PROBABILITY
&
QUEUEING THEORY

(As per SRM INSTITUTE OF SCIENCE AND TECHNOLOGY Syllabus)

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PROBABILITY AND QUEUEING THEORY
UNIT – I : RANDOM VARIABLES

Syllabus

- Review of Probability Concepts - Types of Events, Axioms
- Conditional Probability, Multiplication Theorem, Applications
- Discrete Random Variable
- Continuous Random Variable
- Expectation and Variance
- Moment Generating Function
- Function of Random Variable (One Dimensional Only)
- Chebychev's Inequality

PROBABILITY

Probability (or) Chance: Probably, Chances, Likely, Possible - The terms convey the same meaning.

Example:

1. ***Probably*** your method is correct
2. The ***chances*** of getting ranks Ram and Gothai are equal.
3. It is ***likely*** that Ram may not come for taking his classes today.
4. It is ***possible*** to reach the college by 8.30am.

Ordinary Language: The word probability means uncertainty about happening.

Mathematics or Statistics : A numerical measure of uncertainty is practiced by the important branch of statistics is called the **Theory of Probability.**

Day to Day Life:

- ***Certainty*** - Every day the sun rises in the east
- ***Impossibility*** - It is possible to live without water
- ***Uncertainty*** - Probably Raman gets that job.

In the theory of probability, we represent certainty by 1, impossibility by 0 and uncertainty by a positive fraction which lies between 0 and 1.

Applications : There is no area in **social, physical (or) natural sciences** where the probability theory is not used.

- It is the base of the fundamental laws of statistics.
- It gives solutions to betting of games.
- It is extensively used in business situations characterized by uncertainty.
- It is essential tool in statistical inference and forms the basis of the Decision Theory.

Random Experiment (or) Trial and Event (or) Cases:

An experiment in which the outcome cannot be predicted with certainty is called a random experiment, even though all possible outcomes are known in advance. Tossing a coin is a **random experiment** and getting a head or tail is an **event.**

Favourable Events :

The number of cases favourable to an event in a trial is the number of outcomes which entail the happening of the event.

Example: In tossing 2 coins the cases favourable to the event of getting a head are HT, TH, and HH.

Exhaustive Events : The total number of possible outcomes in any **trial** is known as exhaustive events.

Example: In tossing a coin the possible outcomes are getting a head or tail. Hence we have 2 exhaustive events in throwing a coin.

Mutually Exclusive Event:

Two events are said to be mutually exclusive when the occurrence of one affects the occurrence of the other. In other words, if A & B are mutually exclusive events and if A happens then B will not happen and vice versa.

Example: In tossing a coin the events head or tail are mutually exclusive, since both tail & head cannot appear in the same time.

Equally Likely Events: Two events are said to be equally likely if one of them cannot be expected in preference to the other. ***Example:*** In tossing a coin, head or tail are equally likely events.

Independent Event : Two events are said to be independent when the actual happening of one does not influence in any way the happening of the other. ***Example :*** In tossing a coin, the event of getting a head in the 1^{st} toss is independent of getting a head in the 2^{nd} toss, 3^{rd} toss, etc.

Mathematical Definition of Probability:

If P is the notation for probability of happening of the event, then $P(A) = \frac{\textit{Number of Favourable Cases}}{\textit{Total Number of Exhaustive Cases}} = \frac{m}{n}$

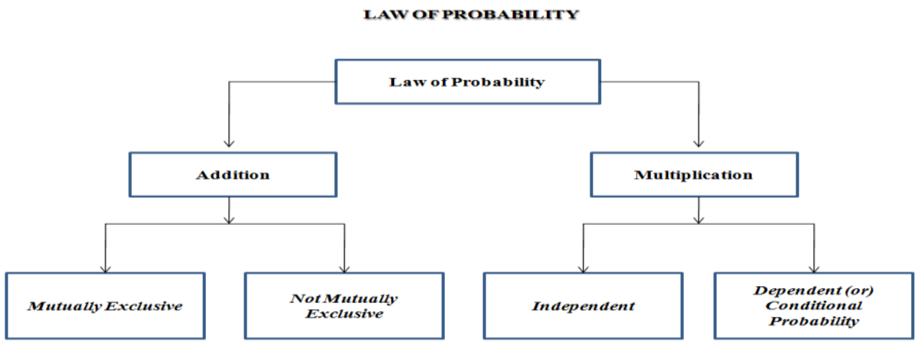
Statistical Definition of Probability:

If in n trials, an event E happens m times, then $P(E) = \lim_{n\to\infty} \frac{m}{n}$

Axiomatic Definition of Probability:

1. For any event A, $P(A) \geq 0$.
2. $P(S) = 1$
3. If $A_1, A_2, A_3, \dots, A_n$ are finite number of disjoint events of S, then
   $P(A_1 \cup A_2 \cup A_3 \cup \dots) = P(A_1) + P(A_2) + P(A_3) + \cdots = \sum P(A_i)$

LAW OF PROBABILITY



ADDITION LAW OF PROBABILITY

Case (i): When events are mutually exclusive

If A and B are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$.

Case (ii): When events are not mutually exclusive

If A and B are any two events, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

MULTIPLICATION LAW OF PROBABILITY

Case (i): When events are independent : The probability that both independent events, A and B will occur is equal to product of the probabilities of each event, then $P(A \cap B) = P(A)\,P(B)$.

Case (ii): When events are dependent (or) conditional probability: If the occurrence of an event A is affected by the occurrence of the another event B, then the events A and B are dependent. $P(A \cap B) = P(A)\,P(B/A) = P(B)\,P(A/B)$

RANDOM VARIABLE

The outcomes of many random experiments may be non-numerical. It is inconvenient to deal with these descriptive outcomes mathematically.

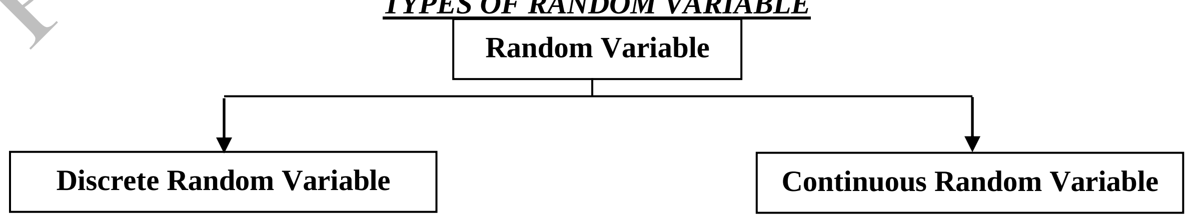
Example: When toss a coin we get two outcomes, namely head or tail. We can assign numerical values; say 1 to head and 0 to tail. This interpretation is easy and attractive from mathematical point of view and also practically meaningful.

Example: Three students sat for an examination & X denotes the number of students who passed. Describe the RV X.

Sample Space S	None	S_1	S_2	S_3	S_1S_2	S_2S_3	S_3S_1	$S_1S_2S_3$
No. of Students who passed X	0	1	1	1	2	2	2	3

$n(S) = 8, \quad P(X=0) = \frac{1}{8}, \quad P(X=1) = \frac{3}{8}, \quad P(X=2) = \frac{3}{8}, \quad P(X=3) = \frac{1}{8}$

TYPES OF RANDOM VARIABLE



DISCRETE RANDOM VARIABLE

A random variable X is discrete, if it assumes only finite number or countably infinite number of values.

Example : (i) *The mark obtained by a student in an examination. It's possible values are 0, 85 or 100.*

(ii) The number of students who are absent for a particular period.

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1. Probability Mass Function (p.m.f.) $\sum_{i=1}^{\infty} P(x_i) = 1$ 2. Mean $E(X) = \sum_{i=1}^{\infty} x_i\, P(x_i)$, $E(X^2) = \sum_{i=1}^{\infty} {x_i}^2\, P(x_i)$
3. Variance $V(X) = E(X^2) - [E(X)]^2$ 4. Cumulative Distribution Function (c.d.f.) $F(X) = P(X \le x) = \sum_{i=1}^{x} P(x_i)$

CONTINUOUS RANDOM VARIABLE

A RV X is continuous, if it takes all possible values between certain limits or in an interval which may be finite or infinite. ***E.g:***(i)The density of milk taken for testing at a farm.(ii)The operating time between two failures of a computer.

1. Probability Density Function (p.d.f.) $\int_{-\infty}^{\infty} f(x)dx = 1$ 2. Mean E$(X) = \int_{-\infty}^{\infty} x\, f(x)dx$, $E(X^2) = \int_{-\infty}^{\infty} x^2\, f(x)dx$
3. Variance $V(X) = E(X^2) - [E(X)]^2$ 4. Cumulative Distribution Function (c.d.f.) $F(X) = P(X \le x) = \int_{-\infty}^{x} f(x)dx$

PROPERTIES OF EXPECTATION

If X and Y are random variables and a, b are constants, then

1. $E(a) = a$ 2. $E(aX) = aE(X)$ 3. $E(aX + b) = aE(X) + b$ 4. $E(X - \bar{X}) = 0$ 5. $|E(X)| \le E(|X|)$
6. $E(X) \ge 0, if\ X \ge 0$ 7. $E(X + Y) = E(X) + E(Y)$ ***(Additive Theorem)***
8. $E(XY) = E(X)E(Y)$ ***(∵ A and B are independent)***
9. $E\big(a\, g(X)\big) = aE\big(g(X)\big)$ 10. $E(g(X) + a) = E\big(g(X)\big) + a$
11. $(E[g(X)\,]) = g[E(X)]$ $[\boldsymbol{g(X)\ is\ linear\ in\ X}]$
12. $P(X \ge a) \le \frac{E(X)}{a}, a > 0$ ***(Markov Inequality)***
13. $P\{|X - E(X)| \ge k\} \ge \frac{\sigma_X^2}{\epsilon^2}$ ***(Chebyshev's Inequality)***

PROPERTIES OF VARIANCE

1. $Var(X) \ge 0$ 2. $E(X^2) \ge [E(X)]^2$ 3. $Var(b) = 0, b\ constant$
4. If X is a random variables, a is constants then $Var(aX) = a^2 Var(X)$
5. If a and b are constants, $Var(aX \pm b) = a^2 Var(X)$
6. If X and Y are two independent RV, a and b are constants then $Var(aX + bY) = a^2 Var(X) + b^2 Var(Y)$

PROPERTIES OF CUMULATIVE DISTRIBUTION FUNCTION

1. If F is the distribution function of the RV X and if $a < b$, then
   $P(a < X \le b) = P(a \le X \le b) = P(a < X < b) = P(a \le X < b) = F(b) - F(a)$
2. If F is the distribution function of one dimensional RV X, then (i) $0 \le F(X) \le 1$ (ii) $F(X) \le F(Y)$, if $x < y$
   In other words, all distribution functions are monotonically non-decreasing and lie between 0 and 1.
3. If F is the distribution function of one dimensional random variable X, then
   $F(-\infty) = \lim_{x \to -\infty} F(X) = 0$ and $F(\infty) = \lim_{x \to \infty} F(X) = 1$ 4. $f(x) = \frac{d}{dx}\big(F(x)\big)$

MOMENTS

Definition: The n^{th} moment about origin of a RV X is defined as the expected value of the n^{th} power of X.

Moments about Origin (Raw Moments)

Discrete : $\mu_n' = E(X^n) = \sum_i x_i^n p_i\ , n \ge 1$. ***Continuous:*** $\mu_n' = E(X^n) = \int_{-\infty}^{\infty} x^n\, f(x)dx\ , n \ge 1$

Moment about Mean (Central Moments)

Discrete : $\mu_n = E[(X - \bar{X})^n] = \sum_i (x_i - \bar{X})^n\, p_i$, ***Continuous:*** $\mu_n = E[(X - \bar{X})^n] = \int_{-\infty}^{\infty} (x - \bar{X})^n\, f(x)dx\ ,\ n \ge 1$

Relationship between moments about origin and moment about mean

$$\mu_r = \mu_r' - rC_1\, \mu\, \mu_{r-1}^1 + rC_2\, \mu^2\, \mu_{r-2}' - \cdots$$

Hence, $\mu_1 = 0$, $\mu_2 = \mu_2' - (\mu_1')^2$, $\mu_3 = \mu_3' - 3\mu_2'\, \mu_1' + 2(\mu_1')^3$, $\mu_4 = \mu_4' - 4\mu_3'\, \mu_1' + 6\mu_2'(\,\mu_1')^2 - 3(\mu_1')^4$

MOMENT GENERATING FUNCTION

Definition : Moment generating function of a random variable about the origin is defined as

Discrete : $M_X(t) = E(e^{tX}) = \sum_x e^{tx}\, p(x)$, ***Continuous :*** $M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx}\, f(x)dx$

Where the integration or summation is taken over the entire range of X, t being a real parameter, assuming that integration or summation is absolutely convergent.

$$M_X(t) = 1 + t\, \mu_1' + \frac{t^2}{2!}\mu_2' + \cdots + \frac{t^r}{r!}\mu_r' , \quad \text{Where } \mu_r' = \text{coefficient of } \frac{t^r}{r!} \text{ in } M_X(t)$$

Note: 1. $\mu_r' = \frac{d^r}{dt^r}[M_X(t)]_{t=0}$ 2. $M_{CX}(t) = M_X(Ct)$, C being a constant. 3. $M_{X=a}(t) = e^{-at}\, M_X(t)$

1. If $X_1, X_2, \ldots X_n$ are n independent RVs, then $M_{X_1 + X_2 + \cdots + X_n}(t) = M_{X_1}(t).M_{X_2}(t) \ldots M_{X_n}(t)$

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PROBLEMS IN DISCRETE RANDOM VARIABLE

1. ***A discrete RV X has the following probability distribution***

x	0	1	2	3	4	5	6	7	8
$p(x)$	a	$3a$	$5a$	$7a$	$9a$	$11a$	$13a$	$15a$	$17a$

(i) Find the value of a (ii) $P(X<3)$ (iii) $P(X\ge 3)$ (iv) $P(0<X<3)$ (v) Find the distribution function of X.

Solution

(i) $\sum_{x=0}^{8} P(x) = 1 \Rightarrow P(0)+P(1)+P(2)+P(3)+P(4)+P(5)+P(6)+P(7)+P(8)=1$

$a+3a+5a+7a+9a+11a+13a+15a+17a=1 \Rightarrow 81a=1 \Rightarrow \boldsymbol{a=\frac{1}{81}}$

(ii) $P(X<3)=P(X=0)+P(X=1)+P(X=2)=a+3a+5a=9a=9\times\frac{1}{81}=\frac{1}{9}$

(iii) $P(X\ge 3)=1-P(X<3)=1-\frac{1}{9}=\frac{8}{9}$

(iv) $P(0<X<3)=P(X=1)+P(X=2)=3a+5a=8a=\frac{8}{81}$

(v)

x	0	1	2	3	4	5	6	7	8
$p(x)$	$\frac{1}{81}$	$\frac{3}{81}$	$\frac{5}{81}$	$\frac{7}{81}$	$\frac{9}{81}$	$\frac{11}{81}$	$\frac{13}{81}$	$\frac{15}{81}$	$\frac{17}{81}$
$F(x)$	$\frac{1}{81}$	$\frac{4}{81}$	$\frac{9}{81}$	$\frac{16}{81}$	$\frac{25}{81}$	$\frac{36}{81}$	$\frac{49}{81}$	$\frac{64}{81}$	1

2. ***A discrete random variable X has the probability function given below:***

x	0	1	2	3	4	5	6	7
$p(x)$	0	K	$2K$	$2K$	$3K$	K^2	$2K^2$	$7K^2+K$

Find (i) The value of K (ii) $P(1.5<X<4.5/X>2)$ (iii) The smallest value of λ for which $P(X\le\lambda)>1/2$.

Solution:

(i) $\sum_{x=0}^{7} P(x)=1 \Rightarrow P(0)+P(1)+P(2)+P(3)+P(4)+P(5)+P(6)+P(7)=1$

$0+K+2K+2K+3K+K^2+2K^2+7K^2+K=1 \Rightarrow 10K^2+9K=1$

$(10K-1)(K+1)=0 \Rightarrow K=\frac{1}{10}, -1 \Rightarrow \boldsymbol{K=\frac{1}{10}}$ **($\because K=-1$, which is meaningless)**

(ii) $P(1.5<X<4.5/X>2)=\frac{P[(1.5<X<4.5)\cap(X>2)]}{P(X>2)}$ $\qquad \because \boldsymbol{P(A/B)=\frac{P(A\cap B)}{P(B)}}$

$P(1.5<X<4.5/X>2)=\frac{P(3)+P(4)}{P(3)+P(4)+P(5)+P(6)+P(7)}=\frac{\left(\frac{5}{10}\right)}{\left(\frac{7}{10}\right)}=\boldsymbol{\frac{5}{7}}$

(iii) $P(X\le\lambda)>\frac{1}{2},\ \lambda=0,\ P(X\le 0)=0\ngtr\frac{1}{2};\ \ \lambda=1,\ P(X\le 1)=\frac{1}{10}\ngtr\frac{1}{2};$

$\lambda=2,\ P(X\le 2)=\frac{3}{10}\ngtr\frac{1}{2};\ \lambda=3,\ P(X\le 3)=\frac{5}{10}\ngtr\frac{1}{2};\ \ \lambda=4,\ P(X\le 4)=\frac{8}{10}>\frac{1}{2}$

The smallest value of λ for which $P(X\le\lambda)>1/2$ is 4.

3. ***If the RV X takes the values 1, 2, 3 & 4 such that $2P(X=1)=3P(X=2)=P(X=3)=5P(X=4)$, find the probability distribution and cumulative distribution function of X.***

Solution: Let $2P(X=1)=3P(X=2)=P(X=3)=5P(X=4)=30K$

x	1	2	3	4
$p(x)$	$15K$	$10K$	$30K$	$6K$

$\sum_{x=1}^{4}P(x)=1 \Rightarrow P(1)+P(2)+P(3)+P(4)=1 \Rightarrow 15K+10K+30K+6K=1 \Rightarrow 61K=1 \Rightarrow \boldsymbol{K=\frac{1}{61}}$

Cumulative distribution function of X

x	1	2	3	4
$p(x)$	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$
$F(x)$	$\frac{15}{61}$	$\frac{25}{61}$	$\frac{55}{61}$	1

4. ***A discrete RV X has the following probability distribution***

x	-2	-1	0	1	2	3
$p(x)$	0.1	K	0.2	$2K$	0.3	$3K$

Find (i) K (ii) $P(X<2)$ (iii) $P(-2<X<2)$ (iv) the cdf of X (v) the mean of X.

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Solution

(i) $\sum_{x=-2}^{3} P(x) = 1 \Rightarrow P(-2) + P(-1) + P(0) + P(1) + P(2) + P(3) = 1 \Rightarrow 6K + 0.6 = 1 \Rightarrow \boldsymbol{K = \frac{1}{15}}$

	-2	-1	0	1	2	3
$p(x)$	$\frac{1}{10}$	$\frac{1}{15}$	$\frac{2}{10}$	$\frac{2}{15}$	$\frac{3}{10}$	$\frac{3}{15}$

(ii) $P(X < 2) = P(-2) + P(-1) + P(0) + P(1) = \frac{1}{10} + \frac{1}{15} + \frac{2}{10} + \frac{2}{15} = \frac{1}{2}$

(iii) $P(-2 < X < 2) = P(-1) + P(0) + P(1) = \frac{1}{15} + \frac{2}{10} + \frac{2}{15} = \frac{2}{5}$

(iv)

x	-2	-1	0	1	2	3
$p(x)$	$\frac{1}{10}$	$\frac{1}{15}$	$\frac{2}{10}$	$\frac{2}{15}$	$\frac{3}{10}$	$\frac{3}{15}$
$F(X)$	$\frac{1}{10}$	$\frac{1}{6}$	$\frac{11}{30}$	$\frac{1}{2}$	$\frac{4}{5}$	1

(v) ***Mean of X***

$$E(X) = \sum_{x=-2}^{3} x\, P(x) = (-2)P(-2) + (-1)P(-1) + 0\,P(0) + 1\,P(1) + 2\,P(2) + 3\,P(3)$$
$$= \left(-2 \times \frac{1}{10}\right) + \left(-1 \times \frac{1}{15}\right) + \left(0 \times \frac{2}{10}\right) + \left(1 \times \frac{2}{15}\right) + \left(2 \times \frac{3}{10}\right) + \left(3 \times \frac{3}{15}\right) = \frac{16}{15}$$

5. ***If X is RV having the density function*** $(x) = \begin{cases} \frac{x}{6}\ \boldsymbol{for\ x = 1, 2, 3} \\ \boldsymbol{0, otherwise} \end{cases}$. ***Find*** $\boldsymbol{E(X^3 + 2X + 7)}$ ***and*** $\boldsymbol{Var\,(4X + 5)}$.

Solution

x	1	2	3
$p(x)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$

$$E(X) = \sum_{x=1}^{3} x\, P(x) = \left(1 \times \frac{1}{6}\right) + \left(2 \times \frac{2}{6}\right) + \left(3 \times \frac{3}{6}\right) = \frac{7}{3}$$
$$E(X^2) = \sum_{x=1}^{3} x^2\, P(x) = \left(1 \times \frac{1}{6}\right) + \left(4 \times \frac{2}{6}\right) + \left(9 \times \frac{3}{6}\right) = 6$$
$$E(X^3) = \sum_{x=1}^{3} x^3\, P(x) = \left(1 \times \frac{1}{6}\right) + \left(8 \times \frac{2}{6}\right) + \left(27 \times \frac{3}{6}\right) = \frac{49}{3}$$
$$E(X^3 + 2X + 7) = E(X^3) + 2E(X) + 7 = \frac{84}{3}$$
$$Var(X) = E(X^2) - [E(X)]^2 = \frac{5}{9}, \quad Var\,(4X + 5) = 4^2 Var(X) = 16 \times \frac{5}{9} = \frac{80}{9}$$

6. ***If X has the distribution function*** $\boldsymbol{F(x)} = \begin{cases} \boldsymbol{0}, & \boldsymbol{x < 1} \\ \boldsymbol{\frac{1}{3}}, & \boldsymbol{1 \le x < 4} \\ \boldsymbol{\frac{1}{2}}, & \boldsymbol{4 \le x < 6} \\ \boldsymbol{\frac{5}{6}}, & \boldsymbol{6 \le x < 10} \\ \boldsymbol{1}, & \boldsymbol{x \ge 10} \end{cases}$

Find (i) The probability distribution of X (ii) $\boldsymbol{P(2 < X < 6)}$ ***(iii) Mean of X (iv) Variance of X.***

Solution: (i) For the given c.d.f., the probability distribution of X is

$$P(X = 1) = F(1) - F(0) = \frac{1}{3} - 0 = \frac{1}{3}, \qquad P(X = 4) = F(4) - F(1) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6},$$
$$P(X = 6) = F(6) - F(4) = \frac{5}{6} - \frac{1}{2} = \frac{1}{3}, \qquad P(X = 10) = F(10) - F(6) = 1 - \frac{5}{6} = \frac{1}{6}$$

x	1	4	6	10
$p(x)$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$

(i) $P(2 < X < 6) = P(X = 4) = \frac{1}{6}$

(ii) $E(X) = \sum_i x_i\, P(x_i) = \left(1 \times \frac{1}{3}\right) + \left(4 \times \frac{1}{6}\right) + \left(6 \times \frac{2}{6}\right) + \left(10 \times \frac{1}{6}\right) = \frac{14}{3}$

$$E(X^2) = \sum_i x_i^2\, P(x_i) = \left(1 \times \frac{1}{3}\right) + \left(16 \times \frac{1}{6}\right) + \left(36 \times \frac{2}{6}\right) + \left(100 \times \frac{1}{6}\right) = \frac{95}{3}$$
$$Var(X) = E(X^2) - [E(X)]^2 = \frac{89}{3}$$

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7. ***When a die is thrown, X denotes the number that turns up. Find $E(X), E(X^2), Var(X)$ and standard deviation.***

Solution: $p = \frac{1}{6}$, X = 1, 2, 3, 4, 5, 6 Here X is a discrete RV

$$E(X) = \sum_i x_i P(x_i) = \left(1\times\frac{1}{6}\right) + \left(2\times\frac{1}{6}\right) + \left(3\times\frac{1}{6}\right) + \left(4\times\frac{1}{6}\right) + \left(5\times\frac{1}{6}\right) + \left(6\times\frac{1}{6}\right) = 3.5$$

$$E(X^2) = \sum_i x_i^2 P(x_i) = \left(1\times\frac{1}{6}\right) + \left(4\times\frac{1}{6}\right) + \left(9\times\frac{1}{6}\right) + \left(16\times\frac{1}{6}\right) + \left(25\times\frac{1}{6}\right) + \left(36\times\frac{1}{6}\right) = \frac{91}{6} = 15.167$$

$$Var(X) = E(X^2) - [E(X)]^2 = 2.9166, \quad S.D. = \sigma_X = \sqrt{Var(X)} = 1.7078$$

8. ***A coin is tossed until a head appears. What is the expectation of the number of tosses required?***

Solution: Let X – No. of tosses required to get the 1st head. The 1st head may appear in the 1st or 2nd … and so on. The events are H, TH, TTH, TTTH, … $p = \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots$

x	1	2	3	4	5	…
$p(x)$	$\frac{1}{2}$	$\frac{1}{2^2}$	$\frac{1}{2^3}$	$\frac{1}{2^4}$	$\frac{1}{2^5}$	…

$$E(X) = \sum_i x_i P(x_i) = \frac{1}{2}\left[1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \cdots\right] = \frac{1}{2}\left(1-\frac{1}{2}\right)^{-2} = 2 \qquad [\because (\mathbf{1-x})^{-2} = \mathbf{1 + 2x + 3x^2} + \cdots]$$

9. ***By throwing a fair dice, a player gains Rs. 20 if 2 turns up, gains Rs. 40 if 4 turns up and loses Rs. 30 if 6 turns up. He never loses or gains if any other number turns up. Find the expected value of money he gains.***

Solution: Let X – money won on an trial. x_i = Amount of money won, if the faces show $i = 1, 2,3,4,5,6$.

	1	2	3	4	5	6
x	0	20	0	40	0	-30
$p(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$E(X) = \sum_i x_i P(x_i) = \left(0\times\frac{1}{6}\right) + \left(20\times\frac{1}{6}\right) + \left(0\times\frac{1}{6}\right) + \left(40\times\frac{1}{6}\right) + \left(0\times\frac{1}{6}\right) + \left(-30\times\frac{1}{6}\right) = 5$$

10. ***A RV X has the probability function $f(x) = \frac{1}{2^x}, x = 1, 2, 3\dots$ Find the (i) moment generating function (ii) Mean***

Solution :

(i) $M_X(t) = \sum_x e^{tx} p(x) = \sum_{x=1}^{\infty} e^{tx} \frac{1}{2^x} = \frac{e^t}{2}\left[1 + \left(\frac{e^t}{2}\right) + \left(\frac{e^t}{2}\right)^2 + \cdots\right] = \frac{e^t}{2}\left(1 - \frac{e^t}{2}\right)^{-1} = \frac{e^t}{2-e^t}$

(ii) $E(X) = \left[\frac{d}{dt} M_X(t)\right]_{t=0} = \left[\frac{d}{dt}\left(\frac{e^t}{2-e^t}\right)\right]_{t=0} = \left[\frac{(2-e^t)e^t - e^t(-e^t)}{(2-e^t)^2}\right]_{t=0} = \frac{(2-e^0)e^0 - e^0(-e^0)}{(2-e^0)^2} = 2$

11. ***If a RV X has moment generating function $M_X(t) = \frac{3}{3-t}$, obtain the standard deviation of X.***

Solution : $M_X(t) = \frac{3}{3-t} = \frac{3}{3\left(1-\frac{t}{3}\right)} = \left(1-\frac{t}{3}\right)^{-1} = 1 + \left(\frac{t}{3}\right) + \left(\frac{t}{3}\right)^2 + \left(\frac{t}{3}\right)^3 + \cdots = 1 + \frac{t}{1!}\left(\frac{1}{3}\right) + \frac{t^2}{2!}\left(\frac{2}{9}\right) + \frac{t^3}{3!}\left(\frac{6}{27}\right) + \cdots$

$$\mu_r' = coefficient\ of\ \frac{t^r}{r!}, \quad \mu_1' = coefficient\ of\ \frac{t^1}{1!} = \frac{1}{3}, \quad \mu_2' = coefficient\ of\ \frac{t^2}{2!} = \frac{2}{9}$$

Variance $= \mu_2' - (\mu_1')^2 = \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$, Standard deviation $= \sqrt{\text{Variance}} = \sqrt{\frac{1}{9}} = \frac{1}{3}$

PROBLEMS IN CONTINUOUS RANDOM VARIABLE

1. ***If $p(x) = \begin{cases} x\, e^{-\frac{x^2}{2}}, & x \ge 0 \\ 0, & x < 0 \end{cases}$ (i) Show that $p(x)$ is a p.d.f. (ii) Find its distribution function $P(x)$.***

Solution

(i) $\int_{-\infty}^{\infty} p(x)dx = \int_{-\infty}^{0} p(x)dx + \int_0^{\infty} p(x)dx = \int_{-\infty}^{0} 0\, dx + \int_0^{\infty} x\, e^{-\frac{x^2}{2}} dx = \int_0^{\infty} x\, e^{-\frac{x^2}{2}} dx$

Put $x^2 = t$, $2x\, dx = dt \Rightarrow x\, dx = \frac{dt}{2}$, $x = 0,\ t = 0$ and $x = \infty,\ t = \infty$

$$\int_{-\infty}^{\infty} p(x)dx = \int_0^{\infty} e^{-\frac{t}{2}} \frac{dt}{2} = \frac{1}{2}\int_0^{\infty} e^{-\frac{t}{2}} dt = \frac{1}{2}\left[\frac{e^{-\frac{t}{2}}}{-\frac{1}{2}}\right]_0^{\infty} = -e^{-\infty} + e^0 = 1 \qquad (\because \mathbf{e^{-\infty} = 0,\ e^0 = 1})$$

$\therefore p(x)$ is a p.d.f. of a RV X.

(ii) $F(X) = P(X \le x) = \int_0^x p(x)dx = \int_0^x x\, e^{-\frac{x^2}{2}} dx = 1 - e^{-\frac{x^2}{2}},\ x \ge 0$

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2. ***If the density function of a continuous RV X is given by*** $(x) = \begin{cases} \boldsymbol{ax}, & \boldsymbol{0 \le x \le 1} \\ \boldsymbol{a}, & \boldsymbol{1 \le x \le 2} \\ \boldsymbol{3a - ax}, & \boldsymbol{2 \le x \le 3} \\ \boldsymbol{0}, & \boldsymbol{otherwise} \end{cases}$ ***. Find (i) a (ii) c.d.f.***

Solution: (i) $\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (3a - ax)dx = 1$

$a\left[\frac{x^2}{2}\right]_0^1 + a[x]_1^2 + \left[3ax - \frac{ax^2}{2}\right]_2^3 = 1 \Rightarrow \frac{a}{2} + a(2-1) + \left(9a - \frac{9a}{2}\right) - \left(6a - \frac{4a}{2}\right) = 1 \Rightarrow \boldsymbol{a = \frac{1}{2}}$

$$f(x) = \begin{cases} \frac{x}{2}, & 0 \le x \le 1 \\ \frac{1}{2}, & 1 \le x \le 2 \\ \frac{3-x}{2}, & 2 \le x \le 3 \\ 0, & otherwise \end{cases}$$

(ii) ***c.d.f. of X:*** $F(X) = P(X \le x) = \int_{-\infty}^{x} f(x)dx$; If $x < 0$, *then* $F(X) = 0$, *since* $f(x) = 0$ *for* $x < 0$

If $0 \le x \le 1$, *then* $F(X) = \int_0^x \frac{x}{2}\,dx = \left[\frac{x^2}{4}\right]_0^x = \frac{x^2}{4}$

If $1 \le x \le 2$, *then* $F(X) = \int_0^x f(x)\,dx = \int_0^1 \left(\frac{x}{2}\right) dx + \int_1^x \left(\frac{1}{2}\right) dx = \left[\frac{x^2}{4}\right]_0^1 + \left[\frac{x}{2}\right]_1^x = \frac{1}{4} + \frac{x}{2} - \frac{1}{2} = \frac{1}{4}(2x-1)$

If $2 \le x \le 3$, *then* $F(X) = \int_0^x f(x)\,dx = \int_0^1 \left(\frac{x}{2}\right) dx + \int_1^2 \left(\frac{1}{2}\right) dx + \int_2^x (3a - ax)\,dx$

$$= \left[\frac{x^2}{4}\right]_0^1 + \left[\frac{x}{2}\right]_1^2 + \left[\frac{3x}{2} - \frac{x^2}{4}\right]_2^x = \frac{1}{4} + \frac{2}{2} - \frac{1}{2} + \left(\frac{3x}{2} - \frac{x^2}{4}\right) - \left(\frac{6}{2} - \frac{4}{4}\right) = \frac{1}{4}(6x - x^2 - 5)$$

If $x \ge 3$, *then* $F(X) = 1$

$$F(x) = \begin{cases} \frac{x^2}{4}, & 0 \le x \le 1 \\ \frac{1}{4}(2x-1), & 1 \le x \le 2 \\ \frac{1}{4}(6x - x^2 - 5), & 2 \le x \le 3 \\ 1, & x \ge 3 \end{cases}$$

3. ***A continuous RV X has a pdf*** $\boldsymbol{f(x) = 3x^2, 0 \le x \le 1}$***. Find a and b such that***
(i) $\boldsymbol{P(X \le a) = P(X > a)}$ ***(ii)*** $\boldsymbol{P(X > b) = 0.05}$
Solution:

(i) $P(X \le a) = P(X > a) \Rightarrow \int_{-\infty}^{a} f(x)dx = \int_a^{\infty} f(x)dx \Rightarrow \int_0^a 3\,x^2 dx = \int_a^1 f(x)dx \Rightarrow 3\left[\frac{x^3}{3}\right]_0^a = 3\left[\frac{x^3}{3}\right]_a^1$

$a^3 = 1 - a^3 \Rightarrow 2a^3 = 1 \Rightarrow a^3 = \frac{1}{2} \Rightarrow \boldsymbol{a} = \left(\frac{1}{2}\right)^{\frac{1}{3}} = \boldsymbol{0.7937}$

(ii) $P(X > b) = 0.05 \Rightarrow \int_b^1 3\,x^2 dx = 0.05 \Rightarrow 3\left[\frac{x^3}{3}\right]_b^1 = 0.05 \Rightarrow 1 - b^3 = 0.05 \Rightarrow \boldsymbol{b = (0.95)^{\frac{1}{3}} = 0.9830}$

4. ***A Continuous RV X that can assume any value between*** $\boldsymbol{x = 2}$ ***and*** $\boldsymbol{x = 5}$ ***has a density function given by*** $\boldsymbol{f(x) = k(1+x)}$***. Find*** $\boldsymbol{P(X < 4)}$***.***

Solution : $\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_2^5 k(1+x)dx = 1 \Rightarrow k\left[x + \frac{x^2}{2}\right]_2^5 = 1 \Rightarrow k\left[\left(5 + \frac{25}{2}\right) - \left(2 + \frac{4}{2}\right)\right] = 1 \Rightarrow \boldsymbol{k = \frac{2}{27}}$

$$P(X < 4) = \frac{2}{27}\int_2^4 (1+x)dx = \frac{2}{27}\left[x + \frac{x^2}{2}\right]_2^4 = \frac{2}{27}\left[\left(4 + \frac{16}{2}\right) - \left(2 + \frac{4}{2}\right)\right] = \frac{16}{27}$$

5. ***A RV X has a pdf*** $\boldsymbol{f(x) = kx^2 e^{-x}, x \ge 0}$***. Find k, mean, variance and*** $\boldsymbol{E(3X^2 - 2X)}$***.***

Solution: $\int_{-\infty}^{\infty} f(x)dx = 1$, $\int_0^{\infty} kx^2 e^{-x} dx = 1$

$\boldsymbol{Differentiation}: u = x^2,\ u' = 2x,\ u'' = 2,\ u''' = 0$

Integration : $v = e^{-x}, v_1 = \frac{e^{-x}}{(-1)}, v_2 = \frac{e^{-x}}{(-1)^2}, v_3 = \frac{e^{-x}}{(-1)^3}$ $(\because \boldsymbol{\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots})$

$k\left[x^2 \frac{e^{-x}}{(-1)} - 2x\frac{e^{-x}}{(-1)^2} + 2\frac{e^{-x}}{(-1)^3}\right]_0^{\infty} = 1 \Rightarrow k[(0 - 0 + 0) - (0 - 0 + 2)] = 1 \Rightarrow \boldsymbol{k = \frac{1}{2}}$ $(\because \boldsymbol{e^{-\infty} = 0,\ e^0 = 1})$

Mean of *X* $E(X) = \int_{-\infty}^{\infty} x\,f(x)dx = \int_0^{\infty} x\left(\frac{1}{2}x^2 e^{-x}\right)dx = \frac{1}{2}\int_0^{\infty} x^3 e^{-x} dx$

$\boldsymbol{Differentiation}: u = x^3,\ u' = 3x^2,\ u'' = 6x,\ u''' = 6,\ u^{iv} = 0$

Integration : $v = e^{-x}, v_1 = \frac{e^{-x}}{(-1)}, v_2 = \frac{e^{-x}}{(-1)^2}, v_3 = \frac{e^{-x}}{(-1)^3}, v_4 = \frac{e^{-x}}{(-1)^4}$ $(\because \int uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots)$

$$E(X) = \frac{1}{2}\left[x^3 \frac{e^{-x}}{(-1)} - 3x^2 \frac{e^{-x}}{(-1)^2} + 6x \frac{e^{-x}}{(-1)^3} - 6 \frac{e^{-x}}{(-1)^4}\right]_0^\infty = 3 \qquad (\because e^{-\infty} = 0,\ e^0 = 1)$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^\infty x^2 \left(\frac{1}{2}x^2 e^{-x}\right) dx = \frac{1}{2}\int_0^\infty x^4 e^{-x} dx$$

Differentiation : $u = x^4,\ u' = 4x^3,\ u'' = 12x^2,\ u''' = 24x,\ u'^{v} = 24,\ u^{v} = 0$

Integration : $v = e^{-x}, v_1 = \frac{e^{-x}}{(-1)}, v_2 = \frac{e^{-x}}{(-1)^2}, v_3 = \frac{e^{-x}}{(-1)^3}, v_4 = \frac{e^{-x}}{(-1)^4}, v_5 = \frac{e^{-x}}{(-1)^5}, v_6 = \frac{e^{-x}}{(-1)^6}$

$$E(X^2) = \frac{1}{2}\left[x^4 \frac{e^{-x}}{(-1)} - 4x^3 \frac{e^{-x}}{(-1)^2} + 12x^2 \frac{e^{-x}}{(-1)^3} - 24x \frac{e^{-x}}{(-1)^4} + 24x \frac{e^{-x}}{(-1)^5}\right]_0^\infty = 12$$

$$V(X) = E(X^2) - [E(X)]^2 = 12 - 9 = 3, \quad E(3X^2 - 2X) = 3E(X^2) - 2E(X) = 3(12) - 2(3) = 30$$

6. ***The prob. distribution function of a RV X is*** $(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$. ***Find the mean and variance.***

Solution

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)dx = \int_{-\infty}^{0} x\, f(x)dx + \int_0^1 x\, f(x)dx + \int_1^2 x\, f(x)dx + \int_2^\infty x\, f(x)dx$$

$$= \int_{-\infty}^{0} 0\, dx + \int_0^1 x\,(x)dx + \int_1^2 x\,(2-x)dx + \int_2^\infty 0\, dx = \int_0^1 x^2 dx + \int_1^2 (2x - x^2)dx$$

$$E(X) = \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_1^2 = \frac{1}{3} + \left(4 - \frac{8}{3}\right) - \left(1 - \frac{1}{3}\right) = 1$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2\, f(x)dx = \int_{-\infty}^{0} x^2\, f(x)dx + \int_0^1 x^2\, f(x)dx + \int_1^2 x^2\, f(x)dx + \int_2^\infty x^2\, f(x)dx$$

$$= \int_0^1 x^3 x + \int_1^2 (2x^2 - x^3)dx = \left[\frac{x^4}{4}\right]_0^1 + \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_1^2 = \frac{1}{4} + \left(\frac{16}{3} - \frac{16}{4}\right) - \left(\frac{2}{3} - \frac{1}{4}\right) = \frac{7}{6}$$

$$V(X) = E(X^2) - [E(X)]^2 = \frac{1}{6}$$

7. ***The distribution function of a RV X is given by*** $F(x) = 1 - (1 + x)e^{-x},\ x \ge 0$. ***Find the density function, mean and variance of X.***

Solution

$$f(x) = \frac{d}{dx}[F(x)] = \frac{d}{dx}[1 - (1 + x)e^{-x}] = [0 - (1 + x)(-e^{-x}) - e^{-x}] = e^{-x} + xe^{-x} - e^{-x} = xe^{-x}, x \ge 0$$

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)dx = \int_0^\infty x^2 e^{-x} dx = \left[x^2 \frac{e^{-x}}{(-1)} - 2x \frac{e^{-x}}{(-1)^2} + 2 \frac{e^{-x}}{(-1)^3}\right]_0^\infty = 2$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2\, f(x)dx = \int_0^\infty x^3 e^{-x} dx = \left[x^3 \frac{e^{-x}}{(-1)} - 3x^2 \frac{e^{-x}}{(-1)^2} + 6x^2 \frac{e^{-x}}{(-1)^3} - 6 \frac{e^{-x}}{(-1)^4}\right]_0^\infty = 6$$

$$V(X) = E(X^2) - [E(X)]^2 = 6 - 4 = 2$$

8. ***The cdf of a continuous RV X is given by*** $F(x) = \begin{cases} 0, & x < 0 \\ x^2, & 0 \le x < \frac{1}{2} \\ 1 - \frac{3}{25}(3 - x)^2, & \frac{1}{2} \le x < 3 \\ 1, & x \ge 3 \end{cases}$

Find the p.d.f. of X and evaluate $P(|X| \le 1)$ ***and*** $P\left(\frac{1}{3} \le X < 4\right)$ ***using both the pdf and cdf.***

Solution: $f(x) = \frac{d}{dx}[F(x)]$; $f(x) = \begin{cases} 0, & x < 0 \\ 2x, & 0 \le x < \frac{1}{2} \\ \frac{6}{25}(3 - x), & \frac{1}{2} \le x < 3 \\ 0, & x \ge 3 \end{cases}$

pdf : $P(|X| \le 1) = P(-1 \le X \le 1) = \int_{-1}^{0} 0\, dx + \int_0^{\frac{1}{2}} 2x\, dx + \int_{\frac{1}{2}}^{1} \frac{6}{25}(3 - x)dx = 2\left[\frac{x^2}{2}\right]_0^{\frac{1}{2}} + \frac{6}{25}\left[3x - \frac{x^2}{2}\right]_{\frac{1}{2}}^{1} = \frac{13}{25}$

cdf: $P(|X| \le 1) = P(-1 \le X \le 1) = F(1) - F(-1) = \frac{13}{25}$

pdf : $P\left(\frac{1}{3} \le X < 4\right) = \int_{\frac{1}{3}}^{\frac{1}{2}} 2x\, dx + \int_{\frac{1}{2}}^{3} \frac{6}{25}(3 - x)dx + \int_3^4 0\, dx = 2\left[\frac{x^2}{2}\right]_{\frac{1}{3}}^{\frac{1}{2}} + \frac{6}{25}\left[3x - \frac{x^2}{2}\right]_{\frac{1}{2}}^{3} = \frac{8}{9}$

cdf: $P\left(\frac{1}{3} \le X < 4\right) = F(4) - F\left(\frac{1}{3}\right) = 1 - \frac{1}{9} = \frac{8}{9}$

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9. ***The first four moments of a distribution about $x = 4$ are 1, 4, 10, 45. Show that the mean is 5, variance is 3, $\mu_3 = 0, \mu_4 = 26$.***

Solution: Let $\mu_1',\ \mu_2',\ \mu_3',\ \mu_4'$ be the first four moments about $X = 4$.

Given $\mu_1' = 1,\ \mu_2' = 4,\ \mu_3' = 10,\ \mu_4' = 45$ about $X = 4$.

$E(X-4) = 1 \Rightarrow E(X) - 4 = 1 \Rightarrow E(X) = 5, \qquad \mu_2 = \mu_2' - (\mu_1')^2 = 4 - 1 = 3$

$\mu_3 = \mu_3' - 3\mu_2'\,\mu_1' + 2(\mu_1')^3 = 10 - 3(4)(1) + 2(1) = 0$

$\mu_4 = \mu_4' - 4\mu_3'\,\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4 = 45 - 4(10)(1) + 6(4)(1) - 3(1) = 26$

10. ***The first three moments about the origin are 5, 26, 78. Show that the first three moments about the value $x = 3$ are $2, 5, -48$.***

Solution: Let $\mu_1',\ \mu_2',\ \mu_3',\ \mu_4'$ be the first four moments about $x = 3$. Given $E(X) = 5, E(X^2) = 26, E(X^3) = 78$

$\mu_1' = E(X-3) = E(X) - 3 = 5 - 3 = 2, \qquad \mu_2' = E(X-3)^2 = E(X^2) - 6E(X) + 9 = 26 - 6(5) + 9 = 5,$

$\mu_3' = E(X-3)^3 = E(X^3) - 9E(X^2) + 27E(X) - 27 = 78 - 9(26) + 27(5) - 27 = -48,$

11. ***If X has probability density function given by $f(x) = \frac{x+1}{2}, -1 \le x \le 1$. Find the 1st four central moments.***

Solution: $\mu_n' = E(X^n) = \int_{-\infty}^{\infty} x^n\, f(x)dx$

$$n = 1,\ \mu_1' = E(X) = \int_{-\infty}^{\infty} x\, f(x)dx = \frac{1}{2}\int_{-1}^{1}(x^2 + x)dx = \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^2}{2}\right]_{-1}^{1} = \frac{1}{3}$$

$$n = 2,\ \mu_2' = E(X^2) = \int_{-\infty}^{\infty} x^2\, f(x)dx = \frac{1}{2}\int_{-1}^{1}(x^3 + x^2)dx = \frac{1}{2}\left[\frac{x^4}{4} - \frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{3}$$

$$n = 3,\ \mu_3' = E(X^3) = \int_{-\infty}^{\infty} x^3\, f(x)dx = \frac{1}{2}\int_{-1}^{1}(x^4 + x^3)dx = \frac{1}{2}\left[\frac{x^5}{5} - \frac{x^4}{4}\right]_{-1}^{1} = \frac{1}{5}$$

$$n = 4,\ \mu_4' = E(X^4) = \int_{-\infty}^{\infty} x^4\, f(x)dx = \frac{1}{2}\int_{-1}^{1}(x^5 + x^4)dx = \frac{1}{2}\left[\frac{x^6}{6} - \frac{x^5}{5}\right]_{-1}^{1} = \frac{1}{5}$$

Moment about Mean (Central Moments)

$$\mu_r = \mu_r' - rC_1\,\mu\,\mu_{r-1}^1 + rC_2\,\mu^2\,\mu_{r-2}' - rC_3\,\mu^3\,\mu_{r-3}' + rC_4\,\mu^4\,\mu_{r-4}' - \cdots$$

$r = 1,\ \mu_1 = 0$

$r = 2,\quad \mu_2 = \mu_2' - (\mu_1')^2 = \frac{2}{9}$

$r = 3,\quad \mu_3 = \mu_3' - 3\mu_2'\,\mu_1' + 2(\mu_1')^3 = -\frac{8}{135}$

$r = 4,\quad \mu_4 = \mu_4' - 4\mu_3'\,\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4 = \frac{48}{405}$

12. ***A RV X has density function given by $f(x) = \begin{cases} 2\,e^{-2x}, & x \ge 0 \\ 0, & x < 0 \end{cases}$. Obtain the (i) moment generating function (ii) Four moments about the origin (iii) Mean (iv) Variance.***

Solution: $M_X(t) = \int_{x=-\infty}^{\infty} e^{tx}\, f(x)dx = \int_{x=0}^{\infty} e^{tx}\, 2\,e^{-2x}\,dx = \int_{x=0}^{\infty} 2\,e^{-(2-t)x}\,dx = 2\left[\frac{e^{-(2-t)x}}{-(2-t)}\right]_0^{\infty} = \frac{2}{2-t}$

$$M_X(t) = \frac{2}{2-t} = \frac{2}{2\left(1-\frac{t}{2}\right)} = \left(1 - \frac{t}{2}\right)^{-1} = 1 + \left(\frac{t}{2}\right) + \left(\frac{t}{2}\right)^2 + \left(\frac{t}{2}\right)^3 + \left(\frac{t}{2}\right)^4 + \cdots = 1 + \frac{t}{1!}\left(\frac{1}{2}\right) + \frac{t^2}{2!}\left(\frac{1}{2}\right) + \frac{t^3}{3!}\left(\frac{3}{4}\right) + \frac{t^3}{4!}\left(\frac{3}{2}\right) + \cdots$$

$\mu_r' = coefficient\ of\ \frac{t^r}{r!}, \qquad r = 1,\ \mu_1' = coefficient\ of\ \frac{t^1}{1!} = \frac{1}{2}$

$r = 2,\ \mu_2' = coefficient\ of\ \frac{t^2}{2!} = \frac{1}{2}, \qquad r = 3,\ \mu_3' = coefficient\ of\ \frac{t^3}{3!} = \frac{3}{4}$

$r = 4,\ \mu_4' = coefficient\ of\ \frac{t^4}{4!} = \frac{3}{2},$ Mean $= \mu_1' = \frac{1}{2}$, Variance $= \mu_2' - (\mu_1')^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

13. ***Find the moment generating function of the RV whose moments are given by $\mu_r' = (r+1)!\,2^r$. Find also mean and variance.***

Solution: $\mu_1' = 2!\,2^1,\quad \mu_2' = 3!\,2^2,\quad \mu_3' = 4!\,2^3,\quad M_X(t) = 1 + \frac{t}{1!}\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\left(\frac{3}{4}\right) + \frac{t^3}{4!}\mu_3' + \cdots$

$M_X(t) = 1 + \frac{t}{1!}(2!\,2^1) + \frac{t^2}{2!}(3!\,2^2) + \frac{t^3}{3!}(4!\,2^3) + \cdots = 1 + 2(2t) + 3(2t)^2 + 4(2t)^3 + \cdots = (1-2t)^{-2}$

Mean $= \mu_1' = 4$, $\mu_2' = 24$, Variance $= \mu_2' - (\mu_1')^2 = 24 - 16 = 8$

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FUNCTION OF ONE DIMENSITIONAL RANDOM VARIABLE

One to One Transformation of Random Variables:

Consider that a random variable X is linearly transformed into an another random variable Y. Let Y be $T(x)$.

A monotonically increasing transformation is one where $T(x_1) < T(x_2)$ for all $x_1 < x_2$. For example, $y = ax, a > 0$

A monotonically decreasing transformation is one where $T(x_1) < T(x_2)$ for all $x_1 > x_2$. For example, $y = ax, a < 0$

If the transformation is monotonically increasing $f_Y(y) = f_X(x)\ \frac{dx}{dy}$

If the transformation is monotonically decreasing $f_Y(y) = f_X(x)\ \left(-\frac{dx}{dy}\right)$

In general, for a linear transformation $\boldsymbol{f_Y(y) = f_X(x)\ \left|\frac{dx}{dy}\right|, where\ x = g^{-1}(y)}$

Non - One to One Transformation of Random Variables:

For a transformation which is non - one to one, the transformation will be broken up into transformations each of which one to one. $\boldsymbol{f_Y(y) = f_X(x_1)\ \left|\frac{dx_1}{dy}\right| + f_X(x_2)\ \left|\frac{dx_2}{dy}\right| + \cdots + f_X(x_n)\ \left|\frac{dx_n}{dy}\right|}$

PROBLEMS IN FUNCTION OF RANDOM VARIABLE

1. ***Consider a RV X with p.d.f. $f(x) = e^{-x}, x \geq 0$ with transformation $y = e^{-x}$. Find the transformed density function.***

   Solution: $f_Y(y) = f_X(x)\ \left|\frac{dx}{dy}\right| = \frac{f_X(x)}{\left|\frac{dy}{dx}\right|} = \frac{e^{-x}}{|-e^{-x}|} = \frac{y}{y} = 1, 0 < y \leq 1$

2. ***Let X be a RV with p.d.f. $f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & otherwise \end{cases}$. Find the p.d.f. of $Y = 8X^3$.***

   Solution: $f_Y(y) = f_X(x)\ \left|\frac{dx}{dy}\right|$, *Let* $y = 8x^3 \Rightarrow x^3 = \frac{y}{8} \Rightarrow x = \left(\frac{y}{8}\right)^{\frac{1}{3}}$, $\quad \frac{dx}{dy} = \frac{1}{3}\left(\frac{y}{8}\right)^{\frac{1}{3}-1}\ \frac{1}{8} = \frac{1}{24}\ \left(\frac{y}{8}\right)^{-\frac{2}{3}}$

   $f_Y(y) = 2x\ \frac{1}{24}\ \left(\frac{y}{8}\right)^{-\frac{2}{3}} = 2\left(\frac{y}{8}\right)^{\frac{1}{3}}\ \frac{1}{24}\ \left(\frac{y}{8}\right)^{-\frac{2}{3}} = \frac{1}{12}\ \left(\frac{y}{8}\right)^{-\frac{1}{3}}$, ***Range:*** $0 < x < 1 \Rightarrow 0 < \left(\frac{y}{8}\right)^{\frac{1}{3}} < 1 \Rightarrow 0 < y < 8$

3. ***If X is uniformly distributed in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ find the pdf of $Y = \tan X$.***

   Solution: Given $f_X(x) = \frac{1}{b-a} = \frac{1}{\left(\frac{\pi}{2}+\frac{\pi}{2}\right)} = \pi$, $\ f_Y(y) = f_X(x)\ \left|\frac{dx}{dy}\right|$, *Let* $y = \tan x \Rightarrow x = \tan^{-1} y$, $\ \frac{dx}{dy} = \frac{1}{1+y^2}$

   $f_Y(y) = \pi\frac{1}{1+y^2}$ ***Range:*** $-\frac{\pi}{2} < x < \frac{\pi}{2} \Rightarrow -\infty < y < \infty$

4. ***If X has an exponential distribution with parameter 1, find the pdf of $Y = \sqrt{X}$.***

   Solution: *Given* $= 1,\ f_X(x) = \lambda\, e^{-\lambda x}, x > 0 = e^{-x}, x > 0,\ \ f_Y(y) = f_X(x)\ \left|\frac{dx}{dy}\right|$, *Let* $y = \sqrt{x} \Rightarrow x = y^2$, $\ \frac{dx}{dy} = 2y$, $\ \ f_Y(y) = 2y\ e^{-y^2}$ ***Range:*** $x > 0 \Rightarrow y > 0$

TCHEBYCHEFF INEQULAITY

Statement :

If X is a RV with $E(X) = \mu$ and $V(X) = \sigma^2$, then $P\{|X-\mu| \geq c\} \leq \frac{\sigma^2}{c^2}$ or $P\{|X-\mu| < c\} \geq 1 - \frac{\sigma^2}{c^2}$, $c > 0$.

Alternative Form : **If we put $c = k\sigma$, where $k > 0$ then Tchebycheff inequality takes the form**

$\boldsymbol{P\left\{\left|\frac{X-\mu}{k}\right| \geq \sigma\right\} \leq \frac{1}{k^2}}$ **or** $\boldsymbol{P\left\{\left|\frac{X-\mu}{k}\right| \leq \sigma\right\} \geq 1 - \frac{1}{k^2}}$

PROBLEMS IN TCHEBYCHEFF INEQULAITY

1. ***A RV X has mean $\mu = 12$ and variance $\sigma^2 = 9$ and an unknown probability distribution. Find $P(6 < X < 18)$.***

   Solution: Since the probability distribution of X is not known, we can not find the value of the required probability. We can find only a lower bound for the probability using Tchebycheff inequality.

   $P\{|X-\mu| \geq c\} \leq \frac{\sigma^2}{c^2}, c > 0$ (or) $\quad P\{|X-\mu| < c\} \geq 1 - \frac{\sigma^2}{c^2},\ c > 0$

   $P\{-c < (X-\mu) < c\} \geq 1 - \frac{\sigma^2}{c^2} \Rightarrow P\{\mu - c < X < \mu + c\} \geq 1 - \frac{\sigma^2}{c^2}$

   $\mu = 12,\ \sigma^2 = 9,\ P\{12 - c < X < 12 + c\} \geq 1 - \frac{9}{c^2}$

   Put $c = 6,\ P\{12 - 6 < X < 12 + 6\} \geq 1 - \frac{9}{6^2}$

   $\boldsymbol{P\{6 < X < 18\} \geq \frac{3}{4}}$

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2. ***A fair die is tossed 720 times. Use Tchebycheff inequality to find a lower bound for the probability of getting 100 to 140 sixes.***

Solution: Let X – no. of sixes obtained when a fair die is tossed 720 times. $p = \frac{1}{6},\ q = \frac{5}{6},\ n = 720$

X follows a binomial distribution with mean $np = 120$ and variance $npq = 100$, that is $\mu = 120, \sigma = 10$

By Tchebycheff inequality $P\{|X - \mu| \le k\sigma\} \ge 1 - \frac{1}{k^2}$

$$P\{|X - 120| \le 10k\} \ge 1 - \frac{1}{k^2}$$

$$P\{-10k < (X - 120) < 10k\} \ge 1 - \frac{1}{k^2}$$

$$P\{120 - 10k < X < 120 + 10k\} \ge 1 - \frac{1}{k^2}$$

$$Put\ k = 2,\ P\{100 < X < 140\} \ge 1 - \frac{1}{4}$$

$$\boldsymbol{P\{100 < X < 140\} \ge \frac{3}{4}}$$

3. ***A discrete RV X takes the values $-1, 0, 1$ with probabilities $\frac{1}{8}, \frac{3}{4}, \frac{1}{8}$ respectively. Evaluate $P\{|X - \mu| \ge 2\sigma\}$ and compare it with the upper bound given by Tchebycheff inequality.***

Solution:

$$E(X) = \sum_{x=-1}^{1} x\, P(x) = \left(-1 \times \frac{1}{8}\right) + \left(0 \times \frac{3}{4}\right) + \left(1 \times \frac{1}{8}\right) = 0$$

$$E(X^2) = \sum_{x=-1}^{1} x^2\, P(x) = \left(1 \times \frac{1}{8}\right) + \left(0 \times \frac{3}{4}\right) + \left(1 \times \frac{1}{8}\right) = \frac{1}{4}$$

$$V(X) = E(X^2) - [E(X)]^2 = \frac{1}{4} - 0 = \frac{1}{4}$$

$$P\{|X - \mu| \ge 2\sigma\} = P\{X \ge 1\} = P(X = -1\ or\ X = 1) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

By Tchebycheff inequality, $P\{|X - \mu| \ge k\sigma\} \le \frac{1}{k^2}$

$$\boldsymbol{P\{|X - \mu| \ge 2\sigma\} \le \frac{1}{4}}$$

All the Best

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