Assignment - Parameter Estimation

(Q1) Let $(X1, X2, \ldots,)$ be a random sample of size n taken from a Nomal Population with parameters: mean $= \theta 1$ and variance $= \theta 2$. find the maximum likelihood Estimates of these two parameters.

To find the maximum Likelihood Estimates (MLEs) of the parameters $\theta 1$ and $\theta 2$, representing the mean and variance respectively of a normal distribution, we start by setting up the likelihood function based on a random sample $X_1, X_2, \ldots, X_n$ from the normal distribution $N(\theta 1, \theta 2)$.

- Step 1: Setup like Likelihood Function

The probability density function of a normally distributed random variable X is given by:

$$f(x|\theta 1, \theta 2) = \frac{1}{\sqrt{2\pi \theta_2}} e^{-\frac{(x-\theta 1)^2}{2\theta_2}}$$

Given the sample $X1, X2, \ldots, Xn$, the joint density function (likelihood function) for the independent observation is:

$$L(\theta_1, \theta_2 | x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{(x_i-\theta_1)^2}{2\theta_2}}$$

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This can be simplified to:

$$L(\theta_1, \theta_2 | x_1, x_2, \ldots, x_n) = (2\pi\theta_2)^{-n/2} e^{-\frac{1}{2\theta_2} \sum_{i=1}^{n} (x_i - \theta_1)^2}$$

⊙ Step 2: Log-Likelihood Function

It is easier to work with the log of the likelihood function:

$$\log L(\theta_1, \theta_2) = -\frac{n}{2} \log(2\pi\theta_2) - \frac{1}{2\theta_2} \sum_{i=1}^{n} (x_i - \theta_1)^2$$

⊙ Step 3: Derive MLE for θ_1 (Mean)

Differentiating the log likelihood function with respect to θ_1 and setting the derivative to zero gives:

$$\frac{d}{d\theta_1}\left(-\frac{1}{2\theta_2} \sum_{i=1}^{n} (x_i - \theta_1)^2 \right) = 0$$

$$\frac{1}{\theta_2} \sum_{i=1}^{n} (x_i - \theta_1) = 0$$

$$\sum_{i=1}^{n} x_i - n\theta_1 = 0$$

$$\theta_1 = \frac{1}{n} \sum_{i=1}^{n} x_i$$

So, the MLE of θ_1 is $\hat{\theta}_1 = \overline{X}$, the sample mean.

- Step 4: Derive MLE for θ_2 (variance)

Differentiating the log-likelihood function with respect to θ_2 and setting the derivative to zero gives :-

$$\frac{d}{d\theta_2}\left(\frac{-n}{2}\log(\theta_2) - \frac{1}{2\theta_2}\sum_{i=1}^{n}(x_i-\theta_1)^2\right) = 0$$

$$-\frac{n}{2\theta_2} + \frac{1}{2\theta_2^2}\sum_{i=1}^{n}(x_i-\theta_1)^2 = 0$$

$$\theta_2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\theta_1)^2$$

So, the MLE of θ_2 is $\hat{\theta}_2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\overline{X})^2$, the sample variance.

- Conclusion:

The MLEs for the parameter of the normal distribution from the given sample are:

$\hat{\theta}_1 = \overline{X}$ for the mean.

$\hat{\theta}_2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{X})^2$ for the variance.

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(Q2) Let $X_1, X_2, \ldots, X_n$ be a random sample from $B(m, \theta)$ distribution, where $\theta \in \Theta = (0,1)$ is unknown and 'm' is a known positive integer. Compute value of θ using the M.L.E

To find the Maximum likelihood estimate of θ in the binomial distribution $B(m, \theta)$, we need to consider that each observation X_i has a binomial distribution with parameters m (known number of trials) and θ (probability of success in each trial). The likelihood function will be then constructed from the probability mass function of these binomially distributed random variables.

⊙ Step 1: Set up the Likelihood Function

Given that each X_i follows a binomial distribution the probability mass function for each X_i is given by:

$$P(X_i = x_i) = \binom{m}{x_i} \theta^{x_i} (1-\theta)^{m-x_i}$$

For a random sample $X_1, X_2, \ldots, X_n$ from $B(m, \theta)$ the joint probability mass function (likelihood function) of observing this particular sample is:

$$L(\theta | x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} \binom{m}{x_i} \theta^{x_i} (1-\theta)^{m-x_i}$$

Given that $\binom{m}{x_i}$ does not depends on θ, it can be treated as constant with respect to θ. Therefore, the likelihood function simplifies to:

$$\mathcal{L}(\theta | x_1, x_2, \ldots, x_n) = \left(\prod_{i=1}^{n} \binom{m}{x_i} \right) \theta^{\sum_{i=1}^{n}} (1-\theta)^{\sum_{i=1}^{n} (m - x_i)}$$

- Step 2: Log - Likelihood Function

To facilitate differentiation and solve for θ, we use the "log-likelihood function":

$$\log L(\theta) = \log \left(\prod_{i=1}^{n} \binom{m}{x_i} \right) + \left(\sum_{i=1}^{n} x_i \right) \log \theta + \left(\sum_{i=1}^{n} (m - x_i) \right) \log (1-\theta)$$

Simplifying, we have:

$$\log L(\theta) = C + \left(\sum_{i=1}^{n} x_i \right) \log \theta + \left(nm - \sum_{i=1}^{n} x_i \right) \log (1-\theta)$$

where C is constant (in terms of θ).

- Step 3: "Differentiate and Find the critical Point"

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Differentiate $\log L(\theta)$ with respect to θ and set this derivative equal to zero:-

$$\frac{d}{d\theta}\log L(\theta) = \frac{\sum_{i=1}^{n} x_i}{\theta} - \frac{nm - \sum_{i=1}^{n} x_i}{1-\theta} = 0$$

$$\sum_{i=1}^{n} x_i(1-\theta) = \theta\left(nm - \sum_{i=1}^{n} x_i\right)$$

$$\sum_{i=1}^{n} x_i - \theta\sum_{i=1}^{n} x_i = \theta nm - \theta\sum_{i=1}^{n} x_i$$

$$\sum_{i=1}^{n} x_i = \theta nm$$

$$\theta = \frac{\sum_{i=1}^{n} x_i}{nm}$$

- **Conclusion:**

The MLE of θ for a Random sample from $B(m,\theta)$ is:

$$\hat{\theta} = \frac{\sum_{i=1}^{n} x_i}{nm}$$

This is the sample mean divided by m, i.e., the average proportion of successes across all trials in the sample.