}{2!}\right) + (-1)\left(\frac{x^3}{3!}\right) + \cdots$$

using Eq. (30.5)

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

■

EXAMPLE 5 Maclaurin series for trigonometric function

Find the first three nonzero terms of the Maclaurin series expansion of $f(x) = \sin 2x$.

$$f(x) = \sin 2x \quad f(0) = 0 \quad f'''(x) = -8\cos 2x \quad f'''(0) = -8$$

$$f'(x) = 2\cos 2x \quad f'(0) = 2 \quad f^{\text{iv}}(x) = 16\sin 2x \quad f^{\text{iv}}(0) = 0$$

$$f''(x) = -4\sin 2x \quad f''(0) = 0 \quad f^{\text{v}}(x) = 32\cos 2x \quad f^{\text{v}}(0) = 32$$

$$f(x) = 0 + 2x + 0 + (-8)\frac{x^3}{3!} + 0 + 32\frac{x^5}{5!} + \cdots$$

$$\sin 2x = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \cdots$$

This series is called an **alternating series,** since every other term is negative.

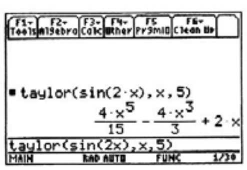

Fig. 30.4

In Fig. 30.4, the TI-89 calculator uses a *Taylor series* (discussed in Section 30.5, starting on page 912) since a Maclaurin series is a special case (expansion about $x = 0$) of a Taylor series. ■

EXAMPLE 6 Maclaurin series – application

Frictional forces in the spring shown in Fig. 30.5 are just sufficient so that the lever does not oscillate after being depressed. Such motion is called *critically damped.* The displacement y as a function of the time t for one case is $y = (1 + t)e^{-t}$. To study the motion for small values of t, a Maclaurin expansion of $y = f(t)$ is to be used. Find the first four terms of the expansion.

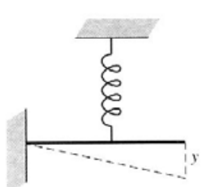

Fig. 30.5

$$f(t) = (1 + t)e^{-t} \qquad f(0) = 1$$

$$f'(t) = (1 + t)e^{-t}(-1) + e^{-t} = -te^{-t} \qquad f'(0) = 0$$

$$f''(t) = te^{-t} - e^{-t} \qquad f''(0) = -1$$

$$f'''(t) = -te^{-t} + e^{-t} + e^{-t} = 2e^{-t} - te^{-t} \qquad f'''(0) = 2$$

$$f^{\text{iv}}(t) = -2e^{-t} + te^{-t} - e^{-t} = te^{-t} - 3e^{-t} \qquad f^{\text{iv}}(0) = -3$$

$$f(t) = 1 + 0 + (-1)\frac{t^2}{2!} + 2\frac{t^3}{3!} + (-3)\frac{t^4}{4!} + \cdots$$

$$(1 + t)e^{-t} = 1 - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{8} + \cdots$$

■

EXERCISES 30.2

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then find the resulting series.

1. In Example 3, in $f(x)$, change the denominator to $2 + x$.

2. In Example 5, in $f(x)$, change $2x$ to $(-2x)$.

In Exercises 3–20, find the first three nonzero terms of the Maclaurin expansion of the given functions.

3. $f(x) = e^x$ **4.** $f(x) = \sin x$

5. $f(x) = \cos x$ **6.** $f(x) = \ln(1 + x)$

7. $f(x) = \sqrt{1 + x}$ **8.** $f(x) = \sqrt[3]{1 + x}$

9. $f(x) = e^{-2x}$ **10.** $f(x) = \dfrac{1}{\sqrt{1 + x}}$

11. $f(x) = \cos 4\pi x$ **12.** $f(x) = e^x \sin x$

13. $f(x) = \dfrac{1}{1 - x}$ **14.** $f(x) = \dfrac{1}{(1 + x)^2}$

15. $f(x) = \ln(1 - 2x)$ **16.** $f(x) = (1 + x)^{3/2}$

17. $f(x) = \cos^2 x$ **18.** $f(x) = \ln(1 + 4x)$

19. $f(x) = \sin(x + \frac{\pi}{4})$ **20.** $f(x) = (2x - 1)^2$

In Exercises 21–28, find the first two nonzero terms of the Maclaurin expansion of the given functions.

21. $f(x) = \tan^{-1} x$ **22.** $f(x) = \cos x^2$

23. $f(x) = \tan x$ **24.** $f(x) = \sec x$

25. $f(x) = \ln \cos x$ **26.** $f(x) = xe^{\sin x}$

27. $f(x) = \sqrt{1 + \sin x}$ **28.** $f(x) = xe^{-x^2}$

In Exercises 29–42, solve the given problems.

W **29.** Is it possible to find a Maclaurin expansion for (a) $f(x) = \csc x$ or (b) $f(x) = \ln x$? Explain.

W **30.** Is it possible to find a Maclaurin expansion for (a) $f(x) = \sqrt{x}$ or (b) $f(x) = \sqrt{1 + x}$? Explain.

31. Find the first three nonzero terms of the Maclaurin expansion for (a) $f(x) = e^x$ and (b) $f(x) = e^{x^2}$. Compare these expansions.

32. By finding the Maclaurin expansion of $f(x) = (1 + x)^n$, derive the first four terms of the binomial series, which is Eq. (19.10). Its interval of convergence is $|x| < 1$ for all values of n.

33. If $f(x) = e^{3x}$, compare the Maclaurin expansion with the linearization for $a = 0$.

34. Find the Maclaurin series for $y = \sinh x$. (See Exercises 55–58 on page 820.)

35. Find the Maclaurin series for $y = \cosh x$. (See Exercises 55–58 on page 820.)

36. Find the Maclaurin series for $f(x) = \cos^2 x$, by using the identity $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$. Compare the result with that of Exercise 17.

37. If $f(x) = x^2$, show that this function is obtained when a Maclaurin expansion is found.

38. If $f(x) = x^4 + 2x^2$, show that this function is obtained when a Maclaurin expansion is found.

39. The displacement y (in cm) of an object hung vertically from a spring and allowed to oscillate is given by the equation $y = 4e^{-0.2t} \cos t$, where t is the time (in s). Find the first three terms of the Maclaurin expansion of this function.

40. For the circuit shown in Fig. 30.6, after the switch is closed, the transient current i (in A) is given by $i = 2.5(1 + e^{-0.1t})$. Find the first three terms of the Maclaurin expansion of this function.

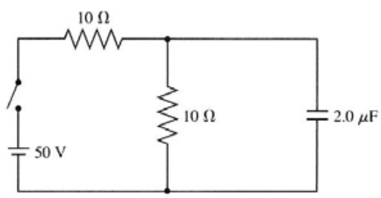

Fig. 30.6

41. The reliability R $(0 \le R \le 1)$ of a certain computer system is $R = e^{-0.001t}$, where t is the time of operation (in min). Express $R = f(t)$ in polynomial form by using the first three terms of the Maclaurin expansion.

42. In the analysis of the optical paths of light from a narrow slit S to a point P as shown in Fig. 30.7, the law of cosines is used to obtain the equation

$$c^2 = a^2 + (a + b)^2 - 2a(a + b)\cos\frac{s}{a}$$

where s is part of the circular arc $\widehat{AB}$. By using two nonzero terms of the Maclaurin expansion of $\cos\frac{s}{a}$, simplify the right side of the equation. (In finding the expansion, let $x = \frac{s}{a}$ and then substitute back into the expansion.)

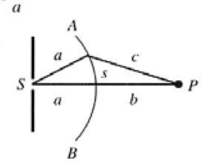

Fig. 30.7

Answer to Practice Exercise

1. $\dfrac{1}{1 + x} = 1 - x + x^2 - x^3 + \cdots$

30.3 Operations with Series

Form New Series from Known Series • Using Functional Notation • Using Algebraic Operations • Differentiating and Integrating • Accuracy of Series

The series found in Exercises 3 to 6 and 32 (the binomial series) of Section 30.2 are of particular importance. They are used to evaluate exponential functions, trigonometric functions, logarithms, powers, and roots, as well as develop other series. For reference, we give them here with their intervals of convergence.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad (\text{all } x) \tag{30.6}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \quad (\text{all } x) \tag{30.7}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \quad (\text{all } x) \tag{30.8}$$

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad (|x| < 1) \tag{30.9}$$

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots \quad (|x| < 1) \tag{30.10}$$

In the next section, we will see how to use these series in finding values of functions. In this section, we see how new series are developed by using the above basic series, and we also show other uses of series.

When we discussed functions in Chapter 3, we mentioned functions such as $f(2x)$ and $f(-x)$. *By using functional notation and the preceding series, we can find the series expansions of many other series without using direct expansion.* This can often save time in finding a desired series.

NOTE ▸

EXAMPLE 1 Series formed using functional notation

Find the Maclaurin expansion of e^{2x}.

From Eq. (30.6), we know the expansion of e^x. Hence,

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Since $e^{2x} = f(2x)$, we have

$$f(2x) = 1 + (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \cdots \qquad \text{in } f(x)\text{, replace } x \text{ by } 2x$$

$$e^{2x} = 1 + 2x + 2x^2 + \frac{4x^3}{3} + \cdots$$

■

EXAMPLE 2 Series formed using functional notation

Find the Maclaurin expansion of $\sin x^2$.

From Eq. (30.7), we know the expansion of $\sin x$. Therefore,

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$f(x^2) = (x^2) - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \cdots \qquad \text{in } f(x)\text{, replace } x \text{ by } x^2$$

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots$$

Direct expansion of this series is quite lengthy. ■

Practice Exercise

1. Using the Maclaurin series for $\ln(1 + x)$, find the first four terms of the Maclaurin expansion of $\ln(1 - 2x)$.

The basic algebraic operations may be applied to series in the same manner they are applied to polynomials. That is, we may add, subtract, multiply, or divide series in order to obtain other series. The interval of convergence for the resulting series is that which is common to those of the series being used. The multiplication of series is illustrated in the following example.

EXAMPLE 3 Series formed by multiplication

Multiply the series expansion for e^x by the series expansion for $\cos x$ to obtain the series expansion for $e^x \cos x$.

Using the series expansion for e^x and $\cos x$ as shown in Eqs. (30.6) and (30.8), we have the following indicated multiplication:

$$e^x \cos x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)$$

By multiplying the series on the right, we have the following result, considering through the x^4 terms in the product.

$$1\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right) \quad x\left(1 - \frac{x^2}{2!}\right) \quad \frac{x^2}{2!}\left(1 - \frac{x^2}{2!}\right) \quad \left(\frac{x^3}{3!} + \frac{x^4}{4!}\right)(1)$$

$$\begin{aligned} e^x \cos x &= 1 - \frac{x^2}{2} + \frac{x^4}{24} + x - \frac{x^3}{2} + \frac{x^2}{2} - \frac{x^4}{4} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots \\ &= 1 + x - \frac{1}{3}x^3 - \frac{1}{6}x^4 + \cdots \end{aligned}$$

■

It is also possible to use the operations of differentiation and integration to obtain series expansions, although the proof of this is found in more advanced texts. Consider the following example.

EXAMPLE 4 Series formed by differentiating

Show that by differentiating the series for $\ln(1 + x)$ term by term, the result is the same as the series for $\dfrac{1}{1 + x}$.

The series for $\ln(1 + x)$ is shown in Eq. (30.9) as

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Differentiating, we have

$$\begin{aligned} \frac{1}{1 + x} &= 1 - \frac{2x}{2} + \frac{3x^2}{3} - \frac{4x^3}{4} + \cdots \\ &= 1 - x + x^2 - x^3 + \cdots \end{aligned}$$

Using the binomial expansion for $\dfrac{1}{1 + x} = (1 + x)^{-1}$, we have

$$\begin{aligned} (1 + x)^{-1} &= 1 + (-1)x + \frac{(-1)(-2)}{2!}x^2 + \frac{(-1)(-2)(-3)}{3!}x^3 + \cdots \\ &= 1 - x + x^2 - x^3 + \cdots \end{aligned}$$

using Eq. (30.10) with $n = -1$

We see that the results are the same. ■

■ For reference, Eq. (12.11) is $re^{j\theta} = r(\cos\theta + j\sin\theta)$.

We can use algebraic operations on series to verify that the definition of the exponential form of a complex number, as shown in Eq. (12.11), is consistent with other definitions. The only assumption required here is that the Maclaurin expansions for e^x, $\sin x$, and $\cos x$ are also valid for complex numbers. This is shown in advanced calculus. Thus,

$$e^{j\theta} = 1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \cdots = 1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \cdots \qquad (30.11)$$

$$j\sin\theta = j\theta - j\frac{\theta^3}{3!} + \cdots \qquad (30.12)$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \cdots \qquad (30.13)$$

■ Eq. (30.14) is known as Euler's Formula. If $\theta = \pi$, we have $e^{j\pi} = -1$, which can be written as

$$e^{j\pi} + 1 = 0$$

This equation connects the five fundamental numbers e, j, π, 1, and 0, and it has been called a "beautiful" equation by mathematicians. (In nontechnical sources, i would appear in place of j (see page 334.))

When we add the terms of Eq. (30.12) to those of Eq. (30.13), the result is the series given in Eq. (30.11). Thus,

$$\boxed{e^{j\theta} = \cos\theta + j\sin\theta} \qquad (30.14)$$

A comparison of Eqs. (12.11) and (30.14) indicates the reason for the choice of the definition of the exponential form of a complex number.

An additional use of power series is now shown. Many integrals that occur in practice cannot be integrated by methods given in the preceding chapters. However, power series can be very useful in giving excellent approximations to some definite integrals.

EXAMPLE 5 Using series for integration

Find the first-quadrant area bounded by $y = \sqrt{1 + x^3}$ and $x = 0.5$.

From Fig. 30.8, we see that the area is

$$A = \int_0^{0.5} \sqrt{1 + x^3}\,dx$$

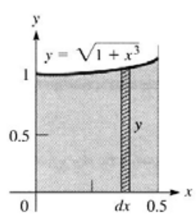

Fig. 30.8

This integral does not fit any form we have used. However, its value can be closely approximated by using the binomial expansion for $\sqrt{1 + x^3}$ and then integrating.

Using the binomial expansion to find the first three terms of the expansion for $\sqrt{1 + x^3}$, we have

$$\sqrt{1 + x^3} = (1 + x^3)^{0.5} = 1 + 0.5x^3 + \frac{0.5(-0.5)}{2}(x^3)^2 + \cdots$$
$$= 1 + 0.5x^3 - 0.125x^6 + \cdots$$

Substituting in the integral, we have

$$A = \int_0^{0.5} (1 + 0.5x^3 - 0.125x^6 + \cdots)\,dx$$
$$= x + \frac{0.5}{4}x^4 - \frac{0.125}{7}x^7 + \cdots \Big|_0^{0.5}$$
$$= 0.5 + 0.007\,812\,5 - 0.000\,139\,5 + \cdots = 0.507\,673 + \cdots$$

We can see that each of the terms omitted was very small. The result shown is correct to four decimal places, or $A = 0.5077$. Additional accuracy can be obtained by using more terms of the expansion. ■

EXAMPLE 6 Using series for integration

Evaluate: $\int_0^{0.1} e^{-x^2}\,dx$.

$$e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \cdots \quad \text{using Eq. (30.6)}$$

$$\int_0^{0.1} e^{-x^2}\,dx = \int_0^{0.1}\left(1 - x^2 + \frac{x^4}{2} - \cdots\right)dx \quad \text{substitute}$$

$$= \left(x - \frac{x^3}{3} + \frac{x^5}{10} - \cdots\right)\Big|_0^{0.1} \quad \text{integrate}$$

$$= 0.1 - \frac{0.001}{3} + \frac{0.000\,01}{10} = 0.099\,667\,7 \quad \text{evaluate}$$

This answer is correct to the indicated accuracy. ■

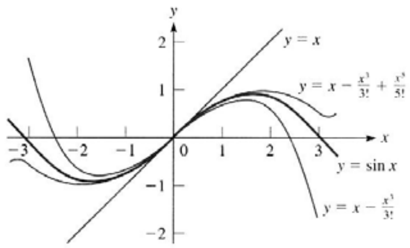

Fig. 30.9

The question of accuracy now arises. The integrals just evaluated indicate that the more terms used, the greater the accuracy of the result. To graphically show the accuracy involved, Fig. 30.9 depicts the graphs of $y = \sin x$ and the graphs of

$$y = x \qquad y = x - \frac{x^3}{3!} \qquad y = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

which are the first three approximations of $y = \sin x$. We can see that each term added gives a better fit to the curve of $y = \sin x$. Also, this gives a graphical representation of the meaning of a series expansion.

We have just shown that the more terms included, the more accurate the result. For small values of x, a Maclaurin series gives good accuracy with a very few terms. In this case, the series *converges* rapidly, as we mentioned earlier. For this reason, a Maclaurin series is of particular use for small values of x. For larger values of x, usually a function is expanded in a Taylor series (see Section 30.5). Of course, if we omit any term in a series, there is some error in the calculation.

EXERCISES 30.3

In Exercises 1 and 2, make the given changes in the indicated examples of this section, and then find the resulting series.

1. In Example 1, change e^{2x} to e^{2x^2}.
2. In Example 3, change e^x to e^{-x}.

In Exercises 3–10, find the first four nonzero terms of the Maclaurin expansions of the given functions by using Eqs. (29.6) to (29.10).

3. $f(x) = e^{3x}$
4. $f(x) = e^{-2x}$
5. $f(x) = \sin\frac{1}{2}x$
6. $f(x) = \sin x^4$
7. $f(x) = x\cos 4x$
8. $f(x) = \sqrt{1 - x^4}$
9. $f(x) = \ln(1 + x^2)$
10. $f(x) = x^2\ln(1 - x)$

In Exercises 11–14, evaluate the given integrals by using three terms of the appropriate series.

11. $\int_0^1 \sin x^2\,dx$
12. $\int_0^{0.4} \sqrt[4]{1 - 2x^2}\,dx$
13. $\int_0^{0.2} \cos\sqrt{x}\,dx$
14. $\int_{0.1}^{0.2} \frac{\cos x - 1}{x}\,dx$

In Exercises 15–28, find the indicated series by the given operation.

15. Find the first four terms of the Maclaurin expansion of the function $f(x) = \frac{2}{1 - x^2}$ by adding the terms of the series for the functions $\frac{1}{1 - x}$ and $\frac{1}{1 + x}$.
16. Find the first four nonzero terms of the expansion of the function $f(x) = \frac{1}{2}(e^x - e^{-x})$ by subtracting the terms of the appropriate series. The result is the series for $\sinh x$. (See Exercise 55 of Section 27.6.)
17. Find the first three terms of the expansion for $e^x \sin x$ by multiplying the proper expansions together, term by term.
18. Find the first three nonzero terms of the expansion for $f(x) = \tan x$ by dividing the series for $\sin x$ by that for $\cos x$.
19. By using the properties of logarithms and the series for $\ln(1 + x)$, find the series for $x^2\ln(1 - x)^2$.

20. By using the properties of logarithms and the series for $\ln(1 + x)$, find the series for $\ln\dfrac{1 + x}{1 - x}$.

21. Find the first three terms of the expansion for $\ln(1 + \sin x)$ by using the expansions for $\ln(1 + x)$ and $\sin x$.

22. Show that by differentiating term by term the expansion for $\sin x$, the result is the expansion for $\cos x$.

23. Show that by differentiating term by term the expansion for e^x, the result is also the expansion for e^x.

24. Find the expansion for $\sin x + x \cos x$ by differentiating term by term the expansion for $x \sin x$.

25. Show that by integrating term by term the expansion for $\cos x$, the result is the expansion for $\sin x$.

26. Show that by integrating term by term the expansion for $-1/(1 - x)$ (see Exercise 13 of Section 30.2), the result is the expansion for $\ln(1 - x)$.

27. By multiplication of series, find the first three terms of the expansion for the displacement of the oscillating object of Exercise 39 on page 904.

28. By using the series for e^x, find the first three terms of the expansion of the electric current given in Exercise 40 on page 904.

In Exercises 29–38, solve the given problems.

29. Evaluate $\int_0^1 e^x\,dx$ directly and compare the result obtained by using four terms of the series for e^x and then integrating.

30. Evaluate $\lim\limits_{x\to 0}\dfrac{\sin x}{x}$ by using the series expansion for $\sin x$. Compare the result with Eq. (27.1).

31. Evaluate $\lim\limits_{x\to 0}\dfrac{\sin x - x}{x^3}$ by using the expansion for $\sin x$.

32. Find the approximate area bounded by $y = \sin x$, $y = 0$, and $x = \pi/6$ by using two terms of the expansion for $\sin x$. Compare the result with that found by direct integration.

33. Find the approximate value of the area bounded by $y = x^2e^x$, $x = 0.2$, and the x-axis by using three terms of the appropriate Maclaurin series.

34. Find the approximate area under the graph of $y = \dfrac{1}{\sqrt{2\pi}}e^{-x^2/2}$ from $x = -1$ to $x = 1$ by using three terms of the appropriate series. See Fig. 30.10 and Fig. 22.11.

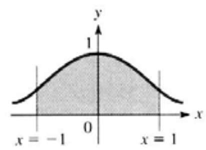

Fig. 30.10

35. The *Fresnel integral* $\int_0^x \cos t^2\,dt$ is used in the analysis of beam displacements (and in optics). Evaluate this integral for $x = 0.2$ by using two terms of the appropriate series.

36. The dome of a sports arena is designed as the surface generated by revolving the curve of $y = 20.0\cos 0.0196x$ $(0 \le x \le 80.0\text{ m})$ about the y-axis. Find the volume within the dome by using three terms of the appropriate series. (This is the same dome as in Exercise 51, page 872. Compare the results).

37. In the theory of relativity, when studying the kinetic (moving) energy of an object, the equation $K = \left[\left(1 - \dfrac{v^2}{c^2}\right)^{-1/2} - 1\right]mc^2$ is used. Here, for a given object, K is the kinetic energy, v is its velocity, and c is the velocity of light. If v is much smaller than c, show that $K = \frac{1}{2}mv^2$, which is the classical expression for K.

38. The charge q on a capacitor in a certain electric circuit is given by $q = ce^{-at}\sin 6at$, where t is the time. By multiplication of series, find the first four nonzero terms of the expansion for q.

In Exercises 39–42, use a graphing calculator to display (a) the given function and (b) the first three series approximations of the function in the same display. Each display will be similar to that in Fig. 30.9 for the function $y = \sin x$ and its first three approximations. Be careful in choosing the appropriate window values.

39. $y = e^x$

40. $y = \cos x$

41. $y = \ln(1 + x)$ $\quad(|x| < 1)$

42. $y = \sqrt{1 + x}$ $\quad(|x| < 1)$

Answer to Practice Exercise

1. $\ln(1 - 2x) = -2x - 2x^2 - \frac{8}{3}x^3 - 4x^4 - \cdots$

30.4 Computations by Use of Series Expansions

Calculating Values of Algebraic, Trigonometric, Exponential, and Logarithmic Functions • Approximations

As we mentioned at the beginning of the previous section, power-series expansions can be used to compute numerical values of exponential functions, trigonometric functions, logarithms, powers, and roots. By including a sufficient number of terms in the expansion, we can calculate these values to any degree of accuracy that may be required.

It is through such calculations that tables of values can be made, and decimal approximations of numbers such as e and π can be found. Also, many of the values found on a calculator or a computer are calculated by using series expansions that have been programmed into the chip which is in the calculator or computer.

EXAMPLE 1 Exponential value

Calculate the value of $e^{0.1}$.

In order to evaluate $e^{0.1}$, we substitute 0.1 for x in the expansion for e^x. The more terms that are used, the more accurate a value we can obtain. The limit of the partial sums would be the actual value. However, since $e^{0.1}$ is irrational, we cannot express the exact value in decimal form.

Therefore, the value is found as follows:

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots \qquad \text{Eq. (30.6)}$$

$$e^{0.1} = 1 + 0.1 + \frac{(0.1)^2}{2} + \cdots \qquad \text{substitute 0.1 for } x$$

$$= 1.105 \qquad \text{using 3 terms}$$

Using a calculator, we find that $e^{0.1} = 1.105\,170\,918$, which shows that our answer is valid to the accuracy shown. ■

EXAMPLE 2 Trigonometric value

Calculate the value of sin 2°.

In finding trigonometric values, we must be careful to ***express the angle in radians.*** Thus, the value of sin 2° is found as follows:

$$\sin x = x - \frac{x^3}{3!} + \cdots \qquad \text{Eq. (30.7)}$$

CAUTION ➧

$$\sin 2° = \left(\frac{\pi}{90}\right) - \frac{(\pi/90)^3}{6} + \cdots \qquad 2° = \frac{\pi}{90}\text{ rad}$$

$$= 0.034\,899\,496\,3 \qquad \text{using 2 terms}$$

A calculator gives the value 0.034 899 496 7. Here, we note that the second term is much smaller than the first. In fact, a good approximation of 0.0349 can be found by using just one term. We now see that $\sin\theta \approx \theta$ for small values of θ, as we noted in Section 8.4. ■

EXAMPLE 3 Trigonometric value

Calculate the value of cos 0.5429.

Since the angle is expressed in radians, we have

$$\cos 0.5429 = 1 - \frac{0.5429^2}{2} + \frac{0.5429^4}{4!} - \cdots \qquad \text{using Eq. (30.8)}$$

$$= 0.856\,249\,5 \qquad \text{using 3 terms}$$

A calculator shows that cos 0.5429 = 0.856 214 082 4. Since the angle is not small, additional terms are needed to obtain this accuracy. With one more term, the value 0.8562139 is obtained. ■

Practice Exercise

1. Using two terms of the appropriate series, calculate the value of cos 2°.

EXAMPLE 4 Logarithmic value

Calculate the value of ln 1.2.

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \qquad \text{Eq. (30.9)}$$

$$\ln 1.2 = \ln(1 + 0.2)$$

$$= 0.2 - \frac{(0.2)^2}{2} + \frac{(0.2)^3}{3} - \cdots = 0.1827$$

To four significant digits, ln(1.2) = 0.1823. One more term is required to obtain this accuracy. ■

We now illustrate the use of series in error calculations and measurement approximations. We also discussed these as applications of differentials. A series solution allows as close a value of the calculated error or approximation as needed, whereas with differentials, the calculation is limited to one term.

EXAMPLE 5 Approximation of value of velocity

The velocity v of an object that has fallen h m is $v = 4.43\sqrt{h}$. Find the approximate error in calculating the velocity of an object that has fallen 100.0 m with a possible error of 2.0 m.

NOTE ➧ If we ***let*** $\boldsymbol{v = 4.43\sqrt{100.0 + x}}$***, where x is the error in h,*** we may express v as a Maclaurin expansion in x:

$$\begin{aligned} f(x) &= 4.43(100.0 + x)^{1/2} & f(0) &= 44.3 \\ f'(x) &= 2.22(100.0 + x)^{-1/2} & f'(0) &= 0.222 \\ f''(x) &= -1.11(100.0 + x)^{-3/2} & f''(0) &= -0.001\,11 \end{aligned}$$

Therefore,

$$v = 4.43\sqrt{100.0 + x} = 4.43 + 0.222x - 0.000\,56x^2 + \cdots$$

Since the calculated value of v for $x = 0$ is 44.3, the error e in the value of v is

$$e = 0.222x - 0.000\,56x^2 + \cdots$$

Calculating, the error for $x = 2.0$ is

$$e = 0.222(2.0) - 0.000\,56(4.0) = 0.444 - 0.002 = 0.442 \text{ m/s}$$

The value 0.444 is that which is found using differentials. The additional terms are corrections to this term. The additional term in this case shows that the first term is a good approximation to the error. Although this problem can be done numerically, a series solution allows us to find the error for any value of x. ■

EXAMPLE 6 Approximation of tangent to earth's surface

From a point on the surface of the Earth, a laser beam is aimed tangentially toward a vertical rod 2 km distant. How far up on the rod does the beam touch? (Assume Earth is a perfect sphere of radius 6400 km.)

From Fig. 30.11, we see that

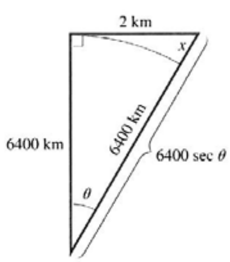

Fig. 30.11

$$x = 6400 \sec\theta - 6400$$

Finding the series for $\sec\theta$, we have

$$\begin{aligned} f(\theta) &= \sec\theta & f(0) &= 1 \\ f'(\theta) &= \sec\theta\tan\theta & f'(0) &= 0 \\ f''(\theta) &= \sec^3\theta + \sec\theta\tan^2\theta & f''(0) &= 1 \end{aligned}$$

Thus, the first two nonzero terms are $\sec\theta = 1 + (\theta^2/2)$. Therefore,

$$\begin{aligned} x &= 6400(\sec\theta - 1) \\ &= 6400\left(1 + \frac{\theta^2}{2} - 1\right) = 3200\,\theta^2 \end{aligned}$$

The first two terms of the expansion for $\tan\theta$ are $\theta + \theta^3/3$, which means that $\tan\theta \approx \theta$, since θ is small (see Section 8.4). From Fig. 30.11, $\tan\theta = 2/6400$, and therefore $\theta = 1/3200$. Therefore, we have

$$x = 3200\left(\frac{1}{3200}\right)^2 = \frac{1}{3200} = 0.0003 \text{ km}$$

This means the 2-km-long beam touches the rod only 30 cm above the surface! ■

EXERCISES 30.4

In Exercises 1 and 2, make the given changes in the indicated examples of this section, and then solve the resulting problems.

1. In Example 1, change $e^{0.1}$ to $e^{-0.1}$.
2. In Example 4, change ln 1.2 to ln 0.8.

In Exercises 3–20, calculate the value of each of the given functions. Use the indicated number of terms of the appropriate series. Compare with the value found directly on a calculator.

3. $e^{0.2}$ (3)	**4.** 1.01^{-1} (4)	**5.** sin 0.1 (2)
6. cos 0.05 (2)	**7.** e (7)	**8.** $e^{-0.5}$ (5)
9. cos $\pi°$ (2)	**10.** sin 8° (3)	**11.** ln 1.4 (4)
12. ln 0.95 (4)	**13.** sin 0.3625 (3)	**14.** cos 1 (4)
15. ln 0.8461 (5)	**16.** ln 1.0534 (3)	**17.** 1.032^6 (3)
18. 0.9982^8 (3)	**19.** $1.1^{-0.2}$ (3)	**20.** 0.96^{-1} (3)

In Exercises 21–24, calculate the value of each of the given functions. In Exercises 21 and 22, use the expansion for $\sqrt{1+x}$, and in Exercises 23 and 24, use the expansion for $\sqrt[3]{1+x}$. Use three terms of the appropriate series.

21. $\sqrt{1.1076}$	**22.** $\sqrt{0.7915}$
23. $\sqrt[3]{0.9628}$	**24.** $\sqrt[3]{1.1392}$

In Exercises 25–28, calculate the maximum error of the values calculated in the indicated exercises. If a series is alternating (every other term is negative), the maximum possible error in the calculated value is the value of the first term omitted.

25. Exercise 5	**26.** Exercise 4
27. Exercise 9	**28.** Exercise 11

In Exercises 29–40, solve the given problems by using series expansions.

29. Evaluate $\sqrt{3.92}$ by noting that $\sqrt{3.92} = \sqrt{4 - 0.08} = 2\sqrt{1 - 0.02}$.
30. Evaluate sin 32° by first finding the expansion for $\sin(x + \pi/6)$.
31. We can evaluate π by use of $\frac{1}{4}\pi = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$ (see Exercise 62 of Section 20.6), along with the series for $\tan^{-1} x$. The first three terms are $\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5$. Using these terms, expand $\tan^{-1}\frac{1}{2}$ and $\tan^{-1}\frac{1}{3}$ and approximate the value of π.
32. Use the fact that $\frac{1}{4}\pi = \tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{1}{3}$ to approximate the value of π. (See Exercise 31.)
33. (W) Explain why $e^x > 1 + x + \frac{1}{2}x^2$ for $x > 0$.
34. Using a calculator, determine how many terms of the expansion for $\ln(1 + x)$ are needed to give the value of ln 1.3 accurate to five decimal places.
35. The time t (in years) for an investment to increase by 10% when the interest rate is 6% is given by $t = \frac{\ln 1.1}{0.06}$. Evaluate this expression by using the first four terms of the appropriate series.
36. The period T of a pendulum of length L is given by
$$T = 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{1}{4}\sin^2\frac{\theta}{2} + \frac{9}{64}\sin^4\frac{\theta}{2} + \cdots\right)$$
where g is the acceleration due to gravity and θ is the maximum angular displacement. If $L = 1.000$ m and $g = 9.800\ \mathrm{m/s^2}$, calculate T for $\theta = 10.0°$ (a) if only one term (the 1) of the series is used and (b) if two terms of the indicated series are used. In the second term, substitute one term of the series for $\sin^2(\theta/2)$.
37. (W) The current in a circuit containing a resistance R, an inductance L, and a battery whose voltage is E is given by the equation $i = \frac{E}{R}(1 - e^{-Rt/L})$, where t is the time. Approximate this expression by using the first three terms of the appropriate exponential series. Under what conditions will this approximation be valid?
38. The image distance q from a certain lens as a function of the object distance p is given by $q = 20p/(p - 20)$. Find the first three nonzero terms of the expansion of the right side. From this expression, calculate q for $p = 2.00$ cm and compare it with the value found by substituting 2.00 in the original expression.
39. At what height above the shoreline of Lake Ontario must an observer be in order to see a point 15 km distant on the surface of the lake? (The radius of the earth is 6400 km.)
40. The efficiency E (in %) of an internal combustion engine in terms of its compression ratio e is given by $E = 100(1 - e^{-0.40})$. Determine the possible approximate error in the efficiency for a compression ratio measured to be 6.00 with a possible error of 0.50. (*Hint:* Set up a series for $(6 + x)^{-0.40}$.)

Answer to Practice Exercise

1. $\cos 2° = 0.999\ 390\ 8$

30.5 Taylor Series

Taylor Series Expansion • More General Than Maclaurin Series • Choice of the Value of a

To obtain accurate values of a function for values of x that are not close to zero, it is usually necessary to use many terms of a Maclaurin expansion. However, we can use another type of series, called a **Taylor series,** *which is a more general expansion than a Maclaurin expansion.* Also, functions for which a Maclaurin series may not be found may have a Taylor series.

The basic assumption in formulating a Taylor expansion is that a function may be expanded in a polynomial of the form

$$\boxed{f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + \cdots} \qquad \textbf{(30.15)}$$

Following the same line of reasoning as in deriving the Maclaurin expansion, we may find the constants $c_0, c_1, c_2, \ldots$. That is, derivatives of Eq. (30.15) are taken, and the function and its derivatives are evaluated at $x = a$. This leads to

$$\boxed{f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)(x - a)^2}{2!} + \cdots} \qquad \textbf{(30.16)}$$

■ Named for the English mathematician Brook Taylor (1685–1737).

Equation (30.16) is the **Taylor series expansion** *of a function.* It converges rapidly for values of x that are close to a, and this is illustrated in Examples 3 and 4.

EXAMPLE 1 Taylor series for e^x

Expand $f(x) = e^x$ in a Taylor series with $a = 1$.

$$f(x) = e^x \qquad f(1) = e \qquad \text{find derivatives and evaluate each at } x = 1$$
$$f'(x) = e^x \qquad f'(1) = e$$
$$f''(x) = e^x \qquad f''(1) = e$$
$$f'''(x) = e^x \qquad f'''(1) = e$$
$$f(x) = e + e(x - 1) + e\frac{(x - 1)^2}{2!} + e\frac{(x - 1)^3}{3!} + \cdots \qquad \text{using Eq. (30.16)}$$
$$e^x = e\left[1 + (x - 1) + \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{6} + \cdots\right]$$

This series can be used in evaluating e^x for values of x near 1. ■

Practice Exercise

1. Expand $f(x) = e^x$ in a Taylor series with $a = 3$.

EXAMPLE 2 Taylor series for $\sqrt{x}$

Expand $f(x) = \sqrt{x}$ in powers of $(x - 4)$.

Another way of stating this is to find the Taylor series for $f(x) = \sqrt{x}$, with $a = 4$. Thus,

$$f(x) = x^{1/2} \qquad f(4) = 2 \qquad \text{find derivatives and evaluate each at } x = 4$$
$$f'(x) = \frac{1}{2x^{1/2}} \qquad f'(4) = \frac{1}{4}$$
$$f''(x) = -\frac{1}{4x^{3/2}} \qquad f''(4) = -\frac{1}{32}$$
$$f'''(x) = \frac{3}{8x^{5/2}} \qquad f'''(4) = \frac{3}{256}$$
$$f(x) = 2 + \frac{1}{4}(x - 4) - \frac{1}{32}\frac{(x - 4)^2}{2!} + \frac{3}{256}\frac{(x - 4)^3}{3!} - \cdots \qquad \text{using Eq. (30.16)}$$
$$\sqrt{x} = 2 + \frac{(x - 4)}{4} - \frac{(x - 4)^2}{64} + \frac{(x - 4)^3}{512} - \cdots$$

This series would be used to evaluate square roots of numbers near 4.

In Fig. 30.12, the TI-89 calculator display of this expansion is shown. In Fig. 30.13, we show the TI-83 or TI-84 graphical view of $y = \sqrt{x}$ and $y = 1 - x/4$ (the first two terms of the series, and the linearization of the function at $x = 4$). We see that each curve passes through (4, 2), and they have nearly equal values of y for values of x near 4. ■

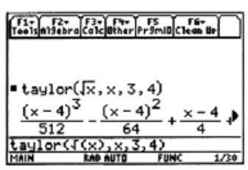

Fig. 30.12

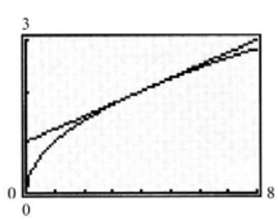

Fig. 30.13

In the last section, we evaluated functions by using Maclaurin series. In the following examples, we use Taylor series to evaluate functions.

EXAMPLE 3 Evaluating square root using Taylor series

By using Taylor series, evaluate $\sqrt{4.5}$.

Using the four terms of the series found in Example 2, we have

$$\begin{aligned}\sqrt{4.5} &= 2 + \frac{(4.5-4)}{4} - \frac{(4.5-4)^2}{64} + \frac{(4.5-4)^3}{512} && \text{substitute 4.5 for } x\\ &= 2 + \frac{(0.5)}{4} - \frac{(0.5)^2}{64} + \frac{(0.5)^3}{512}\\ &= 2.121\,337\,891\end{aligned}$$

The value found directly on a calculator is 2.121 320 344. Therefore, the value found by these terms of the series expansion is correct to four decimal places. ■

In Example 3, we saw that successive terms become small rapidly. If a value of x is chosen such that $x - a$ is larger, the successive terms may not become small rapidly, and many terms may be required. Therefore, ***we should choose the value of a as conveniently close as possible to the x-values that will be used.*** Also, we should note that a Maclaurin expansion for $\sqrt{x}$ cannot be used since the derivatives of $\sqrt{x}$ are not defined for $x = 0$.

CAUTION ♦

EXAMPLE 4 Evaluating sine value using Taylor series

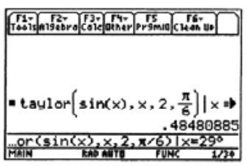

Fig. 30.14

■ Figure 30.14 shows a TI-89 calculator display for this Taylor series evaluation.

Calculate the approximate value of $\sin 29°$ by using three terms of the appropriate Taylor expansion.

Since the value of $\sin 30°$ is known to be $\frac{1}{2}$, we let $a = \frac{\pi}{6}$ (remember, we must use values expressed in radians) when we evaluate the expansion for $x = 29°$ [when expressed in radians, the quantity $(x - a)$ is $-\frac{\pi}{180}$ (equivalent to $-1°$)]. This means that its numerical values are small and become smaller when it is raised to higher powers. Therefore,

$$\begin{aligned}f(x) &= \sin x & f\left(\frac{\pi}{6}\right) &= \frac{1}{2} && \text{find derivatives and evaluate each at } x = \frac{\pi}{6}\\ f'(x) &= \cos x & f'\left(\frac{\pi}{6}\right) &= \frac{\sqrt{3}}{2}\\ f''(x) &= -\sin x & f''\left(\frac{\pi}{6}\right) &= -\frac{1}{2}\end{aligned}$$

$$\begin{aligned}f(x) &= \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\left(x - \frac{\pi}{6}\right)^2 - \cdots && \text{using Eq. (30.16)}\\ \sin x &= \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\left(x - \frac{\pi}{6}\right)^2 - \cdots && f(x) = \sin x\\ \sin 29° &= \sin\left(\frac{\pi}{6} - \frac{\pi}{180}\right) && 29° = 30° - 1° = \frac{\pi}{6} - \frac{\pi}{180}\\ &= \frac{1}{2} + \frac{\sqrt{3}}{2}\left(\frac{\pi}{6} - \frac{\pi}{180} - \frac{\pi}{6}\right) - \frac{1}{4}\left(\frac{\pi}{6} - \frac{\pi}{180} - \frac{\pi}{6}\right)^2 - \cdots && \text{substitute } \frac{\pi}{6} - \frac{\pi}{180} \text{ for } x\\ &= \frac{1}{2} + \frac{\sqrt{3}}{2}\left(-\frac{\pi}{180}\right) - \frac{1}{4}\left(-\frac{\pi}{180}\right)^2 - \cdots\\ &= 0.484\,808\,850\,9\end{aligned}$$

The value found directly on a calculator is 0.484 809 620 2. ■