MA-102
B. Tech. II Sem (2021-2022)
Tutorial sheet-04

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1 Find the row-reduced echelon forms and hence find rank of following matrices:

(i) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

(ii) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

Solulion:-(i)Given Matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$

$$R_1 \to R_1 - R_2,$$
$$\sim \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$R_3 \longleftrightarrow R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$R_2 \to R_2 - R_1$$
$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is the Row Echelon form of A.

And

$$\text{Rank of } A = 3.$$

(ii) Given matrix

$$B = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$$

$R_2 \longrightarrow R_2 - 2R_1$

$R_3 \rightarrow R_3 - 3R_1$

$R_4 \rightarrow R_4 - 6R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$R_2 \longleftrightarrow R_4$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -11 & 5 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -11 & 5 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$R_4 \longrightarrow R_4 + R_3$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -11 & 5 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is Row Echelon form of B.

And

$$\text{Rank of } B = 3.$$

2(i) . Obtain for what values of λ and μ the equations

$$\begin{aligned} x+y+z &= 6 \\ x+2y+3z &= 10 \\ x+2y+\lambda z &= \mu \end{aligned}$$

have
(a)no solution. (b) a unique solution (c) infinitely many solutions.

Solution 2 (i) The Above system of linear equation can be written as.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

The Augmented matrix can be written as

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$R_2 \to R_2 - R_1, \quad R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right]$$

$R_3 \longrightarrow R_3 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right]$$

(a) If rank $A \neq \text{rank}(A \mid b)$ then system of linear equation has no solution.
$\Rightarrow$ For no solution.

$$\text{rank}(A) \neq \text{rank}(A \mid b)$$

If $\lambda = 3, \& \mu \neq 10$ then

$$\text{rank}(A) = 2, \quad \text{rank}(A \mid b) = 3.$$

(b)For unique solution.

$$\text{Rank}(A) = \text{Rank}(A \mid b) = 3.$$

$$\Rightarrow \quad \lambda - 3 \neq 0 \Rightarrow \lambda \neq 3$$

$\Rightarrow$ system of linear equation has unique solution when $\lambda \neq 3$.

(c). For infinitely many solution.

$$\begin{aligned} & \text{Rank}(A) = \text{Rank}(A \mid b) < 3. \\ \Rightarrow \quad & \lambda - 3 = 0 \Rightarrow \lambda = 3 \\ & \mu - 10 = 0 \Rightarrow \mu = 10. \end{aligned}$$

2(ii). Obtain for what values of λ the equation

$$\begin{aligned} & x + y - z = 1 \\ & 2x + 3y + \lambda z = 3 \\ & x + \lambda y + 3z = 2 \end{aligned}$$

have
(a)no solution (b) a unique solution (c) infinitely many solutions.
Solutions-

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & \lambda \\ 1 & \lambda & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

The Augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 2 & 3 & \lambda & 3 \\ 1 & \lambda & 3 & 2 \end{array}\right]$$

$R_2 \to R_2 - 2R_1, \quad R_3 \longrightarrow R_3 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & \lambda + 2 & 1 \\ 0 & \lambda - 1 & 4 & 1 \end{array}\right]$$

$\mathrm{R}_3 \longrightarrow R_3 - (\lambda - 1)R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & \lambda + 2 & 1 \\ 0 & 0 & (\lambda + 3)(\lambda - 2) & 2 - \lambda \end{array}\right]$$

(a) For no solution
rank $(A) \neq \text{rank}(A \mid b)$.
$\Rightarrow \quad \lambda = -3$

(b)For Unique solution
rank$(A) = $ rank$(A \mid b) = 3$.
$\Rightarrow \quad \lambda \neq -3, \lambda \neq 2$.

(c) For Infinitely many solutions

$$\begin{aligned} &\text{rank}(A) = \text{rank}(A \mid b) < 3 \\ \Rightarrow \quad &\lambda = 2. \end{aligned}$$

2(iii). In the following system of linear equations

$$\begin{aligned} ax_1 + x_2 + x_3 &= p \\ x_1 + ax_2 + x_3 &= q \\ x_1 + x_2 + ax_3 &= r \end{aligned}$$

determine all values of a, p, q, r for which the resulting linear system has
(i) unique solution (ii) infinitely many solutions (iii) no solution.
Solution-

$$\left[\begin{array}{ccc|c} a & 1 & 1 & p \\ 1 & a & 1 & q \\ 1 & 1 & a & r \end{array}\right]$$

$$R_1 \longleftrightarrow R_2$$

$$\left[\begin{array}{ccc|c} 1 & a & 1 & q \\ a & 1 & 1 & p \\ 1 & 1 & a & r \end{array}\right]$$

R$_2 \to R_2 - aR_1, \quad R_3 \to R_3 - R_1.$

$$\left[\begin{array}{ccc|c} 1 & a & 1 & q \\ 0 & 1-a^2 & 1-a & p-aq \\ 0 & 1-a & a-1 & r-q \end{array}\right]$$

$$R_2 \longleftrightarrow R_3$$

$$\left[\begin{array}{ccc|c} 1 & a & 1 & q \\ 0 & 1-a & a-1 & r-q \\ 0 & 1-a^2 & 1-a & p-aq \end{array}\right]$$

R$_3 \longrightarrow R_3 - (1+a)R_2$

$$\left[\begin{array}{ccc|c} 1 & a & 1 & q \\ 0 & 1-a & a-1 & r-q \\ 0 & 0 & -(a+2)(a-1) & (p+q)-r(1+a) \end{array}\right]$$

(1) For no solution. $\quad a=-2, a=1$ and $(p+q)-r(1+a) \neq 0$.

(2) For unique solution $\quad a \neq -2, a \neq 1$ and $(p+q)-r(1+a) \neq 0$.

(3) For infinitely many solutions $\quad a=-2, a=1$ and $(p+q)-r(1+a)=0$.

3. Does the system:

$$x+y+z=1$$
$$2x+2y+z=3$$

has a solution for $z=7$? Find the general solution of system by Gauss elimination.

Solution-

put $z=7$ in above equation we get

$$x+y=-6$$
$$2x+2y=-4.$$

We will solve by Gauss eliminiation method.

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6 \\ -4 \end{bmatrix}.$$

Augmented matrix

$$\left[\begin{array}{cc|c} 1 & 1 & -6 \\ 2 & 2 & -4 \end{array}\right]$$

$R_2 \longrightarrow R_2 - R_1$

$$\left[\begin{array}{cc|c} 1 & 1 & -6 \\ 0 & 0 & 8 \end{array}\right]$$

$\Rightarrow \quad x + y = -6$
and $0.x + 0.y = 8$
$\Rightarrow$ Given system of equation has no solution for $z = 7$.

4 Show that the rank of matrix AB is less than or equal to rank of A as well as rank of B, Further prove that rank of AB to equal to rank of A, if B is invertible.

Solution:- As rank of a matrix $M_{m\times n}$ is the dimension of the range $R(M)$ of the matrix M . And range $R(M)$ is defined as

$$R(M) = [y \in R^m | \ y = Mx \ for \ some \ x \in R^n]$$

So we have

$$\text{rank}(AB) = \dim(R(AB))$$

and

$$\text{rank}(A) = \dim(R(A))$$

In general, If a vector space W is a subset of a vector space V,

$$\dim(W) \leqslant \dim(V)$$

Thus, it is sufficient to show that the vector space $R(AB)$ is a subset of the vector space $R(A)$.

Now, consider any vector $y \in R(AB)$, then there exists a vector $x \in R^n$ such that $y = (AB)x$ by the definition of the range.

let $z = Bx \in R^n$

then, we have $y = A(Bx) = Az$

and thus the vector y is in $R(A)$.

Thus $R(AB)$ is a subset of $R(A)$ and we have-

$$\Rightarrow \operatorname{rank}(AB) = \dim(R(AB)) \leq \dim(R(A)) = \operatorname{rank}(A)$$
$$\Rightarrow \operatorname{rank}(AB) \leq \operatorname{rank}(A) \quad .... \boxed{1}$$

Similarly we can prove for

$$\operatorname{rank}(AB) \leq \operatorname{rank}(B)$$

Now the matrix B is nonsingular,i.e. it is invertible. Thus the inverse matrix B^{-1} exists. We apply above result $(equation(1))$ with the matrices AB and B^{-1} instead of A and B. Then we have,

$$\operatorname{rank}((AB)B^{-1}\big) \leq \operatorname{rank}(AB)$$

Now as

$$\operatorname{rank}(A) = \operatorname{rank}\left((AB)B^{-1}\right)$$
$$\Rightarrow \operatorname{rank}(A) \leqslant \operatorname{rank}(AB) \quad ... \boxed{2}$$

from equn (1) & (2)

$$\operatorname{rank}(AB) = \operatorname{rank}(A)$$

5 Suppose that $A_{m\times n}$ has rank k. Show that $\exists B_{m\times k}, C_{k\times n}$ such that rank $(A) = \operatorname{rank}(B) = k$ and $A = BC$.
Solution:
Given that A be an $m \times n$ matrix whose rank is k. By the definition of rank it is **dimension of column space**. Therefore, there are k linearly independent columns in A (equivalently, the dimension of the column space of A is k).

Let $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k\}$ be any basis for the column space of A, let us place them together as column vectors to form the $m \times k$ matrix.

$$B = \begin{bmatrix} \vdots & \vdots & \cdots & \vdots \\ b_1 & b_2 & \cdots & b_k \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix}$$

Therefore, every column vector of A is a linear combination of the columns of C.

To be precise, if $A = \begin{bmatrix} \vdots & \vdots & \cdots & \vdots \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix}$ is an $m \times n$ matrix with $\mathbf{a}_j$ as the j-th column, then every column of A can be written as linear combination of $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k\}$

$$\begin{aligned} a_1 &= c_{11}\mathbf{b}_1 + c_{21}\mathbf{b}_2 + \cdots + c_{k1}\mathbf{b}_k \\ a_2 &= c_{12}\mathbf{b}_1 + c_{22}\mathbf{b}_2 + \cdots + c_{k2}\mathbf{b}_k \\ &\vdots \\ a_j &= c_{1j}\mathbf{b}_1 + c_{2j}\mathbf{b}_2 + \cdots + c_{kj}\mathbf{b}_k \\ &\vdots \\ a_n &= c_{1n}\mathbf{b}_1 + c_{2n}\mathbf{b}_2 + \cdots + c_{kn}\mathbf{b}_k \end{aligned}$$

where c_{ij}'s are the scalar coefficients of $\mathbf{a}_j$ in terms of the basis $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k\}$. Now we can write

$$A = \begin{bmatrix} \vdots & \vdots & \cdots & \vdots \\ b_1 & b_2 & \cdots & b_k \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{k1} & c_{k2} & \cdots & c_{kn} \end{bmatrix}$$

That is $A = BC$.
It is clear that rank of B is also k because it has k linearly independent columns.
So now we have matrices $B_{m\times k}$ and $C_{k\times n}$ such that $A = BC$ and $rank(A) = rank(B) = k$.

6. Find row reduced Echelon form of the following matrices and hence find column space, row space and null space of given matrices.

(a) $A_1 = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \end{bmatrix}$ [(b)] $A_2 = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix}$ [(c)] $A_3 = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 2 \end{bmatrix}$

(d) $A_4 = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 1 \\ 2 & 5 & 2 \end{bmatrix}$

Solution:
(a) $A_1 = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \end{bmatrix}$

$$A_1 = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \end{bmatrix}$$

Applying $R_2 \to R_2 - 2R_1$, we get

$$\to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

RREF of A_1 is $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$
Column Space of A_1= $Span\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \}$
Row Space of $A_1 = Span\,\{(1,1,1),(0,1,0)\}$
Let $W = \{X \in \mathbb{R}^3 | \quad A_1 X = O\}$ where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.
Now,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore, we have, $x_1 + x_2 + x_3 = 0$ and $x_2 = 0$. Thus we have $X = x_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

Therefore, The null space of $A_1 = Span\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \}$

(b)

$$A_2 = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Column Space of A_2= $Span\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}2\\2\\1\end{bmatrix}\}$

Row Space of $A_2 = Span\{(1,2,2),(0,1,2),(0,0,1)\}$

Let $W = \{X \in \mathbb{R}^3 | \quad A_2X = O\}$ where $X = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$.

Now,

$$\begin{bmatrix}1 & 2 & 2\\0 & 1 & 2\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Therefore, we have, $x_1 + x_2 + x_3 = 0$,$x_2 + 2x_3 = 0$ and $x_3 = 0$. Thus we have $X = \begin{bmatrix}0\\0\\0\end{bmatrix}$

Therefore, The null space of $A_2 = \{\begin{bmatrix}0\\0\\0\end{bmatrix}\}$

(c)

$$A_3 = \begin{bmatrix}1 & 2\\1 & 3\\1 & 2\end{bmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\begin{bmatrix}1 & 2\\0 & 1\\0 & 0\end{bmatrix}$$

Column Space of A_3= $Span\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}\}$

Row Space of $A_3 = Span\{(1,2),(0,1)\}$

Let $W = \{X \in \mathbb{R}^2 | A_3X = O\}$ where $X = \begin{bmatrix}x_1\\x_2\end{bmatrix}$.

Now,

$$\begin{bmatrix}1 & 2\\0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Therefore, we have, $x_1 + 2x_2 = 0$,$x_2 = 0$ and $x_1 = 0$. Thus we have $X = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Therefore, The null space of $A_3 = \{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\}$

(d)

$$A_4 = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 1 \\ 2 & 5 & 2 \end{bmatrix}$$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ and $R_4 \to R_4 - 2R_1$, we get

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Applying $R_4 \to R_4 - R_2$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Column Space of $A_4 = Span\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}\}$

Row Space of $A_4 = Span\{(1, 2, 1), (0, 1, 0)\}$

Let $W = \{X \in \mathbb{R}^3 | A_4X = O\}$ where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

Now,

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, we have, $x_1 + 2x_2 + x_3 = 0$ and $x_2 = 0$. Thus we have $X = x_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

Therefore, The null space of $A_4 = Span\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\}$

7. Use Gauss elimination method to find all polynomials $f \in P_2 : f(1) = 2$ and $f(-1) = 6$.
   Solution:
   Let us take $f(x) \in P_2$ as

$$f(x) = a_0 + a_1 x + a_2 x^2, \quad a_0, a_1, a_2 \in \mathbb{R}$$

Then,

$$\begin{aligned} f(1) = 2 &\Rightarrow a_0 + a_1 + a_2 = 2 \quad and \\ f(-1) = 6 &\Rightarrow a_0 - a_1 + a_2 = 6 \end{aligned}$$

Now we can write these equations in matrix form as,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$$

So Augmented matrix is,

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & 6 \end{array}\right]$$

$R_2 \to R_2 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 0 & 4 \end{array}\right]$$

Now, we have equations

$$\begin{aligned} a_0 + a_1 + a_2 = 2 \quad and \\ -2a_1 = 4 \end{aligned}$$

$$\Rightarrow a_1 = -2 \; and \; a_0 + a_2 = 4$$

Put $a_2 = c$ then we have

$$(a_0, a_1, a_2) = (4 - c, -2, c)$$

So, all $f(x) = (4-c) - 2x + cx^2 \in P_2$ where $c \in \mathbb{R}$ are such polynomials in P_2 which satisfies $f(1) = 2$ and $f(-1) = 6$.