MA-102
B. Tech. II Sem (2021-2022)
Tutorial sheet-02

Cholesky Decomposition:

Find Cholesky decomposition for following matrices.

1. $\begin{bmatrix} 1 & -1 & 2 \\ -1 & 5 & -4 \\ 2 & -4 & 6 \end{bmatrix}$

2. $\begin{bmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix}$

Solution

1.
$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 5 & -4 \\ 2 & -4 & 6 \end{bmatrix}$$
$$IA = A$$

$R_2 \to R_2 + R_1$
$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & -2 \\ 2 & -4 & 6 \end{bmatrix}$$
$$IE_1A = A_1$$

$R_3 \to R_3 - 2R_1$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} E_1A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix}$$
$$IE_2E_1A = A_2$$

$R_3 \to R_3 + \frac{1}{2}R_2$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix} E_2E_1A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_3E_2E_1A = U$$
$$A = (E_3E_2E_1)^{-1}U = E_1^{-1}E_2^{-1}E_3^{-1}U = LU$$

$$\begin{aligned}
L &= E_1^{-1}E_2^{-1}E_3^{-1} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -\frac{1}{2} & 1 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
A &= LU = LIU = LDD^{-1}U \\
&= L(\sqrt{D}\sqrt{D})D^{-1}U \\
&= (L\sqrt{D})(\sqrt{D}D^{-1}U) = CC^T
\end{aligned}$$

Let $\qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad$ and $\qquad \sqrt{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{aligned}
L\sqrt{D} &= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & -1 & 1 \end{bmatrix} = C
\end{aligned}$$

$$\begin{aligned}
\sqrt{D}D^{-1}U &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & -2 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} = C^T
\end{aligned}$$

2.

$$A = \begin{bmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix}$$

$$IA = A$$

$R_2 \to R_2 - \frac{3}{5}R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{5} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 25 & 15 & -5 \\ 0 & 9 & 3 \\ -5 & 0 & 11 \end{bmatrix}$$

$$IE_1A = A_1$$

$R_3 \to R_3 + \frac{1}{5}R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{5} & 0 & 1 \end{bmatrix} E_1A = \begin{bmatrix} 25 & 15 & -5 \\ 0 & 9 & 3 \\ 0 & 3 & 10 \end{bmatrix}$$

$$IE_2E_1A = A_2$$

$R_3 \to R_3 - \frac{1}{3}R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{3} & 1 \end{bmatrix} E_2E_1A = \begin{bmatrix} 25 & 15 & -5 \\ 0 & 9 & 3 \\ 0 & 0 & 9 \end{bmatrix}$$

$$E_3E_2E_1A = U$$

$$A = (E_3E_2E_1)^{-1}U = E_1^{-1}E_2^{-1}E_3^{-1}U = LU$$

$$\begin{aligned} L &= E_1^{-1}E_2^{-1}E_3^{-1} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{5} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{5} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{3} & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{5} & 1 & 0 \\ -\frac{1}{5} & \frac{1}{3} & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} A &= LU = LIU = LDD^{-1}U \\ &= L(\sqrt{D}\sqrt{D})D^{-1}U \\ &= (L\sqrt{D})(\sqrt{D}D^{-1}U) = CC^T \end{aligned}$$

$$\text{Let} \qquad D = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} \qquad \text{and} \qquad \sqrt{D} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$
\begin{aligned}
L\sqrt{D} &= \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{5} & 1 & 0 \\ -\frac{1}{5} & \frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 5 & 0 & 0 \\ 3 & 3 & 0 \\ -1 & 1 & 3 \end{bmatrix} = C
\end{aligned}
$$

$$
\begin{aligned}
\sqrt{D}D^{-1}U &= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{25} & 0 & 0 \\ 0 & \frac{1}{9} & 0 \\ 0 & 0 & \frac{1}{9} \end{bmatrix} \begin{bmatrix} 25 & 15 & -5 \\ 0 & 9 & 3 \\ 0 & 0 & 9 \end{bmatrix} \\
&= \begin{bmatrix} 5 & 3 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} = C^T
\end{aligned}
$$

Vector Space:

1. **(i)** Suppose we define addition on $\mathbb{R}^2$ by the rule $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, 0)$. Show that additive identity does not exist in $\mathbb{R}^2$ w.r.t. above rule.

   Solution: The set V contains an additive identity element denoted by 0, such that for any vector v in V, we have $0 + v = v$ and $v + 0 = v$.
   Let $v = (a_1, a_2) \in \mathbb{R}^2$.
   Suppose $(e_1, e_2) \in \mathbb{R}^2$ be the additive identity element such that $(e_1, e_2) + (a_1, a_2) = (a_1, a_2)$.
   As per operation defined above, $(e_1, e_2) + (a_1, a_2) = (e_1 + a_1, 0)$.
   $\Rightarrow (e_1 + a_1, 0) = (a_1, a_2)$
   i.e. $e_1 + a_1 = a_1$ and $a_2 = 0 \Rightarrow e_1 = 0$ and $a_2 = 0$.
   $\Rightarrow (0, e_2)$ is the additive identity element of the element of the form $(a_1, 0)$.
   i.e. additive identity element does not exist for the element of the form (a_1, a_2) where $a_2 \neq 0$.
   So, we can conclude that additive identity element does not exist for all the elements in $\mathbb{R}^2$.
   Hence, additive identity does not exist in $\mathbb{R}^2$ w.r.t. above rule.

   (ii) Suppose we define addition on $\mathbb{R}^3$ by the rule $(a_1, a_2, a_3)+(b_1, b_2, b_3) = (a_1b_1, a_2b_2, a_3b_3)$. Show that we have an additive identity for this operation in $\mathbb{R}^3$ but inverse may not exist for some elements.
   Solution: Let $v = (a_1, a_2, a_3) \in \mathbb{R}^3$.

Suppose $(e_1, e_2, e_3) \in \mathbb{R}^3$ be the additive identity element such that $(e_1, e_2, e_3) + (a_1, a_2, a_3) = (a_1, a_2, a_3)$.
As per operation defined above, $(e_1, e_2, e_3)+(a_1, a_2, a_3) = (e_1a_1, e_2a_2, e_3a_3)$.
$\Rightarrow (e_1a_1, e_2a_2, e_3a_3) = (a_1, a_2, a_3)$
i.e. $e_1a_1 = a_1$, $e_2a_2 = a_2$ and $e_3a_3 = a_3$.
i.e. $e_1 = 1, e_2 = 1$ and $e_3 = 1$.
So, $(e_1, e_2, e_3) = (1, 1, 1)$ is the additive identity element for this operation in $\mathbb{R}^3$.
Now, let (b_1, b_2, b_3) be the inverse of (a_1, a_2, a_3).
$\Rightarrow (a_1, a_2, a_3) + (b_1, b_2, b_3) = (1, 1, 1)$.
$\Rightarrow (a_1b_1, a_2b_2, a_3b_3) = (1, 1, 1)$.
$\Rightarrow b_1 = \dfrac{1}{a_1}, b_2 = \dfrac{1}{a_2}, b_3 = \dfrac{1}{a_3}$.
So, from here we can conclude that, inverse exists only when
$a_1 \neq 0,\ a_2 \neq 0,$ and $a_3 \neq 0$.
i.e. if at least one of the a_i is zero, then inverse does not exists.
Hence, inverse does not exist for some elements.

2. Let $\mathbb{R}^+$ be the set of all positive real numbers. Define operations of addition $\bigoplus$ and the scalar multiplication $\bigotimes$ as follows: $u \bigoplus v = uv$ for all $u, v \in \mathbb{R}^+$ and $\alpha \bigotimes u = u^\alpha$ for all $u \in \mathbb{R}^+$ and $\alpha \in \mathbb{R}$(here $\mathbb{R}^+$ is the field of scalars). Prove that $(\mathbb{R}^+, \bigoplus, \bigotimes)$ is a real vector space.
Solution:
For any $u, v \in \mathbb{R}^+$,
(i) $u \bigoplus v = uv$. Since $uv \in \mathbb{R}^+$, $u \bigoplus v \in \mathbb{R}^+$.
(ii) $u \bigoplus v = uv = vu = v \bigoplus u$ $\qquad [u, v \in \mathbb{R}^+,\ uv = vu]$
Let u, v and $w \in \mathbb{R}^+$,
(iii) $u \bigoplus (v \bigoplus w) = u \bigoplus vw = uvw = uv \bigoplus w = (u \bigoplus v) \bigoplus w$
(iv) For any $u \in \mathbb{R}^+$, $1 \bigoplus u = 1 \cdot u = u$. Similarly, $u \bigoplus 1 = u$. Therefore 1 is additive identity on $\mathbb{R}^+$.
(v) Since $u \in \mathbb{R}^+$,we have $\frac{1}{u} \in \mathbb{R}^+$. Now, $u \bigoplus \frac{1}{u} = u \cdot \frac{1}{u} = 1$ and similarly $\frac{1}{u} \bigoplus u = 1$.
(vi) For any $\alpha \in \mathbb{R}$ and $u \in \mathbb{R}^+$, $\alpha \bigotimes u = u^\alpha$ and $u^\alpha \in \mathbb{R}^+$. Therefore, $\alpha \bigotimes u \in \mathbb{R}^+$.
(vii) For any $u \in \mathbb{R}^+$, $1 \bigotimes u = u^1 = u$.
(viii) For any $\alpha_1, \alpha_2 \in \mathbb{R}$ and $u \in \mathbb{R}^+$,$(\alpha_1\alpha_2) \bigotimes u = u^{\alpha_1\alpha_2} = \{u^{\alpha_2}\}^{\alpha_1} = \alpha_1 \bigotimes (u^{\alpha_2}) = \alpha_1 \bigotimes (\alpha_2 \bigotimes u)$
(ix) For any $\alpha \in \mathbb{R}$ and $u, v \in \mathbb{R}^+$, $\alpha \bigotimes (u \bigoplus v) = \alpha \bigotimes (uv) = \{uv\}^\alpha = u^\alpha v^\alpha = (u^\alpha) \bigoplus (v^\alpha) = (\alpha \bigotimes u) \bigoplus (\alpha \bigotimes v)$.
(x) For any $\alpha_1, \alpha_2 \in \mathbb{R}$ and $u \in \mathbb{R}^+$, $(\alpha_1 + \alpha_2) \bigotimes u = u^{(\alpha_1+\alpha_2)} =$

$u^{\alpha_1}u^{\alpha_2} = (u^{\alpha_1})\bigoplus(v^{\alpha_2}) = (\alpha_1 \bigotimes u)\bigoplus(\alpha_2 \bigotimes v)$.

3. Let $V = \mathbb{R}^2$. Define operations of addition $\bigoplus$ and the saclar multiplication $\bigotimes$ as follows:
   $(a_1, a_2)\bigoplus(b_1, b_2) = (a_1+b_2, a_2+b_1)$ and $\alpha\bigotimes(a_1, a_2) = (\alpha a_1, \alpha a_2), \alpha \in \mathbb{R}$(here $\mathbb{R}$ is the field of scalars).
   Does $(V, \bigoplus, \bigotimes)$ form a real vector space? Give reasons for your assertion.
   Solution: Let us calculate $(a_1, a_2)\bigoplus(b_1, b_2) = (a_1 + b_2, a_2 + b_1)$ and $(b_1, b_2)\bigoplus(a_1, a_2) = (b_1 + a_2, b_2 + a_1)$.
   Since $(a_1 + b_2) \neq (b_1 + a_2)$ and $(a_2 + b_1) \neq (b_2 + a_1)$.
   So $(a_1, a_2)\bigoplus(b_1, b_2) \neq (b_1, b_2)\bigoplus(a_1, a_2)$ which means $(V, \bigoplus, \bigotimes)$ does not form a vector space.

4. Done in class.

5. Done in class.

6. Let $V = C[0, 1]$ be the set of all real valued function defined and continuous on the closed interval $[0, 1]$. Prove that V is a real vector space with respect to pointwise addition and multiplication. Further, determine that which of the following subsets of V are subspaces.

   Solution:
   For $f(x), g(x) \in C[0, 1]$ and $c \in \mathbb{R}$, we define the functions $f + g$ and cf by
   $$(f + g)(x) = f(x) + g(x)$$
   $$(cf)(x) = c(f(x))$$
   We have to show that $V = C[0, 1]$ with the given operations is a vector space. We check the vector space axioms for this V.
   We let f, g, h be arbitrary elements of V. We know from calculus that the sum of any two continuous functions is continuous and that any constant times a continuous function is also continuous. Therefore, the closure of addition and that of scalar multiplication hold.
   Now for all x such that $0 \leq x \leq 1$, we have from the definition and the commutative law of real number addition that
   $$(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x).$$

Since this holds for all x, we conclude that $f+g=g+f$, which is the commutative law of vector addition. Similarly,

$$\begin{aligned}((f+g)+h)(x) &= (f+g)(x)+h(x) = (f(x)+g(x))+h(x) \\ &= f(x)+(g(x)+h(x)) = (f+(g+h))(x)\end{aligned}$$

Since this holds for all x, we conclude that $(f+g)+h = f+(g+h)$, which is the associative law for addition of vectors.

Next, if 0 denotes the constant function with value 0 , then for any $f \in V$ we have that for all $0 \le x \le 1$,

$$(f+0)(x) = f(x)+0 = f(x).$$

Since this is true for all x we have that $f+0=f$, which establishes the additive identity law.
Also, we define $(-f)(x) = -(f(x))$ so that for all $0 \le x \le 1$,

$$(f+(-f))(x) = f(x)-f(x) = 0,$$

from which we see that $f+(-f)=0$. The additive inverse law follows. For the distributive laws, note that for real numbers c, d and continuous functions $f, g \in V$, we have that for all $0 \le x \le 1$,

$$c(f+g)(x) = c(f(x)+g(x)) = cf(x)+cg(x) = (cf+cg)(x),$$

which proves the first distributive law. For the second distributive law, note that for all $0 \le x \le 1$,

$$((c+d)g)(x) = (c+d)g(x) = cg(x)+dg(x) = (cg+dg)(x),$$

and the second distributive law follows. For the scalar associative law, observe that for all $0 \le x \le 1$,

$$((cd)f)(x) = (cd)f(x) = c(df(x)) = (c(df))(x),$$

so that $(cd)f = c(df)$, as required. Finally, we see that

$$(1.f)(x) = 1.f(x) = f(x),$$

from which we have the monoidal law $1.f = f$. Thus, $C[0,1]$ with the prescribed operations is a vector space.

We know that $W(F) \subset V(\mathbb{F})$ is said to be a subspace of V, if $\alpha, \beta \in \mathbb{F}$ and $u, v \in W$, then $\alpha u + \beta v \in W$. Also, zero vector is in W.

(a) $W_1 = \{f \in V : f(1/2) = 0\}$.
Solution:
Clearly, zero vector is in W_1.
Let $f, g \in W_1 \Rightarrow f, g \in V$
As V is vector space $\Rightarrow \alpha f + \beta g \in V$.
If W_1 is subspace then, only we have to show $\alpha f + \beta g \in W_1$ i.e. $(\alpha f + \beta g)\left(\frac{1}{2}\right) = 0$.
As $f, g \in W_1 \Rightarrow f\left(\frac{1}{2}\right) = 0, g\left(\frac{1}{2}\right) = 0$.
$(\alpha f + \beta g)\left(\frac{1}{2}\right) = \alpha f\left(\frac{1}{2}\right) + \beta g\left(\frac{1}{2}\right) = 0 + 0 = 0$.
$\Rightarrow \alpha f + \beta g \in W_1$ i.e. W_1 is subspace of V.

(b) $W_2 = \{f \in V : f(3/4) = 1\}$.
Solution:
Since zero element is not in this subset W_2. So it is not a subspace of V.

(c) $W_3 = \{f \in V : f(0) = f(1)\}$.
Solution:
Clearly, zero vector is in W_3.
Let $f, g \in W_3 \Rightarrow f, g \in V$
As V is vector space $\Rightarrow \alpha f + \beta g \in V$.
If W_3 is subspace then, only we have to show $\alpha f + \beta g \in W_3$ i.e. $(\alpha f + \beta g)(0) = (\alpha f + \beta g)(1)$.
As$f, g \in W_3 \Rightarrow f(0) = f(1) \quad \text{and} \quad g(0) = g(1)$.
$(\alpha f + \beta g)(0) = \alpha f(0) + \beta g(0) = \alpha f(1) + \beta g(1) = (\alpha f + \beta g)(1)$.
$\Rightarrow W_3$ is subspace of V.

(d) $W_4 = \{f \in V : f(x) = 0$ only at a finite number of points $\}$.
Solution:
Since zero polynomial is not in W_4. So, W_4 is not subspace of V.

7. Determine whether each of the following set S forms a subspace of $\mathbb{R}^4$, if addition and multiplication rules are defined in the usual way.

Solution: We know that $W(F) \subset V(\mathbb{F})$ is said to be a subspace of V, if $\alpha, \beta \in \mathbb{F}$ and $u, v \in W$, then $\alpha u + \beta v \in W$. Also, zero vector is in W.

(a) $S = \{(a, b, c, d) \mid a = c + d\}$
Clearly, zero vector is in W.
Let $u = (a_1, b_1, c_1, d_1)\,,\ \ v = (a_2, b_2, c_2, d_2) \in S$ & $\alpha, \beta \in \mathbb{F}$
i.e $a_1 = c_1 + d_1,\ \ a_2 = c_2 + d_2$
$\Rightarrow \quad \alpha\,(a_1, b_1, c_1, d_1) + \beta\,(a_2, b_2, c_2, d_2) = (\alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2, \alpha c_1 + \beta c_2, \alpha d_1 + \beta d_2)$

We know from above,
$a_1 = c_1 + d_1, \quad a_2 = c_2 + d_2$
$\Rightarrow \alpha a_1 = \alpha c_1 + \alpha d_1 \quad \text{and} \quad \beta a_2 = \beta c_2 + \beta d_2$
On adding both terms, we get
$\Rightarrow \alpha a_1 + \beta a_2 = \alpha c_1 + \alpha d_1 + \beta c_2 + \beta d_2$
$\Rightarrow \quad \alpha u + \beta v \in S$
Hence, S is a subspace of $\mathbb{R}^4$.

(b) $S = \{(a, b, c, d) \mid b = c - d \text{ and } a = c + d\}$
Clearly, zero vector is in W.
Let $u = (a_1, b_1, c_1, d_1)\,, \ \ v = (a_2, b_2, c_2, d_2) \in S \ \&\ \alpha, \beta \in \mathbb{F}$
i.e $b_1 = c_1 - d_1,\ \ a_1 = c_1 + d_1$ and $b_2 = c_2 - d_2,\ \ a_2 = c_2 + d_2$
$\Rightarrow \quad \alpha\,(a_1, b_1, c_1, d_1) + \beta\,(a_2, b_2, c_2, d2) = (\alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2, \alpha c_1 + \beta c_2, \alpha d_1 + \beta d_2)$
We know from above,
$b_1 = c_1 - d_1,\ \ a_1 = c_1 + d_1 \ \text{ and } \ b_2 = c_2 - d_2,\ \ a_2 = c_2 + d_2$
$\Rightarrow \alpha b_1 = \alpha c_1 - \alpha d_1,\ \ \alpha a_1 = \alpha c_1 + \alpha d_1$
and $\quad \beta b_2 = \beta c_2 - \beta d_2,\ \ \beta a_2 = \beta c_2 + \beta d_2$
On adding, we get
$\Rightarrow \alpha b_1 + \beta b_2 = \alpha c_1 - \alpha d_1 + \beta c_2 - \beta d_2$
and $\quad \alpha a_1 + \beta a_2 = \alpha c_1 + \alpha d_1 + \beta c_2 + \beta d_2$
$\Rightarrow \quad \alpha u + \beta v \in S$
Hence, S is a subspace of $\mathbb{R}^4$.

(c) $S = \{(a, b, c, d) \mid c = d\}$
Clearly, zero vector is in W.
Let $u = (a_1, b_1, c_1, d_1)\,, \ \ v = (a_2, b_2, c_2, d_2) \in S \ \&\ \alpha, \beta \in \mathbb{F}$
i.e $c_1 = d_1,\ \ c_2 = d_2$
$\Rightarrow \quad \alpha\,(a_1, b_1, c_1, d_1) + \beta\,(a_2, b_2, c_2, d2) = (\alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2, \alpha c_1 + \beta c_2, \alpha d_1 + \beta d_2)$
We know from above,
$c_1 = d_1,\ \ c_2 = d_2$
$\Rightarrow \alpha c_1 = \alpha d_1,\ \ \beta c_2 = \beta d_2$
On adding both terms, we get
$\Rightarrow \alpha c_1 + \beta c_2 = \alpha d_1 + \beta d_2$
$\Rightarrow \quad \alpha u + \beta v \in S$
Hence, S is a subspace of $\mathbb{R}^4$.

(d) $S = \{(-a + c, a - b, b + c, a + b) \mid a, b, c \in \mathbb{R}\}$
Clearly, zero vector is in W.
Let $u = (-a_1 + c_1, a_1 - b_1, b_1 + c_1, a_1 + b_1)\,,$

$v = (-a_2 + c_2, a_2 - b_2, b_2 + c_2, a_2 + b_2) \in S$ & $\alpha, \beta \in \mathbb{F}$

$$\begin{aligned}\Rightarrow \alpha u + \beta v &= \alpha\,(-a_1 + c_1, a_1 - b_1, b_1 + c_1, a_1 + b_1) + \beta\,(-a_2 + c_2, a_2 - b_2, b_2 + c_2, a_2 + b_2)\\ &= (-\alpha a_1 - \beta a_2 + \alpha c_1 + \beta c_2, \alpha a_1 + \beta a_2 - \alpha b_1 - \beta b_2,\\ &\quad \alpha b_1 + \beta b_2 + \alpha c_1 + \beta c_2, \alpha a_1 + \beta a_2 + \alpha b_1 + \beta b_2)\end{aligned}$$

and
$-\alpha a_1 - \beta a_2 + \alpha c_1 + \beta c_2,\ \ \alpha a_1 + \beta a_2 - \alpha b_1 - \beta b_2,\ \ \alpha b_1 + \beta b_2 + \alpha c_1 + \beta c_2 \in \mathbb{R}$
$\Rightarrow \quad \alpha u + \beta v \in S$
Hence, S is a subspace of $\mathbb{R}^4$.

(e) $S = \{(a, b, c, d) \mid a = 1\}$
We know that, if S is a subspace of $\mathbb{R}^4$, then '0'(zero element) must belong to that set S. But, here we can see that '0' $\notin S$.
Because, if $(0, 0, 0, 0) \in S$, then $0 = 1$ which is absurd condition.
Hence, S is not a subspace of $\mathbb{R}^4$.

(f) $S = \{(a, b, c, d) \mid a \leq b\}$
Clearly, zero vector is in W.
Let $u = (1, 2, 4, 5)\,,\ \ v = (2, 4, 5, 6) \in S$
Now, let $\alpha = 1, \beta = -1 \in \mathbb{R}$

$$\begin{aligned}\Rightarrow \alpha\,(1, 2, 4, 5) + \beta\,(2, 4, 5, 6) &= 1.\,(1, 2, 4, 5) + (-1).\,(2, 4, 5, 6)\\ &= (1 - 2, 2 - 4, 4 - 5, 5 - 6)\\ &= (-1, -2, -1, -1)\end{aligned}$$

Now, $-1 \leq -2$ is not possible.
$\Rightarrow \quad \alpha u + \beta v \notin S$
Hence, S is not a subspace of $\mathbb{R}^4$.

(g) $S = \{(a, b, c, d) \mid a = b = c = d\}$
Clearly, zero vector is in W.
Let $u = (a_1, b_1, c_1, d_1)\,,\ \ v = (a_2, b_2, c_2, d_2) \in S$ & $\alpha, \beta \in \mathbb{F}$
i.e $a_1 = b_1 = c_1 = d_1,\ \ a_2 = b_2 = c_2 = d_2$
$\Rightarrow \quad \alpha\,(a_1, b_1, c_1, d_1) + \beta\,(a_2, b_2, c_2, d2) = (\alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2, \alpha c_1 + \beta c_2, \alpha d_1 + \beta d_2)$
We know from above,
$a_1 = b_1 = c_1 = d_1,\ \ a_2 = b_2 = c_2 = d_2$
$\Rightarrow \alpha a_1 = \alpha b_1 = \alpha c_1 = \alpha d_1,\ \ \beta a_2 = \beta b_2 = \beta c_2 = \beta d_2$

On adding both terms, we get
$\Rightarrow \alpha a_1 + \beta a_2 = \alpha b_1 + \beta b_2 = \alpha c_1 + \beta c_2 = \alpha d_1 + \beta d_2$
$\Rightarrow \quad \alpha u + \beta v \in S$
Hence, S is a subspace of $\mathbb{R}^4$.

(h) $S = \{(a, b, c, d) \mid \text{a is an integer}\}$
Clearly, zero vector is in W.
Let $u = (a_1, b_1, c_1, d_1)\,,\ \ v = (a_2, b_2, c_2, d_2) \in S$ & $\alpha, \beta \in \mathbb{F}$
i.e a_1 is an integer, a_2 is an integer
Let $u = (1, 0, 0, 0)\,,\ \ v = (1, 0, 0, 0)\,,\ \ \alpha = \sqrt{2},\ \ \beta = 1$
Then, $\quad \alpha u + \beta v = \sqrt{2}\,(1, 0, 0, 0) + 1.\,(1, 0, 0, 0) = \left(\sqrt{2} + 1, 0, 0, 0\right)$
But, $\sqrt{2} + 1$ is not an integer.
$\Rightarrow \quad \alpha u + \beta v \notin S$
Hence, S is not a subspace of $\mathbb{R}^4$.

(i) $S = \{(a, b, c, d) \mid a^2 - b^2 = 0\}$
Clearly, zero vector is in W.
Let $u = (1, -1, 0, 0)\,,\ \ v = (2, 2, 0, 0) \in S$
Now, let $\alpha = 1, \beta = 1 \in \mathbb{R}$

$$\begin{aligned}\Rightarrow \alpha\,(1, -1, 0, 0) + \beta\,(2, 2, 0, 0) &= 1.\,(1, -1, 0, 0) + 1.\,(2, 2, 0, 0)\\ &= (1 + 2, -1 + 2, 0, 0)\\ &= (3, 1, 0, 0)\end{aligned}$$

Now, $3^2 - 1^2 = 8 \neq 0$.
$\Rightarrow \quad \alpha u + \beta v \notin S$
Hence, S is not a subspace of $\mathbb{R}^4$.

8. Which of the following subsets of P are subspace. Where, P is the real vector space of all polynomials w.r.t usual vector addition and multiplication :
i) $\{p \in P : \deg p \leq 4\}$
ii) $\{p \in P : \deg p = 4\}$
iii) $\{p \in P : \deg p \geq 4\}$
iv) $\{p \in P : p(1) = 0\}$
v)$\{p \in P : p(2) = 1\}$
vi)$\{p \in P : p'(1) = 0\}$

Solution 8(i):
Let $S = \{p \in P : \deg p \leq 4\}$
Let $f(x) = \sum_{i=0}^{4} a_i x^i$ ave $g(x) = \sum_{i=0}^{4} b_i x^i$ be two polynomials in S.
Clearly, S is non-empty since zero polynomial belongs to S.
To check whether S is a subspace of P or not we have to check whether S is closed under addition and scalar multiplication or not.
So $f(x) + g(x) = \sum_{i=0}^{4} (a_i + b_i) x^i$
and $\alpha \cdot f(x) = \sum_{i=0}^{4} \alpha a_i x^i$.
Since addition of two polynomial and multiplication by a scalar does not increase the degree of that polynomial.
Therefore, both $f(x) + g(x),\ \alpha f(x) \in S$.
Hence, S is a subspace of P.

Solution 8(ii): Assume the set $S_1 = \{p \in P : \deg p = 4\}$
Let $f(x) = x^4, g(x) = -x^4 \in S_1$, but $f(x) + g(x) = 0 \notin S_1$.
Therefore, S_1 is not closed under addition.
Hence, S_1 is not a subspace of P.

Solution 8(iii): Consider the set $S_2 = \{p \in P : \deg p \geq 4\}$.
By the similar above example of $f(x)$ and $g(x)$, we can see that $f(x) + g(x) \notin S_2$. So, S_2 is not closed under addition.
Hence, S_2 is not a subspace of P.

Solution 8(iv): Let $S_3 = \{p \in P : p(1) = 0\}$
Clearly, S_3 is nonempty since $0 \in S_3$.
Let $f(x), g(x) \in S_3$, then $f(1) = 0 = g(1)$.
Now, $(f + g)(1) = f(1) + g(1) = 0 + 0 = 0$
and $(\alpha f)(1) = \alpha \cdot f(1) = \alpha \cdot 0 = 0$.
which implies both $f + g, \alpha f \in S_3$.
Therefore, the set S_3 is closed under addition and multiplication.
Hence, S_3 is a subspace of P.

Solution 8(v): Let $S_4 = \{p \in P : p(2) = 1\}$
Since zero polynomial is not in S_4. Hence S_4 is not a subspace of P.

Solution 8(vi): Let $S_5 = \{p \in P : p'(1) = 0\}$
Clearly, S_5 is non empty since zero polynomial belongs to S_5.
Let $f(x), g(x) \in S_5$, then $f'(1) = 0 = g'(1)$.

Now, $(f+g)'(1) = f'(1) + g'(1) = 0 + 0 = 0$
and $(\alpha f)'(1) = \alpha f'(1) = \alpha \cdot 0 = 0 \Rightarrow f+g, \alpha f \in S_5$.
$\Rightarrow S_5$ is closed under addition and scalar multiplication.
Hence, S_5 is a subspace of P.

9. Which of the following subsets of $\mathbb{R}^{2\times 2}$ are subspaces. Note that, $\mathbb{R}^{m\times n}$ is the vector space over real field of all matrices of order $m \times n$ under usual definitions of addition and scalar multiplication of matrices.

   (i) All diagonal matrices.
   (ii) All upper triangular matrices.
   (iii) All symmetric matrices.
   (iv) All invertible matrices.
   (v) All matrices which commute with a given matrix T.
   (vi) All matrices with zero determinant.

   Solution:

   (i) Let, $\mathcal{D}$ be the set of all diagonal matrices of order 2 and $D, E \in \mathcal{D}$ be any two elements.
   $D=\begin{bmatrix} d_{11} & 0 \\ 0 & d_{22} \end{bmatrix}$, $E=\begin{bmatrix} e_{11} & 0 \\ 0 & e_{22}. \end{bmatrix}$, $d_{11}, d_{22}, e_{11}, e_{22} \in \mathbb{R}$

   $\therefore$ D+E=$\begin{bmatrix} d_{11}+e_{11} & 0 \\ 0 & d_{22}+e_{22} \end{bmatrix}$
   Clearly, $D+E \in \mathcal{D}$.
   Also let, $\alpha \in \mathbb{R}$ and $A \in \mathcal{D}$ be any.
   Then,
   $$A = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22}. \end{bmatrix}$$
   for some $a_{11}, a_{22} \in \mathbb{R}$
   $\therefore \alpha A = \begin{bmatrix} \alpha a_{11} & 0 \\ 0 & \alpha a_{22}. \end{bmatrix} \in \mathcal{D}$
   Hence, $\mathcal{D}$ is a vector subspace of $\mathbb{R}^{2\times 2}$.

   (ii) Let, $\mathcal{U}$ be the set of all upper triangular matrices of order 2 and $M, N \in \mathcal{U}$ be any two elements.
   $M=\begin{bmatrix} m_{11} & m_{12} \\ 0 & m_{22} \end{bmatrix}$, $N=\begin{bmatrix} n_{11} & n_{12} \\ 0 & n_{22}. \end{bmatrix}$, $m_{11}, m_{12}, m_{22}, n_{11}, n_{12}, n_{22} \in \mathbb{R}$

$\therefore$ M+N=$\begin{bmatrix} m_{11}+n_{11} & m_{12}+n_{12} \\ 0 & m_{22}+n_{22} \end{bmatrix}$
Clearly $M+N \in \mathcal{U}$.
Also let, $\alpha \in \mathbb{R}$ and $A \in \mathcal{U}$ be any.
Then,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22}. \end{bmatrix}$$

for some $a_{11}, a_{12}, a_{22} \in \mathbb{R}$
$\therefore \alpha A = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} \\ 0 & \alpha a_{22}. \end{bmatrix} \in \mathcal{U}$
Hence, $\mathcal{U}$ is a vector subspace of $\mathbb{R}^{2\times 2}$.

(iii) Let, $\mathcal{S}$ be the set of all symmetric matrices of order 2 and $A, B \in \mathcal{S}$ be any two elements.
A=$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, B=$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}. \end{bmatrix}$, $a_{11}, a_{12}, a_{21}, a_{22}, b_{11}, b_{12}, b_{21}, b_{22} \in \mathbb{R}$
and $a_{12} = a_{21}, b_{12} = b_{21}$

$\therefore$ A+B=$\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}$

Clearly $A+B \in \mathcal{S}$ as $a_{12}+b_{12} = a_{21}+b_{21}$
Also let, $\alpha \in \mathbb{R}$ and $G \in \mathcal{S}$ be any.
Then,

$$G = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}. \end{bmatrix}$$

for some $g_{11}, g_{12}, g_{21}, g_{22} \in \mathbb{R}$ and $g_{12} = g_{21}$

$\therefore \alpha G = \begin{bmatrix} \alpha g_{11} & \alpha g_{12} \\ \alpha g_{21} & \alpha g_{22}. \end{bmatrix} \in \mathcal{S}$ as $\alpha g_{12} = \alpha g_{21}$
Hence, $\mathcal{S}$ is a vector subspace of $\mathbb{R}^{2\times 2}$.

(iv) Let $\mathcal{I}$ be the set of all invertible matrices of order 2.
Then $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & -1. \end{bmatrix}$ are two members of $\mathcal{I}$. Now,

$$A+B = \begin{bmatrix} 2 & 0 \\ 0 & 0. \end{bmatrix}$$

Clearly $A+B \notin \mathcal{I}$
Hence, $\mathcal{I}$ is not a subspace of $\mathbb{R}^{2\times 2}$.

(v) It is given that T is a fixed matrix of order 2. Let, $\mathcal{C}$ be the set of all order 2 matrices which commutes with T. Since the identity matrix of order 2 commutes with T. So, $\mathcal{C}$ is a non empty subset of $\mathbb{R}^{2\times 2}$. Now let, P, Q be any two elements of $\mathcal{C}$. Hence,

$$TP = PT \tag{1}$$
$$TQ = QT \tag{2}$$

Therefore using (1) and (2) we get,

$$\begin{aligned} T(P+Q) &= TP + TQ \\ &= PT + QT \\ &= (P+Q)T. \end{aligned}$$

Hence, $P+Q \in \mathcal{C}$
Similarly, let $A \in \mathcal{C}$ and $\alpha \in \mathbb{R}$.

$$\begin{aligned} T(\alpha A) &= \alpha(TA) \\ &= \alpha(AT) \\ &= (\alpha A)T. \end{aligned}$$

Hence, $\alpha A \in \mathcal{C}$.
Therefore, $\mathcal{C}$ is a vector subspace of $\mathbb{R}^{2\times 2}$.

(vi) Let $\mathcal{N}$ be the set of all order 2 matrices with determinant zero. Then, $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 1. \end{bmatrix}$ are two members of $\mathcal{N}$.

$$\det(A+B) = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \neq 0$$

Hence $A+B \notin \mathcal{N}$, showing that $\mathcal{N}$ is not a vector subspace of $\mathbb{R}^{2\times 2}$.

10. Done in class.

11. Let W_1 and W_2 be subspaces of a vector Space V such that $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$ Show that for each vector u in V there are unique vectors u_1 in W_1 ard u_2 in W_2 such that $u = u_1 + u_2$.

Solutions: It is given that $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$. Let $u \in V$. Then, $u \in V = W_1 + W_2 \Rightarrow u = u_1 + u_2$ for some $u_1 \in W_1, \quad u_2 \in W_2$.

We have to prove the uniqueness. For uniqueness, let $u = w_1 + w_2$ for $w_1 \in W_1$ and $w_2 \in W_2$.

Then $\quad u_1 + u_2 = u = w_1 + w_2.$

$$\Rightarrow \quad u_1 - w_1 = w_2 - u_2 \in W_2.$$

Also, $(u_1 - w_1) \in W_1$. So $(u_1 - w_1) \in W_1 \cap W_2$.
Thus, $(u_1 - w_1) \in \{0\}$. So, $u_1 = w_1$. Similarly we can show, $u_2 = w_2$.
This proves the uniqueness of u_1, u_2. This completes the proof.

12. Let $S = \{(1,2,3),(1,1,-1),(3,5,5)\}$. Determine which of the following are in $L(S)$.

   (a) $(0,0,0)$
   (b) $(1,1,0)$
   (c) $(4,5,0)$
   (d) $(1,-3,8)$

   Solution:- Here we have given set $S = \{(1,2,3),(1,1,-1),(3,5,5)\}$.

   (a) If it is in $L(S)$ then, it can be written as
   $a(1,2,3) + b(1,1,-1) + c(3,5,5) = (0,0,0)$
   Now to find the value of a, b, c we can write it in augmented matrix form as:-

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & 1 & 5 & 0 \\ 3 & -1 & 5 & 0 \end{array}\right]$$

   $R_2 \to R_2 - 2R_1$
   $R_3 \to R_3 - 3R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -4 & -4 & 0 \end{array}\right]$$

$R_2 \to (-1)R_2$
$R_3 \to R_3/(-4)$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So it can be written as :-
$a + b + 3c = 0, b + c = 0$
For simplicity let us consider $c = 1$ then $b = -1, a = -2$.
Therefore, $(0, 0, 0)$ is in $L(S)$.

(b) If it is in $L(S)$ then, it can be written as
$a(1, 2, 3) + b(1, 1, -1) + c(3, 5, 5) = (1, 1, 0)$
Now to find the value of a, b, c we can write it in augmented matrix form as:-

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 2 & 1 & 5 & 1 \\ 3 & -1 & 5 & 0 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & -1 & -1 & -1 \\ 0 & -4 & -4 & -3 \end{array}\right]$$

$R_2 \to (-1)R_2$
$R_3 \to (-1)R_3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 4 & 4 & 3 \end{array}\right]$$

$R_3 \to R_3 - 4R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

So it can be written as :-
$a + b + 3c = 1, b + c = 1,$
and $\quad 0.a + 0.b + 0.c = -1$
which is impossible.
Therefore, $(1, 1, 0)$ is not in $L(S)$.

(c) If it is in $L(S)$ then, it can be written as
$a(1, 2, 3) + b(1, 1, -1) + c(3, 5, 5) = (4, 5, 0)$
Now to find the value of a, b, c we can write it in augmented matrix form as:-

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 2 & 1 & 5 & 5 \\ 3 & -1 & 5 & 0 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & -1 & -1 & -3 \\ 0 & -4 & -4 & -12 \end{array}\right]$$

$R_2 \to (-1)R_2$
$R_3 \to R_3/(-4)$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \end{array}\right]$$

$R_3 \rightarrow R_3 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So it can be written as :-
$a + b + 3c = 4, b + c = 3$
For simplicity let us consider $c = 1$ then $b = 2, a = -1$.
Therefore $(4, 5, 0)$ are in $L(S)$.

(d) If it is in $L(S)$ then it can be written as
$a(1, 2, 3) + b(1, 1, -1) + c(3, 5, 5) = (1, -3, 8)$
Now to find the value of a, b, c we can write it in augmented matrix form as:-

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 2 & 1 & 5 & -3 \\ 3 & -1 & 5 & 8 \end{array}\right]$$

$R_2 \rightarrow R_2 - 2R_1$
$R_3 \rightarrow R_3 - 3R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & -1 & -1 & -5 \\ 0 & -4 & -4 & 5 \end{array}\right]$$

$R_2 \rightarrow (-1)R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 5 \\ 0 & -4 & -4 & 5 \end{array}\right]$$

$R_3 \rightarrow R_3 + 4R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 25 \end{array}\right]$$

So it can be written as :-
$a + b + 3c = 1, b + c = 5,$
$and, 0.a + 0.b + 0.c = 25$
Which is impossible.
Therefore $(1, -3, 8)$ is not in $L(S)$.

13. In the complex vector space $\mathbb{C}^2$, determine whether are not $(1+i, 1-i) \in L\left[(1+i, 1), (1, 1-i)\right]$.
Solution- Here, given vector space is complex vector space i.e. $\mathbb{C}^2(\mathbb{C})$.
Let $a, b \in \mathbb{C}$ such that

$$a(1+i, 1) + b(1, 1-i) = (1+i, 1-i),$$

Now, to find the value of a, b we will use Gauss elimination method.

$$\left(\begin{array}{cc|c} 1+i & 1 & 1+i \\ 1 & 1-i & 1-i \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1-i & 1-i \\ 1+i & 1 & 1+i \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1-i & 1-i \\ 0 & -1 & -1+i \end{array}\right)$$

$$\begin{aligned} a + (1-i)b &= 1-i, \\ -b &= -1+i, \end{aligned}$$

using back substitution, we get,

$$a = 1+i \quad \text{and} \quad b = 1-i.$$

So, $(1+i, 1-i) \in L\left[(1+i, 1), (1, 1-i)\right]$.

14. Let M and N be subsets of the vector space $(V, +, .)$. Define $M + N = \{m + n : m \in M \, and \, n \in N\}$. Then

   (a) $M \subset N \implies L[M] \subset L[N]$

   (b) M is subspace of $V \iff L[M] = M$

   (c) $L[L[M]] = L[M]$.

   Solution- We know that if M is any subset of vector space $(V, +, .)$. Then $L[M]$ is the smallest subspace of V that contain M.
   (**proof:** It is easy to prove that L[M] is subspace of $(V, +, .)$. Now let W be any arbitrary subspace that contains M. Now we have to prove that $L[M] \subset W$. Let $x \in L[M]$, Then $\exists \; m_1, m_2, \cdots m_n \in M$ and $a_1, a_2, \cdots a_n \in F$ such that

$$x = \sum_{i=1}^{n} a_i m_i$$

$$\implies x \in W \, \textit{(by properties of subspace W)}$$

$$\implies L[M] \subset W$$

   Hence, $L[M]$ is the smallest subspace of V that contain M.)
   Solution(a)- Here $M \subset N$ So, $M \subset N \subset L[N]$. But $L[M]$ is the smallest subset that contain M. So, $L[M] \subset L[N]$.
   Solution(b)- Let M is subspace of V. Since $L[M]$ is the smallest subset that contain M. So, $L[M] = M$.Conversely assume that $L[M] = M$ and we know that $L[M]$ is subspace of V. So, M will be subspace of V.
   Solution(c)- we know that $L[M]$ is subspace of V. by (b) we have $L[L[M]] = L[M]$.