(Linear Transformation)

1(i). Find a LT $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(1,0) = (1,1)$ and $T(1,1) = (-1,2)$. Also prove that T maps square with vertices at $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$ into a parallelogram.

1(ii). If possible, find a LT $T : A \to B$ such that
(a) $T(2,3) = (4,5)$, $T(1,0) = (0,0)$, where $A = \mathbb{R}^2$ and $B = \mathbb{R}^2$.

(b) $T(1,1) = (1,0,1)$, $T(0,1) = (1,0,0)$, $T(1,2) = (2,1,1)$ where $A = \mathbb{R}^2$ and $B = \mathbb{R}^3$.

(c) $T(1,0,0) = (2,3)$, $T(0,1,0) = (1,2)$, $T(0,0,1) = (-1,-4)$ where $A = \mathbb{R}^3$ and $B = \mathbb{R}^2$.

(d) $T(1,1,0) = (0,1,1)$, $T(0,0,0) = (0,0,1)$, $T(1,0,1) = (0,0,0)$ where $A = B = \mathbb{R}^3$.

2(i). Find a LT $T : \mathbb{R}^3 \to \mathbb{R}^3$, whose range is spanned by the vectors $(1,0,-1)$ and $(1,2,2)$.

2(ii). Find a nonzero LT $T : \mathbb{R}^2 \to \mathbb{R}^2$, which maps all the vectors on the line $y = x$ onto the origin.

3. Find the range and null space of followings LTs. Also find the rank and nullity wherever applicable:
(i) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x_1, x_2) = (3x_1 + x_2, 0, 0)$.

(ii) $T : \mathbb{R}^4 \to \mathbb{R}^3$ defined by $T(x_1, x_2, x_3, x_4) = (x_1 - x_4, x_2 + x_3, x_3 - x_4)$.

(iii) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1, x_2) = (x_1 + x_2, x_1 + x_2)$.

(iv) $T : \mathcal{P}_3 \to \mathbb{R}^3$ defined by $T(a_0 + a_1x + a_2x^2 + a_3x^3) = (a_0 + a_1 + 2a_3, 2a_1 + a_2, a_3 + a_1)$.

(v) $T : \mathcal{C}(0,1) \to \mathcal{C}(0,1)$ defined by $T(f)x = f(x)\sin x$.

4. Examine whether the following transformations are linear or not. In case of LT, find their matrix representation with respect to given bases B_1 and B_2.
(i) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1, x_2) = (x_1 + x_2, x_2)$; B_1 and B_2 are standard bases.

(ii) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x_1, x_2) = (x_1, x_1 + x_2, x_2)$; B_1 and B_2 are standard bases.

(iii) $T : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $T(x_1 + ix_2, x_3 + ix_4) = (x_1, x_2)$; $B_1 = \{(0,1), (1,1)\}$ and B_2 is standard bases.

(iv) $T : \mathcal{P}_2 \to \mathcal{P}_2$ defined by $T(a_0 + a_1x + a_2x^2) = -a_0 + 2a_1x + (a_2 + a_0)x^2$; B_1 and B_2 are standard bases.

(v) $T : \mathcal{P}_3 \to \mathcal{P}_3$ defined by $T(a_0 + a_1x + a_2x^2 + a_3x^3) = a_0 + a_1(x+1) + a_2(x+1)^2 + a_3(x+1)^3$; $B_2 = \{1,\, 1+x,\, 1+x^2,\, 1+x^3\}$ and B_1 is standard basis.

(vi) $T : \mathcal{P}_2 \to \mathcal{P}_3$ defined by $T(p(x)) = xp(x) + \int_0^x p(t)$; B_1 and B_2 are standard bases.

(vii) $T : \mathcal{P}_2 \to \mathbb{R}^4$ defined by $T(a_0 + a_1x + a_2x^2) = (a_0 + a_2, a_1 - a_0, a_2 - a_1, a_0)$; $B_1 = \{1; 1+x; x+x^2\}$ and $B_2 = \{(1,0,1,0); (1,0,0,0); (0,1,-1,0); (0,0,1,1)\}$.

(viii) $T : \mathbb{R}^{2\times2} \to \mathbb{R}^{2\times2}$ defined by $T(A) = AM, \forall A \in \mathbb{R}^{2\times2}$, where $M = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ is a fixed matrix in $\mathbb{R}^{2\times2}$; B_1 and B_2 are standard bases.

(ix) Repeat part (viii), when $T : \mathbb{R}^{2\times2} \to \mathbb{R}^{2\times2}$ is defined by $T(A) = A + M$.

5. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x_1, x_2, x_3) = (x_1 + x_2, x_1 + 2x_2, 3x_3 + x_2)$. Show that T is invertible and further, find a formula for T^{-1}. Match the result by matrix representation also.

6(i). Find a LT $T : \mathbb{R}^3 \to \mathbb{R}^3$, whose matrix representation is $\begin{bmatrix} 2 & 0 & 0 \\ 2 & -5 & 0 \\ 0 & 2 & 1 \end{bmatrix}$, with respect to standard bases. Find its inverse matrix also.

6(ii). Find a LT $T : \mathbb{R}^3 \to \mathbb{R}^3$, whose matrix representation is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ 1 & 0 & -1 \end{bmatrix}$, with respect to standard bases. Find the matrix of T with respect to basis $\{(1, 1, -1), (1, 2, 0), (1, 0, 1)\}$.

6(iii). Find a LT $T : \mathcal{P}_3 \to \mathbb{R}^3$, whose matrix representation is $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 1 \\ 5 & 4 & 1 & -1 \end{bmatrix}$, with respect to $\{1;\, 1 + x^2;\, x + x^3;\, 1 + x + x^2\}$ and $\{(1, 0, 1), (2, 4, 5), (0, 0, 1)\}$.