NUMERICAL METHODS

There are 2 types of mathematical techniques

analytical | numerical

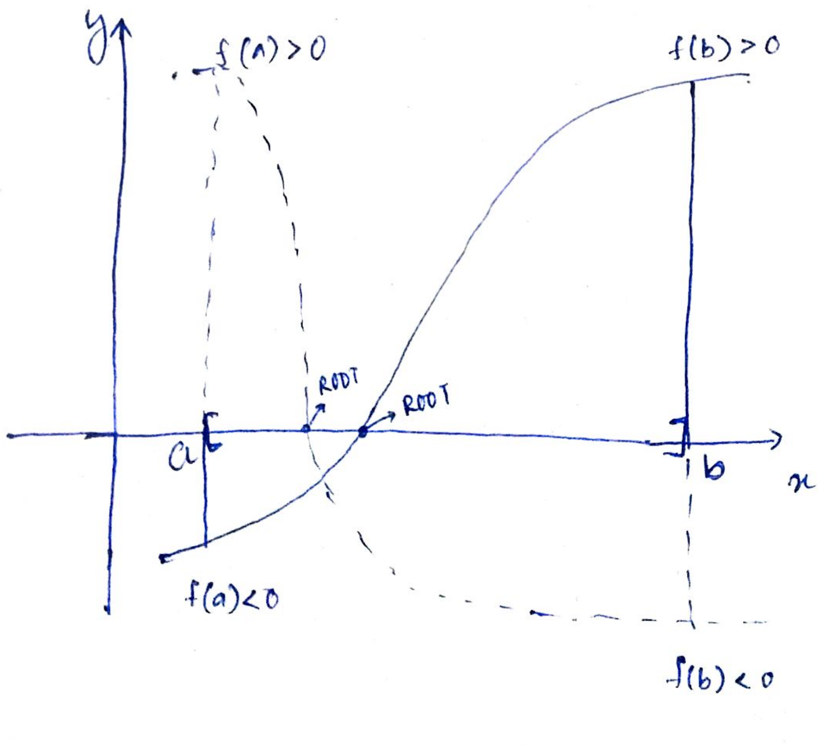
Not all problems are solvable using analytical tech. In such cases, we depend on numerical techniques. The numerical tech. will give us the approx. solution whose deviatⁿ from actual solutⁿ is termed as ERROR.

★ There are error minimising mechanisms in the Numerical Techniques. They can solve a vareity of problems like root finding, interpolation (curve fitting), solving algebraic system of equations, differentiatⁿ & integratⁿ, solution of ordinary & partial diff. eqⁿ.

★ Root Finding:–

- **Intermediate Value Thm:–**

Let $f(x)$ be continuous in $[a,b]$ such that $f(a).f(b)<0$ then, a root of the eqⁿ $f(x) = 0$, will lie in betⁿ a & b.



Q. Solve for root of $x - \cos x = 0$.

$f(x) = x - \cos x$

$f(0) = 0 - \cos 0 = -1 < 0$

$f(1) = 1 - \cos 1 = 0.45 > 0$

} signature changes, i.e root lies in this interval.

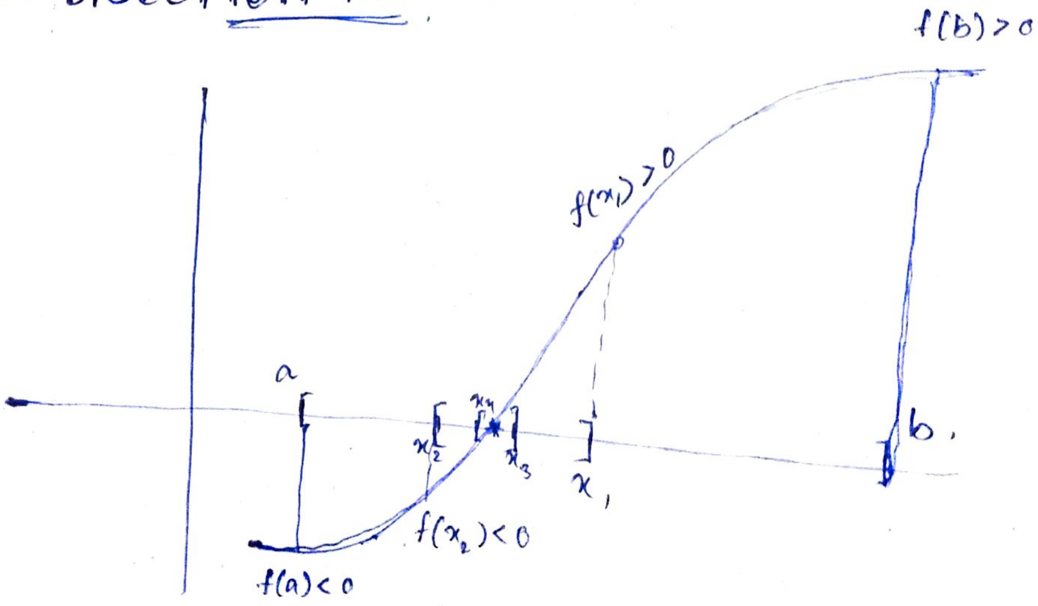
MODE of calc. should be RAD.

Therefore, the interval of root is $[0, 1]$.

★ The Intermediate Value Thm is the basis for finding the roots of given Algebraic or Transcendental (funct^n having e^x, trigonometric, logarithmic, etc) equations.

- There are several methods like Bisection, Regula-Falsi, Newton-Raphson etc to solve Algebraic & Transcendental eq^{n}s.

★ BISECTION METHOD :-



Initial Root =

$$x_1 = \frac{a+b}{2}$$

$$x_2 = \frac{a+x_1}{2}$$

|

* Algo for Bisect^n Method :-

① Let the initial approx. root be the midpt. of the given interval. i.e if the interval is $[a, b]$

$$\Rightarrow \quad x_0 = \frac{a+b}{2}$$

② if $f(x_0) \approx 0$; x_0 is the root.

otherwise,

$f(x_0)$ may be +ve or -ve.

③ We find a new interval of root from the signs of $f(a)$, $f(x_0)$ & $f(b)$.

④ We repeat the above steps till we get a root upto desired accuracy.

Q. Find the sqrt of 2 using bisect^n correct to 3 dec.

$$x = \sqrt{2}.$$

$$x^2 = 2.$$

$$\Rightarrow x^2 - 2 = 0$$

we know,

$\sqrt{2}$ is the root of eq^n $x^2 - 2 = 0$,

$f(0) = -2 < 0$

$\left.\begin{array}{l} f(1) = -1 \\ f(2) = 2. \end{array}\right\}$ root lies bet^n $[1, 2]$.

$f(1.5) = (1.5)^2 - 1 > 0$

$f(1.4) < 0$

$[0 \quad 1] \approx 1$

$[1.4 \quad 1.5] \approx 0.1$

$[1.41 \quad 1.42] \approx 0.01$

→ Initially, we obs. that root lies bet^n $[0 \quad 1]$.

→ furthur, we obs that root lies bet^n $[1.4 \quad 1.5]$

→ We consider this bcoz the smaller interval converges to root in less no. of intervals.

Let the initial approx root be $x_0 = \frac{1.41 + 1.42}{2}$

$= \underline{\underline{1.415}}$

Q: find

Bis

$x^2 - 2 = 0$.

$\therefore f(1.415) > 0$.

$\therefore [1.410 \quad 1.415]$ root lies here

★ $x_1 = \frac{1.41 + 1.415}{2} = 1.4125$

$f(1.4125) < 0$.

$\therefore [1.4125 \quad 1.415] \rightarrow$ root lies here.

$x_2 = \frac{1.4125 + 1.415}{2} = 1.41375$

$f(1.41375) < 0$.

$\therefore$ root lies bet^n $[1.41375 \quad 1.415]$.

$x_3 = \frac{1.41375 + 1.415}{2} = 1.414375$

$f(1.414375) > 0$

$x_4 = \frac{1.41375 + 1.414375}{2}$

$= 1.4140625$

$f(1.4140625) < 0$.

$\therefore \underline{\underline{1.4140625}}$ is the root of 2 ~~up~~ correct upto 3 decimals.

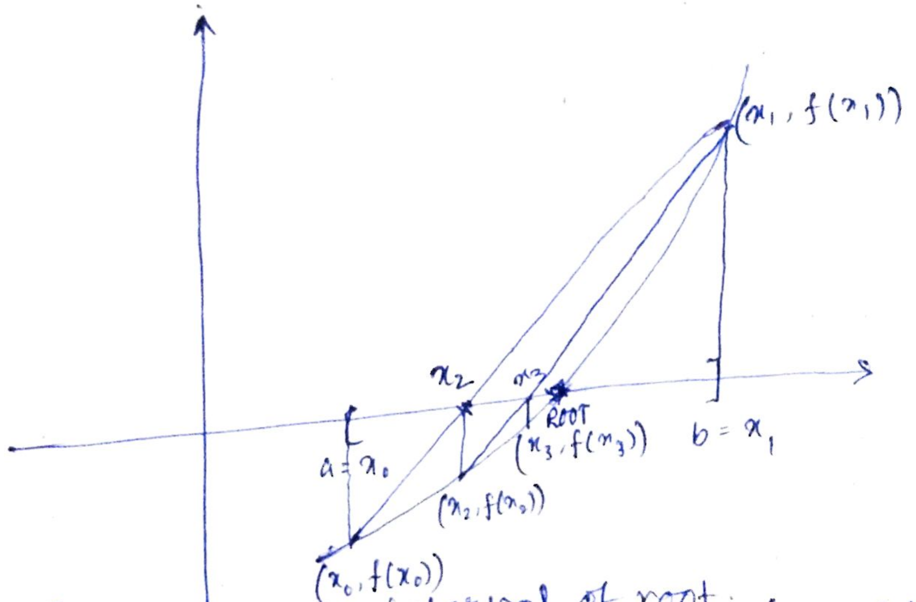
NOTE.

This method is very slow but is a guaranteed convergence method.

★ REGULA FALSI METHOD :-

or

Method of False Position.



Let $[x_0 \; x_1]$ be the interval of root. $(x_1, f(x_1))$

$(x_0, f(x_0))$

$$\cancel{y}^{\,0} - f(x_0) = \left[\frac{f(x_1) - f(x_0)}{x_1 - x_0}\right](x - f(x_0))$$

Since x is the assumed root, there y become 0.

$$\Rightarrow \quad x - x_0 = -\left[\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right] f(x_0)$$

$$\Rightarrow \quad \boxed{x_2 = x_0 - \left[\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right] f(x_0)}$$

NOTE The eq^n of the line joining is.

$$y - f(x_0) = \left[\frac{f(x_1) - f(x_0)}{x_1 - x_0}\right](x - f(x_0))$$

Q. Find the value of sqrt of 2 using FP method by cons. int. as $[1.41 \quad 1.42]$.

$$f(x) = x^2 - 2$$

$f(1.41) = -0.0119 < 0$

$f(1.42) = 0.0164 > 0$

$x_0 = 1.41$ & $x_1 = 1.42$

$$x_2 = 1.41 - \left(\frac{1.42 - 1.41}{(0.0164 + 0.0119)}\right)(-0.0119)$$

$$= 1.41 + \frac{(0.01)(0.0119)}{0.0283}$$

$$= 1.41 + 4.2 \times 10^{-3}$$

$$= 1.414204$$

$f(x_2) = -2.436 \times 10^{-5}$

$[1.414204 \quad 1.42]$.

$$x_3 = 1.414204 - \frac{(1.42 - 1.414204)}{(0.0164 + 2.436 \times 10^{-5})} - 2.436 \times 10^{-5}$$

$$= 1.414204 + \frac{5.796 \times 10^{-3}}{0.01642436} \times 2.436 \times 10^{-5}$$

$$= 1.414204 + 859.64 \times 10^{-8}$$

$$x_3 = 1.414212$$

Q.1) Solve $x \log_{10} x = 1.2$ by bisectⁿ method correct to 4 decimals.

2.7407

Q.2) Using the method of false Positⁿ,
solve the foll. eqⁿ for their roots in the intervals given correct upto 4 decimals.

(i) $xe^x = \cos x$ (0.5 0.6) → 0.5177

(ii) $\tan x + \tanh(x) = 0$ (1.6, 3) → 2.365.

★ NEWTON RAPHSON METHOD :-

This method is used to improve the solⁿ obtained from the prev. methods.

Let x_0 be the approximate solution & let x_1 be the exact solution.

$\therefore x_1 = x_0 + h$

Since, x_1 is the actual root,

$f(x_1) = 0$

$\Rightarrow f(x_0 + h) = 0$

Using Taylor Series :-

$$f(x_0+h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \frac{h^3}{3!} f'''(x_0) + \dots$$

$$= 0.$$

Since, h is very small,
neglecting higher order h values, we have,

$$f(x_0) + hf'(x_0) = 0$$

$$\Rightarrow h = -\frac{f(x_0)}{f'(x_0)}.$$

$\therefore \quad x_1 = x_0 + h.$

$$\Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\Rightarrow \boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

Similarly
for a better accuracy
we treat x_1 as approx. sol^n & x_2 as exact sol^n
proceeding as above
we get,

$$\Rightarrow \boxed{x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}}$$

$\vdots$

in general,

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} \longrightarrow \text{Newton's formula.}$$

NOTE :-

In this method, we need not check the intermediate interval of root as we did in bisect^n & false position methods.

STEPS

1) Use intermediate value thm to find interval of root.

2). In order to choose x_0, find $f(a)$ & $f(b)$.
If absolute value of $f(a)$ is close to zero, then $x_0 = a$ or else $x_0 = b$.

3) This method is not a guaranteed convergence method.

4) The accuracy of the solution is higher than false position method.

HW.

Q. Find a real root of $x = e^{-x}$ using Newton Method?

Q. Find a positive value of $(17)^{1/3}$ correct to 4 dec. using Newton's Method.

★ FIXED-POINT ITERATION METHOD :-

Consider the eq^n; $f(x) = 0$

Step-1

- Write the eq^n as $x = \phi(x)$.
- Several such $\phi(x)$ values may be possible based on the given eq^n.
- Let the interval of root be $I = [a\ b]$

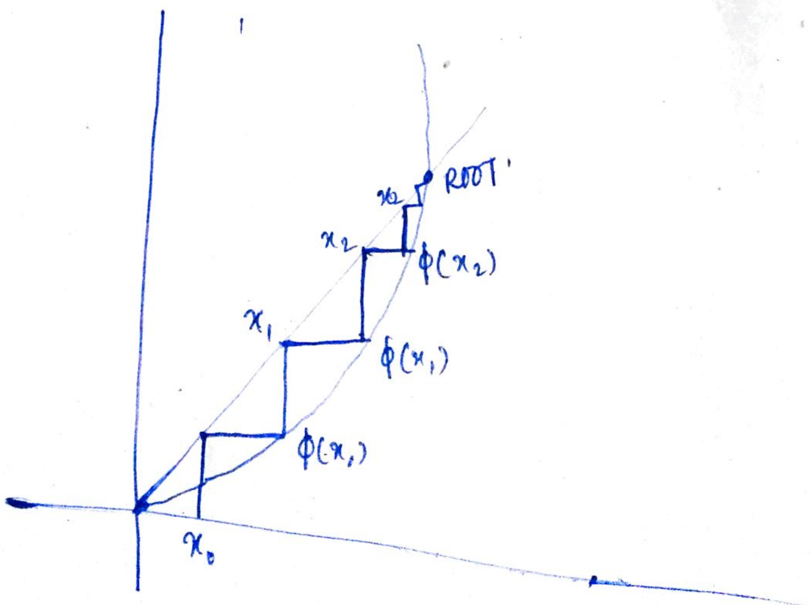
★ Choose that $\phi(x)$ whose absolute value of derivative is less than 1.

i.e $|\phi'(x)| < 1$ in interval of root.

The final formula will be

$x_{n+1} = \phi(x_n)$ where $\phi(x_n)$ is that funct^n whose absolute value of derivate is less than 1 in the interval of root.

★ Choose x_0 in the same way as in Newton's Method.



Q. Solve $x - \cos x$ in iteratⁿ method.

$x - \cos x = 0$

$\Rightarrow x = \cos x$ $\qquad\qquad$ $x = \cos^{-1}(x)$.

$x = \phi_1(x)$ $\qquad\qquad$ $x = \phi_2(x)$.

$I = [0.73 \quad 0.74]$.

We know,

$$|\phi'(x)|_{[a\ b]} < 1.$$

~~case-1~~ Let $\phi(x) = \cos x$.

$\phi'(x) = -\sin x$

$|\phi'(x)| = \sin x$.

$|\phi'(x)|_{x=0.73} = 0.666 < 1$

$|\phi'(x)|_{x=0.74} = 0.674 < 1$

$x_1 = \cos x_0$

$= \cos(0.74) = 0.7384685$

$x_2 = \cos(\quad) = \_\_$

$x_3 = \cos(\quad)$

$\therefore$ We have observed that $|\phi'(x)|$ in the interval $[0.73, 0.74]$ is less then 1.

$\therefore$ the fixed pt. iteratⁿ formula is given by

$$x_{n+1} = \phi(x_n)$$

$$\Rightarrow x_{n+1} = \cos(x_n).$$

$f(x) = x - \cos x$.

$f(0.73) = -0.015$

$f(0.74) = -0.00153 \rightarrow$ closer to zero.

$\therefore x_0 = 0.74$

Case - II.

$\phi(x) = \cos^{-1}(x)$.

$|\phi'(x)| = \dfrac{1}{\sqrt{1-x^2}}$

$|\phi'(x)|_{0.73} = 1.46 > 1$.

Hence condit^n failed!

$\therefore$ it can't use $\phi(x) = \cos^{-1}(x)$ in $[0.73, 0.74]$

8) Find the root of $x^2 - x - 1 = 0$ upto 4 dec. using iterat^n.

$\cancel{\phi(x) = x^2 - x - 1}$.

$\cancel{f(0) = -1}$

$\cancel{f(1) = -1}$

$\cancel{f(2) = 1}$

$\cancel{f(1.61) \longleftrightarrow f(1.62)}$

$\cancel{[1.61, 1.62]}$

$\cancel{|\phi'(x)| = (2x-1)}$.

$\cancel{|\phi'(x)|_{1.61} = 2.22}$

$\cancel{|\phi'(x)|_{1.62} = 2.24}$

$4 - 2 - 1$

$x_0 = 1.61$.

$\cancel{x_1 = x_0^2 - x_0 - 1}$

$\cancel{= -0.0179}$.

$\cancel{x_2 = -0.98177795}$

$\cancel{x_3 = 0.945670}$.

$\cancel{x_4 = -1.0513775}$

$\cancel{x_5 = 1.156772}$

$\cancel{x_6 = -0.818650021}$

$\cancel{x_7 = 0.48883879}$

$x_8 =$

$f(x) = x^2 - x - 1 = 0$

$x = x^2 - 1$

$\phi(x) = x^2 - 1$

$|\phi'(x)|_{1.61} = |2x| > 1$

$|\phi'(x)|_{1.62} = 2x > 1$

~~$x_1 = \phi(x_0) = 1.5921$~~

~~$x_2 = 1.53478$~~

~~$x_3 = 1.355557046$~~

~~$x_4 = 0.83753$~~

$\phi(x) = \sqrt{1+x} = (1+x)^{1/2}$

$\phi'(x) = \frac{1}{2\sqrt{1+x}} = 0.191 < 1$

$\phi'(x)_{1.62} = \frac{1}{2\sqrt{2.62}} = 0.3089$

$\therefore \phi(x) = \sqrt{1+x}$

$x_1 = \sqrt{1+x_0} = \sqrt{1+1.62} = 1.618641$

$x_2 = 1.61822168$

$x_3 = 1.61809198$

$x_4 = 1.6180519$

$x_5 = 1.618039$

Ans $\rightarrow$ 1.618039

Q. Use Newton-Raphson for $x^2 - x - 1 = 0$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow x_{n+1} = x_n - \frac{[x_n^2 - x_n - 1]}{2x_n - 1}$$

$x_0 = 1.62$

$$\Rightarrow x_1 = 1.62 - \frac{[(1.62)^2 - 1.62 - 1]}{2\cdot(1.62) - 1}$$

$$= 1.62 - \frac{4.4 \times 10^{-3}}{2.24}$$

$$= \underline{\underline{1.618035}}.$$

$$x_2 = 1.618035 - \frac{2.261225 \times 10^{-6}}{2.23607}$$

$$= \underline{\underline{1.618033}}$$

Q. $x^2 - 2 = 0$

$\Rightarrow f(x) = x^2 - 2.$

$\Rightarrow \phi(x) = x = \frac{2}{x}.$

$\Rightarrow |\phi'(x)| = \frac{2}{x^2}$

$I = [1.41 \quad 1.42].$

Clearly, $|\phi'(x)| > 1$ in both.

Hence, $\phi'(x) \neq \frac{2}{x^2}$ cant be solved.

To solve this problem,

$$x + x = \frac{2}{x} + x$$

$$\Rightarrow 2x = \frac{x^2 + 2}{x}.$$

$$\Rightarrow x = \frac{x + 2}{2x}.$$

$$\phi(x) = \frac{1}{2}\frac{(x+2)}{x}.$$

$$\phi'(x) = \frac{2x^2 - 2 - x^2}{2x^2} = \frac{x^2 + 2}{2x^2}$$

$$\Rightarrow |\phi'(x)| = \frac{x^2 + 2}{2x^2}$$

clearly, $|\phi'(x)| < 1$ in $[1.41 \quad 1.42]$.

$$\phi(x) = \frac{x^2 + 2}{2x}.$$

$$x_0 = 1.41$$

$$x_1 = 1.4142198$$

$$x_2 = \underline{1.41421362}.$$

Q.1) A funct^n $f(x) = e^x - 3x^2$ has 2 of its real roots near 1 & 4. Find these 2 roots by fixed pt. iterat^n scheme by discussing whether these 2 roots could be obtained by same fixed pt. scheme or not?

Q.2) The sum of 2 nos is 20. If each no. is added to its sqrt, the product of these 2 sums is ~~150~~ 155.55. Find these no. upto 3 dec. accuracy

$f(x) = e^x - 3x^2$

$f(0) = 1$

$f(1) = -0.281$

$f(0.5) = 0.89$

Ans-1.

$f(x) = e^x - 3x^2$

$e^x - 3x^2 = 0 \Rightarrow x^2 = \frac{e^x}{3}$

$\Rightarrow 3x^2 = e^x$

$\Rightarrow x = \pm\sqrt{\frac{e^x}{3}}$

0.742

0.543

$\therefore x = \phi(x)$

[0.91 0.9]

$\phi_1(x) = +\sqrt{\frac{e^x}{3}}$

0.91 0.92

[0.910 0.91]

[0.910 0.911]

$\phi_1'(x) = \frac{1}{\sqrt{3}} \cdot \frac{1 \cdot e^x}{2\sqrt{e^x}} = \frac{1}{2} \cdot \sqrt{\frac{e^x}{3}}$

$\frac{1}{\sqrt{3}} \frac{e^x}{2\sqrt{e^x}}$

$I = [0.910 \quad 0.911]$.

$|\phi'(x)|_{0.91} = 0.45500 < 1$ (closer to zero)

$|\phi'(x)|_{0.911} = 0.45522$

$\sqrt{x} = \frac{1}{2\sqrt{x}}$

$x_0 = 0.91$

$\frac{e^{x/2}}{\sqrt{3}}$

$x_1 = \sqrt{\frac{e^{x_0}}{3}} = 0.910004127$

$\frac{1}{\sqrt{3}}$

$x_2 = \sqrt{\frac{e^{x_1}}{3}} = 0.910006$

$\therefore$ 1st root $= 0.91000$

For 4, $e^x - 3x^2$

$f(4) = 6.598$

$f(3.5) = -3.63$

$I = [3.733 \quad 3.734]$

$$\phi(x) = \sqrt{\frac{e^x}{3}}$$

$$\phi'(x) = \frac{\sqrt{e^x}}{2\sqrt{3}} > 1$$

So, we cant use this scheme.

$$f(x) = e^x - 3x^2$$

$$e^x - 3x^2 = 0$$

$$\Rightarrow e^x = 3x^2$$

$$\Rightarrow x = \log(3x^2).$$

$$\Rightarrow x = \log 3 + 2\log(x)$$

$$\Rightarrow x = 0.4771 + 2\log(x).$$

$$\phi(x) = 0.4771 + 2\log(x).$$

$$\phi'(x) = \frac{2}{x}$$

$$|\phi'(x)|_{3.733} = 0.5357 < 1$$

$$|\phi'(x)|_{3.734} = 0.5356$$

$x_0 = 3.734$

$x_1 = \cancel{0.4771 + 2\log(x_0)}$ $\log_e(3x^2)$.

$= \cancel{0.4771 + 2\log(3.734)}$

$= 3.733036$

$x_2 = 3.73305$

★ # CALCULUS OF FINITE DIFFERENCES :-

Consider $(n+1)$ equally spaced points $x_0, (x_0+h)\cdot x_1$

$x_0 + 2h\ (x_2)$

$x_0 + nh\ (x_n)$

& the corresponding funct'al values

$y_0, y_1, \ldots, y_n$.

Here, the x values are values of the independent variable of the funct^n $y(x)$, known as arguement.

★ The corresponding y values are called entries.

★ The understand the change in dependent var 'y' w.r.t change in independent var 'x' is understood by means of calculus of finite differences whose important operators are given below.

(i) FORWARD DIFFERENCE OPERATOR $[\Delta]$ (ii)

$\downarrow$ DELTA

$$\boxed{\Delta y(x) = y(x+h) - y(x)}.$$

The FD is rep. by Δ.

Simil,

2^{nd} FD is rep by $\Delta^2 y(x) = \Delta[\Delta y(x)]$.

$$\Rightarrow \Delta^2 y(x) = y(x+2h) - 2y(x+h) + y(x).$$

★

x	y	Δy	$\Delta^2 y$
x_0	y_0		
		$y_1 - y_0 = \Delta y_0$	
x_1	y_1		$\Delta y_1 - \Delta y_0 = \Delta^2 y_0$
		$y_2 - y_1 = \Delta y_1$	
x_2	y_2		
		$y_3 - y_2 = \Delta y_2$	$\Delta y_2 - \Delta y_1 = \Delta^2 y_1$
x_3	y_3		

NOTE

(i) There wont be any. Δ for last arguements

(ii) It is simply next value - current value.

(iii) Δ is used For equally spaced arguements.

Q) Const. FDT for foll data :-

Difference Table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
2	10			
4	15	$5 = \Delta y_2$		
		$5 = \Delta y_4$	$0 = \Delta^2 y_2$	
6	20			$11 = \Delta^3 y_2$
8	36	$16 = \Delta y_6$	$11 = \Delta^2 y_4$	

(ii) BACKWARD DIFFERENCE OPERATOR $[\nabla]$.
↳ NABLA.

$$\boxed{\nabla y(x) = y(x) - y(x-h)}$$

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
x_0	y_0			
		$y_1 - y_0 = \nabla y_1$		
x_1	y_1		$\nabla y_2 - \nabla y_1 = \nabla^2 y_2$	
x_2	y_2	$y_2 - y_1 = \nabla y_2$		$\nabla^2 y_3 - \nabla^2 y_2 = \nabla^3 y_3$
			$\nabla y_3 - \nabla y_2 = \nabla^2 y_3$	
x_3	y_3	$y_3 - y_2 = \nabla y_3$		

$\nabla y(x_0) \rightarrow$ not possible

$\nabla y(x_1) = y(x_1) - y(x_0) = y_1$

★ The Backward Diff. Table with appropriate labels for the above data is shown below :-

(iii) SHIFTING OPERATOR [E]

$$E\,y(x) = y(x+h).$$

x	y
x_0	y_0
x_1	y_1
x_2	y_2

$E y_0 = y_1$

$E^2 y_0 = E(E y_0) = E y_1 = y_2$

$E^n y(x_0) = y(x_0 + nh).$

$$E^{-1} y(x) = y(x-h)$$

$$E^{-n} y(x) = y(x-nh).$$

★ NOTE

→ when data is generated from n^{th} degree polynomial, the n^{th} degree constants & all higher differences vanishes.

IMP → We can extend the table by using the above prop. as shown in foll eq.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	2			
		6		
2	8		8	
		14		0.
4	22		8	
		22.		
6	44		8.	
		$\Delta y_6 = 30$		
8	$y_8 = 72$			

HW Estimate the value of y_{14}, y_{+2}, y_{-6} for the above problem?

* INTERPOLATION ..

Let (x_i, y_i) be the tabulated values of the funct^n, $y(x)$ in the interval $[x_0$ to $x_n]$. The process of estimating the values of y at non-tabulated points within the interval $[x_0 ; x_n]$ is called interpolation.

NOTE.

Here, we will fit the given data with a polynomial using which, y values at non-tabulated points can be obtained.

→ There are 2 types of data :-

(i) Equally Spaced

(ii) Unequally spaced.

if Data is equally spaced, we have the foll. 2 formulae:-

★ (i) Newton-Gregory forward Interpolat^n.

(ii) Newton-Gregory Backward Interpolation.

for unequally spaced data, we hv.

★ Newton's Divided Difference Interpolat^n.

NOTE

* if the non-tabulated pt. is at the beginning of the interval i.e close to x_0, we will use Newton-Gr— forward interpolat^n.
* if its close to x_n, we will use Newton-Greg— Backward Interpolat^n.

★ N-Gregory Forward Interpolatⁿ Formula :-

Let the $y = f(x)$, Let $f(a), f(a+h), f(a+2h)$ ... $f(a+nh)$ be the values of y at $x = a, a+h,$ $a+2h$ .... $a+nh$.

- We know that, the degree of the poly. joining $(n+1)$ points is 'n'. Let the poly. be

$$f(x) = A_0 + A_1(x-a) + A_2(x-a)(x-(a+h))$$
$$+ A_3(x-a)(x-(a+h))(x-(a+2h)) + \dots$$
$$\dots + A_n(x-a)(x-a-h) \dots (x-a-(n-1)h) \quad \text{--- (1)}$$

where A_0, A_1, A_2 .... are unknowns to be obtained.

★ We will substitute $x = a, a+h, a+2h$ .... successively in eqn (1) to obtain unknowns.

Put $x = a$ in (1).

$\Rightarrow f(a) = A_0$.

Put $x = a+h$.

$$\Rightarrow f(a+h) = A_0 + A_1(\cancel{a}+h-\cancel{a})$$
$$= A_0 + hA_1$$

$$\Rightarrow f(a+h) = f(a) + A_1 \cdot h$$

$$\Rightarrow \boxed{A_1 = \frac{f(a+h) - f(a)}{h}}$$

$$\Rightarrow \boxed{A_1 = \frac{\Delta f(a)}{1!h}}$$

Put $x = a + 2h$.

$$f(a+2h) = A_0 + A_1(a+2h-a) + A_2(a+2h-a)(a+2h-a-h)$$

$$= A_0 + A_1 \cdot 2h + A_2 \, 2h \cdot h$$

$$= f(a) + \left\{ \frac{\Delta f(a)}{\cancel{h}} \right\} 2\cancel{h} + 2h^2 A_2$$

$$= f(a) + 2\{f(a+h) - f(a)\} + 2h^2 A_2$$

$$f(a+2h) = 2f(a+h) - f(a) + 2h^2 A_2$$

$$\Rightarrow A_2 = \frac{f(a+2h) - 2f(a+h) + f(a)}{2h^2}$$

$$\Rightarrow \boxed{A_2 = \frac{\Delta^2 f(a)}{2h^2}}$$

Proceeding as above, we get :-

$$\boxed{A_3 = \frac{\Delta^3 f(a)}{3!\, h^3}}$$

IMP

$$\boxed{A_n = \frac{\Delta^n f(a)}{n!\, h^n}}$$

* Substituting $A_1, A_2 \dots A_n$ in eq^n (1) we get.

* We know that every no. 'x' can be rep. as $x = x_0 + uh$
$= a + uh$

$$\Rightarrow \boxed{u = \frac{x-a}{h} = \frac{x - x_0}{h}}$$

$$f(x) = f(a+uh) =$$

$$f(a) + \frac{(x-a)}{h}\Delta f(a) + \frac{(x-a)(x-a-h)}{2!\,h^2}\Delta^2 f(a)$$

$$+ \frac{(x-a)(x-a-h)(x-a-2h)}{6h^3}\Delta^3 f(a) + \dots$$

$$\dots \frac{(x-a)(x-a-2h)\dots(x-a-(n-1)h)}{n!\,h^n}\cdot \Delta^n f$$

$$= f(a) + \frac{(uh)}{h}\Delta f(a) + \frac{uh(uh-h)}{2!\,h^2}\Delta^2 f(a) + \dots$$

$$+ \dots \frac{(uh)(uh-h)(uh-2h)\dots(uh-(n-1)h)}{n!\,h^n}$$

$$f(x) =$$

$$f(a+uh) = f(a) + u\cdot\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}$$

$$+ \dots + \frac{u(u-1)(u-2)\dots(u-\overline{n-1})}{n!}\Delta^n f(a).$$

HW.

Q) The populat^n^ of a town is given below, estimate the populat^n^ in 1895

Year	1891	1901	1911	1921	1931
Populat^n^	46	66	81	93	101

$y(1895) = ??$

$h = 10$ $\quad u = \dfrac{x-a}{h} = 0.4$

$a = 1891$

$x = 1895$

Q.

x	y
0	5
1	3
2	7
3	23

The given data is equally spaced with stepsize = 1 & $x_0 = 0$.
We need to find $y(x)$ at $x = 0.5$, since, 0.5 is close to x_0, we use Newton-Gregory forward interpolatⁿ formula, given by :-

$$f(x) = f(x_0) + u\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) + \cdots + \frac{u(u-1)\cdots(u-\overline{n-1})}{n!}\Delta^n f(x_0).$$

where $u = \frac{x - x_0}{h} = \frac{0.5 - 0}{1} = 0.5$.

We const. the forward diff. table as given below.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	$\boxed{5}$ $y(x_0)$			
		$\boxed{-2}$ $\Delta y(x_0)$		
1	3		$\boxed{6}$ $\Delta^2 y(x_0)$	
		4		$\boxed{6}$ $\Delta^3 y(x_0)$.
2	7		12	
		16		
3	23			

$$f(0.5) = 5 + (0.5)(-2) + \frac{(0.5)(-0.5)}{2!}\cdot 6 + \frac{(0.5)(-0.5)(-1.5)}{3!}(6)$$

$$= 5 - 1 - 0.75 + 0.375 = \underline{3.625}$$

Q. Identify the polynomial governing the data in above problem.....

Ans Since, the poly. is req.

$$u = \frac{x - x_0}{h} = \frac{x - 0}{1} = x$$

We substitute $u = x$ & the corresponding diff. from the diff. table in the Newton-Gregory forward interpolat^n formula.

$$f(x) = 5 + x(-2) + \frac{x(x-1)(6)}{2} + \frac{x(x-1)(x-2)}{\cancel{6}}\cancel{6}$$

$$= 5 - 2x + 3x(3x-3) + x(x-1)(x-2)$$

$$= \underline{x^3 - 3x + 5}.$$

★ NEWTON-GREGORY BACKWARD INTERPOLATION

~~Her~~ This tech. is used when the non-tabulated is close to the last pt. of the data.

Here, $\boxed{u = \frac{x - x_n}{h}}$ $x_n \rightarrow$ reference point.

u is always $\underline{-ve}$.

Formula :-

$$f(x) = f(x_n + uh) = f(x_0 + nh + uh).$$

$$= f(x_n) + u\nabla f(x_n) + \frac{u(u+1)}{2!}\nabla^2 f(x_n) + \frac{u(u+1)(u+2)}{3!}\ldots$$

$$+ \ldots + \frac{u(u+1)(u+2)\ldots(u+\overline{n-1})}{n!}\nabla^n f(x_n)$$

Q. F

f(x

Q. The i to 55

8. Find the polynomial governing the data in prev. using N-G Backward itp.

x	y	∇y	$\nabla^2 y$	
0	5			
		-2		
1	3		6	
		4		$\boxed{6}$ $\nabla^3 y(x_n)$
2	7		$\boxed{12}$ $\nabla^2 y(x_n)$	
		$\boxed{16}$		
3	$\boxed{23}$ $y(x_n)$	$\nabla y(x_n)$		

$u = \frac{x-3}{1} = \underline{x-3}$

$f(x) = 23 + (x-3)16 + \frac{(x-3)(x-2)}{2}12 + \frac{(x-3)(x-2)(x-1)}{3!}6$

$= 23 + 16x - 48 + 6(x^2 - 5x + 6) + (x^3 \ldots)$

$= \underline{\underline{x^3 - 3x + 5}}$

5. The foll. table gives the scores obtained by 100 students in a particular subject. Use N. forward formula to find the no. of students who secured more than 55 marks.

Scores	no. of stu
30 - 40	25
40 - 50	35
50 - 60	22
60 - 70	11
70 - 80	7

The given table can be rearranged.

Ans

(X): score less than Scores		(Y) no. of stude	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_0 \to$	less than 40	$\boxed{25}$				
			$\boxed{35}$			
x .	less than 50 (55)	60		$\boxed{-13}$		
	" " 60		22		$\boxed{2}$	
		82		-11		$\boxed{5}$
	" " 70	93	11		7	
				-4		
	" " 80	100	7			

$h = 10$.

Here, $x_0 = 40$ & $x = 55$.

$$u = \frac{x - x_0}{h} = \frac{15}{10} = 1.5$$

$$f(55) = f(x_0) + u\Delta f(x_0) + \frac{u(u-1)}{2}\Delta^2 f(x_0) + \ldots$$

$$= 25 + 1.5(35) + \frac{(1.5)(0.5)}{2}(-13) + \frac{(1.5)(0.5)(-0.5)}{6}(2)$$

$$+ \frac{(1.5)(-1.5)(0.5)(-0.5)}{4!}(5).$$

$$= 25 + 52.5 - 4.875 - 0.125 + 0.1171875$$

$$= 72.617 \approx 73 \text{ students}$$

$\therefore$ Students scoring more than 55 $= 100 - 73$

$= 27$ students

NOTE:

$$\Delta^n y_0 = y_n - {}^nC_1\, y_{n-1} + {}^nC_2\, y_{n-2} - {}^nC_3\, y_{n-3} \cdots (-1)^n y_0$$

$n+0$

$$\Delta y_0 = y_1 - y_0$$

$$\Delta^2 y_0 = y_2 - 2y_1 + y_0.$$

Q. u_x is a funct^n of x for which the 5th diff. are const. &

$$u_1 + u_7 = -786$$

$$u_2 + u_6 = 686$$

$$u_3 + u_5 = 1088.$$

Find u_4 ?

$$\Delta^6 u_1 = 0.$$

$$\Rightarrow \Delta^6 u_1 = u_7 - {}^6C_1 \cdot u_6 + {}^6C_2\, u_5 - {}^6C_3\, u_4 + {}^6C_4\, u_3 - {}^6C_5\, u_2 + {}^6C_6\, u_1 = 0.$$

$$\Rightarrow u_7 - 6u_6 + 15u_5 - 20u_4 + 15u_3 - 6u_2 + u_1 = 0.$$

$$\Rightarrow (u_1 + u_7) - 6(u_6 + u_2) + 15(u_5 + u_3) - 20u_4 = 0$$

$$\Rightarrow 20u_4 = -786 - 6(686) + 15(1088)$$

$$= u_4 = \frac{-4902 + 16320}{20}$$

$$= \frac{11418}{20} = \underline{570.9}$$

★ RELATION BET^N OPERATORS :-

(1) $\Delta = E - 1$.

LHS

$$\Delta f(x) = f(x+h) - f(x)$$

$$= E f(x) - f(x)$$

$$\Delta f(x) = (E-1) f(x).$$

∵ LHS = RHS

$$\Rightarrow \boxed{E = 1 + \Delta}$$

(A)

(2) $\nabla = 1 - E^{-1}$.

LHS

$$\nabla f(x) = f(x) - f(x-h).$$

$$\Rightarrow \quad = f(x) - E^{-1} f(x)$$

$$\Rightarrow \nabla f(x) = (1 - E^{-1}) f(x)$$

$$\Rightarrow \boxed{E^{-1} = 1 - \nabla}$$

(3) $\Delta = E\nabla = \nabla E$.

LHS

$$\Delta f(x) = f(x+h) - f(x). \quad \text{——— (1)}$$

RHS

$$(E\nabla) f(x)$$

$$= E(\nabla f(x))$$

$$= E[f(x) - f(x-h)]$$

$\Rightarrow (E\nabla) f(x) = f(x+h) - f(x)$.

LHS.

RHS ② $(\nabla E) f(x) = \nabla [f(x+h)]$

$= f(x+h) - f(x+h-h)$

$= f(x+h) - f(x)$.

(4) $\boxed{E = e^{hD}}$ where $D = \frac{d}{dx}$

consider $Ef(x) = f(x+h)$.

★ Apply Taylor Series Expansion

$$\Rightarrow Ef(x) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots$$

$$\cdots + \frac{h^n}{n!} f^n(x).$$

$$= f(x) + hDf(x) + \frac{h^2}{2!} D^2 f(x) + \cdots + \frac{h^n}{n!} D^n f(x).$$

$$= \left[1 + hD + \frac{(hD)^2}{2!} + \frac{(hD)^3}{3!} + \cdots + \frac{(hD)^n}{n!}\right] f(x).$$

$$\boxed{Ef(x) = e^{hD} f(x).}$$

★ CENTRAL DIFFERENCE OPERATOR (δ).

The central diff. formula is given by

$$\boxed{y_n - y_{n-1} = \delta y_{n-\frac{1}{2}}}$$

$\Rightarrow \delta y_x = y$

★ AVERAGING OPERATOR :– (μ)

$$\mu\, y(x) = \frac{1}{2}\left[y_{x+\frac{h}{2}} + y_{x-\frac{h}{2}}\right]$$

same,

$$\mu\, f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)\right].$$

Q. Prove that :

$$\delta = E^{1/2} - E^{-1/2}.$$

We know,

$$\delta y_x = y_{x+\frac{h}{2}} - y_{x-\frac{h}{2}}.$$

$$= E^{1/2} y_x - E^{-1/2} y_x.$$

$$\boxed{\delta y_x = \left[E^{1/2} - E^{-1/2}\right] y(x)}$$

$\boxed{\text{Q. Pg. 205 to 224.}}$ → Solve all problems.

8) :-

$= y_{x+\frac{h}{2}} - y_{x-\frac{h}{2}}$

Q. The table given below gives the values of tan x in the interval 0.1 to 0.3.

x	$y = \tan x$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.1	$\boxed{0.1003}$				
		$\boxed{0.0508}$			
0.15	.1511		$\boxed{0.0008}$		
		0.0516		$\boxed{0.0002}$	
0.2	.2027		0.001		$\boxed{0.0002}$
		0.0526		0.0004	
0.25	.2553		0.0014.		
		0.054			
0.3	.3093.				

Estimate the value of 0.12 using forward interpolation?

$h = 0.05$

$x_0 = 0.1$

$$u = \frac{x - x_0}{h} = \frac{0.12 - 0.1}{0.05} = 0.4$$

$$f(1.2) = f(x_0) + u\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) \ldots$$

$$\Rightarrow f(1.2) = 0.1003 + 0.4(0.0508) + \frac{(0.4)(-0.6)}{2}[0.0008]$$

$$+ \frac{0.4(-0.6)(-1.6)}{3!}[0.0002] + \frac{(0.4)(-0.6)(-1.6)(-2.6)}{4!}[0.0002]$$

$$= 0.1003 + 0.02032 - 9.6\times10^{-5} + 1.28\times10^{-5} + 8.32\times10^{-6}$$

$\tan(0.12) = 0.120545$