MA-102
B. Tech. II Sem (2021-2022)
Tutorial sheet-03

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1. **(i)** Check the linear dependence or linear independence of the following sets in respective real vector spaces

   (a) (a) $\left\{e^{x}, e^{2x}\right\}$ in $\mathcal{C}^{\infty}(\mathbb{R})$.
   Solution-
   Let $a, b \in \mathbb{R} \quad$ s.t $\quad a e^{x}+b e^{2x}=0$
   Put $x=0, x=1$ in above equation we get
   $a+b=0$ i.e. $b=-a$
   and $a e+b e^{2}=0$
   Put $b=-a$ in last equation we get $a e-a e^{2}=0$.
   i.e. $a\left(e-e^{2}\right)=0 \Rightarrow \quad a=0 \quad$ Since $e \neq e^{2}$.
   Finally we get $a=0, b=0$
   $\Rightarrow$ Given set is linearly independent.

   (b) $\{x,|x|\}$ in $\mathcal{C}[-1,1]$.
   Solution-
   Let $a, b \in \mathbb{R}$ s.t. $a x+b|x|=0$
   (i) if $x \in[-1,0]$ then $y=-x$ where $y \in \mathbb{N}$
   Above equation becomes

$$
\begin{aligned}
-a y+b y & =0, \\
(a-b) y=0 & \Rightarrow a=b
\end{aligned}
$$

   (ii) if $x \in[0,1]$ then above equation becomes $a x+b x=0$
   i.e. $(a+b) x=0 \Rightarrow a=-b$
   From both cases we have $a=b$ and $a=-b$
   $\Rightarrow a=b=0$
   $\Rightarrow$ Given set is linearly independent.

   (c) $\left\{\left(\frac{1}{2}, \frac{1}{3}, 1\right),(-3,1,0),(1,2,-3)\right\}$ in $\mathbb{R}^{3}$.
   Solution-
   Let $a, b, c \in \mathbb{R}$. s.t.
   $a\left(\frac{1}{2}, \frac{1}{3}, 1\right)+b(-3,1,0)+c(1,2,-3)=0=(0,0,0)$
   $\left(\frac{a}{2}-3 b+c, \frac{a}{3}+b+2 c, a-3 c\right)=0$.
   $\Rightarrow \quad \frac{a}{2}-3 b+c=0$ i.e. $a-6 b+2 c=0$.

$\frac{a}{3} + b + 2c = 0$ i.e. $a + 3b + 6c = 0$
& $\quad a - 3c = 0.$
put $a = 3c$ in above both equations we get
$-6b + 5c = 0$ and $3b + 9c = 0$
$\Rightarrow b = \frac{5}{6}c$ put this value in last equation we get
$3 \cdot \frac{5}{6}c + 9c = 0 \Rightarrow \frac{23}{2}c = 0 \Rightarrow \quad c = 0$
$\Rightarrow \quad b = 0.$ & $a = 0.$
$\Rightarrow$ Given set is L.I.

(d) $\{(1,1,1,0),(3,2,2,1),(1,1,3,-2),(1,2,6,-5)\}$ in $\mathbb{R}^4$.
Solution-
Let a, b, c, d in $\mathbb{R}$ s.t.
$a(1,1,1,0) + b(3,2,2,1) + c(1,1,3,-2) + d(1,2,6,-5) = 0$

$\big(\; a + 3b + c + d, a + 2b + c + 2d, a + 2b + 3c + dd, b - 2c - 5d \; \big) = 0.$
We have

$$\begin{aligned} a + 3b + c + d &= 0 \\ a + 2b + c + 2d &= 0 \\ a + 2b + 3c + 6d &= 0 \\ b - 2c - 5d &= 0 \end{aligned}$$

Above system of linear equations can be written as

$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 2 & 3 & 6 \\ 0 & 1 & -2 & -5 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We will solve it by Gauss Ellimination method. The augmented matrix can be written as

$$\left[\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 0 \\ 1 & 2 & 1 & 2 & 0 \\ 1 & 2 & 3 & 6 & 0 \\ 0 & 1 & -2 & -5 & 0 \end{array}\right]$$

Apply $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ we get

$$\left[\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 5 & 0 \\ 0 & 1 & -2 & -5 & 0 \end{array}\right]$$

Again apply $R_3 \longrightarrow R_3 - R_2$ and $R_4 \longrightarrow R_4 + R_3$

$$\left[\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Above augmented matrix can be written as

$a + 3b + c + d = 0$

$-b + d = 0 \Rightarrow b = d.$

$2c + 4d = 0 \quad \Rightarrow \quad c = -2d.$

$\Rightarrow a + 3d - 2d + d = 0 \Rightarrow a = -2d.$

$\Rightarrow$ The value of a, b & c is depend on d. & d is arbitrary real number.

$\Rightarrow$ Given set is not linearly independent.

(e) $\left\{\left(x, x^3 - x, x^4 + x^2, x + x^2 + x^4 + \frac{1}{2}\right\}\right.$ in $\mathcal{P}_4$.

Solution-

Let $a, b, c, d \in \mathbb{R}$ s.t.

$$ax + b\left(x^3 - x\right) + c\left(x^4 + x^2\right) + d\left(x + x^2 + x^4 + \tfrac{1}{2}\right) = 0.$$

$$(a - b + d)x + (c + d)x^2 + bx^3 + (c + d)x^4 + \frac{d}{2} = 0$$

$$\frac{d}{2} + (a - b + d)x + (c + d)x^2 + bx^3 + (c + d)x^4 = 0.$$

$$\Rightarrow \frac{d}{2} = 0 \Rightarrow d = 0.$$

$$a - b + d = 0.$$

$$c + d = 0.$$

$$b = 0 \quad \Rightarrow a = 0, c = 0$$

$$\Rightarrow \quad a = b = c = d = 0$$

$\Rightarrow$ Given set is L.I.

1(ii) Show that the set S $= \{\sin x, \sin 2x, \cdots, \sin nx\}$ is a LI subset of $\mathcal{C}[-\pi, \pi]$ for every positive integer n.

Solution-

Let $a_1, a_2, a_3, a_4, \ldots a_n, \in \mathbb{R}$ s.t.

$$a_1 \sin x + a_2 \sin 2x + \cdots + a_n \sin nx = 0$$

.

we can write it as $\sum_{i=1}^{n} a_i \sin ix = 0$...(1)

As we know identity $\int_{-\pi}^{\pi} \sin mx \cdot \sin nx dx = \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases}$

Multiply $\sin kx$ in equation (1) we get.

$$\sum_{i=1}^{m} a_i \sin kx \cdot \sin ix = 0$$

Integrate both side from $-\pi$ to π we get

$$\sum_{i=1}^{n} a_i \int_{-\pi}^{\pi} \sin kx \cdot \sin ix dx = 0$$

we get $a_k \pi = 0 \Rightarrow a_k = 0$, where $1 \leq k \leq n \Rightarrow a_1 = a_2 = \cdots = a_n = 0$
$\Rightarrow$ The set S $= \{\sin x, \sin 2x, \cdots, \sin nx\}$ is a LI subset of $\mathcal{C}[-\pi, \pi]$ for every positive integer n.

2. **(i)** If u, v, w are LI vectors of a vector space V, then prove that $u + v, v + w,$ and$w + u$ are also LI.

Solution:

To prove that $u + v, v + w, w + u$ are L.I.

Let for scalars c_1, c_2, c_3

$$c_1 \cdot (u + v) + c_2 \cdot (v + w) + c_3 \cdot (w + u) = 0.$$

$\Rightarrow \quad (c_1 + c_3) \cdot u + (c_1 + c_2) \cdot v + (c_2 + c_3) \cdot w = 0.$

Since u, v, w are L.I, So we have

$$c_1 + c_3 = 0.....(1)$$
$$c_1 + c_2 = 0....(2)$$
$$c_2 + c_3 = 0.....(3)$$

Adding (1), (2), (3) we get, $c_1 + c_2 + c_3 = 0...$ (4).

Subtracting (1), (2), (3) from (4) we get, $c_2 = 0, \quad c_3 = 0, \quad c_1 = 0$ respectively.

$\therefore \quad u + v, v + w$ and $w + u$ are L.I.

(ii) Let S_1 and S_2 be subsets of a vector space V such that $S_1 \subset S_2$. Then prove that

(a) S_1 is LD $\implies$ S_2 is LD.

(b) S_2 is LI $\implies$ S_1 is LI.

Solution:(a)

Let S_1 be LD.Then there is a finite subset P of S_1 which is LD.Now,

$S_1 \subset S_2$ and $P \subset S_1 \implies P \subset S_2$ and P being a finite subset of S_2 and since P is LD so,

S_2 is also LD.

Solution:(b)

Let t_i $(i = 1, 2, ..., n)$ be vectors in S_1 such that $a_1t_1 + a_2t_2 + \ldots + a_nt_n = 0$ for scalars $a_i(i = 1, 2, ..., n)$.Now $t_i \subset S_1$ for all $i = 1, 2, ..., n$ and $S_1 \subset S_2 \implies t_i \subset S_2$ for all $i = 1, 2, \cdots, n$.Now S_2 is LI and thus $a_1t_1 + a_2t_2 + \ldots + a_nt_n = 0$ for scalars $a_i(i = 1, 2, \cdots, n)$ gives $a_i = 0$ for all $i = 1, 2, \cdots, n$. There fore S_1 is LI.

(iii) Let S be LI subset of a vector space V.Let $v \in L[S]. Then \ \{v\} \cup S \ is \ LD.$

Solution:

Since $v \in L[S]$, $v = a_1t_1 + a_2t_2 + \ldots + a_nt_n$ for scalars $a_i(i = 1, 2, ..., n)$ not all zero and vectors $t_i \in S$ $(i = 1, 2, ..., n)$ which gives $v - a_1t_1 - a_2t_2 - \ldots - a_nt_n = 0$ for scalars $1, -a_i(i = 1, 2, ..., n)$ not all zero. This gives {v,t_1,t_2,...,t_n}is LD, which is a subset of $\{v\} \cup S$.Hence $\{v\} \cup S$ is LD(by using (a)).

(iv) Let S be LI subset of a vector space V.Let $v \notin L[S]. Then \ \{v\} \cup S \ is \ LI.$

Solution:

Let $A := \{t_1, t_2, \ldots, t_n\}$ be any finite subset of $\{v\} \cup S$. Then there are two possible cases.

Case(i) $v \notin A$. Then $A \subset S$. S being LI, A is LI.(by using (b))

Case(ii) $v \in A$. without loss of generality,let $v = t_1$.
Let $c_1 \cdot v + c_2 \cdot t_2 + \cdots + c_n \cdot t_n = 0$ for Scalars $c_i(1 \leqslant i \leqslant n)$.....(1)

Let $c_1 \neq 0$. Then $v = -c_1^{-1}c_2t_2 - \cdots - c_1^{-1}c_nt_n$

So $v \in L[s]$, a contradiction.

Hence we must have $c_1 = 0$.

So (1) becomes, $\quad c_2 \cdot t_2 + \cdots + c_n \cdot t_n = 0$.......(2)

S being $LI, \{t_2, \ldots, t_n\}$ is L.I.

Hence (2) gives, $c_i = 0 \quad (2 \leq i \leq n)$.

Then Combining all we get, $c_i = 0 \quad \forall 1 \leq i \leq n$. Hence A is LI.

So in both Cases, A is LI.Thus any finite subset of $\{v\} \cup S$ is LI.Hence $\{v\} \cup S$ is LI.

3. **(i)** In a vector space V, If a **ordered** set $S = \{v_1, v_2, v_3, ..., v_n\}$ is LD with $v_1 \neq 0$ then prove that $\exists$ a vector v_k, $2 \leq k \leq n$ such that $v_k \in L[\{v_1, v_2, ..., v_{k-1}\}]$.
Solution:
Given that set $S = \{v_1, v_2, v_3, ..., v_n\}$ is LD with $v_1 \neq 0$. Then there exist $a_1, a_2, a_3, ..., a_n$ such that $a_i \neq 0$ for some $i \in \{1, 2, ..., n\}$.
Let k be the largest index such that $a_k \neq 0$ then

$$
\begin{aligned}
&a_1v_1 + a_2v_2 + \ldots + a_kv_k = 0 \\
\Rightarrow &v_k = -(a_k^{-1}a_1v_1 + a_k^{-1}a_2v_2 \ldots a_k^{-1}a_{k-1}v_{k-1}) \\
\Rightarrow &v_k \in L[\{v_1, v_2, ..., v_{k-1}\}]
\end{aligned}
$$

(ii) In a vector space V, If a set $S = \{v_1, v_2, v_3, ..., v_n\}$ is LI and $S_1 = \{w_1, w_2, ..., w_m\}$ generates the space V then prove that $n \leq m$.
Solution:
Since $\{w_1, \ldots w_m\}$ spans V, every v_i is a linear combination of $\{w_1, \ldots, w_m\}$.
So there are real numbers c_{ij} such that

$$v_1 = c_{11}w_1 + c_{12}w_2 + \ldots + c_{1m}w_m$$

$$v_2 = c_{21}w_1 + c_{22}w_2 + \ldots + c_{2m}w_m$$

$$v_n = c_{n1}w_1 + c_{n2}w_2 + \ldots + c_{nm}w_m$$

Suppose $m < n$ then the following system of homogeous equations has a non trivial solution:

$$\begin{aligned} c_{11}x_1 + c_{21}x_2 + \ldots + c_{n1}x_n &= 0 \\ c_{12}x_1 + c_{22}x_2 + \ldots + c_{n2}x_n &= 0 \\ \ldots\ldots \\ c_{1m}x_1 + c_{2m}x_2 + \ldots + c_{nm}x_n &= 0 \end{aligned}$$

since there are more variables and less equations. Let $x_1 = r_1, \ldots, x_n = r_n$ be a nontrivial solution. Then

$$
\begin{gathered}
r_1 v_1 + \ldots r_n v_n = \\
r_1 c_{11} w_1 + r_1 c_{12} w_2 + \ldots + r_1 c_{1m} w_m + \\
r_2 c_{21} w_1 + r_2 c_{22} w_2 + \ldots + r_2 c_{2m} w_m + \\
r_n c_{n1} w_1 + r_n c_{n2} w_2 + \ldots + r_n c_{nm} w_m = \\
\left(c_{11} r_1 + c_{21} r_2 + \ldots + c_{n1} r_n\right) w_1 + \\
\left(c_{12} r_1 + c_{22} r_2 + \ldots + c_{n2} r_n\right) w_2 + \\
\left(c_{1m} r_1 + c_{2m} r_2 + \ldots + c_{nm} r_n\right) w_m \\
= 0
\end{gathered}
$$

This contradicts the linear independence of the set $\{v_1, \ldots, v_n\}$, so $n \leq m$

4. Determine whether the following sets are bases for given vector spaces V over field F

THEOREM: Let V be a vector space of finite dimension n. Then:

(a) Any $n+1$ or more vectors in V are linearly dependent.

(b) Any linearly independent set $S = \{u_1, u_2, \cdots u_n\}$ with n elements is a basis of$V$.

(c) Any spanning set $T = \{v_1, v_2, \cdots v_n\}$ of V with n elements is a basis of V.

(i) $\{(2,4,0),(0,2,-2)\}$; $V = \mathbb{R}^3$ and $F = \mathbb{R}$.
Here, dim(V)=3 but the given set has only two elements which means it cannot form bases elements of V.

(ii) $\{(6,4,4),(-2,4,2),(0,7,0)\}$; $V = \mathbb{R}^3$ and $F = \mathbb{R}$.
Let $a, b, c \in \mathbb{R}$ such that

$$
\begin{aligned}
a(6,4,4) + b(-2,4,2) + c(0,7,0) &= (0,0,0) \\
\Longrightarrow \quad (6a - 2b, 4a + 4c + 7c, 4a + 2b) &= (0,0,0) \\
6a - 2b &= 0 \\
4a + 4c + 7c &= 0 \\
4a + 2b &= 0
\end{aligned}
$$

Solving above three equation, we get,

$$a = 0,\ b = 0,\ c = 0.$$

Thus, the given set of 3 vectors are linearly independent (L.I.). Hence, it forms basis for $V = \mathbb{R}^3$.

(iii) $\left\{ \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix} \right\}$; $V = M_{2\times 2}$ and $F = \mathbb{R}$.
Let $a, b, c, d \in \mathbb{R}$ such that

$$a \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 0 \\ 2 & 2 \end{pmatrix} c \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + d \begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} a+c & -a-d \\ 2b-d & 2b+c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Comparing elements in ij^{th} positions, we get

$$\begin{aligned} a + c &= 0, \\ -a - d &= 0, \\ 2b - d &= 0, \\ 2b + c &= 0, \end{aligned}$$

solving above equation, we get

$$a = b = c = d = 0.$$

Thus, given set has L.I. elements and it forms basis for vector space $V = M_{2\times 2}$.

(iv) $\{1, x-2, (x-2)^2, (x-2)^3\}$; $V = P_3$ and $F = \mathbb{R}$.
Let $a, b, c, d \in \mathbb{R}$ such that

$$\begin{aligned} a.1 + b.(x-2) + c.(x-2)^2 + d.(x-2)^3 &= 0, \\ \Longrightarrow\ a + b(x-2) + c(x^2 - 4x + 4) + d(x^3 - 6x^2 + 12x - 8) &= 0, \end{aligned}$$

comparing the coefficients and solve it, we get

$$\begin{aligned} a - 2b + 4c - 8d &= 0, \implies a = 0 \\ b - 4c + 12d &= 0, \implies b = 0 \\ c - 4d &= 0, \implies c = 0 \\ -8d &= 0, \implies d = 0 \end{aligned}$$

Since, vector space V contains polynomials with degree ≤ 3 and $\dim(V)$=4. Here, the elements in given set are L.I. and also forms the basis for V.

(v) $\{x-1, x^2+x-1, x^2-x+1\}$; $V = P_2$ and $F = \mathbb{R}$.
Let $a, b, c \in \mathbb{R}$ such that

$$\begin{aligned} a(x-1) + b(x^2+x-1) + c(x^2-x+1) = 0, \\ \implies -a-b+c = 0 \\ a+b-c = 0 \\ b+c = 0 \end{aligned}$$

Solve all three equation using Gauss elimination method,

$$\begin{pmatrix} -1 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} -1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Since, it has zero rows so the corresponding system will have non-trivial solutions. Thus, elements in the given set are L.D.
Therefore, given set does not forms basis for $V = P_2$.

(vi) $\{(1, i, 1+i), (1, i, 1-i), (i, -i, 1)\}$; $V = \mathbb{C}^3$ and $F = \mathbb{C}$.
Let $a, b, c \in \mathbb{C}$ such that

$$a(1, i, 1+i) + b(1, i, 1-i) + c(i, -i, 1) = 0,$$

To find the value of a, b & c by using Gauss elimination method, we get

$$\begin{pmatrix} 1 & i & 1+i \\ 1 & i & 1-i \\ i & -i & 1 \end{pmatrix} \to \begin{pmatrix} 1 & i & 1+i \\ 0 & 0 & -2i \\ 0 & -1-i & 2-i \end{pmatrix} \to \begin{pmatrix} 1 & i & 1+i \\ 0 & -1-i & 2-i \\ 0 & 0 & -2i \end{pmatrix}$$

Thus,

$$\begin{aligned} a + ib + (1+i)c = 0 \\ (-1-i)b + (2-i)c = 0 \\ -2ic = 0 \end{aligned}$$

$$\implies a = b = c = 0.$$

Thus, the elements in the given sets are L.I. and forms basis for V.

6. Find a basis for the plane $P : x - 2y + 3z = 0$ in $\mathbb{R}^3$. Find a basis for the intersection of P with the xy-plane. Also, find a basis for the space of vectors perpendicular to the plane P.

Solution:

Let us write our plane P as set

$$P = \{(a, b, c) \in \mathbb{R}^3 \mid a - 2b + 3c = 0\}$$

in P, we can write $a - 2b + 3c = 0$ as $a = 2b - 3c$.
So a general element $(a, b, c) \in P$ can be written as

$$\begin{aligned}(2b - 3c, b, c) &= (2b, b, 0) + (-3c, 0, c) \\ &= b(2, 1, 0) + c(-3, 0, 1)\end{aligned}$$

Since $b, c \in \mathbb{R}$, so any element $(a, b, c) \in P$ can be written as an element of $L[\{(2, 1, 0), (-3, 0, 1)\}]$ and vice versa, so

$$P = L[\{(2, 1, 0), (-3, 0, 1)\}]$$

Now we show that set $\{(2, 1, 0), (-3, 0, 1)\}$ is linearly independent. Let $a_1, a_2 \in \mathbb{R}$ such that

$$\begin{aligned} a_1(2, 1, 0) + a_2(-3, 0, 1) \quad &= (0, 0, 0) \\ or, \quad (2a_1 - 3a_2, a_1, a_2) \quad &= (0, 0, 0)\end{aligned}$$

which implies $a_1 = a_2 = 0$, so $\{(2, 1, 0), (-3, 0, 1)\}$ is linearly independent and spans P.
That means set $\{(2, 1, 0), (-3, 0, 1)\}$ is a basis of plane P.

Now let us consider

$$\begin{aligned}(Plane\ P) \quad P &= \{(x, y, z) \in \mathbb{R}^3 \mid x - 2y + 3z = 0\} \\ (xy - plane) \quad Q &= \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\} \\ P \cap Q &= \{(x, y, z) \in \mathbb{R}^3 \mid x - 2y + 3z = 0 \ and \ z = 0\}\end{aligned}$$

We can see that intersection of above two planes is nothing but the line $x = 2y$ in $xy-$plane passing through origin.
Let us take $(a, b, 0) \in P \cap Q$, and since $a = 2b$, we have

$$(a, b, 0) = (2b, b, 0) = b(2, 1, 0)$$

Since $b \in \mathbb{R}$, so $P \cap Q = L[\{(2, 1, 0)\}]$ by previous argument and a singleton set with nonzero element is always linearly independent.

So basis of $P \cap Q$ is $\{(2,1,0)\}$.

As we know direction ratios of vectors perpendicular to plane P are $< 1, -2, 3 >$.
Since plane P contains $(0,0,0)$ so we can write equation of line L perpendicular to plane P and passing through origin, that is,

$$L : \frac{x-0}{1} = \frac{y-0}{-2} = \frac{z-0}{3}.$$

Let us write elements of line L in parametric form as

$$\frac{x-0}{1} = \frac{y-0}{-2} = \frac{z-0}{3} = t$$

$$x = t, \ y = -2t, \ z = 3t$$

Where $t \in \mathbb{R}$, so

$$\begin{aligned} L &= \{(t, -2t, 3t) \in \mathbb{R}^3\} \\ &= \{t(1, -2, 3) \mid t \in \mathbb{R}\} \end{aligned}$$

Now by earlier arguments it is easy to see that $\{(1, -2, 3)\}$ is basis of L.

7(i). Let S $= \{(4,5,6), (a,2,4), (4,3,2)\}$ be a set in $\mathbb{R}^3$. Find the values for a such that $L[\text{ S}] \neq \mathbb{R}^3$.
Solution: If the vectors of S are linearly dependent then $L[S] \neq \mathbb{R}^3$. Therefore, we need check the coefficient matrix

$$[A] = \begin{bmatrix} 4 & a & 4 \\ 5 & 2 & 3 \\ 6 & 4 & 2 \end{bmatrix}$$

If A is singular then $L[S] \neq \mathbb{R}^3$. Therefore,

$$\begin{aligned} \begin{vmatrix} 4 & a & 4 \\ 5 & 2 & 3 \\ 6 & 4 & 2 \end{vmatrix} &= 0 \\ 4(4-12) - a(10-18) + 4(20-12) &= 0 \\ a &= 0 \end{aligned}$$

Therefore, $L[S] \neq \mathbb{R}^3$ if $a = 0$.

7 (ii). For what values of k vectors S $= \{(k+1, -k, k), (2k, 2k-1, k+2), (-2k, k, -k)\}$ form a basis of $\mathbb{R}^3$.

Solution: Similarly as above if S is a can not be a basis if S is linearly dependent.

Now, S is linearly dependent if the coefficient matrix A is singular.

The coefficient matrix is

$$[A] = \begin{bmatrix} k+1 & 2k & -2k \\ -k & 2k-1 & k \\ k & k+2 & -k \end{bmatrix}$$

If $det(A) = 0$ then A is singular. Therefore,

$$\begin{vmatrix} k+1 & 2k & -2k \\ -k & 2k-1 & k \\ k & k+2 & -k \end{vmatrix} = 0$$

$$\begin{vmatrix} k+1 & 0 & -2k \\ -k & 3k-1 & k \\ k & 2 & -k \end{vmatrix} = 0$$

$$\begin{vmatrix} k+1 & 0 & -2k \\ -k & 3k-1 & k \\ 0 & 3k+1 & 0 \end{vmatrix} = 0$$

$$-(3k+1)(k-k^2) = 0$$

$$k = 0, 1, -\frac{1}{3}$$

Therefore S is a basis if $k \neq \{0, 1, -\frac{1}{3}\}$