INTEGRALS PART 2

Sunday, October 27, 2024 4:22 PM

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Definite integrals:

$ax + c$

$\int_a^b f(x)\,dx = y$

→ Find antiderivative of $F(x)$ ← solve indefinite integral

→ Use $F(b) - F(a)$ ← y

Example: $\int_0^1 x^2\,dx$ b / a

$\int_0^1 x^2\,dx = \frac{x^{2+1}}{2+1} = \frac{x^3}{3} + C$ → antiderivative $\int x^n dx = \frac{x^{n+1}}{n+1}$

$\to \left[\frac{x^3}{3}\right]_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3} - 0 = \frac{1}{3}$ → y

$F(b)$ $F(a)$ $F(b) - F(a)$

$F(x) = \frac{x^3}{3}$

$F(b) = \frac{b^3}{3}$

$b = 1 = \frac{1^3}{3}$

Example: $\int_0^2 (3x^2 - 2x + 1)\,dx$

1. Find antiderivative:

$\int_0^2 \underbrace{(3x^2 - 2x + 1)}_{F(x)}\,dx$ →

$x^3 - x^2 + x + C$

$[x^3 - x^2 + x]_0^2$

$F(b)\;[2^3 - 2^2 + 2] - [0^3 - 0^2 + 0]\;F(a)$

$[8 - 4 + 2] - [0]$

$[6] - [0] = 6$

Antiderivative: $\int 3x^2 - \int 2x + \int 1$ 2 3 $F(a) = \frac{x^3}{3}$ $F(a) = \frac{c^3}{3}$

$\to \left[\frac{3x^{2+1}}{2+1}\right] - \left[\frac{2x^{1+1}}{1+1}\right] + [x] + C$ $2x^1$

$\left[\frac{\cancel{3}x^3}{\cancel{3}}\right] - \left[\frac{\cancel{2}x^2}{\cancel{2}}\right] + [x] + C$ → $C = C_1 + C_2 + C_3$

$x^3 - x^2 + x + C$

$F(b) - F(a)$

$F(2) - F(0)$

$\int 1 = 1x + C \to x + C$

$\int a\,dx = ax + C$

$ax + C$

$x^3 + C$

$\int 3x^2 \to x^3 + C_1$

$\int 2x \to x^2 + C_2$

$\int 1 \to x + C_3$

$C_1 + C_2 + C_3 = C$

Common Integrals

Polynomials

$\int dx = x + c$ $\quad \int k\,dx = kx + c$ $\quad \int x^n dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$

$\int \frac{1}{x}\,dx = \ln|x| + c$ $\quad \int x^{-1}\,dx = \ln|x| + c$ $\quad \int x^{-n}dx = \frac{1}{-n+1}x^{-n+1} + c,\ n \neq 1$

$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$ $\quad \int x^{\frac{p}{q}}dx = \frac{1}{\frac{p}{q}+1}x^{\frac{p}{q}+1} + c = \frac{q}{p+q}x^{\frac{p+q}{q}} + c$

Trig Functions

$\int \cos(u)\,du = \sin(u) + c$ $\quad \int \sin(u)\,du = -\cos(u) + c$ $\quad \int \sec^2 u\,du = \tan(u) + c$

$\int \sec(u)\tan(u)\,du = \sec(u) + c$ $\quad \int \csc(u)\cot(u)\,du = -\csc(u) + c$ $\quad \int \csc^2 u\,du = -\cot(u) + c$

$\int \tan(u)\,du = -\ln|\cos(u)| + c = \ln|\sec(u)| + c$ $\quad \int \cot(u)\,du = \ln|\sin(u)| + c = -\ln|\csc(u)| + c$

$\int \sec(u)\,du = \ln|\sec(u) + \tan(u)| + c$ $\quad \int \sec^3(u)\,du = \frac{1}{2}\left(\sec(u)\tan(u) + \ln|\sec(u) + \tan(u)|\right) + c$

$\int \csc(u)\,du = \ln|\csc(u) - \cot(u)| + c$ $\quad \int \csc^3(u)\,du = \frac{1}{2}\left(-\csc(u)\cot(u) + \ln|\csc(u) - \cot(u)|\right) + c$

Exponential & Logarithm Functions

$\int \mathbf{e}^u\,du = \mathbf{e}^u + c$ $\quad \int a^u\,du = \frac{a^u}{\ln(a)} + c$ $\quad \int \ln(u)\,du = u\ln(u) - u + c$

$\int \mathbf{e}^{au}\sin(bu)\,du = \frac{\mathbf{e}^{au}}{a^2+b^2}\left(a\sin(bu) - b\cos(bu)\right) + c$ $\quad \int u\mathbf{e}^u du = (u-1)\mathbf{e}^u + c$

$\int \mathbf{e}^{au}\cos(bu)\,du = \frac{\mathbf{e}^{au}}{a^2+b^2}\left(a\cos(bu) + b\sin(bu)\right) + c$ $\quad \int \frac{1}{u\ln(u)}\,du = \ln|\ln(u)| + c$

Inverse Trig Functions

$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\left(\frac{u}{a}\right) + c$ $\quad \int \sin^{-1}(u)\,du = u\sin^{-1}(u) + \sqrt{1-u^2} + c$

$\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + c$ $\quad \int \tan^{-1}(u)\,du = u\tan^{-1}(u) - \frac{1}{2}\ln\left(1+u^2\right) + c$

$\int \frac{1}{u\sqrt{u^2-a^2}}\,du = \frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right) + c$ $\quad \int \cos^{-1}(u)\,du = u\cos^{-1}(u) - \sqrt{1-u^2} + c$

Hyperbolic Functions

$\int \sinh(u)du = \cosh(u) + c$ $\quad \int \operatorname{sech}(u)\tanh(u)du = -\operatorname{sech}(u) + c$ $\quad \int \operatorname{sech}^2(u)du = \tanh(u) + c$

$\int \cosh(u)du = \sinh(u) + c$ $\quad \int \operatorname{csch}(u)\coth(u)du = -\operatorname{csch}(u) + c$ $\quad \int \operatorname{csch}^2(u)du = -\coth(u) + c$

$\int \tanh(u)du = \ln\left(\cosh(u)\right) + c$ $\quad \int \operatorname{sech}(u)du = \tan^{-1}|\sinh(u)| + c$

Miscellaneous

$\int \frac{1}{a^2-u^2}\,du = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + c$ $\quad \int \sqrt{a^2+u^2}\,du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left|u + \sqrt{a^2+u^2}\right| + c$

$\int \frac{1}{u^2-a^2}\,du = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + c$ $\quad \int \sqrt{u^2-a^2}\,du = \frac{u}{2}\sqrt{u^2-a^2} - \frac{a^2}{2}\ln\left|u + \sqrt{u^2-a^2}\right| + c$

$\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{u}{a}\right) + c$

$\int \sqrt{2au-u^2}\,du = \frac{u-a}{2}\sqrt{2au-u^2} + \frac{a^2}{2}\cos^{-1}\left(\frac{a-u}{a}\right) + c$

Standard Integration Techniques

Techniques of integration:

1. Substitution method: Used when a function contains a function and its derivative; simplifies integral by transforming it into simpler form

Steps:

1. Identify a part of the integrand to set as u; usually a function that has its derivative present
2. Differentiate u with respect to x; find du
3. Substitute u and du into the integral; replace all x-terms
4. Integrate with respect to u
5. Sub back into original variable, if necessary

$\int 3x\cos(x^3+1)\,dx$

dx

$x^3 + 1 = 3x$

$\int x\sin(2x+1)$ u

2

$\int 3x\cos(\underbrace{x^3+1}_{u})$

$\int 3x\cos(u) \to$ $du = 3x\,dx$ u

$\frac{du}{dx} = 3x$

Example: $\int x e^{x^2} dx$ → $x^2 \to 2x$, $x \to 1$

e^{x^2}

$e^{x^2} = u$

e^u

Let $u = x^2$

Derivative of $u\,(du) = 2x$

$du = u\,dx$

$du = 2x\,dx$

$dx = \frac{du}{2x}$

Substitute:

$\int x e^{x^2}\,dx = \int x e^u \frac{du}{2x} = \frac{1}{2}\int \cancel{x} e^u \frac{du}{\cancel{x}}$

$\frac{du}{2x} \to \frac{1}{2} \cdot \frac{du}{x}$

$= \frac{1}{2}\int e^u du \to \int e^u\,du = e^u + C$

from table

$= \frac{1}{2}e^u + C \to u = x^2$

$= \frac{1}{2}e^{x^2} + C$

2. Integration by parts: Derives from product rule of differentiation; useful for integrals involving products of functions.

Steps:

1. Identify parts of integrand to set as u and dv; u is chosen to simplify upon differentiation and dv should be easily integrated
2. Differentiate u to find du; integrate dv to find v
3. Apply integration by parts formula:

u, dv

$$\int u\,dv = uv - \int v\,du$$

Example: $\int x\sin(x)\,dx$

Choose u and dv

$\int \sin(x)$

Let $u = x$

$x \to$

Let $dv = \sin(x)dx$

$\frac{du}{dx} = 1$ $\quad du = dx$

$dv = \sin(x)dx$

$dv = -\cos(x)$ → v $\quad v = -\cos(x)$

Applying the formula:

$\int x\sin(x)\,dx$

$= \underbrace{x}_{u}\underbrace{(-\cos(x))}_{v} - \int -\cos(x)\,dx\ (1)$ $\quad -\cos x$

$-1\cos(x)$

$= -x\cos(x) + \sin(x) + C$ $\to C_1 + C_2$

C_1 C_2 $\quad - \cdot - = +$