

MA-102
B. Tech. II Sem (2021-2022)
Tutorial sheet-02

Cholesky Decomposition:

Find Cholesky decomposition for following matrices.

1. $\begin{bmatrix} 1 & -1 & 2 \\ -1 & 5 & -4 \\ 2 & -4 & 6 \end{bmatrix}$

2. $\begin{bmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix}$

Solution

1.

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 5 & -4 \\ 2 & -4 & 6 \end{bmatrix}$$

$$IA = A$$

$$R_2 \rightarrow R_2 + R_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & -2 \\ 2 & -4 & 6 \end{bmatrix}$$

$$IE_1A = A_1$$

$$R_3 \rightarrow R_3 - 2R_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} E_1A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix}$$

$$IE_2E_1A = A_2$$

$$R_3 \rightarrow R_3 + \frac{1}{2}R_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix} E_2E_1A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_3 E_2 E_1 A = U$$

$$A = (E_3 E_2 E_1)^{-1} U = E_1^{-1} E_2^{-1} E_3^{-1} U = LU$$

$$L = E_1^{-1} E_2^{-1} E_3^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -\frac{1}{2} & 1 \end{bmatrix}$$

$$A = LU = LIU = LDD^{-1}U$$

$$= L(\sqrt{D}\sqrt{D})D^{-1}U$$

$$= (L\sqrt{D})(\sqrt{D}D^{-1}U) = CC^T$$

$$\text{Let } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \sqrt{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$L\sqrt{D} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & -1 & 1 \end{bmatrix} = C$$

$$\sqrt{D}D^{-1}U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} = C^T$$

2.

$$A = \begin{bmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix}$$

$$IA = A$$

$$R_2 \rightarrow R_2 - \frac{3}{5}R_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{5} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 25 & 15 & -5 \\ 0 & 9 & 3 \\ -5 & 0 & 11 \end{bmatrix}$$

$$IE_1A = A_1$$

$$R_3 \rightarrow R_3 + \frac{1}{5}R_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{5} & 0 & 1 \end{bmatrix} E_1A = \begin{bmatrix} 25 & 15 & -5 \\ 0 & 9 & 3 \\ 0 & 3 & 10 \end{bmatrix}$$

$$IE_2E_1A = A_2$$

$$R_3 \rightarrow R_3 - \frac{1}{3}R_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{3} & 1 \end{bmatrix} E_2E_1A = \begin{bmatrix} 25 & 15 & -5 \\ 0 & 9 & 3 \\ 0 & 0 & 9 \end{bmatrix}$$

$$E_3E_2E_1A = U$$

$$A = (E_3E_2E_1)^{-1}U = E_1^{-1}E_2^{-1}E_3^{-1}U = LU$$

$$L = E_1^{-1}E_2^{-1}E_3^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{5} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{5} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{3} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{5} & 1 & 0 \\ -\frac{1}{5} & \frac{1}{3} & 1 \end{bmatrix}$$

$$A = LU = LIU = LDD^{-1}U$$

$$= L(\sqrt{D}\sqrt{D})D^{-1}U$$

$$= (L\sqrt{D})(\sqrt{D}D^{-1}U) = CC^T$$

$$\text{Let } D = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} \quad \text{and} \quad \sqrt{D} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\begin{aligned}
L\sqrt{D} &= \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{5} & 1 & 0 \\ -\frac{1}{5} & \frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 5 & 0 & 0 \\ 3 & 3 & 0 \\ -1 & 1 & 3 \end{bmatrix} = C
\end{aligned}$$

$$\begin{aligned}
\sqrt{D}D^{-1}U &= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{25} & 0 & 0 \\ 0 & \frac{1}{9} & 0 \\ 0 & 0 & \frac{1}{9} \end{bmatrix} \begin{bmatrix} 25 & 15 & -5 \\ 0 & 9 & 3 \\ 0 & 0 & 9 \end{bmatrix} \\
&= \begin{bmatrix} 5 & 3 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} = C^T
\end{aligned}$$

Vector Space:

1. (i) Suppose we define addition on \mathbb{R}^2 by the rule $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, 0)$. Show that additive identity does not exist in \mathbb{R}^2 w.r.t. above rule.

Solution: The set V contains an additive identity element denoted by 0 , such that for any vector v in V , we have $0 + v = v$ and $v + 0 = v$.

Let $v = (a_1, a_2) \in \mathbb{R}^2$.

Suppose $(e_1, e_2) \in \mathbb{R}^2$ be the additive identity element such that $(e_1, e_2) + (a_1, a_2) = (a_1, a_2)$.

As per operation defined above, $(e_1, e_2) + (a_1, a_2) = (e_1 + a_1, 0)$.

$\Rightarrow (e_1 + a_1, 0) = (a_1, a_2)$

i.e. $e_1 + a_1 = a_1$ and $a_2 = 0 \Rightarrow e_1 = 0$ and $a_2 = 0$.

$\Rightarrow (0, e_2)$ is the additive identity element of the element of the form $(a_1, 0)$.

i.e. additive identity element does not exist for the element of the form (a_1, a_2) where $a_2 \neq 0$.

So, we can conclude that additive identity element does not exist for all the elements in \mathbb{R}^2 .

Hence, additive identity does not exist in \mathbb{R}^2 w.r.t. above rule.

- (ii) Suppose we define addition on \mathbb{R}^3 by the rule $(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1b_1, a_2b_2, a_3b_3)$. Show that we have an additive identity for this operation in \mathbb{R}^3 but inverse may not exist for some elements.

Solution: Let $v = (a_1, a_2, a_3) \in \mathbb{R}^3$.

Suppose $(e_1, e_2, e_3) \in \mathbb{R}^3$ be the additive identity element such that
 $(e_1, e_2, e_3) + (a_1, a_2, a_3) = (a_1, a_2, a_3)$.

As per operation defined above, $(e_1, e_2, e_3) + (a_1, a_2, a_3) = (e_1 a_1, e_2 a_2, e_3 a_3)$.
 $\Rightarrow (e_1 a_1, e_2 a_2, e_3 a_3) = (a_1, a_2, a_3)$

i.e. $e_1 a_1 = a_1$, $e_2 a_2 = a_2$ and $e_3 a_3 = a_3$.

i.e. $e_1 = 1$, $e_2 = 1$ and $e_3 = 1$.

So, $(e_1, e_2, e_3) = (1, 1, 1)$ is the additive identity element for this operation in \mathbb{R}^3 .

Now, let (b_1, b_2, b_3) be the inverse of (a_1, a_2, a_3) .

$\Rightarrow (a_1, a_2, a_3) + (b_1, b_2, b_3) = (1, 1, 1)$.

$\Rightarrow (a_1 b_1, a_2 b_2, a_3 b_3) = (1, 1, 1)$.

$\Rightarrow b_1 = \frac{1}{a_1}, b_2 = \frac{1}{a_2}, b_3 = \frac{1}{a_3}$.

So, from here we can conclude that, inverse exists only when
 $a_1 \neq 0$, $a_2 \neq 0$, and $a_3 \neq 0$.

i.e. if at least one of the a_i is zero, then inverse does not exist.

Hence, inverse does not exist for some elements.

2. Let \mathbb{R}^+ be the set of all positive real numbers. Define operations of addition \oplus and the scalar multiplication \otimes as follows: $u \oplus v = uv$ for all $u, v \in \mathbb{R}^+$ and $\alpha \otimes u = u^\alpha$ for all $u \in \mathbb{R}^+$ and $\alpha \in \mathbb{R}$ (here \mathbb{R}^+ is the field of scalars). Prove that $(\mathbb{R}^+, \oplus, \otimes)$ is a real vector space.

Solution:

For any $u, v \in \mathbb{R}^+$,

(i) $u \oplus v = uv$. Since $uv \in \mathbb{R}^+$, $u \oplus v \in \mathbb{R}^+$.

(ii) $u \oplus v = uv = vu = v \oplus u$ [$u, v \in \mathbb{R}^+$, $uv = vu$]

Let u, v and $w \in \mathbb{R}^+$,

(iii) $u \oplus (v \oplus w) = u \oplus vw = uvw = uv \oplus w = (u \oplus v) \oplus w$

(iv) For any $u \in \mathbb{R}^+$, $1 \oplus u = 1 \cdot u = u$. Similarly, $u \oplus 1 = u$.

Therefore 1 is additive identity on \mathbb{R}^+ .

(v) Since $u \in \mathbb{R}^+$, we have $\frac{1}{u} \in \mathbb{R}^+$. Now, $u \oplus \frac{1}{u} = u \cdot \frac{1}{u} = 1$ and similarly $\frac{1}{u} \oplus u = 1$.

(vi) For any $\alpha \in \mathbb{R}$ and $u \in \mathbb{R}^+$, $\alpha \otimes u = u^\alpha$ and $u^\alpha \in \mathbb{R}^+$. Therefore, $\alpha \otimes u \in \mathbb{R}^+$.

(vii) For any $u \in \mathbb{R}^+$, $1 \otimes u = u^1 = u$.

(viii) For any $\alpha_1, \alpha_2 \in \mathbb{R}$ and $u \in \mathbb{R}^+$, $(\alpha_1 \alpha_2) \otimes u = u^{\alpha_1 \alpha_2} = \{u^{\alpha_2}\}^{\alpha_1} = \alpha_1 \otimes (u^{\alpha_2}) = \alpha_1 \otimes (\alpha_2 \otimes u)$

(ix) For any $\alpha \in \mathbb{R}$ and $u, v \in \mathbb{R}^+$, $\alpha \otimes (u \oplus v) = \alpha \otimes (uv) = \{uv\}^\alpha = u^\alpha v^\alpha = (u^\alpha) \oplus (v^\alpha) = (\alpha \otimes u) \oplus (\alpha \otimes v)$.

(x) For any $\alpha_1, \alpha_2 \in \mathbb{R}$ and $u \in \mathbb{R}^+$, $(\alpha_1 + \alpha_2) \otimes u = u^{(\alpha_1 + \alpha_2)} =$

$$u^{\alpha_1} u^{\alpha_2} = (u^{\alpha_1}) \oplus (v^{\alpha_2}) = (\alpha_1 \otimes u) \oplus (\alpha_2 \otimes v).$$

3. Let $V = \mathbb{R}^2$. Define operations of addition \oplus and the scalar multiplication \otimes as follows:

$$(a_1, a_2) \oplus (b_1, b_2) = (a_1 + b_2, a_2 + b_1) \text{ and } \alpha \otimes (a_1, a_2) = (\alpha a_1, \alpha a_2), \alpha \in \mathbb{R} \text{ (here } \mathbb{R} \text{ is the field of scalars).}$$

Does (V, \oplus, \otimes) form a real vector space? Give reasons for your assertion.

Solution: Let us calculate $(a_1, a_2) \oplus (b_1, b_2) = (a_1 + b_2, a_2 + b_1)$ and $(b_1, b_2) \oplus (a_1, a_2) = (b_1 + a_2, b_2 + a_1)$.

Since $(a_1 + b_2) \neq (b_1 + a_2)$ and $(a_2 + b_1) \neq (b_2 + a_1)$.

So $(a_1, a_2) \oplus (b_1, b_2) \neq (b_1, b_2) \oplus (a_1, a_2)$ which means (V, \oplus, \otimes) does not form a vector space.

4. Done in class.

5. Done in class.

6. Let $V = C[0, 1]$ be the set of all real valued function defined and continuous on the closed interval $[0, 1]$. Prove that V is a real vector space with respect to pointwise addition and multiplication. Further, determine that which of the following subsets of V are subspaces.

Solution:

For $f(x), g(x) \in C[0, 1]$ and $c \in \mathbb{R}$, we define the functions $f + g$ and cf by

$$(f + g)(x) = f(x) + g(x)$$

$$(cf)(x) = c(f(x))$$

We have to show that $V = C[0, 1]$ with the given operations is a vector space. We check the vector space axioms for this V .

We let f, g, h be arbitrary elements of V . We know from calculus that the sum of any two continuous functions is continuous and that any constant times a continuous function is also continuous. Therefore, the closure of addition and that of scalar multiplication hold.

Now for all x such that $0 \leq x \leq 1$, we have from the definition and the commutative law of real number addition that

$$(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x).$$

Since this holds for all x , we conclude that $f + g = g + f$, which is the commutative law of vector addition. Similarly,

$$\begin{aligned} ((f + g) + h)(x) &= (f + g)(x) + h(x) = (f(x) + g(x)) + h(x) \\ &= f(x) + (g(x) + h(x)) = (f + (g + h))(x) \end{aligned}$$

Since this holds for all x , we conclude that $(f + g) + h = f + (g + h)$, which is the associative law for addition of vectors.

Next, if 0 denotes the constant function with value 0 , then for any $f \in V$ we have that for all $0 \leq x \leq 1$,

$$(f + 0)(x) = f(x) + 0 = f(x).$$

Since this is true for all x we have that $f + 0 = f$, which establishes the additive identity law.

Also, we define $(-f)(x) = -(f(x))$ so that for all $0 \leq x \leq 1$,

$$(f + (-f))(x) = f(x) - f(x) = 0,$$

from which we see that $f + (-f) = 0$. The additive inverse law follows. For the distributive laws, note that for real numbers c, d and continuous functions $f, g \in V$, we have that for all $0 \leq x \leq 1$,

$$c(f + g)(x) = c(f(x) + g(x)) = cf(x) + cg(x) = (cf + cg)(x),$$

which proves the first distributive law. For the second distributive law, note that for all $0 \leq x \leq 1$,

$$((c + d)g)(x) = (c + d)g(x) = cg(x) + dg(x) = (cg + dg)(x),$$

and the second distributive law follows. For the scalar associative law, observe that for all $0 \leq x \leq 1$,

$$((cd)f)(x) = (cd)f(x) = c(df(x)) = (c(df))(x),$$

so that $(cd)f = c(df)$, as required. Finally, we see that

$$(1.f)(x) = 1.f(x) = f(x),$$

from which we have the monoidal law $1.f = f$. Thus, $C[0, 1]$ with the prescribed operations is a vector space.

We know that $W(F) \subset V(\mathbb{F})$ is said to be a subspace of V , if $\alpha, \beta \in \mathbb{F}$ and $u, v \in W$, then $\alpha u + \beta v \in W$. Also, zero vector is in W .

- (a) $W_1 = \{f \in V : f(1/2) = 0\}$.

Solution:

Clearly, zero vector is in W_1 .

Let $f, g \in W_1 \Rightarrow f, g \in V$

As V is vector space $\Rightarrow \alpha f + \beta g \in V$.

If W_1 is subspace then, only we have to show $\alpha f + \beta g \in W_1$ i.e.

$$(\alpha f + \beta g)\left(\frac{1}{2}\right) = 0.$$

$$\text{As } f, g \in W_1 \Rightarrow f\left(\frac{1}{2}\right) = 0, g\left(\frac{1}{2}\right) = 0.$$

$$(\alpha f + \beta g)\left(\frac{1}{2}\right) = \alpha f\left(\frac{1}{2}\right) + \beta g\left(\frac{1}{2}\right) = 0 + 0 = 0.$$

$$\Rightarrow \alpha f + \beta g \in W_1 \text{ i.e. } W_1 \text{ is subspace of } V.$$

- (b) $W_2 = \{f \in V : f(3/4) = 1\}$.

Solution:

Since zero element is not in this subset W_2 . So it is not a subspace of V .

- (c) $W_3 = \{f \in V : f(0) = f(1)\}$.

Solution:

Clearly, zero vector is in W_3 .

Let $f, g \in W_3 \Rightarrow f, g \in V$

As V is vector space $\Rightarrow \alpha f + \beta g \in V$.

If W_3 is subspace then, only we have to show $\alpha f + \beta g \in W_3$ i.e.

$$(\alpha f + \beta g)(0) = (\alpha f + \beta g)(1).$$

$$\text{As } f, g \in W_3 \Rightarrow f(0) = f(1) \text{ and } g(0) = g(1).$$

$$(\alpha f + \beta g)(0) = \alpha f(0) + \beta g(0) = \alpha f(1) + \beta g(1) = (\alpha f + \beta g)(1).$$

$$\Rightarrow W_3 \text{ is subspace of } V.$$

- (d) $W_4 = \{f \in V : f(x) = 0 \text{ only at a finite number of points}\}$.

Solution:

Since zero polynomial is not in W_4 . So, W_4 is not subspace of V .

7. Determine whether each of the following set S forms a subspace of \mathbb{R}^4 , if addition and multiplication rules are defined in the usual way.

Solution: We know that $W(F) \subset V(\mathbb{F})$ is said to be a subspace of V , if $\alpha, \beta \in \mathbb{F}$ and $u, v \in W$, then $\alpha u + \beta v \in W$. Also, zero vector is in W .

- (a) $S = \{(a, b, c, d) \mid a = c + d\}$

Clearly, zero vector is in W .

Let $u = (a_1, b_1, c_1, d_1), v = (a_2, b_2, c_2, d_2) \in S$ & $\alpha, \beta \in \mathbb{F}$

i.e $a_1 = c_1 + d_1, a_2 = c_2 + d_2$

$$\Rightarrow \alpha(a_1, b_1, c_1, d_1) + \beta(a_2, b_2, c_2, d_2) = (\alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2, \alpha c_1 + \beta c_2, \alpha d_1 + \beta d_2)$$

We know from above,

$$a_1 = c_1 + d_1, \quad a_2 = c_2 + d_2$$

$$\Rightarrow \alpha a_1 = \alpha c_1 + \alpha d_1 \quad \text{and} \quad \beta a_2 = \beta c_2 + \beta d_2$$

On adding both terms, we get

$$\Rightarrow \alpha a_1 + \beta a_2 = \alpha c_1 + \alpha d_1 + \beta c_2 + \beta d_2$$

$$\Rightarrow \alpha u + \beta v \in S$$

Hence, S is a subspace of \mathbb{R}^4 .

(b) $S = \{(a, b, c, d) \mid b = c - d \text{ and } a = c + d\}$

Clearly, zero vector is in W .

$$\text{Let } u = (a_1, b_1, c_1, d_1), \quad v = (a_2, b_2, c_2, d_2) \in S \text{ \& } \alpha, \beta \in \mathbb{F}$$

$$\text{i.e } b_1 = c_1 - d_1, \quad a_1 = c_1 + d_1 \text{ and } b_2 = c_2 - d_2, \quad a_2 = c_2 + d_2$$

$$\Rightarrow \alpha(a_1, b_1, c_1, d_1) + \beta(a_2, b_2, c_2, d_2) = (\alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2, \alpha c_1 + \beta c_2, \alpha d_1 + \beta d_2)$$

We know from above,

$$b_1 = c_1 - d_1, \quad a_1 = c_1 + d_1 \text{ and } b_2 = c_2 - d_2, \quad a_2 = c_2 + d_2$$

$$\Rightarrow \alpha b_1 = \alpha c_1 - \alpha d_1, \quad \alpha a_1 = \alpha c_1 + \alpha d_1$$

$$\text{and } \beta b_2 = \beta c_2 - \beta d_2, \quad \beta a_2 = \beta c_2 + \beta d_2$$

On adding, we get

$$\Rightarrow \alpha b_1 + \beta b_2 = \alpha c_1 - \alpha d_1 + \beta c_2 - \beta d_2$$

$$\text{and } \alpha a_1 + \beta a_2 = \alpha c_1 + \alpha d_1 + \beta c_2 + \beta d_2$$

$$\Rightarrow \alpha u + \beta v \in S$$

Hence, S is a subspace of \mathbb{R}^4 .

(c) $S = \{(a, b, c, d) \mid c = d\}$

Clearly, zero vector is in W .

$$\text{Let } u = (a_1, b_1, c_1, d_1), \quad v = (a_2, b_2, c_2, d_2) \in S \text{ \& } \alpha, \beta \in \mathbb{F}$$

$$\text{i.e } c_1 = d_1, \quad c_2 = d_2$$

$$\Rightarrow \alpha(a_1, b_1, c_1, d_1) + \beta(a_2, b_2, c_2, d_2) = (\alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2, \alpha c_1 + \beta c_2, \alpha d_1 + \beta d_2)$$

We know from above,

$$c_1 = d_1, \quad c_2 = d_2$$

$$\Rightarrow \alpha c_1 = \alpha d_1, \quad \beta c_2 = \beta d_2$$

On adding both terms, we get

$$\Rightarrow \alpha c_1 + \beta c_2 = \alpha d_1 + \beta d_2$$

$$\Rightarrow \alpha u + \beta v \in S$$

Hence, S is a subspace of \mathbb{R}^4 .

(d) $S = \{(-a + c, a - b, b + c, a + b) \mid a, b, c \in \mathbb{R}\}$

Clearly, zero vector is in W .

$$\text{Let } u = (-a_1 + c_1, a_1 - b_1, b_1 + c_1, a_1 + b_1),$$

$$v = (-a_2 + c_2, a_2 - b_2, b_2 + c_2, a_2 + b_2) \in S \text{ \& } \alpha, \beta \in \mathbb{F}$$

$$\begin{aligned} \Rightarrow \alpha u + \beta v &= \alpha(-a_1 + c_1, a_1 - b_1, b_1 + c_1, a_1 + b_1) + \beta(-a_2 + c_2, a_2 - b_2, b_2 + c_2, a_2 + b_2) \\ &= (-\alpha a_1 - \beta a_2 + \alpha c_1 + \beta c_2, \alpha a_1 + \beta a_2 - \alpha b_1 - \beta b_2, \\ &\quad \alpha b_1 + \beta b_2 + \alpha c_1 + \beta c_2, \alpha a_1 + \beta a_2 + \alpha b_1 + \beta b_2) \end{aligned}$$

and

$$\begin{aligned} &-\alpha a_1 - \beta a_2 + \alpha c_1 + \beta c_2, \quad \alpha a_1 + \beta a_2 - \alpha b_1 - \beta b_2, \quad \alpha b_1 + \beta b_2 + \\ &\alpha c_1 + \beta c_2 \in \mathbb{R} \\ \Rightarrow \quad &\alpha u + \beta v \in S \end{aligned}$$

Hence, S is a subspace of \mathbb{R}^4 .

$$(e) \ S = \{(a, b, c, d) \mid a = 1\}$$

We know that, if S is a subspace of \mathbb{R}^4 , then '0'(zero element) must belong to that set S . But, here we can see that '0' $\notin S$. Because, if $(0, 0, 0, 0) \in S$, then $0 = 1$ which is absurd condition. Hence, S is not a subspace of \mathbb{R}^4 .

$$(f) \ S = \{(a, b, c, d) \mid a \leq b\}$$

Clearly, zero vector is in W .

Let $u = (1, 2, 4, 5)$, $v = (2, 4, 5, 6) \in S$

Now, let $\alpha = 1, \beta = -1 \in \mathbb{R}$

$$\begin{aligned} \Rightarrow \alpha(1, 2, 4, 5) + \beta(2, 4, 5, 6) &= 1 \cdot (1, 2, 4, 5) + (-1) \cdot (2, 4, 5, 6) \\ &= (1 - 2, 2 - 4, 4 - 5, 5 - 6) \\ &= (-1, -2, -1, -1) \end{aligned}$$

Now, $-1 \leq -2$ is not possible.

$$\Rightarrow \alpha u + \beta v \notin S$$

Hence, S is not a subspace of \mathbb{R}^4 .

$$(g) \ S = \{(a, b, c, d) \mid a = b = c = d\}$$

Clearly, zero vector is in W .

Let $u = (a_1, b_1, c_1, d_1)$, $v = (a_2, b_2, c_2, d_2) \in S$ & $\alpha, \beta \in \mathbb{F}$

i.e $a_1 = b_1 = c_1 = d_1$, $a_2 = b_2 = c_2 = d_2$

$$\Rightarrow \alpha(a_1, b_1, c_1, d_1) + \beta(a_2, b_2, c_2, d_2) = (\alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2, \alpha c_1 + \beta c_2, \alpha d_1 + \beta d_2)$$

We know from above,

$$a_1 = b_1 = c_1 = d_1, \quad a_2 = b_2 = c_2 = d_2$$

$$\Rightarrow \alpha a_1 = \alpha b_1 = \alpha c_1 = \alpha d_1, \quad \beta a_2 = \beta b_2 = \beta c_2 = \beta d_2$$

On adding both terms, we get

$$\Rightarrow \alpha a_1 + \beta a_2 = \alpha b_1 + \beta b_2 = \alpha c_1 + \beta c_2 = \alpha d_1 + \beta d_2$$

$$\Rightarrow \alpha u + \beta v \in S$$

Hence, S is a subspace of \mathbb{R}^4 .

(h) $S = \{(a, b, c, d) \mid a \text{ is an integer}\}$

Clearly, zero vector is in W .

Let $u = (a_1, b_1, c_1, d_1)$, $v = (a_2, b_2, c_2, d_2) \in S$ & $\alpha, \beta \in \mathbb{F}$

i.e a_1 is an integer, a_2 is an integer

Let $u = (1, 0, 0, 0)$, $v = (1, 0, 0, 0)$, $\alpha = \sqrt{2}$, $\beta = 1$

Then, $\alpha u + \beta v = \sqrt{2}(1, 0, 0, 0) + 1 \cdot (1, 0, 0, 0) = (\sqrt{2} + 1, 0, 0, 0)$

But, $\sqrt{2} + 1$ is not an integer.

$$\Rightarrow \alpha u + \beta v \notin S$$

Hence, S is not a subspace of \mathbb{R}^4 .

(i) $S = \{(a, b, c, d) \mid a^2 - b^2 = 0\}$

Clearly, zero vector is in W .

Let $u = (1, -1, 0, 0)$, $v = (2, 2, 0, 0) \in S$

Now, let $\alpha = 1, \beta = 1 \in \mathbb{R}$

$$\begin{aligned} \Rightarrow \alpha(1, -1, 0, 0) + \beta(2, 2, 0, 0) &= 1 \cdot (1, -1, 0, 0) + 1 \cdot (2, 2, 0, 0) \\ &= (1 + 2, -1 + 2, 0, 0) \\ &= (3, 1, 0, 0) \end{aligned}$$

Now, $3^2 - 1^2 = 8 \neq 0$.

$$\Rightarrow \alpha u + \beta v \notin S$$

Hence, S is not a subspace of \mathbb{R}^4 .

8. Which of the following subsets of P are subspace. Where, P is the real vector space of all polynomials w.r.t usual vector addition and multiplication :

i) $\{p \in P : \deg p \leq 4\}$

ii) $\{p \in P : \deg p = 4\}$

iii) $\{p \in P : \deg p \geq 4\}$

iv) $\{p \in P : p(1) = 0\}$

v) $\{p \in P : p(2) = 1\}$

vi) $\{p \in P : p'(1) = 0\}$

Solution 8(i):

Let $S = \{p \in P : \deg p \leq 4\}$

Let $f(x) = \sum_{i=0}^4 a_i x^i$ and $g(x) = \sum_{i=0}^4 b_i x^i$ be two polynomials in S .

Clearly, S is non-empty since zero polynomial belongs to S .

To check whether S is a subspace of P or not we have to check whether S is closed under addition and scalar multiplication or not.

So $f(x) + g(x) = \sum_{i=0}^4 (a_i + b_i) x^i$

and $\alpha \cdot f(x) = \sum_{i=0}^4 \alpha a_i x^i$.

Since addition of two polynomials and multiplication by a scalar does not increase the degree of that polynomial.

Therefore, both $f(x) + g(x)$, $\alpha f(x) \in S$.

Hence, S is a subspace of P .

Solution 8(ii): Assume the set $S_1 = \{p \in P : \deg p = 4\}$

Let $f(x) = x^4, g(x) = -x^4 \in S_1$, but $f(x) + g(x) = 0 \notin S_1$.

Therefore, S_1 is not closed under addition.

Hence, S_1 is not a subspace of P .

Solution 8(iii): Consider the set $S_2 = \{p \in P : \deg p \geq 4\}$.

By the similar above example of $f(x)$ and $g(x)$, we can see that $f(x) + g(x) \notin S_2$. So, S_2 is not closed under addition.

Hence, S_2 is not a subspace of P .

Solution 8(iv): Let $S_3 = \{p \in P : p(1) = 0\}$

Clearly, S_3 is nonempty since $0 \in S_3$.

Let $f(x), g(x) \in S_3$, then $f(1) = 0 = g(1)$.

Now, $(f + g)(1) = f(1) + g(1) = 0 + 0 = 0$

and $(\alpha f)(1) = \alpha \cdot f(1) = \alpha \cdot 0 = 0$.

which implies both $f + g, \alpha f \in S_3$.

Therefore, the set S_3 is closed under addition and multiplication.

Hence, S_3 is a subspace of P .

Solution 8(v): Let $S_4 = \{p \in P : p(2) = 1\}$

Since zero polynomial is not in S_4 . Hence S_4 is not a subspace of P .

Solution 8(vi): Let $S_5 = \{p \in P : p'(1) = 0\}$

Clearly, S_5 is non empty since zero polynomial belongs to S_5 .

Let $f(x), g(x) \in S_5$, then $f'(1) = 0 = g'(1)$.

Now, $(f + g)'(1) = f'(1) + g'(1) = 0 + 0 = 0$
and $(\alpha f)'(1) = \alpha f'(1) = \alpha \cdot 0 = 0 \Rightarrow f + g, \alpha f \in S_5$.
 $\Rightarrow S_5$ is closed under addition and scalar multiplication.
Hence, S_5 is a subspace of P .

9. Which of the following subsets of $\mathbb{R}^{2 \times 2}$ are subspaces. Note that, $\mathbb{R}^{m \times n}$ is the vector space over real field of all matrices of order $m \times n$ under usual definitions of addition and scalar multiplication of matrices.

- (i) All diagonal matrices.
- (ii) All upper triangular matrices.
- (iii) All symmetric matrices.
- (iv) All invertible matrices.
- (v) All matrices which commute with a given matrix T .
- (vi) All matrices with zero determinant.

Solution:

(i) Let, \mathcal{D} be the set of all diagonal matrices of order 2 and $D, E \in \mathcal{D}$ be any two elements.

$$D = \begin{bmatrix} d_{11} & 0 \\ 0 & d_{22} \end{bmatrix}, E = \begin{bmatrix} e_{11} & 0 \\ 0 & e_{22} \end{bmatrix}, d_{11}, d_{22}, e_{11}, e_{22} \in \mathbb{R}$$

$$\therefore D + E = \begin{bmatrix} d_{11} + e_{11} & 0 \\ 0 & d_{22} + e_{22} \end{bmatrix}$$

Clearly, $D + E \in \mathcal{D}$.

Also let, $\alpha \in \mathbb{R}$ and $A \in \mathcal{D}$ be any.

Then,

$$A = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}$$

for some $a_{11}, a_{22} \in \mathbb{R}$

$$\therefore \alpha A = \begin{bmatrix} \alpha a_{11} & 0 \\ 0 & \alpha a_{22} \end{bmatrix} \in \mathcal{D}$$

Hence, \mathcal{D} is a vector subspace of $\mathbb{R}^{2 \times 2}$.

(ii) Let, \mathcal{U} be the set of all upper triangular matrices of order 2 and $M, N \in \mathcal{U}$ be any two elements.

$$M = \begin{bmatrix} m_{11} & m_{12} \\ 0 & m_{22} \end{bmatrix}, N = \begin{bmatrix} n_{11} & n_{12} \\ 0 & n_{22} \end{bmatrix}, m_{11}, m_{12}, m_{22}, n_{11}, n_{12}, n_{22} \in \mathbb{R}$$

$$\therefore M+N=\begin{bmatrix} m_{11}+n_{11} & m_{12}+n_{12} \\ 0 & m_{22}+n_{22} \end{bmatrix}$$

Clearly $M+N \in \mathcal{U}$.

Also let, $\alpha \in \mathbb{R}$ and $A \in \mathcal{U}$ be any.

Then,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}$$

for some $a_{11}, a_{12}, a_{22} \in \mathbb{R}$

$$\therefore \alpha A = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} \\ 0 & \alpha a_{22} \end{bmatrix} \in \mathcal{U}$$

Hence, \mathcal{U} is a vector subspace of $\mathbb{R}^{2 \times 2}$.

(iii) Let, \mathcal{S} be the set of all symmetric matrices of order 2 and $A, B \in \mathcal{S}$ be any two elements.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, a_{11}, a_{12}, a_{21}, a_{22}, b_{11}, b_{12}, b_{21}, b_{22} \in \mathbb{R}$$

and $a_{12} = a_{21}, b_{12} = b_{21}$

$$\therefore A+B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}$$

Clearly $A+B \in \mathcal{S}$ as $a_{12}+b_{12} = a_{21}+b_{21}$

Also let, $\alpha \in \mathbb{R}$ and $G \in \mathcal{S}$ be any.

Then,

$$G = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$$

for some $g_{11}, g_{12}, g_{21}, g_{22} \in \mathbb{R}$ and $g_{12} = g_{21}$

$$\therefore \alpha G = \begin{bmatrix} \alpha g_{11} & \alpha g_{12} \\ \alpha g_{21} & \alpha g_{22} \end{bmatrix} \in \mathcal{S} \text{ as } \alpha g_{12} = \alpha g_{21}$$

Hence, \mathcal{S} is a vector subspace of $\mathbb{R}^{2 \times 2}$.

(iv) Let \mathcal{I} be the set of all invertible matrices of order 2.

Then $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ are two members of \mathcal{I} . Now,

$$A+B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

Clearly $A + B \notin \mathcal{I}$

Hence, \mathcal{I} is not a subspace of $\mathbb{R}^{2 \times 2}$.

(v) It is given that T is a fixed matrix of order 2. Let, \mathcal{C} be the set of all order 2 matrices which commutes with T . Since the identity matrix of order 2 commutes with T . So, \mathcal{C} is a non empty subset of $\mathbb{R}^{2 \times 2}$.

Now let, P, Q be any two elements of \mathcal{C} . Hence,

$$TP = PT \quad (1)$$

$$TQ = QT \quad (2)$$

Therefore using (1) and (2) we get,

$$\begin{aligned} T(P + Q) &= TP + TQ \\ &= PT + QT \\ &= (P + Q)T. \end{aligned}$$

Hence, $P + Q \in \mathcal{C}$

Similarly, let $A \in \mathcal{C}$ and $\alpha \in \mathbb{R}$.

$$\begin{aligned} T(\alpha A) &= \alpha(TA) \\ &= \alpha(AT) \\ &= (\alpha A)T. \end{aligned}$$

Hence, $\alpha A \in \mathcal{C}$.

Therefore, \mathcal{C} is a vector subspace of $\mathbb{R}^{2 \times 2}$.

(vi) Let \mathcal{N} be the set of all order 2 matrices with determinant zero.

Then, $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are two members of \mathcal{N} .

$$\det(A + B) = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \neq 0$$

Hence $A + B \notin \mathcal{N}$, showing that \mathcal{N} is not a vector subspace of $\mathbb{R}^{2 \times 2}$.

10. Done in class.

11. Let W_1 and W_2 be subspaces of a vector Space V such that $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$ Show that for each vector u in V there are unique vectors u_1 in W_1 and u_2 in W_2 such that $u = u_1 + u_2$.

Solutions: It is given that $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$.
Let $u \in V$. Then, $u \in V = W_1 + W_2 \Rightarrow u = u_1 + u_2$ for some $u_1 \in W_1, u_2 \in W_2$.

We have to prove the uniqueness. For uniqueness, let $u = w_1 + w_2$ for $w_1 \in W_1$ and $w_2 \in W_2$.

Then $u_1 + u_2 = u = w_1 + w_2$.

$$\Rightarrow u_1 - w_1 = w_2 - u_2 \in W_2.$$

Also, $(u_1 - w_1) \in W_1$. So $(u_1 - w_1) \in W_1 \cap W_2$.

Thus, $(u_1 - w_1) \in \{0\}$. So, $u_1 = w_1$. Similarly we can show, $u_2 = w_2$.

This proves the uniqueness of u_1, u_2 . This completes the proof.

12. Let $S = \{(1, 2, 3), (1, 1, -1), (3, 5, 5)\}$. Determine which of the following are in $L(S)$.

- (a) $(0, 0, 0)$
- (b) $(1, 1, 0)$
- (c) $(4, 5, 0)$
- (d) $(1, -3, 8)$

Solution:- Here we have given set $S = \{(1, 2, 3), (1, 1, -1), (3, 5, 5)\}$.

- (a) If it is in $L(S)$ then, it can be written as

$$a(1, 2, 3) + b(1, 1, -1) + c(3, 5, 5) = (0, 0, 0)$$

Now to find the value of a, b, c we can write it in augmented matrix form as:-

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & 1 & 5 & 0 \\ 3 & -1 & 5 & 0 \end{array} \right]$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$R_3 \rightarrow R_3 - 3R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -4 & -4 & 0 \end{array} \right]$$

$$\begin{aligned} R_2 &\rightarrow (-1)R_2 \\ R_3 &\rightarrow R_3/(-4) \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array} \right]$$

$$R_3 \rightarrow R_3 - R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

So it can be written as :-

$$a + b + 3c = 0, b + c = 0$$

For simplicity let us consider $c = 1$ then $b = -1, a = -2$.

Therefore, $(0, 0, 0)$ is in $L(S)$.

(b) If it is in $L(S)$ then, it can be written as

$$a(1, 2, 3) + b(1, 1, -1) + c(3, 5, 5) = (1, 1, 0)$$

Now to find the value of a, b, c we can write it in augmented matrix form as:-

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 2 & 1 & 5 & 1 \\ 3 & -1 & 5 & 0 \end{array} \right]$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$R_3 \rightarrow R_3 - 3R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & -1 & -1 & -1 \\ 0 & -4 & -4 & -3 \end{array} \right]$$

$$R_2 \rightarrow (-1)R_2$$

$$R_3 \rightarrow (-1)R_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 4 & 4 & 3 \end{array} \right]$$

$$R_3 \rightarrow R_3 - 4R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{array} \right]$$

So it can be written as :-

$$a + b + 3c = 1, b + c = 1,$$

$$\text{and } 0.a + 0.b + 0.c = -1$$

which is impossible.

Therefore, $(1, 1, 0)$ is not in $L(S)$.

(c) If it is in $L(S)$ then, it can be written as

$$a(1, 2, 3) + b(1, 1, -1) + c(3, 5, 5) = (4, 5, 0)$$

Now to find the value of a, b, c we can write it in augmented matrix form as:-

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 2 & 1 & 5 & 5 \\ 3 & -1 & 5 & 0 \end{array} \right]$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$R_3 \rightarrow R_3 - 3R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & -1 & -1 & -3 \\ 0 & -4 & -4 & -12 \end{array} \right]$$

$$R_2 \rightarrow (-1)R_2$$

$$R_3 \rightarrow R_3 / (-4)$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \end{array} \right]$$

$$R_3 \rightarrow R_3 - R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

So it can be written as :-

$$a + b + 3c = 4, b + c = 3$$

For simplicity let us consider $c = 1$ then $b = 2, a = -1$.

Therefore $(4, 5, 0)$ are in $L(S)$.

(d) If it is in $L(S)$ then it can be written as

$$a(1, 2, 3) + b(1, 1, -1) + c(3, 5, 5) = (1, -3, 8)$$

Now to find the value of a, b, c we can write it in augmented matrix form as:-

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 2 & 1 & 5 & -3 \\ 3 & -1 & 5 & 8 \end{array} \right]$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$R_3 \rightarrow R_3 - 3R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & -1 & -1 & -5 \\ 0 & -4 & -4 & 5 \end{array} \right]$$

$$R_2 \rightarrow (-1)R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 5 \\ 0 & -4 & -4 & 5 \end{array} \right]$$

$$R_3 \rightarrow R_3 + 4R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 25 \end{array} \right]$$

So it can be written as :-

$$a + b + 3c = 1, b + c = 5,$$

$$\text{and, } 0.a + 0.b + 0.c = 25$$

Which is impossible.

Therefore $(1, -3, 8)$ is not in $L(S)$.

13. In the complex vector space \mathbb{C}^2 , determine whether are not $(1 + i, 1 - i) \in L[(1 + i, 1), (1, 1 - i)]$.

Solution- Here, given vector space is complex vector space i.e. $\mathbb{C}^2(\mathbb{C})$.

Let $a, b \in \mathbb{C}$ such that

$$a(1 + i, 1) + b(1, 1 - i) = (1 + i, 1 - i),$$

Now, to find the value of a, b we will use Gauss elimination method.

$$\left(\begin{array}{cc|c} 1+i & 1 & 1+i \\ 1 & 1-i & 1-i \end{array} \right) \sim \left(\begin{array}{cc|c} 1 & 1-i & 1-i \\ 1+i & 1 & 1+i \end{array} \right) \sim \left(\begin{array}{cc|c} 1 & 1-i & 1-i \\ 0 & -1 & -1+i \end{array} \right)$$

$$a + (1 - i)b = 1 - i,$$

$$-b = -1 + i,$$

using back substitution, we get,

$$a = 1 + i \quad \text{and} \quad b = 1 - i.$$

So, $(1 + i, 1 - i) \in L[(1 + i, 1), (1, 1 - i)]$.

14. Let M and N be subsets of the vector space $(V, +, \cdot)$. Define $M + N = \{m + n : m \in M \text{ and } n \in N\}$. Then

- (a) $M \subset N \implies L[M] \subset L[N]$
- (b) M is subspace of $V \iff L[M] = M$
- (c) $L[L[M]] = L[M]$.

Solution- We know that if M is any subset of vector space $(V, +, \cdot)$. Then $L[M]$ is the smallest subspace of V that contain M .

(proof: It is easy to prove that $L[M]$ is subspace of $(V, +, \cdot)$. Now let W be any arbitrary subspace that contains M . Now we have to prove that $L[M] \subset W$. Let $x \in L[M]$, Then $\exists m_1, m_2, \dots, m_n \in M$ and $a_1, a_2, \dots, a_n \in F$ such that

$$x = \sum_{i=1}^n a_i m_i$$

$$\implies x \in W \text{ (by properties of subspace } W)$$

$$\implies L[M] \subset W$$

Hence, $L[M]$ is the smallest subspace of V that contain M .)

Solution(a)- Here $M \subset N$ So, $M \subset N \subset L[N]$. But $L[M]$ is the smallest subset that contain M . So, $L[M] \subset L[N]$.

Solution(b)- Let M is subspace of V . Since $L[M]$ is the smallest subset that contain M . So, $L[M] = M$. Conversely assume that $L[M] = M$ and we know that $L[M]$ is subspace of V . So, M will be subspace of V .

Solution(c)- we know that $L[M]$ is subspace of V . by (b) we have $L[L[M]] = L[M]$.