Unit - 1 Successive Differentiation

Successive Differentiation:

Let $y = f(x)$ be a fun^n where a variable y is dependent & variable x is independent.
Rate of change of y w.r.t x is denoted by $\frac{dy}{dx}$ called first derivative.

Further the higher order derivatives are $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, ------

In this chapter our interest is to find the n^{th} order derivative denoted by $\frac{d^ny}{dx^n}$.

Remarks:

1. for $y = f(x)$, derivatives may be denoted by Dy, D^2y, D^3y, etc, where $D \equiv \frac{d}{dx}$
2. Another representation for derivatives may be y_1, y_2, y_3 etc.

n^{th} derivative of ~~dubstanced~~ function :-

(1) $y = e^{ax}$

diff. wrt x

$y_1 = ae^{ax}$

$y_2 = a^2 e^{ax}$

⋮

Generalising

$$\boxed{y_n = a^n e^{ax}}$$

(2) $y = a^x$

diff wrt x

$y_1 = a^x \log a$

$y_2 = a^x (\log a)^2$

⋮

Generalising

$$\boxed{y_n = a^x (\log a)^n}$$

(3) $y = (ax+b)^m$

$\Rightarrow y_1 = ma(ax+b)^{m-1}$

$\Rightarrow y_2 = \cancel{m(m-1)} \; m(m-1)\, a^2 (ax+b)^{m-2}$

⋮

Generalising

$y_n = m(m-1)(m-2) \ldots [m-(n-1)] \, a^n (ax+b)^{m-n}$

(4) ~~y = log~~ $\boxed{y_n = \underline{\lfloor n} \, a^n}$

(4) $y = \log(ax+b)$

diff wrt x

$y_1 = \frac{1}{(ax+b)} \cdot a = \frac{(-1)^0 \underline{\lfloor 0} \, a^1}{(ax+b)^1}$

$\cancel{y_2 = \frac{(-1)(-2)a^2}{(ax+b)^2}}$

$y_2 = \frac{-1}{(ax+b)^2} a^2 = \frac{(-1)^1 \underline{\lfloor 1} \, a^2}{(ax+b)^2}$

$y_3 = \frac{(-1)(-2)\, a^3}{(ax+b)^3} = \frac{(-1)^2 \underline{\lfloor 3} \, a^3}{(ax+b)^3}$

Generalising

$$y_n = \frac{(-1)^{n-1} \lfloor n-1 \, a^n}{(ax+b)^n}$$

(5) $y = \sin(ax+b)$

diff wrt x

$y_1 = a\cos(ax+b)$

$= a\sin\left[\frac{\pi}{2} + (ax+b)\right]$

$\left\{ \because \sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta \right\}$

$y_2 = a^2\cos\left[\frac{\pi}{2} + (ax+b)\right]$

$= a^2 \sin\left(\frac{\pi}{2} + \frac{\pi}{2} + (ax+b)\right)$

$y_2 = a^2 \sin\left(\frac{2\pi}{2} + (ax+b)\right)$

Generalising

$y_n = a^n \sin\left(\frac{n\pi}{2} + (ax+b)\right)$

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(6) $y = \cos(ax+b)$.

diff. wrt x

$y_1 = -a\sin(ax+b)$

$y_1 = a\cos\left[\frac{\pi}{2} + (ax+b)\right]$

$y_2 = -a^2\sin\left[\frac{\pi}{2} + (ax+b)\right]$

$y_2 = a^2\cos\left[\frac{2\pi}{2} + (ax+b)\right]$

⋮

Generalising

$y_n = a^n\cos\left[\frac{n\pi}{2} + (ax+b)\right]$

(7) $y = e^{ax}\sin(bx+c)$

diff. wrt x

$y_1 = ae^{ax}\sin(bx+c) + e^{ax}b\cos(bx+c)$

$\Rightarrow y_1 = e^{ax}\left[a\sin(bx+c) + b\cos(bx+c)\right]$

Put $a = r\cos\theta$ $\quad b = r\sin\theta$

square & add

$a^2+b^2 = r^2$

$\boxed{r = (a^2+b^2)^{1/2}}$

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divide

$$\frac{a}{b} = \cot\theta$$

$$\tan\theta = \frac{b}{a}$$

$$\boxed{\theta = \tan^{-1}\left(\frac{b}{a}\right)}$$

Then

$$y_1 = re^{ax}\left[\cos\theta \sin(bx+c) + \sin\theta \cos(bx+c)\right]$$

$$y_1 = re^{ax} \sin(\theta + (bx+c))$$

* $\sin(A+B) = \sin A \cos B + \cos A \sin B$

$$\Rightarrow y_2 = r\left[ae^{ax}\sin(\theta + (bx+c)) + be^{ax}\cos(\theta + (bx+c))\right]$$

$$y_2 = r^2 e^{ax} \sin(\theta + a\theta + bx + c)$$

$$\Rightarrow y_2 = r^2 e^{ax} \sin\left[2\theta + bx + c\right]$$

Generalising

$$y_n = r^n e^{ax} \sin\left[n\theta + bx + c\right]$$

where $r = (a^2+b^2)^{1/2}$ & $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

(8) $D^n\{e^{ax}\cos(bx+c)\} = ?$

Sol^n:

let $y = e^{ax}\cos(bx+c)$

$\Rightarrow y_1 = ae^{ax}\cos(bx+c) - be^{ax}\sin(bx+c)$

$\Rightarrow y_1 = e^{ax}[a\cos(bx+c) - b\sin(bx+c)]$

let $a = r\cos\theta$, $b = r\sin\theta$

squaring & adding

$$\boxed{r = \sqrt{a^2+b^2}}$$

dividing, $\tan\theta = \frac{b}{a} \Rightarrow \theta = \tan^{-1}\left(\frac{b}{a}\right)$

This gives

$y_1 = re^{ax}[\cos\theta\cos(bx+c) - \sin\theta\sin(bx+c)]$

$y_1 = re^{ax}\cos(\theta+bx+c)$

$y_2 = r[ae^{ax}\cos(\theta+bx+c) - be^{ax}\sin(\theta+bx+c)]$

$= re^{ax}[r\cos\theta\cos(\theta+bx+c) - r\sin\theta\sin(\theta+bx+c)]$

$y_2 = r^2e^{ax}\cos(2\theta+bx+c)$

generalising

$$\boxed{y_n = r^n e^{ax}\cos(n\theta+bx+c)}$$

H.K. DAS

Ques Find n^{th} derivative of $\frac{1}{x^2+5x+6}$

let $y = \frac{1}{x^2+5x+6}$

$$y = \frac{1}{(x+2)(x+3)}$$

$$y = \frac{1}{x+2} - \frac{1}{x+3}$$

$$\Rightarrow y_1 = \frac{(-1)}{(x+2)^2} - \frac{(-1)}{(x+3)^2}$$

$$\Rightarrow y_2 = \frac{(-1)(-2)}{(x+2)^3} - \frac{(-1)(-2)}{(x+3)^3}$$

$$= (-1)^2 2! \left[\frac{1}{(x+2)^3} - \frac{1}{(x+3)^3}\right]$$

Generalising

$$y_n = (-1)^n n! \left[\frac{1}{(x+2)^{n+1}} - \frac{1}{(x+3)^{n+1}}\right]$$

(2)

$$D^n\left[\frac{1}{2x^2+5x+2}\right] = ?$$

Let $y = \dfrac{1}{2x^2+5x+2}$

$$y = \frac{1}{(2x+1)(x+2)} = \frac{1}{3}\left[\frac{2}{2x+1} - \frac{1}{x+2}\right]$$

$$y_1 = \frac{1}{3}\left[\frac{2^2(-1)}{(2x+1)^2} - \frac{(-1)}{(x+2)^2}\right]$$

$$\Rightarrow y_2 = \frac{1}{3}\left[\frac{2^3(-1)(-2)}{(2x+1)^3} - \frac{(-1)(-2)}{(x+2)^3}\right]$$

$$\Rightarrow y_2 = \frac{1}{3}(-1)^2 \lfloor 2 \left[\frac{2^3}{(2x+1)^3} - \frac{1}{(x+2)^3}\right]$$

Generalising

$$y_n = \frac{1}{3}(-1)^n \lfloor n \left[\frac{2^{n+1}}{(2x+1)^{n+1}} - \frac{1}{(x+2)^{n+1}}\right]$$

(3) $D^n(\tan^{-1}x) = ?$

Sol^n: Let $y = \tan^{-1}x$

$$y_1 = \frac{1}{1+x^2} = \frac{1}{x^2+1} \quad (\because i^2 = -1)$$

$$= \frac{1}{x^2 - i^2}$$

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$$y_1 = \frac{1}{(x-i)(x+i)}$$

$$\Rightarrow y_1 = \frac{1}{2i}\left[\frac{1}{(x-i)} - \frac{1}{(x+i)}\right]$$

$$y_2 = \frac{1}{2i}\left[\frac{(-1)}{(x-i)^2} - \frac{(-1)}{(x+i)^2}\right]$$

$$\Rightarrow y_3 = \frac{1}{2i}(-1)^2 2!\left[\frac{1}{(x-i)^3} - \frac{1}{(x+i)^3}\right]$$

Generalising

$$y_n = \frac{1}{2i}(-1)^{n-1}\lfloor n-1 \left[\frac{1}{(x-i)^n} - \frac{1}{(x+i)^n}\right]$$

To get rid of 'i', procedure as follows:

Put $x = r\cos\theta$, $1 = r\sin\theta$

Squaring & adding

$$r = \sqrt{x^2+1}$$

dividing: $\theta = \tan^{-1}\left(\frac{1}{x}\right)$

Now $(x \pm i)^n = (r\cos\theta \pm ir\sin\theta)^n$

$= r^n(\cos\theta \pm i\sin\theta)^n$

$= r^n(e^{\pm i\theta})^n \quad (\because \cos\theta \pm i\sin\theta = e^{\pm i\theta})$

$= r^n e^{\pm in\theta}$

That gives

$$y_n = \frac{1}{2i}(-1)^{n-1}(n-1)!\left[\frac{1}{r^n e^{-in\theta}} - \frac{1}{r^n e^{in\theta}}\right]$$

$$y_n = \frac{(-1)^{n-1}(n-1)!}{r^n}\left[\frac{e^{in\theta} - e^{-in\theta}}{2i}\right]$$

$$= \frac{(-1)^{n-1}(n-1)!\sin(n\theta)}{r^n}$$

$$\boxed{\text{Note: } \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}, \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}}$$

(4) $D^n(\sin 2x \cos 3x) = ?$

Let $y = \frac{1}{2}(2\sin 2x \cos 3x)$

$y = \frac{1}{2}[\sin(5x) + \sin(-x)]$

$y = \frac{1}{2}(\sin 5x - \sin x)$

Use $D^n[\sin(ax+b)] = a^n \sin\left[\frac{n\pi}{2} + ax + b\right]$

Then, diff. y 'n times';

$$y_n = \frac{1}{2}\left[5^n \sin\left(\frac{n\pi}{2} + 5x\right) - 1^n \sin\left(\frac{n\pi}{2} + x\right)\right]$$

$$y_n = \frac{1}{2}\left[5^n \sin\left(\frac{n\pi}{2} + 5x\right) - \sin\left(\frac{n\pi}{2} + x\right)\right]$$

Q. find $D^n[e^x(\sin 2x + \cos 3x)]$

let $y = e^x(\sin 2x + \cos 3x)$

$y = e^x \sin 2x + e^x \cos 3x$

Use:

$$D^n[e^{ax}\sin(bx+c)] = r^n e^{ax} \sin[n\theta + bx + c]$$

$$\& \; D^n[e^{ax}\cos(bx+c)] = r^n e^{ax} \cos(n\theta + bx + c)$$

when $r = \sqrt{a^2+b^2}$ &

$\theta = \tan^{-1}\left(\frac{b}{a}\right)$

$r = (a^2+b^2)^{1/2}$

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Remark: It is convinient to take a function type x^n or polynomial as v.

diff y 'n times'

$$\boxed{y_n = (\sqrt{5})^n e^x \sin[n\tan^{-1}(2) + 2x] + (\sqrt{10})^n e^x \cos(n\tan^{-1}(3) + 3x]}$$

Leibnitz Theoram:

We already know that

$$D(u.v) = uv_1 + vu_1$$

Again $D^2(u.v) = D[D(u.v)]$

$$D(u.v) = u_1v + v_1u = {}^1C_0 D^1(u)D^0(v) + {}^1C_1 D^0(u)$$

Again $D^2(u.v) = D[D(u.v)] = D[uv_1 + vu_1]$

$$= u.v_2 + u_1v_1 + v_1u_1 + vu_2$$
$$= u_2v + 2u_1v_1 + uv_2$$
$$= {}^2C_0 D^2(u)D^0(v) + {}^2C_1 D^1(u)D^1(v) + {}^2C_2 D^0(u)$$

General:

$$D^n(u.v) = {}^nC_0 D^n(u)D^0(v) + {}^nC_1 D^{n-1}(u)D^1(v) + {}^nC_2 D^{n-2}(u)D^2(v) + \ldots {}^nC_n D^0(u)D^n(v)$$

Ques: Find the n^{th} derivative of the following

1. $x^2.\sin x$
2. $e^x(2x+3)^3$
3. $x^2.e^x.\cos x$
4. $x^2(e^x + \cos 3x)$

— 1

1 Let $y = \underbrace{\sin x}_{u} \cdot \underbrace{x^2}_{v}$

Let $u = \sin x$, $v = x^2$

Now,

$D^n(u) = D^n(\sin x)$

$= \sin\left(\frac{n\pi}{2} + x\right)$

diff. 1 by L.T

$D^n(y) = {}^nC_0 D^n(\sin x) D^0(x^2)$
$+ {}^nC_1 D^{n-1}(\sin x) D^1(x^2) +$
${}^nC_2 D^{n-2}(\sin x) D^2(x^2)$

$= \sin\left(\frac{n\pi}{2} + x\right) \cdot x^2 + n \sin\left[\frac{(n-1)\pi}{2} + x\right] \cdot 2x$

$+ \frac{n(n-1)}{\cancel{2}} \sin\left(\frac{(n-2)\pi}{2} + x\right) \cdot \cancel{2}$

3 $y = x^2 e^x \cos x$

$y = \underbrace{e^x \cos x}_{u} \; \underbrace{x^2}_{v}$

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$$D^n(u) = D^n[e^x \cos x]$$
$$= r^n e^x \cos[n\theta + x]$$

$$= e^{ax} \cdot r^n \cos(n\theta + bx + c)$$

where $r = \sqrt{a^2+b^2}$ & $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

$\cancel{D^n(u) = D^n(e^x \cos x) = r^n}$

then $D^n(u) = D^n(e^x \cos x)$

$$= (\sqrt{2})^n e^x \cos\left[\frac{n\pi}{4} + x\right]$$

diff. ① by L.T

$$D^n(y) = D^n[e^x \cos x \cdot x^2]$$

$$\Rightarrow y_n = {}^nC_0 D^n(e^x \cos x) \cdot D^0(x^2) + {}^nC_1 D^{n-1}(e^x \cos x) D^1(x^2)$$
$$+ {}^nC_2 D^{n-2}(e^x \cos x) D^2(x^2)$$

$$y_n = 2^{n/2} e^x \cos\left(\frac{n\pi}{4} + x\right) \cdot x^2 + n \cdot 2$$

Ques. If $y = x^n . \log x$ then prove that
$$y_{n+1} = \frac{\lfloor n}{x}$$

Sol^n: Note:
y_{n+1} indicates that for n^{th} derivative Leibnitz theorm should be applied on y_1.

Now
$$y = x^n \log x$$
$$y_1 = x^n \left(\frac{1}{x}\right) + \log x . nx^{n-1}$$
$$= \frac{1}{x}\left[nx^n \log x + x^n\right]$$
$$= \frac{1}{x}\left[ny + x^n\right]$$

$$xy_1 = ny + x^n$$

diff. 'n' times by L.T
$$D^n(y_1 . x) = nD^n(y) + D^n(x^n)$$
$$\left[{}^nC_0 D^n(y_1) D^0(x) + {}^nC_1 D^{n-1}(y_1) D^1(x)\right] = ny_n + \lfloor n$$

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$=0$

$\Rightarrow$ $y_{n+1} \cdot x + \cancel{n y_n (1)} = \cancel{n y_n} + L_n$

$$\boxed{y_{n+1} = \frac{L_n}{x}}$$

proved

Qn $y = x^2 \cdot e^{2x}$ then prove that

$(y_n)_0 = n(n-1)\,2^{n-2}$

Note : $(y_n)_0$ is the value of y_n at $x=0$

let $u = e^{2x}$, $v = x^2$

we know that

$D^n(u) = e^{2x} \cdot 2^n$

diff (1) by LT

$D^n(y) = {}^nC_0 D^n(e^{2x}) D^0(x^2) + {}^nC_1 D^{n-1}(e^{2x}) D^1(x^2)$
$+ {}^nC_2 D^{n-2}(e^{2x}) D^2(x^2)$

$y_n = 2^n e^{2x} \cdot x^2 + n 2^{\frac{n-1}{2}} \cdot e^{2x} \cdot x^2 + \frac{n(n-1)}{\cancel{2}} 2^n e^{2x} \cancel{(2)}$

put $x=0$

$(y_n)_0 = n(n-1)\,2^n$

Ques-1 If $y = a\cos(\log x) + b\sin(\log x)$

then prove that
$x^2 \times y_{n+2} + (2n+1) \times x \times y_{n+1} + (n^2+1)y_n$

Ques-2 If $y = (x^2-1)^n$, prove that
$(x^2-1) \times y_{n+2} + 2x \times y_{n+1} - n \times (n+1) \times y_n = 0$

Ques- If $y = \frac{\sin^{-1}x}{\sqrt{1-x^2}}$, prove that

$(1-x^2) y_{n+1} \cancel{+} 1 - (2n+1) x y_n - n^2 y_{n-1} = 0$

Ques If $y = \cancel{\sin^{-1}x} \sin^{-1}(3x-4x^3)$, prove that
$(1-x^2) y_{n+2} - (2n+1) x y_{n+1} - n^2 y_n = 0$

Ques- If $y = \tan^{-1}\left(\frac{a-x}{a+x}\right)$, prove that
$(a^2+x^2) y_{n+2} + \cancel{2x} 2x(n+1)(y_{n+1}) +$
$n(n+1) y_n = 0$

Ques- If $y^{1/m} + y^{-1/m} = 2x$
prove that $(x^2-1) y_{n+2} + (2n+1) x y_{n+1}$
$+ (n^2-m^2) y_n = 0$

Ques- If $y = \sin[\log(x^2 + 2x + 1)]$, prove that

$(x+1)^2 y_{n+2} + (2n+1) \times (x+1) \times (y_{n+1}) + (n^2 + 4) y_n = 0$

① $y = a\cos(\log x) + b\sin(\log x)$

$y_1 = \frac{-a\sin(\log x)}{x} + \frac{b\cos(\log x)}{x}$

$xy_1 = -a\sin(\log x) + b\cos(\log x)$

$xy_2 + y_1 = \frac{-a\cos(\log x)}{x} - \frac{b\sin(\log x)}{x}$

$x^2y_2 + xy_1 = -[a\cos(\log x) + b\sin(\log x)]$

$x^2y_2 + xy_1 + y = 0$

diff. 'n' times by using L.T

$D^n[y_2 x^2] + D^n(y_1 x) + D^n(y) = 0$

$\left[{}^nC_0 D^n(y_2) D^0(x^2) + {}^nC_1 D^{n-1}(y_2) D'(x^2) + {}^nC_2 D^{n-2}(y_2) D^2(x^2)\right] + \left[{}^nC_0 D^n(y_1) D^0(x) + {}^nC_1 D^{n-1}(y_1) D'(x)\right] + y_n = 0$

$\left[y_{n+2} \cdot x^2 + n y_{n+1}(2x) + \frac{n(n-1)}{2!} y_n (2)\right] + \left[y_{n+1} \cdot x + n y_n\right] + y_n = 0$

$x^2 y_{n+2} + (2n+1) x \cdot y_{n+1} + (n^2 - n + n + 1) y_n = 0$

Proved.

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(2) $y = \frac{\sin^{-1}x}{\sqrt{1-x^2}}$

$(1-x^2)y^2 = (\sin^{-1}x)^2$

diff. wrt x

$(1-x^2)2yy_1 - 2xy^2 = 2\sin^{-1}x \times \frac{1}{\sqrt{1-x^2}}$

or

$(1-x^2)yy_1 - xy^2 = y$

$(1-x^2)y_1 - xy = 1$

diff. 'n' times using L.T

$D^n[y_1(1-x^2)] - D^n[y.x] = D^n(1)$

$\left[{}^nC_0 D^n(y_1) D^0(1-x^2) + {}^nC_1 D^{n-1}(y_1) D^1(1-x^2) + {}^nC_2 D^{n-2}(y_1) D^2(1-x^2)\right] - \left[{}^nC_0 D^n(y) D^0(x) + {}^nC_1 D^{n-1}(y) D^1(x)\right] = 0$

$\left[y_{n+1}(1-x^2) + ny_n(-2x) + \frac{n(n-1)}{2!} y_{n-1}(-2)\right] - \left[y_n . x + ny_{n-1}\right] = 0$

$(1-x^2)y_{n+1} - (2n+1)xy_n - n(1+n-1)y_{n-1} = 0$

$(1-x^2)y_{n+1} - (2n+1)xy_n - n^2 y_{n-1} = 0$

$$(a \pm b) = (b \pm a)$$

$\sqrt{x^2-1}$ $\sqrt{x^2-1}$

(7) $$y^{1/m} + y^{-1/m} = 2x$$

$$y^{1/m} + \frac{1}{y^{1/m}} = 2x$$

let $y^{1/m} = z$

$$z + \frac{1}{z} = 2x$$

$$z^2 - 2xz + 1 = 0$$

$$z = \frac{2x \pm \sqrt{4x^2-4}}{2}$$

$$z = x \pm \sqrt{x^2-1}$$

$$\Rightarrow y^{1/m} = x \pm \sqrt{x^2-1}$$

$$y = \left[x \pm \sqrt{x^2-1}\right]^m$$

diff. wrt x

$$\frac{dy}{dx} = m\left[x \pm \sqrt{x^2-1}\right]^{m-1}\left[1 \pm \frac{x}{\sqrt{x^2-1}}\right]$$

$$y_1 = \pm m \frac{\left[x \pm \sqrt{x^2-1}\right]^{m}}{\left[x \pm \sqrt{x^2-1}\right]} \times \frac{\sqrt{x^2-1} \pm x}{\sqrt{x^2-1}}$$

$$y_1 = \frac{\pm m \left[x \pm \sqrt{x^2-1}\right]^m}{\sqrt{x^2-1}}$$

Caution: $\frac{a \pm b}{b \pm a} = \pm 1$

Sq

$$y_1^2(x^2-1) = m^2\left[x \pm \sqrt{x^2-1}\right]^{2m}$$

Or

$$y_1^2(x^2-1) = m^2 y^2$$

diff. again

$$2y_1 y_2 (x^2-1) + 2x y_1^2 = m^2 2y y_1$$

or

$$y_2(x^2-1) + x y_1 = m^2 y$$

diff. 'n' times by L.T

$$\left[y_{n+2}(x^2-1) + n y_{n+1}(2x) + \frac{n(n-1)}{2!} y_n (2) \right]$$

$$+ y_{n+1} \cdot x + n y_n = m^2 y_n$$

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Ques 1 If $y = \left[x + \sqrt{1+x^2}\right]^m$

prove that (ii) $(y_{2n})_0 = m^2\left[m^2-(2n-2)^2\right].\left[m^2-(2n-4)^2\right]\ldots\left[m^2-2^2\right]m^2$

$(y_{2n})_0$

(iii) $(y_{2n+1})_0 = \left[m^2-(2n-1)^2\right].\left[m^2-(2n-3)^2\right]\ldots\left[m^2-1^2\right]m$

Qn-2 If $y = \sin(m\sin^{-1}x)$

prove that

(1) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-m^2)y_n = 0$

(2) $(y_n)_0 = \begin{cases} \left[(n-2)^2-m^2\right]\left[(n-4)^2-m^2\right]\ldots(1-m^2)m, & \text{if } m \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$

(2) $\because y = \sin(m\sin^{-1}x)$ —— (1)

diff. wrt x

$y_1 = \cos(m\sin^{-1}x) \times \frac{m}{\sqrt{1-x^2}}$ —— (2)

Squaring

$y_1^2(1-x^2) = m^2\cos^2(m\sin^{-1}x)$

diff again

$(1-x^2)2y_1y_2 - 2xy_1^2 = m^2 \times 2\cos(m\sin^{-1}x) \times \left[-\sin(m\sin^{-1}x)\right] \times \frac{m}{\sqrt{1-x^2}}$

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or $(1-x^2)2y_1y_2 - xy_1^2 = -m^2yy_1$

$y_2(1-x^2) - y_1x + m^2y = 0$ ——— ③

diff. 'n' times by L.T

$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-m^2)y_n = 0$

Put $x=0$ in ①, ②, ③

$(y)_0 = 0$

& $(y_1)_0 = m$

& $(y_2)_0 + m^2(y)_0 = 0 \Rightarrow (y_2)_0 = -m^2 \cdot 0 = 0$

~~Put~~ Put $x=0$ in ④

$(y_{n+2})_0 = (n^2-m^2)(y_n)_0$

Put ~~n=~~ $n=1$

$(y_3)_0 = (1^2-m^2)m$

Put $n=2$

$(y_4)_0 = (2^2-m^2) \cdot 0 = 0$

Put $n=3$

$(y_5)_0 = (3^2-m^2)(1^2-m^2) \cdot m$

Put $n=4$, $(y_6)_0 = 0$

Generalising

$$(y_n)_0 = \begin{cases} 0 & \text{, if } n \text{ is even} \\ [(n-2)^2-m^2][(n-4)^2-m^2] + \_\_ (1^2-m^2)m, & \text{if } n \text{ is odd} \end{cases}$$

(1) $y = [x+\sqrt{1+x^2}]^m$

diff.

$$y_1 = m[x+\sqrt{1+x^2}]^{m-1} \times \left[1+\frac{x}{\sqrt{1+x^2}}\right]$$

or $y_1 = \dfrac{m[x+\sqrt{1+x^2}]^m}{\sqrt{1+x^2}}$

$$y_1\sqrt{1+x^2} = my$$

Squaring

$$y_1^2(1+x^2) = m^2y^2$$

⚠ Caution: L.T can't be applied due to powers on y.

diff again

$$2y_1y_2(1+x^2) + 2xy_1^2 = 2m^2yy_1$$

Ques If $y = e^{m\cos^{-1}x}$, Find $(y_n)_0$.

Ques If $y = \cos(m\sin^{-1}x)$. Prove that

(i) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$

(ii) Hence find $(y_n)_0$.

Partial Differentiation

Let $u = f(x,y)$ be a func of two variables.

Here u is a dependent varible while x & y are independent variables.

We can differentiate u wrt x (treating y as constant) & wrt y (treating x as cons

The partial derivatives of first order are $\frac{\partial u}{\partial x}$ & $\frac{\partial u}{\partial y}$

The partial derivatives of second order are $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial y^2}$, $\frac{\partial^2 u}{\partial x \partial y}$, $\frac{\partial^2 u}{\partial y \partial x}$

Note that: $\frac{\partial^2 u}{\partial x \partial y}$ & $\frac{\partial^2 u}{\partial y \partial x}$ are not equal in general.

For example if $u = \sin x \cos y$

$$\frac{\partial u}{\partial x} = \cos x \cos y$$

$$\frac{\partial u}{\partial y} = -\sin x \sin y$$

$$\frac{\partial^2 u}{\partial x^2} = -\sin x \cos y$$

$$\frac{\partial^2 u}{\partial y^2} = -\sin x \cos y$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(-\sin x \sin y\right)$$

$$\frac{\partial^2 u}{\partial x \partial y} = -\cos x \sin y$$

& $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(\cos x \cos y\right)$

$$= -\cos x \sin y$$

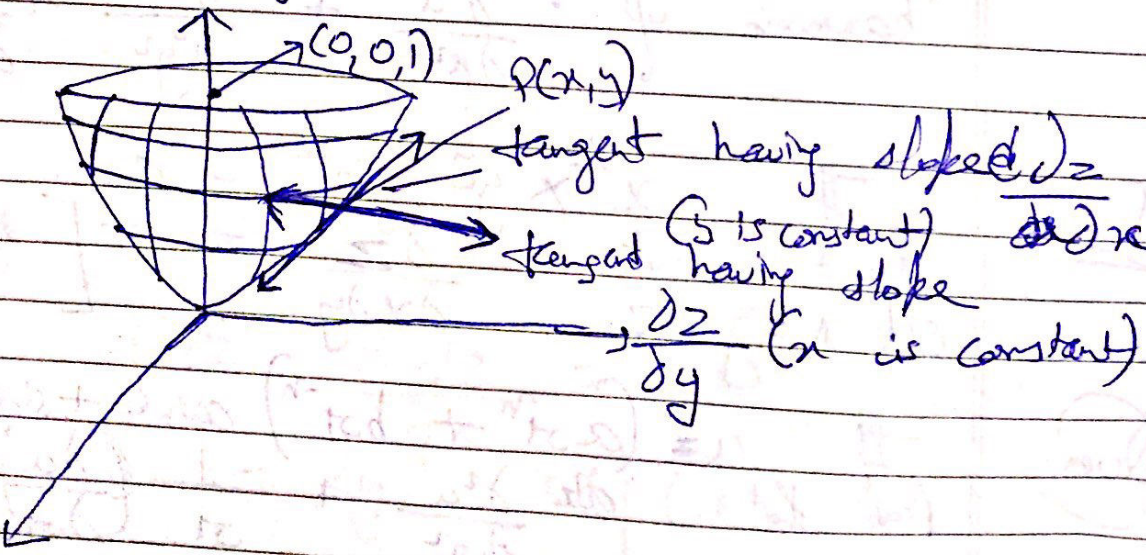
Remarks:

① Consider a rectangle plate placed on x,y plane & suppose $u(x,y)$ denotes the temperature at the point $P(x,y)$ on the plate.

$\frac{\partial u}{\partial x}$ gives the change in temperature u along parallel to x axis, and vice-versa.

② If we replace u by z then the function $z = f(x,y)$ represents a surface in 3-d space.

For example: $z = x^2+y^2$; $0 \le z \le 1$ is surface of hemisphere.



Ques- If $z^3 - zx - y = 0$

find $\frac{\partial^2 z}{\partial x \partial y}$.

Ques- If $z = f(x^2 y)$ prove that $x\frac{\partial z}{\partial x} = 2y\frac{\partial z}{\partial y}$

Ques- If $u = f(x+ky) + g(x-ky)$

prove that $\frac{\partial^2 u}{\partial y^2} = k^2 \frac{\partial^2 u}{\partial x^2}$.

Ques- If $u = (x^2+y^2+z^2)^{n/2}$ find n such that u is harmonic.

(Hint: A fn $u(x,y,z)$ is said to be harmonic if $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$)

Ques- If $x^x \times y^y \times z^z = c$.

prove that at $x=y=z$ $\frac{\partial^2 z}{\partial x \partial y} = -[x \log(ex)]^{-1}$

Ques- If $u = (ar^n + br^{-n}) \cos n\theta + \sin n\theta$.

Prove that $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\left(\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$

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(iii) $z^3 - zx - y = 0$

$\frac{\partial^2 z}{\partial x \partial y}$

(b) $x^2 y$ / x

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① $z^3 - zx - y = 0$

diff. wrt y { $z = f(x,y)$ as $\partial^2 z / \partial x \partial y = ?$ }

$$3z^2 \frac{\partial z}{\partial y} \; x\frac{\partial z}{\partial y} - 1 = 0$$

$$\Rightarrow \frac{\partial z}{\partial y} = \frac{1}{3z^2 - x}$$

diff. partially $\frac{\partial z}{\partial y}$ wrt x.

② $z = f(x^2 y)$

diff. partially wrt x

$$\frac{\partial z}{\partial x} = f'(x^2 y) \times 2xy$$

diff. partially wrt y

$$\frac{\partial^2 z}{\partial^2 y} = f''(x^2 y) \times x^2$$

L.H.S. $\cancel{x\frac{\partial z}{}}$ $\frac{x\,\partial z}{\partial x} = 2x^2 y\, f'(x^2 y)$

$$= 2y \times x^2 f'(x^2 y)$$

$$2y \frac{\partial z}{\partial y} = \text{RHS}$$

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(3) $u = f(x+ky) + g(x-ky)$

diff. partially w.r.t 'x' & 'y'

$$\frac{du}{dx} = f'(x+ky) + g'(x-ky)$$

&

$$\frac{du}{dy} = k f'(x+ky) - k g'(x-ky)$$

diff. again wrt 'x' & 'y'

$$\frac{d^2u}{dx^2} = f''(x+ky) + g''(x-ky)$$

& $$\frac{d^2u}{d^2y} = k^2 f''(x+ky) + k^2 g''(x-ky)$$

or $$\frac{d^2u}{dy^2} = k^2 \frac{d^2u}{dx^2}$$

(4) $$u = (x^2+y^2+z^2)^{n/2}$$

Pre Requisite: Note $r^2 = x^2+y^2+z^2$

diff. partially wrt x

$$2r\frac{dr}{dx} = 2x$$

$$\boxed{\frac{dr}{dx} = \frac{x}{r}}$$

$$\boxed{\frac{dr}{dy} = \frac{y}{r}} \qquad \frac{dr}{dz} = \frac{z}{r}$$

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Then $u = (r^2)^{n/2} = r^n$

diff partially w.r.t x

$$\frac{du}{dx} = nr^{n-1}\frac{dr}{dx}$$

$$= nr^{n-1}\frac{x}{r} = nr^{n-2}x$$

diff again w.r.t x

$$\frac{d^2u}{dx^2} = n\left[r^{n-2}.1 + x(n-2)r^{n-3}\frac{dr}{dx}\right]$$

$$= n\left[r^{n-2} + (n-2)r^{n-4}x^2\right]$$

Similarly

$$\frac{d^2u}{dy^2} = n\left[r^{n-2} + (n-2)r^{n-4}y^2\right]$$

$$\frac{d^2u}{dz^2} = n\left[r^{n-2} + (n-2)r^{n-4}z^2\right]$$

$\because$ u is harmonic, then

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} = 0$$

$$3nr^{n-2} + n(n-2)r^{n-4}(x^2+y^2+z^2) = 0$$

$$\Rightarrow 3nr^{n-2} + n(n-2)r^{n-4}.r^2 = 0$$

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$\Rightarrow \quad r^{n-2} n\left[3+n-2\right] = 0$

$r^{n-2} n\,(n+1) = 0$

$n = 0, -1$

(5) $x^x y^y z^z = 0$

Taking log $\quad x\log x + y\log y + z\log z = 0$

diff. partially wrt x

$x\left(\frac{1}{x}\right) + \log x \cdot 1 + \left(z\left(\frac{1}{z}\right)\frac{\partial z}{\partial x} + \log z\,\frac{\partial z}{\partial x}\right) = 0$

$\Rightarrow \quad \frac{\partial z}{\partial x} = -\frac{(1+\log x)}{(1+\log z)}$

Similarly

$\frac{\partial z}{\partial y} = -\frac{(1+\log y)}{(1+\log z)}$

diff. partially $\frac{dz}{dy}$ w.r.t 'x'

$\frac{d}{dx}\left(\frac{dz}{dy}\right) = \frac{d^2 z}{dx\,dy}$

$$= -(1+\log y) \times \frac{-1}{(1+\log z)^2} \times \left(\frac{1}{z}\frac{dz}{dx}\right)$$

$$\frac{d^2z}{dx\,dy} = \frac{(1+\log y)}{z(1+\log z)^2} \times \frac{-(1+\log x)}{(1+\log z)}$$

$$\left(\frac{d^2z}{dx\,dy}\right)_{x=y=z} = \frac{(1+\log x)}{x(1+\log x)^2} \times \frac{-(1+\log x)}{(1+\log x)}$$

$$= \frac{-1}{x(1+\log x)}$$

$$= \frac{-1}{x(\log e+\log x)}$$

$$= -\frac{1}{x\log(ex)}$$

$$= -[x\log(ex)]^{-1}$$

Homogenous Function

A fun^c $u(x,y)$ is said to be homogeneous in x & y if

$$u(tx,ty) = t^n u(x,y)$$

Its degree is said to be n.

for eg: $u = \frac{x^2+y^2}{x+y}$ is homogeneous in x & y of degree 1 because

$$u(tx,ty) = \frac{t^2x^2 + t^2y^2}{tx+ty}$$

$$= \frac{t^{\cancel{2}}(x^2+y^2)}{\cancel{t}(x+y)}$$

$$u(tx,ty) = t\frac{(x^2+y^2)}{(x+y)} = t^1 u(x,y)$$

But $u(x,y) = \sin(x^2+y^2)$ is not homogeneous as

$$u(tx,ty) \neq t^n u(x,y).$$

<u>Euler's Theorem :-</u>

If u is a homogenous function in x & y of degree n then

$$x\frac{du}{dx} + y\frac{du}{dy} = nu$$

<u>Proof :</u>

let $u = x^n f\left(\frac{y}{x}\right)$

diff. partially w.r.t x

$$\frac{du}{dx} = nx^{n-1}f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right) \times \frac{-y}{x^2}$$

Multiply by x

$$x\frac{\partial u}{\partial x} = nx^n f\left(\frac{y}{x}\right) - x^{n-1} y f'\left(\frac{y}{x}\right) \quad \text{—— (1)}$$

diff partially w.r.t y

$$\frac{\partial u}{\partial y} = x^n \times f'\left(\frac{y}{x}\right) \times \frac{1}{x}$$

Mult. by 'y'

$$y\frac{\partial u}{\partial y} = x^{n-1} y f'\left(\frac{y}{x}\right) \quad \text{—— (2)}$$

(1) + (2)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n f\left(\frac{y}{x}\right)$$

$$= nu$$

Jacobians :-

If u & v are fun^c of x & y, then the determinant

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}$$

is called Jacobian of u & v w.r.t x & y.

Remark: If a fun^c (P) ~~max~~ maps a ~~set~~ Set S of x-y plane to the image

4. S' on u-v plane then $J = \frac{\partial(u,v)}{\partial(x,y)}$

represents the ratio of area of image S' to that of domain S.

Properties:-

1. If $J = \frac{\partial(u,v)}{\partial(x,y)}$ is the Jacobian of variables u,v w.r.t ~~x & y~~ x and y then the Jacobian of x & y w.r.t u & v is given by

$$J^* = \frac{1}{J} = \frac{\partial(x,y)}{\partial(u,v)}$$

2. If u & v are functions of x & y, further x & y are functions of r & θ then

$$J = \frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)}$$

3. If $J = \frac{\partial(u,v)}{\partial(x,y)} = 0$

then u & v are functionally dependent on each other ~~x & y~~, i.e. there exist a relationship between them.

(4) If u, v, w are related to x, y, z by pre implicit equation f_1, f_2, f_3.

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}$$

Ques If $u = xyz$
$v = x^2y^2z^2$
$w = x+y+z$

Then find $J = \dfrac{\partial(u,v,w)}{\partial(x,y,z)}$.

Ques-2 If $x = a(u+v)$, $y = b(u-v)$
& $u = r^2\cos 2\theta$, $v = r^2 \sin 2\theta$
find $\dfrac{\partial(x,y)}{\partial(r,\theta)}$.

Ques-3 If $x = e^v . \sec u$, $y = e^v \tan u$
Prove that $J . J^* = 1$.

Ques-4 Examine wheather u & v are functionally dependent. If yes, then find the relation b/w them.

(i) $u = (x-y)(x+y)$; $v = \dfrac{(x+y)}{x}$

(ii) $u = \dfrac{x+y}{1-xy}$, $v = \tan^{-1}x + \tan^{-1}y$

(iii) $u = xy + yz + zx$, $v = x^2+y^2+z^2$, $w = x+y+z$

Ques-5 If $u+v+w = x+y+z$

$uv+vw+wu = x^2+y^2+z^2$

$uvw = \frac{1}{3}(x^3+y^3+z^3)$

Find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$.

Ques-6 If $u^2+v^2+w^2 = x^3+y^3+z^3$

$u^3+v^3+w^3 = x+y+z$

$u+v+w = x^2+y^2+z^2$

Find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$.

Ques-7 If u,v,w are the roots of equation in λ

$(\lambda-x)^3+(\lambda-y)^3+(\lambda-z)^3 = 0$

Find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$.

① $u = xyz$, $v = x^2y^2z^2$, $w = x+y+z$

$J = \frac{J(u,v,w)}{J(x,y,z)}$

$$= \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w \end{vmatrix}$$

$$\begin{vmatrix} yz & xz & xy \\ 2xy^2z^2 & 2yx^2z^2 & 2zx^2y^2 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2xyz \begin{vmatrix} yz & xz & xy \\ yz & xz & xy \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 0$$

(2) $x = a(u+v),\ y = b(u-v)\ :\ u = r^2\cos 2\theta,\ v = r^2\sin 2\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \frac{\partial(x,y)}{\partial(u,v)} \times \frac{\partial(u,v)}{\partial(r,\theta)} \quad \text{--- (1)}$$

Now

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

$$= \begin{vmatrix} a & a \\ b & -b \end{vmatrix} = -2ab$$

& $$\frac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} u_r & u_\theta \\ v_r & v_\theta \end{vmatrix}$$

$$= \begin{vmatrix} 2r\cos 2\theta & -2r^2\sin 2\theta \\ 2r\sin 2\theta & 2r^2\cos 2\theta \end{vmatrix} = 4r^3$$

Put in (1), $$\frac{\partial(u,v)}{\partial(r,\theta)} = -2ab \times 4r^3 = -8abr^3$$

(3) $J = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$

$$= \begin{vmatrix} e^v \sec u \tan u & e^v \sec u \\ e^v \sec^2 u & e^v \tan u \end{vmatrix}$$

$$= e^{2v} \sec u \begin{vmatrix} \tan u & 1 \\ \sec^2 u & \tan u \end{vmatrix}$$

$$= e^{2v} \sec u \left[\tan^2 u - \sec^2 u\right]$$

$$= e^{2v} \sec u$$

To express u & v in terms of x & y;

$$\frac{e^v \sec u}{e^v \tan u} = \frac{x}{y} \Rightarrow \sin u = \frac{y}{x} \Rightarrow \boxed{u = \sin^{-1}\left(\frac{y}{x}\right)}$$

Also, $(e^v \sec u)^2 - (e^v \tan u)^2 = x^2 - y^2$

$$e^{2v}\left[\sec^2 u - \tan^2 u\right] = x^2 - y^2$$

$$e^{2v} = x^2 - y^2$$

$$\boxed{v = \frac{1}{2}\log(x^2 - y^2)}$$

Now, $J^* = \dfrac{\partial(u,v)}{\partial(x,y)}$

$$= \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{1}{\sqrt{1-\frac{y^2}{x^2}}} \times \dfrac{-y}{x^2} & \dfrac{1}{\sqrt{1-\frac{y^2}{x^2}}} \times \dfrac{1}{x} \\ \dfrac{1}{2(x^2-y^2)} \times 2x & \dfrac{1}{2(x^2-y^2)} \times -2y \end{vmatrix}$$

$$= \frac{1}{(x^2-y^2)} \times \frac{1}{\sqrt{x^2-y^2}} \begin{vmatrix} \dfrac{-y}{x} & 1 \\ x & -y \end{vmatrix}$$

$$= \frac{1}{(x^2-y^2)\sqrt{x^2-y^2}} \times \frac{(y^2-x^2)}{x}$$

$$= \frac{-1}{x\sqrt{x^2-y^2}}$$

L.H.S. $= J.J^* = -e^{2v} \sec u \times \dfrac{1}{x\sqrt{x^2-y^2}}$

$$-(x^2-y^2) \times \left(\frac{1}{\sqrt{1-\frac{y^2}{x^2}}} \right) \times \frac{-1}{x\sqrt{x^2-y^2}}$$

$= 1 = R.H.S$

(4) (iii)

$$u = xy + yz + zx$$
$$v = x^2 + y^2 + z^2$$
$$w = x + y + z$$

$$J = \frac{\partial(u, v, w)}{\partial(x, y, z)}$$

$$= \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$$= \begin{vmatrix} y+z & x+z & y+x \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2 \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2(x+y+z) \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 0$$

$\therefore$ u, v and w are functionally dependent.

Relationship:

$$\omega^2 = x^2+y^2+z^2+2(xy+yz+zx)$$

$$\boxed{\omega^2 = v + 2u}$$

(iii) $u = \dfrac{x+y}{1-xy}$, $v = \tan^{-1}x + \tan^{-1}y$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{1+y^2}{(1-xy)^2} & \dfrac{1+x^2}{(1-xy)^2} \\ \dfrac{1}{1+x^2} & \dfrac{1}{1+y^2} \end{vmatrix} = 0$$

$\therefore$ u & v are functionally dependent.

Relationship:

$$v = \tan^{-1}x + \tan^{-1}y$$

$$v = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$v = \tan^{-1}u$$

$$\boxed{u = \tan v}$$

(7) $(\lambda-x)^3 + (\lambda-y)^3 + (\lambda-z)^3 = 0$

$$[\lambda^3 - x^3 - 3\lambda x(\lambda-x)] + [\lambda^3 - y^3 - 3\lambda y(\lambda-y)] + [\lambda^3 - z^3 - 3\lambda z(\lambda-z)] = 0$$

$$3\lambda^3 - 3\lambda^2(x+y+z) + 3\lambda(x^2+y^2+z^2) - (x^3+y^3+z^3) = 0$$

This is cubic equation in λ;

$\therefore$ u, v and w are roots of it; then

Sum of roots: ~~u+v+~~

$$u+v+w = \frac{-b}{a} = \frac{3(x+y+z)}{3}$$

$$u+v+w = x+y+z \quad \text{—— (1)}$$

Sum of Product of roots:

$$uv + vw + wu = \frac{c}{a} = \frac{3(x^2+y^2+z^2)}{3}$$

$$uv+vw+wu = x^2+y^2+z^2 \quad \text{—— (2)}$$

Product of roots:

$$uvw = \frac{-d}{a} = \frac{x^3+y^3+z^3}{3}$$

Or

$$3uvw = x^3+y^3+z^3 \quad \text{—— (3)}$$

Then

$$f_1 \equiv u+v+w \cancel{+} -x-y-z = 0$$

$$f_2 \equiv uv+vw+wu - x^2-y^2-z^2 = 0$$

$$f_3 \equiv 3uvw - x^3 - y^3 - z^3 = 0$$

Now,
$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}} \quad \text{--- (4)}$$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} = \begin{vmatrix} -1 & -1 & -1 \\ -2x & -2y & -2z \\ -3x^2 & -3y^2 & -3z^2 \end{vmatrix}$$

$$= -6\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= -6(x-y)(y-z)(z-x)$$

& $$\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} = \begin{vmatrix} 1 & 1 & 1 \\ v+w & u+w & u+v \\ 3vw & 3uw & 3uv \end{vmatrix}$$

$$= 3\begin{vmatrix} 1 & 1 & 1 \\ v+w & u+w & u+v \\ vw & uw & uv \end{vmatrix}$$

$C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$

$$= 3\begin{vmatrix} 0 & 0 & 1 \\ v-u & w-v & u+v \\ w(v-u) & u(w-v) & uv \end{vmatrix}$$

$$= 3(v-u)(w-v)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & u+v \\ w & u & uv \end{vmatrix}$$

$$= 3(v-u)(w-v)(u-w)$$

eqⁿ (4)

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{2(x-y)(y-z)(z-x)}{(v-u)(w-v)(u-w)}$$

Extrema of function of two variables :

Let $u = f(x,y)$ be a function of two variables.

A point $P(a,b)$ is said to be

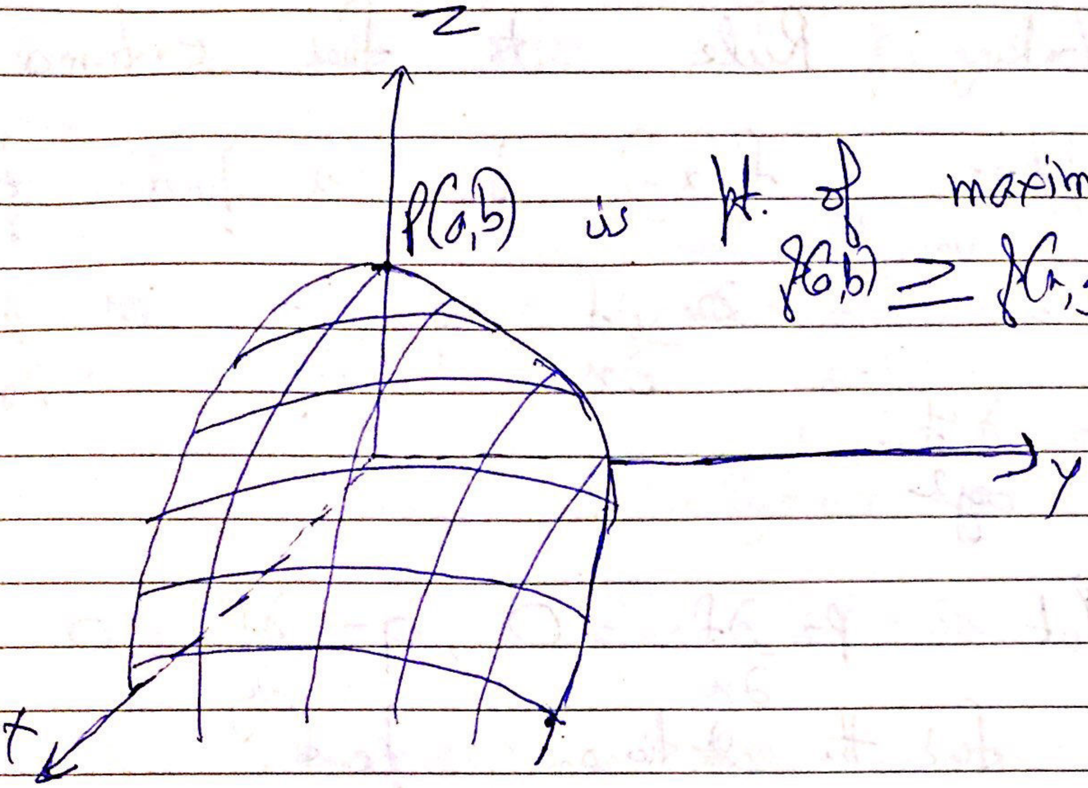
(i) A point of maxima if $f(a,b) \geq f(x,y)$

(ii) A point of minima if $f(a,b) \leq f(x,y)$

(iii) A point of neither maxima nor minima, if $f(a,b) \geq f(x,y)$ as well as $f(a,b) \leq f(x,y)$ in the neighbourhood of the point $P(a,b)$

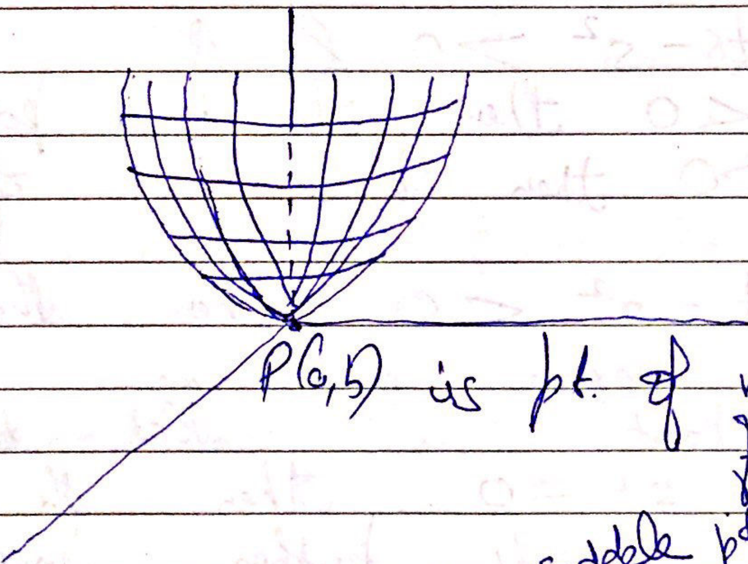
Such a point is called saddle point.

(i)



$f(a,b)$ is pt. of maxima as $f(a,b) \geq f(x,y)$

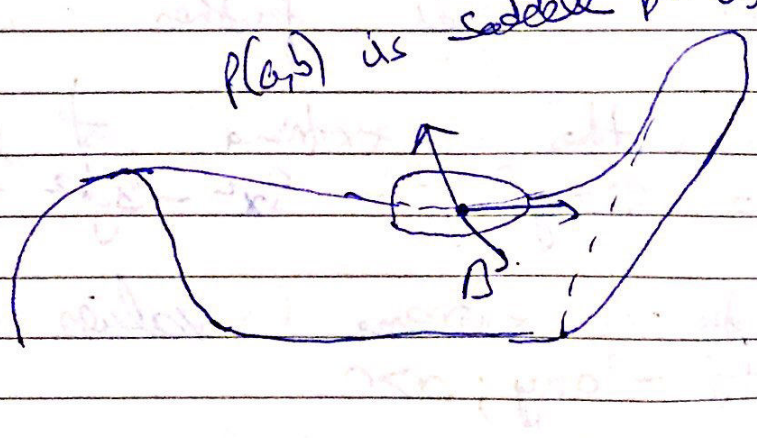
(ii)



$f(a,b)$ is pt. of minima as $f(a,b) \leq f(x,y)$

(iii)

$f(a,b)$ is saddle pt. as $f(a,b) \geq f(x,y)$ along AA' & $f(a,b) \leq f(x,y)$ along BB'



Working Rule to find extrima :-

Suppose $f(x,y)$ be a function of two variables.

(1) Find $p = \frac{\partial f}{\partial x}$, $q = \frac{\partial f}{\partial y}$, $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$

$t = \frac{\partial^2 f}{\partial y^2}$

(2) Put $p = \frac{\partial f}{\partial x} = 0$, $q = \frac{\partial f}{\partial y} = 0$

To find the stationary points.

(3) If $rt - s^2 > 0$ & if

(i) $r < 0$ then it is point of maxima.

(ii) $r > 0$ then it is point of minima.

(4) If $rt - s^2 < 0$ then there is neither maxima nor minima

Such a point is called saddle point.

If $rt - s^2 = 0$ then the case is doubtful & needs further investigation.

Ques-1 Find the extrima of the function

$f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 7$

Ques-2 Find the extreme values of a

$x^3 + y^3 - 3axy$; $a > 0$

Ques-3 Show that the minimum value of

$f(x,y) = xy + \frac{a^3}{x} + \frac{a^3}{y}$ is $3a^2$.

Que. 4 Find the extrema of $f(x,y) = x^4+y^4-2x^2+4xy-2y^2$.

Que. 5 Find the minimum value of $x^2+y^2+z^2$ with the constraint $x+y+z = 3a$

(Hint: Here $f(x,y) = x^2+y^2+(3a-x-y)^2$)

① $f(x,y) = x^3+3xy^2-3x^2-3y^2+7$

Here $p = \frac{\partial f}{\partial x} = 3x^2+3y^2-6x$

$q = \frac{\partial f}{\partial y} = 6xy-6y$, $r = \frac{\partial^2 f}{\partial x^2} = 6x-6$

$s = \frac{\partial^2 f}{\partial x \partial y} = 6y$, $t = \frac{\partial^2 f}{\partial y^2} = 6x-6$

Put $p = \frac{\partial f}{\partial x} = 0$ & $q = \frac{\partial f}{\partial y} = 0$

$x^2+y^2-2x = 0$ — ① & $xy-y=0$

$y(x-1) = 0$

$\Rightarrow y = 0,\ x = 1$

If $y=0$, eqn ① gives $\Rightarrow x(x-2) = 0$

$x = 0, 2$

$\therefore (0,0)$ & $(2,0)$ are stationary points.

If $x=1$; eqn ① gives; $y^2-1=0 \Rightarrow y = \pm 1$

$(1,1)$ & $(1,-1)$ are st. pts.

Stationary pts	r	s	t	$rt-s^2$	Conclusion	Value of $f(x,y)$
$(0,0)$	$-6(<0)$	0	-6	$36(>0)$	pt. of maxima	$f_{max} = f(0,0) = 7$
$(2,0)$	$6(>0)$	0	6	$36(>0)$	pt. of minima	$f_{min} = f(2,0) = 3$
$(1,1)$	0	6	0	$-36(<0)$	Saddle pt.	
$(1,-1)$	0	-6	0	$-36(<0)$	Saddle pt.	

(2) $f(x,y) = x^3 + y^3 - 3axy, \quad a>0$

$p = \frac{\partial f}{\partial x} = 3x^2 - 3ay \; ; \; q = \frac{\partial f}{\partial y} = 3y^2 - 3ax$

$r = \frac{\partial^2 f}{\partial x^2} = 6x, \quad s = \frac{\partial^2 f}{\partial x \partial y} = -3a, \quad t = \frac{\partial^2 f}{\partial y^2} = 6y$

Put $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$

$x^2 - ay = 0 \quad \text{——— (1)}$ & $y^2 - ax = 0 \quad \text{——— (2)}$

from (1) $y = \frac{x^2}{a}$, put in (2)

$\frac{x^4}{a^2} - ax = 0$

$x(x^3 - a^3) = 0$

$x = 0, \quad (x-a)(x^2 + a^2 + ax) = 0$

$x = 0, \quad x = a, \quad x = \frac{-a \pm ia\sqrt{3}}{2}$ (Rejected)

If $x=0$, eq^n (2), $y^2=0 \Rightarrow y=0$ [$(0,0)$ is st. pt]

If $x=a$, eq^n (2), $y^2-a^2=0 \Rightarrow y=\pm a$; $(a,\pm a)$ are st. pts.

! Caution $(a,-a)$ doesn't satisfy eq^n (1), the only st. point is (a,a)

St. pt	r	s	t	$rt-s^2$	Conclusion	Value
$(0,0)$	0	$-3a$	0	$-9a^2$	Saddle pt	—
(a,a)	$6a(>0)$	$-3a$	$6a$	$27a^2$	Pt of min	

$f_{min} = f(a,a) = -a$

(4) $f(x,y) =$

$(0,0)$

(a,a)

Double Integration

Let $f(x,y)$ be a function of two variables which is defined in closed & bounded region R. The double integration of the function $f(x,y)$ over the region R is defined as

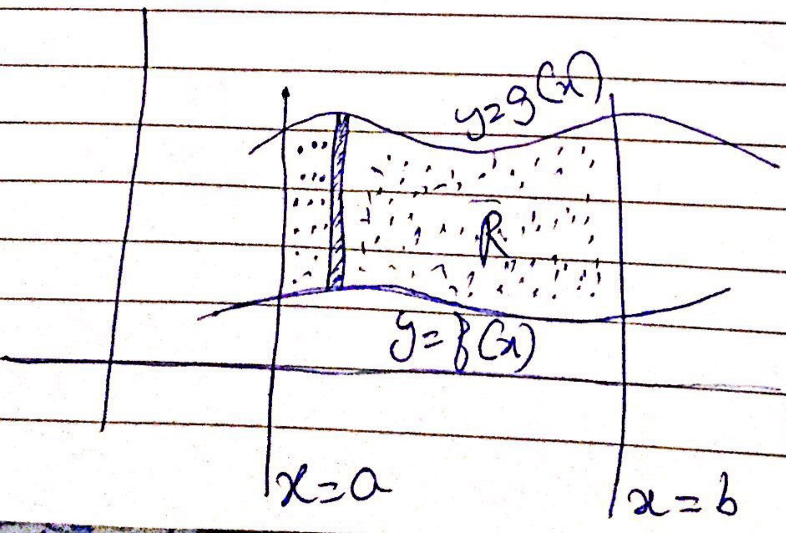
$$\iint_R f(x,y)\,dx\,dy$$

Note : 1) $\iint_R f(x,y)\,dx\,dy$ represent the sum of values of the function $f(x,y)$ over the region R.

2) $\iint_R dx\,dy$ represent the area of the region R.

There are two cases :-

Case 1: If the region R is bounded by curves $y = f(x)$ & $y = g(x)$ between the lines $x = a$ & $x = b$



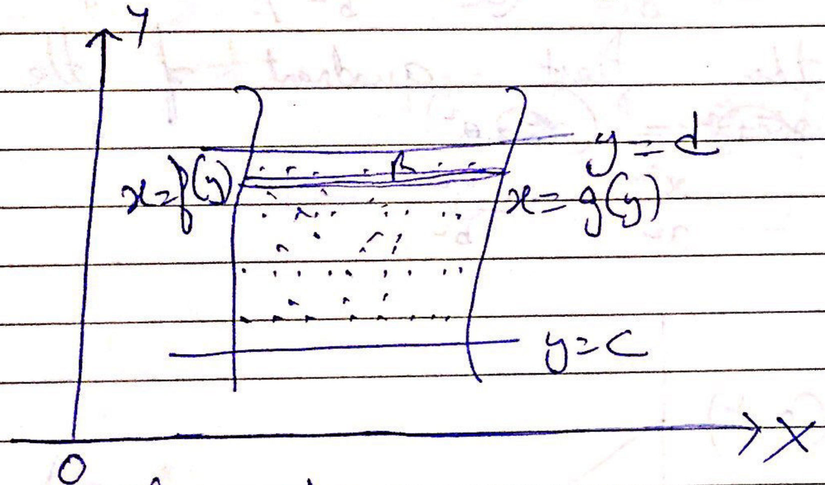
$$\iint f(x,y)\,dx\,dy$$

Limit of y : $f(x)$ to $g(x)$ {base & top of strip}

limit of x : a to b {movement of strip}

Then $\iint_R f(x,y)\,dx\,dy = \int_{x=a}^{b}\int_{y=f(x)}^{g(x)} f(x,y)\,dy\,dx$

Case 2: If the region R is bounded by the curves $x=f(y)$ and $x=g(y)$ between the lines $y=c$ & $y=d$.



limit of x : $f(y)$ to $g(y)$

limit of y : c to d

Then $\iint_R f(x,y)\,dx\,dy = \int_{y=c}^{d}\int_{x=f(y)}^{g(y)} f(x,y)\,dx\,dy$

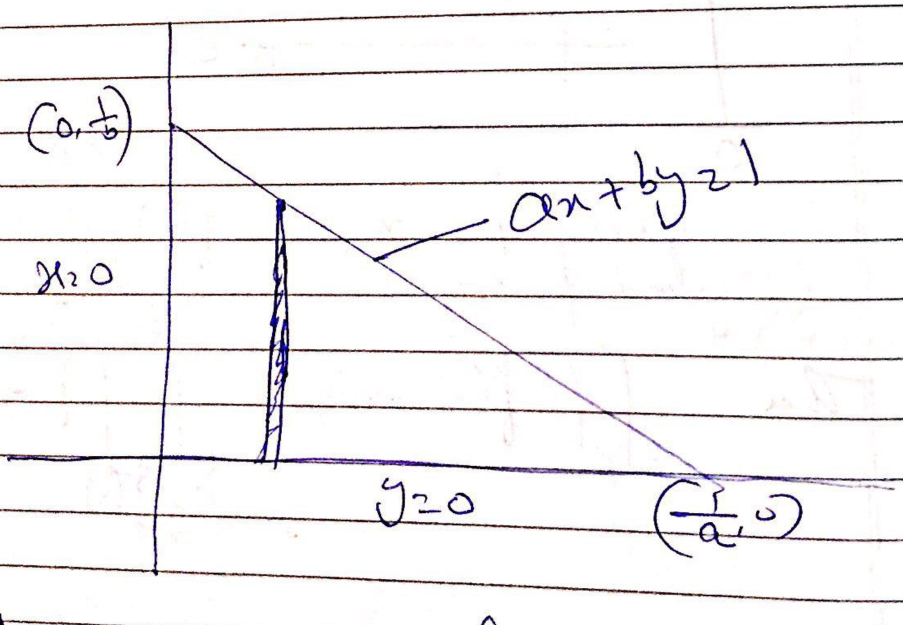
Ques- Evaluate $\iint_R e^{ax+by}\,dx\,dy$, where R is the region bounded by the triangle with the lines $x=0$, $y=0$, and $ax+by=1$, $a>0, b>0$

Ques- Evaluate $\iint_R \frac{xy}{\sqrt{1-y^2}}\,dx\,dy$ where R is the region in the first quadrant of the circle $x^2+y^2=1$.

Ques- Evaluate $\iint xy\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{\frac{n}{2}}dx\,dy$ over the first quadrant of the ellipse $\cancel{x^2+y^2=(x^2+a^2)}$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

①



Limits of y: 0 to $\left(\frac{1-ax}{b}\right)$

Limits of x: 0 to $\frac{1}{a}$

$$I = \int_{x=0}^{\frac{1}{a}} \int_{y=0}^{\left(\frac{1-ax}{b}\right)} e^{ax+by}\, dy\, dx$$

$$= \int_{x=0}^{\frac{1}{a}} \int_{y=0}^{\frac{1-ax}{b}} e^{ax} \cdot e^{by}\, dy\, dx$$

$$= \int_{x=0}^{\frac{1}{a}} e^{ax} \left[\frac{e^{by}}{b} \right]_{y=0}^{\frac{1-ax}{b}} dx$$

$$= \frac{1}{b} \int_{0}^{\frac{1}{a}} e^{ax} \left[e^{1-ax} - 1 \right] dx$$

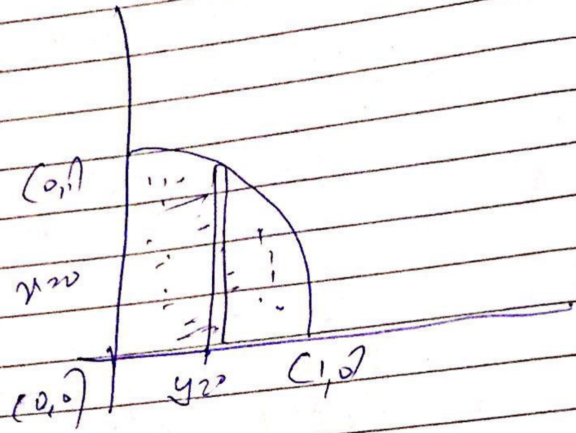
$$= \frac{1}{b} \int_{x=0}^{\frac{1}{a}} (e^{1} - e^{ax})\, dx$$

$$= \frac{1}{b} \left[ex - \frac{e^{ax}}{a} \right]_{0}^{1/a}$$

$$= \frac{1}{b} \left[\left(\frac{e}{a} - \frac{e}{a} \right) - \left(0 - \frac{1}{a} \right) \right]$$

$$= \frac{1}{ab}$$

② Region R,



limits of y : 0 to $\sqrt{1-x^2}$

limit of x : 0 to 1

Now :

$$I_2 = \int_{x=0}^{1} x \int_{y=0}^{\sqrt{1-x^2}} \left(\frac{y\,dy}{\sqrt{1-y^2}}\right) dx$$

M. Imp.

$$\boxed{\text{Note : } \int [f(x)]^n f'(x) = \frac{[f(x)]^{n+1}}{n+1} + c}$$

$$= \frac{-1}{2} \int_{x=0}^{1} x \int_{y=0}^{\sqrt{1-x^2}} \left[(1-y^2)^{-1/2} (-2y)\,dy\right] dx$$

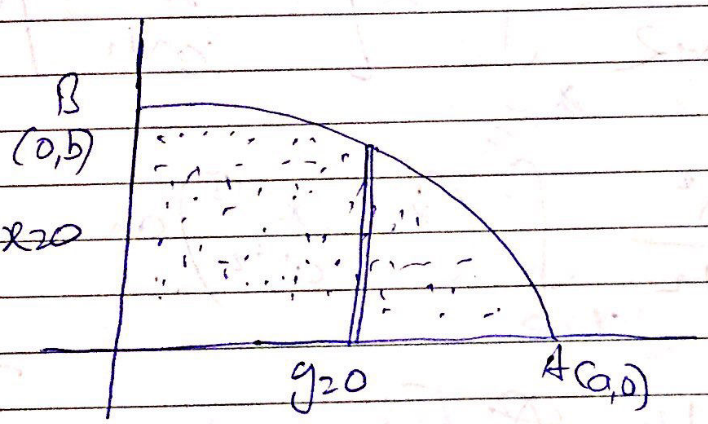
$$= \frac{-1}{2} \int_{x=0}^{1} \left[\frac{(1-y^2)^{1/2}}{1/2}\right]_{y=0}^{\sqrt{1-x^2}} dx$$

$$I = -\int_{x=0}^{1} x(x-1)\,dx$$

$$I = \int_{x=0}^{1} (x-x^2)\,dx$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

(3) Region R



limit of y : 0 to $\frac{b}{a}\sqrt{a^2-x^2}$

limit of x : 0 to a

Evaluation of I :

$$I = \int_{x=0}^{a} \int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} xy\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{n/2} dy\,dx$$

$$\frac{2b^2}{2}\int_{x=0}^{a} x \int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{n/2}\left(\frac{2y}{b^2}\right) dy\, dx$$

$$= \frac{b^2}{2}\int_{x=0}^{a} x \left[\frac{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{\frac{n}{2}+1}}{\frac{n}{2}+1}\right]_{0}^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$I = \frac{b^2}{n+2}\int_{x=0}^{a} x\left[1-\frac{x^{n+2}}{a^{n+2}}\right] dx$$

$$I = \frac{b^2}{n+2}\int_{x=0}^{a}\left(x-\frac{x^{n+3}}{a^{n+2}}\right) dx$$

$$= \frac{b^2}{n+2}\left[\frac{a^2}{2}-\frac{a^2}{n+4}\right]$$

$$= \frac{b^2a^2}{(2n+8)}$$

Que 3

Que

Ques

Que 4

1

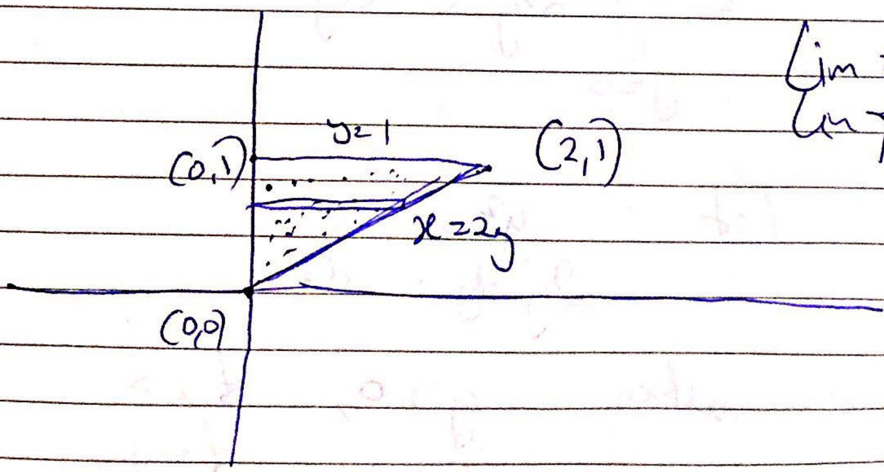
Que Evaluate $\iint e^{y^2}\,dx\,dy$ over the region bounded by triangle with vertices $(0,0)$; $(0,1)$; and $(2,1)$.

Que 2 Evaluate $\iint x^2\,dx\,dy$ over the region bounded by rectangular hyperbola $xy = 16$ and the lines $y=x$, $y=0$ and $x=8$.

Que 3 Evaluate $\iint \frac{dx\,dy}{x^4+y^2}$ over the region bounded by $y \geq x^2$ & $x \geq 1$

Que 4 Evaluate $\iint (a-x)^2\,dx\,dy$ over the right half of circle $x^2+y^2=a^2$

(1)



Lim of x: 0 to $2y$
Lim of y: 0 to 1.

Caution:

(1) Here the integrant e^{y^2} is difficult of integrate w.r.t y first which

Outer limits are those of motion.

implies the inner limits of integration consist of y.

In order to have limits of y as outer limits, the motion of the strip must be along y axis for which the strip has to be horizontal.

$$I = \int_{y=0}^{1}\int_{x=0}^{2y} e^{y^2}\,dx\,dy$$

$$= \int_{y=0}^{1} e^{y^2}\,[x]_{x=0}^{2y}\,dy$$

$$= \int_{y=0}^{1} 2y\, e^{y^2}\,dy$$

Let $y^2 = t$

$2y\,dy = dt$

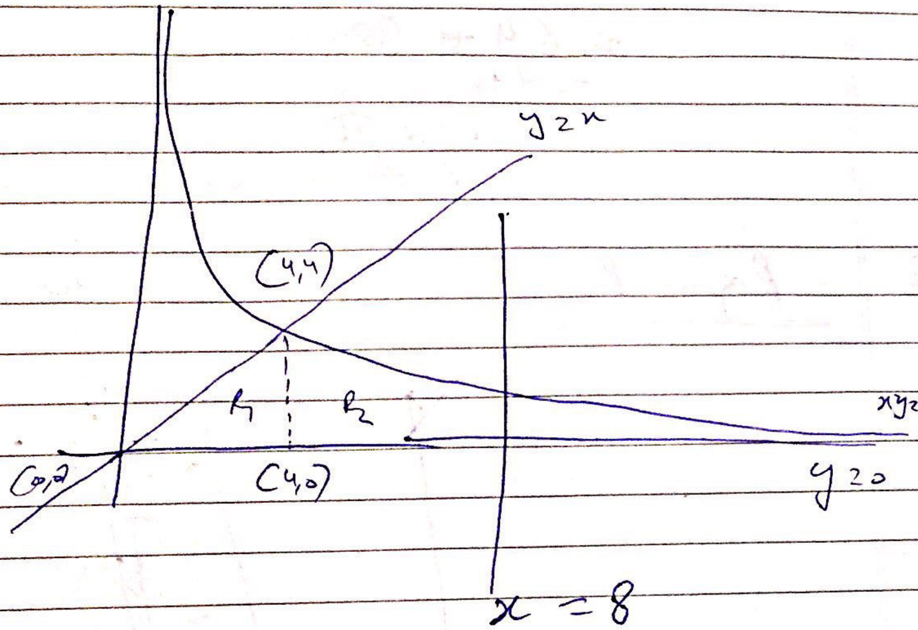
when $y=0,\ t=0$

$y=1,\ t=1$

$$I = \int_0^1 e^t\,dt$$

$= [e^t$

(2)



for R_1,

limit of x : 0 to 4

limit of y : 0 to x

for R_2,

limit of x : 4 to 8

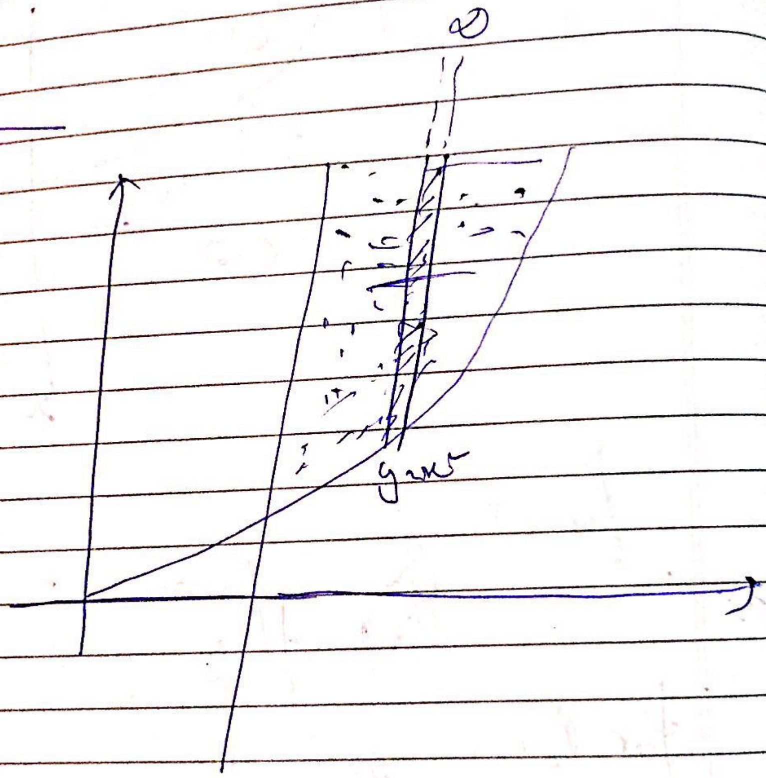
limit of y : 0 to $\frac{16}{x}$

$$I = \iint_{R_1} x^2 \, dx \, dy + \iint_{R_2} x^2 \, dx \, dy$$

$$= \int_{x=0}^{4} \int_{y=0}^{x} x^2 \, dy \, dx + \int_{x=4}^{8} \int_{y=0}^{\frac{16}{x}} x^2 \, dy \, dx$$

$= 64 + 384$
$= 448$

③ Region R:



⚠ Caution:

Here $\frac{1}{x^4+y^2}$ is difficult to integrate wrt x first which implies that limits of x must be outside limits, which in turn tells us that movement must be along x-axis, that is vertical strip should be drawn.

limits of y : x^2 to ∞

limits of x : 1 to ∞

$$I = \int_{x=1}^{\infty}\int_{y=x^2}^{\infty} \frac{1}{y^2+(x^2)^2}\, dy\, dx$$

$$= \int_{x=1}^{\infty} \frac{1}{x^2}\left[\tan^{-1}\left(\frac{y}{x^2}\right)\right]_{y=x^2}^{\infty} dx$$

$$= \int_{1}^{\infty} \frac{1}{x^2}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) dx$$

$$= \frac{\pi}{4}\int_{1}^{\infty} \frac{dx}{x^2}$$

$$= \pi/4$$

$\frac{\pi}{2}$ R

Change of order of integration :-

Consider a double integral

$$I = \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x,y)\, dx\, dy$$

Sometimes it is difficult to integrate wrt x first but easier to integrate wrt y in that case change of order of integration is used, it consists of following steps :-

Step 1) Draw region corresponding to the original limits of the integral (keeping in mind the fact that the inner limits decides the nature of strip & the outer limits decides the motion of strips.

Step 2) Change the nature of strip by rotating it through 90° angle inside ~~side~~ of the region.

Step 3) find the New limits of x & y & evaluate the given ~~at~~ integral I) with then new limits

Ques-1

Ques-2

Ques-3

Ques-4

Ques-5

1

Ques-1 $$\int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y}\, dy\, dx$$

Ques-2 $$\int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\, dy\, dx$$

Ques-3 $$\int_0^{\infty}\int_0^{x} x e^{-x^2/y}\, dy$$

Ques-4 Prove:

$$\int_0^{1}\int_0^{\sqrt{1-y^2}} \frac{\cos^{-1}x}{\sqrt{1-x^2}} \frac{dx\, dy}{\sqrt{1-x^2-y^2}} = \frac{\pi^3}{16}$$

Ques-5 Prove:

$$\int_0^{a}\int_0^{a-\sqrt{a^2-y^2}} \frac{xy \log(x+a)}{(x-a)^2}\, dx\, dy = \frac{a^2}{8}(1+2\log a)$$

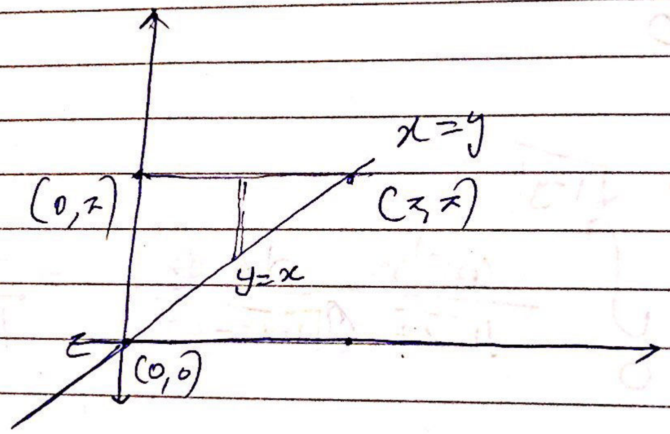
① $$I = \int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y}\, dy\, dx$$

⚠ Caution: Clearly the inner limits are of y but the integrant $\frac{\sin y}{y}$ is difficult to in integrant wrt y. first & easier to integrate wrt x.

Hence change of order of integration must be used here:

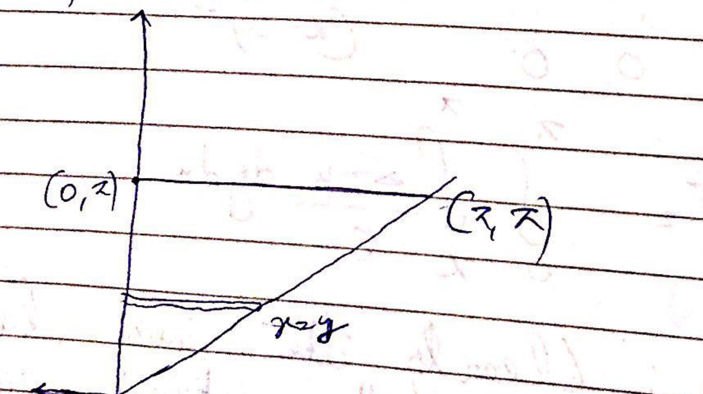
$$I = \int_{0}^{\pi}\int_{x}^{\pi} \frac{\sin y}{y}\,dy\,dx$$

Limit of y : x to π
Limit of x : 0 to π



2

Change of



$$\int_{y=0}^{\pi}\int_{x=0}^{y} \frac{\sin y}{y}\,dx\,dy$$

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$$\int_{y=0}^{x} \frac{\sin y}{\cancel{y}} \cancel{y}\, dy$$

$$-\left[\cos y\right]_0^{x}$$

$$-(\cos x - \cos 0)$$

$$-(-1-1)$$

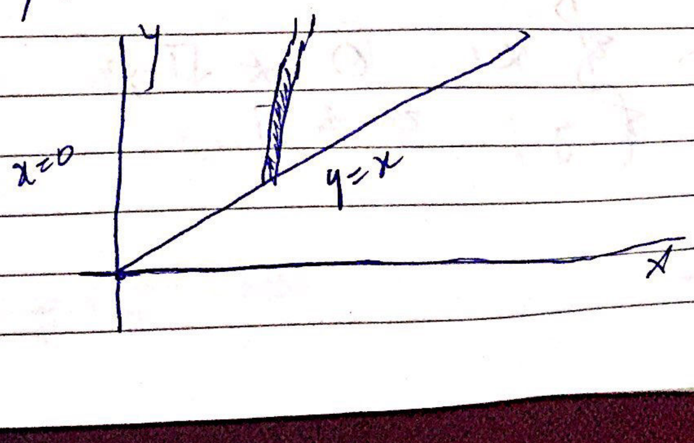
$$2$$

② $$I = \int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\, dy\, dx$$

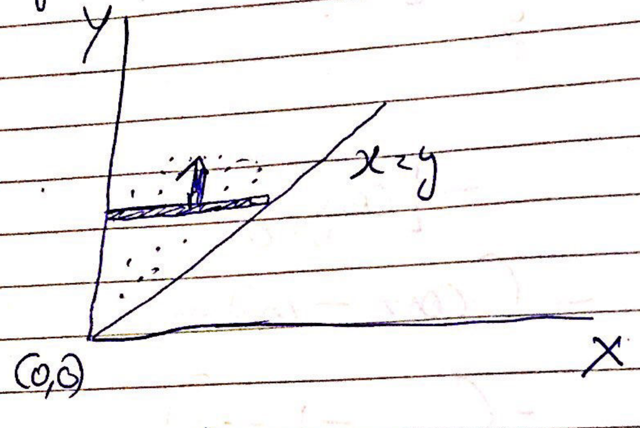
Clearly we can see that ~~of~~ integrating $\frac{e^{-y}}{y}$ wrt y is difficult that integrating it wrt x.

Lim of $y = x$ to ∞

Lim of $x = 0$ to ∞



Change of order of integration :



limit of x : 0 to y.
limit of y : 0 to ∞

find of I

$$I = \int_{y=0}^{\infty}\int_{x=0}^{y} \frac{e^{-y}}{y}\,dx\,dy$$

$I = 1$

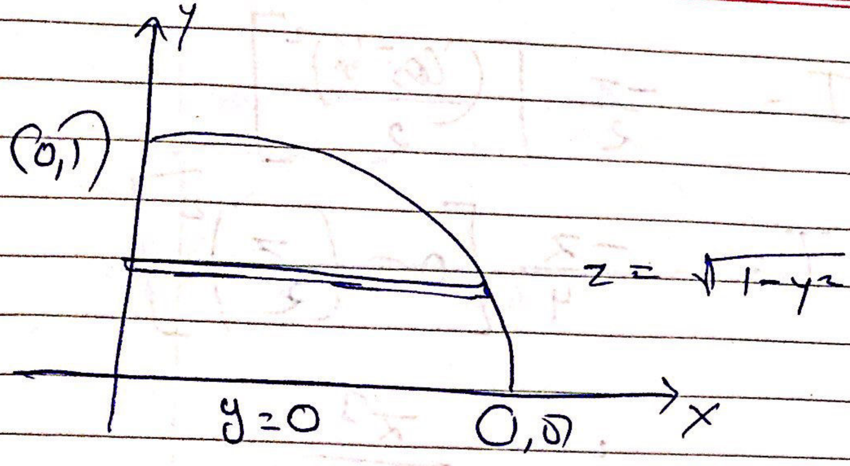
(b)

(4)

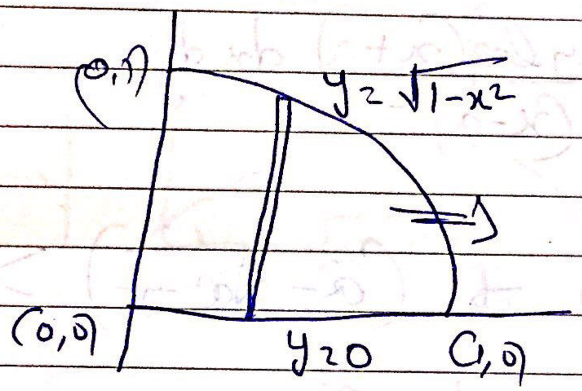
$$I = \int_{0}^{1}\int_{0}^{\sqrt{1-y^2}} \frac{\cos^{-1}x}{\sqrt{1-x^2}}\,\frac{1}{\sqrt{1-x^2-y^2}}\,dx\,dy$$

(c)

lim of x : 0 to $\sqrt{1-y^2}$
lim of y : 0 to 1



(b) Change of



Limit of y : 0 to $\sqrt{1-x^2}$

x : 0 to 1.

(c)

$$I = \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} \frac{\cos^{-1}x}{\sqrt{1-x^2}}\left[\frac{1}{(\sqrt{1-x^2})^2}\frac{dy}{(y^2)}\right]dx$$

$$I = \int_{x=0}^{1} \sin^{-1}\left(\frac{y}{\sqrt{1-x^2}}\right)\Bigg]_{y=0}^{\sqrt{1-x^2}}$$

$$I = \int_{x=0}^{1} \frac{\cos^{-1}x}{\sqrt{1-x^2}}\left(\frac{\pi}{2}-0\right)dx$$

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$$I = \frac{-\pi}{2}\left[\frac{(\cos^{-1}x)^2}{2}\right]$$

$$I = \frac{-\pi}{4}\left[0 - \left(\frac{\pi}{2}\right)^2\right]$$

$$= \frac{\pi^3}{16}$$

(5) $\displaystyle\int_0^a \int_0^{a-\sqrt{a^2-y^2}} \frac{xy\log(x+a)}{(x-a)^2}\,dx\,dy$ (b)

lim $\begin{cases} x : 0 \text{ to } \left(a-\sqrt{a^2-y^2}\right) > 0, \text{ if } a>0 \\ y : 0 \text{ to } a \end{cases}$

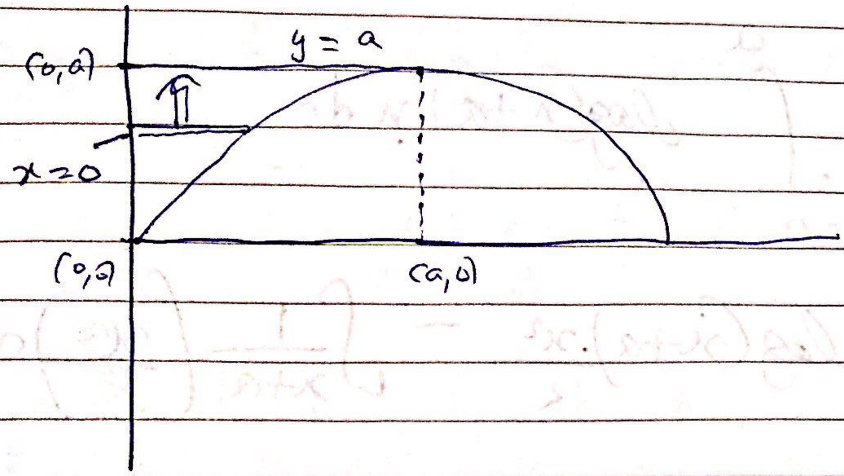
$x = a - \sqrt{a^2-y^2}$

$x - a = -\sqrt{a^2-y^2}$

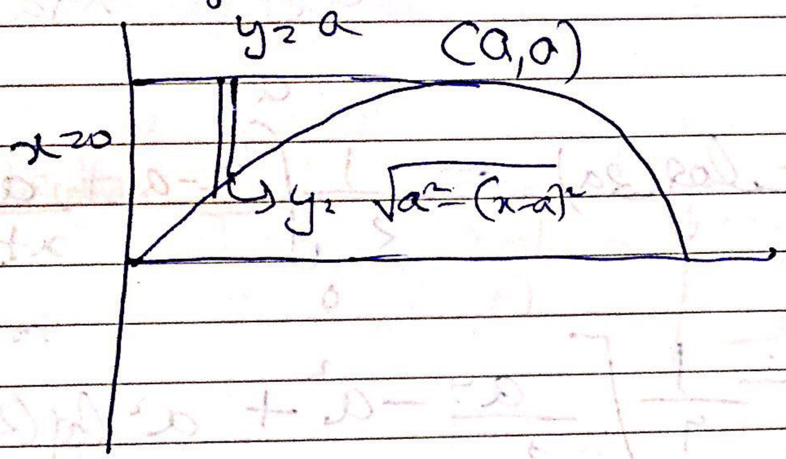
$\Rightarrow (x-a)^2 = a^2 - y^2$

$(x-a)^2 + y^2 = a^2$

Circle. (c)



(b) Change of order of Integration



limits of y : $\sqrt{2ax-x^2}$ to a
" " x : x to a

(c) Evaluation of I :

$$I = \int_{x=0}^{a} \int_{y=\sqrt{2ax-x^2}}^{a} [y\,dy] \frac{x \log(x+a)}{(x+a)^2} dx$$

$$= \frac{1}{2} \int_{x=0}^{a} (a^2 - 2ax + x^2) \frac{x \log(x+a)}{(x-a)^2} dx$$

$$= \frac{1}{2}\int_{x=0}^{a} \log(x+a)\, x\, dx$$

$$= \frac{1}{2}\left[\log(x+a)\cdot\frac{x^2}{2} - \int \frac{1}{x+a}\left(\frac{x^2}{2}\right)dx\right]_0^a$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\log(x+a) - \frac{1}{2}\int \frac{x^2-a^2+a^2}{x+a}\,dx\right]_0^a$$

$$\frac{1}{2}\left[\frac{a^2}{2}\log(2a) - \frac{1}{2}\int_0^a x-a+\frac{a^2}{x+a}\,dx\right]$$

$$\frac{a^2}{4}\log(2a) - \frac{1}{4}\left[\frac{a^2}{2} - a^2 + a^2\log(2)\right]$$

$$\frac{a^2}{4}\log(2a) + \frac{a^2}{8} - \frac{1}{4}a^2\log(2)$$

$$\frac{a^2}{8}\left[1 + 2\log(2a) - 2\log(2)\right]$$

$$\frac{a^2}{8}\left[1 + 2\log\left(\frac{2a}{2}\right)\right]$$

$$\frac{a^2}{8}(1 + 2\log a)$$

$a[\log x+a]_0$

$\log(2a)-\log(a)$

$\log\left(\frac{2a}{a}\right)$

Changing into Polar Coordinates:

Consider $I = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)\, dy\, dx$

Sometimes it is difficult to integrate wrt x as well as y first then changing into polar coordinates may work.

The ~~process~~ process is as follows:-

Put $x = r\cos\theta$, $y = r\sin\theta$ then the Jacobian

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r$$

Replace ~~dx~~ dx, dy by $|J|\, dr\, d\theta$

or $r\, dr\, d\theta$

then $I = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$

The outer limits are always of θ as they are always constant.

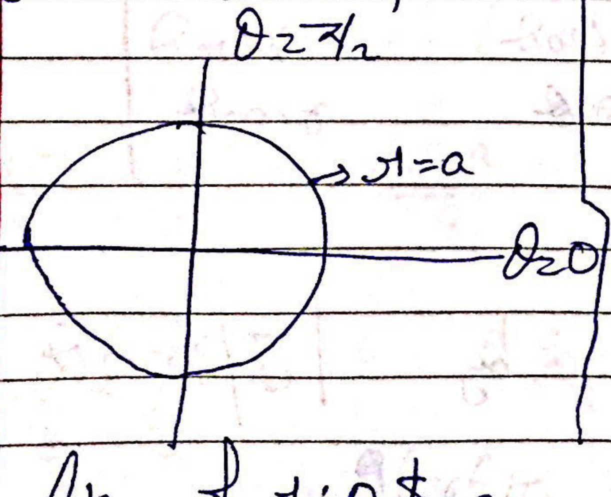
<u>Basic Facts about r-θ plane :</u>

Similar to the coordinate axes (x-axis, y-axis), we have ~~axes~~ initial line $\theta = 0$ and $\theta = \frac{\pi}{2}$ (pole).

~~&~~ Similar to the horizontal or vertical strip inside the region of x-y plane, we draw an elementary strip which is radial in nature.

Equation of substandard circle

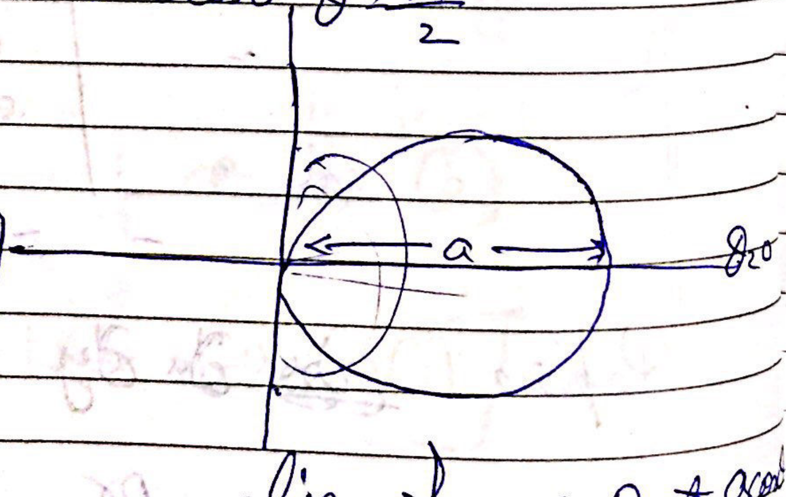
$r = a$ i.e. $x^2 + y^2 = a^2$



lim of r : 0 to a

lim of θ : 0 to 2π

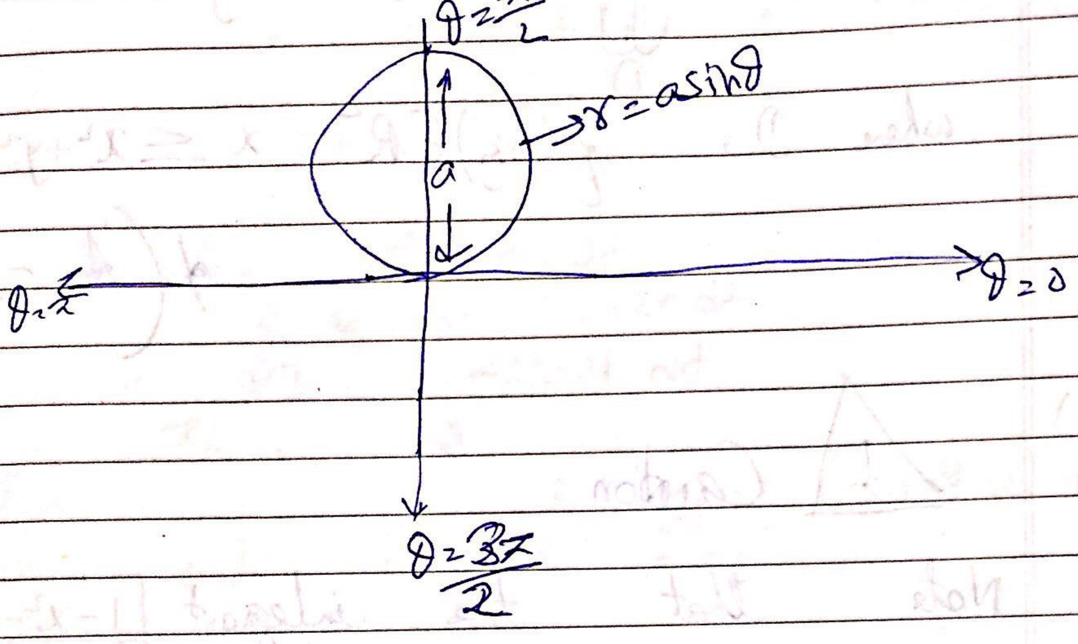
$r = a\cos\theta$



lim of r : 0 to $a\cos\theta$

lim of θ : $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

$r = a\sin\theta$



lim of r: 0 to $a\sin\theta$

lim of θ: 0 to π

Ques- Evaluate $\iint \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\,dx\,dy$ over the first quadrant of circle $x^2+y^2=1$.

Ques Evaluate $\iint \frac{4xy}{x^2+y^2}\left(e^{-x^2-y^2}\right)dx\,dy$ over the region bounded by the circle $x^2+y^2-x=0$ in first quadrant. (Ans- $\frac{1}{e}$)

Ques Evaluate $\iint \frac{x^2y^2}{x^2+y^2}\,dx\,dy$ over the region bounded by the circles $x^2+y^2=a^2$ & $x^2+y^2=b^2$, $a>b$. (Ans- $\frac{\pi}{16}(a^4-b^4)$)

Que— Evaluate $\iint_D x^2+y^2 \, dx\,dy$

where $D = \{(x,y) \in R^2 : x \le x^2+y^2 \le 2x\}$

$\left(\text{Ans} - \frac{4}{9}\right)$

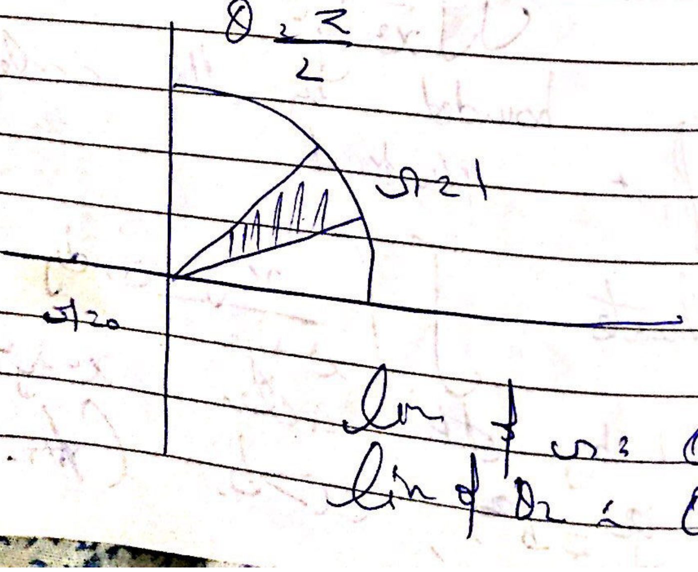
Ans-1 ⚠ Caution:

Note that the integrand $\sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}$ is difficult to integrate wrt x as well as y hence changing into polar coordinates may work.

Put $x = r\cos\theta$, $y = r\sin\theta$
& $dx\,dy = r\,dr\,d\theta$

Circle: $x^2+y^2=1$
$r^2\cos^2\theta + r^2\sin^2\theta = 1$

$r = 1$



lim of r: 0 to 1
lim of θ: 0 to $\frac{\pi}{2}$

Then

$$I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} \sqrt{\frac{1-r^2}{1+r^2}}\, r\,dr\,d\theta$$

Put $r^2 = \cos 2t$

$\cancel{2} r\,dr = \cancel{2}(-\sin 2t\,dt)$

$r\,dr = -\sin 2t\,dt$

$$I = \int_{\theta=0}^{\pi/2} \int_{t=\pi/4}^{0} \sqrt{\frac{1-\cos 2t}{1+\cos 2t}}\,(-\sin 2t\,dt)\,d\theta$$

$$= \int_{\theta=0}^{\pi/2} d\theta \times \int_{t=0}^{\pi/4} \tan t \sin 2t\,dt$$

$$= \frac{\pi}{2} \times \int_{t=0}^{\pi/4} 2\sin^2 t\,dt$$

$$= \pi \int_{t=0}^{\pi/4} \left(\frac{1-\cos 2t}{2}\right) dt$$

$$= \frac{\pi}{2}\left[\frac{\pi}{4} - \frac{1}{2}\right] \underline{A}$$

(2) Put $x = r\cos\theta$, $y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

Circle: $x^2+y^2-x = 0$

$r^2\cos^2\theta + r^2\sin^2\theta - r\cos\theta = 1$

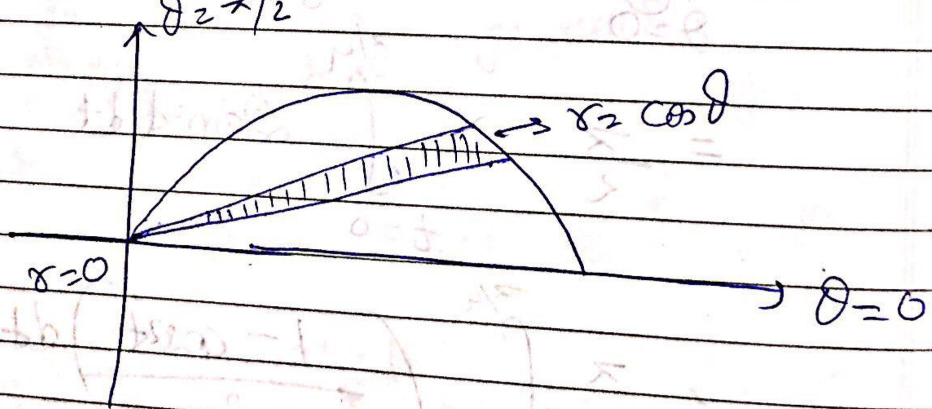
$r^2 - r\cos\theta = 0$

$r(r-\cos\theta) = 0$

$r = 0$ $\quad\quad$ $r = \cos\theta$

$\downarrow$

pt. circle.



limits of r: 0 to $\cos\theta$

limits of θ: 0 to $\frac{\pi}{2}$

$$I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{\cos\theta} \frac{4r^2\cos^2\theta\sin\theta\, e^{-r^2}}{r^2}(r\,dr\,d\theta)$$

$$= -\int_{\theta=0}^{\pi/2}\int_{r=0}^{\cos\theta} e^{-r^2}(-2r)\,dr\,(\sin 2\theta\, d\theta$$

*Note: $\boxed{\int e^{f(x)} f'(x)\,dx = e^{f(x)} + C}$

$$I = -\int_{\theta=0}^{\pi/2} \left[e^{-r^2}\right]_{r=0}^{\cos\theta} \sin 2\theta\, d\theta$$

$$= -\int_{0}^{\pi/2} \left(e^{-\cos^2\theta} - 1\right)(2\sin\theta\cos\theta)\,d\theta$$

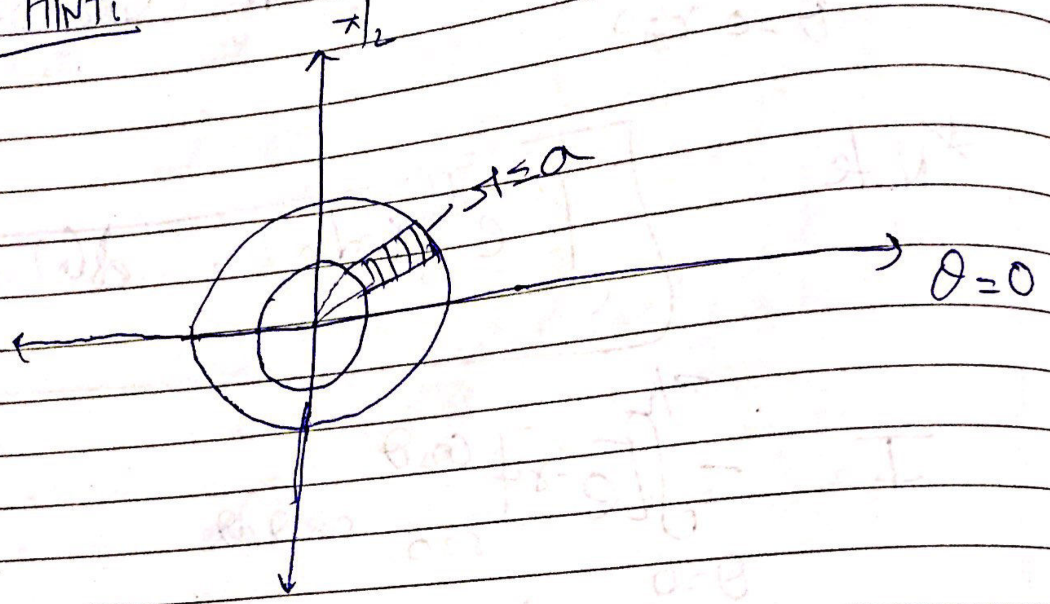
$$= \int_{0}^{\pi/2} \sin 2\theta - \int_{0}^{\pi/2} e^{-\cos^2\theta}(2\sin\theta\cos\theta)\,d\theta$$

$$= 1 - \left[e^{-\cos^2\theta}\right]_{0}^{\pi/2}$$

$$= 1 - (1 - e^{-1})$$

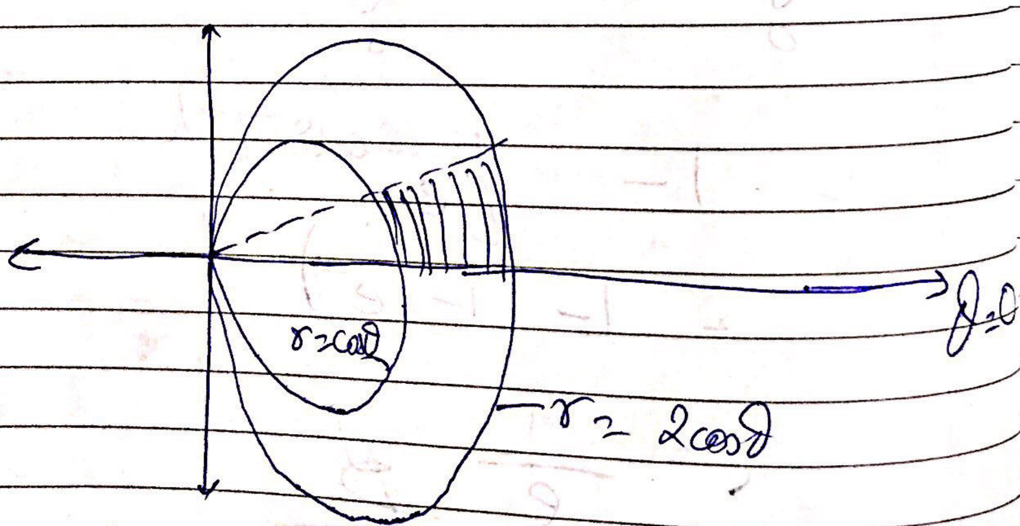
$$= \frac{1}{e} \quad \text{Ans}$$

③ HINT:



lim of r: b to a
lim of θ: 0 to 2π

④ HINT:



lim of r: $\cos\theta$ to $2\cos\theta$
lim of θ = $\frac{-\pi}{2}$ to $\frac{\pi}{2}$

$$D = \{(x,y) \in \mathbb{R}^2 : x \leq x^2 + y^2 \leq 2x\}$$

Change of Variables

Consider $I = \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x,y)\,dx\,dy$

Sometimes it is difficult to integrate wrt x as well as y. Also changing into polar coordinates may not work. In that case changing in ~~variables~~ variables into new variables (u,v) may work.

The process is as follows:

Suppose $x = g(u,v)$ and $y = h(u,v)$ be the given transformation then

$$I = \int_{v_1}^{v_2} \int_{u_1}^{u_2} f(g(u,v), h(u,v))\,|J|\,du\,dv$$

where $|J| = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$

Remark: Change of variables is also used when the region R has curved boundaries.

Ques- Evaluate $\iint \cos\left(\frac{x-y}{x+y}\right) dx\,dy$ over the region bounded by lines $x=0$, $y=0$, and $x+y=1$ Given the transformations $x-y=u$ and $x+y=v$.

(Ans. $\frac{\sin 1}{2}$)

Ques- Using the transformations $x^2-y^2=u$ and $2xy=v$ find $\iint (x^2+y^2)\,dx\,dy$ over the region in Ist quadrant bounded by $x^2-y^2=1$, $x^2-y^2=2$, $xy=2$ and $xy=4$.

(Ans- 1)

Ques- Using the transformation $x+y=u$, $y=uv$ prove that $\int_0^1\int_0^{1-x} e^{y/(x+y)}\,dy\,dx = \frac{(e-1)}{2}$

Ques- Using the transformation $x=u(1+v)$, $y=v(1+u)$, $u\geq 0$, $v\geq 0$. Evaluate

$$\int_0^2\int_0^y \left[(x-y)^2+2(x+y)+1\right]^{-\frac{1}{2}} dx\,dy$$

(Ans- $\log 4-\frac{1}{2}$)

Ques- Evaluate $\iint xy\,dy\,dx$ by changing the variables over the region in Ist quadrant, bounded by the hyperbolas $x^2-y^2=a^2$,

$x^2-y^2=b^2$ and circles $x^2+y^2=c^2$,
$x^2+y^2=d^2$, where $0<a<b<c<d$.

(Ans $\dfrac{(a^2-b^2)(c^2-d^2)}{8}$

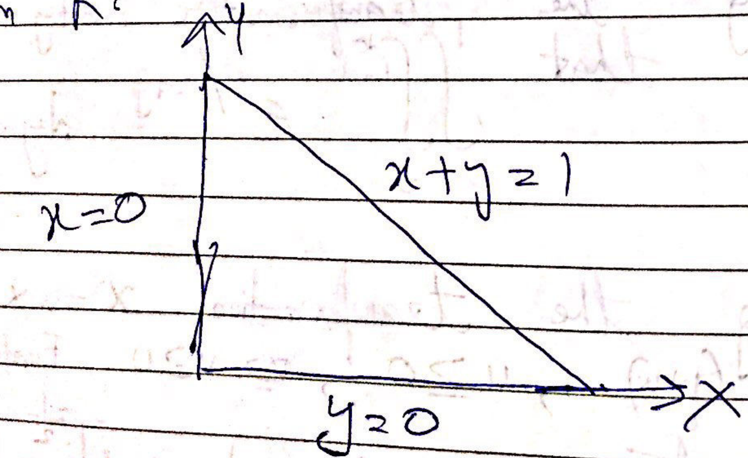
(HINT: Take $u=x^2-y^2$, $v=x^2+y^2$)

(a) $x-y=u, \quad x+y=v$

$$|J^*| = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$$

then $|J| = \dfrac{1}{2}$

(b) Region R:



(c) Transformed Region:

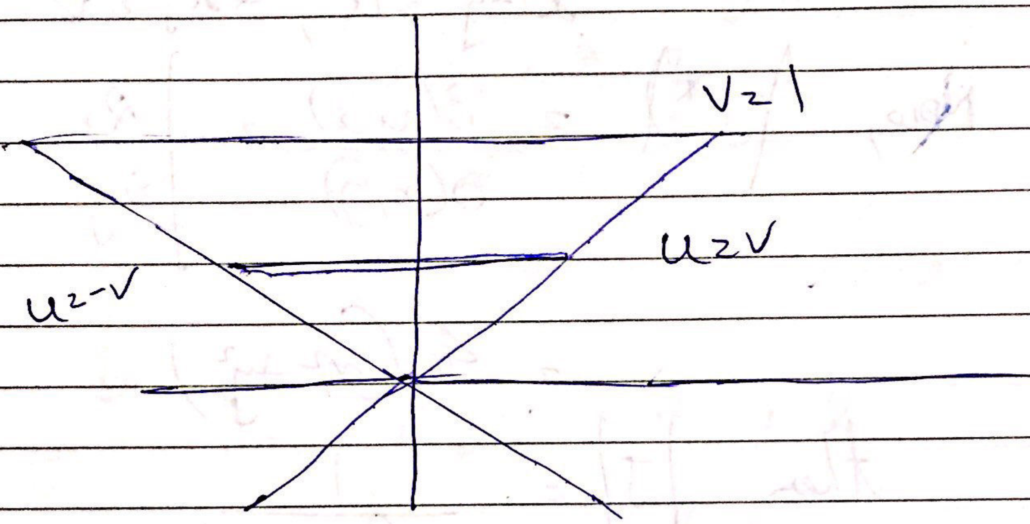
$x=0$ is mapped into $u=-y$
$v=y$

$\Rightarrow \boxed{u=-v}$

$y=0$ is mapped into $u=x$, $v=x$

$\boxed{u=v}$

$x+y=1$ is mapped into $v=1$



lim of u is $-v$ to v

lim of v is 0 to 1

(d) Evaluation of I

$$I = \int_{v=0}^{1}\int_{u=-v}^{v} \cos\left(\frac{u}{v}\right)\left(\frac{1}{2}\,du\,dv\right)$$

$$= \frac{1}{2}\int_{v=0}^{1}\left[\frac{\sin(u/v)}{1/v}\right]_{u=-v}^{v} dv$$

$$= \frac{1}{2}\int_{v=0}^{1} 2\sin(1)\, v\, dv$$

$$= \frac{\sin(1)}{2}$$

② (a) Jacobian:

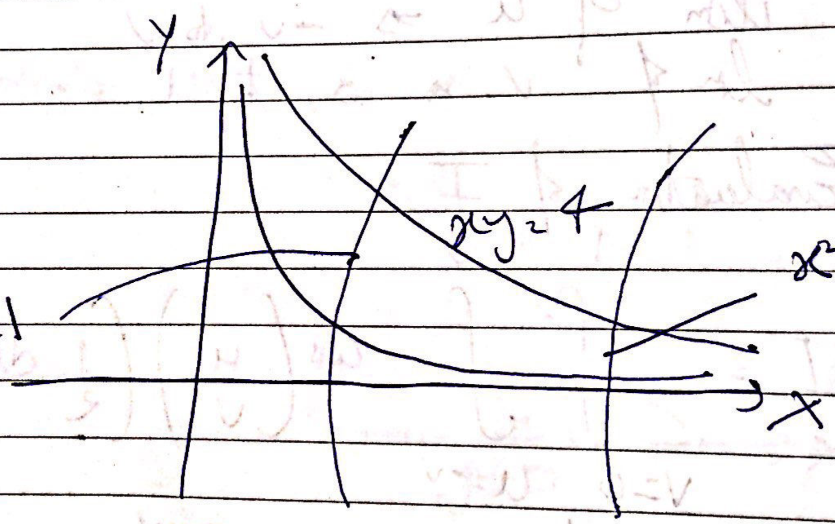
$x^2-y^2=u, \quad 2xy=v$

Now, $|J^*| = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix}$

$$= 4(x^2+y^2)$$

then $|J| = \frac{1}{4(x^2+y^2)}$

(b)



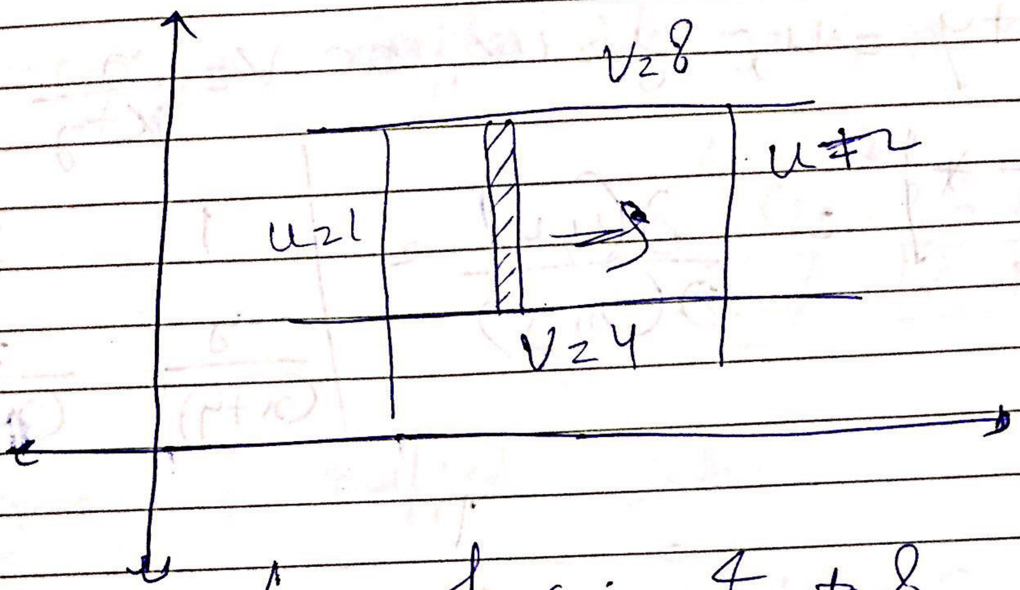
(c) Transformed regions

$x^2-y^2=1$ is mapped into $\boxed{u=1}$

$x^2-y^2=2$ " " " $\boxed{u=2}$

$xy=2$ " " " $\boxed{v=4}$

$xy = 4$ is mapped into $\boxed{v = 8}$



Limit of v : 4 to 8

Limit of u : 1 to 2

(d) <u>Evaluation of I_1</u>

$$I = \int_{u=1}^{2} \int_{v=4}^{8} \cancel{(x^2+y^2)} \frac{1}{4\cancel{(x^2+y^2)}} \, du \, dv$$

$$= \frac{1}{4} \int_{u=1}^{2} du \times \int_{v=4}^{8} dv$$

$= 1$

③ (a) Jacobian:

$x+y = u, \quad y = uv$ or $v = \frac{y}{x+y}$

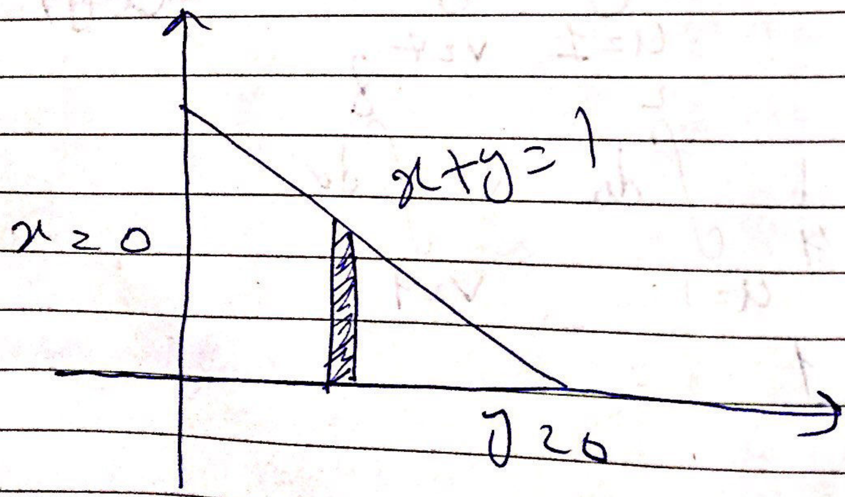
$$|J^*| = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ \frac{-y}{(x+y)^2} & \frac{x}{(x+y)^2} \end{vmatrix}$$

$$= \frac{1}{x+y}$$

ⓑ Region R:

limits of y : 0 to $(1-x)$

limits of x : 0 to 1



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(a) Transformed Region:

$x=0$ is mapped into $y=u,\ y=uv$

$u=uv$

$u(1-v)=0$

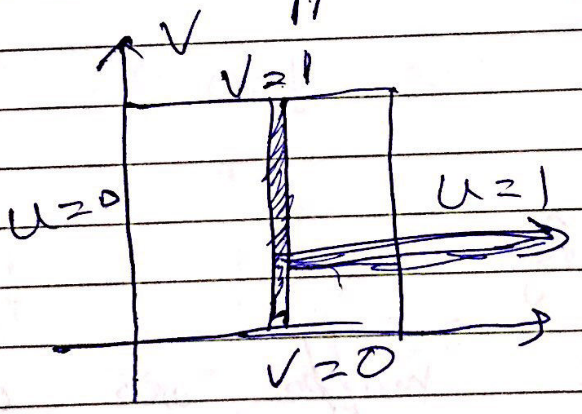
$\boxed{u=0}$ $\boxed{v=1}$

$y=0$ is mapped into

$u=x,\ uv=0$

$u=0,\ \boxed{v=0}$

$x+y=1$ is mapped into $\boxed{u=1}$



(b) Evaluation of I:

$$I=\int_{v=0}^{1}\int_{u=0}^{1} e^{v}u\,du\,dv$$

$$=\int_{v=0}^{1} e^{v}dv \times \int_{u=0}^{1} u\,du=\frac{e-1}{2}$$

(iii)

$x = u(1+v)$; $y = v(1+u)$

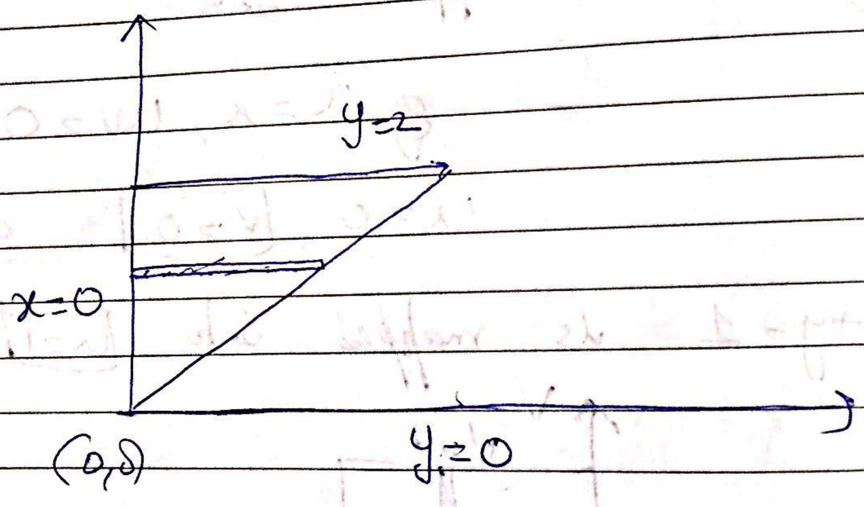
(4)

(a) Jacobian :

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix} = 1+u+v$$

(b) Region R:

Limits of x : 0 to y
Limits of y : 0 to 2



(c) Transformed Region :

(I) $x=0$ is mapped onto $u(1+v)=0$

$\Rightarrow \boxed{u=0}$ $\quad v=-1$ ↓ Rejected as $v>0$

(II) $x=y$ is mapped $u(1+v) = v(1+u)$

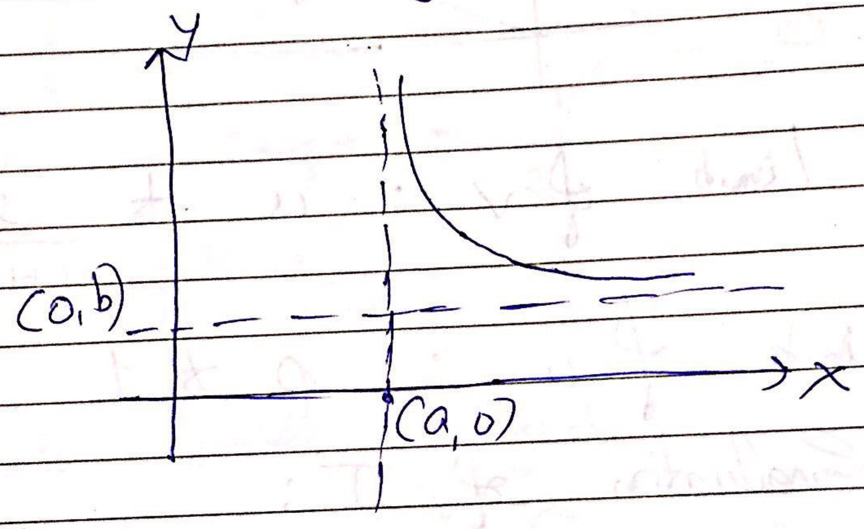
$\boxed{u=v}$

(iii) $y = 2$ is mapped $\boxed{v(1+u) = 2}$ #

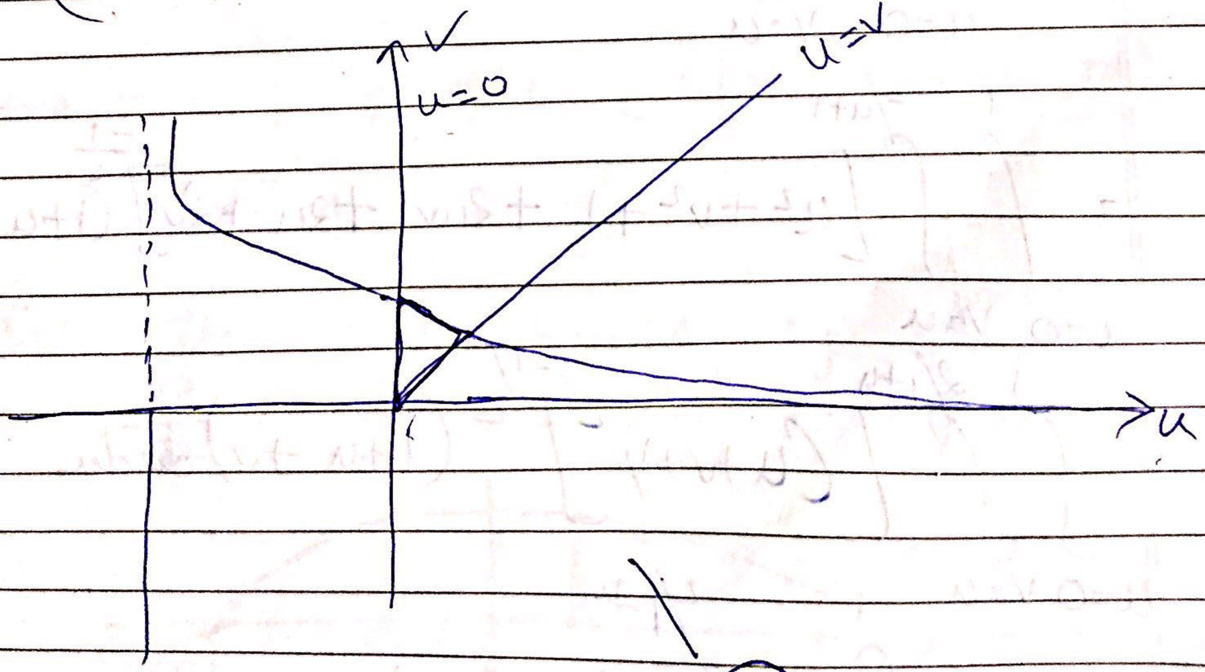
$1+u = \frac{2}{v}$

$u = \frac{2}{v} - 1$

NOTE $(x-a)(y-b) = k$, $(k>0)$



$(u+1)\,v = 2$



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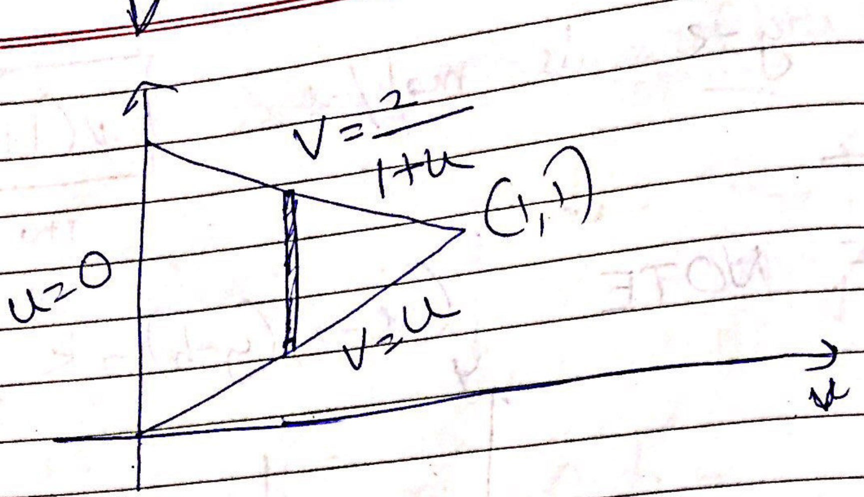

Limits of v : u to $\frac{2}{1+u}$

du dv

Limits of u : 0 to 1

(d) Evaluation of I :

$$I = \int_{u=0}^{1}\int_{v=u}^{2/u+1} \left[(u-v)^2+2(u+v+2uv)+1\right]^{-\frac{1}{2}}(1+u+v)\,dv\,du$$

$$= \int_{u=0}^{1}\int_{v=u}^{2/u+1} \left[u^2+v^2+1+2uv+2u+2v\right]^{-\frac{1}{2}}(1+u+v)\,dv\,du$$

$$= \int_{u=0}^{1}\int_{v=u}^{2/1+u} \left[(u+v+1)^2\right]^{-1/2}(1+u+v)\,dv\,du$$

$$\int_{u=0}^{1}\int_{v=0}^{2/1+u} 1\,dv\,du$$

Quan-

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$$I = \int_{u=0}^{1} \left(\frac{2}{1+u} - u \right) du$$

$$= \left[2\log(1+u) - \frac{u^2}{2} \right]_0^1$$

$$= 2\log 2 - \frac{1}{2}$$

$$= \log 4 - \frac{1}{2}$$

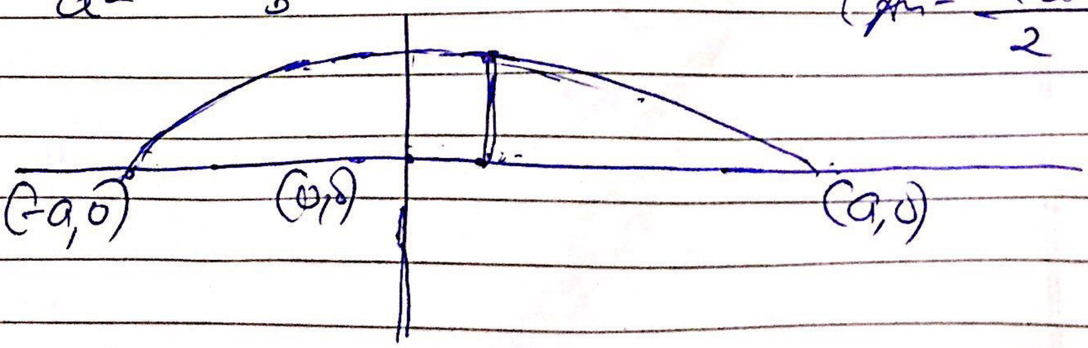
Area as Double Integral:

$$\text{Area} = \int_{y_1}^{y_2}\int_{x_1}^{x_2} dx\, dy \quad \text{(Cartesian Coordinates)}$$

$$\text{Area} = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} r\, dr\, d\theta \quad \text{(Polar Coordinates)}$$

Ques- Find the area bounded by ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ above x-axis.

(Ans- $\frac{\pi ab}{2}$)



$\frac{2ab+2}{4}$

Que- Find the area bounded by the parabola $y^2 = 4x$ and line $2x - 3y + 4 = 0$ $\left(\text{Ans} = \frac{1}{3}\right)$

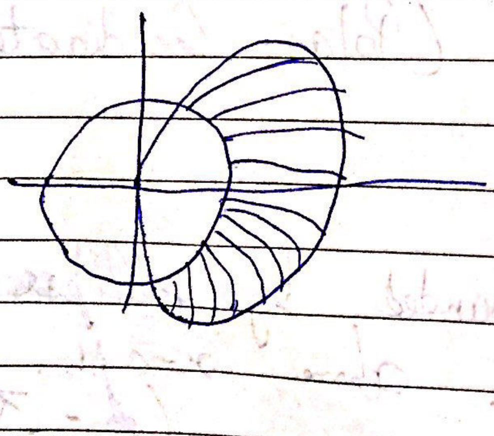
Ques- Find the larger area bounded by circle $x^2+y^2 = 64a^2$ & parabola $y^2 = 12ax$.

Ans- $\frac{16a^2}{3}(8\pi - \sqrt{3})$

Ques- Find the area b/w circles $r = 2\sin\theta$, $r = 4\sin\theta$ (Ans- 3π)

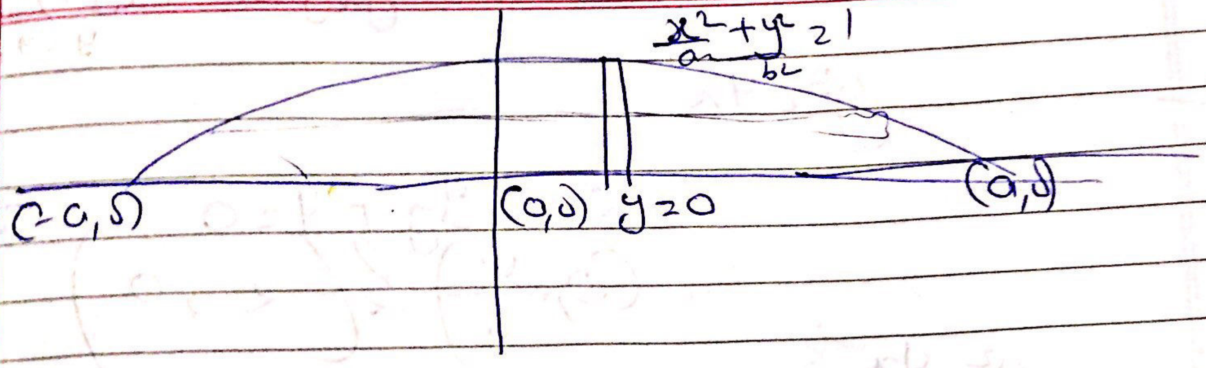
Ques- Find the area of crescent bounded by circles $r = \sqrt{3}$, $r = 2\cos\theta$

Ans- $\left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right)$



$\frac{y^2}{b^2} \quad \frac{b^2}{a^2}(a^2 - x^2) \quad \frac{x^2}{a^2}$

(1)



lim of x : 0 to a

lim of y : 0 to $\frac{b}{a}\sqrt{a^2-x^2}$

$$I = 2\int_{x=0}^{a}\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} dy\,dx$$

$$I = 2\int_{x=0}^{a}\left[y\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$I = 2\,\frac{b}{a}\int_{x=0}^{a}\sqrt{a^2-x^2}\,dx$$

$$I = 2\,\frac{b}{a}\left[\frac{1}{2}x\sqrt{a^2-x^2}+\frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$I = 2\,\frac{b}{a}\left[\frac{a}{a}\,\frac{1}{2}a^2\left(\frac{\pi}{2}\right)\right]$$

$$I = \frac{\pi ab}{2}$$

Ans

$y^2 - 6y + 8 = 0$

$y^2 - 2y - 4y + 8 = 0$

$y(y-2) - 4(y-2) = 0$

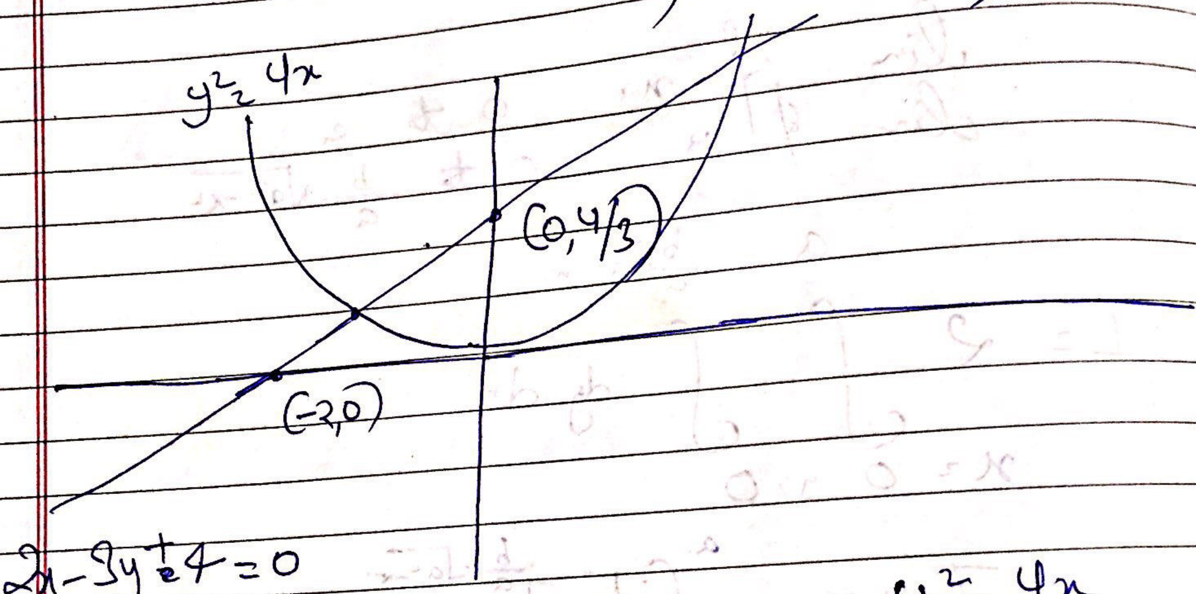
$y = 4, 2$

Q.

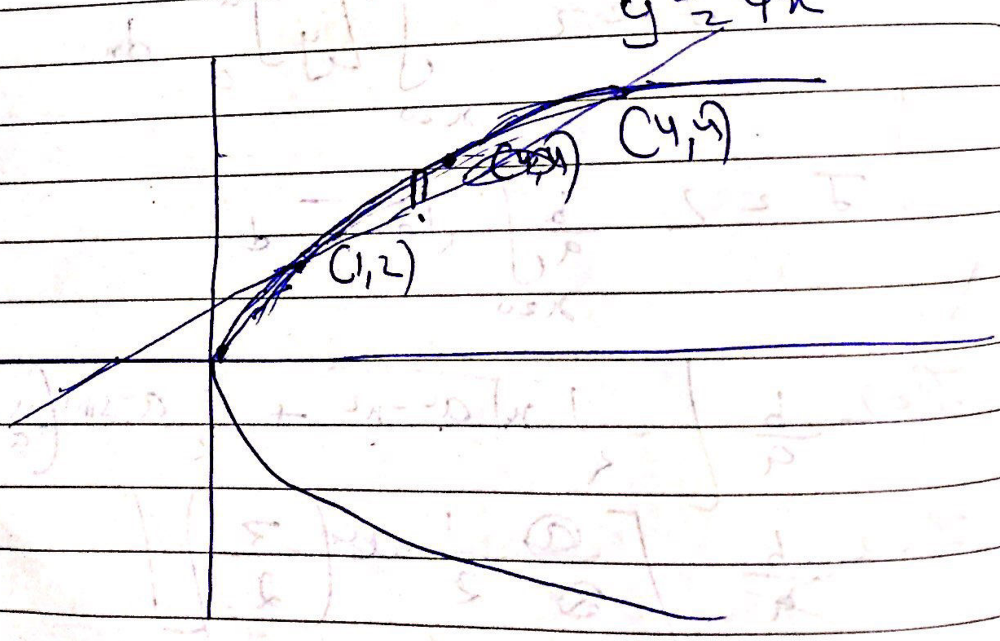
$y^2 = 4x$

$2x - 3y + 4 = 0$

$\left(0, \frac{4}{3}\right)$ & $(-2, 0)$



$\frac{2x+4}{3}$



2a

lim of x : 1 to 4

lim of y : $\frac{2x+4}{3}$ to $2\sqrt{x}$

$$\int_{x=1}^{4}\int_{y=\frac{2x+4}{3}}^{2\sqrt{x}} dy\,dx$$

$$\int_{x=1}^{4}\left[y\right]_{\frac{2x+4}{3}}^{2\sqrt{x}} dx$$

$$\int_{x=1}^{4}\left[2\sqrt{x}-\frac{2x+4}{3}\right]dx$$

$$\frac{1}{3}\int_{1}^{4}\left(6\sqrt{x}-2x-4\right)dx$$

$$\frac{1}{3}\left[\left[\frac{6}{2\sqrt{x}}\right]_1^4-2\left[\frac{x^2}{2}\right]_1^4-4\left[x\right]_1^4\right]dx$$

$$\frac{1}{3}\left[3\left[\frac{1}{2}-\frac{1}{1}\right]-\left[16-1\right]-4\left[4-1\right]\right]$$

$$\frac{1}{3}\left[3\left(\frac{-1}{2}\right)-15-12\right]$$

2a $$\frac{1}{3}\left[\frac{-3-\cancel{8}\,54}{2}\right]=\frac{1}{6}\left(-57\right)$$

Triple Integration

Let $f(x, y, z)$ be a fun^c of three variables defined over volume V bounded by one or more surfaces S. Then the triple integration of $f(x,y,z)$ over the volume V is defined as

$$\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y,z)\,dx\,dy\,dz$$

Note : The outermost limits must be constant and the most complex limits should be taken as the innermost limits.

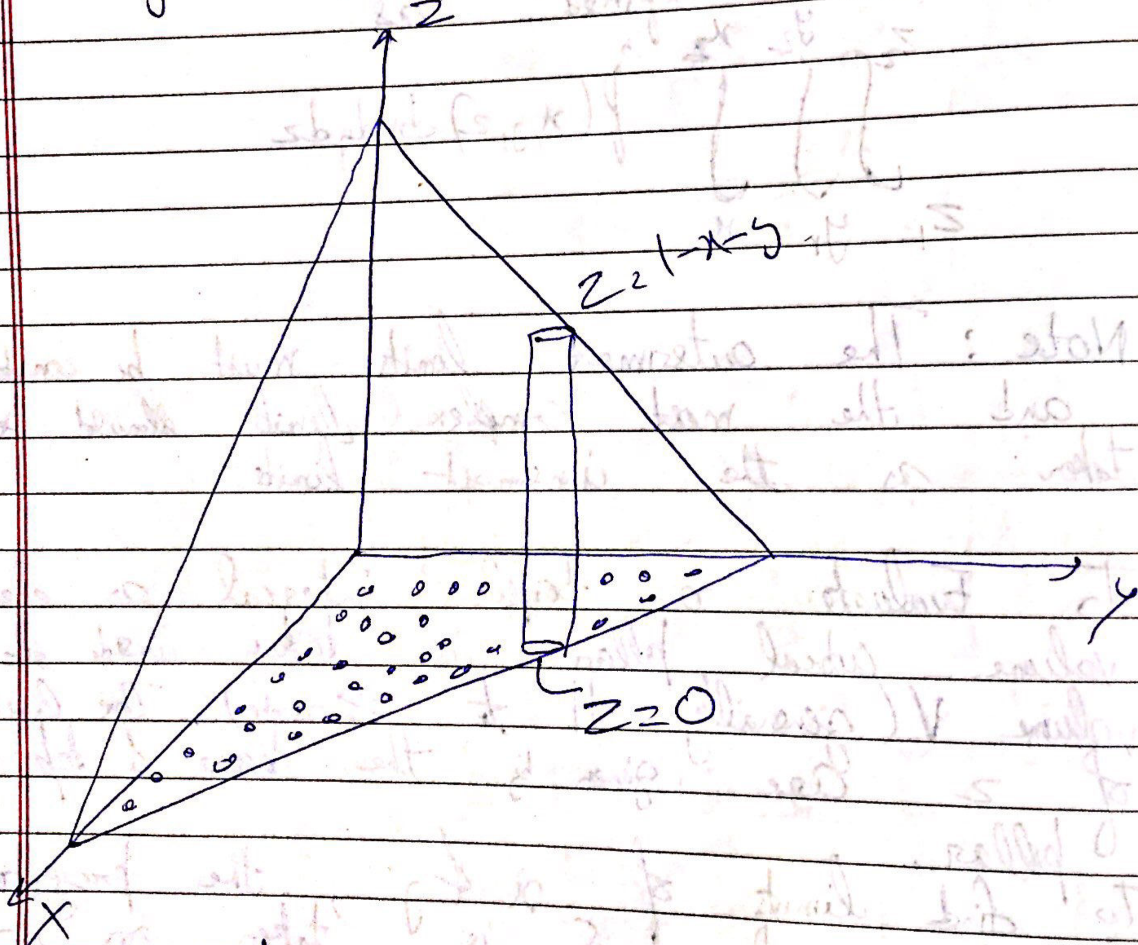
To Evaluate a triple integral an elementary volume cubical pillar is taken inside the volume V (generally $\parallel$ to z-axis), The limits of z are given by the base & top of pillar.

To find limits of x & y, the projection of the surface S is taken on x-y plane, which is called region R.

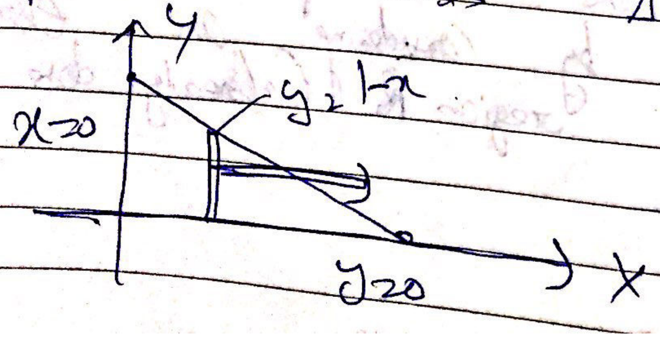
Now the limits of x & y can be easily found by considering an elementary strip inside region R (already done in last section)

Ques- Evaluate $\iiint_V xyz \, dx\,dy\,dz$,

where V is the volume bounded by the planes $x=0, y=0, z=0$ and $x+y+z=1$



To find limit of x & y, take projection on X-Y plane which is ΔOAB



Limit of y : 0 to $1-x$.

Limits of x : 0 to 1

$$I = \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} xyz \, dz \, dy \, dx$$

Q. $\iiint xy \, dx\, dy\, dz$

$$\int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} xy \, dx\, dy\, dz$$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} xy(1-x-y)\, dy\, dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} (xy - x^2y - xy^2)\, dy\, dx$$

$$= \int_{x=0}^{1} \left[\frac{x(1-x)^2}{2} - \frac{x^2(1-x)^2}{2} - \frac{x}{3}(1-x)^3 \right] dx$$

$$= \int_{x=0}^{1} \left[\frac{x(1-x)^3}{2} - \frac{x}{3}(1-x)^3 \right] dx$$

$$\frac{1}{6}\int_{x=0}^{1} x(1-x)^3\,dx$$

$$\frac{1}{6}\int_{0}^{1} x^{2-1}(1-x)^{4-1}\,dx$$

$$\frac{1}{6} B(2,4) = \frac{1}{6}\left|\frac{\Gamma 2\,\Gamma 4}{\Gamma 2+4}\right|$$

$$= \frac{1}{6}\,\frac{\lfloor 1 \times \lfloor 3}{\lfloor 5} =$$

Beta function :-

① $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = B(m,n)$

② $B(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}$

③ $\Gamma n = (n-1)!$; eg $\Gamma 5 = 4! = 24$

Gamma 'n'

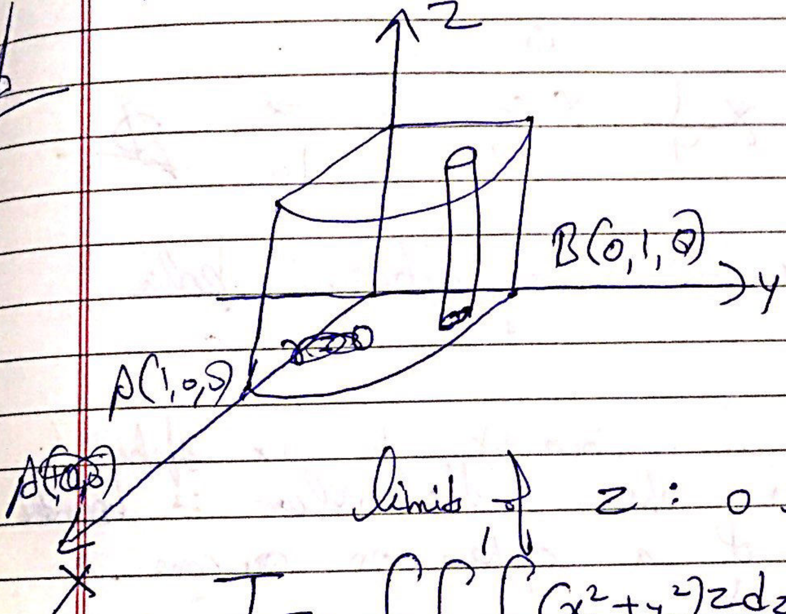
④ $\Gamma \frac{1}{2} = \sqrt{\pi}$

⑤ $\Gamma \frac{5}{2} = \frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi}$;

$\Gamma \frac{7}{2} = \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi}$

Que. Evaluate $\iiint (x^2+y^2)\, z \, dx\, dy\, dz$

over the volume bounded by cylinder $x^2+y^2=1$, $0 \le z \le 1$ in the positive octet.

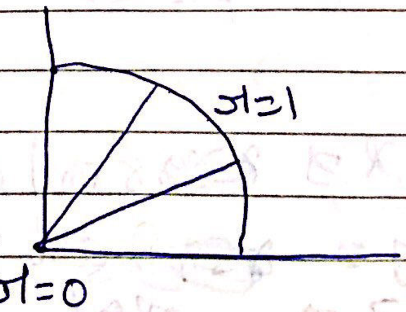


limits of z : 0 to 1

$$I = \iint_R \int_{z=0}^{1} (x^2+y^2)\, z\, dz\, dx\, dy$$

$$= \frac{1}{2} \iint_R (x^2+y^2)\, dx\, dy$$

To find limits of x & y, take portion R on XY - plane which is quadrant of circle $x^2+y^2=1$ or $r=1$ (Polar form)



limits of r : 0 to 1

limits of θ : 0 to $\frac{\pi}{2}$

$$I = \frac{1}{2}\int_{\theta=0}^{\pi/2}\int_{r=0}^{1} (r^2)(r\,dr\,d\theta)$$

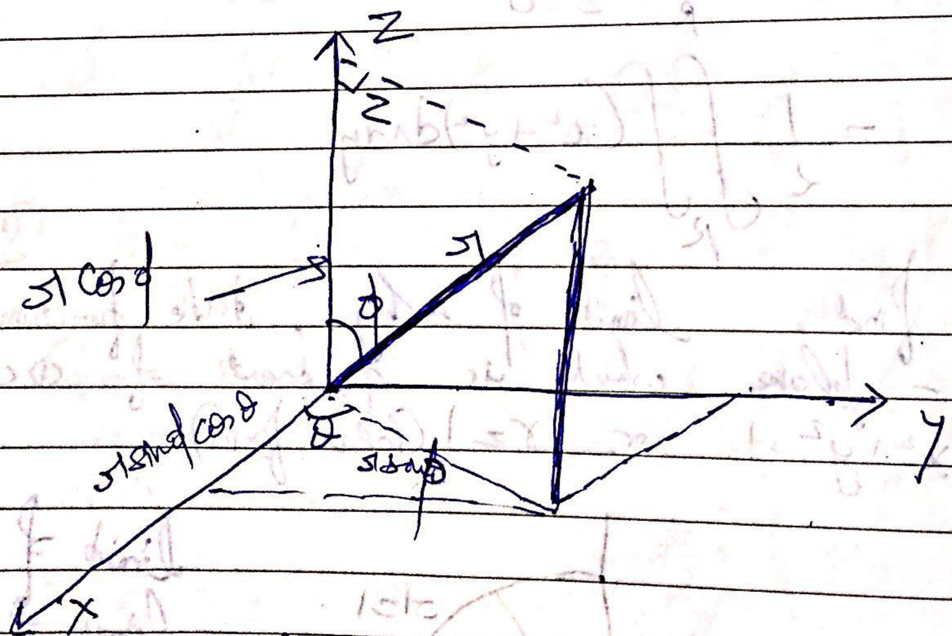
$$\frac{1}{2}\int_{r=0}^{1} r^3\,dr \times \int_{0}^{\pi/2} d\theta$$

$$\frac{1}{2} \times \frac{1}{4} \times \frac{\pi}{2} = \frac{\pi}{16}$$

Triple Integration using spherical polar coordinates:-

Sometimes it is convenient to use spherical polar coordinates when the volume is bounded by a part of a sphere or cone.

Let $P(x, y, z)$ be a point in 3-space



$x = r\sin\phi\cos\theta$
$y = r\sin\phi\sin\theta$
$z = r\cos\phi$

} spherical polar coordinates

$$|J| = \frac{\partial(u,v,w)}{\partial(r,\theta,\phi)} = r^2\sin\phi \quad \text{(Check yourself)}$$

Remarks :

(1) For an ~~oc~~ octet of sphere $x^2+y^2+z^2=a^2$.

limits of r : 0 to a

limits of θ : 0 to $\frac{\pi}{2}$.

limits of ϕ = 0 to $\frac{\pi}{2}$

(2) For a hemisphere above the $x-y$ plane

limits of r : 0 to a

limits of θ = 0 to 2π

limits of ϕ = 0 to $\frac{\pi}{2}$

(3) For a sphere $x^2+y^2+z^2=a^2$

limits of r : 0 to a

limits of θ = 0 to 2π

limits of ϕ : 0 to π

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(1)

Qu Evaluate $\iiint \frac{dx\,dy\,dz}{\sqrt{a^2-x^2-y^2-z^2}}$ over the region bounded by sphere $x^2+y^2+z^2=a^2$ (Ans- $\pi^2 a^2$)

Qu Evaluate $\iiint \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^2}$ over the region bounded by spheres $x^2+y^2+z^2=a^2$ & $x^2+y^2+z^2=b^2$ where $a>b>0$.

Qu Evaluate $\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\int_{z=0}^{\sqrt{a^2-x^2-y^2}} xyz\,dz\,dy\,dx$

Ans- $\left(\frac{a^6}{48}\right)$

Qu Evaluate $\int_0^\infty\int_0^\infty\int_0^\infty \frac{dx\,dy\,dz}{(1+x^2+y^2+z^2)^2}$

(Ans- $\frac{\pi^2}{8}$)

Q Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1} \frac{dx\,dy\,dz}{\sqrt{x^2+y^2+z^2}}$

by transforming into ~~dso~~ transforming polar coordinates. $\left(\frac{\pi}{4}\times\left(\sqrt{e}-1\right)\right)$

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(1) Put $x = r\sin\phi\cos\theta$
$y = r\sin\phi\sin\theta$
$z = r\cos\phi$

and $dx\,dy\,dz = r^2\sin\phi\,dr\,d\theta\,d\phi$

Here volume is bounded by sphere
$x^2+y^2+z^2 = a^2$

lim of r : 0 to a
lim of θ : 0 to 2π
lim of ϕ : 0 to π

$$I = \int_{\phi=0}^{\pi}\int_{\theta=0}^{2\pi}\int_{r=0}^{a} \frac{r^2\sin\phi\,dr\,d\theta\,d\phi}{\sqrt{a^2-\cancel{r^2}}}$$

$$I = \int_{\phi=0}^{\pi}\sin\phi\,d\phi \times \int_{\theta=0}^{2\pi} d\theta \times \int_{r=0}^{a}\frac{r^2}{\sqrt{a^2-r^2}}\,dr$$

$$I = 2\times 2\pi \times I_1 \text{ (say)} \quad \text{—— (1)}$$

Now, $$I_1 = \int_0^a \frac{r^2}{\sqrt{a^2-r^2}}\,dr$$

let $\cancel{x}$ $r = a\sin t \Rightarrow dr = a\cos t\,dt$

$$I_1 = \int_0^{\pi/2}\frac{a^2\sin^2 t}{\sqrt{a^2-a^2\sin^2 t}}(a\cos t\,dt)$$

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$$= a^2 \int_0^{\pi/2} \cancel{a^2} \sin^2 t \, dt$$ (2)

$$\boxed{\# \int_0^{\pi/2} \sin^p\theta \cos^q\theta = \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}}}$$ (3)

$$= a^2 \left[\frac{\Gamma\frac{3}{2}\,\Gamma\frac{1}{2}}{2\,\Gamma 2} \right]$$

$$= a^2 \left[\frac{\frac{1}{2}\sqrt{\pi}\,\sqrt{\pi}}{2 \times 1!} \right] = \frac{\pi a^2}{4}$$

Put in (1)

$$I = \cancel{2} \times 2\pi \times \frac{\pi a^2}{4}$$

$$\boxed{I = \pi^2 a^2}$$

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② (Hint : lim of r : b to a)
(Everything same)

(Ans $4\pi\left(\frac{1}{b}-\frac{1}{a}\right)$)

③ lim of z : 0 to $\sqrt{a^2-x^2-y^2}$
lim of y : 0 to $\sqrt{a^2-x^2}$
lim of x : 0 to a

Clearly, $z=\sqrt{a^2-x^2-y^2}$

Squaring

$x^2+y^2+z^2=a^2$ (Sphere)

Since all the limits of $x, y, \& z$ are positive ($x\geq 0$, $y\geq 0$, $z\geq 0$)

Hence it is an Octant of sphere.

Let $x = r\sin\phi\cos\theta$
$y = r\sin\phi\sin\theta$
$z = r\cos\phi$

$dx\,dy\,dz = r^2\sin\phi\,dr\,d\theta\,d\phi$

lim of r : 0 to a
lim of θ : 0 to $\frac{\pi}{2}$
lim of ϕ : 0 to $\frac{\pi}{2}$

$$I = \int_{\phi=0}^{\pi} \sin\phi \, d\phi \times \int_{\theta=0}^{2\pi} d\theta \times \int_{r=b}^{a} \frac{r^2}{(r^2)^2} dr$$

$$= 2 \times 2\pi \times \left(\frac{1}{b} - \frac{1}{a}\right)$$

$$\underline{I} = 4\pi \left(\frac{1}{b} - \frac{1}{a}\right)$$

$$I = \int_{\phi=0}^{\pi/2} \sin^3\phi \cos\phi \, d\phi \times \int_{\theta=0}^{\pi/2} \sin\theta \cos\theta \, d\theta \times \int_{r=0}^{a} r^5 dr$$

$$= \left(\frac{\Gamma 2 \, \Gamma 1}{2\Gamma 3}\right) \times \left(\frac{\Gamma 1 \, \Gamma 1}{2\Gamma 2}\right) \times \frac{a^6}{6}$$

$$I = \frac{a^6}{48}$$

(4) lim of x : 0 to ∞
lim of y : 0 to ∞
lim of z : 0 to ∞

Since $x \geq 0$, $y \geq 0$, $z \geq 0$, it can be treated as the positive octant of the sphere centered at origin & and having infinite radius.

Put $x = r\sin\phi\cos\theta$
$y = r\sin\phi\sin\theta$
$z = r\cos\phi$

$dx\,dy\,dz = r^2\sin\phi\,dr\,d\theta\,d\phi$

lim of r : 0 to ∞
lim of θ = 0 to $\pi/2$
lim of ϕ = 0 to $\pi/2$

$$I = \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} \frac{r^2\sin\phi\,dr\,d\theta\,d\phi}{(1+r^2)^2}$$

$$I = \int_{\phi=0}^{\pi/2} \sin\phi\,d\phi \times \int_{\theta=0}^{\pi/2} d\theta \times \int_{r=0}^{\infty} \frac{r^2}{(1+r^2)^2}\,dr$$

$$I = 1 \times \frac{\pi}{2} \times I_1 \text{ (say)} \quad \text{—— (1)}$$

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where $I_1 = \int_0^\infty \frac{x^2\,dx}{(1+x^2)^2}$

Put $x = \tan t \Rightarrow dx = \sec^2 t\,dt$

$$I_1 = \int_0^{\pi/2} \frac{\tan^2 t \sec^2 t\,dt}{\sec^4 t}$$

$$= \int_0^{\pi/2} \sin^2 t\,dt$$

$$I_1 = \frac{\pi}{4}$$

eqn (1), $\boxed{I = \frac{\pi^2}{8}}$

(5) lim of z : $\sqrt{x^2+y^2}$ to 1

lim of y : 0 to $\sqrt{1-x^2}$

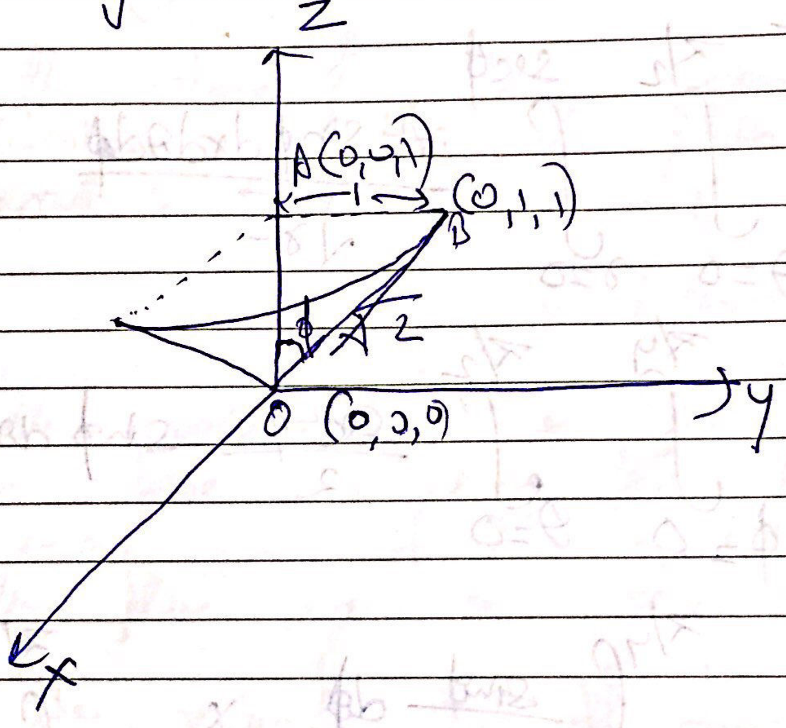
lim of x : to 1

Clearly $z = \sqrt{x^2+y^2}$

$z^2 = x^2+y^2$ (Cone)

Since $x \geq 0,\ y \geq 0,\ z \geq 0$

Hence it is a cone in the positive octant only



let $x = r\sin\phi\cos\theta$
$y = r\sin\phi\sin\theta$
$z = r\cos\phi$
$dx\,dy\,dz = r^2\sin\phi\,dr\,d\theta\,d\phi$

lim of r: 0 to * $\sec\phi$

lim of θ = 0 to $\pi/2$

lim of ϕ = 0 to # $\frac{\pi}{4}$

$$\left\{ \begin{array}{l} * \; z = 1 \text{ gives} \\ r\cos\phi = 1 \\ r = \sec\phi \end{array} \right\}$$

$\#$ $\sin\phi = \frac{AB}{OB}$

$\sin\phi = \frac{1}{\sqrt{2}}$

$\phi = \frac{\pi}{4}$

$$I = \int_{\phi=0}^{\pi/4} \int_{\theta=0}^{\pi/2} \int_{r=0}^{\sec\phi} \frac{r^2 \sin\phi \, dr \, d\theta \, d\phi}{\sqrt{r^2}}$$

$$= \int_{\phi=0}^{\pi/4} \int_{\theta=0}^{\pi/2} \frac{\sec^2\phi}{2} \sin\phi \, d\theta \, d\phi$$

$$= \int_0^{\pi/4} \frac{\sin\phi}{\cos\phi} d\phi \times \int_0^{\pi/2} d\theta$$

$$= \frac{\pi}{2} \int_1^{1/\sqrt{2}} \frac{-dt}{t^2}$$

$$= \frac{\pi}{2} \left(\frac{1}{\sqrt{2}} - 1 \right)$$

Volume as Triple Integral :

$$\text{Volume} = \int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} dx\,dy\,dz \quad \text{(Cartesian coordinates)}$$

$$\text{Volume} = \iiint r^2 \sin\phi \, dr\, d\theta\, d\phi \quad \text{(Spherical polar coordinates)}$$

Que Find the volume of the region bounded by the tetrahedron b/w the planes $x=0, y=0, z=0$, & $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

(Ans $\frac{abc}{6}$)

Ques-2 Find the volume of the prism with triangular base on $x-y$ plane bounded by lines x-axis, $y=x$ & $x=2$, with the top of the plane lies on $z=5-x-y$.

Ques-3 Find the volume cut off from the sphere $x^2+y^2+z^2=a^2$ by a cone $x^2+y^2=z^2$, $z\geq 0$.

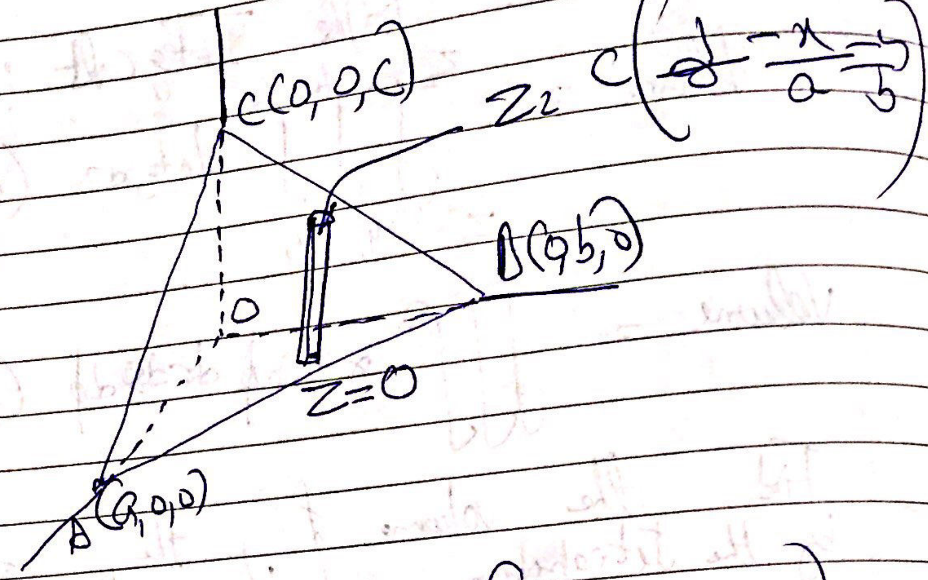
(Ans $\frac{\pi a^3}{3}\times(2-\sqrt{2})$)

Ques-4 Find the volume bounded by cone $x^2+y^2=z^2$ & paraboloid $x^2+y^2=z$

(Ans $\frac{\pi}{6}$)

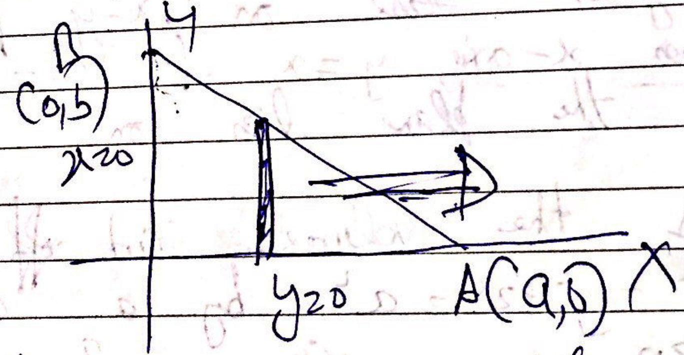
1

2



lim of z: 0 to $c\left(1-\frac{x}{a}-\frac{y}{b}\right)$

To find lim of x & y, take projection on XY-plane, which is ΔOAB

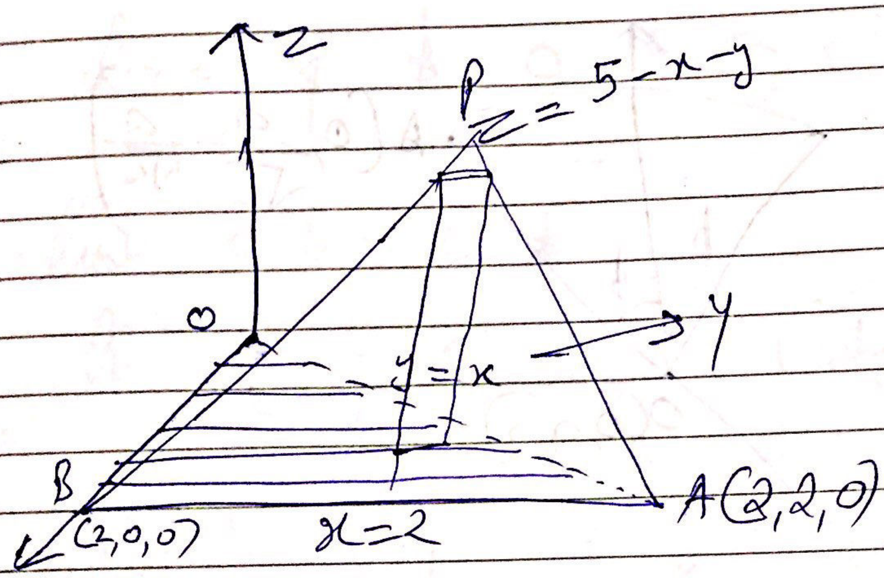


lim of y: 0 to $b\left(1-\frac{x}{a}\right)$

lim of x: 0 to a

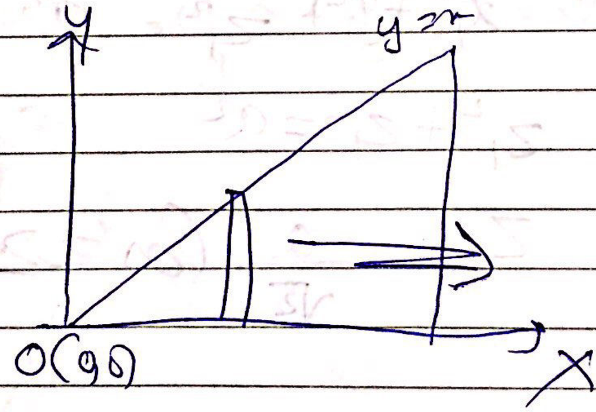
$$V=\int_{x=0}^{a}\int_{y=0}^{b\left(1-\frac{x}{a}\right)}\int_{z=0}^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dz\,dy\,dx$$

①



lim of z : 0 to $5-x-y$

To find lim of x & y take projection on XY plane which is $\triangle OAB$.

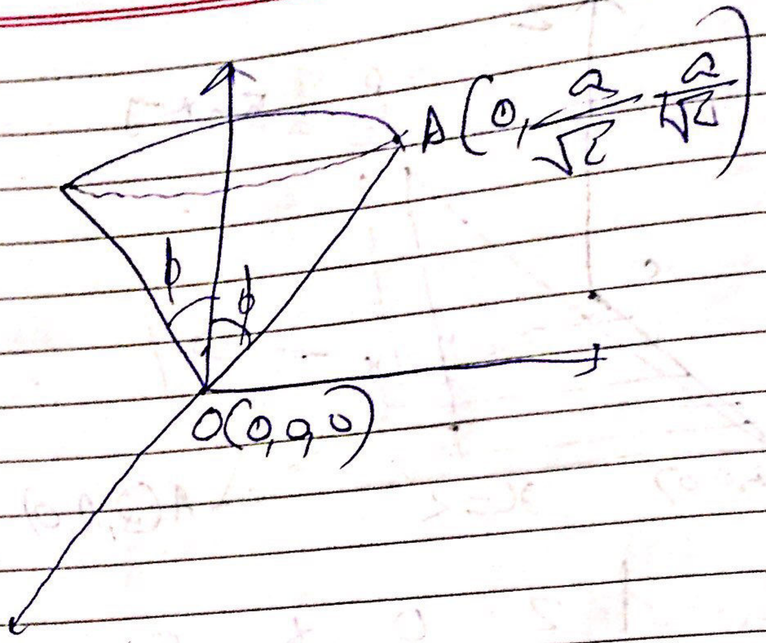


lim of y : 0 to x

lim of x : 0 to 2

$$V = \int_{x=0}^{2}\int_{y=0}^{x}\int_{z=0}^{5-x-y} dz\,dy\,dx$$

(2)



To find $A(0, y_1, z_1)$

Cone: $x^2 + y^2 = z^2$

Sphere: $x_1^2 + y_1^2 + z_1^2 = a^2$

$\Rightarrow z_1^2 + z_1^2 = a^2$

$z_1 = \frac{a}{\sqrt{2}}$ (as $z \geq 0$)

Now,

$x_1^2 + y_1^2 = z_1^2 \Rightarrow 0^2 + y_1^2 = \left(\frac{a}{\sqrt{2}}\right)^2$

$y_1 = \frac{a}{\sqrt{2}}$

In ΔOAB, $\sin\phi = \frac{AB}{OA} = \frac{a/\sqrt{2}}{a}$

$\Rightarrow \phi = \frac{\pi}{4}$

(4)

classmate

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limits of ϕ : 0 to $\frac{\pi}{4}$ {1st octet}

limits of θ : 0 to $\frac{\pi}{2}$

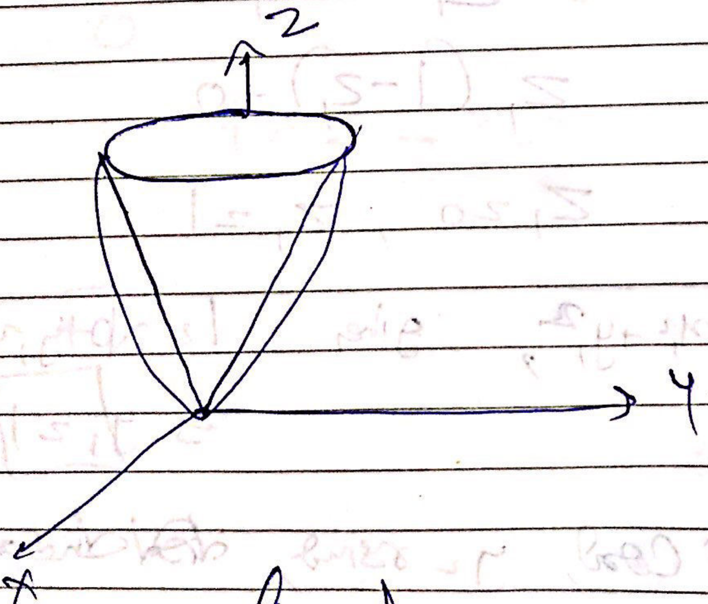
lim of r : 0 to a (why?)

$$V = 4\int_{\phi=0}^{\pi/4}\int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r^2 \sin\phi \, dr \, d\theta \, d\phi$$

$$V = 4\frac{a^3}{3}\left(\frac{\pi}{2}\right)\left(1-\frac{1}{\sqrt{2}}\right)$$

$$= \frac{a^3\pi}{3}\left(2-\sqrt{2}\right)$$

(4)



lim of z : x^2+y^2 to $\sqrt{x^2+y^2}$

$$V = 4\iint\int_{z=x^2+y^2}^{\sqrt{x^2+y^2}} dz \, dR$$

limits of ϕ : 0 to $\frac{\pi}{4}$ {1st octet}

limits of θ : 0 to $\frac{\pi}{2}$

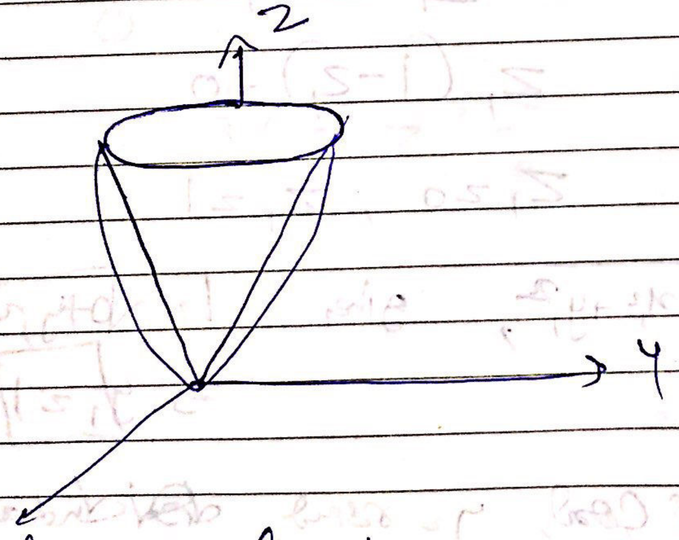
lim of r : 0 to a (why?)

$$V = 4 \int_{\phi=0}^{\pi/4} \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r^2 \sin\phi \, dr \, d\theta \, d\phi$$

$$V = 4 \frac{a^3}{3} \left(\frac{\pi}{2}\right)\left(1 - \frac{1}{\sqrt{2}}\right)$$

$$= \frac{a^3 \pi}{3} \left(2 - \sqrt{2}\right)$$

(4)



lim of z : $\frac{x^2+y^2}{\sqrt{x^2+y^2}}$ to $\sqrt{x^2+y^2}$

$$V = 4 \iint_R \int_{z=x^2+y^2} dz \, dR$$

$$4\iint_R \sqrt{x^2+y^2} - (x^2+y^2)\, dR$$

Take projection in XY-plane which is a quadrant of circle $x^2+y^2 = 1$ or $r=1$ (Relation) V_2

To find $A(0, y_1, z_1)$

Paraboloid : $z_1 = x_1^2+y_1^2$ & Cone $z_1 = \sqrt{x_1^2+y_1^2}$

$\Rightarrow z_1 = z_1^2$

$z_1(1-z_1) = 0$

$z_1 = 0, \; z_1 = 1$

$I_1 : \sqrt{x_1^2+y_1^2}$, given $1 = \sqrt{0+y_1^2}$

$\Rightarrow \boxed{y_1 = 1}$

Put $x = r\cos\theta$, $y = r\sin\theta$, $dxdy = r\,dr\,d\theta$

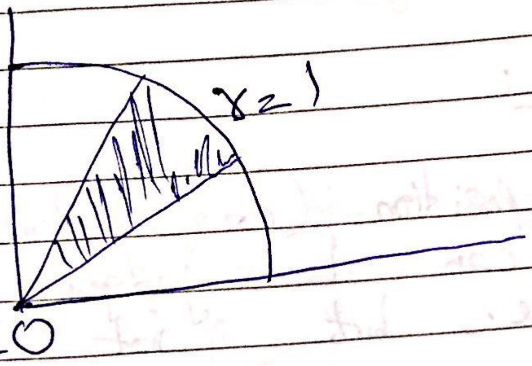
lim of r : 0 to 1

lim of θ : 0 to $\frac{\pi}{2}$

$4 \times 5 = 20$
ques marks

5 ques $\times \frac{1}{2}$ mark $= 60$

$\cancel{20}1 \times 20 = \underline{20}$



$$V = 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{1}\left(\sqrt{r} - r^2\right) r\,dr\,d\theta$$

$$4\int_0^{\pi/2}\left[\frac{1}{3} - \frac{1}{4}\right] d\theta$$

$$4 \times \frac{1}{2} \times \frac{\pi}{2} = \pi/6 \ \underline{\underline{A}}$$