PROBABILITY
&
QUEUEING THEORY

(As per SRM UNIVERSITY Syllabus)

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SMALL SAMPLES $(n < 30)$

- **F – test**
- $\chi^2 -$ ***test (Chi Square – test)***

F – Test

$$F = \frac{S_X^2}{S_Y^2}, \text{ if } S_X^2 > S_Y^2 \quad \text{(or)} \quad F = \frac{S_Y^2}{S_X^2}, \text{ if } S_X^2 < S_Y^2,$$

$$S_X^2 = \frac{\sum(x-\bar{x})^2}{n_1-1},\ S_Y^2 = \frac{\sum(y-\bar{y})^2}{n_2-1},\ S_1^2 = \frac{n_1 s_1^2}{n_1-1},\ S_2^2 = \frac{n_2 s_2^2}{n_2-1},$$

degree of freedom : $(\nu_1 = n_1 - 1,\ \nu_2 = n_2 - 1)$.

Applications: (i) To test whether two independent samples have been drawn from the normal populations with the same variance σ^2 (ii) To test whether two independent estimates of the population variance are homogeneous or not.

PROBLEMS

1. ***If one sample of 8 observations the sum of the squares of deviations of the sample values from the sample mean was 84.4 and in the other sample of 10 observations it was 102.6. Test whether this difference is significant at 5% level.***

Solution: $n_1 = 8,\ n_2 = 10,\ \sum(x-\bar{x})^2 = 84.4,\ \sum(y-\bar{y})^2 = 102.6$

Null Hypothesis : $H_0: \sigma_X^2 = \sigma_Y^2$, Alternate Hypothesis : $H_1: \sigma_X^2 \neq \sigma_Y^2$

$S_X^2 = \frac{\sum(x-\bar{x})^2}{n_1-1} = \frac{84.4}{8-1} = 12.057,\ S_Y^2 = \frac{\sum(y-\bar{y})^2}{n_2-1} = \frac{102.6}{10-1} = 11.4,\quad F = \frac{S_X^2}{S_Y^2} = \frac{12.057}{11.4} = 1.057$,

d.f. = $(n_1 - 1,\ n_2 - 1) = (7, 9)$ at 5% LOS = 3.29. Calculate value F < Tabulated F. $\mathbf{H_0}$ **is accepted**.

2. ***Two random samples of 11 and 9 items show the sample standard deviations of their weights as 0.8 and 0.5 respectively. Assuming that the weight distributions are normal, test the hypothesis that the true variances are equal, against the alternative hypothesis that they are not.***

Solution: $n_1 = 11,\ n_2 = 9,\ s_1^2 = 0.8,\ s_2^2 = 0.5$, Null Hypothesis: $H_0: \sigma_X^2 = \sigma_Y^2$, Alternate Hypothesis : $H_1: \sigma_X^2 \neq \sigma_Y^2$

$S_1^2 = \frac{n_1 s_1^2}{n_1-1} = \frac{11(0.8)^2}{11-1} = 0.704,\ S_2^2 = \frac{n_2 s_2^2}{n_2-1} = \frac{9(0.5)^2}{9-1} = 0.28125,\ F = \frac{S_1^2}{S_2^2} = \frac{0.704}{0.28125} = 2.503$,

d.f. = $(n_1 - 1,\ n_2 - 1) = (10,\ 8)$ at 5% LOS = 3.34. Calculate value F < Tabulated F. $\mathbf{H_0}$ **is accepted**.

3. ***The time taken by workers in performing a job by Method I and Method II is given below***

Method I	***20***	***16***	***26***	***27***	***23***	***22***	***-***
Method II	***27***	***33***	***42***	***35***	***32***	***34***	***38***

Do the data show that the variances of time distribution form population from which these samples are drawn do not differ significantly?

Solution: $n_1 = 6,\ n_2 = 7,\ \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n_1} = \frac{134}{6} = 22.3,\ \bar{y} = \frac{\sum_{j=1}^{n} y_j}{n_2} = \frac{241}{7} = 34.4$

x	$x-\bar{x}$	$(x-\bar{x})^2$	y	$y-\bar{y}$	$(y-\bar{y})^2$
20	-2.3	5.29	27	-7.4	54.76
16	-6.3	39.69	33	-1.4	1.96
26	3.7	13.69	42	7.6	57.76
27	4.7	22.09	35	0.6	0.36
23	0.7	0.49	32	-2.4	5.76
22	-0.3	0.09	34	-0.4	0.16
$\sum x = 134$		$\sum(x-\bar{x})^2 = 81.34$	38	3.6	12.96
			$\sum y = 241$		$\sum(y-\bar{y})^2 = 132.72$

Null Hypothesis : $H_0: \sigma_X^2 = \sigma_Y^2$, Alternate Hypothesis : $H_1: \sigma_X^2 \neq \sigma_Y^2$

$S_X^2 = \frac{\sum(x-\bar{x})^2}{n_1-1} = \frac{81.34}{6-1} = 16.268,\ S_Y^2 = \frac{\sum(y-\bar{y})^2}{n_2-1} = \frac{133.72}{7-1} = 22.29,\ F = \frac{S_Y^2}{S_X^2} = \frac{22.29}{16.268} = 1.37$,

d.f. = $(n_2 - 1,\ n_1 - 1) = (6,\ 5)$ at 5% LOS = 4.95. Calculate value F < Tabulated F. $\mathbf{H_0}$ **is accepted**.

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4. ***Two horses A and B were tested according to the time (in seconds) to run a particular track with the following***

Horse A	***28***	***30***	***32***	***33***	***33***	***29***	***34***
Horse B	***29***	***30***	***30***	***24***	***27***	***29***	***-***

Test whether the two horses have the same running capacity.

Solution: $n_1 = 7,\ n_2 = 6,\ \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n_1} = \frac{219}{7} = 31.28,\ \bar{y} = \frac{\sum_{j=1}^{n} y_j}{n_2} = \frac{169}{6} = 28.2$

$\boldsymbol{x}$	$\boldsymbol{x-\bar{x}}$	$(\boldsymbol{x-\bar{x}})^2$	$\boldsymbol{y}$	$\boldsymbol{y-\bar{y}}$	$(\boldsymbol{y-\bar{y}})^2$
28	-3.28	10.75	29	0.8	0.64
30	-1.28	1.64	30	1.8	3.24
32	0.72	0.52	30	1.8	3.24
33	1.72	2.96	24	-4.2	17.64
33	1.72	2.96	27	-1.2	1.44
29	-2.28	5.20	29	0.8	0.64
34	2.72	7.40	-	-	-
$\boldsymbol{\sum x = 219}$		$\boldsymbol{\sum(x-\bar{x})^2 = 21.2}$	$\boldsymbol{\sum y = 169}$		$\boldsymbol{\sum(y-\bar{y})^2 = 108}$

Null Hypothesis : $H_0: \sigma_X^2 = \sigma_Y^2$, Alternate Hypothesis : $H_1: \sigma_X^2 \neq \sigma_Y^2$

$S_X^2 = \frac{\sum(x-\bar{x})^2}{n_1-1} = \frac{31.43}{7-1} = 5.238,\ S_Y^2 = \frac{\sum(y-\bar{y})^2}{n_2-1} = \frac{26.84}{6-1} = 5.368\ ,\ F = \frac{S_Y^2}{S_X^2} = \frac{5.368}{5.238} = 1.02\ ,$

degree of freedom $= (n_2 - 1,\ n_1 - 1) = (5,\ 6)$ at 5% LOS = 4.39.

Calculate value F < Tabulated F. $\mathbf{H_0}$ **is accepted**. ***Conclusion:*** The two horses have the same running capacity.

χ^2 – *test*

The χ^2distribution function is one of the most extensively used distribution functions in statistics.

Application (or uses) of χ^2***distribution***

1. To test the goodness of fit. 2. To test the independence of attributes.
3. To test if the hypothetical value of the population variance is σ^2.
4. To test the homogeneity of independent estimates of the population variance.
5. To test the homogeneity of independent estimates of the population correlation coefficient.

Condition for validity of χ^2 ***– test***

1. The sample observations should be independent.
2. Constraints on the cell frequencies, if any, should be linear, e.g., $\sum O_i = \sum E_i$
3. N, the total frequency should be reasonably large, say, greater than 50.
4. No theoretical cell frequency should be less than 5. (The chi square distribution is frequency is less than t) . If any theoretical cell frequency is less than 5, then for the application of χ^2 test, it is pooled with the preceding or succeeding frequency so that the pooled frequency is more than 5 and finally adjust for the degree of freedom lost in pooling.
5.

χ^2 – *test of Goodness of fit*

This is a powerful test for testing the significance of the discrepancy between theory and experiment – discovered by Karl Pearson in 1900. It helps us to find if the deviation of the experiment from theory is just by chance or it is due to the in adequacy of the theory to fit the observed data.

$\boldsymbol{\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}}$, where O_i – set of observed frequencies, E_i- Set of expected frequencies. d.f. $= n - 1$

1. ***The following table gives the number of aircraft accidents that occurred during the various days of the week. Test whether the accidents are uniformly distributed over the week.***

Days	***Mon***	***Tue***	***Wed***	***Thu***	***Fri***	***Sat***
No. of Accidents	***14***	***18***	***12***	***11***	***15***	***14***

Solution: $\mathrm{H_0}$: Accidents occur uniformly over the week. The expected no. of accidents on any day $= \frac{83}{6} = 14$.

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O	E	$O-E$	$(O-E)^2$	$\frac{(O-E)^2}{E}$
14	14	0	0	0
18	14	4	16	1.143
12	14	-2	4	0.286
11	14	-3	9	0.643
15	14	1	1	0.071
14	14	0	0	0
				2.143

$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i} = \mathbf{2.143}$, d.f. $= n - 1 = 5$ at 5% LOS = 11.07, Calculated $\chi^2 <$ tabulated χ^2,

$\boldsymbol{H_0}$ **is accepted.** The accidents are uniformly distributed over the week.

2. ***The theory predicts the proportion of beans, in the four groups A, B, C, and D should be 9 : 3 : 3 : 1. In an experiment with 1600 beans the numbers in the four groups were 882, 313, 287, and 118. Does the experimental result support the theory.***

Solution: H_0 : The experimental result support the theory.

O	E	$O-E$	$(O-E)^2$	$\frac{(O-E)^2}{E}$
882	$\frac{9}{16} \times 1600 = 900$	-18	324	0.36
313	$\frac{3}{16} \times 1600 = 300$	13	169	0.563
287	$\frac{3}{16} \times 1600 = 300$	-13	169	0.563
118	$\frac{1}{16} \times 1600 = 100$	18	324	3.24
				4.726

$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i} = \mathbf{4.726}$, d.f. $= n - 1 = 4 - 1 = 3$ at 5% LOS = 7.81

Calculated $\chi^2 <$ tabulated χ^2, $\boldsymbol{H_0}$ **is accepted.** The experimental results support the theory.

3. ***Fit a binomial distribution for the following data and also test the goodness of fit. Find the parameters of the distribution.***

x	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**Total**
f	**5**	**18**	**28**	**12**	**7**	**6**	**4**	**80**

Solution: To find the binomial frequency distribution $N(q+p)^n$, which fits the given data, we require $N, n \text{ and } p$.
We assume $N =$ total frequency $= 80$ and $n =$ no. of trials $= 6$ from the given data.

x	0	1	2	3	4	5	6	Total
f	5	18	28	12	7	6	4	80
fx	0	18	56	36	28	30	24	192

$\bar{x} = \frac{\sum fx}{\sum f} = \frac{192}{80} = 2.4$, $np = 2.4 \Rightarrow 6p = 2.4$, $p = 0.4$, $q = 0.6$, N $P(X = x) = N\, nC_x p^x q^{n-x}$, $x = 0,1, \dots n$

$80\, P(X = 0) = 80 \times 6C_0\,(0.4)^0(0.6)^{6-0} = 3.73$, $80\, P(X = 1) = 80 \times 6C_1\,(0.4)^1(0.6)^{6-1} = 14.93$

$80\, P(X = 2) = 80 \times 6C_2\,(0.4)^2(0.6)^{6-2} = 24.88$, $80\, P(X = 3) = 80 \times 6C_3\,(0.4)^3(0.6)^{6-3} = 22.12$

$80\, P(X = 4) = 80 \times 6C_4\,(0.4)^4(0.6)^{6-4} = 11.06$, $80\, P(X = 5) = 80 \times 6C_5\,(0.4)^5(0.6)^{6-5} = 2.95$

$80\, P(X = 6) = 80 \times 6C_6\,(0.4)^6(0.6)^{6-6} = 0.33$

X	0	1	2	3	4	5	6	***Total***
E_i	4	15	25	22	11	3	0	***80***

The 1st class is combined with the second and the last 2 classes are combined with the last but 2nd class in order to make the expected frequency in each class ≥ 5. Thus, after regrouping, we have

O_i	*23*	*28*	*12*	*17*
E_i	19	25	22	14

H_0 : The given distribution is approximately binomial distribution.

O	E	$O-E$	$(O-E)^2$	$\frac{(O-E)^2}{E}$
23	19	4	16	0.8421
28	25	3	9	0.36
12	22	-10	100	4.5455
17	14	3	9	0.6429
				6.39

$\chi^2 = \sum_{i=1}^{n} \frac{(O_i-E_i)^2}{E_i} = \mathbf{6.39}$, d.f. $= n-k = 4-2 = 2$ at 5% LOS = 5.99,

Calculated $\chi^2 >$ tabulated χ^2, $\boldsymbol{H_0}$ **is rejected.** The binomial fit for the given distribution is not satisfactory.

4. ***Fit a Poisson distribution for the following distribution and also test the goodness of fit.***

x	***0***	***1***	***2***	***3***	***4***	***5***	***Total***
f	***142***	***156***	***69***	***27***	***5***	***1***	***400***

Solution : $P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$, $x = 0,1,\dots\infty$; $\lambda = \bar{x} = \frac{\sum fx}{\sum f} = \frac{400}{400} = 1$, $N = 400$

x	0	1	2	3	4	5	Total
f	142	156	69	27	5	1	400
fx	0	156	138	81	20	5	400

Theoretical frequencies are given by $N\,P(X=x) = \frac{N\,e^{-\lambda}\lambda^x}{x!} = \frac{400\,e^{-1}\,1^x}{x!}, x = 0,1,\dots\infty$

$400\,P(X=0) = \frac{400\,e^{-1}\,1^0}{0!} = 147.15$, $400\,P(X=1) = \frac{400\,e^{-1}\,1^1}{1!} = 147.15$

$400\,P(X=2) = \frac{400\,e^{-1}\,1^2}{2!} = 73.58$, $400\,P(X=3) = \frac{400\,e^{-1}\,1^3}{3!} = 24.53$

$400\,P(X=4) = \frac{400\,e^{-1}\,1^4}{4!} = 6.13$, $400\,P(X=5) = \frac{400\,e^{-1}\,1^5}{5!} = 1.23$

O_i	142	156	69	27	5	1
E_i	147	147	74	25	6	1

The last 3 classes are combined into one, so that the expected frequency in that class may be ≥ 10.

O_i	142	156	69	33
E_i	147	147	74	32

O	E	$O-E$	$(O-E)^2$	$\frac{(O-E)^2}{E}$
142	147	-5	25	0.17
156	147	9	81	0.55
69	74	-5	25	0.34
33	32	1	1	0.03
				1.09

$\chi^2 = \sum_{i=1}^{n} \frac{(O_i-E_i)^2}{E_i} = \mathbf{1.09}$, d.f. $= n-k = 4-2 = 2$ at 5% LOS = 5.99

Calculated $\chi^2 >$ tabulated χ^2, $\boldsymbol{H_0}$ **is accepted.** The Poisson fit for the given distribution is satisfactory.

χ^2 – *test of Independence of Attributes*

Literally, an attribute means a quality or characteristic. E.g.: drinking, smoking, blindness, honesty, etc.

a	b	$(a+b)$
c	d	$(c+d)$
$(a+c)$	$(b+d)$	N

$E(a) = \frac{(a+c)(a+b)}{N}$	$E(b) = \frac{(b+d)(a+b)}{N}$	$(a+b)$
$E(c) = \frac{(a+c)(c+d)}{N}$	$E(d) = \frac{(b+d)(c+d)}{N}$	$(c+d)$
$(a+c)$	$(b+d)$	N

Degree of freedom = $(r-1)(c-1)$, where r – number of rows, c – number of colums.

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5. ***On the basis of information given below about the treatment of 200 patients suffering from a disease, state whether the new treatment is comparatively superior to the conventional treatment.***

	Favourable	***Not Favourable***	***Total***
New	***60***	***30***	***90***
Conventional	***40***	***70***	***110***
	100	***100***	***200***

Solution: H_0 : New and conventional treatment are independent.

$E(60) = \frac{90 \times 100}{200} = 45$	$E(30) = \frac{90 \times 100}{200} = 45$	***90***
$E(40) = \frac{110 \times 100}{200} = 55$	$E(70) = \frac{110 \times 100}{200} = 55$	***110***
100	***100***	***200***

$\boldsymbol{O}$	$\boldsymbol{E}$	$\boldsymbol{O-E}$	$\boldsymbol{(O-E)^2}$	$\boldsymbol{\frac{(O-E)^2}{E}}$
60	45	15	225	5
30	45	-15	225	5
40	55	-15	225	4.09
70	55	15	225	4.09
				18.18

$\boldsymbol{\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i} = 18.18}$**,** d.f. = $(r-1)(c-1) = (2-1)(2-1) = 1$ ***at*** 5% LOS = 3.841.

Calculated $\chi^2 >$ tabulated χ^2, $\boldsymbol{H_0}$ **is rejected.** New and conventional treatment are not independent.

6. ***Given the following contingency table for hair colour and eye colour. Find the value of chi square. Is there good association between the two.***

		Hair Colour			***Total***
		Fair	***Brown***	***Black***	
Eye Colour	***Blue***	***15***	***5***	***20***	***40***
	Grey	***20***	***10***	***20***	***50***
	Brown	***25***	***15***	***20***	***60***
	Total	***60***	***30***	***60***	***150***

Solution : H_0 : The two attributes Hair colour and eye colour are independent.

$\boldsymbol{O}$	$\boldsymbol{E}$	$\boldsymbol{O-E}$	$\boldsymbol{(O-E)^2}$	$\boldsymbol{\frac{(O-E)^2}{E}}$
15	16	-1	1	0.0625
5	8	-3	9	1.125
20	16	4	16	1
20	20	0	0	0
10	10	0	0	0
20	20	0	0	0
25	24	1	1	0.042
15	12	3	9	0.75
20	24	-4	16	0.666
				3.6458

$E(15) = \frac{60 \times 40}{150} = 16$	$E(5) = \frac{30 \times 40}{150} = 8$	$E(20) = \frac{60 \times 40}{150} = 16$	***40***
$E(20) = \frac{60 \times 50}{150} = 20$	$E(10) = \frac{30 \times 50}{150} = 10$	$E(20) = \frac{60 \times 50}{150} = 20$	***50***
$E(25) = \frac{60 \times 60}{150} = 24$	$E(15) = \frac{30 \times 60}{150} = 12$	$E(20) = \frac{60 \times 60}{150} = 24$	***60***
60	***30***	***60***	***150***

$\boldsymbol{\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i} = 3.6458}$**,** d.f. = $(r-1)(c-1) = (3-1)(3-1) = 4$ ***at*** 5% LOS = 9.488.

Calculated $\chi^2 <$ tabulated χ^2, $\boldsymbol{H_0}$ **is accepted.** The hair colour and eye colour are independent.

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F test Table

5% and 1% points of F.

v_2 \ v_1	1	2	3	4	5	6	8	12	24	∞
2	18.51	19.00	19.16	19.25	19.30	19.32	19.37	19.41	19.45	19.50
	98.49	99.00	99.17	99.25	99.30	99.33	99.36	99.42	99.46	99.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.84	8.74	8.64	8.53
	34.12	30.82	29.46	28.71	28.24	27.91	27.49	27.05	26.60	26.12
4	7.71	6.94	6.59	6.39	6.26	6.16	6.04	5.91	5.77	5.63
	21.20	18.00	16.69	15.98	15.52	15.21	14.80	14.37	13.93	13.46
5	6.61	5.79	5.41	5.19	5.05	4.95	4.82	4.68	4.53	4.36
	16.26	13.27	12.06	11.39	10.97	10.67	10.27	9.89	9.47	9.02
6	5.99	5.14	4.76	4.53	4.39	4.28	4.15	4.00	3.84	3.67
	13.74	10.92	9.78	9.15	8.75	8.47	8.10	7.72	7.31	6.88
7	5.59	4.74	4.35	4.12	3.97	3.87	3.73	3.57	3.41	3.23
	12.25	9.55	8.45	7.85	7.46	7.19	6.84	6.47	6.07	5.65
8	5.32	4.46	4.07	3.84	3.69	3.58	3.44	3.28	3.12	2.93
	11.26	8.65	7.59	7.01	6.63	6.37	6.03	5.67	5.28	4.86
9	5.12	4.26	3.86	3.63	3.48	3.37	3.23	3.07	2.90	2.71
	10.56	8.02	6.99	6.42	6.06	5.80	5.47	5.11	4.73	4.31
10	4.96	4.10	3.71	3.48	3.33	3.22	3.07	2.91	2.74	2.54
	10.04	7.56	6.55	5.99	5.64	5.39	5.06	4.71	4.33	3.91
12	4.75	3.88	3.49	3.26	3.11	3.00	2.85	2.69	2.50	2.30
	9.33	6.93	5.95	5.41	5.06	4.82	4.50	4.16	3.78	3.36
14	4.60	3.74	3.34	3.11	2.96	2.85	2.70	2.53	2.35	2.13
	8.86	6.51	5.56	5.03	4.69	4.46	4.14	3.80	3.43	3.00
16	4.49	3.63	3.24	3.01	2.85	2.74	2.59	2.42	2.24	2.01
	8.53	6.23	5.29	4.77	4.44	4.20	3.89	3.55	3.18	2.75
18	4.41	3.55	3.16	2.93	2.77	2.66	2.51	2.34	2.15	1.92
	8.28	6.01	5.09	4.58	4.25	4.01	3.71	3.37	3.01	2.57
20	4.35	3.49	3.10	2.87	2.71	2.60	2.45	2.28	2.08	1.84
	8.10	5.85	4.94	4.43	4.10	3.87	3.56	3.23	2.86	2.42
25	4.24	3.38	2.99	2.76	2.60	2.49	2.34	2.16	1.96	1.71
	7.77	5.57	4.68	4.18	3.86	3.63	3.32	2.99	2.62	2.17
30	4.17	3.32	2.92	2.69	2.53	2.42	2.27	2.09	1.89	1.62
	7.56	5.39	4.51	4.02	3.70	3.47	3.17	2.84	2.47	2.01
40	4.08	3.23	2.84	2.61	2.45	2.34	2.18	2.00	1.79	1.51
	7.31	5.18	4.31	3.83	3.51	3.29	2.99	2.66	2.29	1.81
60	4.00	3.15	2.76	2.52	2.37	2.25	2.10	1.92	1.70	1.39
	7.08	4.98	4.13	3.65	3.34	3.12	2.82	2.50	2.12	1.60

Regards!
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Chi-Square Test Table

Values of χ^2 with probability P and df ν

ν \ P	0.99	0.95	0.50	0.30	0.20	0.10	0.05	0.01
1	0.0002	0.004	0.46	1.07	1.64	2.71	3.84	6.64
2	0.020	0.103	1.39	2.41	3.22	4.60	5.99	9.21
3	0.115	0.35	2.37	3.66	4.64	6.25	7.82	11.34
4	0.30	0.71	3.36	4.88	5.99	7.78	9.49	13.28
5	0.55	1.14	4.35	6.06	7.29	9.24	11.07	15.09
6	0.87	1.64	5.35	7.23	8.56	10.64	12.59	16.81
7	1.24	2.17	6.35	8.38	9.80	12.02	14.07	18.48
8	1.65	2.73	7.34	9.52	11.03	13.36	15.51	20.09
9	2.09	3.32	8.34	10.66	12.24	14.68	16.92	21.67
10	2.56	3.94	9.34	11.78	13.44	15.99	18.31	23.21
11	3.05	4.58	10.34	12.90	14.63	17.28	19.68	24.72
12	3.57	5.23	11.34	14.01	15.81	18.55	21.03	26.22
13	4.11	5.89	12.34	15.12	16.98	19.81	22.36	27.69
14	4.66	6.57	13.34	16.22	18.15	21.06	23.68	29.14
15	5.23	7.26	14.34	17.32	19.31	22.31	25.00	30.58
16	5.81	7.96	15.34	18.42	20.46	23.54	26.30	32.00
17	6.41	8.67	16.34	19.51	21.62	24.77	27.59	33.41
18	7.02	9.39	17.34	20.60	22.76	25.99	28.87	34.80
19	7.63	10.12	18.34	21.69	23.90	27.20	30.14	36.19
20	8.26	10.85	19.34	22.78	25.04	28.41	31.41	37.57
21	8.90	11.59	20.34	23.86	26.17	29.62	32.67	38.93
22	9.54	12.34	21.34	24.94	27.30	30.81	33.92	40.29
23	10.20	13.09	22.34	26.02	28.43	32.01	35.17	41.64
24	10.86	13.85	23.34	27.10	29.55	33.20	36.42	42.98
25	11.52	14.61	24.34	28.17	30.68	34.68	37.65	44.31
26	12.20	15.38	25.34	29.25	31.80	35.56	38.88	45.64
27	12.88	16.15	26.34	30.32	32.91	36.74	40.11	46.96
28	13.56	16.93	27.34	31.39	34.03	37.92	41.34	48.28
29	14.26	17.71	28.34	32.46	35.14	39.09	42.56	49.59
30	14.95	18.49	29.34	33.53	36.25	40.26	43.77	50.89

Regards!

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QUEUEING THEORY

Syllabus

- Introduction to Markovian queueing models
- Single Server Model with Infinite system capacity - $(M/M/1) : (\infty/FIFO)$
- Single Server Model with Finite System Capacity – $(M/M/1) : (K/FIFO)$

INTRODUCTION

History **:** A.K.Erlang (1909) – "The Theory of probabilities and telephone conversations".
All of us have experienced the annoyance of having to wait in line.
Example: 1. We wait in line in our cars in traffic jams. 2. We wait in line at supermarket to check out.
Why then is there waiting? There is more demand for service that there is facility for service available.
Why is this so? 1. There may be a shortage of available servers. 2. There may be a space limit to the amount of service that can be provided.
Question : 1. How long must a customer wait? 2. How many people will form in the line?
Answer: Queuing theory attempts to answer these questions through detailed mathematical analysis.
Customer**:** The term **'**Customer**'** is used in a general sense and does not imply necessarily a human customer.
E.g.: 1. An Air plane waiting in line to take off. 2. A Computer program waiting to be run as a time shared basis.

Characteristics of Queuing Process

1. Arrival pattern of Customers 2. Service pattern of Servers 3. Queue discipline 4. System capacity
5. Number of service channels 6. Number of service stages

1. ***Arrival Pattern of Customer :*** (i) Bulk or Batches (ii) Balked (iii) Reneged (iv) Jockey

(i) ***Bulk or Batches*** **:** More than one arrival can be entering the system simultaneously, the input is said to occur in bulk or batches.

(ii) ***Balked*** : If customer decides not to enter the queue upon arrival, he is said to have balked.

(iii) ***Reneged :*** A Customer may enter the queue, but after a time lose patience and decide to leave. In this case he is said to reneged.

(iv) ***Jockey :*** Two or more parallel waiting lines, customers may switch from one to another.

2. ***Service Pattern of Services :*** If the system is empty, the service facility is idle. Service may also be deterministic (or) probabilistic. Service may also be single (or) batch one generally thinks of one customer being served at a time by a given server, but there are many situations where customer may be served simultaneously by the same server. **E.g.:** 1. Computer with parallel processing. 2. People boarding a train.

The service rate may depend on the number of customer waiting for service. A server may work faster if sees that the queue is building up (or) conversely, he may get flustered and became less efficient. The situation in which service depends on the no. of customers waiting is referred to as state dependent service.

3. Queue Discipline :

(i) First Come First Served (FCFS) or First In First Out (FIFO)
(ii) Last Come First Served (LCFS) or Last In First Out (LCFO)
(iii) Random Selection for Services (RSS) (iv) Priority (a) Preemptive (b) Non- Preemptive

(a) Preemptive : The customer with the highest priority is allowed to enter service immediately even if a customer with lower priority is already in service when the higher priority customer enters system.

(b) Non - preemptive : The highest priority customer goes to the head the queue but cannot get into service until the customer presently in service is completely, even through this customer has a lower priority.

4. ***System Capacity :*** (i) Finite (ii) Infinite

(i) Finite: A queue with limited waiting room, so that when the time reaches a certain length, no further customer are allowed to enter until space becomes available by a service completion.

(ii) Infinite: A queue with unlimited waiting room.

5. ***Number of Service Channels*** **:** (i) Single channel system (ii) Multiple channel systems.

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(i) ***Single Channel System***

(ii) Multiple Channel System

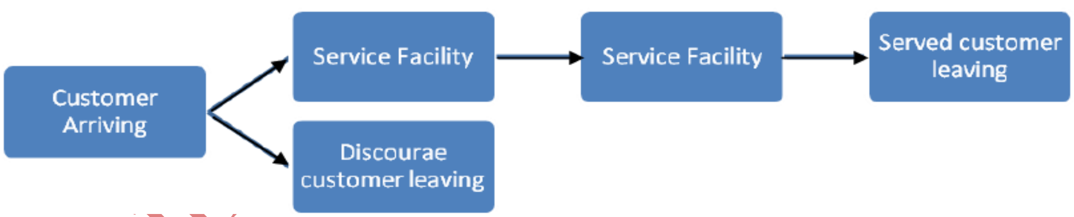
Eg: 1. Barber Shop 2. Supermarket 3. Ticket Counters.

6. **Number of Service Stages :** (i) Single stage (ii) Multiple stage

(i) Single Stage

Eg: 1. Barber Shop 2. Supermarket 3. Theater.

(ii) Multiple Stage



Eg: 1. Medical History 2. Bank A/c opening 3.Canteen

Kendall Notation (A/B/X/Y/Z)

A - Inter arrival time : M - Exponential, D - Deterministic, Ek - Erlang type k,
Hk – Hyper exponential type k, Ph - phase type, G – General.

B - Service time : M - Exponential, D - Deterministic, Ek - Erlang type k,
Hk - Hyper exponential type k, Ph – phase type, G – General.

X - No. of parallel servers : 1, 2 …∞

Y - System capacity : 1, 2 …∞

Z - Queue discipline : FCFS, LCFS, RSS, PR, GD

Queuing Models : 1. Probabilistic or stochastic models 2. Deterministic models 3. Mixed models

Probabilistic model: When there is uncertainty in both arrival rate and service rate (i.e. not treated a customer or not know) and are assumed to be random variables.

Deterministic model: Both arrival rate and service rate are constants (exactly known).

Mixed model: When either the arrival rate or the service rate is exactly known and the other is not known.

We look for probabilistic model only

1. $(M/M/1) : (\infty/FIFO)$

2. $(M/M/1) : (K/FIFO)$

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SINGLE SERVER MODEL WITH INFINITE SYSTEM CAPACITY - $(M/M/1):(\infty/FIFO)$

1. $P_0 = 1 - \frac{\lambda}{\mu}$ and $P_n = \left(\frac{\lambda}{\mu}\right)^n P_0$, λ = arrival rate, μ = service rate
2. Probability that the system is busy $= 1 - P_0 = \frac{\lambda}{\mu}$
3. Expected number of customers in the system : $L_s = \frac{\lambda}{\mu-\lambda}$
4. Expected number of customers in the queue: $L_q = \frac{\lambda^2}{\mu(\mu-\lambda)}$
5. Expected number of customers in non empty queues: $L_n = \frac{\mu}{\mu-\lambda}$
6. Expected waiting time a customer in the system: $W_s = \frac{1}{\mu-\lambda}$
7. Average waiting time that a customer in the queue: $W_q = \frac{\lambda}{\mu(\mu-\lambda)}$
8. Probability that the number of customers in the system exceeds k : $P(n > k) = \left(\frac{\lambda}{\mu}\right)^{k+1}$
9. Probability that the number of customers in the system greater than or equal to k : $P(n \geq k) = \left(\frac{\lambda}{\mu}\right)^{k}$
10. Probability that the waiting time of a customer in the system exceeds t : $P(w > t) = e^{-(\mu-\lambda)t}$
11. Probability that the waiting time of a customer in the queue exceeds t : $P(w > t) = \frac{\lambda}{\mu} e^{-(\mu-\lambda)t}$
12. Probability density function of the waiting time in the system: $f(w) = (\mu - \lambda)e^{-(\mu-\lambda)\,w}$
    Which is the probability density function of an exponential distribution with parameter $\mu - \lambda$.
13. Probability density function of the waiting time in the queue: $g(w) = \begin{cases} \frac{\lambda}{\mu}(\mu-\lambda)e^{-(\mu-\lambda)\,w}, & w > 0 \\ 1 - \frac{\lambda}{\mu}, & w = 0 \end{cases}$

Little's formula: $L_s = \lambda W_s$, $L_q = \lambda W_q$, $W_s = W_q + \frac{1}{\mu}$, $L_s = L_q + \frac{\lambda}{\mu}$

SINGLE SERVER MODEL WITH INFINITE SYSTEM CAPACITY - $(M/M/1):(\infty/FIFO)$

1. Customers arrive at a one man barber shop according to a Poisson process with a mean inter arrival time of 20 minutes. Customers spend an average of 15 minutes in the barber chair. If an hour is used as the unit of time,

(i) What is the probability that a customer need not wait for a hair cut?
(ii) What is the expected number of customers in the barber shop and in the queue?
(iii) How much time can a customer expect to spend in the barbershop?
(iv) Find the average time that a customer spends in the queue?
(v) What is the probability that there will be more than 6 customers?
(vi) What is the probability that there will be 6 or more customers waiting for service?
(vii) What is the probability that the waiting time in the (a) system (b) queue, is greater than 12 minutes?

Solution: $\frac{1}{\lambda} = \frac{20}{30} = \frac{1}{3} \Longrightarrow \boldsymbol{\lambda = 3}$ customers/hour, $\frac{1}{\mu} = \frac{15}{60} = \frac{1}{4} \Longrightarrow \boldsymbol{\mu = 4}$ customers/hour.

(i) $P(a\ customer\ need\ not\ wait) = P(no\ customer\ in\ the\ system)$: $P_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{3}{4} = 0.25$

(ii) Expected number of customers in the barber shop : $L_s = \frac{\lambda}{\mu-\lambda} = \frac{3}{4-3} = 3$ customers

Expected number of customers in the queue: $L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{3^2}{4(4-3)} = \frac{9}{4} = 2.25$ customers

(iii) Expected time a customer spend in the barbershop: $W_s = \frac{1}{\mu-\lambda} = \frac{1}{4-3} = 1$ hour

(iv) Average time that a customer spends in the queue: $W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{3}{4(4-3)} = \frac{3}{4} = 0.75$ hour

(v) The probability that there will be more than 6 customers:

$$P(n > k) = \left(\frac{\lambda}{\mu}\right)^{k+1} \Rightarrow P(n > 6) = \left(\frac{3}{4}\right)^{6+1} = 0.1335$$

(vi) The probability that there will be 6 or more customers waiting for service:

$$P(n \geq k) = \left(\frac{\lambda}{\mu}\right)^{k} \Rightarrow P(n \geq 6) = \left(\frac{3}{4}\right)^{6} = 0.1779$$

(vii) The probability that the waiting time in the system is greater than 12 minutes?

$$P(w > t) = e^{-(\mu-\lambda)t} \Rightarrow P(w > 12) = e^{-(4-3)\times\frac{12}{60}} = e^{-0.2} = 0.8187$$

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(viii) The probability that the waiting time in the queue, is greater than 12 minutes?

$$P(w > t) = \frac{\lambda}{\mu}\, e^{-(\mu-\lambda)t} \Rightarrow P(w > 12) = \left(\frac{3}{4}\right) e^{-(4-3)\times\frac{12}{60}} = \left(\frac{3}{4}\right) e^{-0.2} = 0.61405$$

2. ***If People arrive to purchase cinema tickets at the average rate of 6 per minute at a one man counter, and it takes an average of 7.5 seconds to purchase a ticket. If a person arrives 2 minutes before the picture starts and it takes exactly 1.5 minutes to reach the correct seat after purchasing the ticket,***

(i) can he expect to be seated for the start of the picture?

(ii) What is the probability that he will be seated for the start of the picture?

(iii) How early must he arrive in order to be 99% sure of being seated for the start of the picture?

Solution: $\lambda = 6$ /minute, $\mu = 8$ /minute

(i) $W_s = \frac{1}{\mu-\lambda} = \frac{1}{8-6} = \frac{1}{2} = 0.5$ minute

E(total time required to purchase the ticket and to reach the seat) $= 0.5 + 1.5 = 2$ min

(ii) $P(\text{total time } < 2 \text{ minute}) = P(w < t) = 1 - P(w > t) = 1 - e^{-(\mu-\lambda)t}$

$$P\left(w < \frac{1}{2}\right) = 1 - e^{-(8-6)\times\frac{1}{2}} = 1 - e^{-1} = 0.63$$

(iii) $P(w < t) = 99\% = 0.99 \Rightarrow 1 - P(w > t) = 0.99 \Rightarrow P(w > t) = 0.01 \Rightarrow e^{-(\mu-\lambda)t} = 0.01 \Rightarrow e^{-(8-6)\,t} = 0.01 \Rightarrow e^{-2\,t} = 0.01 \Rightarrow -2t = \ln(0.01) \Rightarrow -2t = -4.6 \Rightarrow t = 2.3$ minute

$P(\text{ticket purchasing time} < 2.3) = 0.99$

$P[\text{total time to get the ticket and to go to the seat} < (2.3 + 1.5)] = 0.99$

∴ The person must arrive at least 2.64 minutes early so as to be 99% sure of seeing the start of the picture.

3. ***The arrivals at the counter in a bank occur in accordance with a Poisson process at an average rate of 8 per hour. The duration of service of a customer has an exponential distribution with a mean of 6 minutes. Find the probability that an arriving customer (i) Has to wait (ii) Finds 4 customers in the system (iii) Has to spend less than 15 minutes in the bank.***

Solution: $\lambda = 8$ / hour, $\mu = \frac{1}{6}$/minute $= 10$/hour

(i) Probability that a customer has to wait = Probability that the system is busy $= \frac{\lambda}{\mu} = \frac{8}{10} = 0.8$

(ii) Probability that there are 4 customers in the system $= P_4 = \left(\frac{8}{10}\right)^4 \left(1 - \frac{8}{10}\right) = 0.08192$

(iii) Probability that a customer has to spend less than 15 minutes in the bank: $P(w > t) = e^{-(\mu-\lambda)t}$

$$P(W_s < 15 \text{ minutes}) = P\left(W_s < \frac{1}{4} \text{ hour}\right) = 1 - P\left(W_s > \frac{1}{4} \text{ hour}\right) = 1 - e^{-(10-8)\left(\frac{1}{4}\right)} = 0.3935$$

4. ***Arrivals at a telephone booth are considered to be Poisson with an average time of 12 minutes between one arrival and the next. The length of the phone call is assumed to be distributed exponentially with mean 4 minutes.***

(i) Find the average number of persons waiting in the system.

(ii) What is the probability that a person arriving at the booth will have to wait in the queue?

(iii) Also estimate the fraction of the day when phone will be in use.

(iv) What is the Probability that it will take him more than 10 minutes for a person to wait and complete his call?

(v) The telephone department will install a second booth when convinced that an arrival would expect to wait aleast 3 minutes for phone. By how much should the flow of arrivals increase in order to justify a second booth?

Solution: Given mean inter arrival time $\frac{1}{\lambda} = 12$ minutes, $\lambda = \frac{1}{12}$ per minute;

Mean service time $\frac{1}{\mu} = 4$ minutes, $\mu = \frac{1}{4}$ per minute

$$W_s = \frac{1}{\mu-\lambda} = \frac{1}{\left(\frac{1}{4}-\frac{1}{12}\right)} = 6$$

(i) The average number of persons waiting in the system is : $L_s = \lambda\, W_s = \frac{1}{12} \times 6 = \frac{1}{2}$ persons.\

(ii) Probability of waiting in the system = P (Channel is busy) $= \frac{\lambda}{\mu} = \frac{1}{3}$

(iii) P(Phone is use) = P(Phone is busy) $= \frac{\lambda}{\mu} = \frac{1}{3}$

(iv) $P(w > t) = e^{-(\mu-\lambda)t} \Rightarrow P(w > 10) = e^{-\left(\frac{1}{4}-\frac{1}{12}\right)10} = e^{-\frac{5}{3}}$

(v) The second phone will be installed if $E(W) > 3 \Rightarrow W_q > 3$

$\frac{\lambda}{\mu(\mu-\lambda)} > 3 \Rightarrow \lambda > 3\mu(\mu-\lambda) \Rightarrow \lambda > 3\left(\frac{1}{4}\right)\left(\frac{1}{4}-\lambda\right) \Rightarrow \lambda > \frac{3}{16} - \frac{3\lambda}{4} \Rightarrow \frac{7\lambda}{4} > \frac{3}{16} \Rightarrow \lambda > \frac{3}{28}$

Hence the increase in arrival rate should be atleast $= \frac{3}{28} - \frac{1}{12} = \frac{1}{42}$ per minute.

5. ***Customers arrive at a watch repair shop according to a Poisson process at a rate of one per every 10 minutes and the service time is an exponential random variable with mean 8 minutes.***
   (i) Find the average number of customers in the shop.
   (ii) Find the average time a customer spends in the shop.
   (iii) Find the average number of customers in the queue.
   (iv) What is the probability that the server is idle?

Solution: $\lambda = 6$ /hour, $\mu = \frac{60}{8} = \frac{15}{2}$ /hour.

(i) The average number of customers in the shop : $L_s = \frac{\lambda}{\mu-\lambda} = \frac{6}{\frac{15}{2}-6} = 4$ customers

(ii) The average time a customer spends in the shop : $W_s = \frac{1}{\mu-\lambda} = \frac{1}{\frac{15}{2}-6} = \frac{2}{3}$ hour

(iii) The average number of customers in the queue: $L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{6^2}{\frac{15}{2}\left(\frac{15}{2}-6\right)} = \frac{16}{5}$

(iv) $P(\text{system is empty})$: $P_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{6}{\left(\frac{15}{2}\right)} = \frac{1}{5}$

6. ***A departmental store has a single cashier. During the rush hours, customers arrive at the rate of 20 customers per hour. The average no. of customers that can be processed by the cashier is 24 per hour. Calculate the following***
   (i) What is the probability that the cashier is idle?
   (ii) What is the average number of customers in the queueing system.
   (iii) What is the average time a customer spends in the system.
   (iv) What is the average number of customers in the queue?
   (v) What is the average time a customer spends in the queue, waiting for service?

Solution: $\lambda = 20$ /hour, $\mu = 24$ /hour.

(i) $P(\text{cashier is idle}) = P_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{20}{24} = 0.1674$

(ii) Average number of customers in the queueing system : $L_s = \frac{\lambda}{\mu-\lambda} = \frac{20}{24-20} = 5$ customers

(iii) Average time a customer spend in the system: $W_s = \frac{1}{\mu-\lambda} = \frac{1}{24-20} = \frac{1}{4}$ hour

(iv) Average number of customers in the queueing system : $L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{20^2}{24(24-20)} = 4.167$ customers

(v) Average time that a customer spends in the queue: $W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{20}{24(24-20)} = \frac{20}{96}$ hour

SINGLE SERVER MODEL WITH FINITE SYSTEM CAPACITY - $(M/M/1) : (k/FIFO)$

$$P_0 = \begin{cases} \dfrac{\left(1-\frac{\lambda}{\mu}\right)}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}, & \lambda \neq \mu \\ \dfrac{1}{k+1}, & \lambda = \mu \end{cases} \quad \textit{and} \quad P_n = \begin{cases} \left(\frac{\lambda}{\mu}\right)^n P_0, & \lambda \neq \mu \\ \dfrac{1}{k+1}, & \lambda = \mu \end{cases}$$

Average number of customers in the system : $L_s = \begin{cases} \left(\frac{\lambda}{\mu-\lambda}\right) - \dfrac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}, & \lambda \neq \mu \\ \dfrac{k}{2}, & \lambda = \mu \end{cases}$

Average number of customers in the queue : $L_q = L_s - \frac{\lambda'}{\mu}$

Effective arrival rate : $\lambda' = \mu(1 - P_0)$

Average waiting time of a customers in the system : $W_s = \frac{L_s}{\lambda'}$

Average waiting time of a customers in the queue : $W_q = \frac{L_q}{\lambda'}$

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1. ***Patients arrive at a clinic according to Poisson distribution at a rate of 60 patients per hour. The waiting room does not accommodate more than 14 patients. Investigation time per patient is exponential with mean rate of 40 per hour.***
   (i) Determine the effective arrival rate at the clinic.
   (ii) What is the probability that an arriving patient will not wait?
   (iii) What is the expected waiting time until a patient is discharged from the clinic?

   Solution: $\lambda = 60$ patients/hr, $\mu = 40$ patients/hr, $k = 14 + 1 = 15$ (14 waiting patients + 1 patient under investigation)

   (i) Effective arrival rate $\lambda' = \mu(1 - P_0)$, Where $P_0 = \dfrac{\left(1-\frac{\lambda}{\mu}\right)}{\left[1-\left(\frac{\lambda}{\mu}\right)^{k+1}\right]}$, $\lambda \neq \mu$

   $P_0 = \dfrac{\left(1-\frac{60}{40}\right)}{1-\left(\frac{60}{40}\right)^{15+1}} = 0.0007624$, $\lambda' = 40(1 - 0.0007624) = 39.9695$ per hour.

   (ii) P(a patient will not wait) $= P_0 = 0.0007624$

   (iii) Expected waiting time until a patient is discharged from the clinic: $W_s = \dfrac{L_s}{\lambda'}$

   $L_s = \left(\dfrac{\lambda}{\mu-\lambda}\right) - \dfrac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}} = \left(\dfrac{60}{40-60}\right) - \dfrac{(15+1)\left(\frac{60}{40}\right)^{15+1}}{1-\left(\frac{60}{40}\right)^{15+1}} = 13$ patients, $W_s = \dfrac{14}{39.9695} = 0.3203$ hour

2. ***At a railway station, only one train is handled at a time. The railway yard is sufficient only for two trains to wait while other is given signal to leave the station. Trains arrive at the station at an average rate of 6 per hour and the railway station can handle them on an average of 12 per hour. Assuming Poisson arrivals and exponential service distribution, find the steady state probabilities for the number of trains in the system. Also find the average waiting time of a new train coming into the yard. If the handling rate is reduced to half, what is the effect of the above results?***

   Solution: Case (i): $\lambda = 6$ trains/hr, $\mu = 12$ trains/hr, $k = 2 + 1 = 3$

   Steady state probabilities for the number of trains in the system $= P_1, P_2$ *and* P_3

   $P_n = \left(\dfrac{\lambda}{\mu}\right)^n P_0$, Where $P_0 = \dfrac{\left(1-\frac{\lambda}{\mu}\right)}{\left[1-\left(\frac{\lambda}{\mu}\right)^{k+1}\right]} = \dfrac{\left(1-\frac{6}{12}\right)}{1-\left(\frac{6}{12}\right)^4} = 0.5333$, $\lambda \neq \mu$

   $P_1 = \left(\dfrac{6}{12}\right)^1(0.5333) = 0.2667$, $P_2 = \left(\dfrac{6}{12}\right)^2(0.5333) = 0.1333$, $P_3 = \left(\dfrac{6}{12}\right)^3(0.5333) = 0.0667$

   Average waiting time of a new train coming in the yard : $W_s = \dfrac{L_s}{\lambda'}$

   $\lambda' = \mu(1 - P_0) = 12(1 - 0.5333) = 5.6004$

   $L_s = \left(\dfrac{\lambda}{\mu-\lambda}\right) - \dfrac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}} = \left(\dfrac{6}{12-6}\right) - \dfrac{(3+1)\left(\frac{6}{12}\right)^{3+1}}{1-\left(\frac{6}{12}\right)^{3+1}} = 0.7333$ train, $W_s = \dfrac{0.7333}{5.6004} = 0.1309$ hour

   Case (ii): If the handling rate is reduced to half, then $\lambda = 6$ trains/hr, $\mu = 6$ trains/hr, $k = 2 + 1 = 3$

   Steady state probabilities for the number of trains in the system $= P_1, P_2$ *and* P_3

   $P_n = \dfrac{1}{k+1}$, $P_1 = \dfrac{1}{3+1} = \dfrac{1}{4}$, $P_2 = \dfrac{1}{4}$, $P_3 = \dfrac{1}{4}$

   Average waiting time of a new train coming in the yard : $W_s = \dfrac{L_s}{\lambda'}$

   $P_0 = \dfrac{1}{k+1} = \dfrac{1}{4}$, $\lambda' = \mu(1 - P_0) = 6\left(1 - \dfrac{1}{4}\right) = 4.5$, $L_s = \dfrac{k}{2} = \dfrac{3}{2} = 1.5$ train, $W_s = \dfrac{1.5}{4.5} = 0.3333$ hours

3. ***A petrol pump with only one pump can accommodate 5 cars. The arrival of cars is Poisson with a mean rate of 10 per hour. The service time is exponentially distributed with a mean 2 minutes. How many cars are in the petrol pump on an average? What is the probability of a newly arriving customer finding the system full and leaving without availing service?***

   Solution: $\lambda = 10$/hr, $\mu = 30$/hr, $k = 5$, $P_0 = \dfrac{\left(1-\frac{\lambda}{\mu}\right)}{\left[1-\left(\frac{\lambda}{\mu}\right)^{k+1}\right]} = \dfrac{\left(1-\frac{10}{30}\right)}{\left[1-\left(\frac{10}{30}\right)^{5+1}\right]} = 0.667$, $\lambda \neq \mu$

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$$L_s = \left(\frac{\lambda}{\mu-\lambda}\right) - \frac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}} = \left(\frac{10}{30-10}\right) - \frac{(5+1)\left(\frac{10}{30}\right)^{5+1}}{1-\left(\frac{10}{30}\right)^{5+1}} = 0.492$$

$P(\text{System full}) = P(5 \text{ cars in the system}) = \left(\frac{\boldsymbol{\lambda}}{\boldsymbol{\mu}}\right)^{\mathbf{5}} \boldsymbol{P_0} = \left(\frac{10}{30}\right)^{\mathbf{5}} (0.667) = 0.00274.$

4. ***A one person barber shop has 6 chairs to accommodate people waiting for a hair cut. Assume that customers who arrive when all the 6 chairs are full leave without entering the barber shop. Customers arrive at the average rate of 3 per hour and spend an average of 15 minutes in the barber chair.***
   - ***(i) What is the probability that a customer can get directly into the barber chair upon arrival?***
   - ***(ii) What is the expected number of customers waiting for a hair cut?***
   - ***(iii) How much time can a customer expect to spend in the barber shop?***
   - ***(iv) What fraction of potential customers are turned away?***

Solution: $\lambda = 3/\text{hr}, \mu = 4/\text{hr}, k = 6 + 1 = 7$

$$P_0 = \frac{\left(1-\frac{\lambda}{\mu}\right)}{\left[1-\left(\frac{\lambda}{\mu}\right)^{k+1}\right]} = \frac{\left(1-\frac{3}{4}\right)}{\left[1-\left(\frac{3}{4}\right)^{7+1}\right]} = 0.2778, \quad \lambda \neq \mu \ ,$$

Effective arrival rate $\lambda' = \mu(1 - P_0) = 4(1 - 0.2778) = 2.89/hour$

(i) Probability of empty system = $P_0 = 0.2778$

(ii) $L_s = \left(\frac{\lambda}{\mu-\lambda}\right) - \frac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}} = \left(\frac{3}{4-3}\right) - \frac{(7+1)\left(\frac{3}{4}\right)^{7+1}}{1-\left(\frac{3}{4}\right)^{7+1}} = 2.11, \qquad L_q = L_s - \frac{\lambda'}{\mu} = 2.11 - \frac{2.89}{4} = 1.3875$

(iv) $W_s = \frac{L_s}{\lambda'} = \frac{2.11}{2.89} = 0.7301/hour$

(v) $P(a\ customer\ is\ turned\ away) = P(system\ is\ full) = P_7 = \left(\frac{\lambda}{\mu}\right)^7 P_0 = \left(\frac{3}{4}\right)^7 (0.2778) = 0.037$

Hence 3.7% of potential customers are turned away.

<u>All the Best</u>

Regards!

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UNIT – 5 : MARKOV CHAINS

Syllabus

- Introduction to Stochastic process, Markov process, Markov chain one step & n-step Transition Probability.
- Transition Probability Matrix and Applications
- Chapman Kolmogorov theorem (Statement only) – Applications.
- Classification of states of a Markov chain – Applications

INTRODUCTION

Random Processes or Stochastic Processes : A random process is a collection of random variables $\{X(s, t)\}$ which are functions of a real variable t (time). Here $s \in S$ (sample space) and $t \in T$ (index set) and each $\{X(s, t)\}$ is a real valued function. The set of possible values of any individual member is called state space.

Classification : Random processes can be classified into 4 types depending on the continuous or discrete nature of the state space S and index set T.

1. Discrete random sequence : If both S and T are discrete
2. Discrete random process : If S is discrete and T is continuous
3. Continuous random sequence : If S is continuous and T is discrete
4. Continuous random process : If both S and T are continuous.

Markov Process : If, for $t_1 < t_2 < t_3 < \cdots < t_n$, we have $P\{X(t) \le x/X(t_1) = x_1, X(t_2) = x_2 \ldots X(t_n) = x_n\} = P\{X(t) \le x/X(t_n) = x_n\}$ then the process $\{X(t)\}$ is called a Markov process. That is, if the future behaviour of the process depends only on the present state and not on the past, then the random process is called a Markov process.

Markov Chain : If, for all n, $P\{X_n = a_n/X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2} \ldots X_0 = a_0\} = P\{X_n = a_n/X_{n-1} = a_{n-1}\}$ then the process $\{X_n; n = 0,1,2 \ldots\}$ is called a Markov chain.

One Step Transition Probability : The conditional probability $P_{ij}(n-1, n) = P\left(X_n = a_j/X_0 = a_i\right)$ is called one step transition probability from state a_i to state a_j in the n^{th} step.

Homogeneous Markov Chain : If the one step transition probability does not depend on the step.
That is, $\boldsymbol{P_{ij}(n-1, n) = P_{ij}(m-1, m)}$ the Markov chain is called a homogeneous markov chain or the chain is said to have stationary transition probabilities.

n - Step Transition Probability : $P_{ij}^{(n)} = P\left(X_n = a_j/X_0 = a_i\right)$

Chapman Kolmogorov Equations : If P is the tpm of a homogeneous Markov chain, the n^{th} step tpm $P^{(n)}$ is equal to P^n. That is, $\left[P_{ij}^{(n)}\right] = \left[P_{ij}\right]^n$.

Regular Matrix : A stochastic matrix P is said to be a regular matrix, if all the entries of P^m are positive. A homogeneous Markov chain is said to be regular if its tpm is regular.

Classification of States of a Markov Chain

Irreducible Chain and Non - Irreducible (or) Reducible: If for every i, j we can find some n such that $P_{ij}^{(n)} > 0$, then every state can be reached from every other state, and the Markov chain is said to be irreducible. Otherwise the chain is non – irreducible or reducible.

Return State : State i of a Markov chain is called a return state, if $P_{ij}^{(n)} > 0$ for some $n > 1$.

Periodic State and Aperiodic State : The period d_i of a return state i is the greatest common divisor of all m such that $P_{ij}^{(n)} > 0$. That is, $d_i = GCD\left(m: P_{ii}^{(m)} > 0\right)$. State i is periodic with period d_i if $d_i > 1$ and aperiodic if $d_i = 1$.

Recurrent (Persistent) State and Transient : If $\sum_{n=1}^{\infty} f_{ii}^{(n)} = 1$, the return to state i is certain and the state i is said to be persistent or recurrent. Otherwise, it is said to be transient.

Null Persistent and Non – null Persistent State : $\mu_{ii} = \sum_{n=1}^{\infty} n\, f_{ii}^{(n)}$ is called the mean recurrence time of the state i.
If μ_{ii} is finite, the state i is non null persistent. If $\mu_{ii} = \infty$ the state i is null persistent.

Ergodic State: A non null persistent and aperiodic state are called ergodic.

Theorem used to classify states

1. If a Markov chain is irreducible, all its states are of the same type. They are all transient, all null persistent or all non null persistent. All its states are either aperiodic or periodic with the same period.
2. If a Markov chain is finite irreducible, all its states are non null persistent.

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PROBLEMS

7. *A man either drives a car or catches a train to go to office each day. He never goes 2 days in a row by train but if he drives one day, then the next day he is just as likely to drive again as he is to travel by train. Now suppose that on the first day of the week, the man tossed a fair die and drove to work if and only if a 6 appeared. (i) Find the probability that he takes a train on the 3rd day (ii) Find the probability that he drives to work in the long run.*

Solution : Let T – Train and C - Car. If today he goes by train, next day he will not go by train.

$P(Train \ \to Train) = 0, P(Train \ \to Car) = 1, P(Car \ \to Train) = \frac{1}{2}, P(Car \ \to Car) = \frac{1}{2}$

$P = \begin{array}{c|cc} & T & C \\ T & 0 & 1 \\ C & \frac{1}{2} & \frac{1}{2} \end{array}$. The first day of the week, the man tossed a fair die and drove to work if and only if a 6 appeared.

Initial state probability distribution is obtained by throwing a die.

Probability of going by car = $\frac{1}{6}$ and Probability of going by train = $1 - \frac{1}{6} = \frac{5}{6}$

The 1st day sate distribution is $P^{(1)} = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} \end{bmatrix}$

The 2nd day sate distribution is $P^{(2)} = P^{(1)}P = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{12} & \frac{11}{12} \end{bmatrix}$

The 3rd day sate distribution is $P^{(3)} = P^{(2)}P = \begin{bmatrix} \frac{1}{12} & \frac{11}{12} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{11}{24} & \frac{13}{24} \end{bmatrix}$

(i) *P(*he travels by train on 3rd day*)* = $\frac{11}{24}$

(ii) The limiting form or long run probability distribution. $\boldsymbol{\pi p = \pi}$

$$\begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix}$$

$$\frac{\pi_2}{2} = \pi_1 \Rightarrow \pi_2 = 2\,\pi_1 \quad (1)$$

$$\pi_1 + \frac{\pi_2}{2} = \pi_2 \Rightarrow \pi_2 = 2\,\pi_1 \quad (2)$$

$$\pi_1 + \pi_2 = 1 \quad (3)$$

Sub. (1) in (3), $\pi_1 + 2\,\pi_1 = 1 \Rightarrow \boldsymbol{\pi_1} = \frac{\mathbf{1}}{\mathbf{3}}$ (4)

Sub. (4) in (3), $\frac{1}{3} + \pi_2 = 1 \Rightarrow \boldsymbol{\pi_2} = \frac{\mathbf{2}}{\mathbf{3}}$ **,** P(driving in the long run) = $\frac{2}{3}$.

8. *A college student X has the following study habits. If he studies one night, he is 70% sure not to study the next night. If he does not study one night, he is only 60% sure not to study the next night also. Find (i) the transition probability matrix (ii) how often he studies in the long run.*

Solution : Let S – Studying and N - Not Studying. If he studies one night, next night he is 70% not studying.

$P(Studying \ \to Not\ Studying) = 0.7,\ P(Studying \ \to Studying) = 0.3,$

$P(Not\ Studying \to Not\ Studying) = 0.6,\ P(Not\ Studying \ \to Studying) = 0.4$,

$P = \begin{array}{c|cc} & S & N \\ S & 0.3 & 0.7 \\ N & 0.4 & 0.6 \end{array}$. The limiting form or long run probability distribution. $\boldsymbol{\pi p = \pi}$

$$\begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix} \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix}$$

$$0.3\,\pi_1 + 0.4\,\pi_2 = \pi_1 \Rightarrow 0.4\,\pi_2 = 0.7\,\pi_1 \quad \Rightarrow 4\,\pi_2 = 7\pi_1 \quad (1)$$

$$0.7\,\pi_1 + 0.6\,\pi_2 = \pi_2 \Rightarrow 0.4\,\pi_2 = 0.7\,\pi_1 \quad \Rightarrow 4\,\pi_2 = 7\pi_1 \quad (2)$$

$$\pi_1 + \pi_2 = 1 \quad (3)$$

Sub. (1) in (3), $\frac{4}{7}\pi_2 + \pi_2 = 1 \Rightarrow \boldsymbol{\pi_2} = \frac{\mathbf{7}}{\mathbf{11}}$ (4)

Sub. (4) in (3), $\pi_1 + \frac{\mathbf{7}}{\mathbf{11}} = 1 \Rightarrow \boldsymbol{\pi_1} = \frac{\mathbf{4}}{\mathbf{11}}$ **,** P(he studies in the long run) = $\frac{\mathbf{4}}{\mathbf{11}}$.

9. *Suppose that the probability of a dry day following a rainy day is $\frac{1}{3}$ and that the probability of a rainy day following a dry day is $\frac{1}{2}$. Given that May 1 is a dry day, find the prob. that (i) May 3 is also a dry day (ii) May 5 is also a dry day.*

Solution : Let D – Dry day and R - Rainy day.

$$P = \begin{array}{c} \\ D \\ R \end{array}\!\!\begin{array}{c} \begin{array}{cc} D & R \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \end{array}$$

. Initial state probability distribution is the probability distribution on May 1. Since May 1 is a dry day.

P(D) = 1 and P(R) = 0.

$P^{(1)} = [1\ \ 0]$, $P^{(2)} = P^{(1)}\ P = [1\ \ 0]\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, $P^{(3)} = P^{(2)}\ P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{5}{12} & \frac{7}{12} \end{bmatrix}$,

$\mathbf{P}(\boldsymbol{May\ 3\ is\ a\ dry\ day}) = \boldsymbol{\frac{5}{12}}$

$P^{(4)} = P^{(3)}\ P = \begin{bmatrix} \frac{5}{12} & \frac{7}{12} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{29}{72} & \frac{43}{72} \end{bmatrix}$, $P^{(4)}\ P = \begin{bmatrix} \frac{29}{72} & \frac{43}{72} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{173}{432} & \frac{259}{432} \end{bmatrix}$,

$\boldsymbol{P(May\ 5\ is\ a\ dry\ day) = \frac{173}{432}}$

10.A salesman territory consists of 3 cities A, B and C. He never sells in the same city on successive days. If he sells in city A, then the next day, he sells in city B. However, if he sells in either B or C, the next day he is twice as likely to sell in city A as in the other city. In the long run, how often does he sell in each of the cities?

Solution : The tpm of the given problem is $P = \begin{array}{c} \\ A \\ B \\ C \end{array}\!\!\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix} \end{array}$. The limiting form or long run prob. distribution.

$\boldsymbol{\pi p = \pi}$

$$[\pi_1\ \ \pi_2\ \ \pi_3]\begin{bmatrix} 0 & 1 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix} = [\pi_1\ \ \pi_2\ \ \pi_3]$$

$$\frac{2}{3}\,\pi_2 + \frac{2}{3}\,\pi_3 = \pi_1 \ \Rightarrow\ 2\,\pi_2 + 2\,\pi_3 = 3\pi_1 \Rightarrow 3\pi_1 - 2\,\pi_2 - 2\,\pi_3 = 0 \qquad (1)$$

$$\pi_1 + \frac{1}{3}\,\pi_3 = \pi_2 \ \Rightarrow\ 3\pi_1 + \pi_3 = 3\pi_2 \Rightarrow 3\pi_1 - 3\pi_2 + \pi_3 = 0 \qquad (2)$$

$$\frac{1}{3}\,\pi_2 = \pi_3 \ \Rightarrow\ \pi_2 = 3\pi_3 \qquad (3)$$

$$\pi_1 + \pi_2 + \pi_3 = 1 \qquad (4)$$

Sub. (3) in (1), $3\pi_1 - 2\,\pi_2 - 2\,\pi_3 = 0 \Rightarrow 3\pi_1 - 2\,(3\pi_3) - 2\,\pi_3 = 0 \Rightarrow 3\pi_1 = 8\,\pi_3 \Rightarrow \pi_1 = \frac{8}{3}\ \pi_3$ (5)

Sub. (3) and (5) in (4), $\pi_1 + \pi_2 + \pi_3 = 1 \Rightarrow \frac{8}{3}\pi_3 + 3\pi_3 + \pi_3 = 1 \Rightarrow \pi_3 = \frac{3}{20}$ (6)

Sub. (6) in (3), $\pi_2 = \frac{9}{20}$, Sub. (6) in (5), $\pi_1 = \frac{8}{20}$, $\boldsymbol{\pi_1 = 0.40, \pi_2 = 0.45, \pi_3 = 0.15}$

Thus in the long run, he sells 40% of the time in city A, 45% of the time in the city B and 15% of the time in city C.

11.A fair die is tossed repeatedly. If X_n denotes the maximum of the numbers occurring in the first n tosses, find the transition probability matrix P of the Markov chain $\{X_n\}$. Find also P^2 and $P(X_2 = 6)$.

Solution: The state space is $\{1, 2, 3, 4, 5, 6\}$. X_n = maximum of the numbers occurring in the first n trials.

X_{n+1} = maximum of the numbers occurring in the first $(n+1)$ trials = $\max[X_n, number\ in\ the\ (n+1)^{th}\ trial]$.

Let us see how the First Row of the tpm is filled.

$X_n = 1$

$X_{n+1} = 1\ if\ 1\ appears\ in(n+1)^{th}\ trial$

$\quad\quad = 2\ if\ 1\ appears\ in(n+1)^{th}\ trial$

$\quad\quad = 3\ if\ 1\ appears\ in(n+1)^{th}\ trial$

$\quad\quad = 4\ if\ 1\ appears\ in(n+1)^{th}\ trial$

$\quad\quad = 5\ if\ 1\ appears\ in(n+1)^{th}\ trial$

$\quad\quad = 6\ if\ 1\ appears\ in(n+1)^{th}\ trial$

Now, in the $(n+1)^{th}$ trial, each of the numbers 1, 2, 3, 4, 5, 6 occurs with probability $\frac{1}{6}$.

Let us see how the Second Row of the tpm is filled.

Here $X_n = 2$

$If\ (n+1)^{th}\ trial\ results\ in\ 1\ or\ 2, X_{n+1} = 2$

$If\ (n+1)^{th}\ trial\ results\ in\ 3, X_{n+1} = 3$

$If\ (n+1)^{th}\ trial\ results\ in\ 4, X_{n+1} = 4$

$If\ (n+1)^{th}\ trial\ results\ in\ 5, X_{n+1} = 5$

$If\ (n+1)^{th}\ trial\ results\ in\ 6, X_{n+1} = 6$

If $X_n = 2, P(X_{n+1} = 2) = \frac{2}{6}$ and $P(X_{n+1} = k) = \frac{1}{6},\ k = 3,4,5,6.$

Proceeding similarly, the tpm is

$$\begin{array}{cc} & (n+1)^{th}\ state \\ P = n^{th} & \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{3}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & \frac{4}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \end{bmatrix} \end{array}, P^2 = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{3}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & \frac{4}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \end{bmatrix} \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{3}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & \frac{4}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} \frac{1}{36} & \frac{3}{36} & \frac{5}{36} & \frac{7}{36} & \frac{9}{36} & \frac{11}{36} \\ 0 & \frac{4}{36} & \frac{5}{36} & \frac{7}{36} & \frac{9}{36} & \frac{11}{36} \\ 0 & 0 & \frac{9}{36} & \frac{7}{36} & \frac{9}{36} & \frac{11}{36} \\ 0 & 0 & 0 & \frac{16}{36} & \frac{9}{36} & \frac{11}{36} \\ 0 & 0 & 0 & 0 & \frac{25}{36} & \frac{11}{36} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The initial probability distribution is $P^{(0)} = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}$

$P(X_2 = 6) = \sum_{i=1}^{6} P(X_2 = 6/X_0 = i)P(X_0 = i) = \frac{1}{6}\sum_{i=1}^{6} P_{i6}^2 = \frac{1}{6}[P_{16}^2 + P_{26}^2 + P_{36}^2 + P_{46}^2 + P_{56}^2 + P_{66}^2]$

$P(X_2 = 6) = \frac{1}{6}\left[\frac{11}{36} + \frac{11}{36} + \frac{11}{36} + \frac{11}{36} + \frac{11}{36} + 1\right] = \frac{91}{216}$

12.A gambler has Rs. 2. He bets Re. 1 at a time and wins Re. 1 with probability $\frac{1}{2}$. He stops playing if he loses Rs. 2 or wins Rs. 4.

(i) What is the tpm of the related Markov chain?

(ii) What is the probability that he has lost his money at the end of 5 plays?

(iii) What is the probability that the game lasts more than 7 plays?

Solution: Let X_n denote the amount with the player at the end of the n^th^ round of the play. The states are the amounts the player is having. The game ends if he looses all the money or wins Rs. 4.

$X_n = 0$ or wins Rs. 4 ans so he has Rs. 2 + 4 = 6. $X_n = 0$ or $X_n = 6$.

$\therefore \{X_n\}, n = 0,1,2,3,\ldots$ is a Markov chain with state space $\{0,1,2,3,4,5,6\}$ as X_n can take only the values 0, 1, 2, 3, 4, 5 and 6.

(i) The tpm is

$$\begin{array}{cc} & X_n\ state \\ P = X_{n-1} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Initially the player had Rs. 2, so the initial probability distribution is $P^{(0)} = [0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0]$.

(ii) To find the probability he has lost the money at the end of 5 plays we have to find $P(X_5 = 0)$.

The entry in the state 0 in $P^{(5)}$

We know

$$P^{(1)} = P^{(0)}P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \end{bmatrix}$$

$$P^{(2)} = P^{(1)}P = \begin{bmatrix} \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} & 0 & 0 \end{bmatrix}; \quad P^{(3)} = P^{(2)}P = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & 0 & \frac{3}{8} & 0 & \frac{1}{8} & 0 \end{bmatrix}$$

$$P^{(4)} = P^{(3)}P = \begin{bmatrix} \frac{3}{8} & 0 & \frac{5}{6} & 0 & \frac{1}{4} & 0 & \frac{1}{16} \end{bmatrix}; \quad P^{(5)} = P^{(4)}P = \begin{bmatrix} \frac{3}{8} & \frac{5}{32} & 0 & \frac{9}{32} & 0 & \frac{1}{8} & \frac{1}{16} \end{bmatrix} \qquad \therefore P(X_5 = 0) = \frac{3}{8}$$

(iii) P(the game lasts for more than 7 plays) = P(the system is neither in state 0 nor in state 6 at the end of the 7th round)

$$P^{(6)} = P^{(5)}P = \begin{bmatrix} \frac{29}{64} & 0 & \frac{7}{32} & 0 & \frac{13}{64} & 0 & \frac{1}{8} \end{bmatrix}; \quad P^{(7)} = P^{(6)}P = \begin{bmatrix} \frac{29}{64} & \frac{7}{64} & 0 & \frac{27}{128} & 0 & \frac{13}{128} & \frac{1}{8} \end{bmatrix}$$

P(the game lasts for more than 7 plays) $= 1 - \left(\frac{29}{64} + \frac{1}{8}\right) = 1 - \frac{37}{64} = \frac{27}{64}$

13. The transition probability matrix of a Markov chain $\{X_n\}, n = 1, 2, \ldots$ having 3 states 1, 2, 3 is $P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$ and the initial distribution $P^{(0)} = (0.7, 0.2, 0.1)$.

Find (i) $P(X_2 = 3, X_1 = 3, X_0 = 2)$ (ii) $P(X_2 = 3)$ (iii) $P(X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2)$.

Solution:

(i) $P(X_2 = 3, X_1 = 3, X_0 = 2) = P(X_2 = 3 / X_1 = 3, X_0 = 2) P(X_1 = 3, X_0 = 2)$

$= P(X_2 = 3 / X_1 = 3) P(X_1 = 3, X_0 = 2) = P_{33}^{(1)} P(X_1 = 3, X_0 = 2)$

$= P_{33}^{(1)} P(X_1 = 3 / X_0 = 2) P(X_0 = 2) = P_{33}^{(1)} P_{23}^{(1)} P(X_0 = 2)$

$= (0.3)(0.2)(0.2) = 0.012$

(ii) $P(X_2 = 3) = P(X_2 = 3, X_0 = 1) + P(X_2 = 3, X_0 = 2) + P(X_2 = 3, X_0 = 3)$

$= P(X_2 = 3 / X_0 = 1) P(X_0 = 1) + P(X_2 = 3 / X_0 = 2) P(X_0 = 2) + P(X_2 = 3 / X_0 = 3) P(X_0 = 3)$

$= P_{13}^{(2)} P(X_0 = 1) + P_{23}^{(2)} P(X_0 = 2) + P_{33}^{(2)} P(X_0 = 3)$

$= (0.26)(0.7) + (0.34)(0.2) + (0.29)(0.1) = 0.279$

$$P^2 = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{bmatrix}$$

(iii) $P(X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2) = P(X_3 = 2 / X_2 = 3, X_1 = 3, X_0 = 2) P(X_2 = 3, X_1 = 3, X_0 = 2)$

$= P(X_3 = 2 / X_2 = 3) P(X_2 = 3, X_1 = 3, X_0 = 2)$

$= P_{32}^{(1)} P(X_2 = 3 / X_1 = 3, X_0 = 2) P(X_1 = 3, X_0 = 2)$

$= P_{32}^{(1)} P(X_2 = 3 / X_1 = 3) P(X_1 = 3, X_0 = 2)$

$= P_{32}^{(1)} P_{33}^{(1)} P(X_1 = 3 / X_0 = 2) P(X_0 = 2) = P_{32}^{(1)} P_{33}^{(1)} P_{23}^{(1)} P(X_0 = 2)$

$= (0.4)(0.3)(0.2)(0.2) = 0.0048$

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14. The transition probability matrix of a Markov chain $\{X_n\}, n = 1, 2, \ldots$ having 3 states 1, 2, 3 is $P = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix}$ & the initial distribution $P^{(0)} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. Find (i) $P(X_3 = 2/X_2 = 1)$ (ii) $P(X_2 = 2)$ (iii) $P(X_2 = 2, X_1 = 1, X_0 = 2)$ (iv) $P(X_3 = 1, X_2 = 2, X_1 = 1, X_0 = 2)$

Solution:

(i) $P(X_3 = 2/X_2 = 1) = P_{12}^{(1)} = \frac{1}{4}$

(ii) $P(X_2 = 2) = P(X_2 = 2, X_0 = 1) + P(X_2 = 2, X_0 = 2) + P(X_2 = 2, X_0 = 3)$

$= P(X_2 = 2/ X_0 = 1)P(X_0 = 1) + P(X_2 = 2/ X_0 = 2)P(X_0 = 2) + P(X_2 = 2/ X_0 = 3) P(X_0 = 3)$

$= P_{12}^{(2)} P(X_0 = 1) + P_{22}^{(2)} P(X_0 = 2) + P_{32}^{(2)} P(X_0 = 3) = \left(\frac{5}{16}\right)\left(\frac{1}{3}\right) + \left(\frac{8}{16}\right)\left(\frac{1}{3}\right) + \left(\frac{9}{16}\right)\left(\frac{1}{3}\right) = \frac{1}{6}$

$$P^2 = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{10}{16} & \frac{5}{16} & \frac{1}{16} \\ \frac{5}{16} & \frac{8}{16} & \frac{3}{16} \\ \frac{3}{16} & \frac{9}{16} & \frac{4}{16} \end{bmatrix}$$

(iii) $P(X_2 = 2, X_1 = 1, X_0 = 2) = P(X_2 = 2/ X_1 = 1, X_0 = 2)P(X_1 = 1, X_0 = 2)$

$= P(X_2 = 2/ X_1 = 1)P(X_1 = 1, X_0 = 2) = P_{12}^{(1)} P(X_1 = 1, X_0 = 2)$

$= P_{12}^{(1)} P(X_1 = 1/X_0 = 2)P(X_0 = 2) = P_{12}^{(1)} P_{21}^{(1)} P(X_0 = 2)$

$= \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{3}\right) = 0.0208$

(iv) $P(X_3 = 1, X_2 = 2, X_1 = 1, X_0 = 2) = P(X_3 = 1/ X_2 = 2, X_1 = 1, X_0 = 2)P(X_2 = 2, X_1 = 1, X_0 = 2)$

$= P(X_3 = 1/ X_2 = 2)P(X_2 = 2, X_1 = 1, X_0 = 2)$

$= P_{21}^{(1)} P(X_2 = 2/ X_1 = 1, X_0 = 2)P(X_1 = 1, X_0 = 2)$

$= P_{21}^{(1)} P(X_2 = 2/ X_1 = 1)P(X_1 = 1, X_0 = 2)$

$= P_{21}^{(1)} P_{12}^{(1)} P(X_1 = 1, X_0 = 2) = P_{21}^{(1)} P_{12}^{(1)} P(X_1 = 1/ X_0 = 2)P(X_0 = 2)$

$= P_{21}^{(1)} P_{12}^{(1)} P_{21}^{(1)} P(X_0 = 2) = \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{3}\right) = 0.0052$

15. Find the nature of the states of the Markov chain with the transition probability matrix $P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$.

Solution : $P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$, $P^2 = P\,P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$,

$P^3 = P^2\,P = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} = P$, $P^4 = P^3\,P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} = P^2$

$P_{11}^{(2)} > 0,\ P_{12}^{(1)} > 0,\ P_{13}^{(2)} > 0,$ $P_{21}^{(1)} > 0,\ P_{22}^{(2)} > 0,\ P_{23}^{(3)} > 0$, $P_{31}^{(2)} > 0,\ P_{32}^{(1)} > 0,\ P_{33}^{(2)} > 0$

The chain is irreducible. Also since there are only 3 states, the chain is finite. i.e., the chain is finite & irreducible.

All the states are non null persistent.

State 1: $P_{11}^{(2)} > 0, P_{11}^{(4)} > 0, P_{11}^{(6)} > 0, \ldots$, $Period\ of\ state\ 1 = GCD(2, 4, 6 \ldots) = 2$

State 2: $P_{22}^{(2)} > 0, P_{22}^{(4)} > 0, P_{22}^{(6)} > 0 \ldots$, $Period\ of\ state\ 2 = GCD(2, 4, 6 \ldots) = 2$

State 3: $P_{33}^{(2)} > 0, P_{33}^{(4)} > 0, P_{33}^{(6)} > 0, \ldots$ $Period\ of\ state\ 3 = GCD(2, 4, 6 \ldots) = 2$

All the states 1, 2, 3 have period 2. That is, they are periodic. All the states are non null persistent, but the states are periodic. ***Hence all the states are not ergodic.***

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16. Three boys A, B, C are throwing a ball to each other. A always throw the ball to B &B always throws to C but C is just as likely to throw the ball to B as to A. Show that the process is Markovian. Find the transition matrix & classify the states.

Solution:

$$P = \begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \end{array},$$

$$P^2 = P\,P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad P^3 = P^2\,P = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

$$P^4 = P^3\,P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}, \quad P^5 = P^4\,P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & \frac{3}{8} & \frac{1}{2} \end{bmatrix}$$

$P_{11}^{(3)} > 0,\ P_{12}^{(1)} > 0,\ P_{13}^{(2)} > 0\,, \quad P_{21}^{(2)} > 0,\ P_{22}^{(2)} > 0,\ P_{23}^{(1)} > 0\,, \quad P_{31}^{(1)} > 0,\ P_{32}^{(1)} > 0,\ P_{33}^{(2)} > 0$

The chain is irreducible. Also since there are only 3 states, the chain is finite. That is, the chain is finite and irreducible. All the states are non null persistent.

1st state A: $P_{11}^{(3)} > 0, P_{11}^{(5)} > 0, \ldots$ $\qquad$ $Period\ of\ A = GCD(3, 5, \ldots) = 1$

2nd state B: $P_{22}^{(2)} > 0, P_{22}^{(3)} > 0, P_{22}^{(4)} > 0 \ldots$ $\qquad$ $Period\ of\ B = GCD(2, 3, 4\ \ldots) = 1$

3rd state C: $P_{33}^{(2)} > 0, P_{33}^{(3)} > 0, P_{33}^{(4)} > 0, \ldots$ $\qquad$ $Period\ of\ C = GCD(2, 3, 4\ \ldots) = 1$

All the states A, B, C have period 1. That is, they are aperiodic.

All the states are aperiodic and non null persistent, they are ergodic.

All the Best

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