MATHS ASSIGNMENT - 2

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Q1. $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \log x \cdot \sin(\log x)$

let $x\frac{dy}{dx} = Dy$ and $x^2\frac{d^2y}{dx^2} = D(D-1)y$

$[D(D-1) + D+1]y = \log x \cdot \sin(\log x)$

$\Rightarrow [D^2+1]y = \log x \cdot \sin(\log x)$

Now, $e^z = x$

$\therefore \log x = z$

$\Rightarrow [D^2+1]y = z\sin z$

All the complementary function by replacing D by M (auxiliary eqtn):

$\Rightarrow (M^2+1)y = 0$

$\Rightarrow M^2 = -1$

$\Rightarrow M = \pm i$ [Roots are complex $\alpha = 0$ and $\beta = 1$]

$\therefore y_{C.F} = C_1\cos z + C_2 \sin z$

Now, the PI can be written as:

$PI = \frac{1}{D^2+1} z\sin z \Rightarrow$ Imaginary

$\Rightarrow \frac{1}{D^2+1} ze^{iz} \Rightarrow$ Imaginary

$\Rightarrow \frac{1}{(D+i)^2+1} ze^{iz}$

$\Rightarrow \frac{1}{D^2+2iD+i^2+1} ze^{iz} = \frac{1}{D^2+2iD+\cancel{1}-\cancel{1}} ze^{iz}$

(2)

$$\Rightarrow \frac{1}{2iD\left(1+\frac{D}{2i}\right)} ze^{iz} = \frac{ze^{iz}}{2iD}\left(1+\frac{D}{2i}\right)^{-1}$$

$$\Rightarrow \frac{e^{iz}}{2iD}\left(z-\frac{1}{2i}\right) = \frac{ie^{iz}}{2i^2D}\left(z-\frac{1}{2i}\right)$$

$$\Rightarrow \frac{ie^{iz}}{-2}\left(\frac{z}{D}-\frac{1}{2iD}\right) = \frac{ie^{iz}}{-2}\left(\frac{z^2}{2}+\frac{i^2 z}{2i}\right)$$

$$\Rightarrow \frac{ie^{iz}}{-2}\left(\frac{z^2+iz}{2}\right) = \frac{zi(\cos z + i\sin z)(z+i)}{-4}$$

$$\Rightarrow \frac{z}{4}(z\cos z + \sin z)$$

$$\therefore y = y_{C.F} + y_{P.I}$$

$$\Rightarrow y = C_1\cos z + C_2 \sin z + \frac{z}{4}(\sin z - z\cos z)$$

$$\Rightarrow y = C_1\cos(\log x) + C_2\sin(\log x) + \frac{\log x}{4}(\sin(\log x) - \log x.\cos(\log x))$$

$$\Rightarrow \boxed{y = C_1\cos(\log x) + C_2\sin(\log x) + \frac{\log x}{4}(\sin(\log x) - \log x.\cos(\log x))}$$ //

Q2. $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$

$$\Rightarrow (D^2-1)y = \frac{2}{1+e^x}$$

Complementary function replacing D by M. (Auxiliary Eqtn):

$\Rightarrow (M^2-1) = 0$

$\Rightarrow M^2 = -1$

$\Rightarrow M = \pm i$

$\therefore$ The roots are complex, $\alpha = 0$ & $\beta = 1$.

$\therefore y_{C.F} = C_1 e^{x} + C_2 e^{-x}$

$P.I = u y_1 + v y_2$

$\Rightarrow y_1 = e^{x},\ y_2 = e^{-x}$ and $f(x) = \dfrac{2}{1+e^{x}}$

Now, $y_1 y_2' - y_2 y_1' = e^{x}(-e^{-x}) - e^{x}(e^{-x})$

$\Rightarrow -1-1$

$\Rightarrow -2$

$\therefore u = \int \dfrac{1}{e^{x}(e^{x}+1)}\, dx$ | let $e^{x} = t$, $e^{x} = \dfrac{dt}{dx} \Rightarrow dx = \dfrac{dt}{e^{x}}$

$\Rightarrow u = \int \dfrac{1}{t^2(t+1)}\, dt$

By partial fractions $\left[\dfrac{A}{t^2} - \dfrac{B}{t} + \dfrac{C}{1+t}\right]$

$\therefore u \int \left(\dfrac{1}{t^2} - \dfrac{1}{t} + \dfrac{1}{t+1}\right) dt$

$\Rightarrow u = \dfrac{-1}{t} - \log t + \log(1+t) + C_1$

$\Rightarrow u = -e^{-x} - x + \log(1+e^{x}) + C_1$ —— 1

$\therefore v \int \dfrac{y_1 f(x)}{y_1 y_2' - y_2 y_1'}\, dx \Rightarrow v = \int \dfrac{e^{x} \cdot \dfrac{2}{1+e^{x}}}{-2}\, dx$

$\Rightarrow v = -\int \dfrac{e^{x}}{1+e^{x}}\, dx$ | let $t = 1+e^{x}$, $dt/dx = e^{x} \Rightarrow dt \cdot e^{x} dx$

$\Rightarrow v = -\int \dfrac{dt}{t} \Rightarrow v = -\log t + C_2$

$\Rightarrow v = -\log(1+e^{x}) + C_2$ —— 2

Now, $Y = uY_1 + vY_2$ (4)

$\therefore Y = e^{x}\left(-e^{-x} - x + \log(1+e^{x}) + C_1\right) + e^{-x}\left(-\log(1+e^{x}) + C_2\right)$

$\Rightarrow \boxed{Y = C_1 e^{x} + C_2 e^{x} - 1 - xe^{x} + \left[(e^{x} - e^{-x})(\log(1+e^{x}))\right]}$ //

Q3. Expanding $f(x,y)$ about $(1,1)$

$$f(x,y) = f(1,1) + \left[(x-1) f_x(1,1) + (y-1) f_y(1,1)\right]$$
$$+ \frac{1}{2!}\left[(x-1)^2 f_{xx}(1,1) + 2(x-1)(y-1) f_{xy}(1,1) + (y-1)^2 f_{yy}(1,1)\right] + \ldots$$

Now, $f(x,y) = \tan^{-1}\frac{y}{x}$; $f(1,1) = \tan^{-1}(1) \Rightarrow \frac{\pi}{4}$

$\therefore f_x = \frac{1}{1+y^2/x^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2}$; $f_x(1,1) = -1/2$

$\therefore f_y = \frac{1}{1+y^2/x^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}$; $f_y(1,1) = 1/2$

$\therefore f_{xx} = \frac{2xy}{(x^2+y^2)^2}$; $f_{xx}(1,1) = 1/2$

$\therefore f_{yy} = \frac{-2xy}{(x^2+y^2)^2}$; $f_{yy}(1,1) = -1/2$

$\therefore f_{xy} = \frac{(x^2+y^2)(1-x)(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

$\Rightarrow f_{xy} = 0$

Now,

$\boxed{\tan^{-1}(x/y) = \pi/4 + 1/1!\left[-1/2(x-1) + 1/2(y-1)\right] + 1/2!\left[1/2(x-1)^2 - 1/2(y-1)^2\right]}$ //

Q4. $g(x,y) = \varphi(u,v)$ where $u = x^2 - y^2$ & $2yx = v$

(5)

$$\Rightarrow \partial\varphi = \frac{\partial\varphi}{\partial u}\cdot\partial u + \frac{\partial\varphi}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\Rightarrow \frac{\partial\varphi}{\partial x} = \frac{\partial\varphi}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial\varphi}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial x} = 2x, \ \frac{\partial v}{\partial x} = 2y$$

$$\Rightarrow \frac{\partial\varphi}{\partial x} = \frac{\partial\varphi}{\partial u}(2x) + \frac{\partial\varphi}{\partial v}(2y) \quad \text{—— (1)}$$

$$\Rightarrow \frac{\partial}{\partial x}\cdot\frac{\partial}{\partial u}(2x) + \frac{\partial}{\partial v}(2y) \quad \text{—— (2)}$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial x}\right) = \frac{\partial}{\partial x}\left(2x\frac{\partial}{\partial u} + 2y\cdot\frac{\partial}{\partial v}\right)$$

$$\Rightarrow \frac{2\partial}{\partial u} + 2x\cdot\frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial u}\right) + \frac{2\partial\varphi}{\partial v}\cdot(0) + 2y\frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial v}\right)$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial x}\right) = \frac{2\partial\varphi}{\partial u} + 2x\left[2x\frac{\partial}{\partial u} + 2y\frac{\partial}{\partial v}\right]\left[\frac{\partial\varphi}{\partial u}\right] + 2y\left[2x\frac{\partial}{\partial v} + 2y\frac{\partial}{\partial v}\right]\left(\frac{\partial\varphi}{\partial v}\right)$$

$$\Rightarrow \frac{\partial^2\varphi}{\partial x^2} = \frac{2\partial\varphi}{\partial u} + \frac{4x^2\partial^2\varphi}{\partial u^2} + 4\frac{xy\,\partial\varphi}{\partial x\partial y} + 4xy\frac{\partial^2 y}{\partial y\partial x} + 4y^2\frac{\partial^2\varphi}{\partial v^2} \quad \text{—— (3)}$$

Similarly, $\frac{\partial\varphi}{\partial y} = \frac{\partial\varphi}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial\varphi}{\partial u}\cdot\frac{\partial u}{\partial y}$

$$\Rightarrow \frac{\partial u}{\partial y} = -2y, \ \frac{\partial v}{\partial y} = 2x$$

$$\Rightarrow \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial v}(2x) + \frac{\partial \psi}{\partial u}(-2y)$$

$$\Rightarrow \frac{\partial}{\partial y} = 2x\frac{\partial}{\partial v} - 2y\frac{\partial \psi}{\partial u}.$$

$$\Rightarrow \frac{\partial}{\partial y} = 2x\frac{\partial}{\partial v} - 2y\frac{\partial}{\partial u} \quad \text{—— (4)}$$

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial y}\left[2x\frac{\partial}{\partial v} - 2y\frac{\partial}{\partial u}\right]$$

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) = \frac{-2\partial \psi}{\partial u} - 2y\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial u}\right) + 2\frac{\partial \psi}{\partial v}\cdot(0) + 2x\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial v}\right)$$

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) = -\frac{2\partial \psi}{\partial u} - 2y\left[2x\frac{\partial}{\partial v} - 2y\frac{\partial}{\partial u}\right]\left(\frac{\partial \psi}{\partial u}\right) + 2x\left[2x\frac{\partial}{\partial v} - 2y\frac{\partial}{\partial u}\right]\left[\frac{\partial \psi}{\partial v}\right]$$

$$\Rightarrow \frac{\partial^2 \psi}{\partial y^2} = \frac{-2\partial \psi}{\partial u} + 4y^2\frac{\partial^2 \psi}{\partial u^2} - 4xy\frac{\partial^2 \psi}{\partial v \partial u} - 4xy\frac{\partial^2 \psi}{\partial u \partial v} + 4x^2\frac{\partial^2 \psi}{\partial v^2} \quad \text{—— (5)}$$

Adding eqtns (4) & (5),

$$\Rightarrow \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 4(x^2+y^2)\left[\frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2}\right]$$

Now, as $g(x,y).\psi(u,v)$

$$\Rightarrow \boxed{\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 4(x^2+y^2)\left[\frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2}\right]} \;//$$

$\therefore$ Hence Proved.

Q5. let $P(x,y,z)$ be a point on the sphere
$x^2+y^2+z^2=24$ and Q be $(1,2,-1)$

Distance $PQ = \sqrt{(x-1)^2+(y-2)^2+(z+1)^2}$

PQ is minimum or maximum. If $f(x,y,z)$ is minimum or maximum.

let $\phi(x,y,z) = x^2+y^2+z^2-24=0$

Auxiliary function →

$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$

where λ is lagrange's multiplier

$F(x,y,z) = (x-1)^2+(y-2)^2+(z+1)^2+\lambda(x^2+y^2+z^2-24)$

$F_x = 2(x-1)+2\lambda x$

$F_y = 2(y-2)+2\lambda y$

$F_z = 2(z+1)+2\lambda z$

$F_\lambda = \phi$

finding stationary points,

$F_x = 0 \Rightarrow 2(x-1)+2\lambda x = 0$

$\Rightarrow x-1 = -\lambda x$

$\Rightarrow -\lambda = \frac{x-1}{x} = 1-\frac{1}{x}$

$F_y = 0 \Rightarrow 2(y-2)+2\lambda y = 0$

$\Rightarrow y-2 = -\lambda y$

$\Rightarrow -\lambda = \frac{y-2}{y} = 1-\frac{2}{y}$

$F_z = 0 \Rightarrow 2(z+1) + 2\lambda z = 0$

$\Rightarrow z+1 = -\lambda z$

$\Rightarrow -\lambda = \frac{z+1}{z} = 1+\frac{1}{z}$

$\therefore 1-\frac{1}{x} = 1-\frac{2}{y} = 1+\frac{1}{z}$

$\Rightarrow 1-\frac{1}{x} = 1-\frac{2}{y} \Rightarrow \frac{1}{x} = \frac{2}{y} \Rightarrow y = 2x$

$\Rightarrow 1-\frac{1}{x} = 1+\frac{1}{z} \Rightarrow -\frac{1}{x} = \frac{1}{z} \Rightarrow -z = x$

$\Rightarrow 1-\frac{2}{y} = 1+\frac{1}{z} \Rightarrow \frac{-2}{y} = \frac{1}{z} \Rightarrow -2z = y$

$\therefore 2x = y = -2z$ or $x = y/2 = -z$

Now, $x^2+y^2+z^2 = 24$

$\Rightarrow x^2+4x^2+x^2 = 24$

$\Rightarrow 6x^2 = 24 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

When $x=2$, $y=4$, $z=-2$

and when $x=-2$, $y=-4$, $z=2$

The stationary points are $P_1(2,4,-2)$ & $P_2(-2,-4,2)$

$P_1Q = \sqrt{1+4+1} = \sqrt{6}$ & $P_2Q = \sqrt{9+36+9} = 3\sqrt{6}$

Hence, The shortest distance is $\sqrt{6}$ and longest distance is $3\sqrt{6}$.