Unit - 3 Matrices

Echelon Form of Matrix :

A matrix A is said to be in its echelon form if the following conditions hold :

1. Matrix A should be upper triangular i.e the elements below the principle diagonal are all zero.

2. The no of leading zeroes must increase as we run down the column.

Note : The zeroes at the beginning of each row before a non-zero entry are called leading zeroes.

Remark:

Any given matrix A can be reduced into its equi echelon form by applying elementary row operations.

For eg:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ; \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} , \begin{bmatrix} 0 & 2 & 5 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

etc. are equi echelon form

but $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & -1 \end{bmatrix}$ is not an echelon form.

Rank of a matrix :

Rank of a matrix A is defined as the number of linearly independent rows in it.

To determine the rank of a given matrix A, we reduce it into its echelon form by applying elementary row operations in it. The no. of non-zero rows in the echelon of matrix A gives the rank of A.

for eg :

Consider $A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 0 & 6 \\ 5 & 4 & 16 \end{bmatrix}$

$R_2 \to R_2 - 3R_1$

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & -6 & -9 \\ 5 & 4 & 16 \end{bmatrix}$$

$R_3 \to R_3 - 5R_1$

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & -6 & -9 \\ 0 & -6 & \cancel{-24}\ -9 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$= \begin{bmatrix} 1 & 2 & 5 \\ 0 & -6 & -9 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank (A) = no of non - zero rows
= 2

Que- Find the rank of matrix

$$A = \begin{bmatrix} 1 & 3 & 5 & 1 \\ 7 & -1 & 2 & 2 \\ 9 & 5 & 12 & 4 \end{bmatrix}$$

$R_2 \to R_2 - 7R_1$, $R_3 \to R_3 - 9R_1$

$$\begin{bmatrix} 1 & 3 & 5 & 1 \\ 0 & -22 & -33 & -5 \\ 0 & -22 & -33 & -5 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 3 & 5 & 1 \\ 0 & -22 & -33 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank(A) = 2

$$\omega = \frac{-1}{2} \pm \frac{i\sqrt{3}}{2}$$

Qus Find rank of matrix $A = \begin{bmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{bmatrix}$

where ω is cube root of unity.

Hint : $\omega^3 = 1$

$\omega^4 = \omega$

$\omega^5 = \omega^2$

$1 + \omega + \omega^2 = 0$

$$A = \begin{bmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{bmatrix}$$

$$R_2 \to R_2 - \omega R_1$$

$$A = \begin{bmatrix} 1 & \omega & \omega^2 \\ 0 & 0 & 1-\omega^3 \\ \omega^2 & 1 & \omega \end{bmatrix}$$

$$R_3 \to R_3 - \omega^2 R_1$$

$$A = \begin{bmatrix} 1 & \omega & \omega^2 \\ 0 & 0 & 1-\omega^3 \\ 0 & 1-\omega^3 & \omega - \omega^4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \omega & \omega^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Rank}(A) = 1$$

Determination of Rank by reduction to normal form :

Matrix A is said to be reduced to its normal form if on applying elementary row/column ~~trans~~ operations, the following form ~~is~~ is obtained.

1. All non-diagonal elements must be zero.
2. Diagonal elements must be either 1 or 0.

For eg:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ etc.}$$

Are normal forms.

Also we can say that the normal form of matrix A contains an identity matrix as its sub matrix.

Rank:

A matrix A is said to be of rank r if its normal form contains an identity matrix of order r as its sub matrix.

For

Consider

Since

I_r

For eg:

Rank :-

Consider $A = \begin{bmatrix} 2 & 3 \\ 8 & 12 \end{bmatrix}$

$R_2 \to R_2 - 4R_1$

$$\begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}$$

$C_2 \to 2C_2 - 3C_1$

$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$R_1 \to \frac{R_1}{2}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

I_1

$\therefore$ Rank(A) = 1

Since the normal form of A contain I_1 as its sub matrix. Hence Rank(A) = 1

Qn Find rank of matrix by reducing it to normal form.

$$A = \begin{bmatrix} 3 & 5 & 7 \\ 1 & 0 & 2 \\ \cancel{5}8 & 15 & 19 \end{bmatrix}$$

$R_2 \to 3R_2 - R_1$, $R_3 \to 3R_3 - 8R_1$

$$A = \begin{bmatrix} 3 & 5 & 7 \\ 0 & -5 & -1 \\ 0 & 5 & 1 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$A = \begin{bmatrix} 3 & 5 & 7 \\ 0 & -5 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to 5C_3 - C_2$

$$A = \begin{bmatrix} 3 & 5 & 30 \\ 0 & -5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to C_3 - 10C_1$

$$A = \begin{bmatrix} 3 & 5 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Qn

$R_1 \to R_1 + R_2$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to \frac{1}{3} R_1$, $R_2 \to \frac{-1}{5} R_2$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Rank}(A) = 2.$$

Q4 Find the rank of matrix A by reducing it normal form.

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 2 & 1 & 0 & 3 \\ 1 & 0 & 3 & 2 \end{bmatrix}$$

$R_2 \to 3R_2 - 2R_1$, $R_3 \to 3R_3 - R_1$

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & -1 & -2 & 9 \\ 0 & -2 & 8 & 6 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & -1 & -2 & 9 \\ 0 & 0 & 12 & -12 \end{bmatrix}$$

$C_4 \rightarrow C_4 + C_3$

$$\begin{bmatrix} 3 & 2 & 1 & 1 \\ 0 & -1 & -2 & 7 \\ 0 & 0 & 12 & 0 \end{bmatrix}$$

$C_4 \rightarrow C_4 + 7C_2$

$$\begin{bmatrix} 3 & 2 & 1 & 15 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 12 & 0 \end{bmatrix}$$

$C_4 \rightarrow C_4 - 5C_1$

$$\begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 12 & 0 \end{bmatrix}$$

$C_3 \rightarrow 2C_3 - C_2$

$$\begin{bmatrix} 3 & 2 & 0 & 0 \\ 0 & -1 & -3 & 0 \\ 0 & 0 & 24 & 0 \end{bmatrix}$$

$C_2 \rightarrow 3C_2 - 2C_1$

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 24 & 0 \end{bmatrix}$$

$C_3 \rightarrow C_3 - C_2$

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 24 & 0 \end{bmatrix}$$

$R_1 \rightarrow \frac{1}{3}R_1$, $R_2 \rightarrow \frac{-1}{3}R_2$, $R_3 \rightarrow \frac{1}{24}R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$Rank(A) = 3$

Rank of a matrix by the method of minors :-

Suppose A is a matrix then matrix A is said to have rank r if the following conditions hold :

(i) Every minor of order greater than r must vanish.

(ii) There exist atleast one minor of order r which doesnot vanish.

Consider a matrix

$$A = \begin{bmatrix} 3 & 5 & 7 \\ 1 & 0 & 2 \\ 8 & 15 & 19 \end{bmatrix}$$

Minors of order 3:

$$\Delta_1 = \begin{vmatrix} 3 & 5 & 7 \\ 1 & 0 & 2 \\ 8 & 15 & 19 \end{vmatrix}$$

$$= 3(0-30) - 5(19-16) + 7(15-0)$$

$$= 0$$

Minor of order 2:

$$\Delta_2 = \begin{vmatrix} 3 & 5 \\ 1 & 0 \end{vmatrix}$$

$$= 0 - 5 = -5 \neq 0$$

Rank $(A) = 2$

Qn: Find X if rank $(A) = 1$

when

$$A = \begin{bmatrix} 3 & X & X \\ X & 3 & X \\ X & X & 3 \end{bmatrix}$$

Que Find all values of μ such that rank(A) = 3, where $A = \begin{bmatrix} \mu & -1 & 0 & 0 \\ 0 & \mu & -1 & 0 \\ 0 & 0 & \mu & -1 \\ -6 & 11 & -6 & 1 \end{bmatrix}$

Q-2 Under what condition, the rank of matrix A is 3? Is it possible that rank(A) = 1? Give reasons, where $A = \begin{bmatrix} 2 & 4 & 2 \\ 3 & 1 & 2 \\ 1 & 0 & x \end{bmatrix}$

(1) Since rank(A) = 1
Hence minors of order greater than 1 must vanish

Minor of order 3: $\begin{vmatrix} 3 & x & x \\ x & 3 & x \\ x & x & 3 \end{vmatrix} = 0$

$$3(9-x^2) - x(3x-x^2) + x(x^2-3x) = 0$$

$$\left[2x^3 - 9x^2 + 27 = 0 \right]$$

$x = 3$ satisfies it.

$$x-3 \overline{\big) 2x^3 - 9x^2 + 27} \big(2x^2 - 3x - 9$$

$$\cancel{2x^3} - 6x^2$$

$$-3x^2 + 27$$

$$-3x^2 + 9x$$

$$27 - 9x$$

$$27 - 9x$$

$(x-3)(2x^2-3x-9)=0$

$(x-3)(2x^2-6x+3x-9)=0$

$(x-3)(2x(x-3)+3(x-3))=0$

$(x-3)(2x+3)(x-3)=0$

$\Rightarrow x = 3, 3, \frac{-3}{2}$

Minor of order 2:

$$\Delta_1 = \begin{vmatrix} 3 & x \\ x & 3 \end{vmatrix} = 0$$

$9-x^2=0$

$x = 3, -3$

&

$$\Delta_2 = \begin{vmatrix} x & x \\ 3 & x \end{vmatrix} = 0$$

$x^2-3x=0$

$x(x-3)=0$

$x=0, 3$

&

$$\Delta_3 = \begin{vmatrix} x & 3 \\ x & x \end{vmatrix} = 0$$

$x^2-3x=0$

$x=0, 3$

$\Rightarrow x=3$ (common soln) is the value of x which Rank $(A) = 1$

(2) Since Rank $(A) = 3$ $\Rightarrow$ Minors of order greater than 3 must vanish.

Minors of order 4.

$$\Delta = \begin{vmatrix} \mu & -1 & 0 & 0 \\ 0 & \mu & -1 & 0 \\ 0 & 0 & \mu & -1 \\ -6 & 11 & -6 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \mu \begin{vmatrix} \mu & -1 & 0 \\ 0 & \mu & -1 \\ 11 & -6 & 1 \end{vmatrix} - (-1) \begin{vmatrix} 0 & -1 & 0 \\ 0 & \mu & -1 \\ -6 & -6 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \mu \left[\mu(\mu - 6) + 1(0 + 11) \right] + 1 \left[1(0 - 6) \right] = 0$$

$$\Rightarrow \mu^3 - 6\mu^2 + 11\mu - 6 = 0$$

$\mu = 1, 2, 3$

If $\mu = 1$, then a minor of order 3 i.e. $\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$

If $\mu = 2$ " " " $\begin{vmatrix} 2 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{vmatrix} = 8 \neq 0$

If $\mu = 3$ " " " $\begin{vmatrix} 3 & -1 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & 3 \end{vmatrix} = 9 \neq 0$

$\mu = 1, 2, 3$ are the required values

③ Since Rank(A) = 3 → All ~~(A)~~ minors of order greater than 3 must vanish.

And they must exist a minor of order 3 that doesnot vanish.

$$\Delta = \begin{vmatrix} 2 & 4 & 2 \\ 3 & 1 & 2 \\ 1 & 0 & x \end{vmatrix} \neq 0$$

$$2(x) - 4(3x-2) + 2(-1) \neq 0$$

$$2x - 12x + 8 - 2 \neq 0$$

$$6 \neq 10x$$

$$x \neq \frac{3}{5}$$

x can be any real number ~~exec~~ except $\frac{3}{5}$.

$$x \in R - \left\{\frac{3}{5}\right\}$$

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Remarks :

① If A is a square matrix of order n & $\det(A) = 0$ then A cannot have full rank i.e. $\text{rank}(A) \neq n$.

② If A is square matrix & $\det(A) \neq 0$ then A has full rank.

③ If $[A]_{m\times n}$ then $\text{rank}(A) \leq \min\{m, n\}$

④ A & B are two matrices such that their Product $C = AB$ is defined then $\text{rank}(C) <= \min\{\text{Rank}(A), \text{Rank}(B)\}$

Solution of Non-Homogeneous System of Equation:-

Consider a system of 3 equations

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

The given system can be represented as

$$AX = B$$

where $A =$ (Coefficient matrix) $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

To solve the system we construct an augmented matrix $C = [A \vdots B]$

$$C = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

Apply elementary row operations to reduce matrix C into its Echelon form.

Then three cases may arise :

① Case 1 :

If $Rank(A) = Rank(C) = r = n$ (no. of variables) then the system is consistent and has a unique solution.

Case 2.

If $Rank(A) = Rank(C) = r < n$ (no. of variables) then the system is consistent & has infinitely many solutions.
(Only $n-r+1$ are linearally independent solutions).

Case 3 :

If $rank(A) \neq rank(C)$ then the

System is inconsistent and has no solution.

Ques) Test the consistency & solve

$$x + y + z = 3$$
$$3x - 2y + z = 2$$
$$5x - y + 2z = 6$$

Augmented matrix $C = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 3 & -2 & 1 & 2 \\ 5 & -1 & 2 & 6 \end{array}\right]$

$R_2 \to R_2 - 3R_1, \; R_3 \to R_3 - 5R_1$

$C = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -5 & -2 & -7 \\ 0 & -6 & -3 & -9 \end{array}\right]$

$R_3 \to 5R_3 - 6R_2$

$C = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -5 & -2 & -7 \\ 0 & 0 & -3 & -3 \end{array}\right]$

Rank (A) = Rank (C) = 3

System is consistent and has unique solution.

This gives $-3z = -3 \Rightarrow \boxed{z = 1}$

$-5y - 2z = -7$

$-5y = -5 \Rightarrow \boxed{y = 1}$

$x + 1 + 1 = 3 \Rightarrow \boxed{x = 1}$

Ques Test the consistency of system

$$x + 2y + 3z = 7$$
$$3x + y + 2z = 6$$
$$2x - y - z = -1$$

Augmented Matrix

$$C = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 3 & 1 & 2 & 6 \\ 2 & -1 & -1 & -1 \end{array}\right]$$

$R_2 \rightarrow R_2 - 3R_1$
$R_3 \rightarrow R_3 - 2R_1$

$$C = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 0 & -5 & -7 & -15 \\ 0 & -5 & -7 & -15 \end{array}\right]$$

$R_3 \rightarrow R_3 - R_2$

$$C = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 0 & -5 & -7 & -15 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since $\text{Rank}(A) = \text{Rank}(C) = \underset{r}{2} < \underset{n}{3}$ (No. of variables)

Hence, the system is consistent & has infinitely many solutions.

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This gives

$$-x+2y+3z=7$$
$$-5y-7z=-15$$
$$-5y+7z=15$$

Assume $n-r=3-2$
$=1$

as arbitary constant. ($n \to$ no. of variables, $r \to$ rank)

let $\boxed{z=k}$ (say)

$$5y+7k=15$$
$$\boxed{y=\frac{15-7k}{5}}$$

$$x+2\left(\frac{15-7k}{5}\right)+3k=7$$
$$x+\frac{(30-14k)}{5}+3k=7$$

$$5x+30-14k+15k=35$$
$$5x=35-30-k$$

$$\boxed{x=\frac{5-k}{5}}$$

$$\boxed{z=k}$$

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Ques- Test the consistency & solve

$2x - 3y + 2z = 1$
$5x + 4y + z = 10$
$3x + 7y - z = 8$

$$C_2 \left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 5 & 4 & 1 & 10 \\ 3 & 7 & -1 & 8 \end{array}\right]$$

$R_2 \rightarrow 2R_2 - 5R_1$
$R_3 \rightarrow 2R_3 - 3R_1$

$$C_2 \left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 23 & -8 & 15 \\ 0 & 23 & -8 & 13 \end{array}\right]$$

$R_3 \rightarrow R_3 - R_2$

$$C_2 \left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 23 & -8 & 15 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

$\underset{2}{\text{Rank}(A)} \neq \underset{3}{\text{Rank}(C)}$

The system is inconsistent

Que. Find the values λ & μ such that the system of equations

$$x + y + z = 3$$
$$2x - 3y + 4z = 3$$
$$3x + y + \lambda z = \mu$$

has
(i) a unique solution
(ii) infinitely many solution
(iii) no solution.

Augmented matrix

$$C = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & -3 & 4 & 3 \\ 3 & 1 & \lambda & \mu \end{array}\right]$$

$$R_2 \to R_2 - 2R_1$$
$$R_3 \to R_3 - 3R_1$$

$$C = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -5 & 2 & -3 \\ 0 & -2 & \lambda-3 & \mu-9 \end{array}\right]$$

$$R_3 \to 5R_3 - 2R_2$$

$\cancel{5\lambda-15-4}$

$\cancel{5\mu-45+6}$

$$C = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -5 & 2 & -3 \\ 0 & 0 & 5\lambda-19 & 5\mu-39 \end{array}\right]$$

(i) For unique ~~sol~~ solution

Rank (A) = Rank (C) = 3

$5\lambda - 19 \neq 0$

for all values of μ

$\boxed{\lambda \neq \frac{19}{5} \text{ for all values of } \mu}$

(ii) for infinitely many solutions

Rank (A) = Rank (C) < 3

Rank (A) = Rank (C) = 2

$\therefore$ for which

$5\lambda - 19 = 0$ (1)

and $5\mu - 39 = 0$

$\boxed{\lambda = \frac{19}{5}, \mu = \frac{39}{5}}$

(iii) for no solution

Rank (A) $\neq$ Rank (C)

i.e.

Rank (A) = 2

Rank (C) = 3

$5\lambda - 19 = 0$ & $5\mu - 39 \neq 0$

$\boxed{\lambda = \frac{19}{5} \text{ \& } \mu \neq \frac{39}{5}}$

Ques. If $x+2y+z=3$, $x+y+z=\lambda$, $3x+y+3z=\lambda^2$ for what values of λ, the system is consistent.

Ques. Show that if $\lambda \neq 0$ then the system
$2x+y=a$, $x+\lambda y-z=b$, $y+2z=c$
has a unique solution for all values of a, b & c.

If $\lambda = 0$, Determine the relationship satisfied by a, b & c so that the system is consistent.

Find a solution if $\lambda=0$ & $a=b=-c$
$=1$

① Augmented matrix, $C = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 1 & 1 & 1 & \lambda \\ 3 & 1 & 3 & \lambda^2 \end{array}\right]$

$R_2 \rightarrow R_2 - R_1$, $R_3 \rightarrow R_3 - 3R_1$

$C = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & -1 & -2 & \lambda-3 \\ 0 & -5 & -6 & \lambda^2-9 \end{array}\right]$

$\lambda^2-9-5\lambda+15$

$R_3 \rightarrow R_3 - 5R_2$

$C = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & -1 & -2 & \lambda-3 \\ 0 & 0 & 4 & \lambda^2-5\lambda+6 \end{array}\right]$

Rank (A) = 3

$Rank(A) = 2$

for the system to be consistent we must have $rank(A) = rank(C)$

but $rank(A) = 2 \Rightarrow rank(C) = 2$

for which

$$\lambda^2 - 5\lambda + 6 = 0$$
$$(\lambda - 2)(\lambda - 3) = 0$$
$$\lambda = 2, \quad \lambda = 3$$

Clearly for $\lambda = 2$, $rank(A) = rank(C) = 2$ (from the last matrix)

Also for $\lambda = 3$, $rank(A) = rank(C) = 2$ (from the last matrix)

Q2 Augmented matrix

$$C = \left[\begin{array}{ccc|c} 2 & 1 & 0 & a \\ 1 & \lambda & -1 & b \\ 0 & 1 & 2 & c \end{array}\right]$$

$R_2 \leftrightarrow R_3$

$$C = \left[\begin{array}{ccc|c} 2 & 1 & 0 & a \\ 0 & 1 & 2 & c \\ 1 & \lambda & -1 & b \end{array}\right]$$

$R_3 \to 2R_3 - R_1$

$$C = \left[\begin{array}{ccc|c} 2 & 1 & 0 & a \\ 0 & 1 & 2 & c \\ 0 & 2\lambda-1 & -2 & 2b-a \end{array}\right]$$

$R_3 \to R_3 - (2\lambda-1)R_2$

$$C = \left[\begin{array}{ccc|c} 2 & 1 & 0 & a \\ 0 & 1 & 2 & c \\ 0 & 0 & -4\lambda & 2b-a-(2\lambda-1)c \end{array}\right]$$

for unique solution

Rank (A) = Rank (C) = 3

for which

$-4\lambda \neq 0$

$\lambda \neq 0$

for all values of a, b, c.

If $\lambda = 0$

$$C = \left[\begin{array}{ccc|c} 2 & 1 & 0 & a \\ 0 & 1 & 2 & c \\ 0 & 0 & 0 & 2b-a+c \end{array}\right]$$

Rank $(A) = 2$

for consistent. Rank(A) = Rank(C)

Rank $(C) = 2$

$$\boxed{2b - a + c = 0}$$

If $\lambda = 0$ & $a = 1, b = 1, c = -1$

$$\left[\begin{array}{ccc|c} 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Rank(A) = 2
Rank(C) = 2

Rank (A) = Rank(C) = 2 < 3 (no. of variables)

System of is consistent and has infinitely many many solution.

This gives

$2x + y = 1$

$y + 2z = -1$

Assume $n - r = 3 - 2 = 1$ as arbitrary constant

Let $z = k$ (say)

$$\boxed{y = -1 - 2k}$$

$2x - 1 - 2k = 1$

$2x - 2k = 2$

$$x - k = 1$$

$$\boxed{x = 1 + k}$$

Remark.

A system of 3 equation can be considered as the eq^n of 3 planes.

Then,

(i) If unique sol^n exist, then 3 planes meet at a point.

(ii) If infinitely many sol^n exist, then 3 planes intersect in a line.

(iii) If there is no solution then the three planes dont have common intersection point or a line.

Solution of a homogeneous system of equation:

Consider a system of three equations.

$$a_{11}x + a_{12}y + a_{13}z = 0$$
$$a_{21}x + a_{22}y + a_{23}z = 0$$
$$a_{31}x + a_{32}y + a_{33}z = 0$$

Augmented Matrix for the above system is

$$C = \left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \end{array}\right]$$

Reduce matrix C into its Echelon form by applying elementary row operations

Two cases arise:

(1) If $Rank(A) = Rank(C) = r = n$ (no. of variables)

then the system is consistent & has unique solution given by $x = y = z = 0$ (trivial solution)

(2) Case (2)

If $Rank(A) = Rank(C) = r < n$ (no. of variables)

then the system is consistent & has infinitely many solutions. Also called as non-trivial solution.

Q1 Solve the system

$$3x - y - z = 0$$
$$x + y + 2z = 0$$
$$5x + y + 3z = 0$$

Q2

$$x + y - z + w = 0$$
$$x - y - 2z - w = 0$$
$$3x + y + w = 0$$

Qu-3 ~~Find~~ Find the value of b such that the system

$$2x+y-2z=0,\quad x+y+3z=0,$$
$$4x+3y+bz=0$$ has

(i) a trivial solution.

(ii) a non-trivial ~~so~~ solution.

Also find ~~the~~ a non-trivial solution.

Qu-4 ~~a~~ Find the values of λ such that

$$(\lambda-1)x+(3\lambda+1)y+2\lambda z=0$$
$$(\lambda-1)x+(4\lambda-2)y+(\lambda+3)z=0$$
$$2x+(3\lambda+1)y+3(\lambda-1)z=0$$ has

a ~~solution~~ non-trivial solution. Also find the solution in each case.

① Augmented matrix

$$C=\begin{bmatrix} 3 & -1 & -1 & : & 0 \\ 1 & 1 & 2 & : & 0 \\ 5 & 1 & 3 & : & 0 \end{bmatrix}$$

$R_2 \to 3R_2 - R_1$, $R_3 \to 3R_3 - 5R_1$

$$C=\begin{bmatrix} 3 & -1 & -1 & : & 0 \\ 0 & 4 & 7 & : & 0 \\ 0 & 8 & 14 & : & 0 \end{bmatrix}$$

3+5
9+5

~~R_2~~ $R_3 \to R_3 - 2R_2$

$$C = \left[\begin{array}{ccc:c} 3 & -1 & -1 & 0 \\ 0 & 4 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Rank(A) = 2

Rank(C) = 2

Rank(A) = Rank(C) = 2 < n(3)

The system has infinitely many solution.

~~Put $z = k$~~

~~$3x - y - k = 0$~~

Assume $(n-r) = 1$ as arbitrary constants. This gives

$$3x - y - z = 0$$
$$4y + 7z = 0$$

Put $\boxed{z = k}$

$$4y + 7k = 0$$
$$4y = -7k \Rightarrow \boxed{y = \frac{-7k}{4}}$$

$$3x + \frac{7k}{4} - k = 0$$

$12x + 7k - 4k = 0$

$3k = -12x$

$12x = -3k$

$$\boxed{x = -\frac{k}{4}}$$

(2) $x + y - z + w = 0$

$x - y - 2z - w = 0$

$3x + y + w = 0$

Augmented Matrix

$$C = \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 1 & -1 & -2 & -1 & 0 \\ 3 & 1 & 0 & 1 & 0 \end{array}\right]$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 3R_1$

$$C = \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & -2 & -1 & -2 & 0 \\ 0 & -2 & 3 & -2 & 0 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$C = \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & -2 & -1 & -2 & 0 \\ 0 & 0 & 4 & 0 & 0 \end{array}\right]$$

Since, Rank(A) = 3 , Rank(C) = 3

②

Rank(A) = Rank(C) = 3 ~~~~ < 4 (no of variables)

~~The system will have~~ ~~unique solution~~

The system is consistent & has infinitely many solution

Here $r = 3$, $n = 4$

$(n-r) = 1$ variable as arbitrary constant.

This gives

$x + y - z + w = 0$

$-2y - z - 2w = 0$

$4z = 0$

$\Rightarrow \boxed{z = 0}$

Let $w = k$ (say)

$-2y - 2k = 0$

$\boxed{y = -k}$

$x - k + k = 0$

$\boxed{x = 0}$

$x = 0$, $y = -k$, $z = 0$, $w = k$

where k is the arbitrary constant in the req. solution

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(2) Aug Augmented Matrix @

$$C = \left[\begin{array}{ccc|c} 2 & 1 & -2 & 0 \\ 1 & 1 & 3 & 0 \\ 4 & 3 & b & 0 \end{array}\right]$$

@ $R_2 \to 2R_2 - R_1$

$R_3 \to R_3 - 2R_1$

$$C = \left[\begin{array}{ccc|c} 2 & 1 & -2 & 0 \\ 0 & 1 & 8 & 0 \\ 0 & 1 & b+4 & 0 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$C = \left[\begin{array}{ccc|c} 2 & 1 & -2 & 0 \\ 0 & 1 & 8 & 0 \\ 0 & 0 & b-4 & 0 \end{array}\right]$$

(i) a trivial solution.

$Rank(A) = Rank(C) = 3$

$b - 4 \neq 0$

$\boxed{b \neq 4}$

b can be any real value except 4

(ii) non trivial. $Rank(A) = Rank(C) < 3$

$Rank(A) = Rank(C) = 2$

$\boxed{b = 4}$

(4)

for $b = 4$

$$C = \left[\begin{array}{ccc|c} 2 & 1 & -2 & 0 \\ 0 & 1 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This gives

$2x + y - 2z = 0$

$y + 8z = 0$

Let

$n = 3$, $n = 3$, $r = 2$

Assume $(n - r) = 1$ as arbitrary constants

Let $\boxed{z = k}$ (say)

$y = -8z$

$\boxed{y = -8k}$

$2x - 8k - 2k = 0$

$\boxed{x = 5k}$

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$3-\lambda$

(4)
$$\left[\begin{array}{ccc|c} \lambda-1 & 3\lambda+1 & 2\lambda & 0 \\ \lambda-1 & 4\lambda-2 & \lambda+3 & 0 \\ 2 & 3\lambda+1 & 3(\lambda-1) & 0 \end{array}\right]$$

$R_1 \longleftrightarrow R_3$

$$\left[\begin{array}{ccc|c} 2 & 3\lambda+1 & 3(\lambda-1) & 0 \\ \lambda-1 & 4\lambda-2 & \lambda+3 & 0 \\ \lambda-1 & 3\lambda+1 & 2\lambda & 0 \end{array}\right]$$

$R_3 \longrightarrow R_3 - R_2$

$R_2 \longrightarrow 2R_2 - R_1(\lambda-1)$

$2\lambda+6 - 3(\lambda+1)(\lambda-1)$
$2\lambda+6 - 3(\lambda^2+1-2\lambda)$
$2\lambda+6 - 3\lambda^2 - 3 + 6\lambda$
$-3\lambda^2 + 8\lambda + 3$

$$= \left[\begin{array}{ccc|c} 2 & 3\lambda+1 & 3(\lambda-1) & 0 \\ 0 & -3\lambda^2+10\lambda-3 & -3\lambda^2+8\lambda+3 & 0 \\ 0 & 3-\lambda & \lambda-3 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 2 & 3\lambda+1 & 3\lambda-3 & 0 \\ 0 & (-3\lambda+1)(\lambda-3) & -(3\lambda+1)(\lambda-3) & 0 \\ 0 & -(\lambda-3) & (\lambda-3) & 0 \end{array}\right]$$

$R_3 \longrightarrow (-3\lambda+1)R_3 + R_2$

$$\left[\begin{array}{ccc|c} 2 & 3\lambda+1 & 3\lambda-3 & 0 \\ 0 & (-3\lambda+1)(\lambda-3) & (-3\lambda+1)(\lambda-3) & 0 \\ 0 & 0 & -6\lambda(\lambda-3) & 0 \end{array}\right]$$

for non-trivial solution

Rank (A) = Rank (C) = 2 < 3

for which $-6\lambda(\lambda-3) = 0$

$\Rightarrow \boxed{\lambda = 0,\ \lambda = 3}$

Linearly dependent & independent vectors:

An $n \times 1$ column ~~vector~~ matrix or $1 \times n$ row ~~vector~~ matrix is called a vector.

A set of n vectors $X_1, X_2 \text{———} X_n$ are said to linearly independent if $C_1X_1 + C_2X_2 + \text{———} + C_nX_n = 0$ gives $C_1 = C_2 = C_3 = \text{———} = C_n = 0$ where $C_1, C_2, \text{———} C_n$ are constant.

Note: If any $C_i \neq 0$ then we say that $X_1, X_2, \text{———} X_n$ are linearly dependent.

Que- Examine whether the given set of vectors are linearly independent or not. If they are linearly dependent find the relationship b/w them.

$X_1 = [3,1,1]$ $X_2 = [2,0,-1]$ $X_3 = [4,2,1]$

$X_1 = [2,2,1]$ $X_2 = [1,3,1]$ $X_3 = [1,2,2]$

$X_1 = [1,2,-1,0]^T$ $X_2 = [1,3,1,2]^T$ $X_3 = [4,2,1,0]^T$

$X_1 = [3,1,-4]$ $X_2 = [2,2,-3]$ $X_3 = [0,-4,1]$

Let $C_1X_1 + C_2X_2 + C_3X_3 = 0$

$C_1[3,1,1] + C_2[2,0,-1] + C_3[4,2,1] = 0$

$[3C_1 + 2C_2 + 4C_3, C_1 + 2C_3, C_1 - C_2 + C_3]$
$= [0,0,0]$

$$\left.\begin{array}{l} 3C_1 + 2C_2 + 4C_3 = 0 \\ C_1 + 2C_3 = 0 \\ C_1 - C_2 + C_3 = 0 \end{array}\right\} \text{This is homogeneous system}$$

Augmented matrix:

$$C = \left[\begin{array}{ccc:c} 3 & 2 & 4 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right]$$

$R_2 \to 3R_2 - R_1$, $R_3 \to 3R_3 - R_1$

$$C \sim \left[\begin{array}{ccc:c} 3 & 2 & 4 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & -5 & -1 & 0 \end{array}\right]$$

$-2C_2 - 2C_3 = 0$
$-C_3 = 0 \quad C_3 = 0$

$R_3 \to 2R_3 - 5R_2$

$$\left[\begin{array}{ccc:c} 3 & 2 & 4 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

Rank (A) = Rank (C) = 3 = no. of var.

∴ Unique soln. given by

$C_1 = C_2 = C_3 = 0$ [Trivial soln.]

∴ vector are L.I.

let $C_1X_1 + C_2X_2 + C_3X_3 = O$

$C_1[1, 2, -1, 0]^T + C_2[1, 3, 1, 2]^T + C_3[4, 2, 1, 0]^T = [0, 0, 0, 0]^T$

$[C_1 + C_2 + 4C_3, 2C_1 + 3C_2 + 2C_3, -C_1 + C_2 + C_3, 2C_2]^T = [0, 0, 0, 0]^T$

This gives

$C_1 + C_2 + 4C_3 = 0$
$2C_1 + 3C_2 + 2C_3 = 0$
$-C_1 + C_2 + C_3 = 0$
$2C_2 = 0$

Augmented Matrix

$$C = \left[\begin{array}{ccc:c} 1 & 1 & 4 & 0 \\ 2 & 3 & 2 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$; $R_3 \to R_3 + R_1$

$$\left[\begin{array}{ccc:c} 1 & 1 & 4 & 0 \\ 0 & 1 & -6 & 0 \\ 0 & 2 & 5 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right]$$

$R_3 \to R_3 - 2R_2$; $R_4 \to R_4 - 2R_2$

$$\left[\begin{array}{ccc:c} 1 & 1 & 4 & 0 \\ 0 & 1 & -6 & 0 \\ 0 & 0 & 17 & 0 \\ 0 & 0 & 12 & 0 \end{array}\right]$$

$R_4 \to 17R_4 - 12R_3$

$$= \left[\begin{array}{ccc:c} 1 & 1 & 4 & 0 \\ 0 & 1 & -6 & 0 \\ 0 & 0 & 17 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore$ Rank (A) = Rank (C) = 3 = no. of variables

$\therefore$ Unique sol^n eq^{n}s given by
$c_1 = c_2 = c_3 = 0$ [Trivial solution]
$\therefore$ vectors are linearly independent.

Ans Let $c_1 x_1 + c_2 x_2 + c_3 x_3 = 0$

$$c_1 [3, 1, -4] + c_2 [2, 2, -3] + c_3 [0, -4, 1] = [0, 0, 0]$$

$$\Rightarrow [3c_1 + 2c_2 + \cancel{0},\ c_1 + 2c_2 \cancel{+} -4c_3,\ -4c_1 + -3c_2 + c_3] = [0, 0, 0]$$

This gives

$$\left.\begin{aligned} 3c_1 + 2c_2 &= 0 \\ c_1 + 2c_2 - 4c_3 &= 0 \\ -4c_1 \cancel{+} -3c_2 + c_3 &= 0 \end{aligned}\right\}$$ this is homogeneous system.

Augmented matrix:

$$C = \left[\begin{array}{ccc:c} 3 & 2 & 0 & 0 \\ 1 & 2 & -4 & 0 \\ -4 & -3 & 1 & 0 \end{array}\right]$$

$R_2 \rightarrow 3R_2 - R_1$, $R_3 \rightarrow 3R_3 + 4R_1$

$$= \begin{bmatrix} 3 & 2 & 0 & : & 0 \\ 0 & 4 & -12 & : & 0 \\ 0 & -1 & 3 & : & 0 \end{bmatrix}$$

$R_3 \to 4R_3 + R_2$

$$= \begin{bmatrix} 3 & 2 & 0 & : & 0 \\ 0 & 4 & -12 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$\because$ Rank (A) = Rank (C) = $2 < 3$ (no. of variable)

$\therefore$ Infinite many sol^n. [non-trivial sol^n] exist.

$\therefore$ Vectors are L.D.

Relationship:

$$3C_1 + 2C_2 = 0$$

$$4C_2 - 12C_3 = 0$$

Assume $(n-r) = 3-2 = 1$ a variable as arbitrary constant

Let $\boxed{C_3 = K}$ (say)

$$4C_2 - 12C_3 = 0$$

$$\boxed{C_2 = 3k}$$

$$3C_1 + 6k = 0 \Rightarrow \boxed{C_1 = -2k}$$

Again $C_1X_1 + C_2X_2 + C_3X_3 = 0$

$(-2k)X_1 + (3k)X_2 + (k)X_3 = 0$

$$\boxed{-2X_1 + 3X_2 + X_3 = 0}$$

is the required solution.

Remark

A set of $(n+1)$ or more vectors in R^n are always linearly dependent. For example:

$X_1 = [1, 3]$, $X_2 = [2, 5]$, $X_3 = [7, 1]$

are always linearly dependent.

$X_1 = [a, b, c]$ $X_2 = [p, q, r]$ $X_3 = [u, v, w]$

$X_4 = [x, y, z]$ are always dependent.

Two vectors X_1 & X_2 are linearly dependent if & only if one is the scalar multiple of other.

For eg: $X_1 = [3, 1, 2]$ & $X_2 = [6, 2, 4]$ are linearly dependent as $X_2 = 2X_1$

~~Geom~~

If X_1, X_2 & X_3 are three vectors in ~~R^2~~ R^3 and if X_1, X_2, X_3 are linearly dependent this implies that there must a plane containing these vectors which will pass through the origin.

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Eigen Values:

Suppose A is a square matrix of order n. A real scalar λ (maybe complex) is called an Eigen value of A if it satisfies the equation

$$\det(A-\lambda I) = 0$$

which is called characteristic equation

The characteristic is a polynomial equation in λ of degree n.

On solving it, we get n roots which are called Eigen values of matrix A.

For example:

If $A = \begin{bmatrix} 0 & 3 \\ -1 & -4 \end{bmatrix}$

Characteristic eq^n:

$$\det(A-\lambda I) = 0$$

$$\Rightarrow \det\left[\begin{bmatrix} 0 & 3 \\ -1 & -4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right] = 0$$

$$\Rightarrow \det \begin{bmatrix} -\lambda & 3 \\ -1 & -4-\lambda \end{bmatrix} = 0$$

$$4\lambda + \lambda^2 + 3 = 0$$

$$\lambda^2 + 4\lambda + 3 = 0$$

$$(\lambda+1)(\lambda+3) = 0$$

$$\underbrace{\lambda = -1, \lambda = -3}_{\text{Eigen values}}$$

Remark:

① Sum of eigen values of = tr(A)
(trace)
(sum of elements of principle diagonal)

② Product of Eigen values = det(A)

③ If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the Eigen values of matrix A of order n then $\lambda_1^k, \lambda_2^k, \ldots \lambda_n^k$ are the eigen values of matrix A^k, where K is any rational number.

④ If λ & μ are Eigen values of matrices A & A^k then $(\lambda+\mu)$ is an eigen value of $(A+A^k)$.

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(5) The Eigen values of a triangular or a diagonal matrix are the element of its diagonal.

(6) For a 2×2 matrix A, the characteristic eq^n is given by

$$\boxed{\lambda^2 - tr(A)\lambda + \det(A) = 0}$$

For a 3×3 matrix A, the characteristic eq^n is given by.

$$\lambda^3 - tr(A)\lambda^2 + (A_{11}+A_{22}+A_{33})\lambda - \det(A) = 0$$

where A_{ii} = Cofactor of diagonal element a_{ii}

Qu. Find the sum & product of the Eigen values of $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$

Qu-2. The product of two of the Eigen values of $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ is 16.

Find the third Eigen value

$6(9-1)+2(-6+2)+2(2-2)$
$48-8-8$
$= 32$

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$3(ab)+3I$

Ques 3 If $A = \begin{bmatrix} 3 & -3 & 0 \\ 0 & a & 1 \\ 0 & 0 & b \end{bmatrix}$

If sum of the eigen values is 100 & product is 30. Then find $a^2 + b^2$

Ques 4 Two eigen values of $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

are $\frac{1}{5}$ times the third. Find its Eigen values

Ques 5 A singular matrix of order 3 has eigen values 2 & 3. Find its third eigen values

Ques 6 Find a & b such that $A = \begin{bmatrix} a & 4 \\ 1 & b \end{bmatrix}$

has Eigen values = 3 & -2.

Ques 7 If A has eigen values 1, 1 and 5 then find the eigen values of

(1) A^T (2) A^{-1} (3) $5A^{-1}$ (4) $A^2+4A+3I$

(1) Sum of eigen values $= tr(A)$
$= 2$

Product of eigen values $= \det(A)$
$= 1(-2) - 2(-4)$
$= -2+8 = 6$

92
52 7
6 679
6 7 9 ×
2^2 3
9409

(2) Let the third eigen value be λ.

Product of eigen value = det (A)

$16\lambda = 32$

$\boxed{\lambda = 2}$

Third eigen value = 2.

(3) Sum of eigen value = tr (A)

$100 = 3 + a + b$

$a + b = 97 \longrightarrow (1)$

Product of eigen value = det (A)

$30 = 3ab$

$ab = 10 \longrightarrow (2)$

$a^2 + b^2 = (a+b)^2 - 2ab$

$= (97)^2 - 2 \times 10$

$= 9409 - 20$

$= 9389$

(4)

(4) Let the third Eigen values be λ

then the other Eigen values $\frac{\lambda}{5}, \frac{\lambda}{5}$

Sum of Eigen values = Tr(A)

$$\lambda + \frac{\lambda}{5} + \frac{\lambda}{5} = 7$$

$$5\lambda + 2\lambda = 35$$

$$7\lambda = 35$$

$$\boxed{\lambda = 5}$$

Third Eigen value $= \lambda = 5$

First and second ~~and~~ Eigen values $\frac{\lambda}{5} = 1$.

(5) Let λ be the third Eigen value.

Product of Eigen values = det (A)

But det (A) = 0 (Singular Matrix)

$$\lambda \times 2 \times 3 = 0 \Rightarrow 6\lambda = 0$$

$$\boxed{\lambda = 0}$$

Third Eigen value = 0.

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(6) Sum of eigen value $=$ Tr(A) (7)

$$3-2 = a+b$$

$$a+b=1 \quad \text{——} \quad (1)$$

Product of eigen value $=$ det A)

$$-6 = ab-4$$

$$ab = -2 \quad \text{——} \quad (2)$$

$$b = \frac{-2}{a}$$

$$a - \frac{2}{a} = 1$$

$$a^2 - 2 = a$$

$$a^2 - a - 2 = 0$$

$$a^2 - 2a + a - 2 = 0$$

$$a(a-2) + 1(a-2) = 0$$

$$(a+1)(a-2) = 0$$

$$\boxed{a=-1} \qquad \boxed{a=2}$$

$$\boxed{b=2} \qquad \boxed{b=-1}$$

(1) E.V of $A \rightarrow 1, 1, 5$

(ii) E.V of $A^T \rightarrow 1, 1, 5$

(iii) E.V of $A^{-1} \rightarrow 1, 1, \frac{1}{5}$

(iv) E.V of $5A^{-1} \rightarrow 5, 5, 1$

(v) E.V of $A^2 + 4A + 3I$

E.V of $A^2 \rightarrow 1, 1, 25$
E.V of $4A \rightarrow 4, 4, 20$
E.V of $3I \rightarrow 3, 3, 3$

E.V of $A^2 + 4A + 3I \rightarrow 8, 8, 48$

Eigen values & diagonlisation:

Suppose A is square matrix of order n. A scalar λ is called an eigen value of matrix A, if it satisfies the characteristic equation $(\det(A - \lambda I) = 0)$

On solving it we get eigen values $\lambda_1, \lambda_2, \ldots \lambda_n$

For each eigen value λ_i, there exist a non-zero vector V_i satisfying the relation $\boxed{AV_i = \lambda_i V_i}$

or $Av_i - \lambda_i v_i = 0$

$$\boxed{(A-\lambda_i)v_i = 0}$$ (v)

A square matrix A is said to be ~~diago~~ ~~diagon~~ diagonalisable if there (1)
a non-singular matrix P such that

$D = P^{-1}AP$ is a diagonal matrix (2)
given by $D = \text{diag}(\lambda_1, \lambda_2, \ldots \lambda_n)$

The steps of diagonalisation of matrix A (3)
are as follows:

(i) Write characteristic equation of matrix A which (4)
is

$$\det(A-\lambda I) = 0$$

(ii) Find the Eigen values $\lambda_1, \lambda_2, \ldots \lambda_n$
on solving the characteristic equation

(iii) For each λ_i, we find an
eigen vector v_i given by

$$(A-\lambda_i I)v_i = 0$$

(iv) Construct a non-singular matrix P
having $v_1, v_2, \ldots v_n$ as its column
Matrix P is called model matrix.

(v) There exist a diagonal matrix D satisfying $D = P^{-1}AP$ where $D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$

Remarks:

① A square matrix of order n is diagonlisable if & only if it has n linearly independent eigen vectors

② A matrix having distinct eigen values is always diagonalisable

③ A matrix A having repeated eigen values may or may not be diagonalisable.

④ A diagonal matrix is always diagonalisable.

Q-1 Find the Eigen values & Eigen vectors of $A = \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}$

Also find a non-singular matrix P such that $P^{-1}AP$ is a diagonal matrix.

Q-2 For a 2×2 matrix A, if $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ are eigen vectors corresponding to the eigen values 1 & 4 respectively. Find matrix A.

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Que If $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ are eigen vectors of A, Find the eigen values of A, if $A = \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}$.

Que If $A = \begin{bmatrix} 3 & -4 \\ 2 & -6 \end{bmatrix}$. Find the eigen value & eigen vectors of A. Also find a nonsingular matrix P such that $D = P^{-1}AP$ is a diagonal

Que Q. Check the diagonalizability of the following matrices. If these are diagonalizable find matrix P s.t.

$D = P^{-1}AP$ is a diagonal.

(i) $A = \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 2 & 1 & -2 \\ 2 & 3 & -4 \\ 1 & 1 & -1 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 3 & -1 & 1 \\ 7 & -5 & 1 \\ 6 & -6 & 2 \end{bmatrix}$

(iv) $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & [illegible] \end{bmatrix}$

Q. $A = \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}$

Solⁿ :-

Ch eqⁿ $\longrightarrow \det(A - \lambda I) = 0$

$* \ \lambda^2 - tr(A)\lambda + \det(A) = 0$

$\lambda^2 - 5\lambda + 4 = 0$

$(\lambda - 1)(\lambda - 4) = 0$

$\Rightarrow \lambda_1 = 1 \ , \ \lambda_2 = 4$ (eigen values)

for $\lambda_1 = 1$; $(A - \lambda_1 I) v_1 = 0$

$(A - I) v_1 = 0$

$\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Augmented matrix :

$C = \left[\begin{array}{cc:c} 1 & -1 & 0 \\ -2 & 2 & 0 \end{array}\right]$

$R_2 \longrightarrow R_2 + 2R_1$

$C = \left[\begin{array}{cc:c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

Rank(A) = Rank(C) = 1 < 2 (no. of variables

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 7 & -5 & 1 \\ 6 & -6 & 2 \end{bmatrix}$$

Ch eq^n :-

$$\det(A - \lambda I) = 0$$

$$\Rightarrow \lambda^3 - 0\lambda^2 + (-4+0-8)\lambda - (-16) = 0$$

$$\lambda^3 - 12\lambda + 16 = 0$$

clearly

$\lambda = 2, 2, -4$ are eigen values

① for $\lambda = -4$ $\quad (A + 4\lambda) v_1 = 0$

$$\begin{bmatrix} 7 & -1 & 1 \\ 7 & -1 & 1 \\ 6 & -6 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented matrix :

$$C = \left[\begin{array}{ccc:c} 7 & -1 & 1 & 0 \\ 7 & -1 & 1 & 0 \\ 6 & -6 & 6 & 0 \end{array}\right]$$

$R_2 \to R_2 - R_1$ & $R_3 \to 7R_3 - 6R_1$

$$C = \left[\begin{array}{ccc:c} 7 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -36 & 36 & 0 \end{array}\right]$$

$R_2 \longleftrightarrow R_3$

$$C = \left[\begin{array}{ccc|c} 7 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore$ Rank(A) = Rank(C) = 2 < 3 (no. of variables)

$\therefore$ $n-r = 3-2 = 1$ eigen vector exists.

thus given

$7x - y + z = 0$

$-y + z = 0$

Let $z = k$ (say)

then

$y = k$ & $x = 0$

$$v_1 = \begin{bmatrix} 0 \\ k \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \text{ for } k = 1.$$

(b) for $\lambda = 2$, $(A - 2I)v = 0$

$$C = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 2 & -2 & 1 & 0 \\ 6 & -6 & 0 & 0 \end{array}\right]$$

where v_2 $\begin{bmatrix} x \\ y \end{bmatrix}$

$R_2 \to R_2 - 7R_1$, $R_3 \to R_3 - 6R_1$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & -6 & 0 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Rank (A) = Rank (C) = 2 < 3 (no. of variable)

So,

$n - r = 3 - 2 = 1$ Eigen value exists

Only 2 L.I eigen ~~value~~ vector exists for matrix $A_{3\times3}$

∴ A is not diagonalisable

(5) (ii) $A = \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix}$

Ch. eqn $\det(A - \lambda I) = 0$

$\lambda^2 - tr(A)\lambda + \det(A) = 0$

$\lambda^2 - 8\lambda + 16 = 0$

$\lambda = 4, 4$ are eigen values.

(a) for $\lambda = 4$, $(A - \lambda I)v = 0$

$(A - 4I)v = 0$

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Augmented Matrix:

$$C = \left[\begin{array}{cc:c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$R_2 \rightarrow R_2 - R_1$

$$C = \left[\begin{array}{cc:c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\text{Rank}(A) = \text{Rank}(C) = 1 < 2$ (no of variables)

$n - r = 2 - 1 = 1 \Rightarrow$ L.I. eigen value exist.

since matrix A is of order 2 & only 1 L.I. eigenvector exists Hence it is not diagonalizable

$-6(-3+\lambda)+6(-4+3)$

$4(-1)-6(-1)+6(-1)$

$$A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$$

<u>Che eq^n</u> $\det(A-\lambda I)=0$

-4

$$\lambda^3 - 4\lambda^2 + [(-1)+(-6)+6]\lambda - (-4) = 0$$

$$\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$$

$\lambda = 1, 4, -1$ are eigen values.

(5) (iii) $A = \begin{bmatrix} 2 & 1 & -2 \\ 2 & 3 & -4 \\ 1 & 1 & -1 \end{bmatrix}$

$$\lambda^3 - 4\lambda^2 + (1+0+4)\lambda - (2) = 0$$

$$\lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$$

$\lambda = 1, 1, 2$ are eigen values.

for $\lambda_1 = 2$, $(A - 2I)v_1 = 0$

$$v_1 \quad \begin{bmatrix} 0 & 1 & -2 \\ 2 & 1 & -4 \\ 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented Matrix:

$$C = \left[\begin{array}{ccc|c} 0 & 1 & -2 & 0 \\ 2 & 1 & -4 & 0 \\ 1 & 1 & -3 & 0 \end{array}\right] \text{ where } v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$R_1 \longleftrightarrow R_2$

$$C = \left[\begin{array}{ccc|c} 2 & 1 & -4 & 0 \\ 0 & 1 & -2 & 0 \\ 1 & 1 & -3 & 0 \end{array}\right]$$

$R_3 \longrightarrow 2R_3 - R_1$

$$= \begin{bmatrix} 2 & 1 & -4 & : 0 \\ 0 & 1 & -2 & : 0 \\ 0 & 1 & -2 & : 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$= \begin{bmatrix} 2 & 1 & -4 & : 0 \\ 0 & 1 & -2 & : 0 \\ 0 & 0 & 0 & : 0 \end{bmatrix}$$

Rank (A) = Rank (C) < 3 (no of variables)

$n - r = 3 - 2 = 1$ L.I eigenvector

Then eq given

$$2x + y - 4z = 0$$

$$y - 2z = 0$$

Let $z = k$ (say)

then $y = 2k$

& $x = k$

$$\therefore V_1 = \begin{bmatrix} k \\ 2k \\ k \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \text{ at } k = 1$$

(b) for $\lambda = 1$, $(A - 1I)v_2 = 0$

$$C = \left[\begin{array}{ccc:c} 1 & 1 & -2 & 0 \\ 2 & 2 & -4 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$$

where $v_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$

$$= \left[\begin{array}{ccc:c} 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Rank(C) = Rank(A) = 1 < 3 (no of variables)

$n - r = 3 - 1 = 2$ L.I eigen vectors exist.

$x + y - 2z = 0$ ——— (A)

let $z = k_1$

$y = k_2$

$x + k_2 - 2k_1 = 0$

$x = 2k_1 - k_2$

$y = k_2$

$z = k_1$ $v = \begin{bmatrix} 2k_1 - k_2 \\ 0k_1 + k_2 \\ k_1 + 0k_2 \end{bmatrix}$

$$= k_1 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Clearly A is diagonalizable

Modal matrix:

$$P = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$P^{-1} = \frac{1}{-1} \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -3 \\ 2 & 1 & -4 \end{bmatrix}$$

$\therefore D = P^{-1} A P$

$$= \begin{bmatrix} 1 & 1 & -2 \\ -1 & -1 & 3 \\ -2 & -1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & -2 \\ 2 & 3 & -4 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & \\ 2 & 0 & \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \text{diag}(2,1,1)$$

Remark:

from eqⁿ Ⓑ

ⓐ put $y=1$, $z=0$ then $x=-1$

ⓑ for $y=0$, $z=1$, then $x=3$

Ⓑ

Here $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ & $v_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ are eigen vectors

& $\lambda_1 = 1$ & $\lambda_2 = 4$ are eigen values

$A = ?$

Modal Matrix $= \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$ & $D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

then $P^{-1} = \frac{1}{3} \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$

$\therefore D = P^{-1}AP$

& Premultiply by P,

$PD = PP^{-1}AP$

$PD = IAP$

Post multiply by P^{-1}

$PDP^{-1} = APP^{-1}$

$$\boxed{A = PDP^{-1}}$$

(3) Here $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ & $v_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ are

Eigen vectors,

of $A = \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}$

$\lambda_1 = ?$, $\lambda_2 = ?$

Modal Matrix, $P = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$

& $P^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$

$D = P^{-1}AP$

$= \frac{1}{3}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\lambda_1 = 1 \ \& \ \lambda_2 = 1$$

Diagonalisation of symmetric matrices :

In the last section we see that a matrix A having repeated eigen values may or may not be diagonalised. But ~~with~~ if a matrix A is symmetric ($A^T = A$) then A is always diagonalisable irrespective of the ~~multiplicity~~ multiplicity of its eigen values.

Following facts for symmetric matrix A are useful.

① The eigen values of symmetric matrix A are always real.

② The eigen vectors of a symmetric matrix A are always orthogonal i.e. eigen $v_i \cdot v_j = 0$

The modal matrix P for a symmetric matrix A is an orthogonal matrix. i.e. ($P^{-1} = P^T$)

The steps of diagonalisation of a symmetric matrix A are as follows :-

Q Write characteristic equation
$\det(A - \lambda_i I) = 0$

find the eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$ on solving characteristic equation

For each eigen value λ_i, we find a non-zero vector v_i satisfying

$$(A - \lambda_i I) v_i = 0$$

which is called an eigen vector.

* In order to obtain the remaining eigen vectors, sometimes the condition of orthogonality of the eigen vectors have to be used which is

$$v_i^T \cdot v_j = 0$$

Construct a non-singular matrix P having $\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_n$ as its columns.

where

$$\hat{v}_i = \frac{v_i}{|v_i|}$$

Matrix P is orthogonal in nature.
i.e. $P^{-1} = P^T$

Then there exist a diagonal matrix D

Satisfying $\boxed{D = P^{-1}AP = P^{T}AP}$

where $D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$

Qr Given the symmetric matrices. Find an orthogonal matrix P such that $P^{-1}AP$ is a diagonal.

(a)
$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

(b)
$$\Rightarrow \begin{bmatrix} -11 & -8 & 4 \\ -8 & -1 & -2 \\ 4 & -2 & -4 \end{bmatrix}$$

(a)
$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Ch eq^n $\rightarrow$ $\det(A - \lambda I) = 0$

$$\lambda^3 - 18\lambda^2 + (5 + 20 + 20)\lambda - 0 = 0$$

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0$$

$$\lambda(\lambda^2 - 18\lambda + 45) = 0$$

$$\lambda(\lambda - 3)(\lambda - 15) = 0$$

$\lambda = 0, 3, 15$

$\lambda_1 = 0, \quad \lambda_2 = 3, \quad \lambda_3 = 15$

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(ii) for $\lambda_1 = 0$; $(A - \lambda_1 I) v_1 = 0$

$A v_1 = 0$

$$C = \left[\begin{array}{ccc:c} 8 & -6 & 2 & 0 \\ -6 & 7 & -4 & 0 \\ 2 & -4 & 3 & 0 \end{array}\right]$$

where $v_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$R_2 \to 4R_2 + 3R_1$, $R_3 \to 4R_3 - R_1$

$$C = \left[\begin{array}{ccc:c} 8 & -6 & 2 & 0 \\ 0 & 10 & -10 & 0 \\ 0 & 10 & -10 & 0 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$C = \left[\begin{array}{ccc:c} 8 & -6 & 2 & 0 \\ 0 & 10 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Rank(A) = Rank(C) < 3 (no. of variables)

$n - r = 3 - 2 = 1$ L.I eigen vector.

$8x - 6y + 2z = 0$

$10y - 10z = 0$

$y - z = 0$

Let $z = \frac{k}{2}$ (say)

then $y = k \Rightarrow x = \frac{k}{2}$

$v_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ for $k = 2$

(b) for $\lambda = 3$, $(A - 3I) v_2 = 0$

$$C = \left[\begin{array}{ccc|c} 5 & -6 & 2 & 0 \\ -6 & 4 & -4 & 0 \\ 2 & -4 & 0 & 0 \end{array}\right]$$

where $v_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$R_2 \to 5R_2 + 6R_1$

$R_3 \to 5R_3 - 2R_1$

$$C = \left[\begin{array}{ccc|c} 5 & -6 & 2 & 0 \\ 0 & -16 & -8 & 0 \\ 0 & -8 & -4 & 0 \end{array}\right]$$

$R_3 \to 2R_3 - R_2$, $R_2 \to \frac{1}{8} R_2$

$$C = \left[\begin{array}{ccc|c} 5 & -6 & 2 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Rank (A) = Rank (C) = 2 < 3

$n - r = 1$ L.I eigen vector exists

$$5x - 6y + 2z = 0$$
$$-2y - z = 0$$

let $z = k$ (say)

then $y = \frac{-k}{2}$ & $x = -k$

$$v_2 = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} \text{ for } k = 2.$$

(iii) for $\lambda_3 = 15$

$$(A - 15I) v_3 = 0$$

$$\therefore C = \begin{bmatrix} -7 & -6 & 2 & : & 0 \\ -6 & -8 & -4 & : & 0 \\ 2 & -4 & -12 & : & 0 \end{bmatrix}$$

$R_2 \longrightarrow 7R_2 - 6R_1$; $R_3 \longrightarrow 7R_3 + 2R_1$

$$\therefore C = \begin{bmatrix} -7 & -6 & 2 & : & 0 \\ 0 & -20 & -40 & : & 0 \\ 0 & -20 & -80 & : & 0 \end{bmatrix}$$

$R_3 \longrightarrow R_3 - 2R_2 \quad , \quad R_2 \longrightarrow \frac{1}{20} R_2$

$$= \left[\begin{array}{ccc:c} -7 & -6 & 2 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore$ Rank(A) = Rank(C) = 2 < 3

$n - r = 1$ L.I. vec eigen vector exists

let $z = k$ (say)

then $y = -2k$

$x = 2k$

$$v_3 = \begin{bmatrix} 2k \\ -2k \\ k \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \quad \text{for} \quad k = 1$$

Modal Matrix:

$$P = \begin{bmatrix} 1/3 & -2/3 & 2/3 \\ 2/3 & -1/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \end{bmatrix}$$

$$P^T = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ -2/3 & -1/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$

$\therefore \quad D = P^{-1}AP$

$= P^{T}AP$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

III :

$$A = \begin{bmatrix} 11 & -8 & 4 \\ -8 & -1 & -2 \\ 4 & -2 & -4 \end{bmatrix}$$

Ch. eq^n : $\det(A - \lambda I) = 0$

$\Rightarrow \lambda^3 - 6\lambda^2 + (0 - 60 - 75)\lambda - 400 = 0$

$\lambda^3 - 6\lambda^2 - 135\lambda - 400 = 0$

$\lambda = -5, -5, \& 16$

(a) for $\lambda = 16$

~~$A - 16I$~~ $\cancel{(A + 5I)} \cancel{= 0}$

$(A - 16I)v_1 = 0$

$$\therefore \quad C = \left[\begin{array}{ccc|c} -5 & -8 & 4 & 0 \\ -8 & -17 & -2 & 0 \\ 4 & -2 & -20 & 0 \end{array}\right] \text{ where } v_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$R_2 \to 5R_2 - 8R_1$, $R_3 \to 5R_3 + 4R_1$

$$= \left[\begin{array}{ccc|c} -5 & -8 & 4 & \\ 0 & -21 & -42 & 0 \\ 0 & -42 & -84 & 0 \\ & & & 0 \end{array}\right]$$

$R_3 \to R_3 - 2R_2$, $R_2 \to \frac{1}{21} R_2$

$$= \left[\begin{array}{ccc|c} -5 & -8 & 4 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\text{Rank}(A) = \text{Rank}(C) = 2 < 3$

$\therefore$ $n - r = 1$ L.I eigen vectors exists.

$$-5x - 8y + 4z = 0$$
$$-y - 2z = 0$$

$-5x - 8y + 4z = 0$ & $y + 2z = 0$

Let $z = k$

then $y = -2k$ & $x = 4k$

$$v_1 = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} \text{ if } k = 1$$

(2) for $\lambda_2 = -5$ $(A + 5I)v_2 = 0$

$$C = \left[\begin{array}{ccc|c} 16 & -8 & 4 & 0 \\ -8 & 4 & -2 & 0 \\ 4 & -2 & 1 & 0 \end{array}\right] \text{ where } v_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$R_2 \longrightarrow 2R_2 + R_1$

$R_3 \longrightarrow 4R_3 - R_1$

$$C = \left[\begin{array}{ccc:c} 16 & -8 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore$ Rank(A) = Rank(C) = $1 < 3$

$n - r = 3 - 1 = 2$ L.I eigen vector exi.

$16x - 8y + 4z = 0$

let ~~x=k~~ $y = 1$, $z = 0$ ~~x2~~

then

$x = \frac{1}{2}$

$$v_2 = \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

let $v_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be the 3^{rd} eigen vetr

Clearly $v_3^T v_1 = 0$ & $v_3^T v_2 = 0$

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} = 0 \ \& \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad 4a - 2b + c = 0 \quad \& \quad a + 2b = 0$

$$C = \left[\begin{array}{ccc|c} 4 & -2 & 1 & 0 \\ 1 & 2 & 0 & 0 \end{array}\right]$$

$R_2 \longrightarrow 4R_2 - R_1$

$$\left[\begin{array}{ccc|c} 4 & -2 & 1 & 0 \\ 0 & 10 & -1 & 0 \end{array}\right]$$

$$\left.\begin{array}{l} 4a - 2b + c = 0 \\ 10b - c = 0 \end{array}\right\}$$

Let $c = k$, then $b = \frac{k}{10}$, $a = \frac{-k}{5}$

$\hat{v}_1 = \frac{\vec{v}_1}{|v_1|}$

$\therefore \quad v_3 = \begin{bmatrix} -2 \\ 1 \\ 10 \end{bmatrix}$ if $k = 10$

$$P = \begin{bmatrix} \frac{4}{\sqrt{21}} & \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{105}} \\ \frac{-2}{\sqrt{21}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{105}} \\ \frac{1}{\sqrt{21}} & 0 & \frac{10}{\sqrt{105}} \end{bmatrix}$$

$\quad \hat{v}_1 \quad \hat{v}_2 \quad \hat{v}_3$

$D = P'AP = \text{diag}\,[16, -5, -5]$, $P' = V$

Power of Matrix :

Let A be a square matrix of order n & P is the ~~model~~ modal matrix satisfying $D = P^{-1}AP$ where $D = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$

We have

$$A = PDP^{-1}$$

Now,

$$A^2 = A \cdot A$$
$$= (PDP^{-1})(PDP^{-1})$$
$$= PD(P^{-1}P)DP^{-1}$$
$$A^2 = PD^2P^{-1}$$

Generalising $\boxed{A^k = PD^kP^{-1}.}$

Remark:

$$\boxed{f(A) = Pf(D)P^{-1}}.$$

Q. Given $A = \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}$. Find

(i) A^6

(ii) $A^{1/2}$

(iii) $f(A)$ if $f(x) = x^3 - x^2$

Solution :

Here $P = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$

$D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

$P^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$

(i) $A^6 = PD^6P^{-1}$

$= \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}^6 \frac{1}{3}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$

$= \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4096 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$

$= \frac{1}{3}\begin{bmatrix} 1 & -4096 \\ 1 & 8192 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$

$= \frac{1}{3}\begin{bmatrix} 4098 & -4095 \\ -8190 & 8193 \end{bmatrix}$

$A^6 = \begin{bmatrix} 1366 & -1363 \\ -2730 & 2731 \end{bmatrix}$

(ii) $A^{1/2} = PD^{1/2}P^{-1}$

$= \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\frac{1}{3}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$

$= \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$ {Taking square root}

$= \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -2 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 4/3 & -1/3 \\ -2/3 & 5/3 \end{bmatrix}$$

$$f(A) = P f(D) P^{-1}$$

$$= P\left(\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}^3 - \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}^2 \right) P^{-1}$$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 48 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -16 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & -16 \\ -32 & 32 \end{bmatrix}$$

Cayley Hamilton Theoram:

The statement of the theorem is "Every square matrix its Characteristic polynomial equation." i.e.

If A is a square matrix and its characteristic equation is $p(\lambda) = 0$

Then the theorem says that $f(A) = 0$

Q. Verify Cayley Hamilton theorem for matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

Also find A^{-1} and A^{-2}.

Qu. If $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

Find the value of

$$A^8 - 5A^7 + 7A^6 - 3A^5 + 8A^4 - 5A^3 + 8A^2 - 2A + I$$

Q. If $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

Find $A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2$ as a linear polynomial in A.

(1) Ch eq^n: $\det(A - \lambda I) = 0$

$\lambda^2 - 4\lambda + 3 = 0$

$\lambda^2 - 3\lambda - \lambda + 3 = 0$

$\lambda(\lambda - 1) - 3(\lambda - 1) = 0$

$(\lambda - 3)(\lambda - 1) = 0$

$\lambda = 3, 1$

By ~~cal~~ cayley hamilton Theorem

$A^2 - 4A + 3I = 0$

~~Sol~~ Since $A^2 = A \cdot A$

$$= \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 8 \\ 0 & 9 \end{bmatrix}$$

L.H.S.

$A^2 - 4A + 3I$

$$\begin{bmatrix} 1 & 8 \\ 0 & 9 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 8 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 \\ 0 & 12 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$= 0$ (Null Matrix)

Since,

$A^{-1} = ?$

$$A^2 - 4A + 3I = 0$$

$$3I = 4A - A^2$$

$$I = \frac{1}{3}(4A - A^2)$$

Pre-multiply by A^{-1}

$$A^{-1}I = \frac{1}{3}\left(4A^{-1}A - A^{-1}A^2\right)$$

$$A^{-1} = \frac{1}{3}\left[4I - A\right] \quad \text{——— ①}$$

$$A^{-1} = \frac{1}{3}\left(\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\right) \quad \text{— ①}$$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & -2/3 \\ 0 & 1/3 \end{bmatrix}$$

$A^{-2} = ?$

~~Pe~~ Pre multiply eqn ① by A^{-1}

$$A^{-2} = \frac{1}{3}\left(4A^{-1}I - A^{-1}A\right)$$

$$\frac{1}{3}\left(4\begin{bmatrix} 1 & -2/3 \\ 0 & 1/3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \frac{1}{3}\begin{bmatrix} 3 & -8/3 \\ 0 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -8/9 \\ 0 & 1/9 \end{bmatrix}$$

(2) $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

$$\lambda^3 - 5\lambda^2 + (2+3+2)\lambda - 3 = 0$$

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

By Cayley hamilton theorem

$$A^3 - 5A^2 + 7A - 3I = 0 \quad \text{—— (1)}$$

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 3A^3 + 8A^2 - 2A + I =$$

$$A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 7A - 3I) \cancel{+ A^2 + A + I}$$

$$= 0 + 0 + A^2 + A + I$$

$= A^2 + A + I$ } it is always polynomial in A of degree (n-1) where A is order of n

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

$3+2=5$

$\det(A-\lambda I)=0$

$\lambda^2 - tr(A)\lambda + \det(A) = 0$

$\lambda^2 - 4\lambda + 5 = 0$

$\cancel{\lambda^2 + 5\lambda + \lambda} + 5 = 0$

$\cancel{\lambda(\lambda - 5)}$

By Cayley Hamilton theorem

$A^2 - 4A + 5I = 0$ —— (1)

$A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2$

$A^4(A^2 - 4A + 5I) + 3A^4 - A(A^2 - 4A + 5I)$
$-11A^3 + 10A^2 + 5A$

$0 + 3A^4 - 0 - 11A^3 + 10A^2 + 5A$

$3A^4 - 11A^3 + 10A^2 + 5A$

$3A^2(A^2 - 4A + 5I) + A^3 - 5A^2 + 5A$

$A^3 - 5A^2 + 5A$

$A(A^2 - 4A + 5I) - A^2$

$-A^2$

$-\left(\begin{bmatrix} 1 & 2 \\ & \end{bmatrix}\begin{bmatrix} 1 & 2 \\ & 3 \end{bmatrix}\right)$ 0

$A^4(A^2-4A+5I)+3A^2(A^2-4A+5I)+(-1)(A^2-4A+5I)$
$-4A+5I$

$= 0+0+0-4A+5I$

$= 5I-4A$ (Linear Polynomial in A)