Ques. Find the root of $x^2-x-1=0$ correct to 4-decimal using fixed point iteration.

$$x^2-x-1=0$$

$$x=\sqrt{x+1} \quad , \quad x = x^2-1$$

$$\phi_1(x) = \sqrt{x+1} \quad , \quad \phi_2(x) = x^2-1$$

$$x^2-x=1$$
$$x(x-1)=1$$
$$x = 1+1/x$$
$$\phi_3(x) = 1+1/x$$

$f(0) = -1 < 0$

$f(1) = -1$

$f(2) = 4-2-1 = 1 > 0$

$[1, 2]$

$f(1.5) = -0.25 < 0$

$[1.5, 2]$

$f(1.75) = 0.3125 > 0$

$(1.5, 1.75)$

$f(1.6) < 0$

$(1.6, 1.75)$

$f(1.61) < 0$

$f(1.62) > 0$

Interval : $[1.61, 1.62]$

Case i): let $\phi(x) = \sqrt{x+1}$

$$\phi'(x) = \frac{1}{2\sqrt{x+1}}$$

$$|\phi'(x)| = \frac{1}{2\sqrt{x+1}}$$

$$|\phi'(x)|_{x=1.61} = 0.3094 < 1$$

$$\therefore x_{n+1} = \phi(x_n)$$

i.e., $x_{n+1} = \sqrt{x_n+1}$

$f(x) = x^2 - x - 1$

$f(1.61) = -0.0179$

$f(1.62) = 4.4 \times 10^{-3} = 0.004$

$\therefore$ Consider, $x_0 = \cancel{0.0179}$ 1.62

$x_1 = \sqrt{x_0 + 1} = 1.61864$

$x_2 = 1.61822$

$x_3 = 1.61809$

$x_4 = 1.61805$

$\therefore$ root $= 1.6180$

case ii) let $\phi(x) = x^2 - 1$

$\phi'(x) = 2x$

$|\phi'(x)|_{x=1.61} = 3.22 > 1$

$|\phi'(x)|_{x=1.62} = 3.24 > 1$

$\therefore$ failure of $|\phi'(x)| \leq 1$

Case iii) let $\phi(x) = 1 + 1/x$

$\phi'(x) = -1/x^2$

$|\phi'(x)| = 1/x^2$

$|\phi'(x)|_{x=1.61} < 1$

$|\phi'(x)|_{x=1.62} < 1$

$\therefore x_{n+1} = \phi(x_n)$

i.e., $x_{n+1} = 1 + 1/x_n$

& $x_1 = 1 + \frac{1}{1.62} = 1.61728$

$x_2 =$

Ques. Solve $x^2 - x - 1 = 0$ by using Newton Raphson method
$[1.61, 1.62]$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f(x) = x^2 - x - 1$$

$$f'(x) = 2x$$

$$x_{n+1} = x_n - \frac{x_n^2 - x_n - 1}{2x_n - 1}$$

$$x_0 = 1.62$$

x_1

$= 1.618035$

$x_2 = 1.61$

Ques. Find the root of 2 using fixed point iteration.

$$x^2 - 2 = 0$$

$$x^2 = 2 \Rightarrow x = 2/x$$

$$x + x = 2/x + x$$

$$2x = \frac{2 + x^2}{x}$$

$$x = \frac{1}{2}\left[\frac{2 + x^2}{x}\right]$$

$$\phi(x) = \frac{1}{2}\left[\frac{2 + x^2}{x}\right]$$

$$\phi'(x) = \frac{1}{2}\left[\frac{x(2x) - (2 + x^2)}{x^2}\right]$$

$$|\phi'(x)| = \frac{1}{2}\left[\frac{x^2 - 2}{x^2}\right]$$

clearly, $|\phi'(x)| < 1$ in $[1.41 \quad 1.42]$

$$x_{n+1} = \phi(x_n)$$

$$x_{n+1} = \frac{1}{2}\left[\frac{2 + x_n^2}{x_n}\right]$$

$$x_0 = 1.41$$

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System of Linear Algebraic Equations.

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots\cdots a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots\cdots a_{2n}x_n &= b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\} \quad (A)$$

$$AX = B$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots\cdots & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots\cdots & a_{mn} \end{bmatrix}_{m\times n}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} ; \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$$

Diagonally Dominance.

Consider the system

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

It is said to be diagonally dominance

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

iff

$$|a_1| \geq |b_1| + |c_1|$$

$$|b_2| \geq |a_2| + |c_2|$$

$$|c_3| \geq |a_3| + |b_3|$$

Note: Sometimes the given system may not be diagonally dominant, but can be made diagonally dominant by interchanging the equations as shown in following example :

$$12x + 16y + z = 24$$
$$3x + 18y + 24z = 16$$
$$24x + 8y + 3z = 14$$

~~making the~~ Rearranging the eqn as -

$$24x + 8y + 3z = 14$$
$$12x + 16y + z = 24$$
$$3x + 18y + 24z = 16$$

Now, the system is diagonally dominant.

Note : In numerical technique, the system should be ~~...~~ diagonally dominant otherwise the soln will not converge.

Gauss-Jacobi Method

Make the system diagonally dominant

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

$$x^{(0)} = y^{(0)} = z^{(0)} = 0$$

$$x^{(n+1)} = \frac{1}{a_1}[d_1 - b_1y^{(n)} - c_1z^{(n)}]$$

$$y^{(n+1)} = \frac{1}{b_2}[d_2 - a_2x^{(n)} - c_2z^{(n)}]$$

$$z^{(n+1)} = \frac{1}{c_3}[d_3 - a_3x^{(n)} - b_3y^{(n)}]$$

Gauss-Seidel Method

Gauss-Seidel Eq^n takes the recent available values of the variable in its scheme which is shown below.

$$x^{(n+1)} = \frac{1}{a_1}\left[d_1 - b_1 y^{(n)} - c_1 z^{(n)}\right]$$

$$y^{(n+1)} = \frac{1}{b_2}\left[d_2 - a_2 x^{(n+1)} - c_2 z^{(n)}\right]$$

$$z^{(n+1)} = \frac{1}{c_3}\left[d_3 - a_3 x^{(n+1)} - b_3 y^{(n+1)}\right]$$

Ques. Solve the following system of eq^{ns} :-

$$2x + 10y + z = 51$$
$$10x + y + 2z = 44$$
$$x + 2y + 10z = 61$$

the given system is not diagonally dominant but can be made by Rearranging the eq^n, i.e., interchanging the rows.

New system is -

$$10x + y + 2z = 44$$
$$2x + 10y + z = 51$$
$$x + 2y + 10z = 61$$

we use Gauss-Seidel method whose iterations scheme is given below:

$$x^{(n+1)} = \frac{1}{a_1}\left[d_1 - b_1 y^{(n)} - c_1 z^{(n)}\right]$$

$$y^{(n+1)} = \frac{1}{b_2}\left[d_2 - a_2 x^{(n+1)} - c_2 z^{(n)}\right]$$

$$z^{(n+1)} = \frac{1}{c_3}\left[d_3 - a_3 x^{(n+1)} - b_3 y^{(n+1)}\right]$$

$$x^{(n+1)} = \frac{1}{10}\left[44 - y^{(n)} - 2z^{(n)}\right]$$

$$y^{(n+1)} = \frac{1}{10}\left[51 - 2x^{(n+1)} - z^{(n)}\right]$$

$$z^{(n+1)} = \frac{1}{10}\left[61 - x^{(n+1)} - 2y^{(n+1)}\right]$$

Consider, $x^{(0)} = y^{(0)} = z^{(0)} = 0$

the sol^n table is shown below

$X^{(n)} = [x^{(n)}\ y^{(n)}\ z^{(n)}]$

Iteration (n)	x	y	z
$X^{(0)}$ initial value	0	0	0
$X^{(1)}$	4.4	4.22	4.816
$X^{(2)}$	3.0148	4.01544	4.995432
$X^{(3)}$	2.99936	4.00058	4.99994
$X^{(4)}$	2.99999	4.00001	5.0000017

$\therefore\ x \simeq 3$

$y \simeq 4$

$z \simeq 5$

Que. $f(x) = e^x - 3x^2$ find these two roots by fixed point iteration by discussing whether these two roots could be obtained by same fixed point scheme R_0.

Ques. The sum of two nos. is 20. If each no. is added to its square root, the product of these two sums is 155.55. Find these nos. upto 3 decimal accuracy.

$$x + y = 20 \quad \& \quad y = 20 - x$$

$$(x + \sqrt{x}) \times (y + \sqrt{y}) = 155.55$$

$$(x + \sqrt{x}) \times \left[(20 - x) + \sqrt{20 - x}\right] - 155.55 = 0$$

$f(0) = -155.55 < 0$

$f(2) < 0$

$f(7) > 0$

$f(6) < 0$

$f(6.5) < 0 = -0.1315$

$f(6.6) > 0 = 0.8794$

$[6.5 \quad 6.6]$

$\therefore x_0 = 6.5$

$x_1 = 6.6$

$$x_2 = x_0 - \left[\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right] f(x_0)$$

$$= 6.5 - \left[\frac{6.6 - 6.5}{0.8794 + 0.1315}\right](-0.1315)$$

$$= 6.5130$$

$f(6.5130) = 0.0015 > 0$

$$x_3 = 6.5 - \left[\frac{6.5130 - 6.5}{0.0015 + 0.1315}\right](-0.1315)$$

$$= 6.5128$$

$f(6.5128) = -0.0049 < 0$

$$x_4 = 6.5128 - \left[\frac{6.5130 - 6.5128}{0.0015 + 0.0049}\right](-0.0049)$$

$$= 6.5129$$

$\therefore$ Root upto 3 decimal accuracy $= 6.512$

Ques. The following system of eq^n is designed to determine the concentrations in a series of couple reactors as a function of the amount of mass ~~factor~~ input to each reactor

$$-3C_1 + 18C_2 - 6C_3 = 1200$$
$$15C_1 - 3C_2 - C_3 = 3800$$
$$-4C_1 - C_2 + 12C_3 = 2350$$

Obtain the concentration values correct to 2-decimal using Gauss-Seidel iteration technique with initial conc^n as

$$[C_1^{(0)}, C_2^{(0)}, C_3^{(0)}] = [300, 220, 310]$$

The given system is not diagonally dominant

∴ Interchanging the rows

So, new system is

$$15C_1 - 3C_2 - C_3 = 3800$$

$$-3C_1 + 18C_2 - 6C_3 = 1200$$

$$-4C_1 - C_2 + 12C_3 = 2350$$

$$C_1^{(n+1)} = \frac{1}{15}[3800 + 3C_2^{(n)} + 1C_3^{(n)}]$$

$$C_2^{(n+1)} = \frac{1}{18}[1200 + 3C_1^{(n+1)} + 6C_3^{(n)}]$$

$$C_3^{(n+1)} = \frac{1}{12}[2350 + 4C_1^{(n+1)} + 1C_2^{(n+1)}]$$

$$C_1^{(1)} = \frac{1}{15}[3800 + 3\times 220 + 310]$$

$$= 318$$

$$C_2^{(1)} = \frac{1}{18}[1200 + 3\times 318 + 6\times 310]$$

$$= 223$$

$$C_3^{(1)} = \frac{1}{12}[2350 + 4\times 318 + 223]$$

$$= 320.416$$

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	C_1	C_2	C_3
x^0	300	220	310
x^1	318	223	320.416
x^2	319.294	226.687	321.155
x^3	320.081	227.065	321.44[illegible]9
x^4	320.176	227.179	321.490
x^5	320.201	227.196	321.500
x^6	320.205	227.200	321.501
x^7	320.206	227.201	321.502

Ques. Solve the system

$$\begin{bmatrix} 10 & -2 & -2 \\ -1 & 10 & -2 \\ -1 & -1 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 8 \end{bmatrix}$$

$$10x - 2y - 2z = 6$$

$$-x + 10y - 2z = 7$$

$$-x - y + 10z = 8$$

$$x^{(n+1)} = \frac{1}{10}\left[6 + 2y^{n} + 2z^{n}\right]$$

$$y^{(n+1)} = \frac{1}{10}\left[7 + x^{n+1} + 2z^{n}\right]$$

$$z^{(n+1)} = \frac{1}{10}\left[8 + x^{n+1} + y^{n+1}\right]$$

$$x^{n} = y^{n} = z^{n} = 0$$

	x	y	z
X^{0}			

Ques. Find the solution of

$$e^{0.5x} - \sqrt{x} = 3.$$

$$f(x) = e^{0.5x} - \sqrt{x} - 3$$

$$f(0) = -3$$

$$f(1) = -2.3512 < 0$$

$$f(2) = -1.6959 < 0$$

$$f(3) = -0.2503 < 0$$

$$f(4) = 2.3890 > 0$$

$$f(3.5) = 0.8837 > 0$$

$$f(3.25) = 0.2756 > 0$$

$$f(3.22) = 0.2083 > 0$$

$$f(3.15) = 0.0559 > 0$$

$$f(3.13) = 0.0134 > 0$$

$$f(3.11) = -0.0284 < 0$$

$$f(3.12) = -0.0075$$

$$[3.12, 3.13]$$

$$\therefore x_0 = 3.12$$

$$f'(x) = 0.5e^{0.5x} - \frac{1}{2\sqrt{x}}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{e^{0.5x_n} - \sqrt{x_n} - 3}{0.5e^{0.5x_n} - 1/2\sqrt{x_n}}$$

$$= \frac{0.5x_n e^{0.5x_n} - \frac{1}{2}\sqrt{x_n} - e^{0.5x_n} + \sqrt{x_n} + 3}{0.5e^{0.5x_n} - 1/2\sqrt{x_n}}$$

$$= \frac{e^{0.5x_n}(0.5x_n - 1) + \frac{1}{2}\sqrt{x_n} + 3}{0.5e^{0.5x_n} - \frac{1}{2\sqrt{x_n}}}$$

$$x_1 = 3.12 + \frac{0.0075}{2.0963}$$

$$= 3.1235$$

$$x_2 = 3.1235 + \frac{1.86 \times 10^{-4}}{2.1006}$$

$$= 3.1235$$

Calculus of Finite Differences

Consider n+1 equally spaced point

$$x_0,\ x_0+h\ (x_1),\ x_0+2h\ (x_2), \ldots\ldots\ x_0+nh\ (x_n)$$

and the corresponding functional values

$$y_0, y_1, y_2, \ldots\ldots y_n$$

Here, the x values are values of the independent variable of the function $y(x)$, known as arguments. The corresponding y values are called as entries.

To understand the change in the dependent variable y wrt change in independent variable x is understood by the means of calculus of finite differences whose important operators are given below :-

i) Forward Difference Operator $[\Delta]$
ii) Backward Difference Operator $[\nabla]$ Nabla
iii) Shifting operator (E)

Forward Difference Operator

$$\Delta y(x) = y(x+h) - y(x)$$

Similarly

$$\Delta^2 y(x) = \Delta[\Delta y(x)]$$
$$= \Delta[y(x+h) - y(x)]$$
$$= \Delta y(x+h) - \Delta y(x)$$

$$\Delta^2 y = y(x+2h) - y(x+h) - y(x+h) + y(x)$$
$$= y(x+2h) - 2y(x+h) + y(x)$$

$$\Delta^2 y_0 = y(x_2) - 2y(x_1) + y(x_0)$$
$$= y_2 - 2y_1 + y_0$$

Note: There won't be any Δ value for last argument.
It is simply, next value minus current value.
Δ is used for equally spaced argument.

Ques. Construct the forward difference table for the following data :-

x :	2	4	6	8
y :	10	15	20	36

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
2	10			
		$\Delta y_2 = 5$		
			$\Delta^2 y_2 = 0$	
4	15			$\Delta^3 y_2 = 11$
		$\Delta y_4 = 5$		
			$\Delta^2 y_4 = 11$	
6	20			
		$\Delta y_6 = 16$		
8	36			

Backward Difference Operator

$$\nabla y(x) = y(x) - y(x-h)$$

$y(x_0) \longrightarrow$ not possible

$\nabla y(x_1) = y(x_1) - y(x_0)$

$= y_1 - y_0$

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
x_0	y_0			
		$y_1 - y_0 = \nabla y_1$		
x_1	y_1		$\Delta y_2 - \nabla y_1 = \nabla^2 y_2$	$\Delta^2 y_3 - \nabla^2 y_2 = \nabla^3 y_3$
		$y_2 - y_1 = \Delta y_2$		
x_2	y_2		$\Delta y_3 - \nabla y_2 = \nabla^2 y_3$	
		$y_3 - y_2 = \Delta y_3$		
x_3	y_3			

The backward difference table for the above data is :-

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x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
2	10			
		$5 = \nabla y_4$		
4	15		$0 = \nabla^2 y_6$	
		$5 = \nabla y_6$		$11 = \nabla^3 y_8$
6	20		$11 = \nabla^2 y_8$	
		$16 = \nabla y_8$		
8	36			

Shifting Operator

$$E y(x) = y(x+h)$$

x	y
x_0	y_0
x_1	y_1
x_2	y_2

$$E y_0 = y_1$$

$$E^2 y_0 = E[E y_0] = E[y_1] = y_2$$

$$E^n y(x_0) = y(x_0 + nh)$$

$$E^{-1} y(x) = y(x-h)$$

$$E^n y(x) = y(x+nh)$$

$$E^{-n} y(x) = y(x-nh)$$

Note: when the data is generated from the n^{th} degree polynomial, the n^{th} differences are constants and all higher differences vanishes.

We can extend the table by using the above property as shown in the following example.

x	y	Δy	$\Delta^2 y$
0	2		
		6	8
2	8	14	
			8
4	22	22	
			$8 = \Delta y_6$
6	44	$y_8 - 22 = \Delta y_6$	
8	y_8		

$\therefore \; y_8 = 22 + 8$

$= 30$

Ques. Estimate the value of y_{14}, y_{-2}, y_{-6}

Interpolation

Let (x_i, y_i) be the ~~[illegible]~~ tabulated value of a function $y(x)$ of in the interval $[x_0\ x_n]$. The process of estimating the values of y at non-tabulated points within the interval $[x_0\ x_n]$ is called interpolation.

Note: In interpolation, we will fit the given data with a polynomial, using which y values at non-tabulated points can be obtained.

There are two types of data:
equally spaced, and
unequally spaced.

If the data is equally spaced, we have the following two formula
i) Newton-Gregory forward Interpolation
ii) " " Backward "

In case of unequally spaced data, we have
Newton's Divided difference interpolation.

Note: If a non-tabulated point is at the beginning of the interval, i.e., close to x_0, we'll use Newton-Gregory forward Interpolation formula.
If it is close to x_n, we'll use Newton-Gregory Backward Interpolation formula.

Newton-Gregory Forward Interpolation Formula

Let the function $y = f(x)$.
Let $f(a), f(a+h), f(a+2h), \ldots\ldots f(a+nh)$ be the values of y at $x = a, a+h, a+2h, \ldots\ldots a+nh$.
We know that, the degree of the polynomial joining $n+1$ points is n.
Let the polynomial be ~~$f(x)$~~

$$f(x) = A_0 + A_1(x-a) + A_2(x-a)(x-a-h) + A_3(x-a)(x-a-h)(x-a-2h) + \ldots\ldots + A_n(x-a)(x-a-h)\ldots\ldots(x-a-\overline{n-1}\,h) \quad \text{——— (i)}$$

where, $A_0, A_1, A_2, \ldots\ldots A_n$ are unknowns to be obtained.

we'll substitute $x = a, a+h, a+2h, \ldots\ldots a+nh$ successively in eq. (i) to obtain unknowns.

Put $x = a$

$\Rightarrow \quad A_0 = f(a)$

Put $x = a+h$

$$\Rightarrow f(a+h) = A_0 + A_1(a+h-a) + 0$$
$$= A_0 + A_1(h)$$
$$= f(a) + A_1(h)$$

$$\Rightarrow A_1 = \frac{f(a+h) - f(a)}{h}$$

$$A_1 = \frac{\Delta f(a)}{h} \quad \Rightarrow \quad A_1 = \frac{\Delta f(a)}{1!\,h}$$

Put $x = a+2h$

$$f(a+2h) = A_0 + A_1(a+2h-a) + A_2(a+2h-a)(a+2h-a-h) + 0$$
$$= A_0 + A_1(2h) + A_2(2h)h$$
$$= f(a) + \left[\frac{\Delta f(a)}{h}\right]\cdot 2h + 2h^2 A_2$$
$$= f(a) + 2[f(a+h) - f(a)] + 2h^2 A_2$$

$$\Rightarrow f(a+2h) = 2f(a+h) - f(a) + 2h^2 A_2$$

$$\Rightarrow A_2 = \frac{f(a+2h) - 2f(a+h) + f(a)}{2h^2}$$

$$A_2 = \frac{\Delta^2 f(a)}{2h^2}$$
$$\Rightarrow A_2 = \frac{\Delta^2 f(a)}{2!\,h^2}$$

Proceeding as above, we get

$$A_3 = \frac{\Delta^3 f(a)}{3!\,h^3}$$

$\vdots$

$$\therefore \boxed{A_n = \frac{\Delta^n f(a)}{n!\,h^n}}$$

~~Substituting $A_0, A_1, A_2, \ldots A_n$ in eq. (i), we get~~

We know that, every no. x can be represented as

$$x = x_0 + uh$$
$$= a + uh$$

$$u = \frac{x - x_0}{h}$$

i.e., $u = \dfrac{x-a}{h}$

Substituting $A_0, A_1, A_2, \ldots . A_n$ in eq.(i), we get

$$f(x) = f(a+uh) = f(a) + (x-a)\frac{\Delta f(a)}{h} + (x-a)(x-a-h).\frac{\Delta^2 f(a)}{2!\,h^2}$$
$$+ (x-a)(x-a-h)(x-a-2h).\frac{\Delta^3 f(a)}{3!\,h^3} + \ldots\ldots +$$
$$(x-a)(x-a-h)\ldots\ldots(x-a-\overline{n-1}\,h).\frac{\Delta^n f(n)}{n!\,h^n}$$
$$= f(a) + (uh)\frac{\Delta f(a)}{h} + (uh)(uh-h)\frac{\Delta^2 f(a)}{2!\,h^2} + \ldots$$
$$\ldots + (uh)(uh-h)(uh-2h)\ldots(uh-\overline{n-1}\,h)\frac{\Delta^n f(a)}{n!\,h^n}$$

$$\left(\because\ u = \frac{x-a}{h}\right)$$

$$f(a+uh) = f(a) + u.\Delta f(a) + u(u-1)\frac{\Delta^2 f(a)}{2!} + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a)$$
$$+ \ldots\ldots + u(u-1)(u-2)\ldots(u-\overline{n-1})\frac{\Delta^n f(a)}{n!}$$

$$\boxed{f(x) = f(a+uh) = f(a) + u.\Delta f(a) + \frac{u(u-1)}{2!}.\Delta^2 f(a) + \ldots + \frac{u(u-1)(u-2)\ldots(u-\overline{n-1})}{n!}.\Delta^n f(a)}$$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
a	$f(a)$			
		$\Delta f(a)$		
			$\Delta^2 f(a)$	
				$\Delta^3 f(a)$
$a+h$	$f(a+h)$			
		$\Delta f(a+h)$		
			$\Delta^2 f(a+h)$	
$a+2h$	$f(a+2h)$			
		$\Delta f(a+2h)$		
$a+3h$	$f(a+3h)$			

Ques. The population of a town was given below, estimate the population in a year 1895

year, x :	1891	1901	1911	1921	1931
Population, y : (in thousands)	46	66	81	93	101

[Ans : 54.85 thousands]

$y(1895) = ?$

$h = 10$

$a = 1891$

$x = 1895$

$$u = \frac{x-a}{h}$$

$$= \frac{1895-1891}{10}$$

$$= 0.4$$

(a)

$\frac{\ldots n-1)}{\ldots} . \Delta^n f(a)$

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Ques. Estimate the value of $y(0.5)$ from the following table

x	y
0	5
1	3
2	7
3	23

The given data set is equally spaced ~~al~~ with step size $h = 1$
and $x_0 = 0$
we need to find.
$y(x)$ at $x = 0.5$
$\because$ 0.5 is close to x_0, we use Newton-Gregory forward interpolation formula, given by

$$f(x) = f(x_0) + u\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) + \dots + \frac{u(u-1)\dots(u-\overline{n-1})}{n!}\Delta^n f(x_0)$$

where, $u = \frac{x - x_0}{h}$

$= \frac{0.5 - 0}{1}$

$= 0.5$

We construct the forward difference table as given below:

	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
$x_0 \rightarrow$	0	$\boxed{5}$ $y(x_0)$			
			$\boxed{-2}$ $\Delta y(x_0)$	$\boxed{16}$ $\Delta^2 y(x_0)$	$\boxed{6}$ $\Delta^3 y(x_0)$
	1	3			
			4		
	2	7		12	
			16		
	3	23			

$$\therefore f(0.5) = 5 + 0.5(-2) + \frac{0.5(0.5-1)}{2}\cdot 6 + \frac{0.5(0.5-1)(0.5-2)}{6} \times 6$$

$= 5 - 1 - 0.25 \times 3 + 0.25 \times 1.5$

$= 4 - 0.75 + 0.375$

$= 4.375 - 0.75 = 3.625$

Ques. Identify the polynomial governing the data in the above data.

Since, the polynomial is required

$$u = \frac{x - x_0}{h}$$

$$= \frac{x - 0}{1}$$

$$= x$$

We substitute $u = x$ and the corresponding differences from the difference table in the Newton-Gregory forward Interpolation formula.

$$f(x) = 5 + x(-2) + \frac{x(x-1)}{2} \times 6 + \frac{x(x-1)(x-2)}{6} \times 6$$

$$y(x) = x^3 - 3x + 5$$

Newton-Gregory Backward Interpolation Formula.

imp. [Proof from the book]

This technique is used when the non-tabulated point is close to the last point of the data.

Here, $u = \frac{x - x_n}{h}$ (i.e., the ~~diff~~ reference point is x_n)

~~To the~~ u is negative

The formula is

$$\boxed{f(x) = f(x_n + uh) = f(\underbrace{x_0 + nh}_{x_n} + uh) = f(x_n) + u\nabla f(x_n) + \frac{u(u+1)}{2!}\nabla^2 f(x_n) + \ldots + \frac{u(u+1)(u+2)\ldots(u+\overline{n-1})}{n!}\nabla^n f(x_n)}$$

Ques. Find the polynomial governing the previous problem using Newton-Gregory Backward Interpolation

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
0	5			
		-2		
1	3		6	
		4		$\boxed{6}$
2	7		$\boxed{12}$	
		$\boxed{16}$		
3	$\boxed{23}$			

$$U = \frac{x - x_n}{h}$$

$$= \frac{x - 3}{1}$$

$$U = x - 3$$

$$f(x) = 23 + (x-3)16 + \frac{(x-3)(x-3+1)\,12}{2} + \frac{(x-3)(x-3+1)(x-3+2) \times 6}{6}$$

$$= 23 + 16x - 48 + (x^2 - 5x + 6)6 + (x^2 - 5x + 6)(x-1)$$

$$= 23 + 16x - 48 + 6x^2 - 30x + 36 + x^3 - 5x^2 + 6x - x^2 + 5x - 6$$

$$= x^3 - 9x$$

Ques. The following table shows marks obtained by 100 students in a particular subject. Use Newton forward formula to find the no. of students who secures more than 55 marks.

Scores :	30 - 40	40 - 50	50 - 60	60 - 70	70 - 80
No. of stu :	25	35	22	11	7

The given table can be rearranged as follows :

(x) Scores less than Scores	(Y) No. of students	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
less than 40	$\boxed{25}$				
		$\boxed{35}$			
" " 50	60		$\boxed{-13}$		
				$\boxed{2}$	
		22			$\boxed{5}$
60	82		-21		
		11		7	
70	93		-4		
		7			
80	100				

Here, $x_0 = 40$

and $x = 55$

$$\therefore U = \frac{x - x_0}{h}$$

$$= \frac{55 - 40}{10}$$

$$= 1.5$$

$$f(55) = 25 + 1.5 \times 35 + \frac{1.5(1.5-1)}{2} \times (-13) + \frac{1.5(1.5-1)(1.5-2)}{6} \times 2$$

$$+ \frac{1.5(1.5-1)(1.5-2)(1.5-3)}{24} \times 5$$

$$= 25 + 52.5 - 4.875 - 0.125 + 0.117$$

$$= 72.617$$

$$= 73$$

$y(55) = 73$

No. of students with marks less than 55 = 73

$\therefore$ No. of students with " greater than 55 = Total student − 73 = 100 − 73

$\therefore$ " " " = 27

Note: $\Delta^n y_0 = y_n - {}^nC_1 y_{n-1} + {}^nC_2 y_{n-2} - {}^nC_3 y_{n-3} \cdots\cdots (-1)^n y_0$

Ques. U_x is a function of x for which the 5^{th} differences are constants and $U_1 + U_7 = -786$

$U_2 + U_6 = 686$

$U_3 + U_5 = 1088$

Find U_4.

$\Delta^6 U_1 = 0$

$\Rightarrow U_7 - {}^6C_1 U_6 + {}^6C_2 U_5 - {}^6C_3 U_4 + {}^6C_4 U_3 - {}^6C_5 U_2 + {}^6C_6 U_1 = 0$

$U_7 - 6U_6 + 15U_5 - 20U_4 + 15U_3 - 6U_2 + U_1 = 0$

$(U_7 + U_1) - 6(U_2 + U_6) + 15(U_3 + U_5) - 20U_4 = 0$

$-786 - 6\times 686 + 15\times 1088 = 20U_4$

$U_4 = 570.9$

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Relation between Operators.

i) $\boxed{\Delta = E - 1}$

LHS : $\Delta f(x) = f(x+h) - f(x)$

$= Ef(x) - f(x)$

$\Delta f(x) = (E-1)f(x)$

$\Delta = E - 1$

ii) $\boxed{\nabla = 1 - E^{-1}}$

LHS : $\nabla f(x) = f(x) - f(x-h)$

$= f(x) - E^{-1} f(x)$

$= f(x)(1-E^{-1})$

$\nabla = 1-E^{-1}$

$E^{-1} = 1 - \nabla$

Ques. Prove that $\Delta = E\nabla = \nabla E$

LHS : $\Delta f(x) = f(x+h) - f(x)$

$(RHS)_1$: $E\nabla = E(\nabla f(x))$

$= E(f(x) - f(x-h))$

$= f(x+h) - f(x)$

Hence, Proved.

$(RHS)_2$: $\nabla E = \nabla(Ef(x))$

$= \nabla(f(x+h))$

$= f(x+h) - f(x)$

Hence, Proved.

Ques. $E = e^{hD}$ where $D = \frac{d}{dx}$

Consider, $E f(x) = f(x+h)$

$$= f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots + \frac{h^n}{n!} f^n(x)$$

$$= f(x) + hDf(x) + \frac{h^2}{2!} D^2 f(x) + \frac{h^3}{3!} D^3 f(x) + \dots + \frac{h^n D^n}{n!} f(x)$$

$$= \left[1 + hD + \frac{h^2}{2!} D^2 + \frac{(hD)^3}{3!} + \dots + \frac{(hD)^n}{n!}\right] f(x)$$

$$E f(x) = e^{hD} f(x)$$

$$\therefore \quad E = e^{hD}$$

Central Difference Operator

x	y	δy	$\delta^2 y$
x_0	y_0		
		$\delta y_{1/2} = y_1 - y_0$	
x_1	y_1		$\delta^2 y_1$
		$\delta y_{3/2} = y_2 - y_1$	
x_2	y_2		$\delta^2 y_2$
		$\delta y_{5/2} = y_3 - y_2$	
x_3	y_3		
		$\delta y_{7/2} = y_4 - y_3$	
x_4	y_4		

The central difference formula is given by :

$$\boxed{y_n - y_{n-1} = \delta y_{n-1/2}}$$

Averaging Operator

$$\mu y(x) = \frac{1}{2}\left[y_{x+h/2} + y_{x-h/2} \right]$$

or $\mu f(x) = \frac{1}{2}\left[f(x+h/2) + f(x-h/2)\right]$

Ques. Prove that $\delta = E^{1/2} - E^{-1/2}$ — Pg: 205-229

$$\delta y_x = y_{x+h/2} - y_{x-h/2}$$

$$= E^{1/2} y_x - E^{-1/2} y_x$$

$$\delta y_x = (E^{1/2} - E^{-1/2}) y_x$$

Ques. The table given below, use the value of $\tan x$ in the interval 0.1 to 0.3

x	$y = \tan x$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.1	0.1003				
		0.0508			
0.15	0.1511		0.0008		
		0.0516		0.0002	
0.2	0.2027		0.0010		0.0002
		0.0526		0.0004	
0.25	0.2553		0.0014		
		0.0540			
0.3	0.3093				

Ques. Estimate the value of $\tan(0.12)$ by Newton-Gregory forward Interpolation.

$h = 0.05$, $x_0 = 0.1$, $x = 0.12$, $u = \frac{x - x_0}{h} = \frac{0.02}{0.05} = 0.4$

$$f(0.12) = 0.1003 + 0.4 \times 0.0508 + \frac{0.4 \times (-0.6)}{2} \times 0.0008$$

$$+ \frac{0.4 \times (-0.6) \times (-1.6)}{6} \times 0.0002 + \frac{0.4 \times (-0.6) \times (-1.6) \times (-2.6)}{24} \times 0.002$$

$$= 0.1205840$$

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Divided Differences

We use divided differences on unequally spaced data.

[Note : This is the master formula which can be applied on every data set]

Note :

$$\Delta y_0 = [x_0 \; x_1] = \frac{y_1 - y_0}{x_1 - x_0}$$

$$\Delta y_1 = [x_1 \; x_2] = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\Delta^2 y_0 = [x_0 \; x_1 \; x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}$$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_0	y_0	$\frac{y_1 - y_0}{x_1 - x_0} = \Delta y_0$	$\frac{\Delta y_1 - \Delta y_0}{x_2 - x_0} = \Delta^2 y_0$	$\frac{\Delta^2 y_1 - \Delta^2 y_0}{x_3 - x_0} = \Delta^3 y$
x_1	y_1	$\frac{y_2 - y_1}{x_2 - x_1} = \Delta y_1$	$\frac{\Delta y_2 - \Delta y_1}{x_3 - x_1} = \Delta^2 y_1$	
x_2	y_2	$\frac{y_3 - y_2}{x_2 - x_1} = \Delta y_2$		
x_3	y_3			

Que. Construction of divided difference table for the given data :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	6	$\frac{9-6}{3-0} = 1$	$\frac{6-1}{4-0} = 5/4$	$\frac{-1/3 - 5/4}{6-0} = -0.264$	0.145
3	9	$\frac{15-9}{4-3} = 6$	$\frac{5-6}{6-3} = -1/3$	$\frac{8/3 + 1/3}{7-3} = 3/4$	-0.221
4	15	$\frac{25-15}{6-4} = 5$	$\frac{13-5}{7-4} = 8/3$	$\frac{-9/4 - 8/3}{10-4} = -0.819$	
6	25	$\frac{38-25}{7-6} = 13$	$\frac{4-13}{10-6} = -9/4$		
7	38	$\frac{50-38}{10-7} = 4$			
10	50				

Properties of Divided differences

i) The divided difference are symmetric about its argument, i.e.,

$$[x_0 \ x_1] = [x_1 \ x_0]$$

Proof : consider $[x_0 \ x_1] = \dfrac{y_1 - y_0}{x_1 - x_0}$

and $[x_1, x_0] = \dfrac{y_0 - y_1}{x_0 - x_1} = \dfrac{y_1 - y_0}{x_1 - x_0}$

ii) The n^{th} divided differences of a n^{th} degree polynomial are constant.

Newton's Divided Difference Interpolation Formula

Let $y_0, y_1, \ldots\ldots y_n$ be the values of $y = f(x)$

at the Points $x_0, x_1, \ldots\ldots x_n$

Consider the first divided difference $[x \ x_0] = \dfrac{y - y_0}{x - x_0}$

$$\Rightarrow y - y_0 = (x - x_0)[x \ x_0]$$

$$y = y_0 + (x - x_0)[x \ x_0] \text{ ——— (i)}$$

$\Delta^5 y$

Consider the second divided difference $[x \ x_0 \ x_1] = \dfrac{[x \ x_0] - [x_0 \ x_1]}{x - x_1}$

$$\rightarrow [x \ x_0] = [x_0 \ x_1] + (x - x_1)[x \ x_0 \ x_1]$$

$\therefore$ from eq. (i)

-0.037

$$y = y_0 + (x - x_0)[x_0 \ x_1] + (x - x_0)(x - x_1)[x \ x_0 \ x_1] \text{ — (ii)}$$

Consider $[x \ x_0 \ x_1 \ x_2] = \dfrac{[x \ x_0 \ x_1] - [x_0 \ x_1 \ x_2]}{x - x_2}$

$$\Rightarrow [x \ x_0 \ x_1] = [x_0 \ x_1 \ x_2] + (x - x_2)[x \ x_0 \ x_1 \ x_2]$$

$\therefore$ from eq. (ii)

$$y = y_0 + (x - x_0)[x_0 \ x_1] + (x - x_0)(x - x_1)[x_0 \ x_1 \ x_2] + (x - x_0)(x - x_1)(x - x_2)[x \ x_0 \ x_1 \ x_2] \text{ — (iii)}$$

$\vdots$

Proceeding like above, we get

$$\boxed{y = y_0 + (x-x_0)\Delta y_0 + (x-x_0)(x-x_1)\Delta^2 y_0 + (x-x_0)(x-x_1)(x-x_2)\Delta^3 y_0 + \dots\dots + (x-x_0)(x-x_1)(x-x_2)\dots(x-x_{n-1})\Delta^n y_0}$$

Ques. Construct the interpolated polynomial for the following data :

x :	0	1	2	3
y :	5	3	7	23

The divided difference is given below

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	$\boxed{5}\ y_0$			
		$\frac{3-5}{1-0} = \boxed{-2}\ \Delta y_0$		
1	3		$\frac{4+2}{2-0} = \boxed{3}\ \Delta^2 y_0$	
		$\frac{7-3}{2-1} = 4$		$\frac{6-3}{3-0} = \boxed{1}\ \Delta^3 y_0$
2	7		$\frac{16-4}{3-1} = 6$	
		$\frac{23-7}{3-2} = 16$		
3	23			

Applying the formula,

$$y = y_0 + (x-x_0)\Delta y_0 + (x-x_0)(x-x_1)\Delta^2 y_0 + \dots + (x-x_0)(x-x_1)\dots(x-x_{n-1})\Delta^n y_0$$

$$y = 5 + (x-0)(-2) + (x-0)(x-1)3 + (x-0)(x-1)(x-2)1$$

$$= 5 - 2x + (x^2-x)3 + x(x^2-3x+2)$$

$$= 5 - \cancel{2x} + \cancel{3x^2} - 3x + x^3 - \cancel{3x^2} + \cancel{2x}$$

$$= x^3 - 3x + 5$$

Numerical Differentiation

We learn the procedures to find the derivatives at tabulated or non-tabulated points till second derivative. If the point at which the derivative is required is close to x_0, we use the derivative formula generated from Newton-Gregory forward Interpolation. If those points are close to x_n, we use the derivative formulae generated from Newton-Gregory backward Interpolation.

Derivative formulae generated using Newton-Gregory Forward Interpolation.

The Newton-Gregory forward Interpolation formula is given by -

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots + \frac{u(u-1)(u-2)\cdots(u-\overline{n-1})}{n!}\Delta^n y_0 \quad \text{——— (i)}$$

where, $u = \frac{x-a}{h}$ ——— (ii)

$$\frac{dy}{du} = 0 + \Delta y_0 + \left(\frac{2u-1}{2}\right)\Delta^2 y_0 + \left(\frac{3u^2-6u+2}{3!}\right)\Delta^3 y_0 + \cdots$$

$$\frac{du}{dx} = \frac{1}{h}$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{du}}{du/dx}$$

$$= \frac{1}{h}\left[\Delta y_0 + \left(\frac{2u-1}{2}\right)\Delta^2 y_0 + \left(\frac{3u^2-6u+2}{3!}\right)\Delta^3 y_0 + \cdots\right] \text{——— (iii)}$$

which gives the first derivative at non-tabulated point close to x_0.

Note: Substituting $u=0$,

$$\frac{dy}{dx} = \frac{1}{x}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 + \cdots\right] \text{——— (iv)}$$

which gives the first derivative at tabulated point.

To compute second derivative, differentiate eq. (iii) wrt u and multiply with du/dx

because $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{du}\left(\frac{dy}{dx}\right) \cdot \frac{du}{dx}$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \left(\frac{6u^2-18u+11}{12}\right)\Delta^4 y_0 + \cdots\cdots\right] \text{——} (v)$$

which gives the second derivative at non-tabulated point.

Substituting $u=0$, at above formula, we get

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \cdots\cdots\right] \text{——} (vi)$$

which gives second derivative at tabulated point.

Derivative formulae using Newton-Gregory Backward Interpolation

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \left(\frac{2u+1}{2}\right)\nabla^2 y_n + \left(\frac{3u^2+6u+2}{6}\right)\nabla^3 y_n + \cdots\cdots\right] \text{——} (vii)$$

Substituting $u=0$ at above formula, we get

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \cdots\cdots\right] \text{——} (viii)$$

eq. (vii) and (viii) gives the formulae for the first derivative at non-tabulated and tabulated point respectively using backward interpolation.

Similarly,

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + (u+1)\nabla^3 y_n + \left(\frac{6u^2+18u+11}{12}\right)\nabla^4 y_n + \cdots\cdots\right] \text{——} (ix)$$

Substituting $u=0$ at above formula, we get

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots\cdots\right] \text{——} (x)$$

represents second derivative at non-tabulated and tabulated point respectively using backward interpolation.

Note: the above formula are valid for equally spaced data. In case of unequally spaced data, we construct the interpolating polynomial using Newton's divided difference formula. And differentiate it based on the requirement and substitute the points at which the derivative are required.

Ques. Find $\frac{dy}{dx}$ at $x=0.1$ from the following data :

x :	0.1	0.2	0.3	0.4
y :	0.9975	0.99	0.9776	0.9604

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0.1	0.9975	$\boxed{-0.0075}$		
			$\boxed{-0.0049}$	
0.2	0.99	-0.0124		$\boxed{0.0001}$
			-0.0048	
0.3	0.9776	-0.0172		
0.4	0.9604			

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 + \dots\right]$$

$$= \frac{1}{0.1}\left[-0.0075 - \frac{1}{2}(-0.0049) + \frac{1}{3}(0.0001)\right]$$

$$= -0.0501$$

Note: From the above problem, $y'(0.2)$ is calculated using a formula

$$\left.\frac{dy}{dx}\right|_{x=0.2} = \frac{1}{0.1}\left[-0.0124 - \frac{1}{2}(-0.0048)\right]$$

[by removing the first row and taking the next values]

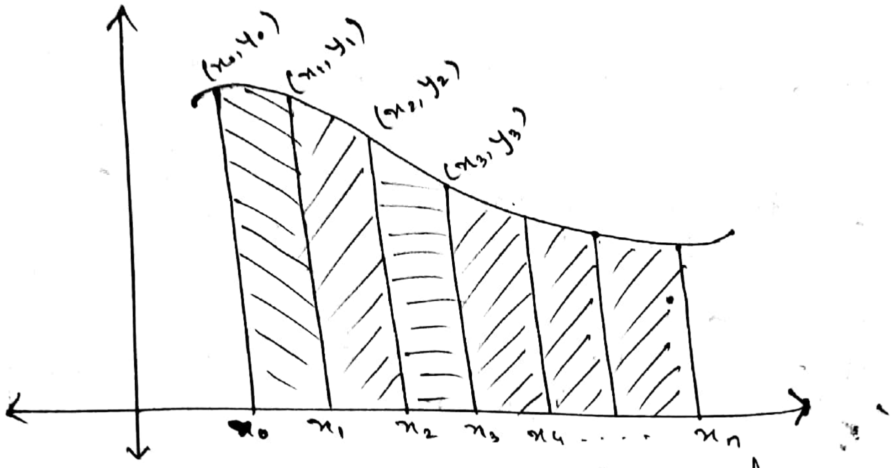
Ques. The table given below gives the velocity v of a body during the time t. Find the acceleration at $t=1.1$ and 1.15

t :	1	1.1	1.2	1.3	1.4
v :	43.1	47.7	52.1	56.4	60.8

[at $t=1.1$, 45.166]

Numerical Integral

Given a set of tabulated values of the integro $f(x)$. To determine $\int_{x_0}^{x_n} f(x)\,dx$ is called numerical integration.



We subdivide the given interval into a no. of subintervals of equal width 'h' and replace the function tabulated at the points of subdivision by any of the intepolating Polynomial Like Newton-Gregory forward Interpolation, Newton-Gregory backward Interpolation.

General Quadrature formula / Newton-Cotes Quadrature formula

Let $I = \int_a^b y\,dx$ where y takes the values $y_0, y_1, y_2, \ldots, y_n$ for $x = x_0, x_1, x_2, \ldots x_n$

Let the interval of integration be (a,b) be divided into n equal subintervals, each of width $h = \frac{b-a}{n}$

So that $x_0 = a$, $x_1 = a+h$, $x_2 = a+2h$, ..., $x_n = a+nh$.

$\therefore$ I becomes

$$I = \int_{x_0}^{x_0+nh} f(x)\,dx$$

Since, every x can be written as $x = x_0 + rh$ and $dx = h\,dr$

I becomes

$$I = \int_0^n f(x_0+rh).\,h\,dr$$

$$= h\int_0^n f(x_0+rh)\,dr$$

Using, Newton-Gregory forward Interpolation formula,

$$I = h\int_0^n \left[y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \ldots\right] dr$$

$$I = h\left[ry_0 + \frac{r^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{r^3}{3} - \frac{r^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{r^4}{4} - r^3 + r^2\right)\Delta^3 y_0 + \dots\right]^n$$

$$= nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \dots\right] \quad \text{— (i)}$$

eq. (i) is called as general quadrature formula or Newton-cotes Quadrature formula.

Note: The no. of important reductions were derived from the above formula, which can be directly used in the integration.

- The important among them are —

Trepetoidal rule, Simpson's $\frac{1}{3}$rd rule, simpson's $3/8$th rule
($n = 1$) ($n = 2$) ($n = 3$)

Trepetoidal Rule ($n = 1$)

Put $n = 1$ in the eq. (i) and taking the curve joining the Points (x_0, y_0) (x_1, y_1) as the polynomial of degree 1, all high order differences should become zero. Therefore,

$$\int_{x_0}^{x_n} y(x)\,dx = \int_{x_0}^{x_1} y\,dx + \int_{x_1}^{x_2} y\,dx + \int_{x_2}^{x_3} y\,dx + \dots + \int_{x_{n-1}}^{x_n} y\,dx$$

Consider,

$$\int_{x_0}^{x_1} y\,dx = 1 \cdot h\left[y_0 + \frac{1}{2}\Delta y_0 + 0\right]$$

$$= h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right]$$

$$= h\left[\frac{2y_0 + y_1 - y_0}{2}\right]$$

$$= \frac{h}{2}\left[y_0 + y_1\right]$$

Consider,

$$\int_{x_1}^{x_2} y\,dx = 1 \cdot h\left[y_1 + \frac{1}{2}\Delta y_1 + 0\right]$$

$$= h\left[y_1 + \frac{1}{2}(y_2 - y_1)\right]$$

$$= \frac{h}{2}\left[y_1 + y_2\right]$$

...]

~~Consider~~,
$$\int_{x_{n-1}}^{x_n} y\,dx = h/2\,[y_{n-1} + y_n]$$

Adding all these values, we get

$$\int_{x_0}^{x_n} y\,dx = h/2\,[y_0+y_1] + h/2\,[y_1+y_2] + h/2\,[y_2+y_3] + \ldots + h/2\,[y_{n-1}+y_n]$$

$$\boxed{\int_{x_0}^{x_n} y\,dx = h/2\left[(y_0+y_n) + 2\,(y_1+y_2+y_3+\ldots+y_{n-1})\right]}$$

This formula is called Trepetoidal Rule.

Similarly, by taking n=2 and n=3, we get Simpson's 1/3rd rule and Simpson's 3/8th rule which are shown below -

Simpson's 1/3rd Rule (n=2)

$$\int_{x_0}^{x_n} f(x)\,dx = h/3\left[(y_0+y_n) + 4(y_1+y_3+y_5+\ldots+y_{n-1}) + 2\,(y_2+y_4+\ldots+y_{n-2})\right]$$

Simpson's 3/8th Rule (n=3)

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{3h}{8}\left[(y_0+y_n) + 3(y_1+y_2+y_4+y_5+y_7+y_8+\ldots+y_{n-2}+y_{n-1}) + 2\,(y_3+y_6+y_9+\ldots+y_{n-3})\right]$$

Note: Simpson's 1/3rd Rule is applicable when the no. of subterms are even and Simpson's 3/8th Rule is applicable when no. of subterms are multiple of 3.

Ques. Estimate the value of π

$$\int_0^1 \frac{1}{1+x^2}\,dx$$

i) Simpson's 1/3rd rule $h = 1/4 = 0.25$

ii) " 3/8th rule $h = 1/6$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \left[\tan^{-1}x\right]_0^1$$

$$= \tan^{-1}(1) - \tan^{-1}(0)$$

$$= \pi/4$$

i) x $\qquad y = \frac{1}{1+x^2}$

$x_0 = 0 \qquad y_0 = 1$

$x_1 = 0.25 \qquad y_1 = 0.9411$

$x_2 = 0.5 \qquad y_2 = 0.8$

$x_3 = 0.75 \qquad y_3 = 0.64$

$x_4 = 1 \qquad y_4 = 0.5$

$$\int_0^1 \frac{1}{1+x^2}dx = \frac{0.25}{3}\left[(1+0.5) + 4\times(0.9411 + 0.64) + 2\times 0.8\right]$$

$= \cancel{\frac{0.25}{3} \times 6.8644}$

$\neq \cancel{0.5720}$ $\quad 0.78539$

$\pi/4 = 0.78539$

$\pi = 3.141568$

ii) $h = 1/6$

x	$y = \frac{1}{1+x^2}$	
0	0	$= y_0$
0.17	0.2570	
0.34	0.8963	
0.51	0.7935	
0.68	0.6838	
0.85	0.5805	
1	0.5	$= y_n$

x	$y = 1/(1+x^2)$	
0	1	$= y_0$
1/6	0.9729	
2/6	0.9	
3/6	0.8	
4/6	0.6923	
5/6	0.59	
1	0.5	$= y_n$

$$= \frac{1/6 \times 3}{8}\left[(1+0.5) + 3(3.1552) + 2(0.8)\right]$$

$= \frac{1}{16}[12.5556]$

$= \cancel{0.5954} \qquad 0.785305 \qquad 0.78540$

$\cancel{\pi/4 = 0.5954}$

$\cancel{\pi = 2.3816}$

$\pi/4 = 0.78540$

$\cancel{\pi = 3.1414}$

$\pi = 3.14158$

Trapezoidal $(n=1)$

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+y_3+\cdots\cdots+y_{n-1})\right]$$

Simpson's $\frac{1}{3}^{rd}$ rule $(n=\text{even})$

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+y_5+\cdots)+2(y_2+y_4+y_6+\cdots)\right]$$

Simpson's $3/8^{th}$ rule $(n=\text{odd})$

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+y_4+y_5+y_7+\cdots)+2(y_3+y_6+y_9+\cdots)\right]$$

Ques. $\int_{a=0}^{b=1} x^3\,dx$ using Simpson's $\frac{1}{3}^{rd}$ and Trapezoidal rule with 4 subinterval.

$\hookrightarrow y(x)$

$n=4$

$h = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4} = 0.25$

x :	0	0.25	0.5	0.75	1
y :	0	0.015125	0.125	0.4218	1

Trapezoidal : $\int_0^1 x^3\,dx = \frac{0.25}{2}\left[(0+1)+2(y_1+y_2+y_3)\right]$

$= 0.2656$

Simpson's $\frac{1}{3}^{rd}$ rule :

$$\int_0^1 x^3\,dx = \frac{0.25}{3}\left[(0+1)+4(0.015625+0.4218)+2(0.125+1)\right]$$

$= 0.25$

$$\int_0^1 x^3\,dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{4} = 0.25$$

$\therefore$ Simpson's $\frac{1}{3}$rd rule is more accurate.

Note : Consider $h = 0.1$

x	$y = x^3$
0	0
0.1	0.001
0.2	0.008
0.3	0.027
0.4	0.064
0.5	0.125
0.6	0.216
0.7	0.343
0.8	0.512
0.9	0.729
1	1

Trapezoidal rule :

$$\int_0^1 x^3\,dx = \frac{0.1}{2}\left[(0+1) + 2(2.025)\right]$$

$$= \frac{0.1}{2} \times 5.05$$

$$= 0.2525$$

Ques. The speed in m/s of a car, in t sec after its start is shown in the following table

t :	0	12	24	36	48	60	72	84	96	108
v :	0	3.6	10.08	18.9	21.6	18.54	10.26	5.40	4.50	5.40

y_0

Using Simpson Find the distance travelled by the car in 2 mins.

$h = 12$ $\quad \frac{11+1}{2} = 6 =$ even

$\therefore$ Simpson's $\frac{1}{3}$rd rule will be applied

$$\frac{12}{3}\left[(0+9) + 4(51.84) + 2(46.44)\right]$$

$= \cancel{4[310.68]}\ 4[309.24]$

$= \cancel{1242.72}$

$= 1236.96$ Ans : 1236.96 m

[Pg. 450 onwards questions]

Ques. from the following value of f(x), find the position of centroid of the area under the curve and the x-axis.

x :	0	1/4	1/2	3/4	1
y :	1	4	8	4	1

Note: Centroid of the plane area under the curve $y = f(x)$ is given by $(\bar{x}, \bar{y})$

$$\bar{x} = \frac{\int_0^1 xy\,dx}{\int_0^1 y\,dx} \quad ; \quad \bar{y} = \frac{\int_0^1 \frac{y^2}{2}\,dx}{\int_0^1 y\,dx}$$

xy :	0	1	4	3	1
$y^2/2$:	1	8	32	8	0.5

$\frac{5+1}{2} = 3$ = odd

∴ Simpson's 3/8th rule will be applied.

$h = 1/4$ ~~hence~~

$$\int_0^1 xy\,dx = \frac{1/4 \times 3}{8}\left[(0+1) + 3(1+4) + 2\times 3\right]$$

$$= 0.09375\,[22]$$

$$= \cancel{0.21972}\ 2.0625$$

120

9

y_n

$$\int_0^1 y\,dx = \frac{1/4\times 3}{8}\left[(1+1) + 3(4+8) + 2\times 4\right]$$

$$= 0.09375\,(46)$$

$$= \cancel{4.59375}\ 4.3125$$

$$\int_0^1 \frac{y^2}{2}\,dx = \frac{1/4\times 3}{8}\left[(1+0.5) + 3(8+32) + 2\times 8\right]$$

$$= 0.09375\,(137.5)$$

$$= 12.8906$$

$$\bar{x} = \frac{2.0625}{4.3125}$$

$$= 0.4782$$

$$\bar{y} = \frac{12.8906}{4.3125}$$

$$= 2.9891$$

Ordinary Differential Equations (ODE)

Initial Value Problem.

The problem of finding the soln of a differential equation with the conditions given at the same point is called initial value problem.

The general initial value problem of order 1 is given by -

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$$

Note: → Without initial condition, the problem (ODE) cannot be solved in numerical method.

→ The solution of a numerical method is a particular solution, i.e., a numeric value.

The following methods are used to find the solution of ODEs:-

i) Picard's Successive Approximation Method

ii) Taylor Series

iii) Euler's Method

Picard's Successive Approximation Method.

Consider $\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$

$$dy = f(x,y)\,dx$$

Integrating, ~~w~~

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x,y)\,dx$$

$$\Rightarrow y - y_0 = \int_{x_0}^{x} f(x,y)\,dx$$

$$\boxed{y = y_0 + \int_{x_0}^{x} f(x,y)\,dx}$$

The first approximate sol^n is

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

The second approximate sol^n is

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\,dx$$

$\vdots$

$$\therefore \boxed{y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})\,dx}$$

Ques. Find the value of $y(0.2)$ from the eq^n : (use 3 approximations)

$$\frac{dy}{dx} - x + y = 0, \quad y(0) = 1$$

Standard form is : $\frac{dy}{dx} = x - y, \quad y(0) = 1$

$$\therefore f(x,y) = x - y; \quad x_0 = 0; \quad y_0 = 1$$

$$y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})\,dx$$

The first approximation is

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

$$= 1 + \int_0^x (x-1)\,dx$$

$$= 1 - x + x^2/2$$

The second approximation is

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)}_0)\,dx$$

$$= 1 + \int_0^1 [x - (1 - x + x^2/2)]\,dx$$

$$= 1 + \int_0^1 (-1 + 2x - x^2/2)\,dx$$

$$= 1 - x + x^2 - x^3/6$$

$$y^{(3)} = y_0 + \int_{x_0}^{x_n} f(x, y^{(2)})\, dx$$

$$= 1 + \int_0^1 [x - (1 - x + x^2 - x^3/6)]\, dx$$

$$= 1 + \int_0^x (-1 + 2x - x^2 + x^3/6)\, dx$$

$$= 1 + \left[-x + x^2 - x^3/3 + \frac{x^4}{24}\right]_0^x$$

$$= 1 - x + x^2 - x^3/3 + x^4/24$$

The solution table is shown below -

x	$y^{(1)}$	$y^{(2)}$	$y^{(3)}$
0.2	0.82	0.83866	0.8374

$\frac{dy}{dx} + y = x$, $y(0) = 1$

$(D+1)y = x$, $y(0) = 1$

$f(D)\, y = Q$ AE : $m+1 = 0 \Rightarrow m = -1$

$y_c = C_1 e^{-x}$

$y_p = \frac{1}{D+1} \cdot x = (1+D)^{-1} x$

$= (1 - D + D^2 - D^3 \ldots) x$

$y_p = x - 1$

$\therefore y(x) = C_1 e^{-x} + x - 1$

$y(0) = 1 \Rightarrow C_1 - 1 = 1 \Rightarrow C_2 = 2$

$y(x) = 2e^{-x} + x - 1$

$y(0.2) =$ ______

Euler's Method

Consider $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$

$$y_1 - y_0 = \left(\frac{dy}{dx}\right)_{(x_0, y_0)} (x_1 - x_0)$$

$$\Rightarrow y_1 = y_0 + h f(x_0, y_0)$$

$$y_2 = y_1 + h f(x_1, y_1)$$

$$\boxed{y_{n+1} = y_n + h f(x_n, y_n)}$$

Ques. Solve $\frac{dy}{dx} = x-y$, $y(0) = 1$ in the interval $[0, 1]$

$f(x, y) = x - y$; $x_0 = 0$; $y_0 = 1$

let $h = 0.25$

$y_{n+1} = y_n + hf(x_n, y_n)$

∴ The sol^n table is :

x	y
0	1
0.25	$y_0 + hf(x_0, y_0) = 1 + (0.25)(0-1) = 0.75$
0.5	$y_1 + hf(x_1, y_1) = 0.75 + (0.25)(0.25 - 0.75) = 0.625$
0.75	$0.625 + (0.25)(0.5 - 0.625) = 0.59375$
1	$0.75 + (0.25)(0.75 - 0.59375) = \cancel{0.7810625}$

Taylor Series Method

Consider $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$ —— (i)

Hence, $y(x_0)$ is the sol^n for (i)

The taylor series representation of $y(x)$ about the initial Point x_0 is

$$y(x) = y(x_0) + (x - x_0) y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \ldots \quad \text{—— (ii)}$$

Replacing x with x_1 —— (iii)

$$y_1 = y(x_1) = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots$$

derivative of y at the point x_0 ; not a derivative of y_0

—— (iv)

$$y_2 = y(x_2) = y_1 + hy_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \ldots$$

Note: We observe that the Euler's formula is the first two terms of the Taylor's formula and hence, the accuracy of the sol^n is less compared to Taylor Series formula with more than 2 terms.

→ Euler's formula will give better result when we decrease the value of h.

Ques. Solve $\frac{dy}{dx} = x+y$, $y(1) = 0$ upto 1.2 with $h=0.1$

Compare your sol^n with analytical sol^n.

Here, $h = 0.1$; $x_0 = 1$; $y_0 = 0$. $f(x,y) = x+y$

To find $y(01.1)$, i.e., y_1

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0'''' + \dots$$

$$y' = x+y \Rightarrow y_0' = x_0 + y_0 = 1+0 = 1$$

$$y'' = 1+y' \Rightarrow y_0'' = 1+y_0' = 1+1 = 2$$

$$y''' = y'' \Rightarrow y_0''' = y_0'' = 2$$

$$y'''' = y''' \Rightarrow y_0'''' = y_0''' = 2$$

$$\therefore y_1 = 0 + (0.1).1 + \frac{(0.1)^2}{2!}\times 2 + \frac{(0.1)^3}{3!}\times 2 + \frac{(0.1)^4}{4!}\times 2$$

[neglecting higher order derivative]

$$= 0.1 + 0.01 + \frac{0.001}{3} + \frac{0.0001}{12}$$

$$= 0.11 + 3.33\times10^{-4} +$$

$$= 0.11034$$

To find $y(1.2)$

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \frac{h^4}{4!}y_1'''' + \dots$$

$$y_1' = x_1 + y_1 = 1.1 + 0.11034 = 1.21034$$

$$y_1'' = 1+y_1' = 2.21034$$

$$y_1''' = y_1'' = 2.21034$$

$$y_1'''' = y_1''' = 2.21034$$

$$\therefore y_2 = y(1.2) = 0.11034 + (0.1)(1.21034) + \frac{(0.1)^2}{2!}\times(2.21034) + \frac{(0.1)^3}{3!}\times(2.21034) + \frac{(0.1)^4}{4!}\times(2.21034)$$

$$= 0.231374 + 0.0110517 + 3.6839\times10^{-4} + 9.20975\times10^{-6}$$

$$= 0.2428032$$

$$\frac{dy}{dx} = x + y$$

$$\frac{dy}{dx} - y = x \Rightarrow (D-1)y = x$$

AE: $\Rightarrow m - 1 = 0$

$m = 1$,

$y_c = c_1 e^x$

$$y_p = \frac{1}{D-1} \cdot x$$

$= -(1-D)^{-1} \cdot x$

$= -[1 + D + D^2 + \dots]x$

$= -x - 1$

$y(x) = c_1 e^x - x - 1$

$y(1) = 0$

$\Rightarrow c_1 e^{-1} - 1 - 1 = 0$

$\Rightarrow c_1 e = 2$

$c_1 = 2/e = 0.73575$

$\therefore y(x) = (0.73575)e^x - x - 1$

$y(1.1) = 0.110314$

$y(1.2) = 0.242776$

Modified Euler's Method

Modified Euler's formula will give a better approximation than the Euler's formula.

$$y_{n+1} = y_n + hf\left(x_n + h/2,\ y_n + h/2\, f(x_n, y_n)\right)$$

Ques. Apply modified Euler's Rule on $\frac{dy}{dx} = e^x + xy$, $y(0) = 0$ to find $y(0.1)$ & $y(0.2)$

$y_0 = 0$; $x_0 = 0$; $f(x, y) = e^x + xy$; $h = 0.1$

$y_1 = y_0 + hf(x_0 + h/2,\ y_0 + h/2\, f(x_0, y_0))$

$= 0 + (0.1) f(0 + 0.1/2,\ 0 + h/2 \cdot 1)$

$= 0.1 \times f(0.05, 0.05)$

$= 0.1[e^{0.05} + 0.05 \times 0.005] = 0.10587$

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x	y
0	0
0.1	0.10537
0.2	0.2239

$$y_2 = y_1 + hf\left(x_1 + h/2,\ y_1 + h/2\, f(x_1, y_1)\right)$$

4th Order Runge-Kutta Method (RK-4 method)

Here, we get Δy values instead of y values and accordingly y values can be calculated.

Step-I] Consider, $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$

Step-II] Compute the following 4 variables
To find y_1,

$$K_1 = hf(x_0, y_0)$$

$$K_2 = hf(x_0 + h/2,\ y_0 + K_1/2)$$

$$K_3 = hf(x_0 + h/2,\ y_0 + K_2/2)$$

$$K_4 = hf(x_0 + h,\ y_0 + K_3)$$

$$\Delta y_0 = \frac{1}{6}\left[K_1 + 2(K_2 + K_3) + K_4\right]$$

$$\boxed{y_1 = y_0 + \Delta y_0}$$

Note: In order to find y_2, we use (x_1, y_1) pair

$$K_1 = hf(x_1, y_1)$$

$$K_2 = hf(x_1 + h/2, y_1 + k_1/2)$$

$$K_3 = hf(x_1 + h/2, y_1 + K_2/2)$$

$$K_4 = hf(x_1 + h, y_1 + K_3)$$

$$\Delta y_0 = \frac{1}{6}[K_1 + 2(K_2 + K_3) + K_4]$$

$$y_2 = y_1 + \Delta y_1$$

Ques. Solve $\frac{dy}{dx} = x + y$, $y(0) = 1$ by RK-4 Method from $x = 0$ to 0.4 with $h = 0.1$. [Ans: $y_1 = 1.11034$; $y_2 = 1.24280$; $y_3 = 1.3997$; $y_4 = 1.5836$]