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18MAB101T

Calculus and Linear Algebra

Unit-3

Prepared by

Dr. S. ATETHAN

Assistant Professor

Department of Mathematics

College of Engineering and Technology

SRM INSTITUTE OF SCIENCE AND TECHNOLOGY

Kattankulathur 603203, Chengalpattu District.

1. Introduction

A differential equation is a mathematical equation which involves a function and its derivatives.

For example,

$$(i) \frac{d^2y}{dx^2} + 5 \frac{dy}{dx} + 6y = e^{-3x}$$

$$(ii) \frac{d^2y}{dx^2} + 3 \left(\frac{dy}{dx} \right)^2 + 2y = \sin 3x$$

$$(iii) \left(\frac{d^2y}{dx^2} \right)^2 + 5 \frac{dy}{dx} + 6y = 5x$$

$$(iv) \frac{d^3y}{dx^3} + \frac{dy}{dx} = e^{-x}$$

are few differential equations.

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2. Linear Differential Equations of Second Order with Constant Co-efficients

An equation of the form

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = F(x)$$

where a_0, a_1, \dots, a_n are constants, is said to be a LINEAR DIFFERENTIAL EQUATION of degree ' n ' with constant coefficients.

Let $\frac{d}{dx} = D$, $\frac{d^2}{dx^2} = D^2 + \cdots + \frac{d^n}{dx^n} = D^n$. Then the above equation can be written as

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_{n-1} D + a_n) y = F(x)$$
$$\phi(D)y = F(x) \quad (2.1)$$

The general or Complete solution of (2.1) consists of two parts namely the Complementary Function (C.F.) and Particular Integral (P.I.). i.e. The General Solution is

$$y = C.F. + P.I. = y_c + y_p$$

Complementary Function

Definition 2.0.1 (Complimentary Function). The general solution of $\phi(D) = 0$ is called as Complementary Function and it is denoted by y_c .

Definition 2.0.2 (Auxiliary Equation). An equation of the form $\phi(m) = 0$ is called as an Auxiliary Equation.

Depends on the roots of the polynomial $\phi(m) = 0$, i.e. the roots of the auxiliary equation, we have the following cases to write the Complimentary Function.

Case 1:

The roots of the auxiliary equation are real and distinct, then we write the roots of $\phi(m) = 0$ as m_1 and m_2 and the C.F. becomes, $y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$.

Generalized form of C.F. in this case: If m_1, m_2, \dots, m_n be the real and distinct roots of the auxiliary equation, then the C.F. becomes, $y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$.

Case 2:

The roots of the auxiliary equation are real and equal, then we write the roots of $\phi(m) = 0$ as $m_1 = m_2 = m$ and the C.F. becomes, $y_c = (c_1 + c_2 x) e^{m x}$.

Generalized form of C.F. in this case: If $m_1 = m_2 = \dots = m_n (= m)$, then the C.F. becomes, $y_c = (c_1 + c_2 x + c_3 x^2 + \dots + c_n x^{n-1}) e^{m x}$.

Case 3:

The roots of the auxiliary equation are complex conjugates i.e. $m = \alpha \pm i\beta$, then the

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C.F. becomes,

$$\begin{aligned}y_c &= e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) \\(\text{OR}) \quad y_c &= c_1 e^{\alpha x} \cos(\beta x + c_2) \\(\text{OR}) \quad y_c &= c_1 e^{\alpha x} \sin(\beta x + c_2)\end{aligned}$$

Note: For repeated complex roots, say $m = \alpha \pm i\beta$, $\alpha \pm i\beta$, the C.F. becomes, $y_c = e^{\alpha x}[(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x]$.

Case 4:

The roots of the auxiliary equation are surds, i.e. $m = \alpha \pm \sqrt{\beta}$, then the C.F. becomes,

$$\begin{aligned}y_c &= e^{\alpha x}(c_1 \cosh \sqrt{\beta} x + c_2 \sinh \sqrt{\beta} x) \\(\text{OR}) \quad y_c &= c_1 e^{\alpha x} \cosh(\sqrt{\beta} x + c_2) \\(\text{OR}) \quad y_c &= c_1 e^{\alpha x} \sinh(\sqrt{\beta} x + c_2)\end{aligned}$$

Note: For repeated complex roots, say $m = \alpha \pm \sqrt{\beta}$, $\alpha \pm \sqrt{\beta}$, the C.F. becomes, $y_c = e^{\alpha x}[(c_1 + c_2 x) \cos \sqrt{\beta} x + (c_3 + c_4 x) \sin \sqrt{\beta} x]$.

Particular Integral

Definition 2.0.3 (Particular Integral). *The general solution of $\phi(D)y = F(x)$ is called as Particular Integral and it is denoted by y_p .*

Based on the function on the RHS of the equation $\phi(D)y = F(x)$, i.e. based on $F(x)$, the following cases can be considered while evaluating the Particular Integral (P.I.).

Let us see some short cut methods of evaluating P.I. when $F(x)$ is of the following form:

A. $F(x) = e^{ax}$

B. $F(x) = \sin ax$ or $\cos ax$

C. $F(x) = x^m$ (polynomial function)

D. $F(x) = e^{ax}\chi(x)$, where $\chi(x) = x^m$ or $\sin ax$ or $\cos ax$

E. $F(x) = x^m\chi(x)$, where $\chi(x) = \sin ax$ or $\cos ax$

CASE A: $F(x) = e^{ax}$

We know that

$$y_p = \frac{1}{\phi(D)} F(x)$$

Now,

$$\begin{aligned} y_p &= \frac{1}{\phi(D)} e^{ax} \\ &= \frac{1}{\phi(a)} e^{ax}, \text{ if } \phi(a) \neq 0 \quad \boxed{\text{Directly replace } D \text{ by } a} \end{aligned}$$

If $\phi(a) = 0$, then we rewrite $\phi(D)$ as the product of its factors and then we have

$$y_p = x \frac{1}{\phi'(D)\psi(D)} e^{ax}, \text{ with } \psi(a) \neq 0$$

(OR)

$$y_p = x^2 \frac{1}{\phi''(D)} e^{ax}$$

CASE B: $F(x) = \sin ax$ or $\cos ax$

We know that

$$y_p = \frac{1}{\phi(D)} F(x)$$

Let us consider $\phi(D) = \psi(D^2)$ (i.e. considering only the quadratic part), then the above equation becomes

$$\begin{aligned} y_p &= \frac{1}{\psi(D^2)} \sin ax \text{ (or) } \frac{1}{\psi(D^2)} \cos ax \\ &= \frac{1}{\psi(-a^2)} \sin ax \text{ (or) } \frac{1}{\psi(-a^2)} \cos ax \quad \boxed{\text{Replace } D^2 \text{ by } -a^2 \text{ if } \psi(-a^2) \neq 0} \end{aligned}$$

$\psi(a^2) = 0$, i.e. when y_p is of the form $y_p = \frac{1}{D^2 + a^2} \sin ax$ (or) $\frac{1}{D^2 + a^2} \cos ax$, then

$$y_p = \frac{x}{2} \int \sin ax \text{ (or) } \frac{x}{2} \int \cos ax$$

CASE C: $F(x) = x^k$, $k \in \mathbb{Z}^+$

We know that

$$\begin{aligned}y_p &= \frac{1}{\phi(D)} F(x) \\&= \frac{1}{\phi(D)} x^k\end{aligned}$$

Now, taking the lowest degree term (may be constant term) and write the the denominator in the form of $[1 + \phi(D)]$, then we have

$$\begin{aligned}y_p &= \frac{1}{[1 + \phi(D)]} F(x) \\&= [1 + \phi(D)]^{-1} x^k\end{aligned}$$

Expanding this relation upto k^{th} derivative using BINOMIAL EXPANSION and hence we get the desired y_p .

Few important Binomial Expansions:

1. $(1 - D)^{-1} = 1 + D + D^2 + \dots$
2. $(1 + D)^{-1} = 1 - D + D^2 - D^3 + \dots$
3. $(1 - D)^{-2} = 1 + 2D + 3D^2 + \dots$
4. $(1 + D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \dots$
5. $(1 - D)^{-3} = 1 + 3D + 6D^2 + \dots$

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$$6. (1 + D)^{-3} = 1 - 3D + 6D^2 - \dots$$

CASE D: $F(x) = e^{ax}\chi(x)$, $\chi(x) = \cos ax$ (or) $\sin ax$ (or) x^k

We know that

$$\begin{aligned} y_p &= \frac{1}{\phi(D)} F(x) \\ &= \frac{1}{\phi(D)} e^{ax} \chi(x) \\ &= e^{ax} \frac{1}{\phi(D+a)} \chi(x) \quad \text{Replace } D \text{ by } D+a \end{aligned}$$

Now, depends upon $\chi(x) = \cos ax$ (or) $\sin ax$ (or) x^k , we proceed by using CASE a (or) b.

CASE E: $F(x) = x^k \chi(x)$, $\chi(x) = \cos ax$ (or) $\sin ax$

We know that

$$\begin{aligned} y_p &= \frac{1}{\phi(D)} F(x) \\ &= \frac{1}{\phi(D)} x^k \chi(x) \end{aligned}$$

Now we consider the following subcases.

SUBCASE E1:

Let $k = 1$, then

$$y_p = \left[x - \frac{\phi'(D)}{\phi(D)} \right] \frac{1}{\phi(D)} \chi(x)$$

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SUBCASE E2:

Let $k \neq 1$ (may be $k = 1$ also considered in this case)

$$y_p = \frac{1}{\phi(D)} x^k \chi(x)$$

$$y_p = \frac{1}{\phi(D)} x^k \cos ax \text{ (or) } \sin ax$$

We know that $e^{i\theta} = \cos \theta + i \sin \theta$, then

$$\cos \theta = \operatorname{Re}(e^{i\theta}), \sin \theta = \operatorname{Im}(e^{i\theta})$$

We may also write the above description as

$$\cos \theta = R.P.(e^{i\theta}), \sin \theta = I.P.(e^{i\theta})$$

Now, if $\chi(x) = \cos ax$, then

$$\begin{aligned} y_p &= \frac{1}{\phi(D)} x^k \cos ax \\ &= \frac{1}{\phi(D)} x^k \operatorname{Re}(e^{iax}) \\ &= \operatorname{Re} \left[\frac{1}{\phi(D)} x^k e^{iax} \right] \\ &= \operatorname{Re} \left[e^{iax} \frac{1}{\phi(D + ia)} x^k \right] \end{aligned}$$

Using the previous cases, we will proceed to solve the above y_p by substituting $e^{iax} = \cos ax + i \sin ax$ and then collecting the terms in the Real Part.
if $\chi(x) = \sin ax$, then

$$\begin{aligned} y_p &= \frac{1}{\phi(D)} x^k \sin ax \\ &= \frac{1}{\phi(D)} x^k \operatorname{Im}(e^{iax}) \\ &= \operatorname{Im} \left[\frac{1}{\phi(D)} x^k e^{iax} \right] \\ &= \operatorname{Im} \left[e^{iax} \frac{1}{\phi(D + ia)} x^k \right] \end{aligned}$$

Using the previous cases, we will proceed to solve the above y_p by substituting $e^{iax} = \cos ax + i \sin ax$ and then collecting the terms in the Imaginary Part.

Example 2.1. Solve: $\frac{d^3 y}{dx^3} - 7 \frac{dy}{dx} - 6y = x^2 + \sin x + e^{4x}$

Hints/Solution:

Given equation is of the form $(D^3 - 7D - 6)y = x^2 + \sin x + e^{4x}$
The auxiliary equation is $m^3 - 7m - 6 = 0 \implies m = -1, -2, 3$.

$$\therefore \text{C.F. } y_c = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x}$$

$$\begin{aligned} (P.I.)_1 &= \frac{1}{D^3 - 7D - 6} x^2 \\ &= -6 \left[1 - \left(\frac{1}{6} D^3 - \frac{7}{6} D \right) \right]^{-1} (x^2) \\ &= -6 \left[1 + \left(\frac{1}{6} D^3 - \frac{7}{6} D \right) + \left(\frac{1}{6} D^3 - \frac{7}{6} D \right)^2 \right] (x^2) \\ &= -6 \left[1 - \frac{7}{6} D + \frac{49}{36} D^2 \right] (x^2) \\ &= -6 \left[x^2 - \frac{7x}{3} + \frac{49}{18} \right] \\ &= \frac{7x}{3} - 6x^2 - \frac{49}{18} \end{aligned}$$

$$\begin{aligned}
 (P.I.)_2 &= \frac{1}{D^3 - 7D - 6} \sin x \\
 &= \frac{1}{-D - 7D - 6} \sin x \\
 &= -\frac{1}{8D + 6} \frac{8D - 6}{8D - 6} \sin x \\
 &= -\frac{8D - 6}{64D^2 - 36} \sin x \\
 &= -\frac{8 \cos x - 6 \sin x}{-64 - 36} \\
 &= \frac{8 \cos x - 6 \sin x}{100}
 \end{aligned}$$

$$\begin{aligned}
 (P.I.)_3 &= \frac{1}{D^3 - 7D - 6} e^{4x} \\
 &= \frac{1}{64 - 28 - 6} e^{4x} \\
 &= \frac{1}{30} e^{4x}.
 \end{aligned}$$

Hence the complete solution is given by

$$\begin{aligned}
 y = C.F. + P.I. &= y_c + y_p = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x} + \frac{7x}{3} - 6x^2 - \frac{49}{18} + \\
 &\quad \frac{8 \cos x - 6 \sin x}{100} + \frac{1}{30} e^{4x}.
 \end{aligned}$$

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3. Linear ODE with variable co-efficients

Methodology to convert ODE with variable co-efficients as ODE with constant co-efficients

If the given ODE is of the form $[X^2 D^2 + XD + 1]y = G(X)$

Let $X = e^z$ and $\theta = \frac{d}{dz}$.

$$\therefore z = \log X \text{ and } \frac{dy}{dX} = \frac{dy}{dz} \cdot \frac{dz}{dX} = \frac{dy}{dz} \cdot \frac{1}{X} \implies X \frac{dy}{dX} = \frac{dy}{dz}$$

i.e. $X \frac{dy}{dX} = XD = \frac{dy}{dz} = \theta y$ and Differentiating $X \frac{dy}{dX} = \frac{dy}{dz}$ w.r.to X , we have

$$\begin{aligned} X \frac{d^2 y}{dX^2} + \frac{dy}{dX} &= \frac{d^2 y}{dz^2} \frac{dz}{dX} \\ \implies X^2 \frac{d^2 y}{dX^2} + X \frac{dy}{dX} &= \frac{d^2 y}{dz^2} \\ \implies X^2 \frac{d^2 y}{dX^2} &= \frac{d^2 y}{dz^2} - \frac{dy}{dz} = (\theta^2 - \theta)y \end{aligned}$$

$$\text{i.e. } X^2 \frac{d^2 y}{dX^2} = X^2 D^2 y = \theta(\theta - 1)y$$

Similarly we have $X^3 \frac{d^3 y}{dX^3} = X^3 D^3 y = \theta(\theta - 1)(\theta - 2)y$,

$X^4 \frac{d^4 y}{dX^4} = X^4 D^4 y = \theta(\theta - 1)(\theta - 2)(\theta - 3)y$ and so on.

Example 3.1. Solve: $(1+x)^2 \frac{d^2 y}{dx^2} + (1+x) \frac{dy}{dx} + y = \log(1+x) + \cos[\log(1+x)]$

Hints/Solution:

Given equation is of the form $[(1+x)^2 D^2 + (1+x)D + 1]y = \log(1+x) + \cos[\log(1+x)]$

Let $X = 1+x$, then the ODE becomes $[X^2 D^2 + XD + 1]y = \log X + \cos[\log X]$

Let $X = e^z$ and $\theta = \frac{d}{dz}$.

$$\therefore z = \log X \text{ and } \frac{dy}{dX} = \frac{dy}{dz} \cdot \frac{dz}{dX} = \frac{dy}{dz} \cdot \frac{1}{X} \implies X \frac{dy}{dX} = \frac{dy}{dz}$$

$$\text{i.e. } X \frac{dy}{dX} = XD = \frac{dy}{dz} = \theta y \text{ and } X^2 \frac{d^2 y}{dX^2} = X^2 D^2 y = \theta(\theta - 1)y$$

Now, the ODE takes the form

$$[\theta(\theta - 1) + \theta + 1]y = (\theta^2 + 1)y = z + \cos z$$

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The auxiliary equation is $m^2 + 1 = 0 \implies m = \pm i$.

\therefore C.F. $y_c = e^{0z}(c_1 \cos z + c_2 \sin z) = c_1 \cos z + c_2 \sin z$

$$\begin{aligned}(P.I.)_1 &= \frac{1}{\theta^2 + 1} z \\&= [1 + \theta^2]^{-1} z \\&= [1 - \theta^2 + \theta^4 - \dots] z \\&= z\end{aligned}$$

$$\begin{aligned}(P.I.)_2 &= \frac{1}{\theta^2 + 1} \cos z \\&= \frac{1}{-1 + 1} \cos z \\&= \frac{z}{2\theta} \cos z \\&= \frac{z}{2} \sin z\end{aligned}$$

Hence the complete solution is given by

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$$y = C.F. + P.I. = y_c + y_p = c_1 \cos z + c_2 \sin z + z + \frac{z}{2} \sin z$$

$$= c_1 \cos \log(1+x) + c_2 \sin \log(1+x) \\ + \log(1+x) + \frac{\log(1+x)}{2} \sin \log(1+x)$$

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4. Method of Variation of Parameters

Finding the solution of the ODE in the form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad (4.1)$$

where P , Q and R are the functions of x or constants.
The homogeneous equation corresponding to (4.1) is

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \quad (4.2)$$

Method 1:

Let the general solution of (4.2) be

$$y = Ay_1 + By_2 \quad (4.3)$$

where A and B are constants and $y = y_1(x)$ and $y = y_2(x)$ are independent particular solutions of (4.2).

Now we consider A and B as functions of x and assume (4.3) to be the general solution of (4.1).

Differentiating (4.3) w.r.to x , we have

$$\frac{dy}{dx} = (Au' + Bv') + (A'u + B'v) \quad (4.4)$$

We select A and B in such a way that

$$(A'u + B'v) = 0 \quad (4.5)$$

∴ (4.4) becomes

$$\frac{dy}{dx} = (Au' + Bv') \quad (4.6)$$

Differentiating (??) w.r.to x , we have

$$\frac{d^2y}{dx^2} = (Au'' + Bv'') + (A'u' + B'v') \quad (4.7)$$

Since (4.3) is the solution of (4.1), the equations (4.3), (4.6) and (4.7) satisfies (4.1).
i.e.

$$\begin{aligned} (Au'' + Bv'' + A'u' + B'v') + P(Au' + Bv') + Q(Au + Bv) &= R \\ A(u'' + Pu' + Qu) + B(v'' + Pv' + Qv) + (A'u' + B'v') &= R \end{aligned} \quad (4.8)$$

Since u and v are the solution of equation (4.2), we have

$$\begin{aligned} (u'' + Pu' + Qu) &= 0 \\ (v'' + Pv' + Qv) &= 0 \end{aligned}$$

Substituting these in (4.8), we get

$$A'u' + B'v' = R \quad (4.9)$$

Solving (4.5) and (4.9), we get the values of A' and B' and then integrating, we get the values of A and B in terms of x . Substituting A and B in (4.3), we get the required general solution of (4.1).

Method 2:

Let the general solution of (4.2) be

$$y_c = Ay_1(x) + By_2(x) \quad (4.10)$$

which is the C.F. of (4.1) with constants A and B .

The P.I. is given by

$$y_p = C(x)y_1(x) + D(x)y_2(x) \quad (4.11)$$

where

$$C(x) = - \int \frac{R \cdot y_2}{W} dx$$
$$D(x) = \int \frac{R \cdot y_1}{W} dx$$

where W is called the Wronskian which is given by

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

Example 4.1. Solve: $\frac{d^2 y}{dx^2} + y = \operatorname{cosec} x$

Hints/Solution:

Given equation is of the form $(D^2 + 1)y = \operatorname{cosec} x$

The auxiliary equation is $m^2 + 1 = 0 \implies m = -\pm i$.

\therefore C.F. $y_c = c_1 \cos x + c_2 \sin x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2 = 1$$

The P.I. is given by

$$y_p = C(x)y_1(x) + D(x)y_2(x) \quad (4.12)$$

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where

$$C(x) = - \int \frac{R \cdot y_2}{W} dx = -x$$

$$D(x) = \int \frac{R \cdot y_1}{W} dx = \log(\sin x)$$

$$P.I. = -x \cos x + \log(\sin x)$$

Hence the complete solution is given by

$$y = C.F. + P.I. = y_c + y_p = c_1 \cos x + c_2 \sin x + -x \cos x + \log(\sin x).$$

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5. Miscellaneous Solved Problems

Example 5.1. Solve the differential equation $(D^2 + 4)y = \sin 2x$

Hints/Solution:

Given equation is of the form $(D^2 + 4)y = \sin 2x$

The auxiliary equation is $m^2 + 4 = 0 \implies m = \pm 2i$.

$\therefore C.F. = c_1 \cos 2x + c_2 \sin 2x$.

$$\begin{aligned}(P.I.) &= \frac{1}{D^2 + 4} \sin 2x \\ &= \frac{x}{4} \cos 2x.\end{aligned}$$

$$y = C.F. + P.I. = c_1 \cos 2x + c_2 \sin 2x + \frac{x}{4} \cos 2x$$

Example 5.2. Solve: $(5 + 2x)^2 \frac{d^2 y}{dx^2} - 6(5 + 2x) \frac{dy}{dx} + 8y = 0$

Hints/Solution:

Let $2x + 5 = e^z$ and $\theta = \frac{d}{dz}$.

$\therefore z = \log(2x + 5)$ and $(2x + 5) \frac{dy}{dx} = \frac{dy}{dz} = 2\theta y$,

$$(2x + 5)^2 \frac{d^2 y}{dx^2} = 2^2 \theta(\theta - 1)y.$$

Now, the ODE takes the form $[2^2\theta(\theta - 1) + 12\theta + 8]y = (4\theta^2 - 16\theta + 8)y = 0$

The A.E. is $m^2 - 4m + 2 = 0 \implies m = 2 \pm \sqrt{2}$. \therefore

$$C.F. = c_1 e^{(2+\sqrt{2})z} + c_2 e^{(2-\sqrt{2})z} = c_1 (2x+5)^{(2+\sqrt{2})} + c_2 (2x+5)^{(2-\sqrt{2})}$$

Example 5.3. Solve the differential equation $(D^2 + 4)y = 4 \tan 2x$

Hints/Solution:

Given equation is of the form $(D^2 + 4)y = 4 \tan 2x$

The auxiliary equation is $m^2 + 4 = 0 \implies m = \pm 2i$.

$$\therefore C.F. = c_1 \cos 2x + c_2 \sin 2x$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2 = 2$$

The P.I. is given by

$y_p = C(x)y_1(x) + D(x)y_2(x)$ where

$$C(x) = - \int \frac{F(x) \cdot y_2}{W} dx = - \int \frac{4 \tan 2x \cdot \sin 2x}{2} dx = - \log(\sec 2x + \tan 2x) + \sin 2x$$

and

$$D(x) = \int \frac{F(x) \cdot y_1}{W} dx = \int \frac{4 \tan 2x \cdot \cos 2x}{2} dx = - \cos 2x$$

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$$\therefore P.I. = -\cos 2x \log(\sec 2x + \tan 2x)$$

Hence the complete solution is given by

$$y = C.F. + P.I. = y_c + y_p = c_1 \cos 2x + c_2 \sin 2x - \cos 2x \log(\sec 2x + \tan 2x).$$

Example 5.4. Solve: $\frac{dx}{dt} + 2y = \sin 2t$; $\frac{dy}{dt} - 2x = \cos 2t$

Hints/Solution:

Eliminating $x(t)$ from the given equations, we get

$$\left(\frac{d^2}{dt^2} + 4\right)y = 0$$

$$\implies y = c_1 \cos 2t + c_2 \sin 2t$$

and hence

$$\frac{dy}{dt} = -2c_1 \sin 2t + 2c_2 \cos 2t.$$

Substituting these in the given equations, we get

$$x(t) = -c_1 \sin 2t + c_2 \cos 2t - \frac{1}{2} \cos 2t$$

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Example 5.5. Solve the differential equations (a) $(D^2 - 2D + 1)y = e^{2x}$ (b) $(D^2 - 5D + 6)y = x^2 + 3$

Hints/Solution:

(a) Given equation is of the form $(D^2 - 2D + 1)y = (D - 2)^2 y = e^{2x}$

The auxiliary equation is $(m - 2)^2 = 0 \implies m = 2, 2.$

$\therefore C.F. = (c_1x + c_2)e^{2x}.$

$$\begin{aligned}(P.I.) &= \frac{1}{(D - 2)^2} e^{2x} \\ &= \frac{x^2}{2} e^{2x}.\end{aligned}$$

$$y = C.F. + P.I. = (c_1x + c_2)e^{2x} + \frac{x^2}{2}e^{2x}$$

(b) Given equation is of the form $(D^2 - 5D + 6)y = x^2 + 3$

The auxiliary equation is $m^2 - 5m + 6 = 0 \implies m = 2, 3.$

$\therefore C.F. = c_1e^{2x} + c_2e^{3x}.$

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$$(P.I.) = \frac{1}{D^2 - 5D + 6} x^2 + 3e^0$$

$$= \frac{1}{6} \left[x^2 + \frac{5}{3}x + \frac{19}{18} + 3 \right]$$

$$= \frac{1}{108} [18x^2 + 30x + 73]$$

$$y = C.F. + P.I. = c_1 e^{2x} + c_2 e^{3x} + \frac{1}{108} [18x^2 + 30x + 73]$$

Example 5.6. Solve the differential equations (a) $(D^2 + 4)y = \sec 2x$ (b) $x^2 \frac{d^2 y}{dx^2} + 4x \frac{dy}{dx} + 2y = x$.

Hints/Solution:

(a) Given equation is of the form $(D^2 + 4)y = \sec 2x$

The auxiliary equation is

$$m^2 + 4 = 0 \implies m = \pm 2i.$$

$$\therefore C.F. = c_1 \cos 2x + c_2 \sin 2x$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2 = 2$$

The P.I. is assumed as $y_p = C(x)y_1(x) + D(x)y_2(x)$

where

$$\begin{aligned} C(x) &= - \int \frac{F(x) \cdot y_2}{W} dx = - \int \frac{\sec 2x \cdot \sin 2x}{2} dx \\ &= \frac{1}{4} \log(\cos 2x) \text{ and} \end{aligned}$$

$$D(x) = \int \frac{F(x) \cdot y_1}{W} dx = \int \frac{\sec 2x \cdot \cos 2x}{2} dx = \frac{1}{2} x$$

$$\therefore P.I. = \frac{1}{4} \log(\cos 2x) \cos 2x + \frac{1}{2} x \sin 2x$$

Hence the complete solution is given by

$$y = C.F. + P.I. = y_c + y_p = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{4} \log(\cos 2x) \cos 2x + \frac{1}{2} x \sin 2x.$$

(b) Given equation is of the form $[x^2 D^2 + 4x D + 2]y = x$

$$\text{Let } x = e^z \text{ and } \theta = \frac{d}{dz}.$$

$$\therefore z = \log x \text{ and } x \frac{dy}{dx} = \frac{dy}{dz} = \theta y,$$

$$x^2 \frac{d^2 y}{dx^2} = \theta(\theta - 1)y.$$

Now, the ODE takes the form $[\theta(\theta - 1) + 4\theta + 2]y = (\theta^2 + 3\theta + 2)y = e^z$

The A.E. is $m^2 + 3m + 2 = 0 \implies m = -1, -2$.

$$\therefore C.F. = c_1 e^{-z} + c_2 e^{-2z} = c_1/x + c_2/x^2$$

$$(P.I. = \frac{1}{\theta^2 + 3\theta + 2} e^z = \frac{x}{[6]})$$

$$y = C.F. + P.I. = c_1/x + c_2/x^2 + \frac{x}{6}$$

Example 5.7. Solve the differential equations (a) $(D^2 - 2D + 1)y = e^x \sin x$ (b)

$$(x + 2)^2 \frac{d^2 y}{dx^2} - (x + 2) \frac{dy}{dx} + y = 0.$$

Hints/Solution:

(a) Given equation is of the form $(D^2 - 2D + 1)y = (D - 1)^2 = e^x \sin x$

The auxiliary equation is $(m - 1)^2 = 0 \implies m = 1, 1$.

$$\therefore C.F. = (c_1 + c_2 x)e^x.$$

$$(P.I.) = \frac{1}{(D - 1)^2} e^x \sin x = -e^x \sin x$$

Hence the complete solution is given by

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$$y = C.F. + P.I. = y_c + y_p = (c_1 + c_2 x)e^x - e^x \sin x.$$

(b) Given equation is of the form $[(2+x)^2 D^2 - (2+x)D + 1]y = 0$

Let $x + 2 = e^z$ and $\theta = \frac{d}{dz}$.

$$\therefore z = \log(x+2) \text{ and } (x+2) \frac{dy}{dx} = \frac{dy}{dz} = \theta y,$$

$$(x+2)^2 \frac{d^2 y}{dx^2} = \theta(\theta-1)y.$$

Now, the ODE takes the form $[\theta(\theta-1) - \theta + 1]y = (\theta^2 - 2\theta + 1)y = 0.$

The A.E. is $m^2 - 2m + 1 = 0 \implies m = 1, 1.$

$$\therefore C.F. = (c_1 x + c_2)e^z = (c_1 \log(x+2) + c_2)(x+2)$$

Example 5.8. Solve the differential equations (a) $(D^2 + 1)y = \operatorname{cosec} x$ (b) $\frac{dx}{dt} + y = e^t$; $x - \frac{dy}{dt} = t.$

Hints/Solution:

(a) Given equation is of the form $(D^2 + 1)y = \operatorname{cosec} x$
 The auxiliary equation is $m^2 + 1 = 0 \implies m = \pm i.$

$$\therefore C.F. = c_1 \cos x + c_2 \sin x$$

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$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2 = 1$$

The P.I. is given by $y_p = C(x)y_1(x) + D(x)y_2(x)$ where

$$\begin{aligned} C(x) &= - \int \frac{F(x) \cdot y_2}{W} dx \\ &= - \int \frac{\operatorname{cosec} x \cdot \sin x}{1} dx \\ &= -x \end{aligned}$$

$$\begin{aligned} D(x) &= \int \frac{F(x) \cdot y_1}{W} dx \\ &= \int \frac{\operatorname{cosec} x \cdot \cos x}{1} dx \\ &= \log(\sin x) \end{aligned}$$

$$\therefore P.I. = -x \cos x + \log(\sin x) \sin x$$

Hence the complete solution is given by

$$y = C.F. + P.I. = y_c + y_p = c_1 \cos x + c_2 \sin x - x \cos x + \log(\sin x) \sin x.$$

(b) Eliminating $x(t)$ from the given equations, we get

$$\left(\frac{d^2}{dt^2} + 1\right)y = e^t - 1$$

$$\Rightarrow y = c_1 \cos t + c_2 \sin t + \frac{1}{2}e^t + 1$$

and hence

$$\frac{dy}{dt} = -c_1 \sin t + 2c_2 \cos t + \frac{1}{2}e^t.$$

Substituting these in the given equations, we get

$$x(t) = -c_1 \sin t + c_2 \cos t - \frac{1}{2}e^t + t$$

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6. Exercise/Practice/Assignment Problems

In all the following problems D represents the operator $\frac{d}{dx}$ and y' represents $\frac{dy}{dx}$.

1. Solve the following differential equations

(a) $(D^2 - 4D + 3)y = \sin 3x + x^2$

Ans: $y = c_1 e^x + c_2 e^{3x} + \frac{1}{30}(2 \cos 3x - \sin 3x) + \frac{1}{3}\left(x^2 + \frac{8}{3}x + \frac{26}{9}\right)$

(b) $y'' - 6y' + 8y = e^{-2x} + 4$

(c) $y'' + 4y = x^4 + \cos^2 x$

Ans: $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{8}(4 - 6x^2 + 2x^4 + x \sin 2x)$

(d) $(D^2 + 4D + 3)y = e^{2x} + 5$

2. Solve the following differential equations

(a) $(D^2 + 5D + 6)y = \cos(-3x) + 6$

(b) $(D^2 - 6D + 8)y = \cos 5x + e^{4x} + \sin 4x$

(c) $(D^2 - 6D + 9)y = e^{3x} + \sin 2x$

(d) $(D^2 + 4D + 3)y = \cos^2 2x$

3. Solve the following differential equations

(a) $(D^2 + 5D + 6)y = x^3 + 2x^2$

(b) $(D^2 - 6D + 8)y = x^4$

$$(c) (D^2 - 6D + 9)y = x^2 + 1$$

$$(d) (D^2 + 4D + 3)y = x^2 - 4x$$

4. Solve the following differential equations

$$(a) (D^2 + 5D + 6)y = \cos^3 3x + e^{-2x} \sin 2x$$

$$(b) (D^2 - 6D + 8)y = e^{-2x} \cos 4x$$

$$(c) (D^2 - 6D + 9)y = e^{3x} x^2$$

$$(d) (D^2 + 4D + 3)y = xe^{2x}$$

5. Solve the following differential equations

$$(a) (D^2 + 5D + 6)y = xe^{-3x} \cos 2x$$

$$(b) (D^2 - 6D + 8)y = x^2 \sin 3x$$

$$(c) (D^2 - 6D + 9)y = x^2 e^{2x} \sin 4x$$

$$(d) (D^2 + 4D + 3)y = xe^{2x} \sin 5x$$

6. Solve the following differential equations

$$(a) (x^2 D^2 - 3xD + 5)y = x^2 \sin(\log x)$$

$$(b) (x^2 D^2 + 4xD + 2)y = \sin(\log x)$$

$$(c) (x^2 D^2 - 3xD)y = x + 1$$

$$(d) (D^2 + \frac{1}{x}D + 5)y = \frac{12 \log x}{x^2}$$

$$(e) ((1+x)^2 D^2 + (1+x)D + 1)y = 2 \sin(\log(x+1))$$

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(f) $((3x + 2)^2 D^2 + 3(3x + 2)D - 36)y = 3x^2 + 4x + 1$

Ans: $y = c_1 e^{2z} + c_2 e^{-2z} + \frac{1}{108} (ze^{2z} + 1)$ with $z = \log(3x + 2)$

(g) $((2x + 1)^2 D^2 - 2(2x + 1)D - 12)y = 6x$

Ans: $y = c_1 e^{3z} + c_2 e^{-z} - \frac{3}{16} (e^z) - \frac{9}{12}$ with $z = \log(2x + 1)$

(h) $((2x + 5)^2 D^2 - 6(2x + 5)D - 8)y = 6x$

Ans: $y = e^{2z} \left[c_1 e^{\sqrt{2z}} + c_2 e^{-\sqrt{2z}} \right] - \frac{3}{4} (e^z) - \frac{15}{8}$ with $z = \log(2x + 5)$

(i) $((1 + 2x)^2 D^2 + 3(1 + 2x)D + 1)y = 8(1 + 2x)^2$

7. Solve the following differential equations using the method of Variation of Parameters

(a) $(D^2 + 1)y = \tan x$

(b) $(D^2 + a^2)y = \sec ax$

(c) $(D^2 - 1)y = e^x \sin x$

(d) $(D^2 + 9)y = 3 \sin 3t$

(e) $(D^2 - 2D + 1)y = \frac{e^x}{x^2 + 1}$

(f) $(D^2 + 2D + 1)y = e^{-x} \cos x$

8. Solve the following system of differential equations

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(a) $\frac{dx}{dt} + y = e^t; \frac{dy}{dt} = t$

(b) $\frac{dx}{dt} + 2x + 3y = 2e^{2t}; \frac{dy}{dt} + 3x + 2y = 0$

(c) $\frac{dx}{dt} + 2y = -\sin t; \frac{dy}{dt} - 2x = \cos t$

(d) $\frac{dx}{dt} - \left(\frac{dx}{dt} - 2\right)y = \cos 2t; \left(\frac{dx}{dt} - 2\right)x + \frac{dy}{dt} = \sin 2t$

(e) $\frac{dx}{dt} + \frac{dy}{dt} + y = 1; \frac{dx}{dt} - \frac{dz}{dt} + 2x + z = 1; \frac{dy}{dt} + \frac{dz}{dt} + y + 2z = 2$

(f) $2\frac{dx}{dt} + \frac{dy}{dt} - 4x - y = e^t; \frac{dx}{dt} + 3x + y = 0$

(g) $\frac{d^2x}{dt^2} - 5x + 3y = \sin t; \frac{d^2y}{dt^2} - 3x + 5y = t$

(h) $\frac{d^2x}{dt^2} - 3x - 4y = 0; \frac{d^2y}{dt^2} + x + y = 0$

(i) $D^2x - 2x - Dy = 2t; Dx + 4Dy - 3y = 0$