Differential Equations

Exact Differential Equation

The total differential of a func^n $f(x,y)$ is given by

$$df = \frac{df}{dx} dx + \frac{df}{dy} dy \quad — (1)$$

Consider the differential Equation

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad — (2)$$

Suppose there exists a function $f(x,y)$ such that

$$\frac{df}{dx} = M(x,y) \quad — (3)$$

$$\frac{df}{dy} = N(x,y) \quad — (4)$$

then the eq^n (2) becomes

$$M\,dx + N\,dy = 0$$

$$\frac{df}{dx} dx + \frac{df}{dy} dy = 0$$

i.e, $df = 0$

upon integrating, we get

$$f(x,y) = c$$

Where c is the arbitary constant

The L.H.S of eqⁿ ② is called exact differential and eqⁿ ③ is called exact differential eqⁿ

Ex- Consider : $xdy + ydx = 0$
we know that, $f = xy$ is a funcⁿ such that $xdy + ydx = df$
i.e $Mdx + Ndy = df$
from the above discussion, the solⁿ is $f = c$,
therefore the solⁿ for this problem is $xy = c$

<u>Note</u> An eqⁿ $Mdx + Ndy = 0$ is called an exact differential eqⁿ, if there exist a funcⁿ $f(x, y)$ such that
$$Mdx + Ndy = df$$
This is called exact differential eqn whose solⁿ is $f = c$

The necessary condition for an eqⁿ to be exact

(i) Partially differentiating eqⁿ ③ w.r.to y

$$\frac{\partial f}{\partial x} = M$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$= \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial N}{\partial x}$$

i.e $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$

Date.
Page No.

* Method to find $f(x,y)$ i.e the sol^n of the exact differential eq^n

Let the given eq^n satisfy the eq^n

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Step1 :- integrating eq^n 3 w.r.to x

$$\frac{\partial f}{\partial x} = M$$

$$f(x,y) = \int M\,dx + g(y) \quad \text{— } (5)$$

where $g(y)$ is an arbitary func^n

Step2 :- to find $g(y)$

Differentiate eq^n 5 w.r.to y and equate it to N. we get

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[\int M\,dx\right] + g(y) = N$$

$$\Rightarrow g'(y) = N - \frac{\partial}{\partial y}\left[\int M\,dx\right] \quad \text{— } (6)$$

Step3 :- by integrating w.r.to y

$$g(y) = \int\left[N - \frac{\partial}{\partial y}\left(\int M\,dx\right)\right] + c_1$$

Step4 :- Substituting eq^n 6 in 5

we get the required sol^n as

$$f(x,y) = \int M\,dx + \int\left[N - \frac{\partial}{\partial y}\left(\int M\,dx\right)\right] + c_1$$

$$= c_2$$

i.e, $f(x,y) = c$

where $c = c_2 - c_1$

Ques Determine which of the following eq^n are exact & solve the

(i) $x\,dy + 2y^2 dx = 0$

comparing with $M dx + N dy = 0$

$\therefore M = 2y^2, \; N = x$

The condition for exactness is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$$\frac{\partial M}{\partial y} = \frac{\partial (2y^2)}{\partial y} = 4y$$

$$\frac{\partial N}{\partial x} = \frac{\partial (x)}{\partial x} = 1 \qquad \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

(ii) $e^y dx + (xe^y + 2y)\,dy = 0$

$\Rightarrow N = xe^y + 2y, \quad M = e^y$

$\frac{\partial M}{\partial y} = e^y, \quad \frac{\partial N}{\partial x} = e^y$

$$M_y = N_x$$

consider, $\frac{\partial f}{\partial x} = M = e^y$

integrating w.r.to x

$$f(x,y) = \int e^y dx + g(y)$$

$$f(x,y) = xe^y + g(y) \quad — (1)$$

differentiating (1) w.r.to y and equating it to 'N' we get

$$\frac{\partial f}{\partial y} = \frac{\partial (xe^y)}{\partial y} + g'(y) = N$$

$$\frac{\partial f}{\partial y} = xe^y + g'(y) = xe^y + 2y$$

$$\therefore g'(y) = 2y$$

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$\Rightarrow$ $g(y) = \int 2y\,dy + c_1$

$= 2y^2/2$

$g(y) = y^2 + c_1$ — (2)

Substituting eq^n (2) in (1)

$f(x,y) = xe^y + y^2 + c_1$

the sol^n is $f(x,y) = c_2$

i.e, $xe^y + y^2 + c_1 = c_2$

$xe^y + y^2 = c$ where, $c = c_2 - c_1$

Que $(3x^2y + y/x)\,dx + (x^3 + \ln x)\,dy = 0$

$M = 3x^2y + y/x$

$\frac{\partial M}{\partial y} = 3x^2 + 1/x$

$N = x^3 + \ln x$

$\frac{\partial N}{\partial x} = 3x^2 + \frac{1}{x}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ exact diff eq^n

consider, $\frac{\partial f}{\partial x} = M = 3x^2y + y/x$

integrating w.r. to x

$f(x,y) = \int (3x^2y + y/x)\,dx + g(y)$

$= \frac{3x^3}{3}y + y\ln x + g(y)$

$f(x,y) = x^3y + y\ln x + g(y)$ — (1)

diff. w.r. to y and equate it to N

$f'(x,y) = x^3 + \ln x + g'(y) = N$

$$x^3 + \ln x + g'(y) = x^3 + \ln x$$
$$g'(y) = 0$$
$$\Rightarrow g'(y) = c_1$$

from eq^n 1
$$f(x,y) = x^3 y + y\ln x + c_1$$

The sol^n is $f = c$
$$x^3 y + y \ln x + c_1 = c$$
$$x^3 y + y \ln x = C \quad \text{where } C = c - c_1$$

Ques $(\cos x - x\cos y)\,dy - (\sin y + y\sin x)\,dx = 0$

$M = -(\sin y + y\sin x)$

$\frac{\partial M}{\partial y} = -\cos y - \sin x$

$N = \cos x - x\cos y$

$\frac{\partial N}{\partial x} = -\sin x - \cos y$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ Exact eq^n

consider, $\frac{\partial f}{\partial x} = M$

integrating M w.r.to x

~~f(x,y) = ...~~ $f(x,y) = -\int(\sin y + y\sin x) + g(y)$

$$f(x,y) = -[x\sin y - y\cos x] + g(y)$$

$$f(x,y) = -x\sin y + y\cos x + g(y) \quad -\ 1$$

diff. w.r.to y and equate it to N

$$f'(x,y) = -x\cos y + \cos x + g'(y)$$

$$-x\cos y + \cos x + g'(y) = \cos x - x\cos y$$

$$g'(y) = 0$$

$$g'(y) = c_1$$

from eq^n (i)

$\therefore f(x,y) = -x\sin y + y\cos x + c_1$

the sol^n is $f(x,y) = c_2$

$\therefore -x\sin y + y\cos x = c$ where $c = c_2 - c_1$

Note:- Short cut formula to find the sol^n of exact diff eq^n

$$\int M dx + \int N dy = c$$

'y' as constant — considering the terms that doesn't contain 'x'

Ex:- $M = -\sin y - y\sin x$, $N = \cos x - x\cos y$

$$-\int(\sin y + y\sin x)dx + \int 0\, dy = c$$

$$-x\sin y + y\cos x = c$$

Sometimes, the integration of M w.r.to x becomes tedious then N can be integrated easily. In that case the formula becomes

$$\int M dx + \int N dy = c$$

considering the terms that doesn't contain 'y' — 'x' as constant

Que Solve $\left(\log(x^2+y^2) + \dfrac{2x^2}{x^2+y^2}\right)dx + \dfrac{2xy}{x^2+y^2}dy = 0$

Sol^n $M = \log(x^2+y^2) + \dfrac{2x^2}{x^2+y^2}$, $N = \dfrac{2xy}{x^2+y^2}$

$$\frac{\partial M}{\partial y} = \frac{2y}{x^2+y^2} + \frac{(-4x^2y)}{(x^2+y^2)^2}, \quad \frac{\partial N}{\partial x} = \frac{2y(x^2+y^2) - 4x^2y}{(x^2+y^2)^2}$$

$$= \frac{2y}{x^2+y^2} + \frac{(-4x^2y)}{(x^2+y^2)^2}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \therefore \text{Exact eq}^n$$

$$\int 0\,dx + \int \frac{2xy}{x^2+y^2}\,dy = c$$

$$x\int \frac{2y}{x^2+y^2}\,dy = c$$

let $x^2+y^2 = t$

$2y\,dy = dt$

$$x\int \frac{1}{t}\,dt = c$$

$$x\log t = c$$

$$x\log(x^2+y^2) = c$$

Ques Find the values of a & b such that $(y+x^3)\,dx + (ax+by^3)\,dy = 0$ is exact. Also find the sol^n

Sol^n

$M = y+x^3$ $\qquad N = ax+by^3$

$\frac{\partial M}{\partial y} = 1$ $\qquad \frac{\partial N}{\partial x} = a$

$$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$1 = a$$

$$\boxed{a = 1}$$

for $a=1$ and b can be any no, the eq^n is exact

$$\int (y+x^3)\,dx + \int by^3\,dy = c$$

$$yx + \frac{x^4}{4} + \frac{by^4}{4} = c$$

$x\,dy + y\,dx$ $\frac{1}{xy}$

Date. ________

Page No. ________

1) Non-exact differential eqⁿ - (Integrating factor technique)

Sometimes $M\,dx + N\,dy = 0$ may not be exact if there exist a function $f(x,y)$ such that $f(x,y)[M\,dx + N\,dy] = 0$ is exact then $f(x,y)$ is called as integrating factor.

There are several methods to calculate integrating factor & the following 3 methods are helpful to deal with non-exact differential eqⁿ.

i) Method of grouping / inspection

The following table is helpful in this method

S.No	Group of terms	IF	exact differential
(i)	$x\,dy + y\,dx$	1	$d(x,y)$
(ii)	$x\,dy + y\,dx$	$1/xy$	$\frac{x\,dy + y\,dx}{xy} = d[\ln(xy)]$
(iii)	$x\,dy + y\,dx$	$\frac{1}{(xy)^n}, n \neq 1$	$\frac{x\,dy + y\,dx}{(xy)^n} = d\left[\frac{(xy)^{1-n}}{1-n}\right]$
(iv)	$x\,dy + y\,dx$	$\frac{1}{x^2+y^2}$	$\frac{x\,dx + y\,dx}{x^2+y^2} = \frac{1}{2} d\{\ln(x^2+y^2)\}$
(v)	$x\,dx + y\,dy$	$\frac{1}{(x^2+y^2)^m}, m \neq 1$	$\frac{x\,dx + y\,dy}{(x^2+y^2)^n} = \frac{1}{2} d\left[\frac{(x^2+y^2)^{1-n}}{1-n}\right]$
(vi)	$x\,dx + y\,dy$	2	$2(x\,dx + y\,dy) = d(x^2+y^2)$
(vii)	$x\,dy - y\,dx$	$1/x^2$	$\frac{x\,dy - y\,dx}{x^2} = d(y/x)$

(viii)	$xdy - ydx$	$1/y^2$	$-d(x/y)$
(ix)	$xdy - ydx$	$1/xy$	$d\{\ln(y/x)\}$
(x)	$xdy - ydx$	$\frac{1}{x^2+y^2}$	$d[\tan^{-1}(y/x)]$

Ques. Solve the eq^n $xdy - ydx + 2x^3 dx = 0$ —(1)

The standard form is $Mdx + Ndy = 0$

$$[2x^3 - y]\,dx + xdy = 0$$

$\because \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ the eq^n is not exact

Dividing by (x^2) in eq^n (1)

$$\frac{xdy - ydx}{x} + 2x\,dx = 0$$

$$\Rightarrow d\left(\frac{y}{x}\right) + 2x\,dx = 0$$

upon integration

$$\frac{y}{x} + x^2 = c'$$

$$\Rightarrow \boxed{y + x^3 = c'x}$$

Q. $xdy - ydx = xy^2 dx$ —(1)

$$xdy - [y + xy^2]\,dx = 0$$

~~$(1+xy)\,ydx - xdy = 0$~~

, divide by y^2

$$\frac{xdy - ydx}{y^2} - x\,dx = 0$$

$$-\frac{(ydx - xdy)}{y^2} - x\,dx = 0$$

$$-d(x/y) - x\,dx = 0$$

upon integrating

$$-x/y - x^2/2 = c$$

Date. ______
Page No. ______

Case 2 (Method)

The IF(F) is a func^n of x alone

$$\left[\frac{My - Nx}{N} = g(x)\right]$$

Let F be the integrating factor

$F\,[Mdx + Ndy] = 0$ —① is exact

$$\frac{\partial (FM)}{\partial y} = \frac{\partial (FN)}{\partial x}$$

$$\Rightarrow F \cdot \frac{\partial M}{\partial y} + M\frac{\partial F}{\partial y} = F\frac{\partial N}{\partial x} + N\frac{\partial F}{\partial x}$$

$$F\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = N \cdot \frac{\partial F}{\partial x} - M \cdot \frac{\partial F}{\partial y}$$

$$\Rightarrow \frac{1}{F}\left[N \cdot \frac{\partial F}{\partial x} - M\frac{\partial F}{\partial y}\right] = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \quad \text{—②}$$

Considering F as a func^n of x alone eq^n becomes

$$\Rightarrow \frac{1}{F} N \frac{dF}{dx} = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \quad \text{—③}$$

$$\Rightarrow \frac{1}{F} \cdot \frac{dF}{dx} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$$

and $\frac{1}{F}\frac{dF}{dx} = \boxed{g(x) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}}$

Consider $\frac{1}{F}\frac{dF}{dx} = g(x)$

$$\frac{dF}{F} = g(x)\,dx$$

integrating by $f = \int g(x)\,dx$

$$\boxed{F = e^{\int g(x)dx}} \quad \text{where } g(x) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$$

Therefore, while considering the IF, we will not write arbitary constant.

Que Solve the eq^n $y(2x^2 - xy + 1)\,dx + (x-y)\,dy = 0$ — (1)

Sol^n then $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

consider $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{2x^2 - 2xy + 1 - 1}{x-y} = \frac{2x(x-y)}{(x-y)} = 2x$, a func^n of x alone

let $F(x,y)$ be the IF

$$\therefore F(x,y) = e^{\int g(x)dx} = e^{\int 2x\,dx} = e^{x^2}$$

Multiply eq^n (1) with the IF (e^{x^2}) we get

$$y e^{x^2}(2x^2 - xy + 1)\,dx + e^{x^2}(x-y)\,dy = 0 \quad — (2)$$

This is of the form $M^* dx + N^* dy = 0$ where

$$M^* = e^{x^2} \cdot M = y e^{x^2}(2x^2 - xy + 1)$$

$$N^* = e^{x^2} \cdot N = e^{x^2}(x-y)$$

$$\frac{\partial M^*}{\partial y} = (2x^2 - 2xy + 1)e^{x^2} = \frac{\partial N^*}{\partial x}$$

Integrating

$$\frac{\partial f}{\partial y} = N^* = e^{x^2}(x-y)$$

integrating ~~Partially diff~~ w.r.t 'y'

$f = e^{x^2}\left(xy - \frac{y^2}{2}\right) + g(x)$

Differentiating w.r.to x & equating to M

$\frac{\partial f}{\partial x} = e^{x^2}\left(xy - \frac{y^2}{2}\right) \cdot 2x + e^{x^2}(y) + \frac{dg}{dx} = ye^{x^2}(2x^2 - xy + 1)$

$\Rightarrow \frac{dg}{dx} = 0 \Rightarrow g(x) = c'$

Therefore, $f = e^{x^2}\left(xy - \frac{y^2}{2}\right) + c'_1$

$\therefore$ The solⁿ is $f = c'$

$e^{x^2}\left(xy - \frac{y^2}{2}\right) + c'_1 = c'_2$

$\Rightarrow e^{x^2}\left(xy - \frac{y^2}{2}\right) = c'$

where, $c' = c'_2 - c'_1$ is arbitary constant

Case 3

When IF is a funcⁿ of y alone $\left[h(y) = \frac{N_x - M_y}{M}\right]$

$I.F. = e^{\int h(y)\, dy}$

Linear Differential eq^ns with constant coefficients

A differential eq^n is said to be linear if its satisfies the following 3 conditions:-

(i) All the derivatives in the eq^n should be of degree 1
(ii) The product of a derivative should not occur either with another derivative or with dependent variable
(iii) No ~~other~~ func^n of dependent variable other than itself present in eq^n

The general form of linear dependent eq^n (n^th order) with constant coefficients is given by

$$a_0 \cdot \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_n y = Q \quad (1)$$

where a_s is constant, $a_0 \neq 0$ & Q is any func^n of 'x'

Ex- $3\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^x$

The operator form

We consider D as $\frac{d}{dx}$

$$D \equiv \frac{d}{dx} \Rightarrow Dy = \frac{dy}{dx}$$

$$D^2 = \frac{d^2}{dx^2} \Rightarrow D^2y = \frac{d^2y}{dx^2}$$

$$D^n = \frac{d^n}{dx^n} \Rightarrow D^ny = \frac{d^ny}{dy^n}$$

Date.
Page No.

eqⁿ ① in its operator form is

$$a_0 \cdot D^n y + a_1 \cdot D^{n-1} y + a_2 \cdot D^{n-2} y + \cdots a_n y = Q$$

$$= [\underbrace{a_0 D^n + a_1 D^{n-1} + a_2 \cdot D^{n-2} + \cdots + a_n}] y = Q$$

$\boxed{f(D) \cdot y = Q}$ where $f(D) = a_0 D^n + a_1 D^{n-1} + a_2 \cdot D^{n-2} + \cdots + a_n$ — ②

eqⁿ ② is called operator funcⁿ of eqⁿ ①

$$f(D) \cdot y = Q$$

$f(D) y = 0$

complementary funcⁿ of $f(D) y = 0$

y_c

$\frac{1}{f(D)} \cdot Q$ = particular integral (y_p)

partical solⁿ of

$$y = y_c + y_p$$

Method to find complementary funcⁿ (CF) or y_c

* **Auxilary eqⁿ**

consider $f(D) \cdot y = Q$. The eqⁿ $f(m) = 0$ is called as auxilary eqⁿ

$$\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 16y = \cos x$$

$$(D^2 - 12D + 16) y = \cos x$$

$$f(D) = D^2 - 12D + 16$$

$$AE = m^2 - 12m + 16 = 0$$

Date.
Page No.

Based on the root of the auxillary eq^n ~~K~~ we have the following cases to find the complementary func^n. of $f(D)y = 0$

case1: when the roots ~~the~~ real & distinct

let $m_1, m_2, \ldots m_n$ be 'n' real & distinct roots

$\Rightarrow$ then $y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots + c_n e^{m_n x}$

Ex- $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^x$

$(D^2 + 5D + 6)y = e^x$

A.E $\Rightarrow m^2 + 5m + 6 = 0$

$(m+2)(m+3) = 0$

$\boxed{m = -2}, \boxed{m = -3}$

real & distinct

$$\boxed{y_c = c_1 e^{-2x} + c_2 e^{-3x}}$$

Case2 : when the roots are real & equal

~~let~~ $m_1, m_2, \ldots m_n$ ~~be r~~

~~$m_1 = m_2$~~

$\Rightarrow$ $y_c = (c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}$

Ex- 3, 3, 4, 5

$$y_c = (c_1 + c_2 x) e^{3x} + c_3 e^{4x} + c_4 e^{5x}$$

Ex- 3, 3, 3, 4

$$y_c = (c_1 + c_2 x + c_3 x^2) e^{3x} + c_4 e^{4x}$$

Date.
Page No.

Case 3 :- When the roots are imaginary $(a \pm ib)$

$\Rightarrow y_c = e^{ax}(c_1 \cos bx + c_2 \sin bx)$

Eg: $3 \pm 5i, 6, 6, 7, 9$

$y_c = e^{3x}(c_1 \cos 5x + c_2 \sin 5x) + (c_3 + c_4 x) e^{6x} + c_5 e^{7x} + c_6 e^{9x}$

Case 4 :- Repeated complex roots

let $a \pm ib$ is equal and repeated twice

$\Rightarrow y_c = e^{ax}[(c_1 + c_2 x) \cos bx + (c_3 + c_4 x) \sin bx]$

thrice :- $y_c = e^{ax}[(c_1 + c_2 x + c_3 x^2) \cos bx + (c_4 + c_5 x + c_6 x^2) \sin bx]$

Case 5 :- When the roots are irrational

let $\alpha \pm \sqrt{\beta}$ be the roots

$\Rightarrow y_c = e^{\alpha x}[c_1 \cosh \sqrt{\beta} x + c_2 \sinh \sqrt{\beta} x]$

Case 6 :- If the irrational roots are repeated twice

$\Rightarrow y_c = e^{\alpha x}[(c_1 + c_2 x) \cosh \sqrt{\beta} x + (c_3 + c_4 x) \sinh \sqrt{\beta} x)$

Ques Solve the eqn

$$\frac{d^3 y}{dx^3} - 7\frac{dy}{dx} - 6y = 0$$

A.E $\Rightarrow m^3 - 7m - 6 = 0$

$m = -1, -2, 3$

$y(x) = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x}$

$m^2 - m - 6$

$m+1 \,)\, m^3 - 7m - 6$

$m^3 + m^2$

$-m^2 - 7m - 6$

$-m^2 - m$

$-6m - 6$

Methods to find particular solⁿ

(i) The inverse operator $\frac{1}{f(D)}$

$\frac{1}{f(D)}$ operated on Q is that func of x free from arbitary constants, which when operated with $f(D)$ gives Q hence the operator $\frac{1}{f(D)}$ is called Inverse operator

Note (i) $\frac{1}{D} Q = \int Q \, dx$

(ii) $\frac{1}{f(D)} Q$ is called particular integral of $f(D)$ operator on $y = Q$ $[f(D)y = Q]$

Imp (iii) $\frac{1}{D-a} Q = e^{ax} \int Q e^{-ax} . dx$

$\frac{1}{D+a} Q = e^{-ax} \int Q e^{ax} dx$

Ques $(D-2) y = e^{x}$

A.E $\Rightarrow m-2=0 \Rightarrow m=2$

$y_c = c_1 e^{2x}$

$y_p = \frac{1}{f(D)} Q = \frac{1}{D-2} . e^{x}$

$= e^{2x} \int e^{x} . e^{-2x} dx$

$= e^{2x} \int e^{-x} dx$

$= -e^{2x} . e^{-x}$

$= -e^{x}$

$y = y_c + y_p$

$= c_1 e^{2x} + (-e^{x})$

Ex $(D^2+5D+6)y = e^x$

AE is $m^2+5m+6=0$

$(m+3)(m+2)=0$

$m=-2,-3$

$y_c = C_1 e^{-3x} + C_2 e^{-2x}$

$y_p = \frac{1}{D^2+5D+6} e^x \Rightarrow \frac{1}{(D+3)(D+2)} \cdot e^x \Rightarrow \left[\frac{A}{D+3} + \frac{B}{D+2}\right] e^x$

$\Rightarrow \frac{A}{D+3} e^x + \frac{B}{D+2} e^x$

The following methods help us to find the complementary func^n based on the nature of Q

Case I when $Q = e^{ax}$ or e^{ax+b}

$y_p = \frac{1}{f(D)} \cdot Q$

$= \frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(a)}$, $f(a) \neq 0$

when $f(a)=0$, $\frac{1}{f(D)} e^{ax} = \frac{x \cdot e^{ax}}{f'(a)}$, $f'(a) \neq 0$

when $f'(a)=0$, $\frac{1}{f(D)} \cdot e^{ax} = \frac{x^2 \cdot e^{ax}}{f''(a)}$, $f''(a) \neq 0$

We repeat this mechanism till the denominator doesnot becomes zero.

Q To find the particular sol^n of $(4D^2+4D-3)y = e^{2x}$

Sol^n

AE:- $4D^2+4D-3=0$

$4D^2+6D-2D-3=0$

$2D(2D+3)-1(2D+3)=0$

$(2D-1)(2D+3)=0$

$\frac{1}{4D^2+4D-3} \cdot e^{2x} = \frac{e^{2x}}{f(2)} = \frac{1}{21} e^{2x}$

Ques $(D^2+3D+2)y = 5$

$y = \frac{1}{D^2+3D+2} \cdot 5 \cdot e^{0x}$

$= \frac{5 \cdot e^{0x}}{f(0)} = 5 \cdot \frac{1}{2} \cdot e^{0x} = \frac{5}{2}$

Ques $(D^3 - 3D^2 + 4)y = e^{2x}$

$f(D) = D^3 - 3D^2 + 4$

$f(2) = 0$

$f'(D) = 3D^2 - 6D$

$f'(2) = 0$

$f''(D) = 6D - 6$

$f''(2) = 6$

$y_p = \frac{x^2 \cdot e^{2x}}{6}$

Case II :- $Q = \sin(ax+b)$ (or) $\cos(ax+b)$ (or) $\sin ax$ (or) $\cos ax$

Step 1 Will replace D^2 with $(-a^2)$, D^4 with (a^4), D^6 with $(-a^6)$ .....

Step 2 By doing so, following possibilities may arise

(i) if the denominator reduces to constant it is the final step which gives particular integral

Ex- $(D^2+6)y = \sin 2x$

$y_p = \frac{1}{D^2+6} \cdot \sin 2x \Rightarrow \frac{1}{2} \sin 2x$

(ii) The denominator reduces to D only then we will integrate the given function Q for one time

Ex- $\frac{1}{(D^3+2D^2+8)}y = \sin 2x$

$\frac{1}{(D \cdot D^2 + 2D^2 + 8)} y = \sin 2x$

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$(-4D-8+8)y$
$= -\frac{1}{4}\int \sin 2x\, dx$
$= -1/4\left[\frac{-\cos 2x}{2}\right] = \frac{\cos 2x}{8}$

Date.
Page No.

(iii) Sometimes the denominator reduces to the form $\alpha D+\beta$. In this case we operate both the num & den by the conjugate no $\alpha D-\beta$

$$\left[\frac{1}{\alpha D+\beta}\cdot\frac{\alpha D-\beta}{\alpha D-\beta}\right]\sin ax$$

$$= \frac{\alpha D-\beta}{\alpha^2 D^2-\beta^2}\sin ax$$

Ex - $(1/2D+5)\sin 2x$

$= \frac{1}{2D+5}\times\frac{2D-5}{2D-5}\sin 2x$

$= \frac{2D-5}{4D^2-25}\sin 2x$

$= \frac{2D-5}{-16-25}\cdot\sin 2x$

$= \frac{2D-5}{-41}\cdot\sin 2x$

$= \frac{-1}{41}[2(2\cos 2x)-5\sin 2x]$ $\because D=\frac{d}{dx}$

Now again replace D^2 by $(-a^2)$ & follow above step

(iv) Cases of failure

If $f(-a^2)=0$ the above method fails

$$\frac{1}{f(D^2)}\cos(ax+b) = x^2\cdot\frac{1}{f''(-a^2)}\cos(ax+b)$$

When $f(-a^2)=0$ we differentiate the ~~denominator~~ denominator & multiply it with x

- When $f'(-a^2)=0$, we differentiate the denominator one more time and multiply the resultant expression again with x. We repeat the process untill we reach a const in a denominator

Cauchy - Euler eqn

The general form of n^{th} order Cauchy

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} x}{dx^{n-1}} + a_2 x^{n-2} \cdot \frac{d^{n-2}}{dx^{n-2}} + \cdots + a_n y = \phi$$

where $a(i)$ are constant

This eqn is the variable coefficient eqn which can be reduce into an constant coefficient eqn, while obtaining its soln

$$x^3 \cdot \frac{d^3 y}{dx^3} + 2x^2 \frac{d^2 y}{dx^2} + 5x \frac{dy}{dx} + 6y = e^{2x}$$

⇒ Solution technique

Let $x = e^z$

$$\Rightarrow z = \log x$$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \cdot \frac{dy}{dz}$$

$$\Rightarrow x \cdot \frac{dy}{dx} = \frac{dy}{dz} = Dy \quad \text{where } D = \frac{d}{dz}$$

Again $\frac{d^2 y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[\frac{1}{x} \cdot \frac{dy}{dz}\right] = \frac{-1}{x^2} \cdot \frac{dy}{dz} + \frac{1}{x} \cdot \frac{d^2 y}{dz^2} \cdot \frac{dz}{dx}$

$$= \frac{-1}{x^2} \cdot \frac{dy}{dz} + \frac{1}{x^2} \cdot \frac{d^2 y}{dz^2}$$

$$= \frac{1}{x^2}\left[\frac{d^2 y}{dz^2} - \frac{dy}{dz}\right]$$

$\Rightarrow x^2 \cdot \frac{d^2y}{dx^2} = D^2y - Dy = D(D-1)y$ where $D = \frac{d}{dz}$

$$x \frac{dy}{dx} = Dy$$

$$x^2 \cdot \frac{d^2y}{dx^2} = D(D-1)y$$

$$x^3 \cdot \frac{d^3y}{dx^3} = D(D-1)(D-2)y$$

$$x^4 \cdot \frac{d^4y}{dx^4} = D(D-1)(D-2)(D-3)y$$

$D = \frac{d}{dz}$ — (A)

Substituting A in cauchy Euler's eq^n

The eq^n will be reduces to a constant coefficient eq^n whose independent variable is z which can be solved by finding compeementary func & particular sol^n. finally we replace z with $\log x$ to find the actual sol^n in terms of x

Q. Solve:- $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$ — (1)

Clearly, this is a cauchy euler eq^n which can be solved by substituting $x = e^z \Rightarrow z = \log x$

we will get,

$$x \cdot \frac{dy}{dx} = Dy$$

$$x^2 \frac{d^2y}{dx^2} = D(D-1)y$$

$$x^3 \cdot \frac{d^3y}{dx^3} = D(D-1)(D-2)y$$

where $D = \frac{d}{dz}$ — (A)

Date.
Page No.

Substituting (A) in (1) we get

$D(D-1)(D-2)y + 2D(D-1)y + 2y = 10(e^z + e^{-z})$

$\Rightarrow [D(D-1)(D-2) + 2D(D-1) + 2]y = 10(e^z + e^{-z})$

$[(D^2-D)(D-2) + 2D^2 - 2D + 2]y = 10(e^z + e^{-z})$

$y(D^3 - D^2 + \cancel{2D} - \cancel{2D^2} + \cancel{2D^2} - \cancel{2D} + 2) = 10(e^z + e^{-z})$

$(D^3 - D^2 + 2)y = 10(e^x + e^{-x})$

AE :- $(m^3 - m^2 + 2) = 0 \Rightarrow m = -1, 1 \pm i$

$\therefore y_c = c_1 e^{-z} + e^z (c_2 \cos z + c_3 \sin z)$

put $z = \log x$

$y_c = \frac{c_1}{x} + x(c_2 \cos(\log x) + c_3 \sin(\log x))$

$y_p = \frac{1}{f(D)} \cdot Q$

$= \frac{1}{D^3 - D^2 + 2} 10(e^z + e^{-x})$

$= 10\left[\frac{1}{D^3 - D^2 + 1} e^z + \frac{1}{D^3 - D^2 + 2} e^{-x}\right]$

$= 10\left[\frac{e^z}{2} + \frac{z \cdot e^{-z}}{5}\right]$ put $z = \log x$

$y_p = 5 e^{\log x} + 2(\log x) \cdot e^{-\log x}$

$y_p = 5x + \frac{2\log x}{5x}$

sol of eq^n (1) = $y_c + y_p$

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$$= \frac{c_1}{x} + x\left(c_2 \cos(\log x) + c_3 \sin(\log x)\right) \quad 5x + \frac{2}{x}(\log x)$$

Legendre Linear Differential eq^n

The general form of eq^n is

$$\left[a_0(ax+b)^n \frac{d^n y}{dx^n} + a_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_2(ax+b)^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_{n-1}(ax+b)\frac{dy}{dx} + a_n y = Q\right] \quad (1)$$

where a_i's, a, b are constants
Q is a func. of independent variable x

This eq^n can be converted into a constant coefficient eq^n
by considering $(ax+b) = e^t$ (2)

$\Rightarrow t = \ln(ax+b)$ (3)

differentiating eq^n (2) w.r.t. ~~to~~ t

$$a \cdot \frac{dx}{dt} = e^t \quad (4)$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= \frac{dy}{dt} \cdot ae^{-t}$$

$$\Rightarrow e^t \cdot \frac{dy}{dx} = a \cdot \frac{dy}{dt} \Rightarrow \boxed{(ax+b) \cdot \frac{dy}{dx} = aDy} \text{ where } D = \frac{d}{dt} \quad (5)$$

Date.
Page No.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx}$$

$$= ae^{-t} \cdot \frac{d}{dt}\left(ae^{-t} \cdot \frac{dy}{dt}\right)$$

$$= ae^{-t}\left\{ae^{-t} \cdot \frac{d^2y}{dt^2} + \frac{dy}{dt} \cdot (-ae^{-t})\right\}$$

$$= ae^{-t}\left[ae^{-t}\left\{\frac{d^2y}{dt^2} - \frac{dy}{dt}\right\}\right]$$

$$= a^2e^{-2t}\left\{D^2y - Dy\right\}$$

$$\Rightarrow \frac{d^2y}{dx^2} = a^2e^{-2t}\, D(D-1)y$$

$$\Rightarrow e^{2t} \cdot \frac{d^2y}{dx^2} = a^2 D(D-1)y$$

$$\Rightarrow (ax+b)^2 \cdot \frac{d^2y}{dx^2} = a^2 D(D-1)y \quad \text{—⑥} \quad \text{where } D \equiv \frac{d}{dt}$$

Similarly, $(ax+b)^3 \dfrac{d^3y}{dx^3} = a^3 D(D-1)(D-2)y$ —⑦, $D \equiv \dfrac{d}{dt}$

Substituting ⑤, ⑥, ⑦ so on in eq^n ① it will be reduces to a constant coefficient eq^n whose sol^n can be obtained by complementary func & particular integral but this sol^n comes in terms of t.

finally we replace t with $\ln(ax+b)$ which becomes the sol^n of eq^n ①

Q Solve $(2x+5)^2\frac{d^2y}{dx^2} - 6(2x+5)\frac{dy}{dx} + 8y = 6x$

This is a legendre differential eq^n

Here $a=2$, $b=5$. let $2x+5 = e^t$

$\Rightarrow t = \ln(2x+5)$

Replace $(2x+5)$ into $\frac{dy}{dx}$

we know that $(ax+b)\frac{dy}{dx} = aDy$

$(ax+b)^2 \cdot \frac{d^2y}{dx^2} = a^2 D(D-1)y$

$$\left.\begin{aligned} &\Rightarrow (2x+5)\frac{dy}{dx} = 2Dy \\ &(2x+5)^2\frac{d^2y}{dx^2} = 4D(D-1)y \end{aligned}\right\} \quad D = \frac{d}{dt} \quad - B$$

Substituting eq^n B in eq^n A

$4D(D-1)y - 6(2Dy) + 8y = 6\left(\frac{e^t - 5}{2}\right)$

$\Rightarrow 4(D^2 - D)y - 12Dy + 8y = 3(e^t - 5)$

$\Rightarrow \{(D^2 - D) - 3D + 2\}y = \frac{3}{4}(e^t - 5)$

$\Rightarrow \boxed{(D^2 - 4D + 2)y = \frac{3}{4}(e^t - 5)}$

Ye $m^2 - 4m + 2 = 0$

$m = \frac{4 \pm \sqrt{16-8}}{2} = \frac{4 \pm 2\sqrt{2}}{2} = 2 \pm \sqrt{2}$

Date.
Page No.

$$y_c = e^{2t}\left[c_1 \cosh\sqrt{2}\, t + c_2 \sinh\sqrt{2}\, t\right]$$

Put $t = \ln(2x+5)$

$$\Rightarrow y_c = e^{2\ln(2x+5)}\left[c_1 \cosh\sqrt{2}\ln(2x+5) + c_2 \sinh\sqrt{2}\ln(2x+5)\right]$$

$$y_c = (2x+5)^2\left[c_1 \cosh\sqrt{2}\ln(2x+5) + c_2 \sinh\sqrt{2}\ln(2x+5)\right]$$

$$y_p = \frac{1}{f(D)}\cdot\phi$$

$$= \frac{1}{D^2-4D+2}\left[\frac{3}{4}(e^t-5)\right]$$

$$= \frac{3}{4}\left[\frac{1}{D^2-4D+2}e^t - 5\cdot\frac{1}{D^2-4D+2}e^{0t}\right]$$

$$= \frac{3}{4}\left[-e^t - \frac{5}{2}\right]$$

$$y_p = -\left(\frac{3}{4}e^t + \frac{15}{8}\right)$$

$$\therefore\ y_p = -\left[\frac{3}{4}(2x+5) + \frac{15}{8}\right]$$

$$y = y_c + y_p$$

$$= (2x+5)^2\left[c_1 \cosh\sqrt{2}(\ln(2x+5)) + c_2 \sinh\sqrt{2}(\ln(2x+5))\right] - \left[\frac{3}{4}(2x+5) + \frac{15}{8}\right]$$

Q- $(x+2)^2y'' + 3(x+2)y' - 3y = 0$ Ans;- $c_1(x+2) + c_2(x+2$

Case 5 Particular sol" when $Q = x.V$ where V is any func of

$$\frac{1}{f(D)}Q = \frac{1}{f(D)}\{xV\}$$

$$\boxed{y_p = x.\frac{1}{f(D)}V - \frac{f'(D)}{(f(D))^2}.V}$$

Case 4 when $Q = e^{ax}.V$ where V is a func of x

$$y_p = \frac{1}{f(D)}Q = \frac{1}{f(D)}e^{ax}.V$$

$$\boxed{y_p = e^{ax}.\frac{1}{f(D+a)}.V}$$

olve $(D^2-2D+1)y = xe^x\sin x$

AE :- $m^2 - 2m + 1 = 0$

$m = 1, 1$

$y_c = (c_1 + c_2x)e^x$

Here Q is $xe^x\sin x$

consider $Q = e^{ax}.V$

$= e^x\{x\sin x\}$

Here $\boxed{a=1}$

Date.
Page No.

$$y_p = \frac{1}{f(D)} \cdot Q = \frac{1}{(D-1)^2} e^x (x\sin x)$$

$$= e^x \cdot \frac{1}{(D+1-1)^2} \cdot x\sin x$$

$$= e^x \cdot \frac{1}{D^2} x\sin x$$

$$= e^x \left[x \cdot \frac{1}{D^2} \sin x - \frac{2D}{D^4} \cdot \sin x \right]$$

$$= e^x \{ x(-\sin x) - 2D(\sin x) \}$$

$$= e^x \{ -x\sin x - 2\cos x \}$$

$$= -e^x (x\sin x + 2\cos x)$$

$$\therefore y = y_c + y_p$$

$$= (c_1 + c_2 x)e^x - e^x(x\sin x + 2\cos x)$$

Ques **Note :-** when Q is not among the 5 func^ns which we have discussed among the particular integral, we use any of the following rules, for finding the particular sol^n

$$\frac{1}{D-a} Q = e^{ax} \int Q e^{-ax} dx$$

$$\frac{1}{D+a} Q = e^{-ax} \int Q \cdot e^{ax} dx$$

Que Solve :- $(D^2 + a^2) y = \sec ax$, find

AE is $m^2 + a^2 = 0$

$\Rightarrow (m + ia)(m - ia) = 0$

$\Rightarrow m = \pm ia$

Date.
Page No.

$y_c = e^{0x}(c_1 \cos ax + c_2 \sin ax)$

$y_c = c_1 \cos ax + c_2 \sin ax$

$y_p = \frac{1}{f(D)} Q = \frac{1}{D^2+a^2} \sec ax$

$= \frac{1}{(D+ia)(D-ia)} \sec ax$

$= \frac{1}{2ia}\left[\frac{1}{D-ia} - \frac{1}{D+ia}\right] \sec ax \longrightarrow (A)$

Consider $\frac{1}{D-ia} \sec ax = e^{iax} \int \sec ax \, e^{-iax} \, dx$

$= e^{iax} \int (\cos ax - i \sin ax) \cdot \sec ax \, dx$

$= e^{iax} \int (1 - i \tan ax) \, dx$

$= e^{iax} \left\{ x + i\left(\frac{\log \cos(ax)}{a}\right) \right\} \longrightarrow (1)$

Similarly

$\frac{1}{D+ia} \sec ax = e^{-iax} \left\{ x - i\left(\frac{\log \cos(ax)}{a}\right) \right\} \longrightarrow (2)$

put (1) & (2) in (A)

$PI = \frac{1}{2ia}\left[e^{iax}\left(x + i\left(\frac{\log \cos ax}{a}\right)\right) - e^{-iax}\left\{x - i\left(\frac{\log \cos ax}{a}\right)\right\}\right]$

$= \frac{1}{2ia}\left[x\left(e^{iax} - e^{-iax}\right) + i\left(\frac{\log \cos ax}{a}\right)\left(e^{iax} + e^{-iax}\right)\right]$

$= \frac{1}{2ia}\left[2ix \sin ax + \frac{i}{a} \log(\cos ax)(2\cos ax)\right]$

$y_p = \frac{1}{a}\left[x \sin ax + \frac{1}{a} \cos ax \log \cos ax\right]$

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Case: When $Q = x^m$ (or) any polynomial

The following steps are followed

(i) Take out the lowest degree term from $f(D)$ as common to make the 1st term as unity so that the binomial theorem for -ve index is applicable. The remaining factor will be of the form $1+\phi(D)$ or $1-\phi(D)$.

Q. $(D^2-3D+5)y = x^3$

$$y_p = \frac{1}{D^2-3D+5} x^3$$

$$= \frac{1}{5\underbrace{\left[1+\left(\frac{D^2-3D}{5}\right)\right]}_{1+\phi(D)}}$$

(ii) Take the factor $[1+\phi(D)$ or $1-\phi(D)]$ to the numerator & expand that in an ascending power of D using the following formulae

$$[1+\phi(D)]^{-1} = 1-\phi(D)+\phi(D)^2-\phi(D)^3+ \cdots$$

$$[1-\phi(D)]^{-1} = 1+\phi(D)+\phi(D)^2+\phi(D)^3+ \cdots$$

(iii) Operate every term of the expansion on Q.

This is shown in the following example.

Ques $(D^2+5D+4)y = x^2+7x+9$

AE :- $m^2+5m+4 = 0$

$m^2+4m+m+4 = 0$

$m(m+4)+(m+4) = 0$

$(m+1)(m+4) = 0$

$m = -1, m = -4$

roots are real & distinct

$y_c = c_1 e^{-x} + c_2 e^{-4x}$

$$y_p = \frac{1}{f(D)} f = \frac{1}{(D^2+5D+4)}(x^2+7x+9)$$

$$= \frac{1}{4\left[1+\left(\frac{D^2+5D}{4}\right)\right]}(x^2+7x+9)$$

$$= \frac{1}{4}\left[1+\left(\frac{D^2+5D}{4}\right)\right]^{-1}(x^2+7x+9)$$

$$= \frac{1}{4}\left[1-\left(\frac{D^2+5D}{4}\right)+\left(\frac{D^2+5D}{4}\right)^2-\left(\frac{D^2+5D}{4}\right)^3 \cdots\right](x^2+7x+9)$$

$$= \frac{1}{4}\left[1-\frac{D^2}{4}-\frac{5D}{4}+\frac{25}{16}D^2\right](x^2+7x+9)$$

$$= \frac{1}{4}\left[x^2+7x+9-\frac{1}{4}(2)-\frac{5}{4}(2x+7)+\frac{25}{16}(2)\right]$$

$$y_p = \frac{1}{4}\left(x^2+\frac{9}{2}x+\frac{23}{8}\right)$$

$f = x^2+7x+9$
$Df = 2x+7$
$D^2f = 2$

Date.
Page No.

$y = y_c + y_p$

$= c_1 e^{-x} + c_2 e^{-4x} + \frac{1}{4}\left[x^2 + \frac{9}{2}x + \frac{23}{8}\right]$

Ans

$\Rightarrow \quad (1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots\cdots$

$(1-x)^{-2} = (1-x)^{-1}(1-x)^{-1}$

$= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots\cdots$

Ques $(D-2)^2 y = 8(e^{2x} + \sin 2x + x^2)$

AE :- $(m-2)^2 = 0$

$(m-2)(m-2) = 0$

$m = 2, 2$

real ~~&~~ equal

$y_c = (c_1 + c_2 x)e^{2x}$

$y_p = \frac{1}{f(D)} Q = 8 \frac{1}{(D-2)^2}(e^{2x} + \sin 2x + x^2)$

$= 8\left\{\frac{1}{(D-2)^2} \cdot e^{2x} + \frac{1}{(D-2)^2}\sin 2x + \frac{1}{(D-2)^2} x^2\right\}$

for $\frac{1}{(D-2)^2} e^{2x} = \frac{1}{f(D)} \cdot e^{ax} = \frac{e^{ax}}{f(a)} \Rightarrow f(a) \neq 0$

if $f(a) = 0$, $\frac{1}{f(D)} e^{ax} = x \cdot \frac{1}{f'(a)} e^{ax}$, $f'(a) \neq 0$

if $f'(a) = 0$, $\frac{1}{f(D)} e^{ax} = \frac{x^2 \cdot e^{ax}}{f''(a)}$

$f(D) = (D-2)^2$

$f(a) = (2-2)^2$

$= 0^2 = 0$

$f'(D) = 2(D-2)$

$f'(a) = 0$

$f''(D) = 2 \neq 0$

~~$f''(a)$~~

$$\frac{1}{(D-2)^2} e^{2x} = \frac{x^2 \cdot e^{2x}}{2}$$

$\Rightarrow$ for $\frac{1}{(D-2)^2} \sin 2x$

$$= \frac{1}{D^2-4D+4} \sin 2x = \frac{1}{-a^2-4D+4} \sin 2x$$

$a=2$

$a^2=4$

$$= \frac{1}{\cancel{-4}-4D\cancel{+4}} \sin 2x$$

$$= \frac{\sin 2x}{4D}$$

$$= \frac{-1}{4} \cdot \frac{1}{D} \sin 2x$$

$$= -\frac{1}{4} \int \sin 2x \, dx$$

$$= \frac{-1}{4} \left[\frac{-\cos 2x}{2} \right]$$

$$= \frac{1}{8} \left(\frac{\cos 2x}{2} \right)$$

$\Rightarrow$ for $\frac{1}{(D-2)^2} x^2 = \frac{1}{\left[-2\left[1-\frac{D}{2}\right]\right]^2} \cdot x^2 = \frac{1}{4\left(1-\frac{D}{2}\right)^2} \cdot x^2$

$$= \frac{1}{4} \left(1 - D/2\right)^{-2} \cdot x^2$$

$$= \frac{1}{4} \left[1 + D + \frac{3D^2}{4} + \frac{4D^3}{8} + \cdots \right] \cdot x^2$$

$$= \frac{1}{4} \left[1 + D + \frac{3D^2}{4} \right] \cdot x^2$$

$f = x^2$

$Df = 2x$

$D^2 f = 2$

Date.
Page No.

$= \frac{1}{4}\left[x^2 + \cancel{x^2}(2x) + \frac{3}{4}(2)\right] = \frac{1}{4}\left[x^2+2x+\frac{3}{2}\right]$

$y_p = 8\left[\frac{x^2 \cdot e^{2x}}{2} + \frac{1}{8}\left(\frac{\cos 2x}{2}\right) + \frac{1}{4}\left(x^2+2x+\frac{3}{2}\right)\right]$

$y = (c_1+c_2x)e^{2x} + 8\left[\frac{x^2 \cdot e^{2x}}{2} + \frac{1}{8}\left(\frac{\cos 2x}{2}\right) + \frac{1}{4}\left(x^2+2x+3/2\right)\right]$

Q. find the particular integral of $y_4 + 2y_2 + y = x^2 \cos$

Variable coefficient linear differential eq^n

The standard form of 2^{nd} order linear differential eq^n with variable coefficients is

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

Note:- The coefficient of y'' is 1

We solve this eq^n by the following methods

Method: To find the complete sol^n of $y'' + Py' + Qy = R$ when a part of complement func^n is known.

(Method of reduction of order)

Let $y = u$ be the part of the complementary func^n of the eq^n $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ — (1)

$$\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = 0 \quad -(2)$$

Let $y = uv$ with a complete sol^n of the eq^n (1)

$\therefore$ we need to find v for the complete sol^n

$$y = uv$$

$\because$

Date.
Page No.

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d^2y}{dx^2} = u \cdot \frac{d^2v}{dx^2} + \frac{du}{dx} \cdot \frac{dv}{dx} + v \cdot \frac{d^2u}{dx^2} + \frac{dv}{dx} \cdot \frac{du}{dx}$$

$$\frac{d^2y}{dx^2} = u \cdot \frac{d^2v}{dx^2} + 2\frac{du}{dx} \cdot \frac{dv}{dx} + v \cdot \frac{d^2u}{dx^2}$$

Substituting y, y' & y'' in eq^n (1) we get

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

$$\Rightarrow \left(u \cdot \frac{d^2v}{dx^2} + 2\frac{du}{dx} \cdot \frac{dv}{dx} + v\frac{d^2u}{dx^2}\right) + P\left(u \cdot \frac{dv}{dx} + v\frac{du}{dx}\right) + Q \cdot uv = R$$

$$u \cdot \frac{d^2v}{dx^2} + \left[2 \cdot \frac{du}{dx} + Pu\right]\frac{dv}{dx} + v\left[\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right] = R$$

$$\Rightarrow u \cdot \frac{d^2v}{dx^2} + \left[2\frac{du}{dx} + Pu\right]\frac{dv}{dx} = R$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left[\frac{2}{u}\frac{du}{dx} + P\right]\frac{dv}{dx} = \frac{R}{u} \quad (3)$$

eq^n (3) is 2nd order variable coefficient eq^n which can be reduced into a 1st order eq^n by substituting $\frac{dv}{dx} = p$

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Date.
Page No.

$$\Rightarrow \frac{dp}{dx} + \left[\frac{2}{u} \cdot \frac{du}{dx} + P\right] p = \frac{R}{u} \quad \text{(4)}$$

eqⁿ (4) is a 1st order eqⁿ which can be solved to get p

but $p = \frac{dv}{dx}$

∴ upon integrating p we get v

The complete solⁿ of eqⁿ (1) is $y = u \cdot v$

Note: Sometimes the part of the complementary funcⁿ u will not be provided us. In that case, we use the following table to find the part of the complementary funcⁿ

Condition	part of C.F
$1 + \frac{P}{a} + \frac{Q}{a^2} = 0$	e^{ax}
$1 + P + Q = 0$	e^{x}
$1 - P + Q = 0$	e^{-x}
$m(m-1) + Pmx + Qx^2 = 0$	x^m
$P + Qx = 0$	x
$2 + 2Px + Qx^2 = 0$	x^2

Date.
Page No.

Q. Solve the eq^n $x^2y'' - (x^2+2x)y' + (x+2)y = x^3 \cdot e^x$
where $y = x$ is a sol^n

Sol^n given $y = x$ is the part of the complementary func^n
The eq^n in the standard form is

$$y'' - \left[1 + \frac{2}{x}\right]y' + \frac{(x+2)}{x^2}y = xe^x \quad \text{--- (1)}$$

Given $\boxed{u = x}$ where u is the part of the CE

$$P = -\left(1 + \frac{2}{x}\right) ; \quad Q = \frac{x+2}{x^2} ; \quad R = xe^x$$

let $y = uv = xv$ be the complete sol^n of eq^n (1)

$$y' = x \cdot \frac{dv}{dx} + v$$

$$y'' = x \cdot \frac{d^2v}{dx^2} + 2 \cdot \frac{dv}{dx}$$

Sustituting y, y' & y'' in eq^n (1)

$$x\frac{d^2v}{dx^2} + 2\frac{dv}{dx} - \left(1 + \frac{2}{x}\right)\left(x\frac{dv}{dx} + v\right) + \left(\frac{1}{x} + \frac{2}{x^2}\right)(x \cdot v) = x \cdot e^x$$

$$\frac{d^2v}{dx^2} - \frac{dv}{dx} = e^x \quad \text{(3)}$$

put $\frac{dv}{dx} = p$

$$\Rightarrow \frac{dp}{dx} - p = e^x \quad \text{--- (4)}$$

Date.
Page No.

$P_I = c_1 e^x$

A E :- $m = 1$ $\Rightarrow$ $P_I = \frac{1}{D-1} e^x = x e^x$

$p = c_1 e^x + x e^x$

$\Rightarrow \frac{dv}{dx} = c_1 e^x + x e^x$

Integrating,

$v = \int (c_1 e^x + x e^x) dx$

$= c_1 e^x + x \int e^x - \iint e^x . dx + c_2$

$v = c_1 e^x + x e^x - e^x + c_2$

$\therefore y = uv = c_1 x e^x + x^2 e^x - x e^x + c_2 x$

$y = x^2 e^x + x e^x (c_1 - 1) + c_2 x$

Note After substituting $y = uv$ in eq^n (1) the coefficient of $v = 0$ otherwise our simplication is wrong

Method 2

To find the complete sol^n of $y'' + Py' + Qy = R$ when it is reduces to normal form (removal of 1st derivative)

Consider $y'' + Py' + Qy = R$

let $y = uv$ be the complete sol^n

$y' = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$

$y'' = u \cdot \frac{d^2v}{dx^2} + 2\frac{du}{dx} \cdot \frac{dv}{dx} + v \cdot \frac{d^2u}{dx^2}$

Date.
Page No.

Sustituting y' & y'' in eq^n 1

$$\frac{d^2v}{dx^2} + \left(\frac{2}{u}\frac{du}{dx} + P\right)\frac{dv}{dx} + v\left[\frac{1}{u}\cdot\frac{d^2u}{dx^2} + \frac{P}{u}\frac{du}{dx} + Q\right] = \frac{R}{u}$$

Choose u such that $\frac{2}{u}\cdot\frac{du}{dx} + P = 0$ — (3)

$$u = e^{-\int \frac{P}{2}dx} \quad \text{— (4)}$$

from (3)

$$\frac{du}{dx} = \frac{-Pu}{2}$$

$$\Rightarrow \frac{d^2u}{dx^2} = \frac{-1}{2}\left[P\frac{du}{dx} + \frac{dP}{dx}u\right]$$

$$= \frac{-1}{2}\left[P\left(\frac{-Pu}{2}\right) + u\cdot\frac{dP}{dx}\right]$$

$$= \frac{P^2u}{4} - \frac{u}{2}\cdot\frac{dP}{dx}$$

The coefficient of v is $\frac{1}{u}\frac{d^2u}{dx^2} + \frac{P}{u}\frac{du}{dx} + Q$ from eq^n (3)

$$= \frac{1}{u}\left[\frac{P^2u}{4} - \frac{u}{2}\cdot\frac{dP}{dx}\right] + \frac{P}{u}\left[-\frac{Pu}{2}\right] + Q$$

$$= Q - \frac{1}{2}P^2 - \frac{1}{2}\frac{dP}{dx} + \frac{P^2}{4}$$

$$= Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4} = I \text{ (say)}$$

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Date.
Page No.

RHS $= \frac{R}{u} = S$ (say)

The eq^n (2) becomes
$$\frac{d^2v}{dx^2} + Iv = S \quad \text{—(5)}$$
This is known as normal form of eq^n (1)
upon solving we get v, the complete sol^n of 1 is $y = u \cdot v$

Steps ~~for~~ sol^n using normal form

(i) write the eq^n in the standard form
(ii) Obtain P, Q, R
(iii) let $y = u \cdot v$ ~~...~~ be a complete sol^n of a given eq^n
(iv) Calculate $u = e^{-\frac{1}{2}\int P dx}$
*(V) Check whether I is constant or constant by x^2
$$I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}$$

if not the method is not applicable

(vi) If I is constant we get a linear diff. eq^n of 2nd order with constant coefficient, if $I = \text{constant}/x^2$

(vii) Solve the normal eq^n to get 'v'

(viii) The final sol^n is $y = uv$

Que $y'' - 4xy' + (4x^2 - 1)y = -3e^{x^2}\sin 2x$

$P = -4x;\ Q = 4x^2 - 1;\ R = -3e^{x^2}\sin 2x$

let $y = uv$ be the complete soln

$u = e^{-1/2\int P\,dx} = e^{-1/2\int -4x\,dx} = e^{x^2}$

$I = 4x^2 - 1 - \frac{1}{2}(-4) - \frac{16x^2}{4}$

$= \cancel{4x^2} - 1 + 2 - \cancel{4x^2}$

$I = 1$

$S = \frac{R}{u} = \frac{-3e^{x^2}\sin 2x}{e^{x^2}} = -3\sin 2x$

The normal form eqn is

$\frac{d^2v}{dx^2} + I\,v = S$

AE :- $m^2 + 1 = 0 \Rightarrow m = \pm i$

$V_c = e^{0x}(c_1\cos x + c_2\sin x)$

$V_c = c_1\cos x + c_2\sin x$

$V_p = \frac{1}{D^2+1}\,-3\sin 2x = -3\cdot\frac{1}{D^2+1}\sin 2x$

$V_p = \sin 2x$

$v = c_1\cos x + c_2\sin x + \sin 2x$

Date.
Page No.

$y = uv$
$= e^{x^2}(c_1 \cos x + c_2 \sin x + \sin 2x)$

Method 3 Changing of independent variable technique

(i) Write the eq^n in the standard form and identify P, Q & R.
let the independent variable x is changing to new variable z where z is a func^n of x

(ii) By changing the independent variable, the new eq^n is $y'' + P_1 y' + Q_1 y = R_1$ — (2)
where $P_1 = \dfrac{\frac{\partial^2 z}{\partial x^2} + P \frac{\partial z}{\partial x}}{\left(\frac{\partial z}{\partial x}\right)^2}$

$Q_1 = \dfrac{Q}{\left(\frac{\partial z}{\partial x}\right)^2}$, $R_1 = \dfrac{R}{\left(\frac{\partial z}{\partial x}\right)^2}$

We choose z such that $Q_1 = \dfrac{Q}{\left(\frac{\partial z}{\partial x}\right)^2} = a^2$, a perfect square

if this z makes the value of P_1 as constant then eq^n (2) reduces to a constant coefficient eq^n which can be solved for y in terms of z. Finally substituting $z = f(x)$, we get the actual sol^n.

Q. Solve $x^6 y'' + 3x^5 y' + a^2 y = 1/x^2$, where a is a constant by changing the independent.

Sol^n

$$y'' + \frac{3}{x} y' + \frac{a^2}{x^6} y = \frac{1}{x^8}$$

$$P = \frac{3}{x}, \quad Q = \frac{a^2}{x^6}, \quad R = \frac{1}{x^8}$$

Let $z = f(x)$ is a new independent variable, then the new eq^n will be

$$y'' + P_1 y' + Q_1 y = R_1$$

where $P_1 = \dfrac{\frac{\partial^2 z}{\partial x^2} + P \cdot \frac{\partial z}{\partial x}}{\left(\frac{\partial z}{\partial x}\right)^2}$, $Q_1 = \dfrac{Q}{\left(\frac{\partial z}{\partial x}\right)^2}$, $R_1 = \dfrac{R}{\left(\frac{\partial z}{\partial x}\right)^2}$

Consider $Q_1 = \dfrac{Q}{\left(\frac{\partial z}{\partial x}\right)^2}$

Choose $\left(\frac{\partial z}{\partial x}\right)^2$ as $1/x^6$

$\therefore$ $Q_1 = a^2$, a perfect square

$$\left(\frac{\partial z}{\partial x}\right)^2 = \frac{1}{x^6}$$

$$\therefore \boxed{\frac{\partial z}{\partial x} = \frac{1}{x^3}}$$

$\Rightarrow z = -x^{-2}/2 = \dfrac{-1}{2x^2} = -\dfrac{x^{-2}}{2}$

Now, $\dfrac{\partial z}{\partial x} = \cancel{+2x/z = -2x(-)} \dfrac{-2 \times x^{-3}}{\cancel{z}} = x^{-3}$

$\dfrac{\partial^2 z}{\partial x^2} = \cancel{+2} -3x^{-4}$

$P_1 = 0$, $Q_1 = a^2$, $R_1 = \dfrac{1/x^8}{1/x^6} = x^{-2}$

Substituting P_1, Q_1 & R_1

$$y'' + a^2 y = -2z$$

which is constant coefficient eq^n

Date. ______________
Page No. ______________

$z = m^2 + \cancel{\;}^2 = 0$

$\cancel{\;}(m^2 + q^2) = 0$

$y_c = (c_1 \cos az + c_2 \sin az)$

Put $z = \frac{-1}{2x^2}$

$\therefore y_c = c_1 \cos\left(\frac{-1}{2} x^{-2} a\right) + c_2 \sin\left(\frac{-1}{2} x^{-2} a\right)$

$y_p = \frac{1}{D^2 + a^2} (-2z)$

$= -2 \cdot \frac{1}{a^2\left[1 + \frac{D^2}{a^2}\right]} \cdot z$

$= \frac{-2}{a^2}\left[1 + \frac{D^2}{a^2}\right]^{-1} z$

$= \frac{-2}{a^2}\left[1 - \frac{D^2}{a^2} - - - \right] z$

$= \frac{-2}{a^2} z$

$= \frac{-\cancel{2}}{a^2}\left(\frac{-1}{\cancel{2}x^2}\right)$

$= \frac{+1}{a^2 x^2}$

$y = y_c + y_p$

$= c_1 \cos\left(\frac{-1}{2} x^{-2} a\right) + c_2 \sin\left(\frac{-1}{2} x^{-2} a\right) + \frac{1}{a^2 x^2}$

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** Method of variation of parameters to solve $y'' + Py' + Qy$

Note This method is generally used when finding particular integral becomes a difficult task.

Consider $y'' + Py' + Qy = R$ — (1)

let the complementary func^n of eq^n (1) be
$$y = c_1 u + c_2 v \quad \text{— (2)}$$

where c_1 and c_2 are constants
and u and v are func^n of x

Since, u and v are parts of complimentary func^n they satisfy $y'' + Py' + Qy = 0$

$$\Rightarrow u'' + Pu' + Qu = 0 \quad \text{— (3)}$$
$$v'' + Pv' + Qv = 0 \quad \text{— (4)}$$

let the complete sol^n of eq^n (1) is given by
$$y = Au + Bv \quad \text{— (5)}$$

where u and v are parts of the complimentary func^n and A \& B are func^n's of x to be chosen such that eq^n (5) satisfies eq^n (1)

$$y_1 = Au_1 + Bv_1 + A_1 u + B_1 v \quad \text{— (6)}$$
$$= Au_1 + Bv_1 + (A_1 u + B_1 v)$$

let us choose A and B such that
$$A_1 u + B_1 v = 0 \quad \text{— (7)}$$

Page No.

Now, eq^n (6) becomes

$$y_1 = Au_1 + Bv_1 \quad —(8)$$

$$y_2 = Au_2 + A_1u_1 + Bv_2 + B_1v_1 \quad —(9)$$

Substituting y, y_1 & y_2 in eq^n (1), we get

$$A_1u_1 + Au_2 + B_1v_1 + Bv_2 + P(Au_1 + Bv_1) + Q(Au + Bv) = R$$

$$\Rightarrow A_1u_1 + B_1v_1 + A(u_2 + Pu_1 + Qu) + B(v_2 + Pv_1 + Qv) = R$$

$$\Rightarrow A_1u_1 + B_1v_1 = R \quad —(10)$$

upon solving eq^n (7) & (10), we get

$$A_1 = \frac{-Rv}{uv_1 - u_1v} = \phi(x) \text{ (say)}$$

$$B_1 = \frac{Ru}{uv_1 - u_1v} = \psi(x) \text{ (say)}$$

Wronskian

(u, v)

$$W(u,v) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}$$

$$= uv' - u'v$$

$$= uv_1 - u_1v$$

$$\therefore A_1 = \frac{-Rv}{W} \quad \text{and} \quad B_1 = \frac{Ru}{W}$$

upon, integrating A_1 and B_1, we get

$$A = \int -\frac{Rv}{W}\,dx + C_1$$

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$$B = \int \frac{Ru}{w}\, dx + c_2$$

~~Upon, integrating A and B, we get~~ finally, $y = Au + Bv$ is the complete sol^n

$$\cancel{A = \int -\frac{Rv}{w}\, dx + c_1}$$

Steps:

(i) find out the parts of C.F

(ii) let them be u and v

(iii) consider $y = Au + Bv$ as the complete sol^n

(iv) A and B are obtained by the following formula

$$A = \int -\frac{Rv}{w}\, dx + c_1$$

$$B = \int \frac{Ru}{w}\, dx + c_2$$

where, w = wronskian of u and v

$= uv_1 - u_1 v$

(v) final sol^n is $y = Au + Bv$

Ques Solve $y'' + a^2 y = \sec ax$

AE is $m^2 + a^2 = 0$

$m = \pm ai$

$\therefore y_c = c_1 \cos ax + c_2 \sin ax$

let $u = \cos ax$ and $v = \sin ax$

let $y = Au + Bv$ be the complete sol^n

Date.
Page No.

$$A = -\int \frac{Rv}{w}\,dx + c_1$$

$$B = \int \frac{Ru}{w}\,dx + c_2$$

$$w(u,v) = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix}$$

$$= a[\cos^2 ax + \sin^2 ax]$$

$$= a$$

$$\therefore\ A = -\int \frac{(\sec ax)\,(\sin ax)}{a}\,dx + c_1$$

$$= \frac{-1}{a}\int \tan ax\,dx + c_1$$

$$= \frac{1}{a^2}\int \frac{1}{\cos ax}\,d(\cos ax)$$

$$= \frac{1}{a^2}\ln(\cos ax) + c_1$$

$$B = \int \frac{\sec ax\,\cos ax}{a}\,dx + c_2$$

$$= \frac{1}{a}\int dx + c_2$$

$$= x/a + c_2$$

$$y = Au + Bv = \left[\frac{1}{a^2}\ln(\cos ax) + c_1\right]\cos ax + \left[x/a + c_2\right]\sin ax$$

Date.
Page No.

Tutorial-

(1) Solve the following diff eqⁿ by variation of parameters

$(x^2D^2 - 2xD + 2)y = x + x^2$

Solⁿ Put $x = e^z$

$xD = D_1 , \quad D_1 = \frac{d}{dz}$

$x^2D^2 = D_1(D_1 - 1)$

$(D_1(D_1-1) - 2D_1 + 2)y = e^z + e^{2z}$

$(D_1^2 - 3D_1 + 2)y = e^z + e^{2z}$

AE :- $m^2 - 3m + 2 = 0$

$(m-2)(m-1) = 0$

$m = 1 , m = 2$

$y_c = c_1 e^z + c_2 e^{2z}$

$y_1(z) = e^z , \quad y_2(z) = e^{2z}$

$y(z) = A(z) y_1(z) + B(z) y_2(z)$ is the complete solⁿ

$A(z) = -\int \frac{y_2(z) \cdot R(z)}{W(z)} dz + d_1$

$B(z) = \int \frac{y_1(z) R(z)}{W(z)} dz + d_2$

where $W(z) = \begin{vmatrix} y_1(z) & y_2(z) \\ y_1'(z) & y_2'(z) \end{vmatrix}$

$= \begin{vmatrix} e^z & e^{2z} \\ e^z & 2e^{2z} \end{vmatrix}$

$= 2e^{2z} \cdot e^z - e^z \cdot e^{2z}$

$= e^{3z}$

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$$A(x) = -\int \frac{e^{2x} \cdot (e^{e^x} + e^{2e^x})}{e^{3x}} dx + d_1$$

$$\left[X = -\int \frac{e^{e^x} \cdot e^{2x} - e^{2x} \cdot e^{2e^x}}{e^{3x}} dx + d_1 \right]$$

$$A(x) = -\int \frac{(e^{3x} + e^{4x})}{e^{3x}} dx + d_1$$

$$= \left[-\left\{ \int \frac{e^{3x}}{e^{3x}} dx + \int \frac{e^{4x}}{e^{3x}} dx \right\} + d_1 \right]$$

$$= \left[-\left\{ \int dx + \int e^x dx \right\} + d_1 \right]$$

$$= -[x + e^x] + d_1$$

$$B(x) = \int \frac{e^x \cdot (e^x + e^{2x})}{e^{3x}} dx + d_2$$

$$= \left[\int \frac{e^{2x}}{e^{3x}} dx + \int \frac{e^{3x}}{e^{3x}} dx \right] + d_2$$

$$= \int e^{-x} dx + \int dx + d_2$$

$$= (-e^{-x} + x) + d_2$$

$$y(x) = \left[-(x + e^x) + d_1 \right] (e^x) + \left[[-e^{-x} + x] + d_2 \right] (e^{2x})$$

$$= -e^x(x + e^x) + e^x d_1 + e^{2x}(-e^{-x} + x) + e^{2x} d_2$$

$$= -x(\log x + x) + x d_1 + x^2 \left(-\frac{1}{x} + \log x\right) + x^2 d_2$$

$$= \left[-x\log x - x^2 - x + x^2 \log x \right] + (x d_1 + x^2 d_2)$$