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Raabe's Test

② Test the following series for Convergence

$$\Sigma \frac{1}{\sqrt{n+1}-1}$$

Sol

$$u_n = \frac{1}{\sqrt{n+1}-1}$$

$$u_{n+1} = \frac{1}{\sqrt{n+2}-1}$$

By Ratio test,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left(\frac{1}{\sqrt{n+2}-1}\left(\sqrt{n+1}-1\right)\right)$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2}\left(\sqrt{\left(1+\frac{1}{n}\right)}-\frac{1}{n^2}\right)}{\cancel{n^2}\left(\sqrt{\left(1+\frac{2}{n}\right)}-\frac{1}{n^2}\right)}$$

$$= \frac{\sqrt{1}}{\sqrt{1}} = 1$$

$k=1$, Ratio test fails

Apply Raabe's test,

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left(\frac{\sqrt{n+2}-1}{\sqrt{n+1}-1}-1\right)$$

$$= \lim_{n\to\infty} n\left(\frac{\left(\sqrt{n+2}\cancel{-1}\right)-\left(\sqrt{n+1}\cancel{-1}\right)}{\sqrt{n+1}-1}\right)$$

$$= \lim_{n\to\infty} n\left(\frac{\sqrt{1+\frac{2}{n}}-\sqrt{1+\frac{1}{n}}}{\sqrt{1+\frac{1}{n}}-\frac{1}{n^2}}\right)$$

$$= 0 < 1$$

The series is divergent

③ Test the following series for Convergence

$$\frac{x}{2}+x^2+\frac{9}{8}x^3+x^4+\frac{25}{32}x^5+\cdots$$

$$\frac{1^2}{2} \quad \frac{2^2}{2^2} \quad \frac{3^2}{2^3} \quad \frac{4^2}{2^4} \quad \frac{5^2}{2^5}$$

Sol

$$u_n = \frac{n^2 x^n}{2^n}$$

$$u_{n+1} = \frac{(n+1)^2 x^{n+1}}{2^{n+1}}$$

Apply Ratio test

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left(\frac{(n+1)^2 x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{n^2 x^n}\right)$$

$$= \lim_{n\to\infty} \left(\frac{\cancel{n^2}\left(1+\frac{1}{n}\right)^2 \cancel{x^n}\cdot x \;\cancel{2^n}}{\cancel{2^n}\cdot 2 \;\cancel{n^2}\cancel{x^n}}\right)$$

$$= \frac{x}{2}$$

a) $\frac{x}{2} < 1$ (or) $x < 2$, Σu_n is Convergent

b) $\frac{x}{2} > 1$ (or) $x > 2$, Σu_n is divergent

c) $\frac{x}{2} = 1$ (or) $x = 2$, ratio test fails

Apply Raabe's test,

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left(\frac{\cancel{2}n^2}{\cancel{2}(n+1)^2}-1\right)$$

$$= \lim_{n\to\infty} n\left(\frac{\cancel{n^2}-\cancel{n^2}-1-2n}{(n+1)^2}\right)$$

$$= \lim_{n\to\infty} \frac{\cancel{n}\left(-2-\frac{1}{n}\right)}{\cancel{n^2}\left(1+\frac{1}{n}\right)^2}$$

$$= -2 < 1$$

$\therefore$ The series Σu_n is divergent, if $x=2$