INTEGRALS PART 3

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Integration techniques continued:

③ Partial fractions: Breaks down rational expressions into simpler fractions that can be integrated easier; Particularly helpful when denominator can be factored into linear/quadratic terms that are irreducible.

$$\int \frac{x}{x^2+bx+7}$$

Steps:

① Factorize the denominator
-> Factor into simpler factors (ideally linear)
-> if there are irreducible quadratic factors -> require diff approach

② Set up the equation
-> write the original fraction as a sum of fractions with unknown coefficients; form of each fraction depends on factors of the denominator

$(x^2-9) \to (x+3)(x-3)(ax+b)$

-> ① Linear factors $(ax+b)$: $\frac{A}{ax+b} + \frac{B}{ax+b} \dots$

$\frac{A}{x+3} + \frac{B}{x-3} + \frac{C}{ax+b} \dots$

② repeated linear factors $(ax+b)^n$: $\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} \dots + \frac{A_n}{(ax+b)^n}$

$(x-2)^2 \to \frac{A}{x-2} + \frac{A_2}{(x-2)^2}$

③ Irreducible quadratic factors (ax^2+bx+c): $\frac{Ax+B}{ax^2+bx+c}$

$(x-3)^3 \to \dots + \frac{A_3}{(x-3)^3}$

③ Determine the coefficients: Multiply both sides of the equation by entire denominator to eliminate fractions; equate coefficients or use specific values of x to solve for unknowns.

$$\cancel{(x+3)(x-2)}\left(\frac{x}{\cancel{(x-2)(x+3)}}\right) = \left(\frac{A}{\cancel{x-2}} + \frac{B}{\cancel{x+3}}\right)\cancel{(x-2)(x+3)}$$

$$x = A(x+3) + B(x-2)$$

④ Integrate each term: once partial fractions are determined, integrate each term seperately

Example: $\int \frac{3x+5}{x^2-x-6}\,dx$

① Factor the denominator:

$[x^2-x-6 \to (x-3)(x+2)]$ Linear factors $ax+b$

$a \cdot c = -6$, $b = -1$ } $-3, 2$

② Set up equation:

Linear: $\frac{A}{ax+b} + \frac{B}{ax+b}$

$$\frac{3x+5}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$$

③ Multiply both sides by denominator:

$$\cancel{(x-3)(x+2)} \times \frac{3x+5}{\cancel{(x-3)(x+2)}} = \left(\frac{A}{\cancel{x-3}} + \frac{B}{\cancel{x+2}}\right) \times (x-3)(x+2)$$

$$3x+5 = A(x+2) + B(x-3)$$

$x+2=0$
$x=-2$
$-2+2=0$

for $x=-2$:

$3(-2)+5 = A\cancel{(-2+2)} + B(-2-3)$
$-6+5 = 0 + B(-5)$
$-1 = 0 - 5B$
$-1 = -5B$
$B = \frac{1}{5}$

$x-3=0$
$x=3$
$3-3=0$

for $x=3$:

$3(3)+5 = A(3+2) + B\cancel{(3-3)} \to 0$
$9+5 = A(5)$
$14 = 5A$
$\frac{14}{5} = A$

$A = \frac{14}{5}$ $B = \frac{1}{5}$

④ Integrate each part:

$$\int \frac{3x+5}{x^2-x-6}\,dx = \int \frac{\frac{14}{5}}{x-3}\,dx + \int \frac{\frac{1}{5}}{x+2}\,dx$$

$$\frac{14}{5}\int \frac{1}{x-3}\,dx + \frac{1}{5}\int \frac{1}{x+2}\,dx$$

$\int \frac{1}{x+2}\,dx$ ($ax+b$) $\to \frac{1}{1}\ln|x+2| + C$

$\ln|x+2| + C_2$

$\int \frac{1}{x-3}\,dx$ ($ax+b$)

$\frac{1}{a}\ln|ax+b| + C$

$\frac{1}{1}\ln|x-3| + C$

$\ln|x-3| + C_1$

$$\frac{14}{5}\ln|x-3| + \frac{1}{5}\ln|x+2| + C \to C_1 + C_2$$

$$\frac{1}{5}\left(14\ln|x-3| + 1\ln|x+2|\right) + C$$

$c=1$ $a=x+2$

$$\frac{1}{5}\left(\ln|x-3|^{14} + \ln|x+2|^1\right) + C$$

$$\frac{1}{5}\ln\left(|x-3|^{14}|x+2|\right) + C$$

$c \cdot \ln(ab)$

Sum of logs: $\ln(a) + \ln(b) = \ln(ab)$

Coefficient as exp: $c \cdot \ln(a) = \ln(a^c)$

Common Integrals

Polynomials

$\int dx = x + c$ $\int k\,dx = kx + c$ $\int x^n dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$

$\int \frac{1}{x}\,dx = \ln|x| + c$ $\int x^{-1}\,dx = \ln|x| + c$ $\int x^{-n}dx = \frac{1}{-n+1}x^{-n+1} + c,\ n \neq 1$

$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$ $\int x^{\frac{p}{q}}dx = \frac{1}{\frac{p}{q}+1}x^{\frac{p}{q}+1} + c = \frac{q}{p+q}x^{\frac{p+q}{q}} + c$

Trig Functions

$\int \cos(u)\,du = \sin(u) + c$ $\int \sin(u)\,du = -\cos(u) + c$ $\int \sec^2 u\,du = \tan(u) + c$

$\int \sec(u)\tan(u)\,du = \sec(u) + c$ $\int \csc(u)\cot(u)\,du = -\csc(u) + c$ $\int \csc^2 u\,du = -\cot(u) + c$

$\int \tan(u)\,du = -\ln|\cos(u)| + c = \ln|\sec(u)| + c$ $\int \cot(u)\,du = \ln|\sin(u)| + c = -\ln|\csc(u)| + c$

$\int \sec(u)\,du = \ln|\sec(u) + \tan(u)| + c$ $\int \sec^3(u)\,du = \frac{1}{2}\left(\sec(u)\tan(u) + \ln|\sec(u) + \tan(u)|\right) + c$

$\int \csc(u)\,du = \ln|\csc(u) - \cot(u)| + c$ $\int \csc^3(u)\,du = \frac{1}{2}\left(-\csc(u)\cot(u) + \ln|\csc(u) - \cot(u)|\right) + c$

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Exponential & Logarithm Functions

$\int \mathbf{e}^u\,du = \mathbf{e}^u + c$ $\int a^u\,du = \frac{a^u}{\ln(a)} + c$ $\int \ln(u)\,du = u\ln(u) - u + c$

$\int \mathbf{e}^{au}\sin(bu)\,du = \frac{\mathbf{e}^{au}}{a^2+b^2}\left(a\sin(bu) - b\cos(bu)\right) + c$ $\int u\mathbf{e}^u du = (u-1)\mathbf{e}^u + c$

$\int \mathbf{e}^{au}\cos(bu)\,du = \frac{\mathbf{e}^{au}}{a^2+b^2}\left(a\cos(bu) + b\sin(bu)\right) + c$ $\int \frac{1}{u\ln(u)}\,du = \ln|\ln(u)| + c$

Inverse Trig Functions

$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\left(\frac{u}{a}\right) + c$ $\int \sin^{-1}(u)\,du = u\sin^{-1}(u) + \sqrt{1-u^2} + c$

$\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + c$ $\int \tan^{-1}(u)\,du = u\tan^{-1}(u) - \frac{1}{2}\ln\left(1+u^2\right) + c$

$\int \frac{1}{u\sqrt{u^2-a^2}}\,du = \frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right) + c$ $\int \cos^{-1}(u)\,du = u\cos^{-1}(u) - \sqrt{1-u^2} + c$

Hyperbolic Functions

$\int \sinh(u)du = \cosh(u) + c$ $\int \operatorname{sech}(u)\tanh(u)du = -\operatorname{sech}(u) + c$ $\int \operatorname{sech}^2(u)du = \tanh(u) + c$

$\int \cosh(u)du = \sinh(u) + c$ $\int \operatorname{csch}(u)\coth(u)du = -\operatorname{csch}(u) + c$ $\int \operatorname{csch}^2(u)du = -\coth(u) + c$

$\int \tanh(u)du = \ln\left(\cosh(u)\right) + c$ $\int \operatorname{sech}(u)du = \tan^{-1}|\sinh(u)| + c$

Miscellaneous

$\int \frac{1}{a^2-u^2}\,du = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + c$ $\int \sqrt{a^2+u^2}\,du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left|u + \sqrt{a^2+u^2}\right| + c$

$\int \frac{1}{u^2-a^2}\,du = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + c$ $\int \sqrt{u^2-a^2}\,du = \frac{u}{2}\sqrt{u^2-a^2} - \frac{a^2}{2}\ln\left|u + \sqrt{u^2-a^2}\right| + c$

$\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{u}{a}\right) + c$

$\int \sqrt{2au-u^2}\,du = \frac{u-a}{2}\sqrt{2au-u^2} + \frac{a^2}{2}\cos^{-1}\left(\frac{a-u}{a}\right) + c$

Standard Integration Techniques