Transforms and Boundary Value Problems
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Introduction

Differential Equation applications have significance in both academic and real life. An equation denotes the relation between two quantity or two functions or two variables or set of variables or between two functions. Differential equation denotes the relationship between a function and its derivatives, with some set of formulas. There are many examples, which signifies the use of these equations. The functions are the one which denotes some sort of operation performed and the rate of change during the performance is the derivative of that operation, and the relation between them is the differential equation. These equations are represented in the form of order of the degree, such as first order, second order, etc. Its applications are common to find in the field of various engineering and science disciplines. We indicate a few such problems in the following list:

- The problem of determining the motion of a projectile, rocket, satellite, or planet.
- The problem of determining the charge or current in an electric circuit.
- The problem of the conduction of heat in a rod or in a slab.
- The rate of growth of a population.
- The problem of determining the vibrations of a wire or a membrane.
- The study of the reactions of chemicals.

What is Partial Differential Equation(PDE)?

Definition: A differential equation involving partial derivatives of one or more dependent variable with respect to more than one independent variable is called a partial differential equation.

If we consider z as a dependent variable which depends on the two independent variables x and y i.e., z = f (x,y), then the following standard symbols are universally used:

$\frac{\partial z}{\partial x}$ = p = partial derivative of z w.r. to x

$\frac{\partial z}{\partial y}$ = q = partial derivative of z w.r. to y

$\frac{\partial^2 z}{\partial x^2} = r, \quad \frac{\partial^2 z}{\partial x \partial y} = s \quad \text{and} \quad \frac{\partial^2 z}{\partial y^2} = t$

Examples:

1. $x^2 p + y^2 q = z$ i.e., $x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = z$
2. $r + 3s + t = 0$ i.e., $\frac{\partial^2 z}{\partial x^2} + 3 \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$

Order and Degree of a PDE

Order: The order of the highest ordered derivative involved in a PDE is called the order of the PDE.

Degree: The greatest power of the highest ordered derivative involved in a PDE is called the degree of the PDE.

The degree of the PDE is the greatest power of its highest derivative occurring in it.

Examples:

(i) $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1,$ order=1, degree=1.

(ii) $\left(\frac{\partial z}{\partial x}\right)^2 + 2\left(\frac{\partial^2 z}{\partial x \partial y}\right) + \left(\frac{\partial z}{\partial y}\right)^2 = 0,$ order=2, degree=1.

(iii) $\left(\frac{\partial^2 z}{\partial x^2}\right)^{\frac{3}{2}} = \frac{\partial z}{\partial x} \Rightarrow \left(\frac{\partial^2 z}{\partial x^2}\right)^3 = \left(\frac{\partial z}{\partial x}\right)^2,$ order=2, degree=3.

Classification of First Order PDE

Linear PDE

A first order PDE is said to be linear if
i. Linear in p, q and z
ii. The coefficients of p, q and z depending on the independent variables x and y only.
i.e., a first order linear PDE can be expressed in the form
$A(x, y)p + B(x, y)q + C(x, y)z = D(x, y)$
Example: $yp - xq = xyz + x$

Semilinear PDE

A first order PDE is said to be semilinear if
i. Linear in the leading (highest order) terms i.e., p and q.
ii. It need not be linear in z.
iii. The coefficients of p and q are functions of the independent variables x and y only.
i.e., a first order semilinear PDE can be expressed in the form
$A(x, y)p + B(x, y)q = C(x, y, z)$
Example: $yp + xq = xyz^2 + x^2y$

Classification of First Order PDE

Quasi-linear PDE

A first order PDE is said to be quasi-linear if
i. Linear in p and q.
ii. The coefficients of p and q are functions of the independent variables x and y as well as dependent variable z.
i.e., a first order quasi-linear PDE can be expressed in the form

$$A(x,y,z)p + B(x,y,z)q = C(x,y,z)$$

Example: $(x^2+z^2)p - xyq = xz^3 + y^2$

Nonlinear PDE

A first order PDE is said to be nonlinear if it is not linear PDE.

Example: $p^2 + q^2 = 1$

Formation of PDEs

Let

$$f(x,\ y,\ z,\ a,\ b) = 0 \tag{1}$$

be the given function, where $z = z(x, y)$ is the dependent variable depends on x and y, a and b are arbitrary constants. Then we can form a PDE from equation (1) in two ways:

- Elimination of arbitrary constants
- Elimination of arbitrary functions

Elimination of arbitrary constants:

Let $f(x,\ y,\ z,\ a,\ b) = 0$ be the given equation where a and b are arbitrary constants. Let us form the PDE by eliminating a and b and we proceed as below.

Step-I: Differentiate equation (1) partially w.r.to x

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0 = \frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} \tag{2}$$

Elimination of arbitrary constants

Step-II: Differentiate equation (1) partially w.r.to y

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} = 0 = \frac{\partial f}{\partial x} + q\frac{\partial f}{\partial z} \tag{3}$$

Eliminating the arbitrary constants a and b from equations (2) and (3).

Finally, we get the PDE in the form

$$\phi(x,\ y,\ z,\ p,\ q) = 0. \tag{4}$$

Note: The equation (1) is called the complete integral or complete solution of the PDE (4).

Elimination of arbitrary constants

Note:

1. If the number of arbitrary constants to be eliminated is equal to the number of independent variables, the process of elimination results in a unique partial differential equation of the first order.
2. If the number of arbitrary constants to be eliminated is more than the number of independent variables, the process of elimination will lead to partial differential equation of second or higher order. In this case, the PDE need not be unique.
3. If the number of arbitrary constants to be eliminated is less than the number of independent variables, the process of elimination will lead to partial differential equation of the first order. In this case, the PDE need not be unique.

Elimination of arbitrary constants

Example 1: Form the PDE from $z = (x^2 + a)(y^2 + b)$.

Solution: Differentiating partially w.r.to x we get

$$\frac{\partial z}{\partial x} = 2x(y^2 + b) = p \tag{5}$$

Differentiating partially w.r.to y we get

$$\frac{\partial z}{\partial y} = 2y(x^2 + a) = q \tag{6}$$

Equation(5) and (6) gives

$$(y^2 + b) = \frac{p}{2x} \quad \text{and} \quad (x^2 + a) = \frac{q}{2y} \ \text{ respectively.} \tag{7}$$

Substituting equation (7) in the original equation
$z = (x^2 + a)(y^2 + b) = \frac{p}{2x} . \frac{q}{2y} \quad \Rightarrow pq = 4zxy$
which is the required partial differential equation.

Elimination of arbitrary constants

Example 2: Find the PDE of all planes cutting equal intercepts from the x-axis and y-axis.

Solution: Let a and c are x and z intercepts respectively. The the equation of plane is given by

$$\frac{x}{a}+\frac{y}{a}+\frac{z}{c}=1. \tag{8}$$

Differentiating partially w.r.to x we get

$$\frac{1}{a}+\frac{1}{c}\frac{\partial z}{\partial x}=0=\frac{1}{a}+\frac{1}{c}p. \tag{9}$$

Differentiating partially w.r.to y we get

$$\frac{1}{a}+\frac{1}{c}\frac{\partial z}{\partial y}=0=\frac{1}{a}+\frac{1}{c}q. \tag{10}$$

Solving (9) and (10) gives

$$\frac{1}{c}p-\frac{1}{c}q=0 \quad \Rightarrow \frac{1}{c}(p-q)=0 \qquad \Rightarrow p-q=0.$$

Elimination of arbitrary constants

Example 3: Form the PDE from $\log(az-1) = x + ay + b$.

Solution:

$$\frac{a}{az-1}\frac{\partial z}{\partial x} = 1 = \frac{ap}{az-1} \tag{11}$$

$$\frac{a}{az-1}\frac{\partial z}{\partial y} = a = \frac{aq}{az-1} \tag{12}$$

Dividing (12) by (11), we get $\frac{q}{p} = a$. From equation (12) we have

$$q = az - 1 = \frac{q}{p}z - 1 \quad \Rightarrow pq = qz - p \quad \Rightarrow p(q+1) = qz.$$

Elimination of arbitrary constants

Example 4: Obtain the PDE of all spheres with centers lies on $z = 0$ and whose radius is constant and equal to r.

Solution: The equation of the spheres with $z = 0$ is $(x-a)^2 + (y-b)^2 + z^2 = r^2$.

$$2(x-a) + 2z\frac{\partial z}{\partial x} = 0 = 2(x-a) + 2zp \quad \text{and}$$
$$2(y-b) + 2z\frac{\partial z}{\partial y} = 0 = 2(y-b) + 2zq$$
$$\Rightarrow (x-a) = -zp \quad \text{and} \quad (y-b) = -zq$$
$$\Rightarrow z^2p^2 + z^2q^2 + z^2 = r^2 = z^2(p^2+q^2+1).$$

Example 5: Form a PDE by eliminating a from $z = x + ay$.

Solution: Differentiating given equation partially w.r.to x, we get $p = 1$.

Differentiating given equation partially w.r.to y, we get $q = a \quad \Rightarrow \quad z = x + qy$

The PDEs are $p = 1$ and $z = x + qy$.

Elimination of arbitrary constants

Example 6: Form a PDE by eliminating a, b and c from $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Solution:

$$\frac{2x}{a^2} + \frac{2z}{c^2}\frac{\partial z}{\partial x} = 0 \qquad \text{or} \qquad c^2 x + a^2 z\frac{\partial z}{\partial x} = 0 \tag{13}$$

$$\frac{2y}{b^2} + \frac{2z}{c^2}\frac{\partial z}{\partial y} = 0 \qquad \text{or} \qquad c^2 y + b^2 z\frac{\partial z}{\partial y} = 0 \tag{14}$$

Again differentiating (13) partially w.r.t y we get

$$0 + a^2\left[z\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\right] = 0 \quad \Rightarrow pq + zs = 0. \tag{15}$$

Problems for practice:

Problems for practice:

1. Form the partial differential equation by eliminating the arbitrary constants a and b from $logz = alogx + \sqrt{1-a^2}logy + b$
   Ans: $p^2x^2 + q^2y^2 = z^2$
2. Find the PDE of the set of all right circular cones whose axes coincides with $z-$axis and given by $x^2 + y^2 = (z-c)^2 \tan^2 \alpha$.
   Ans: $yp = xq$
3. Form a PDE by eliminating a and b, from $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$
   Ans: $2z = px + qy$
4. Form a PDE by eliminating a and b, from $2z = (ax + y)^2 + b$
   Ans: $q = \frac{p}{q}x + y$
5. Form the partial differential equation by eliminating the arbitrary constants from $z = \frac{1}{2}\left(\sqrt{x+a} + \sqrt{y+b}\right)$
   Ans: $8pqz = p + q$

Elimination of arbitrary functions

In order to form a PDE by eliminating arbitrary functions, we may come across two types of functions

- **Form I:** $z = f(x, y)$, where x and y are independent variables.
- **Form II:** $\phi(u, v) = 0$, where $u = u(x, y, z)$, $v = v(x, y, z)$ and x, y and z are independent variables.

Note:

If the partial differential equation is formed by eliminating arbitrary functions, the order of the partial differential equation will be, in general, equal to the number of arbitrary functions eliminated.

Form I:

Example 1: Form the PDE by eliminating the arbitrary function from $z = f(x^2 + y^2)$.

Solution: Differentiating w.r.t x and y separately, we get

$$\frac{\partial z}{\partial x} = 2xf^{'}(x^2 + y^2) = p \quad \text{and} \quad \frac{\partial z}{\partial y} = 2yf^{'}(x^2 + y^2) = q.$$

Dividing them we get

$$\frac{p}{q} = \frac{x}{y} \qquad \Rightarrow yp - xq = 0.$$

Example 2: Form the PDE by eliminating the arbitrary function from $z = f(x - at) + f(x + at)$.
Solution: Differentiating w.r.t x, we get

$$\begin{aligned} \frac{\partial z}{\partial x} &= f^{'}(x - at) + f^{'}(x + at) \\ \frac{\partial^2 z}{\partial x^2} &= f^{''}(x - at) + f^{''}(x + at). \end{aligned} \tag{16}$$

Form I

Similarly differentiating w.r.t t, we get

$$\begin{aligned}\frac{\partial z}{\partial t} &= -af^{'}(x-at)+af^{'}(x+at)\\ \frac{\partial^2 z}{\partial t^2} &= a^2f^{''}(x-at)+a^2f^{''}(x+at). \end{aligned} \tag{17}$$

Form (17) one can see that

$$\begin{aligned}\frac{\partial^2 z}{\partial t^2} &= a^2f^{''}(x-at)+a^2f^{''}(x+at)\\ &= a^2[f^{''}(x-at)+f^{''}(x+at)]\\ &= a^2\frac{\partial^2 z}{\partial x^2}. \quad \text{From (16)}\end{aligned}$$

Therefore the required PDE is $\frac{\partial^2 z}{\partial t^2} = a^2\frac{\partial^2 z}{\partial x^2}$

Form I

Example 3: Form the PDE by eliminating the arbitrary function from $y = f\left(\frac{y}{x}\right)$.

Solution: Differentiating w.r.t x and y separately, we get

$$\frac{\partial z}{\partial x} = f^{'}\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) \quad \text{or} \quad f^{'}\left(\frac{y}{x}\right) = \left(\frac{-x^2}{y}\right)\frac{\partial z}{\partial x} \tag{18}$$

and

$$\frac{\partial z}{\partial y} = f^{'}\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) \quad \text{or} \quad f^{'}\left(\frac{y}{x}\right) = x\frac{\partial z}{\partial y} \tag{19}$$

From (18) and (19) one can find

$$\left(\frac{-x^2}{y}\right)\frac{\partial z}{\partial x} = x\frac{\partial z}{\partial y} \qquad \Rightarrow x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0.$$

Form I

Example 4: Form the PDE by eliminating the arbitrary function from $y = x^2 + 2f\left(\frac{1}{y} + \log x\right)$.

Solution: Differentiating w.r.t x and y separately, we get

$$\begin{aligned}
\frac{\partial z}{\partial x} = p &= 2x + 2f^{'}\left(\frac{1}{y} + \log x\right)\left(\frac{1}{x}\right) \quad \text{and} \\
\frac{\partial z}{\partial y} = q &= 2f^{'}\left(\frac{1}{y} + \log x\right)\left(\frac{-1}{y^2}\right) \\
&\Rightarrow px + qy^2 = 2x^2 \\
&\Rightarrow \frac{p - 2x}{q} = -\frac{y^2}{x}.
\end{aligned}$$

Form I

Example 5: Form the PDE by eliminating the arbitrary function from $y = xy + 2f(x^2 + y^2 + z^2)$.

Solution: Differentiating w.r.t x and y separately, we get

$$\begin{aligned}
\frac{\partial z}{\partial x} = p &= y + 2f^{'}(x^2 + y^2 + z^2)\left(2x + 2z\frac{\partial z}{\partial x}\right) \quad \text{and} \\
\frac{\partial z}{\partial y} = q &= x + 2f^{'}(x^2 + y^2 + z^2)\left(2y + 2z\frac{\partial z}{\partial y}\right) \\
&\Rightarrow p - y = 2f^{'}(x^2 + y^2 + z^2)(2x + 2zp) \\
&\Rightarrow q - x = 2f^{'}(x^2 + y^2 + z^2)(2y + 2zq) \\
&\Rightarrow \frac{p - y}{q - x} = \frac{x + zp}{y + zq}.
\end{aligned}$$

Problems for practice:

Problems for practice:

1. Form the PDE by eliminating the arbitrary functions f and g from $z = f(ax+by) + g(\alpha x + \beta y)$.

   Ans: $b\beta \frac{\partial^2 z}{\partial x^2} - (a\beta + b\alpha)\frac{\partial^z}{\partial x \partial y} + a\alpha \frac{\partial^2 z}{\partial x^2} = 0$

2. Form a PDE by eliminating f and g, from $z = yf(x) + xg(y)$

   Ans: $xy\frac{\partial^z}{\partial x \partial y} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - z$

3. Eliminate the arbitrary function f and obtain the PDE from $z = e^{ay} f(x+by)$.

   Ans: $\frac{\partial z}{\partial y} = az + b\frac{\partial z}{\partial x}$

4. Form the PDE by eliminating the arbitrary functions f from $xy + yz + zx = f\left(\frac{z}{x+y}\right)$

   Ans: $(x+y)(x+2z)p - (x+y)(y+2z)q = z(x-y)$

5. Form the PDE by eliminating the arbitrary functions f and g from $z = xf(ax+by) + g(ax+by)$

   Ans: $b^2 r - 2abs + a^2 t = 0$

Form II

Let $\phi(u,\ v) = 0$ be given function. Then we can construct the PDE as follows:

Step 1: Differentiate u and v w.r.t x, y and z

Step 2: Find

$$
\begin{aligned}
P &= \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y} = \frac{\partial(u, v)}{\partial(y, z)} \\
Q &= \frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z} = \frac{\partial(u, v)}{\partial(z, x)} \\
R &= \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = \frac{\partial(u, v)}{\partial(x, y)}
\end{aligned}
$$

Step 3: Write the PDE $Pp + Qq = R$.

Form II

Example 1: Form the PDE by eliminating the arbitrary function ϕ from $\phi(x+y+z,\ x^2+y^2-z^2)=0$.

Solution: Here $u = x+y+z$ and $v = x^2+y^2-z^2$

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = 1, \quad \frac{\partial u}{\partial z} = 1$$

$$\frac{\partial v}{\partial x} = 2x, \quad \frac{\partial v}{\partial y} = 2y, \quad \frac{\partial v}{\partial z} = -2z.$$

Now

$$P = \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y} = 1.(-2z) - 1.(2y) = -2z - 2y$$

$$Q = \frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z} = 1.(2x) - 1.(-2z) = 2x + 2z$$

$$R = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = 1.(2y) - 1.(2x) = 2y - 2x.$$

Write the PDE as

$$Pp + Qq = R$$

$$\Rightarrow (-2z-2y)p + (2x+2z)q = 2y-2x$$

Form II

Example 2: Form the PDE by eliminating the arbitrary function ϕ from $\phi(x^2 + y^2 + z^2,\ z^2 - 2xy) = 0$.

Solution: Here $u = x^2 + y^2 + z^2$ and $v = z^2 - 2xy$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2y, \quad \frac{\partial u}{\partial z} = 2z$$
$$\frac{\partial v}{\partial x} = -2y, \quad \frac{\partial v}{\partial y} = -2x, \quad \frac{\partial v}{\partial z} = 2z.$$

$$\begin{aligned} P &= (2y)(2z) - (2z)(-2x) = 4z(y + x) \\ Q &= (2z)(-2y) - (2x)(2z) = -4z(y + x) \\ R &= (2x)(-2x) - (2y)(-2y) = 4(y^2 - x^2). \end{aligned}$$

Hence $Pp + Qq = R \Rightarrow p - q = \frac{y-x}{z}$.

Form II

Example 3: Form the PDE by eliminating the arbitrary function f from $f\left(\frac{x-a}{z-c}, \frac{y-b}{z-c}\right) = 0$.

Solution: Here $u = \frac{x-a}{z-c}$ and $v = \frac{y-b}{z-c}$

$$\frac{\partial u}{\partial x} = \frac{1}{z-c}, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial u}{\partial z} = -\frac{x-a}{(z-c)^2}$$
$$\frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = \frac{1}{z-c}, \quad \frac{\partial v}{\partial z} = -\frac{y-b}{(z-c)^2}.$$

$$\begin{aligned}
\Rightarrow P &= (0)\left(-\frac{y-b}{(z-c)^2}\right) - \left(-\frac{x-a}{(z-c)^2}\right)\frac{1}{z-c} = \frac{x-a}{(z-c)^3} \\
Q &= \left(-\frac{x-a}{(z-c)^2}\right)(0) - \frac{1}{z-c}\left(-\frac{y-b}{(z-c)^2}\right) = \frac{y-b}{(z-c)^3} \\
R &= \frac{1}{z-c}.\frac{1}{z-c} - (0)(0) = \frac{1}{(z-c)^2}.
\end{aligned}$$

Hence the PDE is

$$\begin{aligned}
Pp + Qq &= R \\
(x-a)p + (y-b)q &= (z-c).
\end{aligned}$$

Form II

Example 4: Form the PDE by eliminating the arbitrary function ϕ from the relation $\phi(x^2 + y^2 + z^2,\ lx + my + nz) = 0$.

Solution: Here $u = x^2 + y^2 + z^2$ and $v = lx + my + nz$

$$\begin{aligned}
P &= \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y} = (2y).n - (2z).m \\
Q &= \frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z} = (2z).l - (2x).m \\
R &= \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = (2x).m - (2y).n.
\end{aligned}$$

$$Pp + Qq = R \Rightarrow 2(ny - mz)p + 2(lz - nx)q = 2(mx - ly)$$

$$\Rightarrow (ny - mz)p + (lz - nx)q = (mx - ly).$$

Solutions of First Order Non-Linear PDEs

Solution (Integral): The solution of a PDE is a relation between the independent variables and dependent variables which satisfies the given PDE.

There are four types of solutions:

1. **Complete Solution or Complete Integral**
2. **Particular Integral**
3. **Singular Integral**
4. **General Solution or General Integral**.

Complete Integral: A solution which contains as many arbitrary constants as there are independent variables is called complete integral.

Example: If $f(x, y, z, p, q) = 0$, where x and y are independent variables, is a given PDE then the solution of the form $\phi(x, y, z, a, b) = 0$ is a complete integral.

Particular Integral: Let $\phi(x, y, z, a, b) = 0$ be a complete integral of the given PDE where a and b are arbitrary constants. The solution obtained by determining the arbitrary constants in the complete integral or arbitrary functions in the general integral by using some specified condition is called a particular integral.

Example: If $\phi(x, y, z, a, b) = 0$ is a complete integral, then $\phi(x, y, z, 1, 2) = 0$, $\phi(x, y, z, 0, 1) = 0$, $\phi(x, y, z, 3, 9) = 0$ all are possible particular integrals.

Singular Integral:

Definition: The envelope of the complete integral is also a solution of the PDE $f(x, y, z, p, q) = 0$. It is called the singular integral of the PDE.

Envelope: The envelope of a family of curves is a curve such that at each point it touches tangentially one of the family of curves.

Procedure to Find Singular Integral: Let

$$\phi(x, y, z, a, b) = 0 \tag{20}$$

be a complete integral of a given PDE. Differentiating (20) w.r.t a and b, one can obtain

$$\frac{\partial \phi}{\partial a} = 0 \qquad \text{and} \qquad \frac{\partial \phi}{\partial b} = 0. \tag{21}$$

Now eliminating a and b from (20) and (21) we get singular integral.

General Integral

Definition: Any relation of the form $\phi(u, v) = 0$ involving an arbitrary function ϕ connecting with two known functions $u = u(x, y, z)$ and $v = v(x, y, z)$ and satisfies the PDE $f(x, y, z, p, q) = 0$ is called a general integral of the first order PDE.

Procedure to Find General Integral: Let $\phi(x, y, z, a, b) = 0$ be a complete integral of $f(x, y, z, p, q) = 0$. If we take $b = \varphi(a)$ or $a = \varphi(b)$ we get

$$\phi(x, y, z, a, \varphi(a)) = 0 \qquad \text{or} \qquad \phi(x, y, z, \varphi(b), b) = 0 \tag{22}$$

Differentiating (22) w.r.t a we can obtain

$$\frac{\partial \phi}{\partial a} = 0. \tag{23}$$

Now eliminating a from (22) and (23) we get general integral.

Types of First Order Non-Linear PDEs

Let us consider the first order PDEs of the form $f(x,\ y,\ z,\ p,\ q) = 0$ where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$

In order to find the solutions of first order non-linear PDEs, let us classify the given PDE as below:

- **Type I:** $f(p, q) = 0$
- **Type II: (Clairaut's form)**
- **Type III:** $f(z, p, q) = 0$
- **Type IV: (Separable equations)**.

Type I: $f(p, q) = 0$

A partial differential equation which involves p and q only and the variables $x,\ y,\ z$ do not occur explicitly is of the form $f(p,\ q) = 0$. The solution of the PDE can be obtained as below:

- **Step 1 :** Put $p = a$ and $q = b$ which gives $f(a,\ b) = 0$
- **Step 2 :** Assume the solution as $z = ax + by + c$
- **Step 3 :** From $f(a,\ b) = 0$, write $a = \varphi(b)$ or $b = \varphi(a)$
- **Step 4 (Complete Integral):** Write the complete integral as

$$z = ax + \varphi(a)y + c \tag{24}$$

- **Step 5 (Singular solution):** Obtain the singular solution by eliminating a and c from the equations (24) and

$$\frac{\partial z}{\partial a} = 0, \qquad \frac{\partial z}{\partial b} = 0$$

- **Step 6 (General solution):** Assume $c = \psi(a)$ and eliminate a from the following equations

$$z = ax + \varphi(a)y + \psi(a) \qquad \text{and} \qquad \frac{\partial z}{\partial a} = 0 = x + \varphi^{'}(a)y + \psi^{'}(a).$$

Type I

Example 1: Solve $\sqrt{p} + \sqrt{q} = 1$.

Solution: Consider

$$F(p,q) = \sqrt{p} + \sqrt{q} - 1 \Rightarrow F(a,b) = \sqrt{a} + \sqrt{b} - 1 = 0.$$

Write the complete integral as

$$\begin{aligned} z &= ax + by + c \qquad \text{with} \qquad \sqrt{a} + \sqrt{b} = 1. \\ &\Rightarrow \sqrt{b} = 1 - \sqrt{a} \qquad \Rightarrow b = (1 - \sqrt{a})^2 \end{aligned}$$

Therefore, the complete integral is given by

$$z = ax + (1 - \sqrt{a})^2 y + c, \tag{25}$$

where a and c are arbitrary constants.

Type I

For the singular integral, differentiate (25) w.r.t c and equate with 0, we get

$$\frac{\partial z}{\partial c} = 0, \qquad \Rightarrow 1 = 0, \quad \text{which is absurd.}$$

Therefore, there is no singular solution. For the general solution take $c = \psi(a)$, hence we get

$$z \quad = \quad ax + (1 - \sqrt{a})^2 y + \psi(a). \tag{26}$$

Now differentiation of (26) w.r.t a gives

$$\frac{\partial z}{\partial a} = x + 2(1 - \sqrt{a}) \times \frac{-1}{2\sqrt{a}} + \psi^{'}(a) = 0. \tag{27}$$

Eliminating a from (26) and (27), we get general solution.

Type I

Example 2: Solve $p^2 + q^2 = npq$.

Solution: Consider

$$\begin{aligned} F(p,q) &= p^2 + q^2 - npq = 0 \qquad \Rightarrow F(a,b) = a^2 + b^2 - nab = 0 \\ &\Rightarrow \quad b = \frac{a}{2}\left[n \pm \sqrt{n^2 - 4}\right]. \end{aligned}$$

Complete integral:

$$z = ax + \frac{a}{2}\left[n \pm \sqrt{n^2 - 4}\right] y + c.$$

Singular integral: Differentiating the above equation w.r.t c

$$\frac{\partial z}{\partial c} = 0, \qquad \Rightarrow 1 = 0, \quad \text{which is absurd.}$$

Therefore, there is no singular solution.

Type I

For the general solution take $c = \psi(a)$, hence we get

$$z \;=\; ax + \frac{a}{2}\left[n \pm \sqrt{n^2 - 4}\right] y + \psi(a). \tag{28}$$

Now differentiation of (28) w.r.t a gives

$$\frac{\partial z}{\partial a} = x + \frac{1}{2} \times \left[n \pm \sqrt{n^2 - 4}\right] y + \psi^{'}(a) = 0. \tag{29}$$

Eliminating a from (28) and (29), we get general solution.

Type II(Clairaut's form):

Let the first order PDE is of the form $z = px + qy + f(p, q)$ where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$. then the solution can be obtained as below:

- **Complete integral:** Put $p = a$ and $q = b$ and write the complete integral

$$z = ax + by + f(a, b). \tag{30}$$

  where a and b are arbitrary constants.

- **Singular integral:** Obtain the singular integral by eliminating a and b from (30) and the following equations

$$\text{Diff. w.t.t } a \quad \Rightarrow x + \frac{\partial f}{\partial a} = 0 \quad \text{and} \tag{31}$$

$$\text{Diff. w.t.t } a \quad \Rightarrow y + \frac{\partial f}{\partial b} = 0. \tag{32}$$

Type II(Clairaut's form):

- **General integral:** Putting $b = \psi(a)$ in (30), we get

$$z = ax + \psi(a)y + f(a, \psi(a)). \tag{33}$$

Differentiation of (33) w.r.t a gives

$$\frac{\partial z}{\partial a} = x + \psi^{'}(a)y + f^{'}(a) = 0. \tag{34}$$

Eliminating a from (33) and (34), one can get the general integral.

Type II

Example 1: Solve $z = px + qy + \sqrt{1 + p^2 + q^2}$.

Solution: This is of the form $z = px + qy + f(p, q)$ where $f(p, q) = \sqrt{1 + p^2 + q^2}$.

Complete integral: It is given by $z = ax + by + \sqrt{1 + a^2 + b^2}$.

$$z = ax + by + \sqrt{1 + a^2 + b^2}. \tag{35}$$

Singular integral: Differentiate (35) w.r.t a and b and eliminate them as follows:

$$\begin{aligned} x + \frac{a}{\sqrt{1 + a^2 + b^2}} &= 0 \Rightarrow x = -\frac{a}{\sqrt{1 + a^2 + b^2}} \\ &\text{and} \\ y + \frac{b}{\sqrt{1 + a^2 + b^2}} &= 0 \Rightarrow y = -\frac{b}{\sqrt{1 + a^2 + b^2}}. \end{aligned} \tag{36}$$

Type II

From the above equations one can find

$$\begin{aligned}
x^2 + y^2 &= \frac{a^2}{1+a^2+b^2} + \frac{b^2}{1+a^2+b^2} \\
\Rightarrow 1 - x^2 - y^2 &= \frac{1}{1+a^2+b^2} \\
\Rightarrow 1 + a^2 + b^2 &= \frac{1}{1-x^2-y^2} \quad \Rightarrow \sqrt{1+a^2+b^2} = \frac{1}{\sqrt{1-x^2-y^2}}.
\end{aligned}$$

As a result (36) becomes

$$\begin{aligned}
x = -a\sqrt{1-x^2-y^2} \quad &\text{and} \quad y = -b\sqrt{1-x^2-y^2} \\
\Rightarrow a = -\frac{x}{\sqrt{1-x^2-y^2}} \quad &\text{and} \quad b = -\frac{y}{\sqrt{1-x^2-y^2}}.
\end{aligned}$$

Type II

Therefore the singular integral w.r.t the equation (35) is given by

$$\begin{aligned} z &= -\frac{x^2}{\sqrt{1-x^2-y^2}} - \frac{y^2}{\sqrt{1-x^2-y^2}} + \frac{1}{1-x^2-y^2} \\ z &= \frac{1-x^2-y^2}{\sqrt{1-x^2-y^2}} = \sqrt{1-x^2-y^2} \\ &\Rightarrow \quad z^2 = 1-x^2-y^2 \end{aligned}$$

$$\boxed{z^2+x^2+y^2=1}.$$

Type II

Example 2: Solve $z = px + qy + p^2q^2$.

Solution: This is of the form $z = px + qy + f(p, q)$ where $f(p, q) = p^2q^2$.

Complete integral: It is given by

$$z = ax + by + a^2b^2. \tag{37}$$

Singular integral: Differentiate (37) w.r.t a and b and eliminate them as follows:

$$\begin{aligned} x + 2ab^2 &= 0 \quad \Rightarrow x = -2ab^2 \\ &\text{and} \\ y + 2a^2b &= 0 \quad \Rightarrow y = -2a^2b. \end{aligned} \tag{38}$$

Type II

From the above equations one can find

$$\frac{x}{a} = \frac{y}{b} = -2ab = \frac{1}{k} \quad \Rightarrow a = kx \qquad \text{and} \qquad b = ky. \tag{39}$$

Using (39) in (39) we get

$$\begin{aligned} x &= -2k^3x^2y \quad \Rightarrow k^3 = -\frac{1}{2xy} \\ z &= kxy + kxy + k^4x^2y^2 = 2kxy + kx^2y^2\left(-\frac{1}{2xy}\right) = \frac{3}{2}kxy \\ \Rightarrow z^3 &= \frac{27}{8}k^3x^3y^3 = \frac{27}{8}k^3x^3y^3\left(-\frac{1}{2xy}\right) = -\frac{27}{16}x^2y^2 \end{aligned}$$

$$\boxed{16z^3 + 27x^2y^2 = 0}.$$

Type II

Example 3: Solve $z = px + qy + p^2 - q^2$.

Solution: This is of the form $z = px + qy + f(p, q)$ where $f(p, q) = p^2 - q^2$.
Complete integral: It is given by

$$z = ax + by + a^2 - b^2. \tag{40}$$

Singular integral: Differentiate (40) w.r.t a and b and eliminate them as follows:

$$x + 2a = 0 \quad \Rightarrow x = -2a \quad \text{and} \quad y - 2b = 0 \quad \Rightarrow y = 2b.$$

$$\Rightarrow a = -\frac{x}{2} \quad \text{and} \quad b = -\frac{y}{2} \quad \Rightarrow z = -\frac{x^2}{2} + \frac{y^2}{2} + \frac{x^2 - y^2}{4}$$

$$\boxed{4z = y^2 - x^2}.$$

Type II

General integral: Put $b = \psi(a)$ in (40) becomes

$$z = ax + \psi(a)y + a^2 - (\psi(a))^2.$$

Differentiating the above equation w.r.t a we find

$$0 = x + \psi^{'}(a)y + 2a - 2\psi(a)\psi^{'}(a).$$

Eliminating a from the above equations, we will get the general solution.

Type III:Case I

The PDEs of the form $F(z, p, q) = 0$ which does not contain x and y is called Type-III problems. Assume the trial solution $u = x + ay$ and take

$$\begin{aligned} z &= f(x+ay) = f(u) \\ \Rightarrow p &= \frac{\partial z}{\partial x} = \frac{dz}{du}.\frac{\partial u}{\partial x} = \frac{dz}{du} \text{ and } q = \frac{\partial z}{\partial y} = \frac{dz}{du}.\frac{\partial u}{\partial y} = a\frac{dz}{du}. \end{aligned}$$

Then find the solutions as follows:

- **Step I:** Putting $p = \frac{dz}{du}$ and $q = a\frac{dz}{du}$ with $u = x + ay$, we will get the ODE of the form $\frac{dz}{du} = \phi(z, a)$.
- **Step II:** Solve the ODE $\frac{dz}{du} = \phi(z, a)$ as $\frac{dz}{\phi(z,a)} = du$. Integrating this we get $\int \frac{dz}{\phi(z,a)} = u + c \Rightarrow \psi(z, a) = x + ay + c$, where c is arbitrary constant. This is complete integral of the given PDE.
- **Step III:** Obtain the singular and general integral as usual.

Type III:Case I

Example 1: Solve $p(1+q) = qz$.

Solution: This is of the form Type III. Assume $u = x + ay$ and put $p = \frac{dz}{du}$ and $q = a\frac{dz}{du}$ in the given equation.

$$\frac{dz}{du}\left(1 + a\frac{dz}{du}\right) = az\frac{dz}{du} \quad \Rightarrow a\left(\frac{dz}{du}\right)^2 + (1-az)\frac{dz}{du} = 0$$

$$\Rightarrow a\frac{dz}{du} = (1-az) \quad \Rightarrow a\frac{dz}{(1-az)} = du$$

$$\Rightarrow a\int \frac{dz}{(1-az)} = \int du + c \quad \Rightarrow \log(1-az) = x + ay + c.$$

Therefore, the complete integral is

$$\boxed{\log(1-az) = x + ay + c}.$$

Type III:Case I

Singular integral: Differentiate w.r.t a and c we get

$$\frac{z}{(1-az)} = y \qquad\qquad 0 = 1, \qquad \text{which is absurd.}$$

$\therefore$ Singular solution does not exist.
General integral: Put $c = \psi(a)$ in $\log(1-az) = x + ay + c$ differentiate it w.r.t a

$$\frac{z}{(1-az)} = y + \psi^{'}(a)$$

and eliminating a from the above we will get the general integral.

Type III:Case I

Example 2: Solve $9(p^2z+q^2)=4$.

Solution: This is of the form Type III. Assume $u=x+ay$ and put $p=\frac{dz}{du}$ and $q=a\frac{dz}{du}$ in the given equation.

$$9\left[z\left(\frac{dz}{du}\right)^2+a^2\left(\frac{dz}{du}\right)^2\right]=4 \quad \Rightarrow 9\left(\frac{dz}{du}\right)^2(z+a^2)=4$$

$$\Rightarrow \left(\frac{dz}{du}\right)^2=\frac{4}{9(z+a^2)} \quad \Rightarrow \frac{dz}{du}=\frac{2}{3\sqrt{(z+a^2)}}$$

$$\Rightarrow 3\sqrt{(z+a^2)}dz=2du \quad \text{integrating we get} \quad (z+a^2)^{3/2}=u+b$$

$\therefore$ The complete integral is given by

$$\boxed{(z+a^2)^3=(x+ay+b)^2}$$

Type III:Case II

Suppose the equation is of the form $f(x, p, q) = 0$, which contains x, p and q. This equation is also Type III. To solve such equation we use the following steps:

- **Step I:** Assume $q = a$ and write $p = \phi(x, a)$ from $f(x, p, a) = 0$
- **Step II:** Put the values of p and q in $dz = pdx + qdy$ which gives $dz = \phi(x, a)dx + qdy$
- **Step III:** Integrate the above equation and write the complete integral as $z = \phi_1(x, a) + ay + c$, where c is integrating constant and $\phi_1(x, a)$ is integration of $\phi(x, a)$ w.r.t x.

Type III:Case II

Example 1: Solve $q = px + p^2$.

Solution: This is of the form Type III.

$$p^2 + px - q = 0, \quad \text{put } q = a \quad \Rightarrow p^2 + px - a = 0.$$

The above equation is a quadratic equation in term of p. Therefore, $p = \frac{-x \pm \sqrt{x^2+4a}}{2} = \phi(x, a)$.

$$\begin{aligned} dz &= pdx + qdy \\ &= \frac{-x \pm \sqrt{x^2+4a}}{2} dx + ady = \left[\frac{-x}{2} \pm \frac{\sqrt{x^2+4a}}{2} \right] dx + ady. \end{aligned}$$

Integrating this, we get

$$\boxed{z = \frac{-x^2}{4} \pm \frac{1}{2} \left[\sin h^{-1} \left(\frac{x}{2\sqrt{a}} \right) + \frac{x}{2} \sqrt{x^2+4a} \right] + ay + b}$$

Type III:Case III

Suppose the equation is of the form $f(y, p, q) = 0$, which contains y, p and q. This equation is also Type III. To solve such equation we use the following steps:

- **Step I:** Assume $p = a$ and write $q = \phi(y, a)$ from $f(y, p, a) = 0$
- **Step II:** Put the values of p and q in $dz = pdx + qdy$ which gives $dz = adx + \phi(y, a)dy$
- **Step III:** Integrate the above equation and write the complete integral as $z = ax + \phi_1(y, a)y + c$, where c is integrating constant and $\phi_1(y, a)$ is integration of $\phi(y, a)$ w.r.t y.

Type III:Case III

Example 1: Solve $pq = y$.

Solution: Put $p = a \quad \Rightarrow \quad q = \frac{y}{a}$. Now

$$dz = pdx + qdy = adx + \frac{y}{a}dy \qquad \Rightarrow adz = a^2dx + ydy.$$

Integrating this, we get

$$\boxed{az = ax^2 + y^2/2 + c}$$

is the complete integral. There will be no singular integral and the general integral can be obtain by taking $b = \psi(a)$ and eliminating a.

Type IV (Separable Equations):

The first order PDE is said to be separable equation if it can written in the form $f(x,p) = \phi(y,p)$.

For such PDE the solutions can be obtained as below:

- **Step I:** Put $f(x,p) = \phi(y,p) = a$ (say)
- **Step II:** Write $p = f_1(x,a)$ and $q = \phi_1(y,a)$
- **Step III:** Putting in $dz = pdx + qdy$ and integrating, get the complete integral as

$$z = \int f_1(x,a)dx + \int \phi_1(y,a)dy + b.$$

Type IV:

Example 1: Solve $p^2 y(1+x^2) = qx^2$.

Solution: This is separable equation because

$$\frac{p^2(1+x^2)}{x^2} = \frac{q}{y} = a \quad \Rightarrow p^2 = \frac{ax^2}{(1+x^2)} \Rightarrow p = \frac{\sqrt{a}x}{\sqrt{1+x^2}} \quad \text{and} \quad q = ay.$$

$$dz = pdx + qdy = \frac{\sqrt{a}x}{\sqrt{1+x^2}}dx + aydy.$$

Integrating the above equation we get the complete integral as

$$\boxed{z = \sqrt{a(1+x^2)} + \frac{ay^2}{2} + b}$$

Type IV:

Example 2: Solve $p^2 + q^2 = x + y$.

Solution: This is separable equation because

$$p^2 - x = y - q^2 = a \quad \Rightarrow \quad \begin{aligned} p^2 &= x + a \Rightarrow p = \sqrt{x+a} \quad \text{and} \\ q^2 &= y - a \Rightarrow q = \sqrt{y-a}. \end{aligned}$$

Now

$$dz = pdx + qdy = \sqrt{x+a}dx + \sqrt{y-a}dy.$$

Integrating the above equation we get the complete integral as

$$\boxed{z = \tfrac{2}{3}(x+a)^{3/2} + \tfrac{2}{3}(y-a)^{3/2} + b}$$

Type IV:

Example 3: Solve $p^2 + q^2 = x^2 + y^2$.

Solution: This is separable equation because

$$p^2 - x^2 = y^2 - q^2 = a^2 \quad \Rightarrow \quad \begin{array}{l} p^2 = x^2 + a^2 \Rightarrow p = \sqrt{x^2 + a^2} \quad \text{and} \\ q^2 = y^2 - a^2 \Rightarrow q = \sqrt{y^2 - a^2}. \end{array}$$

Now

$$dz = pdx + qdy = \sqrt{x^2 + a^2}dx + \sqrt{y^2 - a^2}dy.$$

Integrating the above equation we get the complete integral as

$$\boxed{z = \frac{a^2}{2} \sin h^{-1} \frac{x}{a} + \frac{x}{2}\sqrt{x^2 + a^2} + \frac{y}{2}\sqrt{y^2 - a^2} - \frac{a^2}{2} \cos h^{-1} \frac{y}{a} + b}$$

Lagrange's Linear Equation

A linear first order PDE of the form

$$Pp + Qq = R, \tag{41}$$

where $P,\ Q,\ R$ are are functions of $x,\ y,\ z$ is called the Lagrange's linear equation. This equation can be solved in two ways

- **Method I (Direct method):**
- **Method II (Method of multipliers):**

Method I:

The following steps are followed to solve (41):

- **Step 1:** Form the auxiliary simultaneous equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$
- **Step 2:** Solve the auxiliary equation to get two independent solutions $u(x, y, z) = a$ and $v(x, y, z) = b$
- **Step 3:** Write the solution as $\phi(u, v) = 0$, where ϕ is an arbitrary function.

Method I:

Example 1: Find the general solution of $px + qy = z$.

Solution: Write the auxiliary equation as

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

Consider the following pair of equations and solve them

$$\frac{dx}{x} = \frac{dy}{y} \quad \text{and} \quad \frac{dy}{y} = \frac{dz}{z}$$
$$\Rightarrow \quad \frac{x}{y} = u \quad \text{and} \quad \frac{y}{z} = v.$$

$\therefore$ The general solution is given by

$$\boxed{\phi(\tfrac{x}{y}, \tfrac{y}{z}) = 0}$$

Method I:

Example 2: Solve $\frac{y^2 z}{x}p + xzq = y^2$.

Solution: Write the auxiliary equation as

$$\frac{xdx}{y^2 z} = \frac{dy}{xz} = \frac{dz}{y^2}$$

Consider the following pair of equations and solve them

$$\frac{xdx}{y^2 z} = \frac{dy}{xz} \quad \Rightarrow x^2 dx = y^2 dy \quad \Rightarrow x^3 - y^3 = u$$

$$\frac{xdx}{y^2 z} = \frac{dz}{y^2} \quad \Rightarrow xdx = ydy \quad \Rightarrow x^2 - y^2 = v$$

$\therefore$ The general solution is given by

$$\boxed{\phi(x^3 - y^3, x^2 - y^2) = 0}$$

Method I:

Example 3: Solve $\tan xp + \tan yq = \tan z$.

Solution: Write the auxiliary equation as

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z} \quad \Rightarrow \cot x dx = \cot y dy = \cot z dz$$

Taking $\cot x dx = \cot y dy$ and integrating one can get

$$\log sinx = \log siny + \log u \ \Rightarrow \frac{sinx}{siny} = u.$$

Similarly taking

$$\cot y dy = \cot z dz \quad \text{and integrating we get} \quad \frac{siny}{sinz} = v.$$

$\therefore$ The general solution is given by

$$\boxed{\phi\left(\frac{sinx}{siny}, \frac{siny}{sinz}\right) = 0}$$

Method II (Method of Multipliers):

The following steps are followed to solve (41):

- **Step 1:** Write the auxiliary equations as $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$
- **Step 2:** Chose the two set of multipliers $(l,\ m,\ n)$ and $(l^{'},\ m^{'},\ n^{'})$ may be constants or functions of $x,\ y,\ z$ and rewrite the auxiliary equations as $\frac{ldx+mdy+ndz}{lP+mQ+nR} = \frac{l^{'}dx+m^{'}dy+n^{'}dz}{l^{'}P+m^{'}Q+n^{'}R}$ such that $lP + mQ + nR = 0$ and $l^{'}P + m^{'}Q + n^{'}R = 0$
- **Step 3:** Then solve $ldx + mdy + ndz = 0$ and write the solution as $u = u(x, y, z) = C_1$. Similarly solve $l^{'}dx + m^{'}dy + n^{'}dz = 0$ and write the solution as $v = v(x, y, z) = C_2$.
- **Step 4:** Write the general solution as $\phi(u, v) = 0$, where ϕ is an arbitrary function.

Method II (Method of Multipliers):

Example 1: Solve $(mz - ny)p + (nx - lz)q = (ly - mx)$.

Solution: Write the auxiliary equation as

$$\frac{dx}{(mz - ny)} = \frac{dy}{(nx - lz)} = \frac{dz}{(ly - mx)}$$

Consider the multipliers as $x,\ y,\ z$

$$\frac{xdx + ydy + zdz}{x(mz - ny) + y(nx - lz) + z(ly - mx)} = \frac{xdx + ydy + zdz}{0}.$$

Solving the following

$$xdx + ydy + zdz = 0 \quad \text{we get} \quad u(x, y, z) = x^2 + y^2 + z^2.$$

Method II (Method of Multipliers):

Similarly considering the multipliers as $l, \; m, \; n$ we have

$$\frac{ldx + mdy + ndz}{l(mz - ny) + m(nx - lz) + n(ly - mx)} = \frac{ldx + mdy + ndz}{0}$$

Solving the following

$$ldx + mdy + ndz = 0 \quad \text{we get} \quad v(x, y, z) = lx + my + nz.$$

$\therefore$ The general solution is given by

$$\boxed{\phi(x^2 + y^2 + z^2, lx + my + nz) = 0}$$

Method II (Method of Multipliers):

Example 2: Solve $x(z^2 - y^2)p + y(x^2 - z^2)q = z(y^2 - x^2)$.

Solution: Write the auxiliary equation as

$$\frac{dx}{x(z^2 - y^2)} = \frac{dy}{y(x^2 - z^2)} = \frac{dz}{z(y^2 - x^2)}$$

Consider the multipliers as $x, \ y, \ z$

$$\frac{xdx + ydy + zdz}{x^2(z^2 - y^2) + y^2(x^2 - z^2) + z^2(y^2 - x^2)} = \frac{xdx + ydy + zdz}{0}.$$

Solving the following

$$xdx + ydy + zdz = 0 \quad \text{we get} \quad u(x, y, z) = x^2 + y^2 + z^2.$$

Method II (Method of Multipliers):

Similarly considering the multipliers as $1/x,\ 1/y,\ 1/z$ we have

$$\frac{1/xdx + 1/ydy + 1/zdz}{(z^2 - y^2) + (x^2 - z^2) + (y^2 - x^2)} = \frac{1/xdx + 1/ydy + 1/zdz}{0}.$$

Solving the following

$$1/xdx + 1/ydy + 1/zdz = 0 \quad \text{we get} \quad v(x, y, z) = xyz.$$

$\therefore$ The general solution is given by

$$\boxed{\phi(x^2 + y^2 + z^2, xyz) = 0}$$

Method II (Method of Multipliers):

Example 3: Solve $(y+z)p+(z+x)q=(x+y)$.

Solution: Write the auxiliary equation as

$$\frac{dx}{(y+z)}=\frac{dy}{(z+x)}=\frac{dz}{(x+y)}$$

It can also be written as below:

$$\frac{dx+dy+dz}{2(x+y+z)}=\frac{dx-dy}{y-x}=\frac{dy-dz}{z-y}.$$ Now taking

$$\frac{dx+dy+dz}{2(x+y+z)}=\frac{dx-dy}{-(x-y)}$$ and integrating we get

$$\log(x+y+z)=-2\log(x-y)+\log u \quad \Rightarrow u=(x+y+z)(x-y)^2$$

Method II (Method of Multipliers):

Similarly considering

$$\frac{dx - dy}{y - x} = \frac{dy - dz}{z - y} \quad \Rightarrow \frac{d(x - y)}{x - y} = \frac{d(y - z)}{y - z}$$

and integrating we get

$$\log(x - y) = \log(y - z) + \log v \quad \Rightarrow v = \frac{(x - y)}{(y - z)}$$

$\therefore$ The general solution is given by

$$\boxed{\phi\left((x + y + z)(x - y)^2, \tfrac{(x-y)}{(y-z)}\right) = 0}$$

Method II (Method of Multipliers):

Example 4: Solve $zp + yq = x$.

Solution: Write the auxiliary equation as

$$\frac{dx}{z} = \frac{dy}{y} = \frac{dz}{x}$$

Considering the following

$$\frac{dx}{z} = \frac{dz}{x} \quad \text{and integrating we get}$$

$$u = x^2 - z^2$$

Method II (Method of Multipliers):

Similarly considering

$$\frac{dx+dy+dz}{x+y+z} = \frac{dy}{y} \quad \Rightarrow \frac{d(x+y+z)}{x+y+z} = \frac{dy}{y}$$

and integrating we get

$$\log(x+y+z) = \log y + \log v \quad \Rightarrow v = \frac{(x+y+z)}{y}$$

$\therefore$ The general solution is given by

$$\boxed{\phi\left(x^2 - z^2, \frac{(x+y+z)}{y}\right) = 0}$$

PDEs of Higher Order with Constant Coefficients

Although PDEs of Higher Order can be classify in many categories but in our course we will discuss only the following types of PDE:

- **Homogeneous Linear PDE:**
- **Non-homogeneous Linear PDE:**

Homogeneous Linear PDE: Equations in which the partial derivatives occurring are all of the same order and the coefficients are constants. Such equations are called homogeneous linear PDEs with constant coefficients.

Non-Homogeneous Linear PDE: Equations in which the partial derivatives occurring are not of the same order and the coefficients are constants are called non-homogeneous linear PDEs with constant coefficients.

Homogeneous Linear PDE:

The standard form of a homogeneous linear partial differential equation of n^{th} order with constant coefficients is

$$a_0\frac{\partial^n z}{\partial x^n}+a_1\frac{\partial^n z}{\partial x^{n-1}\partial y}+a_2\frac{\partial^n z}{\partial x^{n-2}\partial y^2}+\dots\dots\dots++a_n\frac{\partial^n z}{\partial y^n}=F(x,\ y),$$

where $a_0,\ a_1,\ \dots\dots\ a_n$ are the constants and $F(x,\ y)$ is the known functions of x and y is known as homogeneous linear PDE with constants co-efficient.
If we take $\frac{\partial}{\partial x}=D$ and $\frac{\partial}{\partial y}=D'$ in the above equation, then it can be written as

$$\left(a_0D^n+a_1D^{n-1}D'+a_2D^{n-2}{D'}^2+\dots\dots\dots+a_n{D'}^n\right)z=F(x,\ y)$$

which can be further written as

$$f\left(D,\ D'\right)z=F(x,\ y). \tag{42}$$

The method of solving (42) is similar to that of solving ordinary linear differential equations with constant coefficients.

Homogeneous Linear PDE:

The solution of (42) consist of two parts:

- **Complementary function:**
- **Particular Integral:**

Complementary function(C.F):

In order to find the complementary function the following steps may be followed:

- **Step 1:** Consider $f(D,\ D')\,z = 0$
- **Step 2:** Put $D = m$ and $D' = 1$ and write the auxiliary equation as

$$a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + ... + a_n = 0. \tag{43}$$

- **Step 3:** Solve the auxiliary equation (43) and find the roots. Let $m_1,\ m_2,\ ......m_n$ are the n roots of this equation, then write the complementary solution as per the following cases:

Complementary function(C.F):

Case-I: If $m_1 \neq m_2 \neq ..... \neq m_n$ i.e. that is all the roots are distinct, then write the complementary function as:

$$\boxed{z = \phi_1(y + m_1x) + \phi_2(y + m_2x) + \phi_3(y + m_3x) + ... + \phi_n(y + m_nx).}$$

Case II: If $m_1 = m_2 = ... = m_n$ i.e. all (may be some) roots are equal (repeated) then write the complementary function as:

$$\boxed{z = \phi_1(y + m_1x) + x\phi_2(y + m_1x) + x^2\phi_3(y + m_1x) + ... + x^{n-1}\phi_n(y + m_1x).}$$

Note: The degree of x may varies depending up on the repetition of the particular root.

Particular Integral(P.I):

Here we will discuss 6 possible types of functions to construct the P.I:

Type 1: If $F(x,\ y) = e^{ax+by}$, then P.I is given by

$$\boxed{\text{P.I} = \frac{1}{f(D,\ D')} e^{ax+by} = \frac{1}{f(a,\ b)} e^{ax+by}, \qquad f(a,\ b) \neq 0}$$

Note: Suppose $f(a,\ b) = 0$, then multiply the numerator by x and differentiate the denominator by D, as a result we get

$$\boxed{\frac{1}{f(D,\ D')} e^{ax+by} = \frac{x^n}{n!} e^{ax+by}.}$$

Particular Integral(P.I):

Type 2: If $F(x,\ y) = \sin(ax+by)$ or $\cos(ax+by)$, then P.I is given by

$$\begin{aligned} \text{P.I} &= \frac{1}{f\left(D^2,\ DD',\ D'^2\right)} \sin(ax+by)/\cos(ax+by) \\ &= \frac{1}{f\left(-a^2,\ -ab,\ -b^2\right)} \sin(ax+by)/\cos(ax+by), \end{aligned}$$

provided $f\left(-a^2,\ -ab,\ -b^2\right) \neq 0$.

Note: Suppose $f\left(-a^2,\ -ab,\ -b^2\right) = 0$, then follow the previous note.

Particular Integral(P.I):

Type 3: If $F(x,\ y) = x^r y^s$, then P.I is given by

$$\boxed{\text{P.I} = \tfrac{1}{f(D,\ D')} x^r y^s,}$$

where we can expand $[f\,(D,\ D')]^{-1}$ using binomial expansion in power of D and D'.

Type 4: If $F(x,\ y) = \phi(x,y)e^{ax+by}$, then P.I is given by

$$\text{P.I} \quad = \quad \frac{1}{f\,(D,\ D')}\phi(x,y)e^{ax+by} = \frac{e^{ax+by}}{f\,(D+a,\ D'+b)}\phi(x,y),$$

then proceed as type 3.

Particular Integral(P.I):

Type 5: If $F(x,\ y) = \sin ax \sin by$, then P.I is given by

$$\text{P.I} = \frac{1}{f\left(D^2,\ D'^2\right)}.\sin ax \sin by = \frac{\sin ax \sin by}{f\left(-a^2,\ -b^2\right)}, \quad f\left(-a^2,\ -b^2\right) \neq 0.$$

Type 6: If $F(x,\ y) = \cos ax \cos by$, then P.I is given by

$$\text{P.I} = \frac{1}{f\left(D^2,\ D'^2\right)}.\cos ax \cos by = \frac{\cos ax \cos by}{f\left(-a^2,\ -b^2\right)}, \quad f\left(-a^2,\ -b^2\right) \neq 0.$$

Note: If $f\left(-a^2,\ -b^2\right) = 0$ in both type, then proceed as the note given in Type 1.

Homogeneous Linear PDE:

Example 1: Solve $\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial y^3} = e^{x+2y}$.

Solution: The above equation can be written as

$$\left(D^3 - 3D^2D' + 4{D'}^3\right) z = e^{x+2y}, \qquad \text{here}$$

$$f\left(D,\ D'\right) = D^3 - 3D^2D' + 4{D'}^3 \quad \text{and from} \quad e^{x+2y}, \ \ a = 1, \ \ b = 2.$$

Put $D = m$ and $D' = 1$ in the above equation and write the auxiliary equation as

$$m^3 - 3m^2 + 4 = 0 \quad \Rightarrow (m+1)(m-2)^2 = 0 \quad \Rightarrow m = -1,\ 2,\ 2,$$

where 2 is repeated root. Therefore, the C.F. is

$$\boxed{C.F. = \phi_1(y - x) + x\phi_2(y + 2x) + x^2\phi_2(y + 2x).}$$

Homogeneous Linear PDE:

P.I.

$$
\begin{aligned}
P.I. &= \frac{1}{D^3 - 3D^2D' + 4D'^3} \times e^{x+2y}, \\
&\Rightarrow \frac{1}{(1)^3 - 3(1)^2(2) + 4(2)^3} \times e^{x+2y} \\
&= \frac{e^{x+2y}}{27}.
\end{aligned}
$$

$\therefore$ The complete solution is given by:

$$
\boxed{C.F. + P.I. = \phi_1(y-x) + x\phi_2(y+2x) + x^2\phi_3(y+2x) + \frac{e^{x+2y}}{27}.}
$$

Homogeneous Linear PDE:

Example 2: Solve $\left(D^3 - 7DD'^2 - 6D'^3\right) z = e^{2x+y}$.

Solution: The auxiliary equation is

$$m^3 - 7m - 6 = 0 \qquad \Rightarrow (m+1)(m^2 - m - 6) = 0$$
$$\Rightarrow (m+1)(m-3)(m+2) = 0 \qquad \Rightarrow m = -1,\ 3,\ -2$$

$\therefore$ The C.F. is

$$\boxed{C.F. = \phi_1(y - x) + \phi_2(y + 3x) + \phi_3(y - 2x).}$$

$$\begin{aligned} P.I. &= \frac{1}{D^3 - 7DD'^2 - 6D'^3} \times e^{2x+y} \Rightarrow \frac{1}{(2)^3 - 7(2)(1)^2 - 6(1)^3} \times e^{2x+y} \\ &= -\frac{e^{2x+y}}{12}. \end{aligned}$$

$$\boxed{z = C.F. + P.I. = \phi_1(y - x) + \phi_2(y + 3x) + \phi_3(y - 2x) - \frac{e^{2x+y}}{12}.}$$

Homogeneous Linear PDE:

Example 3: Solve $\left(D^2 - 4DD' + 4D'^2\right) z = e^{2x+y}$.

Solution: The auxiliary equation is

$$m^2 - 4m + 4 = 0 \quad \Rightarrow (m-2)^2 \quad \Rightarrow m = 2,\ 2. \therefore \text{ The C.F. is:}$$

$$\boxed{C.F. = \phi_1(y+2x) + x\phi_2(y+2x).}$$

$$\begin{aligned} P.I. &= \frac{1}{D^2 - 4DD' + 4D'^2} \times e^{2x+y} \Rightarrow \frac{1}{(2)^2 - 4(2)(1) + 4(1)^2} \times e^{2x+y} \\ &= -\frac{e^{2x+y}}{0}. \end{aligned}$$

Hence we follow the Note.

Homogeneous Linear PDE:

Differentiate the denominator w.r.t D and multiply the numerator by x

$$P.I. = \frac{x}{2D - 4D'} \times e^{2x+y} = \frac{e^{2x+y}}{0}$$

$\therefore$ Once again differentiating the denominator w.r.t D and multiply the numerator by x, we get

$$P.I. = \frac{x^2}{2} \times e^{2x+y} \Rightarrow \frac{e^{2x+y}}{2}.$$

$\therefore$ The complete integral is given by:

$$\boxed{z = C.F. + P.I. = \phi_1(y + 2x) + x\phi_2(y + 2x) + \tfrac{e^{2x+y}}{2}.}$$

Homogeneous Linear PDE:

Example 4: Solve $\left(D^3 - 2D^2D'\right) z = \sin(x + 2y) + 3x^2y.$

Solution: The auxiliary equation is
$m^3 - 2m^2 = 0 \quad \Rightarrow m^2(m-2) = 0 \quad \Rightarrow m = 0,\ 0,\ 2.$
$\therefore$ The C.F. is:

$$\boxed{C.F. = \phi_1(y) + x\phi_2(y) + \phi_2(y + 2x).}$$

Here we have to find two P.I. corresponding to two terms $\sin(x + 2y)$ and $3x^2y$ as in Type 2 and Type 3 respectively.

$$(P.I.)_1 = \frac{1}{D^3 - 2D^2D'} \times \sin(x + 2y) \text{ and } (P.I.)_1 = \frac{1}{D^3 - 2D^2D'} \times 3x^2y$$

Homogeneous Linear PDE:

$$
\begin{aligned}
(P.I.)_1 &= \frac{1}{D^3 - 2D^2 D'} \times \sin(x+2y) \\
&= \frac{1}{D.D^2 - 2D.DD'} \times \sin(x+2y) \\
&= \frac{1}{D.-(1)^2 - 2D.(-1.2)} \times \sin(x+2y) \\
&= \frac{1}{-D+4D} \times \sin(x+2y) = \frac{1}{3D} \times \sin(x+2y).
\end{aligned}
$$

As $\frac{1}{D}$ is integration w.r.t x, hence:

$$
\boxed{(P.I.)_1 = -\tfrac{1}{3} \times \cos(x+2y).}
$$

Homogeneous Linear PDE:

$$
\begin{aligned}
(P.I.)_2 &= \frac{1}{D^3 - 2D^2D'} \times 3x^2y = \frac{1}{D^3\left(1 - \frac{2D'}{D}\right)} \times 3x^2y \\
&= \frac{1}{D^3}\left(1 - \frac{2D'}{D}\right)^{-1} \times 3x^2y \quad \text{(Using Binomial expansion)} \\
&= \frac{1}{D^3}\left[1 + \frac{2D'}{D} - \frac{4D'^2}{D^2} + - - - - - - - -\right] \times 3x^2y. \\
&\Rightarrow \frac{1}{D^3}\left[3x^2y + \frac{6x^2}{D}\right] = \frac{1}{D^3}\left[3x^2y + 2x^3\right]
\end{aligned}
$$

Integrating w.r.t x thrice, we get $(P.I.)_2 = \frac{x^5y}{20} + \frac{x^6}{60}$. Finally we get

$$
\begin{aligned}
z &= C.F. + (P.I.)_1 + (P.I.)_2 \\
&= \phi_1(y) + x\phi_2(y) + \phi_2(y + 2x) - \frac{1}{3} \times \cos(x + 2y) + \frac{x^5y}{20} + \frac{x^6}{60}.
\end{aligned}
$$

Problems for Practice:

1. Solve the equation $\left(D^3 - D^2D' - 8DD'^2 + 12D'^3\right)z = 0$

   Ans: $z = xf_1(y+2x) + f_2(y+2x) + f_3(y-3x)$

2. Solve the equation $\left(9D^2 + 6DD' + D'^2\right)z = \left(e^x + e^{-2y}\right)^2$

   Ans: $z = xf_1(3y-x) + f_2(3y-x) + \frac{e^{2x}}{36} + \frac{e^{-4y}}{16} + 2e^{x-2y}$

3. Solve the equation $\left(D^3 + D^2D' - 4DD'^2 - 4D'^3\right)z = \cos(2x+y)$

   Ans: $z = f_1(y-x) + f_2(y-2x) + f_3(y+2x) - \frac{x}{4}\cos(2x-y)$

4. Solve the equation $\left(D^2 - 2DD' + D'^2\right)z = x^2y^2e^{x+y}$

   Ans: $z = xf_1(y+x) + f_2(y+x) + \left(\frac{y^2}{12} + \frac{xy}{15} + \frac{x^2}{60}\right)x^4e^{x+y}$

5. Solve the equation $\left(D^2 - D'^2\right)z = e^{x-y}\sin(2x+3y)$

   Ans: $z = f_1(y+x) + f_2(y-x) + \frac{e^{x-y}}{25}\left[\sin(2x+3y) - 2\cos(2x+3y)\right]$

Thank You