18MAB201T : TRANSFORMS AND BOUNDARY VALUE PROBLEMS

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October 30, 2022

CONTENTS:

- Z - Transforms - Elementary properties of Z - Transforms
- Inverse Z - Transform (using partial fraction and residues)
- Initial value theorem and final value theorem
- Convolution theorem
- Formation of difference equations
- Solution of difference equations using Z - Transform

Z - TRANSFORM

INTRODUCTION

- The Z - transform plays a significant role in the Modern communication theory especially in signal processing.
- In signal processing, the Z - transform converts a discrete time - domain signal, which is a sequence of real numbers, into a complex frequency - domain representation.
- The Z - transform and advanced Z - transform were introduced (under the Z - transform name) by E. I. Jury in 1958 in Sampled - Data Control Systems (John Wiley & Sons). The idea contained within the Z - transform was previously known as the "generating function method".
- There are two mainly known Z - transforms, namely, unilateral and bilateral.
- The (unilateral) Z - transform is to discrete - time signals what the one - sided Laplace transform is to continuous - time signals.

Z - TRANSFORM

INTRODUCTION

- In other words, just as Laplace transforms are used to solve differential equations, Z - transforms are used to solve difference equations.
- The Z - transform is an essential part of a structured control system design.
- One of the very nice things about the Z - transform is the ease with which we can write a software from it.
- Using Z - transform, almost by inspection, theoretically correct coding from a transfer function can be done.

DEFINITION : Unilateral Z - transform

Let $\{f(n)\}$ be any sequence defined for $n = 0, 1, 2, \cdots$.

Then the Z - transform of the given sequence $\{f(n)\}$ is defined as

$$\boxed{Z[f(n)] = \bar{f}(z) = \sum_{n=0}^{\infty} f(n)\, z^{-n}}$$

where z is an arbitrary complex variable.

The right hand side of the above equation is a function of z and hence it is denoted by $Z[f(n)] = F(z)$.

This transform is known as Unilateral Z - transform (or) One - Sided Z - transform of the given sequence $\{f(n)\}$.

Z - TRANSFORM

DEFINITION : Unilateral Z - transform for discrete values of t

Let $\{f(t)\}$ be a sequence defined for discrete values of $t = 0, T, 2T, 3T, \cdots$.
i.e., $\{f(t)\}$ is defined for discrete values of t, where $t = nT$,
$n = 0, 1, 2, 3, \cdots$ and T being the sampling period,
Then the Z - transform of the given sequence $\{f(t)\}$ is defined as

$$\boxed{Z[f(t)] = \sum_{n=0}^{\infty} f(nT)\, z^{-n}}$$

i.e., If the sequence is given as $\{f(t)\}$, then replace t by nT and find the Z - Transform of $\{f(nT)\}$.

Z - TRANSFORM

DEFINITION : Bilateral Z - transform

Let $\{f(n)\}$ be a sequence defined for $n = 0, \pm 1, \pm 2, \cdots$.

Then the Z - transform of the given sequence $\{f(n)\}$ is defined as

$$\boxed{Z[f(n)] = \sum_{n=-\infty}^{\infty} f(n)\, z^{-n}}$$

This Z - transform is called as Bilateral Z - transform (or) two - sided Z - transform of the given sequence $\{f(n)\}$.

NOTE :

(1) The domain for the sequence in unilateral Z-transform is $n = \{0, 1, 2, \cdots\}$ whereas for Bilateral Z-transform is $n = \{0, \pm1, \pm2, \cdots\}$.

(2) Throughout this chapter we consider only the unilateral Z - transform.

Basic Formulae :

(1) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots \quad where \ |x| < 1$

(2) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots \quad where \ |x| < 1$

(3) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots \quad where \ |x| < 1$

(4) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots \quad where \ |x| < 1$

(5) $-\log(1-x) = x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \dfrac{x^4}{4} + \cdots$

(6) $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \cdots$

(1). To Find : $Z[1]$

Solution : We know that

$$Z\left[f\left(n\right)\right] = \sum_{n=0}^{\infty} f\left(n\right)\, z^{-n}$$

$$\therefore\ Z\left[1\right] = \sum_{n=0}^{\infty} 1.\left(\frac{1}{z}\right)^{n}$$

$$= 1 + \left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^{2} + \left(\frac{1}{z}\right)^{3} + \cdots \quad where \quad \left|\frac{1}{z}\right| < 1$$

$$= \left(1 - \frac{1}{z}\right)^{-1} \quad ,\ 1 < |z|$$

$$= \left(\frac{z-1}{z}\right)^{-1} \quad ,\ |z| > 1$$

$$\boxed{Z\left[1\right] = \frac{z}{z-1}} \quad ,\ |z| > 1$$

(2). To Find : $Z[a^n]$

Solution : We know that

$$
\begin{aligned}
Z\left[f(n)\right] &= \sum_{n=0}^{\infty} f(n)\, z^{-n} \\
\therefore\ \ Z\left[a^n\right] &= \sum_{n=0}^{\infty} a^n . \left(\frac{1}{z}\right)^n \\
&= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n \\
&= 1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \cdots \quad \textit{where} \ \left|\frac{a}{z}\right| < 1 \\
&= \left(1 - \frac{a}{z}\right)^{-1} \quad , \ |a| < |z| \\
&= \left(\frac{z-a}{z}\right)^{-1} \quad , \ |z| > |a|
\end{aligned}
$$

$$\boxed{Z\left[a^n\right] = \frac{z}{z-a}} \quad , \ |z| > |a|$$

(3). To Find : $Z[n]$

Solution : We know that $Z\left[f\left(n\right)\right] = \sum_{n=0}^{\infty} f\left(n\right) z^{-n}$

$$\begin{aligned}
Z\left[n\right] &= \sum_{n=0}^{\infty} n.\left(\frac{1}{z}\right)^{n} \\
&= 0 + 1\left(\frac{1}{z}\right) + 2\left(\frac{1}{z}\right)^{2} + 3\left(\frac{1}{z}\right)^{3} + 4\left(\frac{1}{z}\right)^{4} + \cdots \\
&= \frac{1}{z}\left[1 + 2\left(\frac{1}{z}\right) + 3\left(\frac{1}{z}\right)^{2} + 4\left(\frac{1}{z}\right)^{3} + \cdots\right] \\
&= \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-2} \\
&= \frac{1}{z}\left(\frac{z-1}{z}\right)^{-2} = \frac{1}{z}\left(\frac{z}{z-1}\right)^{2}
\end{aligned}$$

$$\boxed{Z\left[n\right] = \frac{z}{\left(z-1\right)^{2}}} \quad , \ |z| > 1$$

(4). To Find : $Z[\frac{1}{n}]$, $n > 0$

Solution : We know that $Z\left[f(n)\right] = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

$$
\begin{aligned}
Z\left[\frac{1}{n}\right] &= \sum_{n=1}^{\infty} \left(\frac{1}{n}\right) z^{-n} \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{n}\right) \frac{1}{z^n} \\
&= \frac{1}{1}\left(\frac{1}{z}\right) + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \cdots \\
&= \frac{\left(\frac{1}{z}\right)}{1} + \frac{\left(\frac{1}{z}\right)^2}{2} + \frac{\left(\frac{1}{z}\right)^3}{3} + \cdots \\
&= -\log\left(1 - \frac{1}{z}\right) = -\log\left(\frac{z-1}{z}\right) = \log\left(\frac{z-1}{z}\right)^{-1}
\end{aligned}
$$

$$\boxed{Z\left[\frac{1}{n}\right] = \log\left(\frac{z}{z-1}\right)} \quad //$$

(5). To Find : $Z[\frac{1}{n-1}]$, $n > 1$

Solution : We know that $Z\left[f\left(n\right)\right] = \sum_{n=0}^{\infty} f\left(n\right) z^{-n}$

$$Z\left[\frac{1}{n-1}\right] = \sum_{n=2}^{\infty}\left(\frac{1}{n-1}\right)z^{-n} = \sum_{n=2}^{\infty}\left(\frac{1}{n-1}\right)\left(\frac{1}{z}\right)^{n}$$

$$= \frac{1}{1}\left(\frac{1}{z}\right)^{2} + \frac{1}{2}\left(\frac{1}{z}\right)^{3} + \frac{1}{3}\left(\frac{1}{z}\right)^{4} + \cdots$$

$$= \frac{1}{z}\left[\left(\frac{1}{z}\right) + \frac{1}{2}\left(\frac{1}{z}\right)^{2} + \frac{1}{3}\left(\frac{1}{z}\right)^{3} + \cdots\right]$$

$$= -\frac{1}{z}\log\left(1-\frac{1}{z}\right) = -\log\left(\frac{z-1}{z}\right) = \frac{1}{z}\log\left(\frac{z-1}{z}\right)^{-1}$$

$$\boxed{Z\left[\frac{1}{n-1}\right] = \frac{1}{z}\log\left(\frac{z}{z-1}\right)} \, , \quad |z| > 1 \quad //$$

(6). To Find : $Z[\frac{1}{n!}]$

Solution : We know that $Z\left[f\left(n\right)\right] = \sum_{n=0}^{\infty} f\left(n\right) z^{-n}$

$$\begin{aligned}
Z\left[\frac{1}{n!}\right] &= \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{z}\right)^{n} \\
&= 1 + \frac{1}{1!}\left(\frac{1}{z}\right) + \frac{1}{2!}\left(\frac{1}{z}\right)^{2} + \frac{1}{3!}\left(\frac{1}{z}\right)^{3} + \frac{1}{4!}\left(\frac{1}{z}\right)^{4} + \cdots \\
&= 1 + \frac{\left(\frac{1}{z}\right)}{1!} + \frac{\left(\frac{1}{z}\right)^{2}}{2!} + \frac{\left(\frac{1}{z}\right)^{3}}{3!} + \frac{\left(\frac{1}{z}\right)^{4}}{4!} + \cdots
\end{aligned}$$

$$\boxed{Z\left[\frac{1}{n!}\right] = e^{\frac{1}{z}}} \quad //$$

(7). To Find : The Z - Transform of Unit Impulse Sequence

Solution : The Unit Impulse Sequence is defined by

$$\delta(n) = \begin{cases} 1 & for\ n = 0 \\ 0 & for\ n \neq 0 \end{cases}$$

$$\begin{aligned}
\therefore \quad Z[f(n)] &= \sum_{n=0}^{\infty} f(n) \,.\, z^{-n} \\
Z[\delta(n)] &= \sum_{n=0}^{\infty} \delta(n) \,.\, z^{-n} \\
&= \sum_{n=0}^{\infty} \delta(n) \,.\, \left(\frac{1}{z}\right)^{n} \\
&= \delta(0) \,. + \delta(1) \left(\frac{1}{z}\right) + \delta(2) \,.\left(\frac{1}{z}\right)^{2} + \delta(3) \,.\left(\frac{1}{z}\right)^{3} + \cdots \\
&= 1 + 0 + 0 + 0 + \cdots
\end{aligned}$$

$$\boxed{Z[\delta(n)] = 1} \quad //$$

(8). To Find : The Z - Transform of Unit Step Sequence

Solution : The Unit Step Sequence is defined by

$$u(n) = \begin{cases} 1 & for\ n \geq 0 \\ 0 & for\ n < 0 \end{cases}$$

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n) \,.\, z^{-n}$$

$$Z[u(n)] = \sum_{n=0}^{\infty} u(n) \,.\, \left(\frac{1}{z}\right)^{n}$$

$$= u(0) \,. + u(1) \left(\frac{1}{z}\right) + u(2) \,.\left(\frac{1}{z}\right)^{2} + u(3) \,.\left(\frac{1}{z}\right)^{3} + \cdots$$

$$= 1 + \left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^{2} + \left(\frac{1}{z}\right)^{3} + \cdots$$

$$= \left(1 - \frac{1}{z}\right)^{-1} = \left(\frac{z-1}{z}\right)^{-1}$$

$$\boxed{Z[u(n)] = \frac{z}{z-1}} \quad //$$

Properties of Z - Transform

(1). Linearity Property

$Z\left[af\left(n\right) \pm bg\left(n\right)\right] = aZ\left[f\left(n\right)\right] \pm bZ\left[g\left(n\right)\right]$

(2). Differentiation in z - Domain :

If $Z\left[f\left(n\right)\right] \ = \ F\left(z\right)$, Then

$$\boxed{Z\left[nf\left(n\right)\right] \ = \ -z\frac{d}{dz}\left[F\left(z\right)\right]}$$

(3). First Shifting [Frequency Shifting] Property (or) Damping Rule :

If $Z\left[f\left(n\right)\right] \ = \ F\left(z\right)$, Then

$$\boxed{Z\left[a^n f\left(n\right)\right] \ = \ F\left(\frac{z}{a}\right)}$$

Properties of Z - Transform

(4). Second Shifting [Time Shifting] Property :

If $Z\left[f\left(n\right)\right] \ = \ F\left(z\right)$, Then

(i) $Z\left[f\left(n+1\right)\right] \ = \ zF\left(z\right) - zf\left(0\right)$

(ii) $Z\left[f\left(n+2\right)\right] \ = \ z^2F\left(z\right) - z^2f\left(0\right) - zf\left(1\right)$

(iii) $Z\left[f\left(n+3\right)\right] \ = \ z^3F\left(z\right) - z^3f\left(0\right) - z^2f\left(1\right) - zf\left(2\right)$

(2). Differentiation in z - Domain :

Statement :

If $Z\left[f\left(n\right)\right] = F\left(z\right)$, Then $Z\left[nf\left(n\right)\right] = -z\dfrac{d}{dz}\left[F\left(z\right)\right]$

Proof : We know that

$$F\left(z\right) = Z\left[f\left(n\right)\right] = \sum_{n=0}^{\infty} f\left(n\right) z^{-n}$$

$$i.e., \ F\left(z\right) = \sum_{n=0}^{\infty} f\left(n\right) z^{-n}$$

$$\frac{d}{dz}\left[F\left(z\right)\right] = \sum_{n=0}^{\infty} f\left(n\right)\left(-n.z^{-n-1}\right)$$

$$\frac{d}{dz}\left[F\left(z\right)\right] = -\sum_{n=0}^{\infty} nf\left(n\right) z^{-n}.z^{-1}$$

$$\frac{d}{dz}\left[F\left(z\right)\right] = -\frac{1}{z}\sum_{n=0}^{\infty} nf\left(n\right) z^{-n}$$

$$-z\frac{d}{dz}\left[F\left(z\right)\right] = \sum_{n=0}^{\infty} nf\left(n\right) z^{-n}$$

$$-z\frac{d}{dz}\left[F\left(z\right)\right] = Z\left[nf\left(n\right)\right]$$

$$i.e., \ \boxed{Z\left[nf\left(n\right)\right] = -z\frac{d}{dz}\left[F\left(z\right)\right]}$$

(3). First Shifting [Frequency Shifting] Property (or) Damping Rule :

Statement : If $Z[f(n)] = F(z)$, Then $Z[a^n f(n)] = F\left(\frac{z}{a}\right)$

Proof : We know that $F(z) = Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$

$$
\begin{aligned}
Z[a^n f(n)] &= \sum_{n=0}^{\infty} a^n f(n) z^{-n} \\
&= \sum_{n=0}^{\infty} f(n) \frac{a^n}{z^n} \\
&= \sum_{n=0}^{\infty} f(n) \left(\frac{a}{z}\right)^n \\
&= \sum_{n=0}^{\infty} f(n) \left(\frac{z}{a}\right)^{-n}
\end{aligned}
$$

$$\boxed{Z[a^n f(n)] = F\left(\frac{z}{a}\right)} \quad //$$

(4). Second Shifting [Time Shifting] Property :

(i) Statement : If $Z[f(n)] = F(z)$, Then $Z[f(n+1)] = zF(z) - zf(0)$.

Proof : We know that $F(z) = Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$

$$\therefore \; Z[f(n+1)] = \sum_{n=0}^{\infty} f(n+1) z^{-n}$$

Put $m = n+1 \implies n = m-1$, Also $n = 0 \implies m = 1$

$$\begin{aligned} Z[f(n+1)] &= \sum_{m=1}^{\infty} f(m) z^{-(m-1)} = z \sum_{m=1}^{\infty} f(m) z^{-m} \\ &= z\left(\sum_{m=0}^{\infty} f(m) z^{-m} - f(0)\right) \\ &= z(F(z) - f(0)) \end{aligned}$$

$$\boxed{Z[f(n+1)] = zF(z) - zf(0)} \quad //$$

(4). Second Shifting [Time Shifting] Property :

(ii) Statement : If $Z[f(n)] = F(z)$, Then

$$Z[f(n+2)] = z^2 F(z) - z^2 f(0) - zf(1)$$

Proof : We know that $F(z) = Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$

$$\therefore \quad Z[f(n+2)] = \sum_{n=0}^{\infty} f(n+2) z^{-n}$$

Put $m = n+2 \implies n = m-2$, Also $n = 0 \implies m = 2$

$$Z[f(n+2)] = \sum_{m=2}^{\infty} f(m) z^{-(m-2)} = z^2 \sum_{m=2}^{\infty} f(m) z^{-m}$$

$$= z^2 \left(\sum_{m=0}^{\infty} f(m) z^{-m} - f(0) z^{-0} - f(1) z^{-1} \right)$$

$$= z^2 \left(F(z) - f(0) - f(1) z^{-1} \right)$$

$$\boxed{Z[f(n+2)] = z^2 F(z) - z^2 f(0) - zf(1)} \quad //$$

(4). Second Shifting [Time Shifting] Property :

(iii) Statement : If $Z[f(n)] = F(z)$, Then
$$Z[f(n+3)] = z^3 F(z) - z^3 f(0) - z^2 f(1) - zf(2)$$

Proof : We know that $F(z) = Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$

$$\therefore \quad Z[f(n+3)] = \sum_{n=0}^{\infty} f(n+3) z^{-n}$$

Put $m = n+3 \implies n = m-3$, Also $n = 0 \implies m = 3$

$$\begin{aligned} Z[f(n+3)] &= \sum_{m=3}^{\infty} f(m) z^{-(m-3)} = z^3 \sum_{m=3}^{\infty} f(m) z^{-m} \\ &= z^3 \left(\sum_{m=0}^{\infty} f(m) z^{-m} - f(0) z^{-0} - f(1) z^{-1} - f(2) z^{-2} \right) \\ &= z^3 \left(F(z) - f(0) - f(1) z^{-1} - f(2) z^{-2} \right) \end{aligned}$$

$$\boxed{Z[f(n+3)] = z^3 F(z) - z^3 f(0) - z^2 f(1) - zf(2)} \quad //$$

Initial Value Theorem

Statement : Initial Value Theorem

If $Z[f(n)] = F(z)$, Then $f(0) = \lim_{z\to\infty} F(z)$.

Proof : We know that $F(z) = Z[f(n)] = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

$$F(z) = \sum_{n=0}^{\infty} \frac{f(n)}{z^n}$$

$$F(z) = \frac{f(0)}{z^0} + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \cdots$$

Taking limit $z \to \infty$ on both the sides , We get

$$\lim_{z\to\infty} F(z) = \lim_{z\to\infty} \left[\frac{f(0)}{z^0} + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \cdots \right]$$

$$\lim_{z\to\infty} F(z) = f(0)$$

$$i.e., \boxed{f(0) = \lim_{z\to\infty} F(z)} \quad //$$

Final Value Theorem

Statement : Final Value Theorem

If $Z[f(n)] = F(z)$, Then $\lim\limits_{n \to \infty} f(n) = \lim\limits_{z \to 1} (z-1)F(z)$.

Proof : We know that

(i) $F(z) = Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$

(ii) $Z[f(n+1)] = zF(z) - zf(0)$

$$Now \;\; Z[f(n+1) - f(n)] = \sum_{n=0}^{\infty} [f(n+1) - f(n)] z^{-n}$$

$$Z[f(n+1)] - Z[f(n)] = \sum_{n=0}^{\infty} [f(n+1) - f(n)] z^{-n}$$

$$zF(z) - zf(0) - F(z) = \sum_{n=0}^{\infty} [f(n+1) - f(n)] z^{-n}$$

$$(z-1)F(z) - zf(0) = \sum_{n=0}^{\infty} [f(n+1) - f(n)] z^{-n}$$

Taking limit $z \to 1$ in both the sides , We get

$$
\begin{aligned}
\lim_{z\to 1}\left[(z-1)F(z) - zf(0)\right] &= \lim_{z\to 1}\sum_{n=0}^{\infty}\left[f(n+1) - f(n)\right]z^{-n} \\
\lim_{z\to 1}(z-1)F(z) - f(0) &= \sum_{n=0}^{\infty}\left[f(n+1) - f(n)\right] \\
&= \lim_{n\to\infty}\sum_{m=0}^{n}\left[f(m+1) - f(m)\right] \\
&= \lim_{n\to\infty}\{\left[f(1) - f(0)\right] + \left[f(2) - f(1)\right] + \cdots + \\
&\quad \left[f(n) - f(n-1)\right] + \left[f(n+1) - f(n)\right]\} \\
&= \lim_{n\to\infty}\left[f(n+1) - f(0)\right] \\
\lim_{z\to 1}(z-1)F(z) - f(0) &= \lim_{n\to\infty} f(n) - f(0) \\
\lim_{z\to 1}(z-1)F(z) &= \lim_{n\to\infty} f(n) \\
i.e., \ \boxed{\lim_{n\to\infty} f(n) = \lim_{z\to 1}(z-1)F(z)} & \quad //
\end{aligned}
$$

Problems on Z - Transform

(1). To Find : $Z\left[e^{an}\right]$

Solution : We know that $$Z\left[a^n\right] = \frac{z}{z-a}$$

$$\begin{aligned} Z\left[e^{an}\right] &= Z\left[\left(e^a\right)^n\right] \\ &= \frac{z}{z-e^a} \quad // \end{aligned}$$

(2). To Find : $Z\left[e^{-an}\right]$

Solution : We know that $$Z\left[a^n\right] = \frac{z}{z-a}$$

$$\begin{aligned} Z\left[e^{-an}\right] &= Z\left[\left(e^{-a}\right)^n\right] \\ &= \frac{z}{z-e^{-a}} \quad // \end{aligned}$$

(3). To Find : $Z\left[e^{at}\right]$

Solution : We know that $\quad Z\left[a^n\right] = \dfrac{z}{z-a} \quad$ and $\quad Z\left[f(t)\right] = Z\left[f(nT)\right]$

$$\begin{aligned} Z\left[e^{at}\right] &= Z\left[e^{anT}\right] \\ &= Z\left[\left(e^{aT}\right)^n\right] \\ Z\left[e^{at}\right] &= \frac{z}{z-e^{aT}} \quad // \end{aligned}$$

(4). To Find : $Z\left[e^{-at}+t\right]$

Solution : We know that $\quad Z\left[a^n\right] = \dfrac{z}{z-a} \quad$ and $\quad Z\left[f(t)\right] = Z\left[f(nT)\right]$

$$\begin{aligned} Z\left[e^{-at}+t\right] = Z\left[e^{-anT}+nT\right] &= Z\left[\left(e^{-aT}\right)^n + T.n\right] \\ &= Z\left[\left(e^{-aT}\right)^n\right] + T\,Z\left[n\right] \\ Z\left[e^{-at}+t\right] &= \frac{z}{z-e^{-aT}} + T\frac{z}{(z-1)^2} \quad // \end{aligned}$$

(5). To Find : $Z\left[a^n \cosh n\theta\right]$

Solution : We know that (i) $Z\left[a^n\right] = \dfrac{z}{z-a}$ and (ii) $\cosh x = \dfrac{e^x + e^{-x}}{2}$

$$
\begin{aligned}
Z\left[a^n \cosh n\theta\right] &= Z\left[a^n\left(\frac{e^{n\theta}+e^{-n\theta}}{2}\right)\right] \\
&= \frac{1}{2} Z\left[a^n\left(e^{n\theta}+e^{-n\theta}\right)\right] \\
&= \frac{1}{2} Z\left[a^n e^{n\theta}+a^n e^{-n\theta}\right] \\
&= \frac{1}{2} Z\left[\left(ae^{\theta}\right)^n+\left(ae^{-\theta}\right)^n\right] \\
&= \frac{1}{2}\left\{Z\left[\left(ae^{\theta}\right)^n\right]+Z\left[\left(ae^{-\theta}\right)^n\right]\right\} \\
&= \frac{1}{2}\left\{\frac{z}{z-ae^{\theta}}+\frac{z}{z-ae^{-\theta}}\right\} \\
&= \frac{z}{2}\left\{\frac{1}{z-ae^{\theta}}+\frac{1}{z-ae^{-\theta}}\right\}
\end{aligned}
$$

$$
\begin{aligned}
Z\left[a^n \cosh n\theta\right] &= \frac{z}{2}\left\{\frac{z - ae^{-\theta} + z - ae^{\theta}}{z^2 - aze^{\theta} - aze^{-\theta} + a^2}\right\} \\
&= \frac{z}{2}\left\{\frac{2z - a\left(e^{\theta} + e^{-\theta}\right)}{z^2 - az\left(e^{\theta} + e^{-\theta}\right) + a^2}\right\} \\
&= \frac{z}{2}\left\{\frac{2z - 2a\cosh\theta}{z^2 - 2az\cosh\theta + a^2}\right\}
\end{aligned}
$$

$$
\boxed{Z\left[a^n \cosh n\theta\right] = \frac{z\left(z - a\cosh\theta\right)}{z^2 - 2az\cosh\theta + a^2}} \quad //
$$

(6). To Find : $Z\left[a^n \sinh n\theta\right]$

Solution : We know that (i) $Z\left[a^n\right] = \dfrac{z}{z-a}$ and (ii) $\sinh x = \dfrac{e^x - e^{-x}}{2}$

$$
\begin{aligned}
Z\left[a^n \sinh n\theta\right] = Z\left[a^n\left(\frac{e^{n\theta} - e^{-n\theta}}{2}\right)\right] &= \frac{1}{2} Z\left[a^n\left(e^{n\theta} - e^{-n\theta}\right)\right] \\
&= \frac{1}{2} Z\left[a^n e^{n\theta} - a^n e^{-n\theta}\right]
\end{aligned}
$$

$$
\begin{aligned}
Z\left[a^n \sinh n\theta\right] &= \frac{1}{2} Z\left[\left(ae^{\theta}\right)^n - \left(ae^{-\theta}\right)^n\right] \\
&= \frac{1}{2}\left\{Z\left[\left(ae^{\theta}\right)^n\right] - Z\left[\left(ae^{-\theta}\right)^n\right]\right\} \\
&= \frac{1}{2}\left\{\frac{z}{z-ae^{\theta}} - \frac{z}{z-ae^{-\theta}}\right\} \\
&= \frac{z}{2}\left\{\frac{1}{z-ae^{\theta}} - \frac{1}{z-ae^{-\theta}}\right\} \\
&= \frac{z}{2}\left\{\frac{z-ae^{-\theta}-z+ae^{\theta}}{z^2-aze^{\theta}-aze^{-\theta}+a^2}\right\} \\
&= \frac{z}{2}\left\{\frac{a\left(e^{\theta}-e^{-\theta}\right)}{z^2-az\left(e^{\theta}+e^{-\theta}\right)+a^2}\right\} \\
&= \frac{z}{2}\left\{\frac{2a\sinh\theta}{z^2-2az\cosh\theta+a^2}\right\}
\end{aligned}
$$

$$
\boxed{Z\left[a^n \sinh\theta\right] = \frac{az\sinh\theta}{z^2-2az\cosh\theta+a^2}} \quad //
$$

(7). To Find : $Z\left[r^n \cos n\theta\right]$

Solution : We know that (i) $Z\left[a^n\right] = \dfrac{z}{z-a}$ and (ii) $e^{in\theta} = \cos n\theta + i\sin n\theta$

Put $a = r\,e^{i\theta}$ in (i)

$$\begin{aligned}
Z\left[\left(re^{i\theta}\right)^n\right] &= \frac{z}{z-re^{i\theta}} \\
Z\left[r^n e^{in\theta}\right] &= \frac{z}{z-re^{i\theta}} \\
Z\left[r^n\left(\cos n\theta + i\sin n\theta\right)\right] &= \frac{z}{z-r\left(\cos\theta + i\sin\theta\right)} \\
Z\left[r^n\cos n\theta + i r^n \sin n\theta\right] &= \frac{z}{z-r\cos\theta - ir\sin\theta} \\
Z\left[r^n\cos n\theta\right] + iZ\left[r^n\sin n\theta\right] &= \frac{z}{\left(z-r\cos\theta\right)-ir\sin\theta} \\
&= \frac{z}{\left(z-r\cos\theta\right)-ir\sin\theta} \times \frac{\left(z-r\cos\theta\right)+ir\sin\theta}{\left(z-r\cos\theta\right)+ir\sin\theta} \\
&= \frac{z\left(z-r\cos\theta\right)+i\,rz\sin\theta}{\left(z-r\cos\theta\right)^2+\left(r\sin\theta\right)^2}
\end{aligned}$$

$$
\begin{aligned}
Z\left[r^n \cos n\theta\right] + iZ\left[r^n \sin n\theta\right] &= \frac{z\left(z - r\cos\theta\right) + i\, rz\sin\theta}{z^2 - 2zr\cos\theta + r^2\cos^2\theta + r^2\sin^2\theta} \\
&= \frac{z\left(z - r\cos\theta\right) + i\, rz\sin\theta}{z^2 - 2zr\cos\theta + r^2} \\
Z\left[r^n \cos n\theta\right] + iZ\left[r^n \sin n\theta\right] &= \frac{z\left(z - r\cos\theta\right)}{z^2 - 2zr\cos\theta + r^2} + i\frac{rz\sin\theta}{z^2 - 2zr\cos\theta + r^2}
\end{aligned}
$$

Comparing the real and imaginary parts on both the sides , We get

$$
\boxed{Z\left[r^n \cos n\theta\right] = \frac{z\left(z - r\cos\theta\right)}{z^2 - 2zr\cos\theta + r^2}}
$$

&

$$
\boxed{Z\left[r^n \sin n\theta\right] = \frac{rz\sin\theta}{z^2 - 2zr\cos\theta + r^2}} \quad //
$$

(8). To Find : $Z[\sin n\theta]$

Solution : We know that

$$Z[a^n \sin n\theta] = \frac{az\sin\theta}{z^2 - 2az\cos\theta + a^2} \qquad (1)$$

Put $a = 1$ in (1), We get

$$\boxed{Z[\sin n\theta] = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}} \quad //$$

(9). To Find : $Z\left[\sin\left(\frac{n\pi}{2}\right)\right]$

Solution : We know that $Z[a^n \sin n\theta] = \dfrac{az\sin\theta}{z^2 - 2az\cos\theta + a^2}$ (1)

Put $a = 1$ *and* $\theta = \dfrac{\pi}{2}$ in (1), We get

$$\boxed{Z\left[\sin\left(\frac{n\pi}{2}\right)\right] = \frac{z\sin\frac{\pi}{2}}{z^2 - 2z\cos\frac{\pi}{2} + 1} = \frac{z}{z^2+1}} \quad //$$

(10). To Find : $Z\left[\cos^2\left(\frac{n\pi}{2}\right)\right]$

Solution : We know that $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

$$\therefore\ \cos^2\left(\frac{n\pi}{2}\right) = \frac{1+\cos 2\left(\frac{n\pi}{2}\right)}{2} = \frac{1+\cos n\pi}{2} = \frac{1+(-1)^n}{2}$$

$$\begin{aligned}
Z\left[\cos^2\left(\frac{n\pi}{2}\right)\right] &= Z\left[\frac{1+(-1)^n}{2}\right] = \frac{1}{2}Z\left[1+(-1)^n\right] \\
&= \frac{1}{2}\left\{Z[1]+Z\left[(-1)^n\right]\right\} \\
&= \frac{1}{2}\left\{\frac{z}{z-1}+\frac{z}{z+1}\right\} \\
&= \frac{z}{2}\left\{\frac{1}{z-1}+\frac{1}{z+1}\right\} \\
&= \frac{z}{2}\left\{\frac{z+1+z-1}{z^2-1}\right\} = \frac{z}{2}\left\{\frac{2z}{z^2-1}\right\}
\end{aligned}$$

$$\boxed{Z\left[\cos^2\left(\frac{n\pi}{2}\right)\right] = \frac{z^2}{z^2-1}} \quad //$$

(11). To Find : $Z\left[\sin^3\left(\frac{n\pi}{6}\right)\right]$

Solution : We know that $\sin^3\theta = \frac{1}{4}\left[3\sin\theta - \sin 3\theta\right]$

Also $$Z\left[\sin n\theta\right] = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

$$\begin{aligned}
\therefore\ \sin^3\left(\frac{n\pi}{6}\right) &= \frac{1}{4}\left[3\sin\left(\frac{n\pi}{6}\right) - \sin 3\left(\frac{n\pi}{6}\right)\right] \\
\sin^3\left(\frac{n\pi}{6}\right) &= \frac{3}{4}\sin\left(\frac{n\pi}{6}\right) - \frac{1}{4}\sin\left(\frac{n\pi}{2}\right) \\
\implies \quad Z\left[\sin^3\left(\frac{n\pi}{6}\right)\right] &= \frac{3}{4}Z\left[\sin\left(\frac{n\pi}{6}\right)\right] - \frac{1}{4}Z\left[\sin\left(\frac{n\pi}{2}\right)\right] \\
&= \frac{3}{4}\frac{z\sin\left(\frac{\pi}{6}\right)}{z^2 - 2z\cos\left(\frac{\pi}{6}\right) + 1} - \frac{1}{4}\frac{z}{z^2+1} \\
&= \frac{3}{4}\frac{z\frac{1}{2}}{z^2 - 2z\frac{\sqrt{3}}{2} + 1} - \frac{1}{4}\frac{z}{z^2+1}
\end{aligned}$$

$$\boxed{Z\left[\sin^3\left(\frac{n\pi}{6}\right)\right] = \frac{3}{8}\frac{z}{z^2 - \sqrt{3}\,z + 1} - \frac{1}{4}\frac{z}{z^2+1}} \quad //$$

(12). To Find : $Z\left[\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)\right]$

Solution : WKT $\boxed{Z\left[\cos\left(\frac{n\pi}{2}\right)\right] = \frac{z^2}{z^2+1}}$ & $\boxed{Z\left[\sin\left(\frac{n\pi}{2}\right)\right] = \frac{z}{z^2+1}}$

$$
\begin{aligned}
\textit{Now} \quad \cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right) &= \cos\left(\frac{n\pi}{2}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{\pi}{4}\right) \\
\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right) &= \frac{1}{\sqrt{2}}\left[\cos\left(\frac{n\pi}{2}\right) - \sin\left(\frac{n\pi}{2}\right)\right] \\
\Longrightarrow \quad Z\left[\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)\right] &= \frac{1}{\sqrt{2}} Z\left[\cos\left(\frac{n\pi}{2}\right) - \sin\left(\frac{n\pi}{2}\right)\right] \\
&= \frac{1}{\sqrt{2}}\left\{Z\left[\cos\left(\frac{n\pi}{2}\right)\right] - Z\left[\sin\left(\frac{n\pi}{2}\right)\right]\right\} \\
&= \frac{1}{\sqrt{2}}\left\{\frac{z^2}{z^2+1} - \frac{z}{z^2+1}\right\} = \frac{1}{\sqrt{2}}\left[\frac{z^2-z}{z^2+1}\right]
\end{aligned}
$$

$$\boxed{Z\left[\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)\right] = \frac{z\,(z-1)}{\sqrt{2}\,(z^2+1)}} \quad //$$

(13). To Find : $Z\left[\frac{2n+3}{(n+1)(n+2)}\right]$

Solution : Let $f(n) = \frac{2n+3}{(n+1)(n+2)}$. Using the method of partial fraction

$$\begin{aligned}
\frac{2n+3}{(n+1)(n+2)} &= \frac{A}{(n+1)} + \frac{B}{(n+2)} \\
2n+3 &= A(n+2) + B(n+1) \\
Put\ n = -1 \quad we\, get \quad -2+3 &= A+0 \quad \Longrightarrow \quad A = 1 \\
Put\ n = -2 \quad we\, get \quad -4+3 &= 0-B \quad \Longrightarrow \quad B = 1 \\
\therefore \quad f(n) = \frac{2n+3}{(n+1)(n+2)} &= \frac{1}{(n+1)} + \frac{1}{(n+2)} \\
Z[f(n)] = Z\left[\frac{2n+3}{(n+1)(n+2)}\right] &= Z\left[\frac{1}{n+1}\right] + Z\left[\frac{1}{n+2}\right] \\
&= z\log\left(\frac{z}{z-1}\right) + z^2\log\left(\frac{z}{z-1}\right) - z
\end{aligned}$$

$$\boxed{Z\left[\frac{2n+3}{(n+1)(n+2)}\right] = z(z+1)\log\left(\frac{z}{z-1}\right) - z} \quad //$$

(1). To Find : $Z\left[n^2\right]$

Solution : WKT $\quad Z[nf(n)] = -z\dfrac{d}{dz}(Z[f(n)]) \quad \& \quad Z[n] = \dfrac{z}{(z-1)^2}$

$$\begin{aligned}
Now \ \ Z\left[n^2\right] = Z[n.n] = -z\frac{d}{dz}(Z[n]) &= -z\frac{d}{dz}\left(\frac{z}{(z-1)^2}\right)\\
&= -z\left(\frac{(z-1)^2.1 - z.2(z-1)}{(z-1)^4}\right)\\
&= -z\left(\frac{z^2-2z+1-2z^2+2z}{(z-1)^4}\right)\\
&= z\left(\frac{z^2-1}{(z-1)^4}\right) = \frac{z(z+1)(z-1)}{(z-1)^4}
\end{aligned}$$

(2). To Find : $Z\left[n^3\right]$

Solution : We know that

$(i)\ Z[nf(n)] = -z\frac{d}{dz}(Z[f(n)])$ &

$(ii)\ Z\left[n^2\right] = \frac{z(z+1)}{(z-1)^3}$

$$Now\ \ Z\left[n^3\right] = Z\left[n.n^2\right] = -z\frac{d}{dz}\left(Z\left[n^2\right]\right)$$
$$= -z\frac{d}{dz}\left(\frac{z(z+1)}{(z-1)^3}\right)$$
$$= -z\frac{d}{dz}\left(\frac{(z^2+z)}{(z-1)^3}\right)$$

After the differentiation using $d\left(\frac{u}{v}\right)$, We get

$$\boxed{Z\left[n^3\right] = \frac{z\left(z^2+4z+1\right)}{(z-1)^4}} \quad //$$

(3). To Find : $Z\left[n(n-1)(n-2)\right]$

Solution : We know that $(i)\ Z[n] = \dfrac{z}{(z-1)^2}$ & $(ii)\ Z\left[n^2\right] = \dfrac{z(z+1)}{(z-1)^3}$

$(iii)\ Z\left[n^3\right] = \dfrac{z\left(z^2+4z+1\right)}{(z-1)^4}$.

$$
\begin{aligned}
\textit{Now}\quad Z\left[n(n-1)(n-2)\right] &= Z\left[n^3-3n^2+2n\right] \\
&= Z\left[n^3\right] - 3Z\left[n^2\right] + 2Z[n] \\
&= \frac{z\left(z^2+4z+1\right)}{(z-1)^4} - 3\frac{z(z+1)}{(z-1)^3} + 2\frac{z}{(z-1)^2} \\
&= \frac{z\left(z^2+4z+1\right) - 3z(z+1)(z-1) + 2z(z-1)^2}{(z-1)^4} \\
&= \frac{z^3+4z^2+z-3z^3+3z+2z^3-4z^2+2z}{(z-1)^4}
\end{aligned}
$$

$$\boxed{Z\left[n(n-1)(n-2)\right] = \frac{6z}{(z-1)^4}} \quad //$$

(4). To Find : $Z\left[a^n n\right]$

Solution : We know that $(i)\ Z\left[a^n f(n)\right] = \left(Z\left[f(n)\right]\right)_{z \to \frac{z}{a}}$ & $(ii)\ Z\left[n\right] = \dfrac{z}{(z-1)^2}$

$$Now\ \ Z\left[a^n n\right] = \left(Z\left[n\right]\right)_{z \to \frac{z}{a}} = \left(\frac{z}{(z-1)^2}\right)_{z \to \frac{z}{a}} = \frac{z}{a\left(\frac{z}{a}-1\right)^2}$$

$$\boxed{Z\left[a^n n\right] = \frac{az}{(z-a)^2}} \quad //$$

(5). To Find : $Z\left[\dfrac{a^n}{n!}\right]$

Solution : We know that $(i)\ Z\left[a^n f(n)\right] = \left(Z\left[f(n)\right]\right)_{z \to \frac{z}{a}}$ & $(ii)\ Z\left[\dfrac{1}{n!}\right] = e^{\frac{1}{z}}$

$$\boxed{Z\left[\frac{a^n}{n!}\right] = \left(Z\left[\frac{1}{n!}\right]\right)_{z \to \frac{z}{a}} = \left(e^{\frac{1}{z}}\right)_{z \to \frac{z}{a}} = e^{\frac{a}{z}}} \quad //$$

(6). To Find : $Z\left[a^n \cos\left(\frac{n\pi}{2}\right)\right]$

Solution : We know that $(i)\ Z\left[a^n f(n)\right] = \left(Z\left[f(n)\right]\right)_{z \to \frac{z}{a}}$ &

$(ii)\ Z\left[\cos\left(\frac{n\pi}{2}\right)\right] = \frac{z^2}{z^2+1}$

$$
\begin{aligned}
Z\left[a^n \cos\left(\frac{n\pi}{2}\right)\right] &= \left(Z\left[\cos\left(\frac{n\pi}{2}\right)\right]\right)_{z \to \frac{z}{a}} = \left(\frac{z^2}{z^2+1}\right)_{z \to \frac{z}{a}} \\
&= \frac{\left(\frac{z}{a}\right)^2}{\left(\frac{z}{a}\right)^2 + 1} \\
&= \frac{z^2}{a^2\left(\frac{z^2+a^2}{a^2}\right)}
\end{aligned}
$$

$$
\boxed{Z\left[a^n \cos\left(\frac{n\pi}{2}\right)\right] = \frac{z^2}{z^2+a^2}} \quad //
$$

(7). To Find : $Z\left[a^n \delta\left(n-k\right)\right]$

Solution : We know that $(i)\ \delta\left(n-k\right) = \begin{cases} 1 \ \textit{if}\ n = k \\ 0 \ \textit{if}\ n \neq k \end{cases}$ & $(ii)\ Z\left[\delta\left(n-k\right)\right] = \dfrac{1}{z^k}$

$$\begin{aligned} Z\left[a^n \delta\left(n-k\right)\right] &= \left(Z\left[\delta\left(n-k\right)\right]\right)_{z \to \frac{z}{a}} = \left(\frac{1}{z^k}\right)_{z \to \frac{z}{a}} \\ &= \left(z^{-k}\right)_{z \to \frac{z}{a}} = \left(\frac{z}{a}\right)^{-k} = \left(\frac{a}{z}\right)^{k} \quad // \end{aligned}$$

(8). To Find : $Z\left[a^n u(n)\right]$

Solution : We know that $(i)\ Z\left[a^n f\left(n\right)\right] = \left(Z\left[f\left(n\right)\right]\right)_{z \to \frac{z}{a}}$ & $(ii)\ Z\left[u(n)\right] = \dfrac{z}{z-1}$

$$\begin{aligned} Z\left[a^n u\left(n\right)\right] &= \left(Z\left[u\left(n\right)\right]\right)_{z \to \frac{z}{a}} = \left(\frac{z}{z-1}\right)_{z \to \frac{z}{a}} \\ &= \frac{\frac{z}{a}}{\frac{z}{a}-1} = \frac{z}{a\left(\frac{z-a}{a}\right)} = \frac{z}{z-a} \quad // \end{aligned}$$

INVERSE Z - TRANSFORM

Inverse Z - Transform : Definition

If $Z[f(n)] = F(z)$, Then $f(n)$ is called the inverse Z - Transform of $F(z)$. It is denoted by $Z^{-1}\left[F\left(z\right)\right] = f\left(n\right)$

Some Important Inverse Z - Transform Formulae

(1). $Z^{-1}\left[\dfrac{z}{z-1}\right] \; = \; 1$

(2). $Z^{-1}\left[\dfrac{z}{z-a}\right] \; = \; a^n$

(3). $Z^{-1}\left[\dfrac{z}{(z-1)^2}\right] \; = \; n$

(4). $Z^{-1}\left[\dfrac{z}{(z-a)^2}\right] \; = \; na^{n-1}$

(5). $Z^{-1}\left[\dfrac{z}{(z-a)^3}\right] \; = \; \dfrac{n\left(n-1\right)a^{n-2}}{2!}$

(6). $Z^{-1}\left[\dfrac{z^2}{z^2+a^2}\right] \; = \; a^n\cos\left(\dfrac{n\pi}{2}\right)$

(7). $Z^{-1}\left[\dfrac{z}{z^2+a^2}\right] \; = \; a^{n-1}\sin\left(\dfrac{n\pi}{2}\right)$

Problems on Inverse Z - Transform by Partial Fraction Method

Find the Inverse Z - Transform of the following using partial fraction method:

(1). $Z^{-1}\left[\dfrac{10z}{z^2-3z+2}\right]$

(2). $Z^{-1}\left[\dfrac{z^3}{(z-1)^2(z-2)}\right]$

(3). $Z^{-1}\left[\dfrac{z^2}{(z+2)(z^2+4)}\right]$

(1). To Find : $Z^{-1}\left[\dfrac{10z}{z^2-3z+2}\right]$

Solution : Let

$$F(z) = \frac{10z}{z^2-3z+2} = \frac{10z}{(z-1)(z-2)}$$

$$\boxed{\frac{F(z)}{z} = \frac{10}{(z-1)(z-2)}}$$

Use the method of partial fraction , We get

$$\frac{10}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$\boxed{10 = A(z-2) + B(z-1)}$$

Put $z = 1$, We get $10 = -A + B(0) \implies \boxed{A = -10}$

Put $z = 2$, We get $10 = 0 + B \implies \boxed{B = 10}$

$$
\begin{aligned}
\therefore \quad \frac{10}{(z-1)(z-2)} &= \frac{-10}{z-1} + \frac{10}{z-2} \\
\frac{10}{(z-1)(z-2)} &= 10\left(\frac{1}{z-2} - \frac{1}{z-1}\right) \\
\frac{F(z)}{z} &= 10\left(\frac{1}{z-2} - \frac{1}{z-1}\right) \\
F(z) &= 10\left(\frac{z}{z-2} - \frac{z}{z-1}\right) \\
Z^{-1}[F(z)] &= 10Z^{-1}\left(\frac{z}{z-2} - \frac{z}{z-1}\right) \\
f(n) &= 10\left\{Z^{-1}\left(\frac{z}{z-2}\right) - Z^{-1}\left(\frac{z}{z-1}\right)\right\}
\end{aligned}
$$

$$
\boxed{f(n) = 10\{2^n - 1\}} \quad //
$$

(2). To Find : $Z^{-1}\left[\dfrac{z^3}{(z-1)^2(z-2)}\right]$

Solution : Let
$$F(z) = \frac{z^3}{(z-1)^2(z-2)}$$

$$\boxed{\frac{F(z)}{z} = \frac{z^2}{(z-1)^2(z-2)}}$$

Use the method of partial fraction , We get

$$\frac{z^2}{(z-1)^2(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-1)^2} + \frac{C}{(z-2)}$$
$$\boxed{z^2 = A(z-1)(z-2) + B(z-2) + C(z-1)^2}$$

Put $z = 1$, We get $1 = A(0) + B(-1) + C(0) \implies \boxed{B = -1}$

Put $z = 2$, We get $4 = A(0) + B(0) + C(1) \implies \boxed{C = 4}$

Comparing the coefficients of z^2 on both the sides , We get

$$1 = A + C \implies A = 1 - C = 1 - 4 \implies \boxed{A = -3}$$

$$\begin{aligned}
\therefore \quad \frac{z^2}{(z-1)^2 (z-2)} &= \frac{-3}{(z-1)} + \frac{-1}{(z-1)^2} + \frac{4}{(z-2)} \\
\frac{F(z)}{z} &= \frac{-3}{(z-1)} + \frac{-1}{(z-1)^2} + \frac{4}{(z-2)} \\
F(z) &= -3\frac{z}{(z-1)} - \frac{z}{(z-1)^2} + 4\frac{z}{(z-2)} \\
Z^{-1}[F(z)] &= -3Z^{-1}\left(\frac{z}{(z-1)}\right) - Z^{-1}\left(\frac{z}{(z-1)^2}\right) + 4Z^{-1}\left(\frac{z}{(z-2)}\right) \\
f(n) &= -3(1^n) - n + 4(2)^n \\
\boxed{f(n) = 4(2^n) - n - 3} \quad //
\end{aligned}$$

(3). To Find : $Z^{-1}\left[\dfrac{z^2}{(z+2)(z^2+4)}\right]$

Solution : Let

$$F(z) = \frac{z^2}{(z+2)(z^2+4)}$$

$$\boxed{\frac{F(z)}{z} = \frac{z}{(z+2)(z^2+4)}}$$

Use the method of partial fraction , We get

$$\frac{z}{(z+2)(z^2+4)} = \frac{A}{(z+2)} + \frac{Bz+C}{(z^2+4)}$$

$$\boxed{z = A\left(z^2+4\right) + [Bz+C](z+2)}$$

Put $z = -2$, We get $-2 = A(4+4) + [Bz+C](0) \implies \boxed{A = \frac{-1}{4}}$

Put $z = 0$, We get $0 = 4A + 2C \implies 0 = -1 + 2C \implies \boxed{C = \frac{1}{2}}$

Comparing the coefficients of z^2 on both the sides , We get

$$0 = A + B \implies B = -A \implies \boxed{B = \frac{1}{4}}$$

$$\begin{aligned}
\therefore \quad \frac{z}{(z+2)(z^2+4)} &= \frac{-\frac{1}{4}}{(z+2)} + \frac{\frac{1}{4}z + \frac{1}{2}}{(z^2+4)} \\
\frac{F(z)}{z} &= -\frac{1}{4}\frac{1}{(z+2)} + \frac{1}{4}\frac{z}{(z^2+4)} + \frac{1}{2}\frac{1}{(z^2+4)} \\
F(z) &= -\frac{1}{4}\frac{z}{(z+2)} + \frac{1}{4}\frac{z^2}{(z^2+4)} + \frac{1}{2}\frac{z}{(z^2+4)} \\
Z^{-1}[F(z)] &= -\frac{1}{4}Z^{-1}\left(\frac{z}{z+2}\right) + \frac{1}{4}Z^{-1}\left(\frac{z^2}{z^2+4}\right) + \frac{1}{2}Z^{-1}\left(\frac{z}{z^2+4}\right) \\
f(n) &= -\frac{1}{4}(-2)^n + \frac{1}{4}2^n\cos\left(\frac{n\pi}{2}\right) + \frac{1}{2}2^{n-1}\sin\left(\frac{n\pi}{2}\right)
\end{aligned}$$

$$\boxed{f(n) = -\frac{1}{4}(-2)^n + \frac{1}{4}2^n\cos\left(\frac{n\pi}{2}\right) + \frac{1}{4}2^n\sin\left(\frac{n\pi}{2}\right)} \qquad (or)$$

$$\boxed{f(n) = \frac{2^n}{4}\left[\cos\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right) - (-1)^n\right]} \quad //$$

Inverse Z - Transform by Residue Method

I.Z.T by Residue Method : Formula

$$\boxed{Z^{-1}\left[F\left(z\right)\right] \ = \ \textit{Sum of the residues of } z^{n-1}F\left(z\right) \textit{ at its poles.}}$$

Residue Formula :

The residue of $z^{n-1}F\left(z\right)$ at a pole $z \ = \ a$ of order m is given by

$$\boxed{\mathrm{Res}\left[z^{n-1}F\left(z\right)\ ,\ z=a\right] \ = \ \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left[\left(z-a\right)^{m}.z^{n-1}F\left(z\right)\right]}$$

At pole of order 1 [i.e., at simple pole] $z \ = \ a$, The residue is given by the formula

$$\boxed{\mathrm{Res}\left[z^{n-1}F\left(z\right)\ ,\ z=a\right] \ = \ \lim_{z\to a}\left(z-a\right).z^{n-1}F\left(z\right)}$$

Problems on Inverse Z - Transform by Residue Method

Find the Inverse Z - Transform of the following using residue method :

(1). $Z^{-1}\left[\dfrac{z}{z^2+3z+2}\right]$ (2). $Z^{-1}\left[\dfrac{z(z+1)}{(z-1)^3}\right]$

(3). $Z^{-1}\left[\dfrac{z^3}{(z-1)^2(z-2)}\right]$

(1). To Find : $Z^{-1}\left[\dfrac{z}{z^2+3z+2}\right]$

Solution : Let $$F(z) = \frac{z}{z^2+3z+2} = \frac{z}{(z+1)(z+2)}$$

$$\Longrightarrow \boxed{z^{n-1}F(z) = \frac{z^n}{(z+1)(z+2)}}$$

To Find : The poles of $z^{n-1}F(z)$

Assume that $(z+1)(z+2) = 0$

$\Longrightarrow$ $z = -1$ is a simple pole and $z = -2$ is also a simple pole

To Find : The Residues of $z^{n-1}F(z)$

$$\mathrm{Res}\left[z^{n-1}F(z)\,,\,z=-1\right] = \lim_{z\to -1}(z+1)\,z^{n-1}F(z)$$

$$\mathrm{Res}(-1) = \lim_{z\to -1}(z+1)\frac{z^n}{(z+1)(z+2)}$$

$$= \lim_{z\to -1}\frac{z^n}{(z+2)} = \frac{(-1)^n}{(-1+2)}$$

$$\boxed{\mathrm{Res}(-1) = (-1)^n}$$

$$\mathrm{Res}\left[z^{n-1}F(z)\ ,\ z=-2\right] = \lim_{z\to -2}(z+2)\,z^{n-1}F(z)$$

$$\mathrm{Res}(-2) = \lim_{z\to -2}(z+2)\,\frac{z^n}{(z+1)(z+2)}$$

$$= \lim_{z\to -2}\frac{z^n}{(z+1)}$$

$$= \frac{(-2)^n}{(-2+1)}$$

$$\boxed{\mathrm{Res}(-2) = -(-2)^n}$$

$$\therefore\ Z^{-1}\left[\frac{z}{(z+1)(z+2)}\right] = Z^{-1}[F(z)]$$

$$= \textit{Sum of the residues of } z^{n-1}F(z)$$

$$f(n) = \mathrm{Res}(-1)+\mathrm{Res}(-2)$$

$$\boxed{f(n) = (-1)^n-(-2)^n} \quad //$$

(2). To Find : $Z^{-1}\left[\frac{z(z+1)}{(z-1)^3}\right]$

Solution : Let $F(z) = \frac{z(z+1)}{(z-1)^3}$

$$\implies \boxed{z^{n-1}F(z) = \frac{z^n(z+1)}{(z-1)^3}}$$

To Find : The poles of $z^{n-1}F(z)$
Assume that $(z-1)^3 = 0 \implies z = 1$ is a pole of order $m = 3$
To Find : The Residues of $z^{n-1}F(z)$

$$\mathrm{Res}\left[z^{n-1}F(z)\ ,\ z=1\right] = \frac{1}{2!}\lim_{z\to 1}\frac{d^2}{dz^2}\left[(z-1)^3 z^{n-1}F(z)\right]$$

$$\mathrm{Res}(1) = \frac{1}{2}\lim_{z\to 1}\frac{d^2}{dz^2}\left[(z-1)^3\frac{z^n(z+1)}{(z-1)^3}\right]$$

$$= \frac{1}{2}\lim_{z\to 1}\frac{d^2}{dz^2}\left[z^{n+1}+z^n\right]$$

$$\begin{aligned}
\mathrm{Res}\,(1) &= \frac{1}{2}\lim_{z\to 1}\frac{d}{dz}\left[(n+1)\,z^{n}+nz^{n-1}\right] \\
&= \frac{1}{2}\lim_{z\to 1}\left[n\,(n+1)\,z^{n-1}+n\,(n-1)\,z^{n-2}\right] \\
&= \frac{1}{2}\left[n\,(n+1)+n\,(n-1)\right] \\
&= \frac{1}{2}\left[n^{2}+n+n^{2}-n\right] \\
&= \frac{1}{2}\left[2n^{2}\right]
\end{aligned}$$

$$\boxed{\mathrm{Res}\,(1) = n^{2}}$$

$$\begin{aligned}
\therefore\ Z^{-1}\left[\frac{z\,(z+1)}{(z-1)^{3}}\right] &= Z^{-1}\left[F\,(z)\right] \\
&= \mathit{Sum\ of\ the\ residues\ of}\ z^{n-1}F\,(z) \\
f\,(n) &= \mathrm{Res}\,(1)
\end{aligned}$$

$$\boxed{f\,(n) = n^{2}} \quad //$$

(3). To Find : $Z^{-1}\left[\dfrac{z^3}{(z-1)^2(z-2)}\right]$

Solution : Let
$$F(z) = \frac{z^3}{(z-1)^2(z-2)}$$

$$\Longrightarrow \quad z^{n-1}F(z) = \frac{z^{n-1}z^3}{(z-1)^2(z-2)}$$

$$\Longrightarrow \quad \boxed{z^{n-1}F(z) = \frac{z^{n+2}}{(z-1)^2(z-2)}}$$

To Find : The poles of $z^{n-1}F(z)$

Assume that $(z-1)^2(z-2) = 0$

$\Longrightarrow \quad z = 1$ is a pole of order 2 and

$z = 2$ is a pole of order 1

To Find : The Residues of $z^{n-1}F(z)$

$$\mathrm{Res}\left[z^{n-1}F(z)\ ,\ z=1\right] = \frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}\left[(z-1)^2 z^{n-1}F(z)\right]$$

$$\mathrm{Res}(1) = \lim_{z\to 1}\frac{d}{dz}\left[(z-1)^2\frac{z^{n+2}}{(z-1)^2(z-2)}\right]$$

$$= \lim_{z\to 1}\frac{d}{dz}\left[\frac{z^{n+2}}{(z-2)}\right]$$

$$= \lim_{z\to 1}\left[\frac{(z-2)(n+2)z^{n+1}-z^{n+2}(1)}{(z-2)^2}\right]$$

$$= \left[\frac{(-1)(n+2)-1}{(1)^2}\right]$$

$$\boxed{\mathrm{Res}(1) = -(n+3)}$$

$$\mathrm{Res}\left[z^{n-1}F(z)\,,\ z=2\right] = \lim_{z\to 2}(z-2)\ z^{n-1}F(z)$$

$$\mathrm{Res}(2) = \lim_{z\to 2}(z-2)\ \frac{z^{n+2}}{(z-1)^2(z-2)}$$

$$= \lim_{z\to 2}\left[\frac{z^{n+2}}{(z-1)^2}\right]$$

$$= \frac{2^{n+2}}{(2-1)^2}$$

$$\boxed{\mathrm{Res}(2) = 4(2)^n}$$

$$\therefore\ Z^{-1}\left[\frac{z^3}{(z-1)^2(z-2)}\right] = Z^{-1}[F(z)]$$

$$= \textit{Sum of the residues of } z^{n-1}F(z)$$

$$f(n) = \mathrm{Res}(1) + \mathrm{Res}(2)$$

$$\boxed{f(n) = -(n+3) + 4(2)^n} \quad //$$

Inverse Z - Transform by Convolution Method

Convolution Product : Definition

Let $f\left(n\right)$ and $g\left(n\right)$ be two sequences , then the convolution product of $f\left(n\right)$ and $g\left(n\right)$ is defined by

$$\boxed{f\left(n\right) * g\left(n\right) \ = \ \sum_{k=0}^{n} f\left(k\right) g\left(n-k\right)}$$

Convolution Theorem : Statement

If $Z\left[f\left(n\right)\right] \ = \ F\left(z\right)$ and $Z\left[g\left(n\right)\right] \ = \ G\left(z\right)$, then

$$\boxed{Z\left[f\left(n\right) * g\left(n\right)\right] \ = \ Z\left[f\left(n\right)\right] Z\left[g\left(n\right)\right]} \quad (OR)$$

$$\boxed{Z\left[f\left(n\right) * g\left(n\right)\right] \ = \ F\left(z\right) G\left(z\right)}$$

NOTE : $\boxed{Z^{-1}\left[F\left(z\right) G\left(z\right)\right] = Z^{-1}\left[F\left(z\right)\right] * Z^{-1}\left[G\left(z\right)\right]}$

Find the Inverse Z - Transform of the following using convolution method :

(1). $Z^{-1}\left[\dfrac{z^2}{(z-a)(z-b)}\right]$ (2). $Z^{-1}\left[\dfrac{8z^2}{(2z-1)(4z+1)}\right]$

(3). $Z^{-1}\left[\dfrac{z^2}{(z-a)^2}\right]$

(1). To Find : $Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$

Solution : We know that

(i) $Z^{-1}[F(z)\cdot G(z)] = Z^{-1}[F(z)] * Z^{-1}[G(z)]$

(ii) $f(n) * g(n) = \sum_{k=0}^{n} f(k)\, g(n-k)$

$$
\begin{aligned}
Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] &= Z^{-1}\left[\frac{z}{(z-a)}\cdot\frac{z}{(z-b)}\right] \\
&= Z^{-1}[F(z)\cdot G(z)] \\
&= Z^{-1}[F(z)] * Z^{-1}[G(z)] \\
&= Z^{-1}\left[\frac{z}{z-a}\right] * Z^{-1}\left[\frac{z}{z-b}\right] \\
&= a^n * b^n \\
&= f(n) * g(n) \\
&= \sum_{k=0}^{n} f(k)\, g(n-k)
\end{aligned}
$$

$$
\begin{aligned}
Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] &= \sum_{k=0}^{n} a^k b^{n-k} = \sum_{k=0}^{n} a^k b^n b^{-k} \\
&= b^n \sum_{k=0}^{n} \left(\frac{a}{b}\right)^k = b^n \sum_{k=0}^{n} r^k \\
&= b^n \left(1 + r + r^2 + \cdots + r^n\right) \\
&= b^n \left(\frac{1 - r^{n+1}}{1 - r}\right) \\
&= b^n \left(\frac{1 - \left(\frac{a}{b}\right)^{n+1}}{1 - \frac{a}{b}}\right) \\
&= b^n \left(\frac{b}{b^{n+1}} \times \frac{b^{n+1} - a^{n+1}}{b - a}\right) \\
&= \frac{b^{n+1} - a^{n+1}}{b - a}
\end{aligned}
$$

$$
\boxed{Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = \frac{a^{n+1} - b^{n+1}}{a - b}} \quad //
$$

(2). To Find : $Z^{-1}\left[\dfrac{8z^2}{(2z-1)(4z+1)}\right]$

Solution : We know that (i) $Z^{-1}[F(z)\cdot G(z)] = Z^{-1}[F(z)] * Z^{-1}[G(z)]$

(ii) $f(n) * g(n) = \sum_{k=0}^{n} f(k)\, g(n-k)$

$$
\begin{aligned}
Z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right] &= Z^{-1}\left[\frac{8z^2}{2\left(z-\frac{1}{2}\right)4\left(z+\frac{1}{4}\right)}\right] \\
&= Z^{-1}\left[\frac{z^2}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{4}\right)}\right] \\
&= Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]
\end{aligned}
$$

Where $a = \dfrac{1}{2}$ *and* $b = \dfrac{-1}{4}$

$$
= Z^{-1}\left[\frac{z}{(z-a)}\cdot\frac{z}{(z-b)}\right]
$$

$$
\begin{aligned}
Z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right] &= Z^{-1}\left[F(z)\cdot G(z)\right] \\
&= Z^{-1}\left[F(z)\right] * Z^{-1}\left[G(z)\right] \\
&= Z^{-1}\left[\frac{z}{z-a}\right] * Z^{-1}\left[\frac{z}{z-b}\right] \\
&= a^n * b^n \;=\; f(n) * g(n) \\
&= \sum_{k=0}^{n} f(k)\, g(n-k) \\
&= \sum_{k=0}^{n} a^k b^{n-k} \\
&= \sum_{k=0}^{n} a^k b^n b^{-k} \\
&= b^n \sum_{k=0}^{n} \left(\frac{a}{b}\right)^k \;=\; b^n \sum_{k=0}^{n} r^k \\
&= b^n \left(1 + r + r^2 + \cdots + r^n\right)
\end{aligned}
$$

$$
\begin{aligned}
Z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right] &= b^n\left(\frac{1-r^{n+1}}{1-r}\right)\\
&= b^n\left(\frac{1-\left(\frac{a}{b}\right)^{n+1}}{1-\frac{a}{b}}\right)\\
&= b^n\left(\frac{b}{b^{n+1}}\times\frac{b^{n+1}-a^{n+1}}{b-a}\right)\\
&= \frac{b^{n+1}-a^{n+1}}{b-a}\\
&= \frac{a^{n+1}-b^{n+1}}{a-b}
\end{aligned}
$$

Put $a=\frac{1}{2}$ and $b=\frac{-1}{4}$, We get

$$
= \frac{\left(\frac{1}{2}\right)^{n+1}-\left(\frac{-1}{4}\right)^{n+1}}{\frac{1}{2}+\frac{1}{4}}
$$

$$
\boxed{Z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right] = \frac{4}{3}\left[\left(\frac{1}{2}\right)^{n+1}-\left(\frac{-1}{4}\right)^{n+1}\right]} \quad //
$$

(3). To Find : $Z^{-1}\left[\dfrac{z^2}{(z-a)^2}\right]$

Solution : We know that

(i) $Z^{-1}[F(z)\cdot G(z)] = Z^{-1}[F(z)] * Z^{-1}[G(z)]$

(ii) $f(n) * g(n) = \sum_{k=0}^{n} f(k)\, g(n-k)$

$$\begin{aligned}
Z^{-1}\left[\frac{z^2}{(z-a)^2}\right] &= Z^{-1}\left[\frac{z}{(z-a)}\cdot\frac{z}{(z-a)}\right]\\
&= Z^{-1}[F(z)\cdot G(z)]\\
&= Z^{-1}[F(z)] * Z^{-1}[G(z)]\\
&= Z^{-1}\left[\frac{z}{z-a}\right] * Z^{-1}\left[\frac{z}{z-a}\right]\\
&= a^n * a^n\\
&= f(n) * g(n)
\end{aligned}$$

$$
\begin{aligned}
Z^{-1}\left[\frac{z^2}{(z-a)^2}\right] &= \sum_{k=0}^{n} f(k)\, g(n-k) \\
&= \sum_{k=0}^{n} a^k a^{n-k} \\
&= \sum_{k=0}^{n} a^{k+n-k} \\
&= \sum_{k=0}^{n} a^n \\
&= \left(\underbrace{a^n + a^n + \cdots + a^n}_{(n+1)\ times}\right)
\end{aligned}
$$

$$
\boxed{Z^{-1}\left[\frac{z^2}{(z-a)^2}\right] = (n+1)\, a^n} \quad //
$$

Formation of Difference Equations

Form the difference equations from the following by eliminating the arbitrary constants given :

(1). $u_n = A\,2^n + Bn$ (2). $y\,(n) = a\,2^n + b(-2)^n$

(3). $u_n = (A + Bn)\,(-3)^n$

(1). Form the difference equations from $u_n = A\,2^n + B\,n$ by eliminating the arbitrary constants A and B.

Solution :

$$\begin{aligned} u_n &= A2^n + B\,n \\ u_{n+1} &= A2^{n+1} + B\,(n+1) \\ u_{n+2} &= A2^{n+2} + B\,(n+2) \end{aligned}$$

$$\begin{vmatrix} u_n & 2^n & n \\ u_{n+1} & 2^{n+1} & n+1 \\ u_{n+2} & 2^{n+2} & n+2 \end{vmatrix} = 0 \implies 2^n \begin{vmatrix} u_n & 1 & n \\ u_{n+1} & 2 & n+1 \\ u_{n+2} & 4 & n+2 \end{vmatrix} = 0$$

$$\implies \begin{vmatrix} u_n & 1 & n \\ u_{n+1} & 2 & n+1 \\ u_{n+2} & 4 & n+2 \end{vmatrix} = 0$$

$$u_n\,[2\,(n+2) - 4\,(n+1)] - u_{n+1}\,[n+2-4n] + u_{n+2}\,[n+1-2n] = 0$$

$$\boxed{(1-n)\,u_{n+2} - (2-3n)\,u_{n+1} - 2nu_n = 0}$$

This is the required difference equation of second order. //

(2). Form the difference equations from $y(n) = a\,2^n + b\,(-2)^{\,n}$ by eliminating the arbitrary constants a and b.

Solution : Given that $\quad y(n) = a\,2^n + b\,(-2)^{\,n}$

$$\implies \quad y(n+1) = a\,2^{n+1} + b\,(-2)^{\,n+1} \;\;\&\;\; y(n+2) = a\,2^{n+2} + b\,(-2)^{\,n+2}$$

$$\begin{vmatrix} y(n) & 2^n & (-2)^{\,n} \\ y(n+1) & 2^{n+1} & (-2)^{\,n+1} \\ y(n+2) & 2^{n+2} & (-2)^{\,n+2} \end{vmatrix} = 0$$

$$2^n(-2)^n \begin{vmatrix} y(n) & 1 & 1 \\ y(n+1) & 2 & -2 \\ y(n+2) & 4 & 4 \end{vmatrix} = 0$$

$$\begin{vmatrix} y(n) & 1 & 1 \\ y(n+1) & 2 & -2 \\ y(n+2) & 4 & 4 \end{vmatrix} = 0$$

$$y(n)\,[8+8] - y(n+1)\,[4-4] + y(n+2)\,[-2-2] = 0$$

$$-4y(n+2) + 16y(n) = 0$$

$$\boxed{y(n+2) - 4y(n) = 0} \quad //$$

(3). Form the difference equations from $u_n = (A + Bn)(-3)^{\,n}$ by eliminating the arbitrary constants A and B.

Solution : Given that $u_n = (A + Bn)(-3)^{\,n}$

$$\begin{aligned} u_n &= A(-3)^{\,n} + Bn(-3^n) \\ u_{n+1} &= A(-3)^{\,n+1} + B(n+1)(-3^n) \\ u_{n+2} &= A(-3)^{\,n+1} + B(n+2)(-3^n) \end{aligned}$$

Consider the determinant

$$\begin{vmatrix} u_n & (-3)^n & n(-3)^{\,n} \\ u_{n+1} & (-3)^{n+1} & (n+1)(-3)^{\,n+1} \\ u_{n+2} & (-3)^{n+2} & (n+2)(-3)^{\,n+2} \end{vmatrix} = 0$$

$$(-3)^n(-3)^n \begin{vmatrix} u_n & 1 & n \\ u_{n+1} & -3 & -3(n+1) \\ u_{n+2} & 9 & 9(n+2) \end{vmatrix} = 0$$

$$\begin{vmatrix} u_n & 1 & n \\ u_{n+1} & -3 & -3(n+1) \\ u_{n+2} & 9 & 9(n+2) \end{vmatrix} = 0$$

Expand the determinant along the first column , We get

$$\left.\begin{array}{c} u_n\left[-27(n+2)+27(n+1)\right]-u_{n+1}\left[9(n+2)-9n\right] \\ +u_{n+2}\left[-3(n+1)+3n\right] \end{array}\right\} = 0$$

$$\left.\begin{array}{c} u_n\left[-27n-54+27n+27\right]-u_{n+1}\left[9n+18-9n\right] \\ +u_{n+2}\left[-3n-3+3n\right] \end{array}\right\} = 0$$

$$-3u_{n+2}+18u_{n+1}-27u_n = 0$$

$$\boxed{u_{n+2}-6u_{n+1}+9u_n = 0}$$

This is our required difference equation of second order.

Solution of Difference Equations using Z - Transform

Solve the following difference equations :

(1). $u_{n+2} + 3u_{n+1} + 2u_n = 0$ given that $u_0 = 1\,,\ u_1 = 2$

(2). $y\,(n+3) - 3y\,(n+1) + 2y\,(n) = 0$ given $y\,(0) = 4$, $y\,(1) = 0$ and $y\,(2) = 8$

(3). $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$ given $y_0 = y_1 = 0$

(4). $y_{n+2} + 4y_{n+1} - 5y_n = 24n - 8$ given $y_0 = 3\,,\ y_1 = -5$

Recall : Second Shifting Property

(i). $Z[f(n+1)] = zF(z) - zf(0)$

(ii). $Z[f(n+2)] = z^2F(z) - z^2f(0) - zf(1)$

(iii). $Z[f(n+3)] = z^3F(z) - z^3f(0) - z^2f(1) - zf(2)$

(1). Solve $u_{n+2} + 3u_{n+1} + 2u_n = 0$ given that $u_0 = 1\,,\ u_1 = 2$

Solution: Given difference equation is

$$u_{n+2} + 3u_{n+1} + 2u_n = 0 \tag{1}$$

Also given that $u_0 = 1\,,\ u_1 = 2$
Taking Z - Transform on both the sides of (1), We get

$$\begin{aligned}
Z[u_{n+2} + 3u_{n+1} + 2u_n] &= Z[0] \\
Z[u_{n+2}] + 3Z[u_{n+1}] + 2Z[u_n] &= Z[0] \\
\left[z^2U(z) - z^2u(0) - zu(1)\right] + 3[zU(z) - zu(0)] + 2U(z) &= 0 \\
\left(z^2 + 3z + 2\right)U(z) - z^2(1) - z(2) - 3z(1) &= 0
\end{aligned}$$

$$\left(z^2+3z+2\right)U(z) = z^2+5z$$
$$(z+1)(z+2)U(z) = z(z+5)$$
$$\boxed{U(z) = \frac{z(z+5)}{(z+1)(z+2)}} \qquad (2)$$

To Find : $Z^{-1}[U(z)] = Z^{-1}\left[\frac{z(z+5)}{(z+1)(z+2)}\right]$

$$(2) \implies \frac{U(z)}{z} = \frac{z+5}{(z+1)(z+2)}$$

Using the method of partial fraction , We get

$$\frac{z+5}{(z+1)(z+2)} = \frac{A}{z+1}+\frac{B}{z+2}$$
$$z+5 = A(z+2)+B(z+1)$$

Put $z = -1$, We get $4 = A(1)+B(0) \implies \boxed{A = 4}$

Put $z = -2$, We get $3 = A(0)+B(-1) \implies \boxed{B = -3}$

$$\frac{z+5}{(z+1)(z+2)} = \frac{4}{z+1} - \frac{3}{z+2}$$

$$\frac{U(z)}{z} = \frac{4}{z+1} - \frac{3}{z+2}$$

$$U(z) = 4\frac{z}{z+1} - 3\frac{z}{z+2}$$

Taking inverse Z - transform on both the sides, We get

$$Z^{-1}[U(z)] = 4Z^{-1}\left[\frac{z}{z+1}\right] - 3Z^{-1}\left[\frac{z}{z+2}\right]$$

$$\boxed{u(n) = 4(-1)^n - 3(-2)^n} \quad //$$

(2). Solve $y(n+3) - 3y(n+1) + 2y(n) = 0$ given $y(0) = 4$, $y(1) = 0$ and $y(2) = 8$

Solution: Given difference equation is

$$y(n+3) - 3y(n+1) + 2y(n) = 0 \tag{1}$$

Also given that $y(0) = 4$, $y(1) = 0$ and $y(2) = 8$

Taking Z - Transform on both the sides of (1), We get

$$\begin{aligned}
Z[y(n+3) - 3y(n+1) + 2y(n)] &= Z[0] \\
Z[y(n+3)] - 3Z[y(n+1)] + 2Z[y(n)] &= Z[0] \\
\left.\begin{array}{l} \left[z^3 Y(z) - z^3 y(0) - z^2 y(1) - zy(2)\right] \\ -3[zY(z) - zy(0)] + 2Y(z) \end{array}\right\} &= 0 \\
\left(z^3 - 3z + 2\right) Y(z) - 4z^3 - 8z + 12z &= 0 \\
\left(z^3 - 3z + 2\right) Y(z) &= 4z^3 - 4z \\
(z-1)^2 (z+2) Y(z) &= 4z\left(z^2 - 1\right) \\
Y(z) &= \frac{4z\left(z^2-1\right)}{(z-1)^2 (z+2)}
\end{aligned}$$

$$Y(z) = \frac{4z(z+1)(z-1)}{(z-1)^2(z+2)}$$

$$\boxed{Y(z) = \frac{4z(z+1)}{(z-1)(z+2)}} \qquad (2)$$

To Find : $Z^{-1}[Y(z)] = Z^{-1}\left[\frac{4z(z+1)}{(z-1)(z+2)}\right]$

$$(2) \implies \frac{Y(z)}{z} = \frac{4(z+1)}{(z-1)(z+2)}$$

Using the method of partial fraction , We get

$$\frac{4(z+1)}{(z-1)(z+2)} = \frac{A}{z-1} + \frac{B}{z+2}$$

$$4(z+1) = A(z+2) + B(z-1)$$

Put $z = 1$, We get $8 = A(3) + B(0) \implies \boxed{A = \frac{8}{3}}$

Put $z = -2$, We get $-4 = A(0) + B(-3) \implies \boxed{B = \frac{4}{3}}$

$$
\begin{aligned}
\frac{4(z+1)}{(z-1)(z+2)} &= \frac{8}{3}\frac{1}{z-1} + \frac{4}{3}\frac{1}{z+2} \\
\frac{Y(z)}{z} &= \frac{8}{3}\frac{1}{z-1} + \frac{4}{3}\frac{1}{z+2} \\
Y(z) &= \frac{8}{3}\frac{z}{z-1} + \frac{4}{3}\frac{z}{z+2}
\end{aligned}
$$

Taking inverse Z - transform on both the sides, We get

$$
\begin{aligned}
Z^{-1}[Y(z)] &= \frac{8}{3}Z^{-1}\left[\frac{z}{z-1}\right] + \frac{4}{3}Z^{-1}\left[\frac{z}{z+2}\right] \\
y(n) &= \frac{8}{3}(1) + \frac{4}{3}(-2)^n \\
\Aboxed{y(n) &= \frac{4}{3}\left[2+(-2)^n\right]} \quad //
\end{aligned}
$$

(3). Solve $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$ given $y_0 = y_1 = 0$

Solution: Given difference equation is

$$y_{n+2} + 6y_{n+1} + 9y_n \;=\; 2^n \tag{1}$$

Also given that $y_0 \;=\; 0 \;\&\; y_1 \;=\; 0$

Taking Z - Transform on both the sides of (1), We get

$$\begin{aligned}
Z\left[y_{n+2} + 6y_{n+1} + 9y_n\right] &= Z\left[2^n\right]\\
Z\left[y_{n+2}\right] + 6Z\left[y_{n+1}\right] + 2Z\left[y_n\right] &= Z\left[2^n\right]\\
\left.\begin{array}{l}\left[z^2 Y(z) - z^2 y(0) - zy(1)\right] + \\ 6\left[zY(z) - zy(0)\right] + 9Y(z)\end{array}\right\} &= \frac{z}{z-2}\\
\left(z^2 + 6z + 9\right) Y(z) - 0 - 0 - 0 &= \frac{z}{z-2}\\
(z+3)^2 Y(z) &= \frac{z}{z-2}\\
\boxed{Y(z) \;=\; \frac{z}{(z-2)(z+3)^2}} \qquad (2)
\end{aligned}$$

To Find : $Z^{-1}[Y(z)] = Z^{-1}\left[\dfrac{z}{(z-2)(z+3)^2}\right]$

$$(2) \implies \frac{Y(z)}{z} = \frac{1}{(z-2)(z+3)^2}$$

Using the method of partial fraction , We get

$$\frac{1}{(z-2)(z+3)^2} = \frac{A}{(z-2)} + \frac{B}{(z+3)} + \frac{C}{(z+3)^2}$$

$$1 = A(z+3)^2 + B(z-2)(z+3) + C(z-2)$$

Put $z = 2$, We get $1 = A(25) + B(0) + C(0) \implies \boxed{A = \dfrac{1}{25}}$

Put $z = -3$, We get $1 = A(0) + B(0) + C(-5) \implies \boxed{C = \dfrac{-1}{5}}$

Comparing the coefficients of z^2 on both the sides , We get

$$0 = A + B \implies B = -A \implies \boxed{B = \frac{-1}{25}}$$

$$
\begin{aligned}
\frac{1}{(z-2)(z+3)^2} &= \frac{1}{25}\frac{1}{(z-2)} - \frac{1}{25}\frac{1}{(z+3)} - \frac{1}{5}\frac{1}{(z+3)^2} \\
\frac{Y(z)}{z} &= \frac{1}{25}\frac{1}{(z-2)} - \frac{1}{25}\frac{1}{(z+3)} - \frac{1}{5}\frac{1}{(z+3)^2} \\
Y(z) &= \frac{1}{25}\frac{z}{(z-2)} - \frac{1}{25}\frac{z}{(z+3)} - \frac{1}{5}\frac{z}{(z+3)^2}
\end{aligned}
$$

Taking inverse Z - transform on both the sides, We get

$$
\begin{aligned}
Z^{-1}[Y(z)] &= \frac{1}{25}Z^{-1}\left[\frac{z}{(z-2)}\right] - \frac{1}{25}Z^{-1}\left[\frac{z}{(z+3)}\right] - \frac{1}{5}Z^{-1}\left[\frac{z}{(z+3)^2}\right] \\
y(n) &= \frac{1}{25}2^n - \frac{1}{25}(-3)^n - \frac{1}{5}n(-3)^{n-1} \\
&\boxed{y(n) = \frac{1}{25}2^n - \frac{1}{25}(-3)^n + \frac{1}{15}n(-3)^n} \quad //
\end{aligned}
$$

(4). Solve $y_{n+2}+4y_{n+1}-5y_n=24n-8$ given $y_0=3\,,\,y_1=-5$

Solution: Given difference equation is

$$y_{n+2}+4y_{n+1}-5y_n=24n-8 \tag{1}$$

Also given that $y_0=3\,,\,y_1=-5$

Taking Z - Transform on both the sides of (1), We get

$$\begin{aligned}
Z\left[y_{n+2}+4y_{n+1}-5y\left(n\right)\right] &= Z\left[24n-8\right]\\
Z\left[y_{n+2}\right]+4Z\left[y_{n+1}\right]-5Z\left[y_n\right] &= 24Z\left[n\right]-8Z\left[1\right]\\
\left.\begin{array}{l}\left[z^2Y\left(z\right)-z^2y\left(0\right)-zy\left(1\right)\right]+\\ 4\left[zY\left(z\right)-zy\left(0\right)\right]-5Y\left(z\right)\end{array}\right\} &= 24\frac{z}{\left(z-1\right)^2}-8\frac{z}{z-1}\\
\left(z^2+4z-5\right)Y\left(z\right)-3z^2+5z-12z &= \frac{24z-8z\left(z-1\right)}{\left(z-1\right)^2}\\
\left(z+5\right)\left(z-1\right)Y\left(z\right) &= \frac{24z-8z^2+8z}{\left(z-1\right)^2}+3z^2+7z
\end{aligned}$$

$$\begin{aligned}
(z+5)(z-1)\,Y(z) &= \frac{32z-8z^2+\left(3z^2+7z\right)\left(z^2-2z+1\right)}{(z-1)^2} \\
Y(z) &= \frac{32z-8z^2+3z^4-6z^3+3z^2+7z^3-14z^2+7z}{(z+5)(z-1)^3} \\
Y(z) &= \frac{3z^4+z^3-19z^2+39z}{(z+5)(z-1)^3}
\end{aligned}$$

$$\boxed{Y(z) = \frac{z\left[3z^3+z^2-19z+39\right]}{(z+5)(z-1)^3}} \qquad (2)$$

To Find : $Z^{-1}\left[Y(z)\right] = Z^{-1}\left[\frac{z\left[3z^3+z^2-19z+39\right]}{(z+5)(z-1)^3}\right]$

$$(2) \implies \frac{Y(z)}{z} = \frac{3z^3+z^2-19z+39}{(z+5)(z-1)^3}$$

Using the method of partial fraction , We get

$$\frac{3z^3+z^2-19z+39}{(z+5)(z-1)^3} = \frac{A}{(z+5)} + \frac{B}{(z-1)} + \frac{C}{(z-1)^2} + \frac{D}{(z-1)^3}$$

$$3z^3+z^2-19z+39 = \begin{cases} A(z-1)^3 + B(z+5)(z-1)^2 + \\ C(z+5)(z-1) + D(z+5) \end{cases}$$

Put $z = -5$, We get

$$-375+25+95+39 = A(-6)^3 \implies -216 = -216A \implies \boxed{A = 1}$$

Put $z = 1$, We get

$$3+1-19+39 = D(6) \implies 24 = 6D \implies \boxed{D = 4}$$

Equating the coefficients of z^3 on both the sides , We get

$$3 = A+B \implies B = 3-A = 3-1 \implies \boxed{B = 2}$$

Put $z = 0$, We get

$$\begin{aligned} 39 = -A+5B-5C+5D &\implies -39 = -1+10-5C+20 \\ &\implies 39 = 29-5C \\ &\implies \boxed{C = -2} \end{aligned}$$

$$\frac{3z^3 + z^2 - 19z + 39}{(z+5)(z-1)^3} = \frac{1}{(z+5)} + \frac{2}{(z-1)} - \frac{2}{(z-1)^2} + \frac{4}{(z-1)^3}$$

$$\frac{Y(z)}{z} = \frac{1}{(z+5)} + \frac{2}{(z-1)} - \frac{2}{(z-1)^2} + \frac{4}{(z-1)^3}$$

$$Y(z) = \frac{z}{(z+5)} + 2\frac{z}{(z-1)} - 2\frac{z}{(z-1)^2} + 4\frac{z}{(z-1)^3}$$

Taking inverse Z - transform on both the sides, We get

$$Z^{-1}[Y(z)] = \begin{cases} Z^{-1}\left[\dfrac{z}{z+5}\right] + 2Z^{-1}\left[\dfrac{z}{z-1}\right] \\ -2Z^{-1}\left[\dfrac{z}{(z-1)^2}\right] + 4Z^{-1}\left[\dfrac{z}{(z-1)^3}\right] \end{cases}$$

$$y(n) = (-5)^n + 2 - 2n + 4\frac{n(n-1)}{2}$$

$$y(n) = (-5)^n + 2 - 2n + 2n^2 - 2n$$

$$\boxed{y(n) = (-5)^n + 2n^2 - 4n + 2} \quad //$$


Thank You!