(LI, LD, Basis and Dimension)

1(i). Check the linear dependence or linear independence of the following sets in respective real vector spaces
(a) $\{e^x, e^{2x}\}$ in $\mathcal{C}^\infty(\mathbb{R})$.
(b) $\{x, |x|\}$ in $\mathcal{C}[-1,1]$.
(c) $\{(\frac{1}{2}, \frac{1}{3}, 1), (-3,1,0), (1,2,-3)\}$ in $\mathbb{R}^3$.
(d) $\{(1,1,1,0), (3,2,2,1), (1,1,3,-2), (1,2,6,-5)\}$ in $\mathbb{R}^4$.
(e) $\{(x, x^3 - x, x^4 + x^2, x + x^2 + x^4 + \frac{1}{2}\}$ in $\mathcal{P}_4$.

1(ii). Show that the set $\mathrm{S} = \{\sin x, \sin 2x, \cdots, \sin nx\}$ is a LI subset of $\mathcal{C}[-\pi, \pi]$ for every positive integer n.

2(i). If u, v and w are LI vectors os a vector space V, then prove that $u+v, v+w,$ and $w+u$ are also LI.

2(ii). Let $\mathrm{S}_1, \mathrm{S}_2$ be subsets of a vector space V such that $\mathrm{S}_1 \subset \mathrm{S}_2$. Then prove that
(a) S_1 is LD $\Rightarrow$ S_2 is LD.
(b) S_2 is LI $\Rightarrow$ S_1 is LI.

2(iii). Let S be a LI subset of a vector space V. Let $v \in L[\mathrm{S}]$. Prove that $\{v\} \cup \mathrm{S}$ is a LD set.

2(iv). Let S be a LI subset of a vector space V. Let v does not belong in $L[\mathrm{S}]$. Prove that $\{v\} \cup \mathrm{S}$ is a LI set also.

3(i). In a vector space V, if a **ordered** set $\mathrm{S} = \{v_1, v_2, v_3, \cdots, v_n\}$ is LD **with** $v_1 \neq 0$ then prove that $\exists$ a vector v_k, $2 \leq k \leq n$ such that $v_k \in L[\{v_1, v_2, v_3, \cdots, v_{\mathbf{k-1}}\}]$.

3(ii). In a vector space V, if a set $\mathrm{S} = \{v_1, v_2, v_3, \cdots, v_n\}$ is LI and $\mathrm{S}_1 = \{w_1, w_2, w_3, \cdots, w_m\}$ generates the space V then prove that $n \leq m$.

4. Determine whether the following sets are bases for given vector spaces V over field F
(i) $\{(2,4,0), (0,2,-2)\}$; $\mathrm{V} = \mathbb{R}^3$ and $F = \mathbb{R}$.
(ii) $\{(6,4,4), (-2,4,2), (0,7,0)\}$; $\mathrm{V} = \mathbb{R}^3$ and $F = \mathbb{R}$.
(iii) $\left\{\begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix}\right\}$; $\mathrm{V} = \mathcal{M}_{2\times 2}$ and $F = \mathbb{R}$.
(iv) $\{1, x-2, (x-2)^2, (x-2)^3\}$; $\mathrm{V} = \mathcal{P}_3$ and $F = \mathbb{R}$.
(v) $\{x-1, x^2+x-1, x^2-x+1\}$; $\mathrm{V} = \mathcal{P}_2$ and $F = \mathbb{R}$.
(vi) $\{(1, i, 1+i), (1, i, 1-i), (i, -i, 1)\}$; $\mathrm{V} = \mathbb{C}^3$ and $F = \mathbb{C}$.

5(i). Find the co-ordinates of the following vector of $\mathbb{R}^3$ relative to the ordered basis $B = \{(2,1,0), (2,1,1), (2,2,1)\}$
(i) $(1,2,-1)$ (ii)$(2,0,-1)$ (iii) $(-1,3,1)$

5(ii). Find the relation between the co-ordinates of the vector $(1,5)$ with respect to the ordered bases $B_1 = \{(1,1), (0,1)\}$ and $B_2 = \{(-1,4), (7,6)\}$

6. Find a basis for the plane $P : x - 2y + 3z = 0$ in $\mathbb{R}^3$. Find a basis for the intersection of P with with the xy-plane. Also, find a basis for the space of vectors perpendicular to the plane P.

7(i). Let $\mathrm{S} = \{(4,5,6), (a,2,4), (4,3,2)\}$ be a set in $\mathbb{R}^3$. Find the values for a such that $L[\mathrm{S}] \neq \mathbb{R}^3$.

7(ii). For what values of k vectors $\mathrm{S} = \{(k+1, -k, k), (2k, 2k-1, k+2), (-2k, k, -k)\}$ form a basis of $\mathbb{R}^3$.

8. For each of followings, find a basis (here all vector spaces are real)
(i) $\{(x_1, x_2, x_3)$ in $\mathbb{R}^3 : x_1 - x_3 = 0\}$.
(ii) $\{(x_1, x_2, x_3)$ in $\mathbb{R}^3 : 2x_1 + x_2 + x_3 = 0\}$.
(iii) $\{(x_1, x_2, x_3, x_4)$ in $\mathbb{R}^4 : x_1 + x_2 + 2x_3 = 0, 2x_2 + x_3 = 0$ and $x_1 - x_2 + x_3 = 0\}$.
(iv) $\{a + bx + cx^3$ in $\mathcal{P}_3 : a - 2b + c = 0\}$.

(v) $\{p \text{ in } \mathcal{P}_4 : p(7) = 0 \text{ and } p'(1) = 0\}$.

(vi) $\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ in } \mathbb{R}^{2\times 2} : a - d + c = 0\}$.

(vii) $\{A \text{ in } \mathbb{R}^{4\times 4} : A \text{ is a real symmetric martix}\}$.

(viii) $\{A \text{ in } \mathbb{R}^{5\times 5} : \text{Trace } A = 0\}$.

(ix) $\{A \text{ in } \mathbb{R}^{2\times 2} : A \text{ is a complex Hermitian martix}\}$.

(x) $\{A \text{ in } \mathbb{R}^{m\times n} : \text{ sum of each row of } A = 0\}$.

9(i). Write two bases of $\mathbb{R}^4$ that have no common elements.

9(ii). Write two different bases of $\mathbb{R}^4$ that have the vectors $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$ in common.

9(iii). Find a basis of $L[\{(1, -1, 2, 3), (1, 0, 1, 0), (3, -2, 5, 2)\}]$ which includes the vectors $(1, 1, 0, -1)$.

9(iv). Extend the set $\{(1, 1, -1, 0), (1, 0, 1, 1), (1, 2, 1, 1)\}$ to a basis of $\mathbb{R}^4$.

10. Find a basis for U, W, $U \bigcap W$ and $U + W$ in the following cases for a vector space V.
(i) $U = \{(x_1, x_2, x_3) : x_1 + x_2 - x_3 = 0\}$, $W = \{(x_1, x_2, x_3) : 2x_1 + x_2 = 0\}$, $\mathrm{V} = \mathbb{R}^3$.
(ii) $U = \{a_0 + a_1x + a_2x^2 : a_1 + a_2 = 0\}$, $W = \{a_0 + a_1x + a_2x^2 : 2a_0 + a_1 = 0\}$, $\mathrm{V} = \mathcal{P}_2$.
(iii) $U = \{p : p(2) = 0\}$, $W = \{p : p'(2) = 0\}$, $\mathrm{V} = \mathcal{P}_4$.

11. Find the subspaces $S \bigcap T$, $S + T$ of vector space V. Further, find $\dim(S)$, $\dim(T)$, $\dim(S \bigcap T)$ $\dim(S + T)$ if
(i) $S = L[\{(1, -1, 0), (1, 0, 2)\}]$, $T = L[\{(0, 1, 0), (0, 1, 2)\}]$, $\mathrm{V} = \mathbb{R}^3$.
(ii) $S = L[\{(2, 2, -1, 2), (1, 1, 1, -2), (0, 0, 2, -4)\}]$, $T = L[\{(2, -1, 1, 1), (-2, 1, 3, 3), (3, -6, 0, 0)\}]$, $\mathrm{V} = \mathbb{R}^4$.

Answers

1(i). (a) LI (b) LI (c) LI (d) LD (e) LI
4. (i) No (ii) Yes (iii) Yes (iv) Yes (v) No (vi) Yes

$$\sum a_i \sin(ix) = \sum^n a_i \int_{-\pi}^{\pi} \sin nx \sin ix$$