★ DIVIDED DIFFERENCES :—

We use divided differences on unequally spa[ced]
data.

NOTE
This is the master formula which can be applied on every data set.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_0	y_0			
		$\frac{y_1 - y_0}{x_1 - x_0} = \Delta y_0$	$\frac{\Delta y_1 - \Delta y_0}{x_2 - x_0} = \Delta^2 y_0$	
x_1	y_1			$\frac{\Delta^2 y_1 - \ldots}{x_3 - \ldots}$
		$\frac{y_2 - y_1}{x_2 - x_1} = \Delta y_1$		$= \ldots$
x_2	y_2		$\frac{\Delta y_2 - \Delta y_1}{x_3 - x_1} = \Delta^2 y_1$	
		$\frac{y_3 - y_2}{x_3 - x_2} = \Delta y_2$		
x_3	y_3			

Const. of Divided Diff. Table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	6			
		$\frac{9-6}{3-0} = 1$		
3	9		$\frac{6-1}{4-0} = 1.25$	$\frac{-\frac{1}{3} - 1.25}{6-0} = -0.264$
		$\frac{15-9}{4-3} = 6$		
4	15		$\frac{5-6}{6-3} = \frac{-1}{3}$	
		$\frac{25-15}{6-4} = 5$		$\frac{\frac{8}{3} + \frac{1}{3}}{7-3} = 0.75$
6	25		$\frac{13-5}{7-4} = \frac{8}{3}$	
		$\frac{38-25}{7-6} = 13$		
7	38		$\frac{4-13}{10-6} = \frac{-9}{4}$	$\frac{\frac{-9}{4} - \frac{8}{3}}{10-4} = -0.819$
		$\frac{50-38}{10-7} = 4$		
10	50			

* Properties of Divided Diff :-

1) The div. diff. are symmetric about its arguements i.e $[x_0\ x_1] = [x_0\ x_1]$.

Proof Consider

$$[x_0\ x_1] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{y_1}{x_1 - x_0} - \frac{y_0}{x_1 - x_0}$$

$$= \frac{y_0}{x_0 - x_1} - \frac{y_1}{x_0 - x_1}$$

$$= \frac{y_0 - y_1}{x_0 - x_1} = [x_1\ x_0].$$

This is called as symmetricity.

2) The n^{th} divided diff of an n^{th} degree polynomial are constants.

* Newton's Divided Difference Interpolation Formula :-

* Let $y_0, y_1, y_2 \cdots y_n$ be the values of $y = f(x)$ at the points $x_0, x_1, \cdots x_n$.

* Consider the first divided diff. $[x\ x_0] = \dfrac{y - y_0}{x - x_0}$

$$\Rightarrow y - y_0 = (x - x_0)[x\ x_0].$$

$$\Rightarrow y = y_0 + (x - x_0)[x\ x_0]. \quad \text{—— (1)}$$

* Consider the second div. diff :-

$$[x\ x_0\ x_1] = \frac{[x\ x_0] - [x_0\ x_1]}{x - x_1}$$

$$\Rightarrow [x\ x_0] = [x_0\ x_1] + (x - x_1)[x\ x_0\ x_1]$$

$$\therefore (1) \Rightarrow y - y_0 = (x - x_0)[x_0\ x_1] + (x - x_0)(x - x_1)[x\ x_0\ x_1].$$

(2)

Consider $[x \; x_0 \; x_1 \; x_2] =$

$$\frac{[x \; x_0 \; x_1] - [x_0 \; x_1 \; x_2]}{x - x_2}$$

$$\Rightarrow [x \; x_0 \; x_1] = [x_0 \; x_1 \; x_2] + (x - x_2) \cdot [x \; x_0 \; x_1 \; x_2]$$

$\Rightarrow$ 2 $\Rightarrow$

$$y = y_0 + (x - x_0)[x_0 - x_1] + (x - x_0)(x - x_1)[x_0 \; x_1 \; x_2] + (x - x_0)(x - x_1)(x_1 - x_2)[x \; x_0 \; x_1 \; x_2]$$

3

★ Proceeding like above, we get :-

$$y = y_0 + (x - x_0)\Delta y_0 + (x - x_0)(x - x_1)\Delta^2 y_0 + (x - x_0)(x - x_1)(x - x_2)\Delta^3 y_0 + \cdots + \cdots + (x - x_0)(x - x_1)(x - x_2)\cdots(x - x_{n-1})\Delta^n y_0$$

Q:

	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_0	0	$\boxed{5}$ y_0			
			$\frac{-2}{1} = \boxed{-2}$ Δy_0		
x_1	1	3		$\boxed{3}$ $\Delta^2 y_0$	
			$\frac{4}{2} = 4$		$\boxed{1}$ $\Delta^3 y_0$
x_2	2	7		6	
			16		
x_3	3	23.			

$y = y_0 + (x - x_0)\Delta y_0 + \ldots$

$\Rightarrow y = 5 - 2x + (x)(x-1)\,3 + x(x-1)(x-2)$ $\quad (x^2 - 3x + 2)$

$= 5 - \cancel{2x} + \cancel{3x^2} - 3x + x^3 - \cancel{3x^2} + \cancel{2x}$

$= x^3 - 3x + 5$

* NUMERICAL DIFFERENTIATION :-

We learn the procedures to find the derivatives at tabulated or non-tabulated points (till 2nd derivatives). If the pt at which the derivative is required is close to 'x_0', we use the derivative formula generated from Newton-Gregory forward interpolation.

If those pts are close to x_n, we use the derivative formulae generated from Newton-Gregory Backward Interpolation.

* The derivative formula generated using N-G forward interpolatⁿ :-

The N-G forward Ip formula is given by :-

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots + \frac{u(u-1)(u-2)\ldots(u-n+1)}{n!}\Delta^n y_0 \quad \text{--- (1)}$$

where $u = \dfrac{x - x_0}{h}$ --- (2)

$$\frac{dy}{du} = \Delta y_0 + \left(\frac{2u-1}{2}\right)\Delta^2 y_0 + \left(\frac{3u^2-6u+2}{3!}\right)\Delta^3 y_0 + \cdots$$

$$\frac{du}{dx} = \frac{1}{h}.$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{h}\left[\Delta y_0 + \left(\frac{2u-1}{2}\right)\Delta^2 y_0 + \left(\frac{3u^2-6u+2}{6}\right)\Delta^3 y_0 + \cdots\right.$$

which gives the first derivative at non, tabulated p[oint] close to x_0.

NOTE

Substituting $u = 0$,

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

which gives the first derivative at tabulated point.

★ To compute 2^{nd} derivative,

diff. eq^n (3) wrt u & multiply $\frac{du}{dx}$.

because $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{du}\left(\frac{dy}{dx}\right) \cdot \frac{du}{dx}.$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \left(\frac{6u^2-18u+11}{12}\right)\Delta^4 y_0 + \cdots\right.$$

which gives the derivate -2- at non-tabulated pts.

Substituting $u = 0$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \cdots\right] \quad \text{—— (6)}$$

which gives 2^{nd} derivative at tabulated pts.

* Derivative formula using Newton-Gregory Backward Interpolation :—

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \left(\frac{2u+1}{2}\right)\nabla^2 y_n + \left(\frac{3u^2+6u+2}{3!}\right)\nabla^3 y_n + \cdots\right] \quad (7)$$

Put $u = 0$

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \cdots\right] \quad (8)$$

Eqn (7) & (8) gives the formula for 1st derivative at non-tabulated & tabulated pts respt using backward interpolation.

Similarly,

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + (u+1)\nabla^3 y_n + \left(\frac{6u^2+18u+11}{12}\right)\nabla^4 y_n + \cdots\right] \quad (9)$$

Put $u = 0$;

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots\right] \quad (10)$$

gives 2nd derivative at non-tabulated & tabulated pts using backward Ip.

NOTE

The above formulae are valid for equally spaced data.

For unequally spaced data, we construct the Ip polynomial using N. Div. Diff formula & differentiate it based on the requirement & substitute the points at which the derivatives are required.

Q. Find $\frac{dy}{dx}$ at $x = 0.1$ from the data

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0.1	$\boxed{0.9975}$			
		$\boxed{-0.0075}$		
0.2	0.99		$\boxed{-0.0049}$	
		-0.0124		$\boxed{0.0001}$
0.3	0.9776		-0.0048	
		-0.0172		
0.4	0.9604			

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0\right]$$

$$= \frac{1}{0.1}\left[0.9975 - \frac{1}{2}(-0.0049) + \frac{1}{3}(0.0001)\right]$$

$$= -0.0501$$

from above prob. $y'(0.2)$ is cal. using formula

$$\left.\frac{dy}{dx}\right|_{x=0.2} = \frac{1}{h}\left[0.99 - \frac{1}{2}(-0.0124) + \frac{1}{3}(-0.0048)\right.$$

$$= \frac{1}{0.1}\left[\right.$$

Q. The table given below gives velo. v of a body during time t. find acce. at t = 1.1 & 1.15.

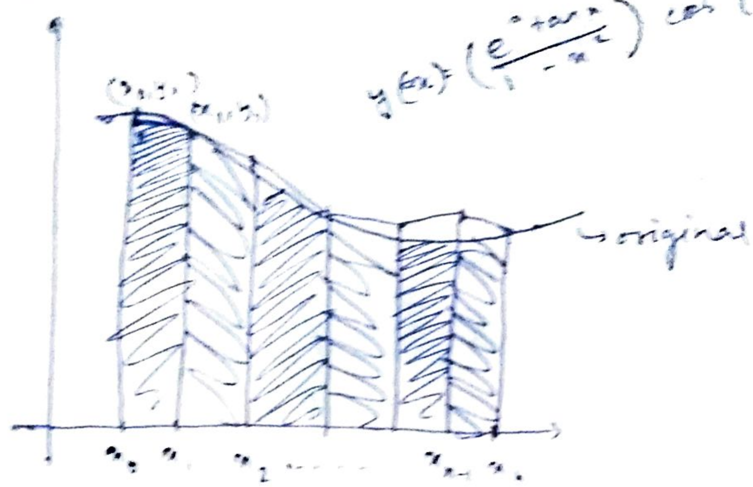
45.166

t:	1	1.1	1.2	1.3	1.4
v:	43.1	47.7	52.1	56.4	60.8

NUMERICAL INTEGRATION.

Given, a set of tabulated values of the integrand $f(x)$. To determine $\int_{x_0}^{x_n} f(x)\,dx$ is called numerical integration.

$\int^{1} \left(\frac{e^{\tan x}}{1-x^2}\right) \cos^{-1}(x)\, dx$



We subdivide the given interval into a no. of sub intervals of equal width 'h' & replace the funct^n tabulated at the pts of subdivision by any of the interpolating polynomial like N-G forward, N-G backward etc.

★ General ~~Ex~~ Quadrature formula/ Newton - Cotes Quadrature formula.

Let $I = \int_a^b y\,dx$; where y takes values $y_0, y_1, \ldots y_n$

for $x = x_0, x_1, x_2 \ldots x_n$. Let the interval of integration (a, b) be divided into 'n' equal sub-intervals each of with $\boxed{h = \frac{b-a}{n}}$, so, then

$x_0 = a$; $x_1 = a + h$; $x_2 = a + 2h$; ---- ; $x_n = a + nh$.

$\therefore \quad I = \int_{x_0}^{x_0+nh} f(x)\cdot dx$

Since, every x can be written as $x = x_0 + rh$.

& $dx = h \cdot dr$.

$$I = h\int_0^n f(x_0 + rh) \cdot dr$$

Using Newton forward Ip formula :-
interpolat^n

$$I = h\int_0^n \left[y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots + \frac{r(r-1)(r-2)\cdots(r-\overline{n-1})}{n!}\Delta^n y_0 \right] dr$$

$$= h\left[ry_0 + \frac{r^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{r^3}{3} - \frac{r^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{r^4}{4} - r^3 + r^2\right)\Delta^3 y_0 \cdots \right]$$

$$\Rightarrow I = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots \right] \quad (1)$$

Eq^n (1) is called General Quadrature formula or Newton - Cotes formula.

NOTE

A no. of imp. reductions who are derived from the above formula which can be directly used in the integration. The imp. among them are :-

* Trapezoidal Rule. (n = 1)
* Simpson's One third Rule (n = 2)
* Simpson's 3/8th Rule (n = 3).

* Trapezoidal Rule (n = 1)

Put n = 1 in formula (1), & taking the curve joining points $(x_0\ y_0)$ $(x_1\ y_1)$ as a polynomial of degree (1), all higher order diff. should become 0.

∴ We write

$$\int_{x_0}^{x_n} y(x) \cdot dx = \int_{x_0}^{x_1} y \cdot dx + \int_{x_1}^{x_2} y \cdot dx + \int_{x_2}^{x_3} y \cdot dx + \cdots + \int_{x_{n-1}}^{x_n} y \cdot dx$$

Consider,

$$\int_{x_0}^{x_1} y \cdot dx = 1 \cdot h \cdot \left[y_0 + \frac{1}{2}\Delta y_0 \right] = h\left[y_0 + \frac{(y_1 - y_0)}{2} \right]$$

$$h\left[\frac{2y_0+y_1-y_0}{2}\right] = \frac{h}{2}\left[y_0+y_1\right].$$

Consider $\int_{x_1}^{x_2} y\,dx = \int 1 \cdot h\left[y_1+\frac{1}{2}\Delta y_1\right] = \frac{h}{2}\left[y_1+y_2\right].$

⋮

We get :-
$$\int_{x_{n-1}}^{x_n} y \cdot dx = \frac{h}{2}\left[y_{n-1}+y_n\right].$$

Adding all values :-

We get :-
$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}\left[y_0+y_1\right] + \frac{h}{2}\left[y_1+y_2\right] + \dots + \frac{h}{2}\left[y_{n-1}+y_n\right].$$

$$\boxed{\int_{x_0}^{x_n} y \cdot dx = \frac{h}{2}\left\{\left[y_0+y_n\right] + 2\left[y_1+y_2+\dots+y_{n-1}\right]\right\}}$$

This formula is called Trapezoidal Rule.

Similarly :-
by taking $n=2$ & $n=3$; we get;
Simpson's One-Third Rule & Simpson's 3/8th rule which are shown below.

Simpson's 1/3rd Rule ($n=2$).

$$\int_{x_0}^{x_n} f(x) \cdot dx = \frac{h}{3}\left[(y_0+y_n) + 4(y_1+y_3+y_5+\dots y_{n-1}) + 2(y_2+y_4+\dots y_{n-2})\right].$$

Simpson's 3/8th Rule ($n=3$).

$$\int_{x_0}^{x_n} f(x) \cdot dx = \frac{3h}{8}\left[(y_0+y_n) + 3(y_1+y_2+y_4+y_5+y_7+y_8+\dots+y_{n-2}+y_{n-1}) + 2(y_3+y_6+y_9+\dots y_{n-3})\right]$$

NOTE:
Simpson's $\frac{1}{3}$rd Rule is applicable when no. of subint are even. & Simpson's $3/8$th rule is applied when no. of subintervals are multiples of 3.

Q. Estimate the value of pi by evaluating $\int_0^1 \frac{1}{1+x^2}\,dx$ using (i) Simpsons $\frac{1}{3}$rd rule with $h = \frac{1}{4}$

(ii) " $3/8$th " " $h = 1/6$.

0.5 ~~[illegible]~~ 0. ~~518~~

Q. $x \cdot e^x - \cos x = 0$.

$f(0) = -1$ $\quad f(0.5) \cdot$ $\quad [0.5\cancel{17} \quad 0.518]$

$f(1) = 2.177$.

x

$x_0 = 0.517$.

-2.3025×10^{-3}

$x_2 = x_0 - \cancel{f(x_0)} \frac{x_1 - x_0}{f(x_1) - f(x_0)} \cdot f(x_0)$.

$x_2 = 0.517 - \frac{0.001}{3.04077 \times \cancel{10^{-3}}} \times (-2.3025 \times \cancel{10^{-3}})$

$= 0.517 + \cdots$

$= 0.51707$

$[2.279 \quad 2.\cancel{28}0]$.

Q. $f(x) = x^3 - 3x - 5$

$f(0) = -5$ $\quad f(1) = 1 - 3 - 5 = -7$.

$f(2) = 8 - 6 - 5 = -3$ $\quad f(3) = 27 - 9 - 5 = \cancel{13}$

$3x^2 - 3$.

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$.

$= 2.279 + \frac{2.36361 \times 10^{-4}}{12.5815}$

$=$

Evaluate $\int_0^1 x^3\,dx$ using Simpson's $1/3^{rd}$ & Trapezoidal rule in 4 sub-intervals. $y(x) = x^3$

$n = 4$

$h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$

x	y
0	0
1/4	0.015625
2/4	0.125
3/4	0.4218
1	1

Use Trapezoidal

$$\int_0^1 x^3\,dx = \frac{1}{8}\left[(0+1) + 2(0.015625 + 0.125 + 0.4218)\right]$$

$= 0.2656$

Use Simpson's $\frac{1}{3}^{rd}$

$$\int_0^1 x^3\,dx = \frac{1}{4\times 3}\left[(0+1) + 4(0.015625 + 0.4218) + 2(0.125)\right]$$

$= 0.25$

consider $h = 0.1$

x	y
0	0
0.1	0.001
0.2	0.008
0.3	0.027
0.4	0.064
0.5	0.125
0.6	0.216
0.7	0.343
0.8	0.512
0.9	0.729
1	1

Trap

$$\int_0^1 x^3\,dx = \frac{0.1}{2}\left[(0+1) + 2(2.025)\right]$$

$= 0.2525$

Q. Estimate the value of π by evaluating $\int_0^1 \frac{1}{1+x^2}dx$ Using Simpsons $1/3^{rd}$ rule.

$h = \frac{1}{4}$.

x	y
0	1
$1/4$	0.941176
$2/4$	0.8
$3/4$	0.64
1	0.5

$$\int \frac{1}{1+x^2} = \left[\tan^{-1}x\right]_0^1$$

$$= \tan^{-1}1 - \tan^{-1}0$$

$$= \frac{\pi}{4}$$

$$\int_0^1 \frac{1}{1+x^2}dx = \frac{0.25}{3}\left[(1+0.5) + 4(0.941176 + 0.64) + 2(0.8)\right]$$

$$= 0.785392.$$

$$\therefore \frac{\pi}{4} = 0.785392$$

$$\pi = 3.141568$$

Simpson's $3/8^{th}$ $\rightarrow h = \frac{1}{6}$.

x	y
0	1
$\frac{1}{6}$	0.972972
$\frac{2}{6}$	0.9
$\frac{3}{6}$	$\boxed{0.8}$
$\frac{4}{6}$	0.692307
$\frac{5}{6}$	0.590163
1	0.5

$$\frac{3}{8} \times \frac{0.1}{6}\left[(1+0.5) + 3(9.466326) + 1.6\right]$$

$$= \frac{3}{8\times 6}\left[12.566326\right]$$

$$= 0.785395$$

Ans

$\pi = 3.1415815$

Q. The speed 'v' m/sec of a car, t sec after it starts is shown in the foll. table :-

t	v
0	0 (y_0)
12	3.6
24	10.08
36	18.9
48	21.6
60	18.54
72	10.26
84	5.40
96	4.50
108	5.40
120	9 (y_n)

Using Simpsons rule find the dist travelled by car in 2 min?

after
Pg - ~~...~~ 450

Ans.

$h = 12$

$$\therefore I = \frac{12}{3}\left[(9) + 4(51.84) + 2(46.44)\right]$$

$$= 4\,[309.24]$$

$$= 1236.96 \text{ m}$$

Q. From the foll. values of f(x), find the posit^n of centroid of the area under the curve & the x-axis

x	y
0	1
1/4	4
1/2	8
3/4	4
1	1

NOTE

Centroid of the plane area under the curve $y = f(x)$ is given by

$$\bar{x} = \frac{\int_0^1 xy\,dx}{\int_0^1 y\,dx} \qquad \bar{y} = \frac{\int_0^1 \frac{y^2}{2}\,dx}{\int_0^1 y\,dx}$$

Ans

x	y	xy	$y^2/2$
0	10	0	1/2
1/4	4	1	8
1/2	8	4	32
3/4	4	3	8
1	1	1	1/2.

★ ORDINARY DIFF. EQUATIONS :-

★ Initial Value Problem :-

The probl^m of finding the sol^n of a DE with the condit^ns given at the same point is called Initial Value Problem.

★ The general initial value problem of order ① is given by :-

$$\frac{dy}{dx} = f(x,y) \; ; \; y(x_0) = y_0.$$

NOTE :-

★ Without initial condit^n, the ODE can't be solved in Numerical Methods.

★ The sol^n of num. methods is a particular sol^n, i.e a numeric value.

• The foll. methods are used to find the sol^n of ODE :-

1) PICCARD'S SUCCESSIVE APPROXIMATION METHOD :-

$$\boxed{y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)}) \cdot dx.}$$

Q. Find the value of $y(0.2)$ from the eq^n $\frac{dy}{dx} - x + y = 0$; $y(0) = 1$ [Use 3 approx].

$$\therefore \frac{dy}{dx} = x - y.$$

$$f(x, y) = x - y \quad ; \quad x_0 = 0 \;\&\; y_0 = 1.$$

★ The first approx. is :-

$$y^{(1)} = y_0 + \int_{x_0}^{x} (x - y_0)\,dx$$

$$= 1 + \int_0^x (x - 1)\,dx$$

$$= 1 - x + \frac{x^2}{2}$$

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\,dx$$

$$= y_0 + \int_0^x x - 1 + x - \frac{x^2}{2}$$

$$= y_0 + \int_0^x \left[-1 + 2x - \frac{x^2}{2} \right] dx$$

$$= 1 - x + x^2 - \frac{x^3}{6}$$

$$-1 + 2x - x^2 + \frac{x^3}{6}$$

$$y^{3} = y_0 + \int_0^x f(x, y^2)\cdot dx$$

$$= 1 + \int_0^x \left[x - 1 + x - x^2 + \frac{x^3}{6} \right] dx$$

$$= 1 + \left[-x + x^2 - \frac{x^3}{3} + \frac{x^4}{24} \right]$$

$$= 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{24}.$$

The sol^n table is shown below :—

x	$y^{(1)}$	$y^{(2)}$	$y^{(3)}$
0.2	0.82	0.83860	0.8394

In interval [0 1]

x	y
0	
1/4	
2/4	
3/4	
1	

Proof

$$\frac{dy}{dx} + y = x$$

$\Rightarrow (D+1)y = x \; ; \; y(0) = 1$

$\Rightarrow f(D)y = Q \; ; \; y(0) = 1$

AE $\Rightarrow m+1 = 0 \Rightarrow m = -1$

$y_c = C_1 . e^{-x}$.

$y_p = \frac{1}{D+1} x = (1+D)^{-1} x$

$= (1 - D + D^2 - D^3 + \ldots) x$

$= x - 1$

G.S

$$\boxed{y(x) = C_1 e^{-x} + x - 1}$$

$y(0) = 1$

for ~~x = 0.2~~

$\Rightarrow C_1 . e^0 + x - 1 = 1$

$\Rightarrow C_1 = 2$

$\therefore y(x) = 2e^{-x} + x - 1$

$\Rightarrow y(0.2) = 0.837461$

2) EULER'S METHOD :-

Consider $\frac{dy}{dx} = f(x, y)$; $y(x_0) = y_0$.

$$y_1 - y_0 = \left(\frac{dy}{dx}\right)_{(x_0, y_0)} (x_1 - x_0)$$

$$\Rightarrow y_1 = y_0 + h \cdot f(x_0, y_0).$$

x	y
x_0	y_0
x_1	$y_1 = y_0 + h \cdot f(x_0, y_0)$.
x_2	$y_2 = y_1 + h \cdot f(x_1, y_1)$
x_n	$\boxed{y_{n+1} = y_n + h \cdot f(x_n, y_n).}$

Q. Solve $\frac{dy}{dx} = (x - y)$ $y(0) = 1$ in the interval $[0\ 1]$.

$f(x, y) = x - y$; $x_0 = 0$; $y_0 = 1$.

Let $h = 0.25$.

$\therefore$ the soln table is.

x	y
0	1
0.25	$y_0 + h \cdot f(x_0, y_0) = 1 + 0.25(0 - 1) = 0.75$.
0.5	$y_1 + h \cdot f(x_1, y_1) = 0.75 + 0.25(0.25 - 0.75) = 0.625$
0.75	$0.625 + 0.25(0.5 - 0.625) = 0.59375$
1	$0.59375 + 0.25(0.75 - 0.59375) = 0.63281$

(iii) TAYLOR SERIES METHOD

Consider $\frac{dy}{dx} = f(x,y)$; $y(x_0) = y_0$ ——— (1)

$y(x)$ is the solⁿ for eqⁿ (1).

★ The taylor series representatⁿ of $y(x)$ about the initial pt x_0 is

$$y(x) = y(x_0) + (x-x_0)\,y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \cdots \quad \text{——— (2)}$$

Replacing x with x_1 :-

$$y(x_1)=y_1 = y_0 + hy'_0 + \frac{h^2}{2!}y''_0 + \frac{h^3}{3!}y'''_0 + \cdots \quad \text{——— (3)}$$

Similarly

$$y(x_2) = y_2 = y_1 + hy'_1 + \frac{h^2}{2!}y''_1 + \frac{h^3}{3!}y'''_1 + \cdots \quad \text{———}$$

$\vdots$

NOTE

★ We observe that the Euler's formula is 1st 2 term of taylor's formula & hence the accuracy of the solution is less compared to Taylor series form with more than 2 terms.

★ Euler's formula will give better result if we dec. the value of h.

Q) Solve $\frac{dy}{dx} = x + y$; $y(1) = 0$ upto 1.2 with $h = 0.1$

Compare with analytical sol^n.

x	y
1	0
1.1	0.11034
1.2	0.242803

$f(x, y) = x + y$

To find $y(1.1) = y_1$

$= y_1 = y_0 + h \cdot y_0' + \frac{h^2}{2} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv}$

$y' = x + y$

$\Rightarrow y_0' = x_0 + y_0 = 1$

$y'' = 1 + y' \Rightarrow y_0'' = 1 + y_0' = 2$

$y''' = y'' \Rightarrow y_0''' = 2$

$y^{iv} = y''' \Rightarrow y_0^{iv} = 2$

$\therefore y_1 = 0 + (0.1) 1 + \frac{0.01}{2} \cdot 2 + \frac{0.001}{6} \cdot 2 + \frac{0.0001}{24} \times 2$

by neglecting hod.

$= 0.11034$

To find $y(1.2) = y_2$

$\Rightarrow y_2 = y_1 + h \cdot y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \frac{h^4}{4!} y_1^{iv} \ldots$

$\Rightarrow y_1' = x_1 + y_1 = 1.1 + 0.11034 = 1.21034$

$\Rightarrow y_1'' = 1 + y_1' = 2.21034$

$y_1''' = 2.21034$ $\quad y_1^{iv} = 2.21034$

$\Rightarrow y_2 = 0.11034 + (0.1)(1.21034) + \frac{0.01}{2} \times \cancel{2.21034}$ 1.10517

$+ \frac{0.001}{6} \times 2.21034 + \frac{0.0001}{24} \times 2.21034$

$= 0.2428032$

★ MODIFIED EULER'S METHOD :-

Modified Euler's formula will give a better approximatn.

$$y_{n+1} = y_n + h \cdot f\left\{x_n + \frac{h}{2}, y_n + \frac{h}{2} \cdot f(x_n, y_n)\right\}$$

Q. Apply MEM :— on

$$\frac{dy}{dx} = e^x + xy \quad ; \quad y(0) = 0 \text{ to find}$$

$y(0.1)$ & $y(0.2)$?

$y_0 = 0$ & $x_0 = 0$

x	y
0	0
0.1	0.10537
0.2	

$y_1 = y_0 + h \cdot f\{x_0 + 0.05, y_0 + 0.05 ...$

$\Rightarrow y_1 = 0 + 0.1 \cdot f(0.05, 0.05)$

$= 0.1 \times (e^{0.05} + 0.0025)$

$= \underline{0.10537}$

$y_2 = y_1 + h \cdot f(x_1 + 0.05, y_1 + 0.05 \times f(x_1, y_1))$

$= 0.10537 + 0.1 \times f(0.15, 0.10537 + 0.5$

* 4^{th} order RUNGE - KUTTA METHOD

Here, we get Δy values instead of y values & accordingly y values can be calculated.

STEP-1 : Consider $\frac{dy}{dx} = f(x, y)$; $y(x_0) = y_0$.

STEP 2 : Compute the foll. 4 variables.

to find (y_1)

$$K_1 = h f(x_0, y_0)$$

$$K_2 = h \cdot f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$$

$$K_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$$

$$K_4 = h f((x_0 + h), y_0 + K_3).$$

$$\Delta y_0 = \frac{1}{6} \cdot [K_1 + 2(K_2 + K_3) + K_4]$$

$$\Rightarrow \boxed{y_1 = y_0 + \Delta y_0}$$

NOTE : In order to find $y_2 \longrightarrow$ we use (x_1, y_1) pair.

$$K_1 = h f(x_1, y_1)$$

$$K_2 = h \cdot f\left(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}\right)$$

$$K_3 = h \cdot f\left(x_1 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$$

$$K_4 = h \cdot f(x_1 + h, y_0 + K_3)$$

$$\Rightarrow \Delta y_1 = \frac{1}{6}[K_1 + 2(K_2 + K_3) + K_4]$$

$$\Rightarrow y_2 = y_1 + \Delta y_1.$$

Q. Solve $\frac{dy}{dx} = x+y$; $y(0) = 1$ by R-K 4 method

for $x = 0$ to 0.4 with $h = 0.1$

Ans

$f(x,y) = x+y$; $x_0 = 0$; $y_0 = 1$.

$h = 0.1$

$y_1 = 1.11034$

$y_2 = 1.24280$

$y_3 = 1.3997$

$y_4 = 1.5836$.

* To find $y(0.1)$.

Solⁿ Table

x	y
x_0 0	1 y_0
x_1 0.1	1.11034 y_1
x_2 0.2	1.22458 y_2

$k_1 = h.f(x_0, y_0) = 0.1 \times (0+1) = 0.1$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1) \times \left(\frac{0.1}{2} + 1.05\right)$

$= 0.11$

$k_3 = hf\left(x_0 + \frac{h}{2} ; y_0 + \frac{k_2}{2}\right) = (0.1) \times (0.05 + 1 + 0.055)$

$= 0.1105$

$k_4 = h.f(x_0 + h, y_0 + k_3) = (0.1) \times (0.1 + 1 + 0.1105)$

$= 0.12105$

$\Delta y_0 = \frac{1}{6}[k_1 + 2(k_2 + k_3) + k_4]$

$= \frac{1}{6} \times [0.1 + 2(0.11 + 0.1105) + 0.12105]$

$= 0.11034$

$\therefore y_1 - y_0 = 0.11034$

$\therefore y_1 = y_0 + \Delta y_0 = 1.11034$

for 2^{nd} iteratⁿ

$k_1 = h \cdot f(x_1, y_1) = (0.1) \times (0.1 + 1.11034)$

$= 0.121034$

$k_2 = h \cdot f(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2})$

$= 0.1 \times (0.1 + 0.05 + 1.11034 + 0.060517)$

$= 0.1270857$

$k_3 = h \cdot f(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2})$

$= 0.1 \times [0.15 + 1.11034 + 0.06354285]$

$= 0.132388255$

$k_4 = h \cdot f(x_1 + h, y_1 + k_3)$

$= 0.1 \times ((0.1 + 0.1) + 1.11034 + 0.132 \ldots)$

$= 0.1442728 \ldots$

$\Delta y_1 = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$

$= 0.1140424 \ldots$

$y_2 - y_1 = \Delta y_1$

$\therefore y_2 = \Delta y_1 + y_1 = \cancel{1.22438}\ 1.24280$

Q. Solve $\frac{dy}{dx} = -xy^2$; $y(0) = 2$ using Taylor's series to find $y(0.1)$ & $y(0.2)$?

x	y	
0	2	y_0
0.1	___	y_1
0.2	___	y_2

$y' = -xy^2$

$\Rightarrow y_0' = -x_0 y_0^2 = 0$

$y'' = -\{2xyy' + y^2\}$

$y_0'' = -4$

$y''' = -[2\{(xy).y'' + (xy'+y).y'\} + 2yy'] = -2[xyy'' + x(y')^2 + 2yy']$

$y_0''' = -2[0 + 0 + 2.2.0] = 0$

$y^{iv} = -2[xyy''' + (xy'+y)y'' + (y')^2 + 2y'y'' + 2(y')^2 + 2yy'']$

$\Rightarrow y_0^{iv} = -2[0 + 0 + yy'' + (y')^2 + 2yy'' + 2(y')^2 + 2yy'']$

calculate $= -2[5yy'' + 3(y')^2]$

$= -2[2\times(-4)\times 5 + 0] = 48$

$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{iv} + \ldots$

$= 2 + \cancel{(0.1)0} + \frac{(0.1)^2}{2}.(-4) + \cancel{\frac{(0.1)^3}{6} \times 0} + \frac{(0.1)^4}{24}.48$

$= 2 - 0.02 + 0.0002$

$= 1.9802$

$y_2 = y_1 + h.y_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \frac{h^4}{4!}y_1^{iv}$

$= 1.9802 + (0.1)\times(-0.392119) + \frac{0.01}{2}\times(-3.805897)$

$+ \frac{0.001}{6}\times 4.582432 + \frac{0.0001}{24}\times 7.5899321$

$$y_1' = -x_1 y_1^2 = -(0.1)(y_1)^2 = -0.392119$$

$$y_1'' = -\{2 x_1 y_1 y_1' + y_1^2\}$$

$$= -\{2 \times (0.1) \times (1.9802) \times (-0.392119) + (1.9802)^2\}$$

$$= -\{-0.1152948 + (1.9802)^2\}$$

$$= -3.805897$$

$$y_1''' = -2\left[x_1 y_1 y_1'' + x_1 \underline{(y_1')^2} + 2 y_1 y_1'\right]$$

$$= -2\{(0.1)(1.9802)(-3.805897) + (0.1) \times (-0.392119)^2 + 2 \times (-0.392119) \times (1.9802)\}$$

$$= -2\{-0.7536437 + 0.0153757 - 1.552948\}$$

$$= 4.582432$$

$$y_1^{IV} = -\left(\underline{2 x_1 y_1 y_1'''} + \underline{2 x_1 y_1' y_1''} + \underline{2 y_1 y_1''} + \underline{2 (y_1')^2} + \underline{4 x_1 y_1''} + \underline{4 (y_1')^2} + 4 y_1 y_1''\right).$$

$$= -\{(2 \times 0.1 \times 1.9802 \times 4.582432) + (2 \times 0.1 \times -0.392119 \times -3.805897)$$
$$+ (2 \times 1.9802 \times -3.805897) + 2 \times (0.392119)^2$$
$$+ 4 \times 0.1 \times (-3.805897) + 4 \times (0.392119)^2$$
$$+ (4 \times -0.392119 \times -3.805897)]$$

$$= -[1.8148263 + 0.2984729 - 15.0728448 + 0.3075146 - 1.5223588 + 0.6150292 + 5.969458]$$

$$= +[7.5899321]$$