Wednesday, October 27, 2021 3:26 PM

Taylor's theorem for functions of Two Variables

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots$$

$\hookrightarrow$ Single Variable

$$f(x+h, y+k) = \left\{ f(x, y+k) + h \frac{\partial}{\partial x} f(x,y+k) + \frac{h^2}{2!} \frac{\partial^2}{\partial x^2} f(x,y+k) + \frac{h^3}{3!} \frac{\partial^3}{\partial x^3} f(x,y+k) + \cdots \right.$$

$$= \left\{ f(x,y) + k \frac{\partial}{\partial y} f(x,y) + \frac{k^2}{2!} \frac{\partial^2}{\partial y^2} f(x,y) + \frac{k^3}{3!} \frac{\partial^3}{\partial y^3} f(x,y) + \cdots \right.$$

$$\Rightarrow \boxed{f(x+h, y+k) = f(x,y) + \left[h\frac{\partial}{\partial x} + k \frac{\partial}{\partial y}\right] f + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f + \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k \frac{\partial}{\partial y}\right)^3 f + \cdots}$$

$\downarrow$ General

Result 1

① $x = a,\ y = b$ ✓

$$f(a+h, b+k) = f(a,b) + \left[h f_x(a,b) + k f_y(a,b)\right] + \frac{1}{2!}\left[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)\right]$$
$$+ \frac{1}{3!}\left[h^3 f_{xxx}(a,b) + 3h^2k f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b) + k^3 f_{yyy}(a,b)\right] + \cdots$$

degree 1 ——

degree 2 ——

degree 3 ——

Result 2

$a = 0,\ b = 0$

$$f(x,y) = f(0,0) + \left[x f_x(0,0) + y f_y(0,0)\right] + \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right] + \cdots$$

This is called Called **Maclaurin's theorem**

In the problem

In the powers of $(x-1)$ and $(y+2)$

$\downarrow \qquad \downarrow$

$(x-a) \qquad (y-b)$

$\boxed{a = 1,\ b = -2}$

Examples:-

① $f = x^3y^3 + 3x^2y$

Find 1) $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$

2) $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y}$

3) $\frac{\partial^3 f}{\partial x^3}, \frac{\partial^3 f}{\partial y^3}, \frac{\partial^3 f}{\partial x \partial y^2}, \frac{\partial^3 f}{\partial x^2 \partial y}$

Sol

1) $\frac{\partial f}{\partial x} = y^3(3x^2) + 3y(2x) = 3x^2y^3 + 6xy$

$\frac{\partial f}{\partial y} = x^3(3y^2) + 3x^2(1) = 3x^3y^2 + 3x^2$

2) ✓ $\frac{\partial^2 f}{\partial x^2} \Rightarrow 6xy^3 + 6y$

$\frac{\partial^2 f}{\partial y^2} \Rightarrow 6x^3y$

$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 9x^2y^2 + 6x$

3) $\frac{\partial^3 f}{\partial x^3} = 6y^3$ | $\frac{\partial^3 f}{\partial y^3} = 6x^3$

$\frac{\partial^3 f}{\partial x \partial y^2} = 18x^2y$ | $\frac{\partial^3 f}{\partial x^2 \partial y} = 18xy^2 + 6$

$\downarrow \qquad \downarrow$

$\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y^2}\right) \qquad \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x \partial y}\right)$

② **Home work**

$f(x,y) = e^x \cos y$

③ **Home work**

$f(x,y) = \tan^{-1}(y/x)$