18MAB201T- Transforms and Boundary Value Problems

Unit IV - Fourier Transforms

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Introduction

We shall discuss the Fourier integral and Fourier transforms which are useful in solving boundary value problems arising in engineering e.g. conduction of heat, theory of communication, wave propagation etc. Fourier series are helpful in problems involving periodic function. In many practical problems, the function is non-periodic. A suitable representation for non-periodic function can be obtained by considering the limiting form of Fourier series when the fundamental period is infinite. In such case, the Fourier series becomes Fourier Integral which can be expressed in terms of Fourier Transforms which transform a non-periodic function.

Introduction

The effect of applying an integral transforms to a partial differential equation is to reduce the number of independent one. The choice of particular transform is decided by the nature of the boundary conditions and the facility with which the transform can be inverted to give $f(x)$.

Integral Transform

If $f(x)$ is defined in (a, b), the integral transform of $f(x)$ with the Kernal $K(s, x)$ is defined by

$$F(s) = \bar{f}(s) = \int_a^b f(x)K(s, x)\,dx$$

if the integral exists.

Note: If a, b are finite, the transform is finite and if a, b are infinite, it is an infinite transform.

Fourier Integral Theorem

If $f(x)$ is piece-wise continuously differentiable and absolutely integrable in $(-\infty, \infty)$, then

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) e^{i(x-t)s} \, dt \, ds$$

Complex Fourier Transform

Let $f(x)$ be a function defined in $(-\infty, \infty)$ and be piecewise continuous in each finite partial interval and absolutely integrable in $(-\infty, \infty)$. Then the complex (or infinite) Fourier transform of $f(x)$ is given by

$$\bar{f}(s) = F(s) = F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\, dx \tag{1}$$

Inversion theorem for Complex Fourier Transform

If $f(x)$ satisfies the Dirichlet's conditions in every finite interval $(-l, l)$ and if it is absolutely integrable in the range and if $F(s)$ denotes the complex Fourier transform of $f(x)$ then at every point of continuity of $f(x)$, we have

$$f(x) = F^{-1}\{F(s)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx}\, ds \tag{2}$$

Both equations (1) and (2) are called as Fourier Transforms pairs.

Properties of Fourier Transforms

Property 1: Linearity Property

Fourier transform is linear. i.e. $F[af(x)+bg(x)] = aF[f(x)] + bF[g(x)]$ where F stands for Fourier transform.

Proof:

By definition $F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{isx}\,dx$

$$F[af(x)+bg(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (af(x)+bg(x))\,e^{isx}\,dx$$

$$= a.\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{isx}\,dx + b.\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{isx}\,dx$$

$$= aF[f(x)] + bF[g(x)]$$

Properties of Fourier Transforms

Property 2: Shifting property (in x)

If $F\{f(x)\} = F(s)$ then $F\{f(x-a)\} = e^{ias}F(s)$.

Proof:

$$\text{By definition} \quad F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\, dx$$

$$\Rightarrow F\{f(x-a)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a) e^{isx}\, dx$$

Putting $x - a = t \Rightarrow dx = dt$

$$F\{f(t)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{is(t+a)}\, dt$$

$$= e^{ias}F(s)$$

Property 3:

If $F\{f(x)\} = F(s)$ then $F\left\{e^{ias} f(x)\right\} = F(s+a)$.

Proof:

By definition $$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\, dx$$

$$F\left\{e^{iax} f(x)\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} f(x) e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(s+a)x}\, dx$$

$$= F(s+a)$$

Property 4: Change of scale property

If $F\{f(x)\} = F(s)$ then $F\{f(ax)\} = \frac{1}{|a|} F\left(\frac{s}{a}\right)$ where $a \neq 0$.

Proof: $F\{f(ax)\} = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(ax) e^{isx}\, dx$

Case (i): $a > 0$

Putting $ax = t \Rightarrow a\,dx = dt$

when $x = -\infty \Rightarrow t = -\infty$ and when $x = \infty \Rightarrow t = \infty$

$$F\{f(ax)\} = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(t) e^{is\left(\frac{t}{a}\right)} \frac{dt}{a}$$

$$= \frac{1}{a\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(t) e^{is\left(\frac{t}{a}\right)}\, dt$$

Change of scale property (Contd.)

$$F\{f(ax)\} = \frac{1}{a} F\left(\frac{s}{a}\right) \tag{3}$$

Case (ii): $a < 0$

Putting $ax = t \Rightarrow a\,dx = dt$

when $x = -\infty \Rightarrow t = \infty$ and when $x = \infty \Rightarrow t = -\infty$

$$F\{f(ax)\} = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(t) e^{is\left(\frac{t}{a}\right)} \frac{dt}{a}$$

$$= -\frac{1}{a\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{is\left(\frac{t}{a}\right)} dt$$

$$= -\frac{1}{a} F\left(\frac{s}{a}\right) \tag{4}$$

From (3) and (4), we get $F\{f(ax)\} = \frac{1}{|a|} F\left(\frac{s}{a}\right)$.

Properties of Fourier Transforms

Property 5: Modulation Theorem

If $F\{f(x)\} = F(s)$ then $F\{f(x)\cos ax\} = \frac{1}{2}[F(s-a)+F(s+a)]$.

Proof:

$$F\{f(x)\cos ax\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cos ax e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\left[\frac{e^{iax}+e^{-iax}}{2}\right] e^{isx}\, dx$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i(s+a)}\, dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i(s-a)}\, dx\right]$$

$$= \frac{1}{2}[F(s-a)+F(s+a)].$$

Property 6: Derivative of transform

If $F\{f(x)\} = F(s)$ then $F\{x^n f(x)\} = (-i)^n \dfrac{d^n}{ds^n} F(s)$.

Proof: By definition

$F\{f(x)\} = F(s) = \dfrac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(x) e^{isx}\, dx$

Differentiating with respect to s both sides n times, we get

$$\begin{aligned}
\frac{d^n}{ds^n} F(s) &= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(x)(ix)^n e^{isx}\, dx \\
&= i^n \left(\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} x^n f(x) e^{isx}\, dx \right) \\
&= i^n F\{x^n f(x)\}
\end{aligned}$$

Derivative of transform (Contd.)

$$
\begin{aligned}
F\left\{x^n f(x)\right\} &= \frac{1}{i^n}\frac{d^n}{ds^n}F(s)\\
&= \left(\frac{1}{i}\right)^n \frac{d^n}{ds^n}F(s)\\
&= \left(\frac{i}{i\times i}\right)^n \frac{d^n}{ds^n}F(s)\\
&= (-i)^n \frac{d^n}{ds^n}F(s)
\end{aligned}
$$

Properties of Fourier Transforms

Property 7: Fourier transform of Derivative

$F\left\{f'(x)\right\} = -isF(s)$ if $f(x) \to \infty$ as $x \to \pm\infty$

Proof:

$$F\left\{f'(x)\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f'(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx}\, d\{f(x)\}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left\{ e^{isx} f(x) \right\}_{\infty}^{\infty} - is \int_{-\infty}^{\infty} f(x) e^{isx}\, dx \right]$$

$$= -isF(s) \text{ if } f(x) \to 0 \text{ as } x \to \pm\infty$$

Properties of Fourier Transforms

Property 8: Fourier transform of an integral function

$$F\left\{\int_a^x f(x)\,dx\right\} = \frac{F(s)}{(-is)}$$

Proof:

Let $\phi(x) = \int_a^x f(x)\,dx$ then $\phi'(x) = f(x)$

$$\begin{aligned} F\left\{\phi'(x)\right\} &= (-is)\overline{\phi}(s) \\ &= (-is)F(\phi(x)) \\ &= (-is)F\left\{\int_a^x f(x)\,dx\right\} \end{aligned}$$

Fourier transform of an integral function (Contd.)

$$\Rightarrow F\left\{\int_{a}^{x} f(x)\,dx\right\} = \frac{1}{(-is)} F\left\{\phi'(x)\right\}$$

$$= \frac{1}{(-is)} F\left(f(x)\right)$$

$$= \frac{F(s)}{(-is)}$$

Properties of Fourier Transforms

Property 9: $F\left\{\overline{f(-x)}\right\} = \overline{F(s)}$, where $\overline{F(s)}$ is the complex conjugate of $F(s)$.

Proof:

By definition $F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(x)e^{isx}\, dx$

Taking complex conjugate, we get

$$\overline{F(s)} = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \overline{f(x)}e^{-isx}\, dx$$

Put $x = -y \Rightarrow dx = -dy$

When $x \to -\infty \Rightarrow y \to \infty$ and $x \to \infty \Rightarrow y \to -\infty$

Property 9 (Contd.)

$$\overline{F(s)} = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} \overline{f(-y)} e^{isy} (-dy)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(-y)} e^{isy} \, dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(-x)} e^{isx} \, dx, \text{ by changing the dummy variable}$$

$$= \overline{F(s)} = F\left\{\overline{f(-x)}\right\}$$

Convolution Theorem or Faltung Theorem

Convolution of two function: The convolution of two functions $f(x)$ and $g(x)$ is defined as

$$f * g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)g(x-t)\,dt$$

Theorem: The Fourier transforms of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transforms.
That is $F\{f(x) * g(x)\} = F(s).G(s) = F\{f(x)\}.F\{g(x)\}$.
Proof:

By definition $F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{isx}\,dx$

Convolution Theorem (Contd.)

$$\Rightarrow F\{f*g\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(f(x)*g(x))\,e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)g(x-t)\,dt\right)e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(x-t)e^{isx}\,dx\right)dt$$

by changing the order of integration

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)F\{g(x-t)\}\,dt$$

Convolution Theorem (Contd.)

$$F\{f(x)*g(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{its}G(s)\,dt$$

by shifting theorem

$$= G(s).\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{its}\,dt$$

$$= G(s).F(s)$$

$$= F(s).G(s) = F\{f(x)\}.F\{g(x)\}$$

Note:

By inversion, $f*g = F^{-1}\{F(s)G(s)\} = F^{-1}\{F(s)\}*F^{-1}\{G(s)\}$.

Parseval's Identity

If $F(s)$ is the Fourier transform of $f(x)$ then

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |F(s)|^2 \, ds$$

Proof:
By convolution theorem, $F\{f(x) * g(x)\} = F(s)G(s)$

$$\Rightarrow f * g = F^{-1}\{F(s)G(s)\}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)g(x-t)\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)G(s)e^{-isx}\, ds$$

Parseval's Identity

Put $x = 0$, we get

$$\int_{-\infty}^{\infty} f(t)g(-t)\,dt = \int_{-\infty}^{\infty} F(s)G(s)\,ds \tag{5}$$

Take $g(-t) = \overline{f(t)} \Rightarrow g(t) = \overline{f(-t)}$

Therefore $G(s) = F\{g(t)\} = F\left\{\overline{f(-t)}\right\} = \overline{F(s)}$ by property 9

Hence equation (5) becomes

$$\int_{-\infty}^{\infty} f(t)\overline{f(t)}\,dt = \int_{-\infty}^{\infty} F(s)\overline{F(s)}\,ds$$

$$\Rightarrow \int_{-\infty}^{\infty} |f(t)|^2\,dt = \int_{-\infty}^{\infty} |F(s)|^2\,ds$$

Example 1:

Find the complex Fourier transform of $f(x) = \begin{cases} x & \text{for } |x| \le a \\ 0 & \text{for } |x| > a \end{cases}$.

Solution:

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x(\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-a}^{a} x \cos sx\, dx + i \int_{-a}^{a} x \sin sx\, dx \right\}$$

Example 1: (Contd.)

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\left\{0 + 2i\int_0^a x\sin sx\,dx\right\}$$

Since the first integral is an odd function and the second integral is an even function.

$$= \frac{2i}{\sqrt{2\pi}}\left[x\left(-\frac{\cos sx}{s}\right) - 1\left(-\frac{\sin sx}{s^2}\right)\right]_0^a$$

$$= \frac{2i}{\sqrt{2\pi}}\left[\frac{-a\cos sa}{s} + \frac{\sin sa}{s^2}\right]$$

$$= \frac{2i}{\sqrt{2\pi}}\left[\frac{\sin sa - as\cos sa}{s^2}\right]$$

Example 2:

Find the Fourier transform of $f(x) = \begin{cases} 1-x^2 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}$. Hence evaluate $\int\limits_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right)\cos\left(\frac{x}{2}\right)\,dx$.

Sol: By definition $F(s) = F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} f(x)e^{isx}\,dx$

$$F(s) = \frac{1}{\sqrt{2\pi}}\int\limits_{-1}^{1}(1-x^2)(\cos sx + i\sin sx)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\int\limits_{-1}^{1}(1-x^2)\cos sx dx + i\int\limits_{-1}^{1}(1-x^2)\sin sx\,dx\right\}$$

Example 2: (Contd.)

$$F(s) = \frac{1}{\sqrt{2\pi}} \left\{ 2\int_0^1 (1-x^2)\cos sx\, dx + 0 \right\}$$

Since the first integral is an even function and the second integral is an odd function.

$$= \frac{2}{\sqrt{2\pi}} \left[(1-x^2)\left(\frac{\sin sx}{s}\right) - (-2x)\left(-\frac{\cos sx}{s^2}\right) \right.$$

$$\left. + (-2)\left(-\frac{\sin sx}{s^3}\right) \right]_0^1 = \frac{2}{\sqrt{2\pi}} \left[\frac{-2\cos s}{s^2} + \frac{2\sin s}{s^3} \right]$$

$$= \frac{-4}{\sqrt{2\pi}} \left[\frac{s\cos s - \sin s}{s^3} \right]$$

Example 2: (Contd.)

To find $\int\limits_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right)\cos\left(\frac{x}{2}\right)\,dx$

Using inverse Fourier Transform $f(x) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} F(s)e^{-isx}\,ds$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\frac{-4}{\sqrt{2\pi}}\left(\frac{s\cos s - \sin s}{s^3}\right)(\cos sx - i\sin sx)\,ds$$

$$= \frac{-2}{\pi}\left\{\int\limits_{-\infty}^{\infty}\left(\frac{s\cos s - \sin s}{s^3}\right)\cos sx\,ds\right.$$

$$\left. - i\int\limits_{-\infty}^{\infty}\left(\frac{s\cos s - \sin s}{s^3}\right)\sin sx\,ds\right\}$$

Example 2: (Contd.)

$$f(x) = \frac{-2}{\pi}\left\{2\int_0^\infty \left(\frac{s\cos s - \sin s}{s^3}\right)\cos sx\,ds - 0\right\}$$

Since the first integral is an even function and the second integral is an odd function.

Put $x = \dfrac{1}{2}$ in the above integral. But $x = \dfrac{1}{2}$ is a point of continuity of $f(x)$.

Therefore value of the integral when $x = \dfrac{1}{2}$ is

$$f\left(\frac{1}{2}\right) = 1 - \frac{1}{4} = \frac{3}{4}.$$

Example 2: (Contd.)

Therefore $\dfrac{3}{4} = -\dfrac{4}{\pi}\int\limits_0^\infty \left(\dfrac{s\cos s - \sin s}{s^3}\right)\cos\left(\dfrac{s}{2}\right)\,ds$

$$\Rightarrow \int\limits_0^\infty \left(\frac{s\cos s - \sin s}{s^3}\right)\cos\left(\frac{s}{2}\right)\,ds = -\frac{3\pi}{16}$$

Hence $\displaystyle\int\limits_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right)\cos\left(\frac{x}{2}\right)\,dx = -\frac{3\pi}{16}$

Example 3:

Find the Fourier transform of $f(x)$ given by $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$ and hence evaluate (i) $\int\limits_{-\infty}^{\infty} \frac{\sin as \cos sx}{s}\, ds$, (ii) $\int\limits_{0}^{\infty} \frac{\sin x}{x}\, dx$ and prove that $\int\limits_{0}^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$.

Sol:

$$F(s) = F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(x) e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int\limits_{-a}^{a} (\cos sx + i \sin sx)\, dx$$

Example 3: (Contd.)

$$F(s) = \frac{1}{\sqrt{2\pi}} \left\{ \int_{-a}^{a} \cos sx\, dx + i \int_{-a}^{a} \sin sx\, dx \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ 2 \int_{0}^{a} \cos sx\, dx + 0 \right\}$$

Since the first integral is an even function and the second integral is an odd function.

$$= \frac{2}{\sqrt{2\pi}} \left[\frac{\sin sx}{s} \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} . \frac{\sin as}{s}$$

Example 3: (Contd.)

To find (i) $\int_{-\infty}^{\infty} \frac{\sin as \cos sx}{s} \, ds$

Using inverse Fourier Transform $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} \, ds$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} . \frac{\sin as}{s} (\cos sx - i \sin sx) \, ds$$

$$1 = \frac{1}{\pi} \left\{ \int_{-\infty}^{\infty} \left(\frac{\sin as}{s} \right) \cos sx \, ds - i \int_{-\infty}^{\infty} \left(\frac{\sin as}{s} \right) \sin sx \, ds \right\}$$

Example 3: (Contd.)

Equating the real part, we have

$$1 = \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin as}{s} \right) \cos sx \, ds$$

Hence

$$\int_{-\infty}^{\infty} \frac{\sin as \cos sx}{s} \, ds = \pi. \qquad (6)$$

To find (ii) $\int_{0}^{\infty} \frac{\sin x}{x} \, dx$

Put $x = 0$ in equation (6), we have

$$\int_{-\infty}^{\infty} \frac{\sin as}{s} \, ds = \pi.$$

Example 3: (Contd.)

$\Rightarrow 2\int\limits_0^\infty \frac{\sin as}{s}\,ds = \pi$. Since the given integral is an even.

$\therefore \int\limits_0^\infty \frac{\sin as}{s}\,ds = \frac{\pi}{2}$. Putting $as = t \Rightarrow a\,ds = dt$

$$\int\limits_0^\infty \frac{\sin t}{(t/a)}\cdot\frac{dt}{a} = \frac{\pi}{2} \Rightarrow \int\limits_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}.$$

Hence $\int\limits_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$.

Example 3: (Contd.)

(iii) To prove that $\int\limits_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$

Using Parseval's identity $\int\limits_{-\infty}^{\infty} |F(s)|^2\, ds = \int\limits_{-\infty}^{\infty} |f(x)|^2\, dx$

$$\Rightarrow \int\limits_{-\infty}^{\infty} \frac{2}{\pi}\left(\frac{\sin as}{s}\right)^2 ds = \int\limits_{-a}^{a} 1.\, dx$$

$$\Rightarrow \frac{2}{\pi}.2\int\limits_0^\infty \left(\frac{\sin as}{s}\right)^2 ds = (x)_{-a}^{a}$$

$$\Rightarrow \frac{4}{\pi}\int\limits_0^\infty \left(\frac{\sin as}{s}\right)^2 ds = 2a$$

Example 3: (Contd.)

$$\Rightarrow \int_0^\infty \left(\frac{\sin as}{s}\right)^2 ds = \frac{a\pi}{2}$$

Putting $as = t \Rightarrow a\,ds = dt$

$$\Rightarrow \int_0^\infty \left(\frac{\sin t}{(t/a)}\right)^2 \cdot \frac{dt}{a} = \frac{a\pi}{2}$$

$$\Rightarrow \int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}.$$

Example 4:

Find the Fourier transform of $f(x) = \begin{cases} a^2 - x^2 & |x| < a \\ 0 & |x| > a \end{cases}$. Hence evaluate (i) $\int\limits_0^\infty \left(\frac{\sin x - x\cos x}{x^3} \right) dx = \frac{\pi}{4}$ and

(ii) $\int\limits_0^\infty \left(\frac{\sin x - x\cos x}{x^3} \right)^2 dx = \frac{\pi}{15}$

Sol:

$$F(s) = F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(x) e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int\limits_{-1}^{1} (a^2 - x^2)(\cos sx + i \sin sx)\, dx$$

Example 4: (Contd.)

$$F(s) = \frac{1}{\sqrt{2\pi}} \left\{ \int_{-a}^{a} (a^2 - x^2) \cos sx dx + i \int_{-a}^{a} (a^2 - x^2) \sin sx\, dx \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ 2 \int_{0}^{a} (a^2 - x^2) \cos sx\, dx + 0 \right\}$$

Since the first integral is an even function and the second integral is an odd function

$$= \frac{2}{\sqrt{2\pi}} \left[(a^2 - x^2) \left(\frac{\sin sx}{s} \right) - (-2x) \left(-\frac{\cos sx}{s^2} \right) \right.$$

$$\left. + (-2) \left(-\frac{\sin sx}{s^3} \right) \right]_0^a$$

Example 4: (Contd.)

$$F(s) = \frac{2}{\sqrt{2\pi}}\left[\frac{-2a\cos as}{s^2} + \frac{2\sin as}{s^3}\right]$$
$$= \frac{4}{\sqrt{2\pi}}\left[\frac{\sin as - as\cos as}{s^3}\right]$$

To find (i) $\int\limits_0^\infty \left(\frac{\sin x - x\cos x}{x^3}\right) dx = \frac{\pi}{4}$

Using Inverse Fourier Transform $f(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} F(s) e^{-isx}\, ds$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \frac{4}{\sqrt{2\pi}} \left(\frac{\sin as - as\cos as}{s^3}\right) (\cos sx - i\sin sx)\, ds$$

Example 4: (Contd.)

$$f(x) = \frac{2}{\pi}\left\{ \int_{-\infty}^{\infty} \left(\frac{\sin as - as\cos as}{s^3} \right) \cos sx\, ds \right.$$

$$\left. - i \int_{-\infty}^{\infty} \left(\frac{\sin as - as\cos as}{s^3} \right) \sin sx\, ds \right\}$$

$$= \frac{2}{\pi}\left\{ 2\int_{0}^{\infty} \left(\frac{\sin as - as\cos as}{s^3} \right) \cos sx\, ds - 0 \right\}$$

Since the first integral is an even function and the second integral is an odd function

Example 4: (Contd.)

Put $x = 0$ in the above integral. But $x = 0$ is a point of continuity of $f(x)$.

$\therefore$ the value of the integral when $x = 0$ is $f(0) = a^2 - 0 = a^2$.

$$\therefore a^2 = \frac{4}{\pi} \int_0^\infty \left(\frac{\sin as - as \cos as}{s^3} \right) ds$$

$$\Rightarrow \int_0^\infty \left(\frac{\sin as - as \cos as}{s^3} \right) ds = \frac{\pi a^2}{4}$$

Putting $as = t \Rightarrow a\, ds = dt$, we get

$$\int_0^\infty \left(\frac{\sin t - t \cos t}{(t/a)^3} \right) \frac{dt}{a} = \frac{\pi a^2}{4} \Rightarrow \int_0^\infty \left(\frac{\sin t - t \cos t}{t^3} \right) dt = \frac{\pi}{4}$$

Hence $\displaystyle\int_0^\infty \left(\frac{\sin x - x \cos x}{x^3} \right) dx = \frac{\pi}{4}$.

Example 4: (Contd.)

To find (ii) $\int\limits_0^\infty \left(\frac{\sin x - x\cos x}{x^3}\right)^2 dx = \frac{\pi}{15}$

Using Parseval's identity $\int\limits_{-\infty}^{\infty} |F(s)|^2\, ds = \int\limits_{-\infty}^{\infty} |f(x)|^2\, dx$

$$\Rightarrow \int\limits_{-\infty}^{\infty} \frac{16}{2\pi}\left(\frac{\sin as - as\cos as}{s^3}\right)^2 ds = \int\limits_{-a}^{a} (a^2 - x^2)^2\, dx$$

$$\Rightarrow \frac{8}{\pi}.2\int\limits_0^{\infty} \left(\frac{\sin as - as\cos as}{s^3}\right)^2 ds = 2.\int\limits_0^{a} (a^2 - x^2)^2\, dx$$

$$\Rightarrow \int\limits_0^{\infty} \left(\frac{\sin as - as\cos as}{s^3}\right)^2 ds = \frac{\pi}{8}.\int\limits_0^{a} (a^4 - 2a^2x^2 + x^4)\, dx$$

Example 4: (Contd.)

$$\Rightarrow \int_0^\infty \left(\frac{\sin as - as\cos as}{s^3}\right)^2 ds = \frac{\pi}{8}\left[a^4x - 2a^2\frac{x^3}{3} + \frac{x^5}{5}\right]_0^a$$

$$\Rightarrow \int_0^\infty \left(\frac{\sin as - as\cos as}{s^3}\right)^2 ds = \frac{\pi a^5}{15}$$

Putting $as = t \Rightarrow a\,ds = dt$, we get

$$\int_0^\infty \left(\frac{\sin t - t\cos t}{(t/a)^3}\right)^2 \frac{dt}{a} = \frac{\pi a^5}{15} \Rightarrow \int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \frac{\pi}{15}.$$

Hence $\displaystyle\int_0^\infty \left(\frac{\sin x - x\cos x}{x^3}\right)^2 dx = \frac{\pi}{15}.$

Example 5:

Find the Fourier transform of $f(x) = \begin{cases} a - |x|, & |x| < a \\ 0, & |x| > a \end{cases}$ and hence deduce that (i) $\displaystyle\int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}$ and (ii) $\displaystyle\int_0^{\infty} \left(\frac{\sin x}{x}\right)^4 dx = \frac{\pi}{3}$.

Sol:

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a - |x|)(\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-a}^{a} (a - |x|) \cos sx dx + i \int_{-1}^{1} (a - |x|) \sin sx\, dx \right\}$$

Example 5: (Contd.)

$$F(s) = \frac{1}{\sqrt{2\pi}} \left\{ 2 \int_0^a (a - |x|) \cos sx \, dx + 0 \right\}$$

Since the first integral is an even function and the second integral is an odd function

$$= \frac{2}{\sqrt{2\pi}} \int_0^a (a - x) \cos sx \, dx$$

$$= \frac{2}{\sqrt{2\pi}} \left[(a - x) \left(\frac{\sin sx}{s} \right) - (-1) \left(-\frac{\cos sx}{s^2} \right) \right]_0^a$$

$$= \frac{2}{\sqrt{2\pi}} \left[\frac{-\cos as}{s^2} + \frac{1}{s^2} \right]$$

Example 5: (Contd.)

$$F(s) = \sqrt{\frac{2}{\pi}} \left(\frac{1 - \cos as}{s^2} \right)$$
$$= \sqrt{\frac{2}{\pi}} \cdot \left(\frac{2\sin^2(as/2)}{s^2} \right)$$

To find (i) $\int\limits_0^\infty \left(\frac{\sin x}{x} \right)^2 dx = \frac{\pi}{2}$

Using Inverse Fourier Transform $f(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} F(s) e^{-isx} \, ds$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left(\frac{2\sin^2(as/2)}{s^2} \right) (\cos sx - i \sin sx) \, ds$$

Example 5: (Contd.)

$$f(x) = \frac{2}{\pi}\left\{\int_{-\infty}^{\infty}\left(\frac{\sin^2(as/2)}{s^2}\right)\cos sx\,ds - i\int_{-\infty}^{\infty}\left(\frac{\sin^2(as/2)}{s^2}\right)\sin sx\,ds\right\}$$

Since the first integral is an even function and

the second integral is an odd function

$$= \frac{2}{\pi}\left\{2\int_{0}^{\infty}\left(\frac{\sin^2(as/2)}{s^2}\right)\cos sx\,ds - 0\right\}$$

Put $x = 0$ in the above integral. But $x = 0$ is a point of continuity of $f(x)$.

Therefore value of the integral when $x = 0$ is $f(0) = a - 0 = a$.

$$\therefore a = \frac{4}{\pi}\int_{0}^{\infty}\left(\frac{\sin^2(as/2)}{s^2}\right) ds$$

Example 5: (Contd.)

$$\Rightarrow \int_0^\infty \left(\frac{\sin^2(as/2)}{s^2} \right) ds = \frac{\pi a}{4}$$

Putting $\dfrac{as}{2} = t$. Therefore $a\,ds = 2\,dt$

$$\int_0^\infty \left(\frac{\sin^2 t}{(2t/a)^2} \right) . \frac{2\,dt}{a} = \frac{\pi a^2}{4} \Rightarrow \int_0^\infty \left(\frac{\sin t}{t} \right)^2 dt = \frac{\pi}{2}$$

Hence $\displaystyle\int_0^\infty \left(\frac{\sin x}{x} \right)^2 dx = \frac{\pi}{2}.$

Example 5: (Contd.)

To prove that (ii) $\int_0^{\infty} \left(\frac{\sin x}{x}\right)^4 dx = \frac{\pi}{3}$

Using Parseval's identity $\int_{-\infty}^{\infty} |F(s)|^2\, ds = \int_{-\infty}^{\infty} |f(x)|^2\, dx$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{2}{\pi} \left(\frac{2\sin^2(as/2)}{s^2}\right)^2 ds = \int_{-a}^{a} (a-|x|)^2\, dx$$

$$\Rightarrow \frac{8}{\pi}.2 \int_0^{\infty} \left(\frac{\sin^2(as/2)}{s^2}\right)^2 ds = 2\int_0^{a} (a-x)^2\, dx$$

$$\Rightarrow \frac{2}{\pi} \int_0^{\infty} \frac{\sin^4(as/2)}{s^4}\, ds = \int_0^{a} (a^2-2ax+x^2)\, dx$$

Example 5: (Contd.)

$$\Rightarrow \frac{8}{\pi}\int_0^\infty \frac{\sin^4(as/2)}{s^4}\,ds = \left(a^2x - 2a\frac{x^2}{2} + \frac{x^3}{3}\right)_0^a$$

$$\Rightarrow \frac{8}{\pi}\int_0^\infty \frac{\sin^4(as/2)}{s^4}\,ds = \frac{a^3}{3}$$

$$\Rightarrow \int_0^\infty \frac{\sin^4(as/2)}{s^4}\,ds = \frac{\pi a^3}{24}$$

Put $\dfrac{as}{2} = t$. Therefore $a\,ds = 2\,dt$

Hence $\displaystyle\int_0^\infty \frac{\sin^4 t}{(2t/a)^4}\cdot\frac{2\,dt}{a} = \frac{\pi a^3}{24} \Rightarrow \int_0^\infty \frac{\sin^4 t}{t^4}\,dt = \frac{\pi}{3}.$

Example 6:

Show that the transformation of $e^{-x^2/2}$ is $e^{-s^2/2}$ by finding the transform of $e^{-a^2x^2}, a > 0$.

Sol:

By the Fourier transform $F(s) = F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{isx}\, dx$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a^2x^2 - isx)}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[a^2x^2 - isx + \frac{i^2s^2}{4a^2} - \frac{i^2s^2}{4a^2}\right]}\, dx$$

Example 6: (Contd.)

$$
\begin{aligned}
F(s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\left(ax-\frac{is}{2a}\right)^2-\frac{i^2s^2}{4a^2}\right]} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax-\frac{is}{2a}\right)^2} . e^{\frac{i^2s^2}{4a^2}} dx \\
&= \frac{e^{\frac{-s^2}{4a^2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax-\frac{is}{2a}\right)^2} dx
\end{aligned}
$$

Putting $t = ax - \dfrac{is}{2a} \Rightarrow dt = a\,dx$

when $x = \infty \Rightarrow t = \infty$ and when $x = -\infty \Rightarrow t = -\infty$

$$\therefore F\{f(x)\} = \frac{e^{-s^2/4a^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} \frac{dt}{a}$$

Example 6: (Contd.)

$\Rightarrow F\{f(x)\} = \dfrac{e^{-s^2/4a^2}}{a\sqrt{2\pi}} .2 \displaystyle\int_0^\infty e^{-t^2}\,dt$; Since the integral is an even function.

Putting $t^2 = u \Rightarrow 2t\,dt = du \Rightarrow dt = \dfrac{du}{2\sqrt{u}}$

$\therefore F\{f(x)\} = \dfrac{2e^{-s^2/4a^2}}{a\sqrt{2\pi}} \displaystyle\int_0^\infty e^{-u}\frac{du}{2\sqrt{u}}$

We know that (Gamma definition) $\Gamma n = \int_0^\infty e^{-x}x^{n-1}\,dx$ and $\Gamma(1/2) = \sqrt{\pi}$

$\Rightarrow F\{f(x)\} = \dfrac{e^{-s^2/4a^2}}{a\sqrt{2\pi}} \int_0^\infty e^{-u}u^{1/2-1}\,du$

Example 6: (Contd.)

$$F\left\{e^{-a^2x^2}\right\} = \frac{e^{-s^2/4a^2}}{a\sqrt{2\pi}}\Gamma(1/2)$$
$$= \frac{e^{-s^2/4a^2}}{a\sqrt{2\pi}}\sqrt{\pi}$$
$$= \frac{e^{-s^2/4a^2}}{a\sqrt{2}}$$

Substituting $a = \frac{1}{\sqrt{2}}$ in above, we get

$$F\left\{e^{-x^2/2}\right\} = e^{-s^2/2}.$$

Example 7:

Find the Fourier transform of $f(x) = \begin{cases} 1 - |x| & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}$ and hence find the value of $\int\limits_0^\infty \frac{\sin^4 t}{t^4} dt$.

Sol: By the Fourier transform $F(s) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(x) e^{isx} \, dx$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int\limits_{-1}^{1} (1 - |x|)(\cos sx + i \sin sx) \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int\limits_{-1}^{1} (1 - |x|) \cos sx dx + i \int\limits_{-1}^{1} (1 - |x|) \sin sx \, dx \right\}$$

Example 7: (Contd.)

$$F(s) = \frac{1}{\sqrt{2\pi}} \left\{ 2 \int_0^1 (1 - |x|) \cos sx \, dx + 0 \right\}$$

Since the first integral is an even function and the second integral is an odd function

$$= \frac{2}{\sqrt{2\pi}} \int_0^1 (1 - x) \cos sx \, dx$$

$$= \frac{2}{\sqrt{2\pi}} \left[(1 - x) \left(\frac{\sin sx}{s} \right) - (-1) \left(-\frac{\cos sx}{s^2} \right) \right]_0^1$$

$$= \frac{2}{\sqrt{2\pi}} \left[\frac{-\cos s}{s^2} + \frac{1}{s^2} \right]$$

Example 7: (Contd.)

$$F(s) = \sqrt{\frac{2}{\pi}}\left(\frac{1-\cos s}{s^2}\right)$$
$$= \sqrt{\frac{2}{\pi}}\cdot\left(\frac{2\sin^2(s/2)}{s^2}\right)$$

To find $\int\limits_0^\infty \frac{\sin^4 t}{t^4}\,dt$

Using Parseval's identity $\int\limits_{-\infty}^{\infty} |F(s)|^2\,ds = \int\limits_{-\infty}^{\infty} |f(x)|^2\,dx$

$$\Rightarrow \int\limits_{-\infty}^{\infty} \frac{2}{\pi}\left(\frac{2\sin^2(s/2)}{s^2}\right)^2 ds = \int\limits_{-1}^{1}(1-|x|)^2\,dx$$

Example 7: (Contd.)

$$\Rightarrow \frac{8}{\pi}.2\int_0^\infty \left(\frac{\sin^2(s/2)}{s^2}\right)^2 ds = 2\int_0^1 (1-x)^2\,dx$$

$$\Rightarrow \frac{8}{\pi}\int_0^\infty \frac{(\sin^2(s/2))^2}{s^4}\,ds = \int_0^1 (1-2x+x^2)\,dx$$

$$\Rightarrow \frac{8}{\pi}\int_0^\infty \frac{\sin^4(s/2)}{s^4}\,ds = \left(x-x^2+\frac{x^3}{3}\right)_0^1$$

$$\Rightarrow \frac{8}{\pi}\int_0^\infty \frac{\sin^4(s/2)}{s^4}\,ds = \frac{1}{3}$$

$$\Rightarrow \int_0^\infty \frac{\sin^4(s/2)}{s^4}\,ds = \frac{\pi}{24}.$$

Example 7: (Contd.)

Put $\dfrac{s}{2} = t \Rightarrow s = 2t$. Therefore $ds = 2\,dt$

Hence $\displaystyle\int_0^\infty \frac{\sin^4 t}{(2t)^4}.2\,dt = \frac{\pi}{24}$

$\Rightarrow \displaystyle\int_0^\infty \frac{\sin^4 t}{t^4}\,dt = \frac{\pi}{3}.$