Unit-5 Sequences and Series

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CONTENTS

Sequence

Definition:

A set of numbers $a_1, a_2, ...a_n, ...$ such that to each positive integer n, there corresponds a number a_n of the set , is called a sequence and it is denoted by $\{a_n\}$. In otherwords, a sequence of real numbers is a function s from the set of natural numbers N into the set of real numbers R.

Examples:

1. If $a_n = \frac{1}{n}$, then the sequence is $1, \frac{1}{2}, \frac{1}{3} ..., \frac{1}{n} ...$
2. If $a_n = (-1)^n$, then the sequence is $-1, 1, -1, ...$
3. If $a_n = k$, then the sequence is $k, k, ...$

Note:

1. A finite sequence has a finite number of terms.
2. A sequence which is not finite, is an infinite sequence.
3. Example 3 is a constant sequence.

Operations on Sequences:

If $\{s_n\}$ and $\{t_n\}$ are sequences then

1. Sum sequence is $\{s_n + t_n\} = \{s_n\} + \{t_n\}$.
2. Product sequence is $\{s_n t_n\} = \{s_n\}.\{t_n\}$.
3. If $c \in R$, then $c\{s_n\} = \{cs_n\}$.
4. $\{\frac{1}{s_n}\}$ is called the reciprocal of $\{s_n\}$.
5. $\{\frac{s_n}{t_n}\}$ is defined as the quotient of sequence $\{s_n\}$ and $\{t_n\}$, $t_n \neq 0$.

Bounded Sequence

A sequence $\{s_n\}$ is said to be bounded if there exist numbers m, M such that $m < a_n < M$ for all $n \in N$. Otherwise it is said to be unbounded.

Example:

1. $\{\frac{1}{n}\}$ which is bounded by 1.
2. $\{2^n\}$ which is unbounded.

Monotonic Sequence:
A sequence $\{s_n\}$ is said to be
(i) Monotonically increasing if $s_{n+1} \geq s_n$ for every n, $s_1 \leq s_2 \leq s_3 .... \leq s_n \leq s_{n+1} \leq ....$
(ii) Monotonically decreasing if $s_{n+1} \leq s_n$ for every n, $s_1 \geq s_2 \geq s_3 .... \geq s_n \geq s_{n+1} \geq ....$
(iii) Monotonic if it is either monotonically increasing or monotonically decreasing.

Example:

1. $1, 4, 7, 10, ...$ is a monotonic sequence.
2. $1, \frac{1}{2}, \frac{1}{3}, ...$ is monotonic sequence.
3. $1, -1, 1, -1, ...$ is not a monotonic sequence.

Limit of a sequence:
Let $\{s_n\}$ be a sequence. l is said to be limit of the sequence $\{s_n\}$, if to each $\varepsilon > 0$ there exists $m \in z^+$ such that $|s_n - l| < \varepsilon$, $\forall n \geq m$. That is $lim_{n\to\infty} s_n = l$.

Convergent Sequence:
A sequence $\{s_n\}$ is said to be convergent if it has a finite limit. That is $lim_{n\to\infty} s_n = l$.
Divergent Sequence:
If $lim_{n\to\infty} s_n = \infty$, $\{s_n\}$ is divergent.
Oscillatory Sequence:
If $lim_{n\to\infty}$ is not unique (oscillates finitely) or $\pm\infty$ (oscillates infinitely) then $\{s_n\}$ is oscillatory sequence.
Examples:

1. $\{\frac{1}{n^2}\}$ is a convergent sequence.
2. $\{n\}$ is a divergent sequence.
3. $\{(-1)^n\}$ oscillates finitely.
4. $\{(-1)^n n^2\}$ oscillates infinitely.

Problems:

Which of the following sequence are convergent?

(i) $\{\frac{n}{n^2+1}\}$ (ii) $\{(-1)^{n+1}\}$ (iii) $\{\frac{n}{n+1}\}$ (iv) $\{1+\frac{(-1)^n}{n}\}$ (v) $\{\left(\frac{1}{2}\right)^n\}$

Solution: (i)

$$\begin{aligned} lim_{n\to\infty} s_n &= lim_{n\to\infty} \frac{n}{n^2+1} \\ &= lim_{n\to\infty} \frac{n}{n\left(n+\frac{1}{n}\right)} \\ &= lim_{n\to\infty} \frac{1}{\left(n+\frac{1}{n}\right)} \\ &= 0 \end{aligned}$$

Hence the sequence is convergent.

(ii) $\{(-1)^{n+1}\}$=1,-1,1,-1,.... is an oscillating sequence.

(iii)

$$\begin{aligned} lim_{n\to\infty} s_n &= lim_{n\to\infty} \frac{n}{n+1} \\ &= lim_{n\to\infty} \frac{n}{n\left(1+\frac{1}{n}\right)} \\ &= lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)} \\ &= 1 \end{aligned}$$

Hence the sequence is convergent.
(iv)

$$\begin{aligned} lim_{n\to\infty} s_n &= lim_{n\to\infty} 1 + \frac{(-1)^n}{n} \\ &= 1 \end{aligned}$$

Hence the sequence is convergent.
(v)The sequence $\{\left(\frac{1}{2}\right)^n\}$ is convergent.

Series

Definition:

$u_1, u_2, ..., u_n...$ is an infinite sequence. The expression $u_1 + u_2 + ... + u_n + ...$ is called the series. It is denoted by $\sum_{n=1}^{\infty} u_n$.

Note:

1. If the number of terms are finite in a series then the series is called a finite series.
2. If the number of terms are infinite in a series then the series is called an infinite series.

Definition:

The sum of a finite number of terms (the first n-terms) of a series is called the n^{th} partial sum of the series. $S_n = u_1 + u_2 + ... + u_n = \sum_{n=1}^{\infty} u_n$.

1. If $lim_{n \to \infty} S_n = S(finite)$, then the series $\sum_{n=1}^{\infty} u_n$ converges.
2. If $lim_{n \to \infty} S_n = \pm\infty$, then the series $\sum_{n=1}^{\infty} u_n$ diverges.
3. If $lim_{n \to \infty} S_n$ is more than one limit (or) $\pm\infty$, then $\sum_{n=1}^{\infty} u_n$ is oscillatory (or) non converges.

Problems:

1.Examine the nature of the series $1+3+5+7+....\infty$

Solution:

The n^{th} partial sum is $S_n = 1+3+5+7+...+n$. It is an arithmetic series with $a = 1$, $d = 2$, $S_n = \frac{n}{2}[2a+(n-1)d] = n^2$. It follows that $lim_{n\to\infty} S_n = \infty$. Hence the series is divergent.

2. Show that the series $1+r+r^2+...\infty$ (i) Converges if $|r| < 1$ (ii) Diverges if $r \geq 1$ and (iii) Oscillatory if $r \leq -1$

Solution:

(i) If $|r| < 1$, the n^{th} partial sum is $S_n = 1+r+r^2+\cdots+r^{n-1}$= $\frac{1(1-r^n)}{1-r}$. $lim_{n\to\infty} r^n = 0$, if $|r| < 1$. Thus the series is convergent.

(ii) If $r > 1$, $lim_{n\to\infty} r^n = \infty$. If $r = 1$, then $S_n = 1+1+\cdots+1$=n. Hence $lim_{n\to\infty} r^n = \infty$. Hence the series is divergent if $r \geq 1$.

(iii) If $r < -1$, then $lim_{n\to\infty} S_n$= $\begin{cases} \infty, & \text{if n is odd} \\ -\infty, & \text{if n is even} \end{cases}$

If $r=-1$, then $S_n = 1-1+1-1+\cdots 1$. $S_n = \begin{cases} 1, & \text{if n is odd} \\ -1, & \text{if n is even} \end{cases}$

Hence the series $1+r+r^2+...\infty$ is oscillatory if $r \leq -1$

3. Examine the converges of the following series.

(i) $5-4-1+5-4-1+\cdots\infty$ (ii) $1+\frac{5}{4}+\frac{6}{4}+\cdots\infty$ (iii) $\sum_{n=1}^{\infty}\frac{1}{n(n+2)}$

(iv) $1+\frac{1}{2}+\frac{1}{2^2}\cdots\infty$ (v) $1+\frac{4}{3}+\left(\frac{4}{3}\right)^2+\cdots\infty$

Solution:

(i) n^{th} partial sum is S_n=5 or 1 or 0. Hence the series is oscillatory.

(ii) The series is arithmetic series $a=1$, $d=\frac{1}{4}$,
$S_n = \frac{n}{2}[2a+(n-1)d] = \frac{n}{8}[7+n]$. Therefore $lim_{n\to\infty}S_n = \infty$. Hence the series is divergent.

(iii)$u_n = \dfrac{1}{n(n+2)}$. By using partial fractions $\dfrac{1}{n(n+2)} = \frac{A}{n}+\frac{B}{n+2}$.
Which implies $1 = A(n+2)+Bn$. When $n=0$, $A=\frac{1}{2}$ and when $n=-2$, $B=\frac{-1}{2}$.

Therefore $u_n = \dfrac{1}{n(n+2)} = \frac{1}{2n} - \frac{1}{2(n+2)}$. The n^{th} partial sum is $S_n = u_1 + u_2 + \cdots + u_n = \frac{1}{2} - \frac{1}{2(n+2)}$. $lim_{n\to\infty} S_n = \frac{1}{2}$. Hence the given series is convergent.

(iv) The n^{th} partial sum is $S_n = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^{n-1}$.
$S_n = \frac{1-\frac{1}{2^n}}{\frac{1}{2}} = 2\left(1 - \frac{1}{2^n}\right)$. $lim_{n\to\infty} S_n = 2$. Hence the given series is convergent.

(v) The n^{th} partial sum is $S_n = 1 + \frac{4}{3} + \left(\frac{4}{3}\right)^2 + \cdots + \left(\frac{4}{3}\right)^{n-1}$.
$S_n = \frac{\left(\frac{4}{3}\right)^n - 1}{\frac{1}{3}} = 3\left(\left(\frac{4}{3}\right)^n - 1\right)$. $lim_{n\to\infty} S_n = \infty$. Hence the given series is divergent.

Properties of Series:

1. Convergence of a series remains unchanged by the replacement, inclusion or omission of a finite number of terms.
2. A series remains convergent, divergent or oscillatory when each term of it is multiplied by a fixed number other than zero.
3. A series of positive terms either converges or diverges to $+\infty$. That is omitting the negative terms the sum of first n terms tends to either a finite limit or $+\infty$
4. Every finite series is a convergent series.

Definition:

If all terms after few positive terms in an infinite series are positive, such a series is a positive term series.

Example: -10-6-1+5+12+20+$\cdots$.

Series of positive terms

Necessary Condition for Convergence:
If a positive term series $\sum_{n=1}^{\infty} u_n$ is convergent, then $lim_{n\to\infty} u_n = 0$.
Note:
Converse of the above theorem is not true.
Example:
The series $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} + \cdots$ is divergent eventhough $lim_{n\to\infty} u_n = lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.
Test for Divergence:
If $lim_{n\to\infty} u_n \neq 0$, the series $\sum_{n=1}^{\infty} u_n$ must be divergent.
Comparison Test for Convergence:
If two positive term series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ be such that $\sum_{n=1}^{\infty} v_n$ converges and $u_n \leq v_n$ for all values of n, then $\sum_{n=1}^{\infty} u_n$ also converges.

Comparison Test for divergence:
If two positive term series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ be such that $\sum_{n=1}^{\infty} v_n$ diverges and $u_n \geq v_n$ for all values of n, then $\sum_{n=1}^{\infty} u_n$ also diverges.

Limit Comparision test:
If two positive term series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ be such that $lim_{n\to\infty}\frac{u_n}{v_n}$ = non zero finite value, then $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ converges or diverges together.

Auxiliary Series:
(a) ***p*-series:**
The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for $p > 1$ and divergent for $p \leq 1$.
(b) **Geometric series:**
The geometric series $\sum_{n=1}^{\infty} r^{n-1}$ is convergent if $r > 1$ and divergent if $r \geq 1$.

Problems:

1. Examine the series $\frac{1}{1.3.5}+\frac{2}{3.5.7}+\frac{3}{5.7.9}+\cdots$ for convergence.

Solution:

To find u_n:

1,2,3,.... is an arithmetic series where $a=1, d=1$,

$t_n=a+(n-1)d=n$

1,3,5,.... is an arithmetic series where $a=1, d=2$,

$t_n=a+(n-1)d=2n-1$

3,5,7,.... is an arithmetic series where $a=3, d=2$,

$t_n=a+(n-1)d=2n+1$

5,7,9,.... is an arithmetic series where $a=5, d=2$,

$t_n=a+(n-1)d=2n+3$

Therefore $u_n=\dfrac{n}{(2n-1)(2n+1)(2n+3)}=\dfrac{1}{n^2(2-\frac{1}{n})(2+\frac{1}{n})(2+\frac{3}{n})}$.

Choose $v_n=\frac{1}{n^2}$.

Problems:

$lim_{n\to\infty}\frac{u_n}{v_n} = lim_{n\to\infty}\dfrac{1}{(2-\frac{1}{n})(2+\frac{1}{n})(2+\frac{3}{n})} = \frac{1}{8} \neq 0$. By comparison test, $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent, implies that $\sum_{n=1}^{\infty}\dfrac{n}{(2n-1)(2n+1)(2n+3)}$ is convergent.

2. Discuss the convergence or divergence of the series
$\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+\cdots\infty$.

Solution:

$u_n = \dfrac{(n+1)}{n^p}$. Choose $v_n = \dfrac{1}{n^{p-1}}$. We have $lim_{n\to\infty}\frac{u_n}{v_n} = lim_{n\to\infty}(1+\frac{1}{n}) = 1 \neq 0$. The series $\sum_{n=1}^{\infty}\dfrac{1}{n^{p-1}}$ is convergent if $p>2$ and it is divergent if $p\leq 2$. Hence the given series $\sum_{n=1}^{\infty}\dfrac{(n+1)}{n^p}$ is convergent if $p>2$ and it is divergent if $p\leq 2$.

3. Determine whether the following series is convergent or divergent $\frac{\sqrt{2}-1}{3^3-1}+\frac{\sqrt{3}-1}{4^3-1}+\frac{\sqrt{4}-1}{5^3-1}+\cdots$.

Solution:

$u_n=\frac{\sqrt{n+1}-1}{(n+1)^3-1}=\frac{\sqrt{1+\frac{1}{n}}-\frac{1}{\sqrt{n}}}{n^{\frac{5}{2}}\left((1+\frac{2}{n})^3-\frac{1}{n^3}\right)}$. Choose $v_n=\frac{1}{n^{\frac{5}{2}}}$. $lim_{n\to\infty}\frac{u_n}{v_n}=$

$lim_{n\to\infty}=\frac{\sqrt{1+\frac{1}{n}}-\frac{1}{\sqrt{n}}}{\left((1+\frac{2}{n})^3-\frac{1}{n^3}\right)}=1\neq 0$. Since $\sum_{n=1}^{\infty}v_n$ is convergent. Hence by comparison test $\sum_{n=1}^{\infty}u_n$ is convergent.

Problems:

4. Examine the nature of series $\sum_{n=1}^{\infty} \dfrac{1}{(a+n)^p(b+n)^q}$ where a, b, p, q all positive.

Solution:

$$u_n = \frac{1}{(a+n)^p(b+n)^q}$$
$$= \frac{1}{n^{p+q}\left(1+\frac{a}{n}\right)^p\left(1+\frac{b}{n}\right)^q}$$
$$\text{Choose} v_n = \frac{1}{n^{p+q}}$$
$$lim_{n\to\infty}\frac{u_n}{v_n} = lim_{n\to\infty}\frac{1}{\left(1+\frac{a}{n}\right)^p\left(1+\frac{b}{n}\right)^q} = 1$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{p+q}}$ is convergent if $p+q>1$ and is divergent if $p+q \leq 1$. Which implies $\sum_{n=1}^{\infty} \dfrac{1}{(a+n)^p(b+n)^q}$ is convergent if $p+q>1$ and is divergent if $p+q \leq 1$.

Problems:

5. Determine whether the following series is convergent or divergent $\sum_{n=1}^{\infty}\left(\sqrt{n^4+1}-\sqrt{n^4-1}\right)$.

Solution:

$$u_n = \left(\sqrt{n^4+1}-\sqrt{n^4-1}\right)\left(\frac{\sqrt{n^4+1}+\sqrt{n^4-1}}{\sqrt{n^4+1}+\sqrt{n^4-1}}\right)$$

$$= \frac{(n^4+1)-(n^4-1)}{\sqrt{n^4+1}+\sqrt{n^4-1}} = \frac{2}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$

$$= \frac{2}{n^2}\frac{1}{\sqrt{1+\frac{1}{n^4}}+\sqrt{1-\frac{1}{n^4}}}$$

Choose $v_n = \frac{1}{n^2}$

$$lim_{n\to\infty}\frac{u_n}{v_n} = lim_{n\to\infty}\frac{2}{2} = 1.$$

Since the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent, $\sum_{n=1}^{\infty}\left(\sqrt{n^4+1}-\sqrt{n^4-1}\right)$ is convergent.

Problems:

6. Determine whether the following series is convergent or divergent $\sum_{n=1}^{\infty}\sqrt{\dfrac{3^n-1}{2^n+1}}$.

Solution:

$$u_n = \sqrt{\frac{3^n-1}{2^n+1}}$$

$$= \left(\sqrt{\frac{3}{2}}\right)^n \sqrt{\frac{1-\frac{1}{3^n}}{1+\frac{1}{2^n}}}$$

$$\text{Let } v_n = \left(\sqrt{\frac{3}{2}}\right)^n$$

$$lim_{n\to\infty}\frac{u_n}{v_n} = lim_{n\to\infty}\sqrt{\frac{1-\frac{1}{3^n}}{1+\frac{1}{2^n}}} = 1$$

Since $\sum_{n=1}^{\infty} v_n$ is divergent, $\sum_{n=1}^{\infty}\sqrt{\dfrac{3^n-1}{2^n+1}}$ is also divergent.

Problems:

7. Examine the nature of the series $\sum_{n=1}^{\infty} sin\left(\frac{1}{n}\right)$.

Solution:

$$u_n = sin\left(\frac{1}{n}\right), \text{ choose } v_n = \frac{1}{n}. \quad lim_{n\to\infty}\frac{u_n}{v_n} = lim_{n\to\infty}\frac{sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. Hence $\sum_{n=1}^{\infty} sin\left(\frac{1}{n}\right)$ is also divergent.
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Cauchy's Integral Test

If $\sum_{n=1}^{\infty} u_n$ is a series of positive terms and if $u(x) = f(x)$ be such that

- $f(x)$ is continuous in $1 < x < \infty$.
- $f(x)$ decreases as x increases, then the series $\sum_{n=1}^{\infty} u_n$ is convergent or divergent according as the integral $\int_1^{\infty} f(x)dx$ is finite or infinite.

Cauchy's Integral Test:

1. Apply Cauchy's integral test to discuss the nature of the harmonic series (p-series) $\sum_{n=1}^{\infty} \frac{1}{n^p}$.

Solution:

Let $u_x = f(x) = \frac{1}{n^p}$. As x increases $f(x)$ decreases.

$$\begin{aligned}
\int_1^{\infty} f(x)dx &= \int_1^{\infty} \frac{1}{x^p}dx \\
&= \int_1^{\infty} x^{-p}dx \\
&= \left[\frac{x^{-p+1}}{-p+1}\right] \\
&= \frac{-1}{p-1}[x^{-p+1}]_1^{\infty} \\
&= \begin{cases} \frac{1}{p-1} & \text{if} p > 1 \\ \infty & \text{if} p \leq 1 \end{cases}
\end{aligned}$$

Therefore $\sum u_n$ is convergent if $p > 1$ and divergent if $p \leq 1$.

2. Find the nature of the series $\sum_{n=2}^{\infty} \frac{1}{n(logn)^p}$.

Solution:

Let $u_x = f(x) = \frac{1}{x(logx)^p}$. As x increases $f(x)$ decreases.

$$\int_2^{\infty} \frac{1}{x(logx)^p} = \int_{log2}^{\infty} \frac{dt}{t^p}$$

$$= \left[\frac{t^{-p+1}}{-p+1}\right]$$

$$= \begin{cases} \text{finite}, & \text{if } p = 1 \\ \text{finite}, & \text{if } p > 1 \\ \text{infinite}, & \text{if } p \leq 1 \end{cases}$$

Therefore $\sum u_n$ is convergent if $p > 1$ and divergent if $p \leq 1$.

3. Discuss the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{nlogn}$.

Solution:

Let $u_x = f(x) = \frac{1}{xlogx}$. As x increases $f(x)$ decreases.

$$\int_2^{\infty} f(x)dx = \int_2^{\infty} \frac{1}{x(logx)} dx$$
$$= \int_{log2}^{\infty} \frac{dt}{t}$$
$$= infinite$$

Therefore $\sum u_n$ is divergent.

4. Examine the convergence of the series $1 + \frac{1}{4^{\frac{2}{3}}} + \frac{1}{9^{\frac{2}{3}}} + \frac{1}{16^{\frac{2}{3}}} +$ ......

Solution:

Let $u_x = f(x) = \frac{1}{x^{\frac{4}{3}}}$. As x increases $f(x)$ decreases.

$$\begin{aligned}\int_1^\infty f(x)dx &= \int_1^\infty \frac{1}{x^{\frac{4}{3}}} dx \\ &= \int_1^\infty x^{\frac{-4}{3}} dx \\ &= -3[x^{\frac{-1}{3}}]_1^\infty \\ &= 3(\textit{finite})\end{aligned}$$

Therefore $\sum u_n$ is convergent.

D'Alembert's Ratio Test:

D'Alembert's Ratio Test
The series $\sum_{n=1}^{\infty} u_n$ of positive terms is convergent if $lim_{n\to\infty}\dfrac{u_{n+1}}{u_n} < 1$ is divergent if $lim_{n\to\infty}\dfrac{u_{n+1}}{u_n} > 1$. If $lim_{n\to\infty}\dfrac{u_{n+1}}{u_n} = 1$, the test fails.

1. Test the convergence of the series $1+\dfrac{1+\alpha}{1+\beta}+\dfrac{(1+\alpha)(1+2\alpha)}{(1+\beta)(1+2\beta)}+$ ......

Solution:
Let $u_n = \dfrac{(1+\alpha)(1+2\alpha)....(1+(n)\alpha)}{(1+\beta)(1+2\beta).....(1+(n)\beta)}$ then
$u_{n+1}=\dfrac{(1+\alpha)(1+2\alpha)....(1+(n)\alpha)(1+(n+1)\alpha)}{(1+\beta)(1+2\beta).....(1+(n)\beta)(1+(n+1)\beta)}$. It follows that
$\dfrac{u_{n+1}}{u_n} = \dfrac{(1+(n+1)\alpha)}{(1+(n+1)\beta)} \Rightarrow lim_{n\to\infty}\dfrac{u_{n+1}}{u_n} = \frac{\alpha}{\beta}$. Thus the series converges if $\frac{\alpha}{\beta} < 1$ and diverges if $\frac{\alpha}{\beta} > 1$. If $\frac{\alpha}{\beta} = 1$, then $\alpha = \beta$. Therefore the series $\sum_{n=1}^{\infty} u_n = 1+1+1+....$ is a divergent series.

2. Test the convergence of the series $x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + ...$

Solution:

Let $u_n = \frac{x^n n^n}{n!}$ then $u_{n+1} = \frac{x^{n+1}(n+1)^{(n+1)}}{(n+1)!}$. It follows that $\frac{u_{n+1}}{u_n} = x\left(\frac{n+1}{n}\right)^n \Rightarrow lim_{n\to\infty}\frac{u_{n+1}}{u_n} = xlim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = xe$. Thus the series converges if $ex < 1$ and diverges if $ex > 1$. If $ex = 1$, then the ratio test fails.

3. Find the nature of the series $\frac{2}{1} + \frac{2.5.8}{1.5.9} + \frac{2.5.8.11}{1.5.9.13} + ...$

Solution:

$u_n = \frac{2.5.8....(3n-1)}{1.5.9....(4n-3)}$ and $u_{n+1} = \frac{2.5.8....(3n-1)(3n+2)}{1.5.9....(4n-3)(4n+1)}$. Thus $\frac{u_{n+1}}{u_n} = \frac{3n+2}{4n+1}$. Hence $lim_{n\to\infty}\frac{u_{n+1}}{u_n} = lim_{n\to\infty}\frac{3+\frac{2}{n}}{4+\frac{1}{n}} = \frac{3}{4} < 1$. Hence the given series is convergent.

Raabe's Test

The positive termed series $\sum_{n=1}^{\infty} u_n$ is convergent or deivergent according as $lim_{n\to\infty}\, n\left(\frac{u_{n+1}}{u_n}\right) - 1 > 1$ or < 1. If D'Alemberts test fails then use Raabe's test.

1. Test the convergence of the series $\frac{2}{3.4} + \frac{2.4}{3.5.6} + \frac{2.4.6}{3.5.7.8} + ...$

Solution:

Let $u_n = \frac{2.4.6.8....2n}{3.5.7....(2n+1)} \frac{1}{2n+2}$ then

$u_{n+1} = \frac{2.4.6.8....2n.2(n+1)}{3.5.7....(2n+1).(2n+3)} \frac{1}{2n+4}$. It follows that

$\frac{u_{n+1}}{u_n} = \frac{\left(1+\frac{1}{n}\right)^2}{\left(1+\frac{3}{2n}\right)\left(1+\frac{4}{2n}\right)} \Rightarrow lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$. Hence the ratio test fails. $n\left(\frac{u_{n+1}}{u_n}\right) - 1 = n\left[\frac{(2n+3)(2n+4)}{(2n+2)^2} - 1\right] \to \frac{3}{2}$ as $n \to \infty$.

Therefore by Raabe's test the given series is convergent.

2. Test the convergence of the series $\sum_{n=1}^{\infty} \dfrac{1.3.5....(2n-1)}{2.4.6....(2n)} x^n$.

Solution:

$\dfrac{u_{n+1}}{u_n} = x\left(\dfrac{2n+1}{2n+2}\right) \Rightarrow lim_{n\to\infty}\dfrac{u_{n+1}}{u_n} = x$. Thus the series converges if $0 < x < 1$ and diverges if $x > 1$. If $x = 1$, then the D'Alemberts ratio test fails. Apply Raabe's test.

$n\left(\dfrac{u_n}{u_{n+1}} - 1\right) = \frac{n}{2n+1} \to \frac{1}{2}$ as $n \to \infty$. Therefore when $x = 1$ the given series is divergent.

Cauchy's root Test

If $\sum_{n=1}^{\infty} u_n$ is a series of positive terms, then the series is convergent or deivergent according as $lim_{n\to\infty} u_n^{\frac{1}{n}} < 1$ or > 1.

1. Test the convergence of the series $\sum_{n=1}^{\infty} \dfrac{1}{(logn)^n}$.

Solution:

Let $u_n = \dfrac{1}{(logn)^n}$. By Cauchy's root test $u_n^{\frac{1}{n}} = \frac{1}{(logn)^n}^{\frac{1}{n}} = \frac{1}{logn}$. It follows that ${u_n}^{\frac{1}{n}} \to 0$ as $n \to \infty$. By Cauchy's root test the given series is convergent.

2. Test the convergence of the series $\sum_{n=1}^{\infty} \dfrac{1}{\left(1+\frac{1}{n}\right)^{n^2}}$.

Solution:

Let $u_n = \dfrac{1}{\left(1+\frac{1}{n}\right)^{n^2}}$. By Cauchy's root test $u_n^{\frac{1}{n}} = \dfrac{1}{\left(1+\frac{1}{n}\right)^{n}}$. It follows that ${u_n}^{\frac{1}{n}} \to \frac{1}{e} < 1$ as $n \to \infty$. By Cauchy's root test the given series is convergent.

Alternating Series
A series in which the terms are alternatively positive or negative is called an alternating series.
Lebnitz's Rule
An alternating series $u_1 - u_2 + u_3 - u_4 + ...$ converges if $u_n - u_{n-1} < 0$ and $lim_{n\to\infty} u_n = 0$. The alternating series is not convergent if one of the condition is satisfied. If $lim_{n\to\infty} u_n \neq 0$, then the given series is oscillatory.
1. Discuss the convergence of the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...$
Solution:
$u_n = \dfrac{1}{n}$, $u_{n-1} = \dfrac{1}{n-1}$. Then $u_n - u_{n-1} = \frac{-1}{n(n-1)} < 0$. $lim_{n\to\infty} u_n = 0$.
Therefore by Lebnitz's Rule the given series is convergent.

2. Examine the nature of the series $\sum_{n=2}^{\infty}\frac{(-1)^{n-1}x^n}{n(n-1)}$, $0<x<1$.

Solution:

$u_n=\frac{x^n}{n(n-1)}$, $u_{n-1}=\frac{x^{n-1}}{(n-2)(n-1)}$. u_n-u_{n-1} is less than zero.

Also $lim_{n\to\infty}u_n$=$lim_{n\to\infty}\frac{x^n}{n(n-1)}=0$. Thus the Lebnitz's Conditions are satisfied. Hence the given series is convergent.

3. Examine the nature of the series $\sum_{n=1}^{\infty}(-1)^{n-1}\left[\sqrt{n^2+1}-n\right]$.

Solution:

$u_n=\sqrt{n^2+1}-n$, $u_{n-1}=\sqrt{n^2-2n+2}-n+1$. Also $u_n-u_{n-1}<0$.

$lim_{n\to\infty}u_n$=$lim_{n\to\infty}\left[\sqrt{n^2+1}-n\right]$= $lim_{n\to\infty}\left[\frac{(\sqrt{n^2+1})^2-n^2}{\sqrt{n^2+1}+n}\right]=0$.

Thus the Lebnitz's Conditions are satisfied. Hence the given series is convergent.

Absolute Convergence and Conditional Convergence:

Absolute Convergence: If the series of arbitrary terms $u_1 + u_2 + ... + u_n + ...$ be such that the series $|u_1| + |u_2| + ... + |u_n| + ...$ is convergent, then then series $\sum_{n=1}^{\infty} u_n$ is absolutely convergent.

Conditional Convergence: If the series $\sum_{n=1}^{\infty} |u_n|$ is divergent but $\sum_{n=1}^{\infty} u_n$ is convergent, then then series $\sum_{n=1}^{\infty} u_n$ is conditionally convergent.

1. Test the series $1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + ...$ for (i) Absolute Convergence (ii) Conditional Convergence.

Solution:

(i)$\sum_{n=1}^{\infty} u_n = 1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + ...$ Thus $\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$. Which harmonic p-series with $p = \frac{3}{2} > 1$. Hence the series $\sum_{n=1}^{\infty} |u_n|$ is convergent which implies $\sum_{n=1}^{\infty} u_n$is absolutely convergent.

(ii) $\sum_{n=1}^{\infty} u_n = 1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + ...$ is an alternating series. $u_n = \frac{1}{n\sqrt{n}}$ and $u_{n-1} = \frac{1}{n-1\sqrt{n-1}}$. Also $u_n - u_{n-1} < 0$ and $lim_{n\to\infty} u_n = 0$. Hence $\sum_{n=1}^{\infty} u_n$ is also convergent. Hence the given series is not conditionally convergent.

Absolute Convergence:

2. Prove that the exponential series $1+\frac{x}{1!}+\frac{x}{2!}+...+\frac{x}{n!}+....$ is absolutely convergent and hence convergent for all values of x.

Solution:

Let $u_n = \dfrac{x^{n-1}}{(n-1)!}$, $u_{n+1} = \dfrac{x^n}{n!}$. Thus $\dfrac{u_{n+1}}{u_n} = \frac{x}{n}$. $lim_{n\to\infty}\left|\dfrac{u_{n+1}}{u_n}\right| =$ $lim_{n\to\infty}\frac{|x|}{n} = 0 < 1, \forall x$. Hence the series is absolutely convergent and hence convergence for all real x.