PROBABILITY
&
QUEUEING THEORY

(As per SRM UNIVERSITY Syllabus)

Dr. P. GODHANDARAMAN
M.Sc., M.A., M.B.A., Ph.D.
Assistant Professor (Senior Grade)
Department of Mathematics
SRM IST, Kattankulathur – 603 203
Email : godhanda@gmail.com
Mobile : 9941740168

Dr. P. Godhandaraman, Assistant Professor (Sr.G.), Dept of Maths, SRM IST, Kattankulathur, Mobile: 9941740168, Email: godhanda@gmail.com

SMALL SAMPLES ($n < 30$)

- **F – test**
- $\chi^2 - test$ (*Chi Square − test*)

F – Test

$$F = \frac{S_X^2}{S_Y^2}, \text{ if } S_X^2 > S_Y^2 \text{ (or) } F = \frac{S_Y^2}{S_X^2}, \text{ if } S_X^2 < S_Y^2 ,$$

$$S_X^2 = \frac{\sum(x - \bar{x})^2}{n_1 - 1}, \ S_Y^2 = \frac{\sum(y - \bar{y})^2}{n_2 - 1}, \ S_1^2 = \frac{n_1 s_1^2}{n_1 - 1}, \ S_2^2 = \frac{n_2 s_2^2}{n_2 - 1},$$

degree of freedom : $(v_1 = n_1 - 1, \ v_2 = n_2 - 1)$.

Applications: (i) To test whether two independent samples have been drawn from the normal populations with the same variance σ^2 (ii) To test whether two independent estimates of the population variance are homogeneous or not.

PROBLEMS

1. **If one sample of 8 observations the sum of the squares of deviations of the sample values from the sample mean was 84.4 and in the other sample of 10 observations it was 102.6. Test whether this difference is significant at 5% level.**

   Solution: $n_1 = 8$, $n_2 = 10$, $\sum(x - \bar{x})^2 = 84.4$, $\sum(y - \bar{y})^2 = 102.6$

   Null Hypothesis : $H_0: \sigma_X^2 = \sigma_Y^2$, Alternate Hypothesis : $H_1: \sigma_X^2 \neq \sigma_Y^2$

   $S_X^2 = \frac{\sum(x - \bar{x})^2}{n_1 - 1} = \frac{84.4}{8 - 1} = 12.057$, $S_Y^2 = \frac{\sum(y - \bar{y})^2}{n_2 - 1} = \frac{102.6}{10 - 1} = 11.4$, $F = \frac{S_X^2}{S_Y^2} = \frac{12.057}{11.4} = 1.057$,

   d.f. $= (n_1 - 1, \ n_2 - 1) = (7, 9)$ at 5% LOS $= 3.29$. Calculate value F $<$ Tabulated F. **H_0 is accepted**.

2. **Two random samples of 11 and 9 items show the sample standard deviations of their weights as 0.8 and 0.5 respectively. Assuming that the weight distributions are normal, test the hypothesis that the true variances are equal, against the alternative hypothesis that they are not.**

   Solution: $n_1 = 11$, $n_2 = 9$, $s_1^2 = 0.8$, $s_2^2 = 0.5$, Null Hypothesis: $H_0: \sigma_X^2 = \sigma_Y^2$, Alternate Hypothesis : $H_1: \sigma_X^2 \neq \sigma_Y^2$

   $S_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{11(0.8)^2}{11 - 1} = 0.704$, $S_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{9(0.5)^2}{9 - 1} = 0.28125$, $F = \frac{S_1^2}{S_2^2} = \frac{0.704}{0.28125} = 2.503$,

   d.f. $= (n_1 - 1, \ n_2 - 1) = (10, 8)$ at 5% LOS $= 3.34$. Calculate value F $<$ Tabulated F. **H_0 is accepted**.

3. **The time taken by workers in performing a job by Method I and Method II is given below**

Method I	20	16	26	27	23	22	-
Method II	27	33	42	35	32	34	38

   Do the data show that the variances of time distribution form population from which these samples are drawn do not differ significantly?

   Solution: $n_1 = 6$, $n_2 = 7$, $\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n_1} = \frac{134}{6} = 22.3$, $\bar{y} = \frac{\sum_{j=1}^{n} y_j}{n_2} = \frac{241}{7} = 34.4$

x	$x - \bar{x}$	$(x - \bar{x})^2$	y	$y - \bar{y}$	$(y - \bar{y})^2$
20	-2.3	5.29	27	-7.4	54.76
16	-6.3	39.69	33	-1.4	1.96
26	3.7	13.69	42	7.6	57.76
27	4.7	22.09	35	0.6	0.36
23	0.7	0.49	32	-2.4	5.76
22	-0.3	0.09	34	-0.4	0.16
$\sum x = 134$		$\sum(x - \bar{x})^2 = 81.34$	38	3.6	12.96
			$\sum y = 241$		$\sum(y - \bar{y})^2 = 132.72$

   Null Hypothesis : $H_0: \sigma_X^2 = \sigma_Y^2$, Alternate Hypothesis : $H_1: \sigma_X^2 \neq \sigma_Y^2$

   $S_X^2 = \frac{\sum(x - \bar{x})^2}{n_1 - 1} = \frac{81.34}{6 - 1} = 16.268$, $S_Y^2 = \frac{\sum(y - \bar{y})^2}{n_2 - 1} = \frac{133.72}{7 - 1} = 22.29$, $F = \frac{S_Y^2}{S_X^2} = \frac{22.29}{16.268} = 1.37$,

   d.f. $= (n_2 - 1, \ n_1 - 1) = (6, 5)$ at 5% LOS $= 4.95$. Calculate value F $<$ Tabulated F. **H_0 is accepted**.

2

Dr. P. Godhandaraman, Assistant Professor (Sr.G.), Dept of Maths, SRM IST, Kattankulathur, Mobile: 9941740168, Email: godhanda@gmail.com

4. *Two horses A and B were tested according to the time (in seconds) to run a particular track with the following*

Horse A	28	30	32	33	33	29	34
Horse B	29	30	30	24	27	29	-

Test whether the two horses have the same running capacity.

Solution: $n_1 = 7$, $n_2 = 6$, $\overline{x} = \frac{\sum_{i=1}^{n} x_i}{n_1} = \frac{219}{7} = 31.28$, $\overline{y} = \frac{\sum_{j=1}^{n} y_j}{n_2} = \frac{169}{6} = 28.2$

x	$x - \overline{x}$	$(x - \overline{x})^2$	y	$y - \overline{y}$	$(y - \overline{y})^2$
28	-3.28	10.75	29	0.8	0.64
30	-1.28	1.64	30	1.8	3.24
32	0.72	0.52	30	1.8	3.24
33	1.72	2.96	24	-4.2	17.64
33	1.72	2.96	27	-1.2	1.44
29	-2.28	5.20	29	0.8	0.64
34	2.72	7.40	-	-	-
$\sum x = 219$		$\sum (x - \overline{x})^2 = 21.2$	$\sum y = 169$		$\sum (y - \overline{y})^2 = 108$

Null Hypothesis : $H_0 : \sigma_X^2 = \sigma_Y^2$, Alternate Hypothesis : $H_1 : \sigma_X^2 \neq \sigma_Y^2$

$S_X^2 = \frac{\sum (x - \bar{x})^2}{n_1 - 1} = \frac{31.43}{7 - 1} = 5.238$, $S_Y^2 = \frac{\sum (y - \bar{y})^2}{n_2 - 1} = \frac{26.84}{6 - 1} = 5.368$, $F = \frac{S_Y^2}{S_X^2} = \frac{5.368}{5.238} = 1.02$,

degree of freedom $= (n_2 - 1, \ n_1 - 1) = (5, \ 6)$ at 5% LOS $= 4.39$.

Calculate value F $<$ Tabulated F. **H_0 is accepted**. *Conclusion:* The two horses have the same running capacity.

$\chi^2 - test$

The χ^2 distribution function is one of the most extensively used distribution functions in statistics.

Application (or uses) of χ^2 distribution
1. To test the goodness of fit. 2. To test the independence of attributes.
3. To test if the hypothetical value of the population variance is σ^2.
4. To test the homogeneity of independent estimates of the population variance.
5. To test the homogeneity of independent estimates of the population correlation coefficient.

Condition for validity of χ^2 – test
1. The sample observations should be independent.
2. Constraints on the cell frequencies, if any, should be linear, e.g., $\sum O_i = \sum E_i$
3. N, the total frequency should be reasonably large, say, greater than 50.
4. No theoretical cell frequency should be less than 5. (The chi square distribution is frequency is less than t) . If any theoretical cell frequency is less than 5, then for the application of χ^2 test, it is pooled with the preceding or succeeding frequency so that the pooled frequency is more than 5 and finally adjust for the degree of freedom lost in pooling.
5.

$\chi^2 - test$ of Goodness of fit

This is a powerful test for testing the significance of the discrepancy between theory and experiment – discovered by Karl Pearson in 1900. It helps us to find if the deviation of the experiment from theory is just by chance or it is due to the in adequacy of the theory to fit the observed data.

$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$, where O_i – set of observed frequencies, E_i- Set of expected frequencies. d.f. $= n - 1$

1. *The following table gives the number of aircraft accidents that occurred during the various days of the week. Test whether the accidents are uniformly distributed over the week.*

Days	Mon	Tue	Wed	Thu	Fri	Sat
No. of Accidents	14	18	12	11	15	14

Solution: H_0 : Accidents occur uniformly over the week. The expected no. of accidents on any day $= \frac{83}{6} = 14$.

O	E	$O-E$	$(O-E)^2$	$\frac{(O-E)^2}{E}$
14	14	0	0	0
18	14	4	16	1.143
12	14	-2	4	0.286
11	14	-3	9	0.643
15	14	1	1	0.071
14	14	0	0	0
				2.143

$\chi^2 = \sum_{i=1}^n \frac{(O_i - E_i)^2}{E_i} = 2.143$, d.f. $= n - 1 = 5$ at 5% LOS $= 11.07$, Calculated $\chi^2 <$ tabulated χ^2,

H_0 **is accepted.** The accidents are uniformly distributed over the week.

2. *The theory predicts the proportion of beans, in the four groups A, B, C, and D should be 9 : 3 : 3 : 1. In an experiment with 1600 beans the numbers in the four groups were 882, 313, 287, and 118. Does the experimental result support the theory.*

Solution: H_0 : The experimental result support the theory.

O	E	$O-E$	$(O-E)^2$	$\frac{(O-E)^2}{E}$
882	$\frac{9}{16} \times 1600 = 900$	-18	324	0.36
313	$\frac{3}{16} \times 1600 = 300$	13	169	0.563
287	$\frac{3}{16} \times 1600 = 300$	-13	169	0.563
118	$\frac{1}{16} \times 1600 = 100$	18	324	3.24
				4.726

$\chi^2 = \sum_{i=1}^n \frac{(O_i - E_i)^2}{E_i} = 4.726$, d.f. $= n - 1 = 4 - 1 = 3$ at 5% LOS $= 7.81$

Calculated $\chi^2 <$ tabulated χ^2, H_0 **is accepted.** The experimental results support the theory.

3. *Fit a binomial distribution for the following data and also test the goodness of fit. Find the parameters of the distribution.*

x	0	1	2	3	4	5	6	**Total**
f	**5**	**18**	**28**	**12**	**7**	**6**	**4**	**80**

Solution: To find the binomial frequency distribution $N(q + p)^n$, which fits the given data, we require N, n and p.

We assume N = total frequency = 80 and n = no. of trials = 6 from the given data.

x	0	1	2	3	4	5	6	Total
f	5	18	28	12	7	6	4	80
fx	0	18	56	36	28	30	24	192

$\bar{x} = \frac{\sum fx}{\sum f} = \frac{192}{80} = 2.4$, $np = 2.4 \Rightarrow 6p = 2.4$, $p = 0.4$, $q = 0.6$, N $P(X = x) = N\, nC_x p^x q^{n-x}$, $x = 0,1,\dots n$

$80\, P(X = 0) = 80 \times 6C_0 (0.4)^0 (0.6)^{6-0} = 3.73$, $80\, P(X = 1) = 80 \times 6C_1 (0.4)^1 (0.6)^{6-1} = 14.93$

$80\, P(X = 2) = 80 \times 6C_2 (0.4)^2 (0.6)^{6-2} = 24.88$, $80\, P(X = 3) = 80 \times 6C_3 (0.4)^3 (0.6)^{6-3} = 22.12$

$80\, P(X = 4) = 80 \times 6C_4 (0.4)^4 (0.6)^{6-4} = 11.06$, $80\, P(X = 5) = 80 \times 6C_5 (0.4)^5 (0.6)^{6-5} = 2.95$

$80\, P(X = 6) = 80 \times 6C_6 (0.4)^6 (0.6)^{6-6} = 0.33$

X	0	1	2	3	4	5	6	**Total**
E_i	4	15	25	22	11	3	0	**80**

The 1ˢᵗ class is combined with the second and the last 2 classes are combined with the last but 2ⁿᵈ class in order to make the expected frequency in each class ≥ 5. Thus, after regrouping, we have

O_i	23	28	12	17
E_i	19	25	22	14

H_0 : The given distribution is approximately binomial distribution.

O	E	$O - E$	$(O - E)^2$	$\frac{(O-E)^2}{E}$
23	19	4	16	0.8421
28	25	3	9	0.36
12	22	-10	100	4.5455
17	14	3	9	0.6429
				6.39

$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i} = \mathbf{6.39}$, d.f. $= n - k = 4 - 2 = 2$ at 5% LOS = 5.99,

Calculated χ^2 > tabulated χ^2, H_0 **is rejected.** The binomial fit for the given distribution is not satisfactory.

4. _Fit a Poisson distribution for the following distribution and also test the goodness of fit._

x	0	1	2	3	4	5	_Total_
f	142	156	69	27	5	1	400

Solution : $P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}$, $x = 0, 1, \dots \infty$; $\lambda = \bar{x} = \frac{\sum fx}{\sum f} = \frac{400}{400} = 1$, $N = 400$

x	0	1	2	3	4	5	Total
f	142	156	69	27	5	1	400
fx	0	156	138	81	20	5	400

Theoretical frequencies are given by $N P(X = x) = \frac{N e^{-\lambda} \lambda^x}{x!} = \frac{400 e^{-1} 1^x}{x!}$, $x = 0, 1, \dots \infty$

$400 P(X = 0) = \frac{400 e^{-1} 1^0}{0!} = 147.15$, $400 P(X = 1) = \frac{400 e^{-1} 1^1}{1!} = 147.15$

$400 P(X = 2) = \frac{400 e^{-1} 1^2}{2!} = 73.58$, $400 P(X = 3) = \frac{400 e^{-1} 1^3}{3!} = 24.53$

$400 P(X = 4) = \frac{400 e^{-1} 1^4}{4!} = 6.13$, $400 P(X = 5) = \frac{400 e^{-1} 1^5}{5!} = 1.23$

O_i	142	156	69	27	5	1
E_i	147	147	74	25	6	1

The last 3 classes are combined into one, so that the expected frequency in that class may be ≥ 10.

O_i	142	156	69	33
E_i	147	147	74	32

O	E	$O - E$	$(O - E)^2$	$\frac{(O-E)^2}{E}$
142	147	-5	25	0.17
156	147	9	81	0.55
69	74	-5	25	0.34
33	32	1	1	0.03
				1.09

$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i} = \mathbf{1.09}$, d.f. $= n - k = 4 - 2 = 2$ at 5% LOS = 5.99

Calculated χ^2 > tabulated χ^2, H_0 **is accepted.** The Poisson fit for the given distribution is satisfactory.

$\chi^2 - $ _test of Independence of Attributes_

Literally, an attribute means a quality or characteristic. E.g.: drinking, smoking, blindness, honesty, etc.

a	b	$(a + b)$
c	d	$(c + d)$
$(a + c)$	$(b + d)$	N

$E(a) = \frac{(a + c)(a + b)}{N}$	$E(b) = \frac{(b + d)(a + b)}{N}$	$(a + b)$
$E(c) = \frac{(a + c)(c + d)}{N}$	$E(d) = \frac{(b + d)(c + d)}{N}$	$(c + d)$
$(a + c)$	$(b + d)$	N

Degree of freedom $= (r - 1)(c - 1)$, where $r - $ number of rows, $c - $ number of colums.

5. **On the basis of information given below about the treatment of 200 patients suffering from a disease, state whether the new treatment is comparatively superior to the conventional treatment.**

	Favourable	Not Favourable	Total
New	60	30	90
Conventional	40	70	110
	100	100	200

Solution: H_0 : New and conventional treatment are independent.

$E(60) = \dfrac{90 \times 100}{200} = 45$	$E(30) = \dfrac{90 \times 100}{200} = 45$	90
$E(40) = \dfrac{110 \times 100}{200} = 55$	$E(70) = \dfrac{110 \times 100}{200} = 55$	110
100	100	200

O	E	$O - E$	$(O - E)^2$	$\dfrac{(O-E)^2}{E}$
60	45	15	225	5
30	45	-15	225	5
40	55	-15	225	4.09
70	55	15	225	4.09
				18.18

$\chi^2 = \sum_{i=1}^{n} \dfrac{(O_i - E_i)^2}{E_i} = \mathbf{18.18}$, d.f. $= (r-1)(c-1) = (2-1)(2-1) = 1$ at 5% LOS $= 3.841$.

Calculated $\chi^2 >$ tabulated χ^2, H_0 **is rejected.** New and conventional treatment are not independent.

6. **Given the following contingency table for hair colour and eye colour. Find the value of chi square. Is there good association between the two.**

		Hair Colour			Total
		Fair	Brown	Black	
Eye Colour	Blue	15	5	20	40
	Grey	20	10	20	50
	Brown	25	15	20	60
	Total	60	30	60	150

Solution: H_0 : The two attributes Hair colour and eye colour are independent.

O	E	$O - E$	$(O - E)^2$	$\dfrac{(O-E)^2}{E}$
15	16	-1	1	0.0625
5	8	-3	9	1.125
20	16	4	16	1
20	20	0	0	0
10	10	0	0	0
20	20	0	0	0
25	24	1	1	0.042
15	12	3	9	0.75
20	24	-4	16	0.666
				3.6458

$E(15) = \dfrac{60 \times 40}{150} = 16$	$E(5) = \dfrac{30 \times 40}{150} = 8$	$E(20) = \dfrac{60 \times 40}{150} = 16$	40
$E(20) = \dfrac{60 \times 50}{150} = 20$	$E(10) = \dfrac{30 \times 50}{150} = 10$	$E(20) = \dfrac{60 \times 50}{150} = 20$	50
$E(25) = \dfrac{60 \times 60}{150} = 24$	$E(15) = \dfrac{30 \times 60}{150} = 12$	$E(20) = \dfrac{60 \times 60}{150} = 24$	60
60	30	60	150

$\chi^2 = \sum_{i=1}^{n} \dfrac{(O_i - E_i)^2}{E_i} = \mathbf{3.6458}$, d.f. $= (r-1)(c-1) = (3-1)(3-1) = 4$ at 5% LOS $= 9.488$.

Calculated $\chi^2 <$ tabulated χ^2, H_0 **is accepted.** The hair colour and eye colour are independent.

F test Table

5% and 1% points of F.

v_2 \ v_1	1	2	3	4	5	6	8	12	24	∞
2	18.51	19.00	19.16	19.25	19.30	19.32	19.37	19.41	19.45	19.50
	98.49	99.00	99.17	99.25	99.30	99.33	99.36	99.42	99.46	99.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.84	8.74	8.64	8.53
	34.12	30.82	29.46	28.71	28.24	27.91	27.49	27.05	26.60	26.12
4	7.71	6.94	6.59	6.39	6.26	6.16	6.04	5.91	5.77	5.63
	21.20	18.00	16.69	15.98	15.52	15.21	14.80	14.37	13.93	13.46
5	6.61	5.79	5.41	5.19	5.05	4.95	4.82	4.68	4.53	4.36
	16.26	13.27	12.06	11.39	10.97	10.67	10.27	9.89	9.47	9.02
6	5.99	5.14	4.76	4.53	4.39	4.28	4.15	4.00	3.84	3.67
	13.74	10.92	9.78	9.15	8.75	8.47	8.10	7.72	7.31	6.88
7	5.59	4.74	4.35	4.12	3.97	3.87	3.73	3.57	3.41	3.23
	12.25	9.55	8.45	7.85	7.46	7.19	6.84	6.47	6.07	5.65
8	5.32	4.46	4.07	3.84	3.69	3.58	3.44	3.28	3.12	2.93
	11.26	8.65	7.59	7.01	6.63	6.37	6.03	5.67	5.28	4.86
9	5.12	4.26	3.86	3.63	3.48	3.37	3.23	3.07	2.90	2.71
	10.56	8.02	6.99	6.42	6.06	5.80	5.47	5.11	4.73	4.31
10	4.96	4.10	3.71	3.48	3.33	3.22	3.07	2.91	2.74	2.54
	10.04	7.56	6.55	5.99	5.64	5.39	5.06	4.71	4.33	3.91
12	4.75	3.88	3.49	3.26	3.11	3.00	2.85	2.69	2.50	2.30
	9.33	6.93	5.95	5.41	5.06	4.82	4.50	4.16	3.78	3.36
14	4.60	3.74	3.34	3.11	2.96	2.85	2.70	2.53	2.35	2.13
	8.86	6.51	5.56	5.03	4.69	4.46	4.14	3.80	3.43	3.00
16	4.49	3.63	3.24	3.01	2.85	2.74	2.59	2.42	2.24	2.01
	8.53	6.23	5.29	4.77	4.44	4.20	3.89	3.55	3.18	2.75
18	4.41	3.55	3.16	2.93	2.77	2.66	2.51	2.34	2.15	1.92
	8.28	6.01	5.09	4.58	4.25	4.01	3.71	3.37	3.01	2.57
20	4.35	3.49	3.10	2.87	2.71	2.60	2.45	2.28	2.08	1.84
	8.10	5.85	4.94	4.43	4.10	3.87	3.56	3.23	2.86	2.42
25	4.24	3.38	2.99	2.76	2.60	2.49	2.34	2.16	1.96	1.71
	7.77	5.57	4.68	4.18	3.86	3.63	3.32	2.99	2.62	2.17
30	4.17	3.32	2.92	2.69	2.53	2.42	2.27	2.09	1.89	1.62
	7.56	5.39	4.51	4.02	3.70	3.47	3.17	2.84	2.47	2.01
40	4.08	3.23	2.84	2.61	2.45	2.34	2.18	2.00	1.79	1.51
	7.31	5.18	4.31	3.83	3.51	3.29	2.99	2.66	2.29	1.81
60	4.00	3.15	2.76	2.52	2.37	2.25	2.10	1.92	1.70	1.39
	7.08	4.98	4.13	3.65	3.34	3.12	2.82	2.50	2.12	1.60

Regards!
Dr. P. Godhanda Raman, *Assistant Professor(Senior Grade), Department of Mathematics, SRM IST, Kattankulathur*
Email : godhanda@gmail.com, Mobile : 9941740168

Dr. P. Godhandaraman, Assistant Professor (Sr.G.), Dept of Maths, SRM IST, Kattankulathur, Mobile: 9941740168, Email: godhanda@gmail.com

Chi-Square Test Table

Values of χ^2 with probability P and df v

P \ v	0.99	0.95	0.50	0.30	0.20	0.10	0.05	0.01
1	0.0002	0.004	0.46	1.07	1.64	2.71	3.84	6.64
2	0.020	0.103	1.39	2.41	3.22	4.60	5.99	9.21
3	0.115	0.35	2.37	3.66	4.64	6.25	7.82	11.34
4	0.30	0.71	3.36	4.88	5.99	7.78	9.49	13.28
5	0.55	1.14	4.35	6.06	7.29	9.24	11.07	15.09
6	0.87	1.64	5.35	7.23	8.56	10.64	12.59	16.81
7	1.24	2.17	6.35	8.33	9.80	12.02	14.07	18.48
8	1.65	2.73	7.34	9.52	11.03	13.36	15.51	20.09
9	2.09	3.32	8.34	10.66	12.24	14.68	16.92	21.67
10	2.56	3.94	9.34	11.78	13.44	15.99	18.31	23.21
11	3.05	4.58	10.34	12.90	14.63	17.28	19.68	24.72
12	3.57	5.23	11.34	14.01	15.81	18.55	21.03	26.22
13	4.11	5.89	12.34	15.12	16.98	19.81	22.36	27.69
14	4.66	6.57	13.34	16.22	18.15	21.06	23.68	29.14
15	5.23	7.26	14.34	17.32	19.31	22.31	25.00	30.58
16	5.81	7.96	15.34	18.42	20.46	23.54	26.30	32.00
17	6.41	8.67	16.34	19.51	21.62	24.77	27.59	33.41
18	7.02	9.39	17.34	20.60	22.76	25.99	28.87	34.80
19	7.63	10.12	18.34	21.69	23.90	27.20	30.14	36.19
20	8.26	10.85	19.34	22.78	25.04	28.41	31.41	37.57
21	8.90	11.59	20.34	23.86	26.17	29.62	32.67	38.93
22	9.54	12.34	21.34	24.94	27.30	30.81	33.92	40.29
23	10.20	13.09	22.34	26.02	28.43	32.01	35.17	41.64
24	10.86	13.85	23.34	27.10	29.55	33.20	36.42	42.98
25	11.52	14.61	24.34	28.17	30.68	34.68	37.65	44.31
26	12.20	15.38	25.34	29.25	31.80	35.56	38.88	45.64
27	12.88	16.15	26.34	30.32	32.91	36.74	40.11	46.96
28	13.56	16.93	27.34	31.39	34.03	37.92	41.34	48.28
29	14.26	17.71	28.34	32.46	35.14	39.09	42.56	49.59
30	14.95	18.49	29.34	33.53	36.25	40.26	43.77	50.89

Regards!
Dr. P. Godhanda Raman, *Assistant Professor(Senior Grade), Department of Mathematics, SRM IST, Kattankulathur*
Email : godhanda@gmail.com, Mobile : 9941740168

Dr. P. Godhandaraman, Assistant Professor (Sr.G.), Dept of Maths, SRM IST, Kattankulathur, Mobile: 9941740168, Email: godhanda@gmail.com

QUEUEING THEORY

INTRODUCTION

History : A.K.Erlang (1909) – "The Theory of probabilities and telephone conversations".

All of us have experienced the annoyance of having to wait in line.

Example: 1. We wait in line in our cars in traffic jams. 2. We wait in line at supermarket to check out.

Why then is there waiting? There is more demand for service that there is facility for service available.

Why is this so? 1. There may be a shortage of available servers. 2. There may be a space limit to the amount of service that can be provided.

Question : 1. How long must a customer wait?   2. How many people will form in the line?

Answer: Queuing theory attempts to answer these questions through detailed mathematical analysis.

Customer: The term 'Customer' is used in a general sense and does not imply necessarily a human customer.

E.g.: 1. An Air plane waiting in line to take off. 2. A Computer program waiting to be run as a time shared basis.

Characteristics of Queuing Process

1. Arrival pattern of Customers 2. Service pattern of Servers 3. Queue discipline 4. System capacity

5. Number of service channels 6. Number of service stages

1. *Arrival Pattern of Customer* : (i) Bulk or Batches (ii) Balked (iii) Reneged (iv) Jockey

  (i) *Bulk or Batch**es :** More than one arrival can be entering the system simultaneously, the input is said to occur in bulk or batches.

  (ii) *Balked* : If customer decides not to enter the queue upon arrival, he is said to have balked.

  (iii) *Reneged :* A Customer may enter the queue, but after a time lose patience and decide to leave. In this case he is said to reneged.

  (iv) *Jockey :* Two or more parallel waiting lines, customers may switch from one to another.

2. *Service Pattern of Services :* If the system is empty, the service facility is idle. Service may also be deterministic (or) probabilistic. Service may also be single (or) batch one generally thinks of one customer being served at a time by a given server, but there are many situations where customer may be served simultaneously by the same server. **E.g.:** 1. Computer with parallel processing. 2. People boarding a train.

   The service rate may depend on the number of customer waiting for service. A server may work faster if sees that the queue is building up (or) conversely, he may get flustered and became less efficient. The situation in which service depends on the no. of customers waiting is referred to as state dependent service.

3. *Queue Discipline :*

  (i) First Come First Served (FCFS) or First In First Out (FIFO)

  (ii) Last Come First Served (LCFS) or Last In First Out (LCFO)

  (iii) Random Selection for Services (RSS) (iv) Priority (a) Preemptive (b) Non- Preemptive

(a) Preemptive : The customer with the highest priority is allowed to enter service immediately even if a customer with lower priority is already in service when the higher priority customer enters system.

(b) Non - preemptive : The highest priority customer goes to the head the queue but cannot get into service until the customer presently in service is completely, even through this customer has a lower priority.

4. *System Capacity :* (i) Finite (ii) Infinite

  (i) Finite: A queue with limited waiting room, so that when the time reaches a certain length, no further customer are allowed to enter until space becomes available by a service completion.
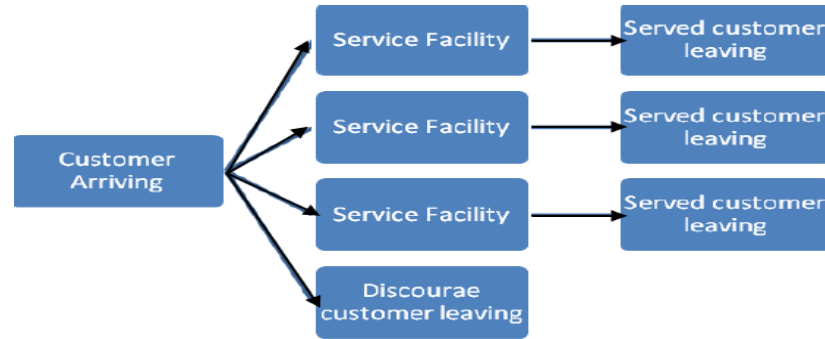
  (ii) Infinite: A queue with unlimited waiting room.

5. *Number of Service Channels* : (i) Single channel system (ii) Multiple channel systems.

  (i) *Single Channel System*

(ii) Multiple Channel System

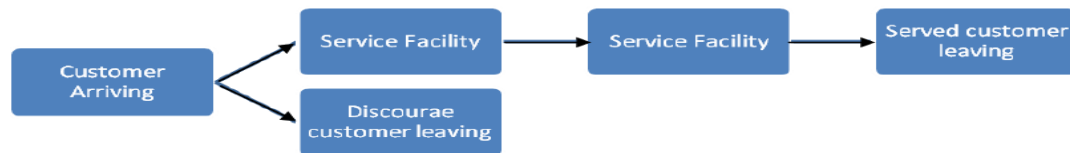**Eg:** 1. Barber Shop 2. Supermarket 3. Ticket Counters.

6. **Number of Service Stages :** (i) Single stage (ii) Multiple stage

   (i) Single Stage

   Eg: 1. Barber Shop 2. Supermarket 3. Theater.

   (ii) Multiple Stage

   Eg: 1. Medical History 2. Bank A/c opening 3.Canteen

Kendall Notation (A/B/X/Y/Z)

A - Inter arrival time   :   M - Exponential, D - Deterministic, Ek - Erlang type k,
                                      Hk – Hyper exponential type k, Ph - phase type, G – General.

B - Service time   :   M - Exponential, D - Deterministic, Ek - Erlang type k,
                                      Hk - Hyper exponential type k, Ph – phase type, G – General.

X - No. of parallel servers   : $1, 2 \ldots \infty$

Y - System capacity   : $1, 2 \ldots \infty$

Z - Queue discipline   :   FCFS, LCFS, RSS, PR, GD

Queuing Models : 1. Probabilistic or stochastic models 2. Deterministic models 3. Mixed models

Probabilistic model: When there is uncertainty in both arrival rate and service rate (i.e. not treated a customer or not know) and are assumed to be random variables.

Deterministic model: Both arrival rate and service rate are constants (exactly known).

Mixed model: When either the arrival rate or the service rate is exactly known and the other is not known.

We look for probabilistic model only

1. $(M/M/1) : (\infty/FIFO)$

2. $(M/M/1) : (K /FIFO)$

SINGLE SERVER MODEL WITH INFINITE SYSTEM CAPACITY - $(M/M/1) : (\infty/FIFO)$

1. $P_0 = 1 - \frac{\lambda}{\mu}$ and $P_n = \left(\frac{\lambda}{\mu}\right)^n P_0$, λ = arrival rate, μ = service rate

2. Probability that the system is busy $= 1 - P_0 = \frac{\lambda}{\mu}$

3. Expected number of customers in the system : $L_s = \frac{\lambda}{\mu-\lambda}$

4. Expected number of customers in the queue: $L_q = \frac{\lambda^2}{\mu(\mu-\lambda)}$

5. Expected number of customers in non empty queues: $L_n = \frac{\mu}{\mu-\lambda}$

6. Expected waiting time a customer in the system: $W_s = \frac{1}{\mu - \lambda}$

7. Average waiting time that a customer in the queue: $W_q = \frac{\lambda}{\mu(\mu - \lambda)}$

8. Probability that the number of customers in the system exceeds k : $P(n > k) = \left(\frac{\lambda}{\mu}\right)^{k+1}$

9. Probability that the number of customers in the system greater than or equal to k : $P(n \geq k) = \left(\frac{\lambda}{\mu}\right)^{k}$

10. Probability that the waiting time of a customer in the system exceeds t : $P(w > t) = e^{-(\mu - \lambda)t}$

11. Probability that the waiting time of a customer in the queue exceeds t : $P(w > t) = \frac{\lambda}{\mu} e^{-(\mu - \lambda)t}$

12. Probability density function of the waiting time in the system: $f(w) = (\mu - \lambda)e^{-(\mu - \lambda)w}$
    Which is the probability density function of an exponential distribution with parameter $\mu - \lambda$.

13. Probability density function of the waiting time in the queue: $g(w) = \begin{cases} \frac{\lambda}{\mu}(\mu - \lambda)e^{-(\mu - \lambda)w}, & w > 0 \\ 1 - \frac{\lambda}{\mu}, & w = 0 \end{cases}$

Little's formula: $L_s = \lambda W_s$, $L_q = \lambda W_q$, $W_s = W_q + \frac{1}{\mu}$, $L_s = L_q + \frac{\lambda}{\mu}$

SINGLE SERVER MODEL WITH INFINITE SYSTEM CAPACITY - $(M/M/1) : (\infty/FIFO)$

1. **Customers arrive at a one man barber shop according to a Poisson process with a mean inter arrival time of 20 minutes. Customers spend an average of 15 minutes in the barber chair. If an hour is used as the unit of time,**
   (i) What is the probability that a customer need not wait for a hair cut?
   (ii) What is the expected number of customers in the barber shop and in the queue?
   (iii) How much time can a customer expect to spend in the barbershop?
   (iv) Find the average time that a customer spends in the queue?
   (v) What is the probability that there will be more than 6 customers?
   (vi) What is the probability that there will be 6 or more customers waiting for service?
   (vii) What is the probability that the waiting time in the (a) system (b) queue, is greater than 12 minutes?

   Solution: $\frac{1}{\lambda} = \frac{20}{30} = \frac{1}{3} \Rightarrow \lambda = 3$ customers/hour, $\frac{1}{\mu} = \frac{15}{60} = \frac{1}{4} \Rightarrow \mu = 4$ customers/hour.

   (i) $P(a\ customer\ need\ not\ wait) = P(no\ customer\ in\ the\ system)$: $P_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{3}{4} = 0.25$

   (ii) Expected number of customers in the barber shop : $L_s = \frac{\lambda}{\mu - \lambda} = \frac{3}{4-3} = 3$ customers

   Expected number of customers in the queue: $L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{3^2}{4(4-3)} = \frac{9}{4} = 2.25$ customers

   (iii) Expected time a customer spend in the barbershop: $W_s = \frac{1}{\mu - \lambda} = \frac{1}{4-3} = 1$ hour

   (iv) Average time that a customer spends in the queue: $W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{3}{4(4-3)} = \frac{3}{4} = 0.75$ hour

   (v) The probability that there will be more than 6 customers:
   $P(n > k) = \left(\frac{\lambda}{\mu}\right)^{k+1} \Rightarrow P(n > 6) = \left(\frac{3}{4}\right)^{6+1} = 0.1335$

   (vi) The probability that there will be 6 or more customers waiting for service:
   $P(n \geq k) = \left(\frac{\lambda}{\mu}\right)^{k} \Rightarrow P(n \geq 6) = \left(\frac{3}{4}\right)^{6} = 0.1779$

   (vii) The probability that the waiting time in the system is greater than 12 minutes?
   $P(w > t) = e^{-(\mu - \lambda)t} \Rightarrow P(w > 12) = e^{-(4-3) \times \frac{12}{60}} = e^{-0.2} = 0.8187$

   (viii) The probability that the waiting time in the queue, is greater than 12 minutes?
   $P(w > t) = \frac{\lambda}{\mu} e^{-(\mu - \lambda)t} \Rightarrow P(w > 12) = \left(\frac{3}{4}\right)e^{-(4-3) \times \frac{12}{60}} = \left(\frac{3}{4}\right)e^{-0.2} = 0.61405$

2. **If People arrive to purchase cinema tickets at the average rate of 6 per minute at a one man counter, and it takes an average of 7.5 seconds to purchase a ticket. If a person arrives 2 minutes before the picture starts and it takes exactly 1.5 minutes to reach the correct seat after purchasing the ticket,**
   (i) can he expect to be seated for the start of the picture?
   (ii) What is the probability that he will be seated for the start of the picture?
   (iii) How early must he arrive in order to be 99% sure of being seated for the start of the picture?

   Solution: $\lambda = 6$ /minute, $\mu = 8$ /minute

(i) $W_s = \frac{1}{\mu - \lambda} = \frac{1}{8-6} = \frac{1}{2} = 0.5$ minute

E(total time required to purchase the ticket and to reach the seat) = $0.5 + 1.5 = 2$ min

(ii) P(total time < 2 minute) $= P(w < t) = 1 - P(w > t) = 1 - e^{-(\mu - \lambda)t}$

$P\left(w < \frac{1}{2}\right) = 1 - e^{-(8-6)\times\frac{1}{2}} = 1 - e^{-1} = 0.63$

(iii) $P(w < t) = 99\% = 0.99 \Rightarrow 1 - P(w > t) = 0.99 \Rightarrow P(w > t) = 0.01 \Rightarrow e^{-(\mu - \lambda)t} = 0.01 \Rightarrow e^{-(8-6)\,t} = 0.01 \Rightarrow e^{-2\,t} = 0.01 \Rightarrow -2t = \ln(0.01) \Rightarrow -2t = -4.6 \Rightarrow t = 2.3$ minute

P(ticket purchasing time < 2.3) $= 0.99$

P[total time to get the ticket and to go to the seat $< (2.3 + 1.5)$] $= 0.99$

∴ The person must arrive at least 2.64 minutes early so as to be 99% sure of seeing the start of the picture.

3. *The arrivals at the counter in a bank occur in accordance with a Poisson process at an average rate of 8 per hour. The duration of service of a customer has an exponential distribution with a mean of 6 minutes. Find the probability that an arriving customer (i) Has to wait (ii) Finds 4 customers in the system (iii) Has to spend less than 15 minutes in the bank.*

Solution: $\lambda = 8$ / hour, $\mu = \frac{1}{6}$/minute = 10/hour

(i) Probability that a customer has to wait = Probability that the system is busy $= \frac{\lambda}{\mu} = \frac{8}{10} = 0.8$

(ii) Probability that there are 4 customers in the system $= P_4 = \left(\frac{8}{10}\right)^4 \left(1 - \frac{8}{10}\right) = 0.08192$

(iii) Probability that a customer has to spend less than 15 minutes in the bank: $P(w > t) = e^{-(\mu - \lambda)t}$

$P(W_s < 15 \text{ minutes}) = P\left(W_s < \frac{1}{4} \text{ hour}\right) = 1 - P\left(W_s > \frac{1}{4} \text{ hour}\right) = 1 - e^{-(10-8)\left(\frac{1}{4}\right)} = 0.3935$

4. *Customers arrive at a watch repair shop according to a Poisson process at a rate of one per every 10 minutes and the service time is an exponential random variable with mean 8 minutes.*
  (i) Find the average number of customers in the shop.
  (ii) Find the average time a customer spends in the shop.
  (iii) Find the average number of customers in the queue.
  (iv) What is the probability that the server is idle?

Solution: $\lambda = 6$ /hour, $\mu = \frac{60}{8} = \frac{15}{2}$ /hour.

(i) The average number of customers in the shop : $L_s = \frac{\lambda}{\mu - \lambda} = \frac{6}{\frac{15}{2} - 6} = 4$ customers

(ii) The average time a customer spends in the shop : $W_s = \frac{1}{\mu - \lambda} = \frac{1}{\frac{15}{2} - 6} = \frac{2}{3}$ hour

(iii) The average number of customers in the queue: $L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{6^2}{\frac{15}{2}\left(\frac{15}{2} - 6\right)} = \frac{16}{5}$

(iv) P(system is empty): $P_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{6}{\left(\frac{15}{2}\right)} = \frac{1}{5}$

5. *A departmental store has a single cashier. During the rush hours, customers arrive at the rate of 20 customers per hour. The average no. of customers that can be processed by the cashier is 24 per hour. Calculate the following*
  (i) What is the probability that the cashier is idle?
  (ii) What is the average number of customers in the queueing system.
  (iii) What is the average time a customer spends in the system.
  (iv) What is the average number of customers in the queue?
  (v) What is the average time a customer spends in the queue, waiting for service?

Solution: $\lambda = 20$ /hour, $\mu = 24$ /hour.

(i) P(cashier is idle) $= P_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{20}{24} = 0.1674$

(ii) Average number of customers in the queueing system : $L_s = \frac{\lambda}{\mu - \lambda} = \frac{20}{24 - 20} = 5$ customers

(iii) Average time a customer spend in the system: $W_s = \frac{1}{\mu - \lambda} = \frac{1}{24 - 20} = \frac{1}{4}$ hour

(iv) Average number of customers in the queueing system : $L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{20^2}{24(24 - 20)} = 4.167$ customers

(v) Average time that a customer spends in the queue: $W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{20}{24(24 - 20)} = \frac{20}{96}$ hour

$$P_0 = \begin{cases} \dfrac{\left(1-\frac{\lambda}{\mu}\right)}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}, & \lambda \neq \mu \\ \dfrac{1}{k+1}, & \lambda = \mu \end{cases} \quad and \quad P_n = \begin{cases} \left(\frac{\lambda}{\mu}\right)^n P_0, & \lambda \neq \mu \\ \dfrac{1}{k+1}, & \lambda = \mu \end{cases}$$

Average number of customers in the system : $L_s = \begin{cases} \left(\dfrac{\lambda}{\mu-\lambda}\right) - \dfrac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}, & \lambda \neq \mu \\ \dfrac{k}{2}, & \lambda = \mu \end{cases}$

Average number of customers in the queue $:$ $L_q = L_s - \dfrac{\lambda'}{\mu}$

Effective arrival rate $:$ $\lambda' = \mu(1 - P_0)$

Average waiting time of a customers in the system $:$ $W_s = \dfrac{L_s}{\lambda'}$

Average waiting time of a customers in the queue $:$ $W_q = \dfrac{L_q}{\lambda'}$

SINGLE SERVER MODEL WITH FINITE SYSTEM CAPACITY - $(M/M/1) : (k/FIFO)$

1. **_Patients arrive at a clinic according to Poisson distribution at a rate of 60 patients per hour. The waiting room does not accommodate more than 14 patients. Investigation time per patient is exponential with mean rate of 40 per hour._**
   (i) Determine the effective arrival rate at the clinic.
   (ii) What is the probability that an arriving patient will not wait?
   (iii) What is the expected waiting time until a patient is discharged from the clinic?
   Solution: $\lambda = 60$ patients/hr, $\mu = 40$ patients/hr, $k = 14 + 1 = 15$ (14 waiting patients + 1 patient under investigation)

   (i) Effective arrival rate $\lambda' = \mu(1 - P_0)$, Where $P_0 = \dfrac{\left(1-\frac{\lambda}{\mu}\right)}{\left[1-\left(\frac{\lambda}{\mu}\right)^{k+1}\right]}$, $\lambda \neq \mu$

   $P_0 = \dfrac{\left(1-\frac{60}{40}\right)}{1-\left(\frac{60}{40}\right)^{15+1}} = 0.0007624$, $\quad \lambda' = 40(1 - 0.0007624) = 39.9695$ per hour.

   (ii) P(a patient will not wait) $= P_0 = 0.0007624$

   (iii) Expected waiting time until a patient is discharged from the clinic: $W_s = \dfrac{L_s}{\lambda'}$

   $L_s = \left(\dfrac{\lambda}{\mu-\lambda}\right) - \dfrac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}} = \left(\dfrac{60}{40-60}\right) - \dfrac{(15+1)\left(\frac{60}{40}\right)^{15+1}}{1-\left(\frac{60}{40}\right)^{15+1}} = 13$ patients, $\quad W_s = \dfrac{14}{39.9695} = 0.3203$ hour

2. **_At a railway station, only one train is handled at a time. The railway yard is sufficient only for two trains to wait while other is given signal to leave the station. Trains arrive at the station at an average rate of 6 per hour and the railway station can handle them on an average of 12 per hour. Assuming Poisson arrivals and exponential service distribution, find the steady state probabilities for the number of trains in the system. Also find the average waiting time of a new train coming into the yard. If the handling rate is reduced to half, what is the effect of the above results?_**
   Solution: Case (i): $\lambda = 6$ trains/hr, $\mu = 12$ trains/hr, $k = 2 + 1 = 3$

   Steady state probabilities for the number of trains in the system $= P_1, P_2$ and P_3

   $P_n = \left(\dfrac{\lambda}{\mu}\right)^n P_0$, Where $P_0 = \dfrac{\left(1-\frac{\lambda}{\mu}\right)}{\left[1-\left(\frac{\lambda}{\mu}\right)^{k+1}\right]} = \dfrac{\left(1-\frac{6}{12}\right)}{1-\left(\frac{6}{12}\right)^4} = 0.5333$, $\lambda \neq \mu$

   $P_1 = \left(\dfrac{6}{12}\right)^1 (0.5333) = 0.2667$, $P_2 = \left(\dfrac{6}{12}\right)^2 (0.5333) = 0.1333$, $P_3 = \left(\dfrac{6}{12}\right)^3 (0.5333) = 0.0667$

   Average waiting time of a new train coming in the yard : $W_s = \dfrac{L_s}{\lambda'}$

   $\lambda' = \mu(1 - P_0) = 12(1 - 0.5333) = 5.6004$

   $L_s = \left(\dfrac{\lambda}{\mu-\lambda}\right) - \dfrac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}} = \left(\dfrac{6}{12-6}\right) - \dfrac{(3+1)\left(\frac{6}{12}\right)^{3+1}}{1-\left(\frac{6}{12}\right)^{3+1}} = 0.7333$ train, $W_s = \dfrac{0.7333}{5.6004} = 0.1309$ hour

   Case (ii): If the handling rate is reduced to half, then $\lambda = 6$ trains/hr, $\mu = 6$ trains/hr, $k = 2 + 1 = 3$

Steady state probabilities for the number of trains in the system $= P_1, P_2$ and P_3

$P_n = \frac{1}{k+1}$, $P_1 = \frac{1}{3+1} = \frac{1}{4}$, $P_2 = \frac{1}{4}$, $P_3 = \frac{1}{4}$

Average waiting time of a new train coming in the yard : $W_s = \frac{L_s}{\lambda'}$

$P_0 = \frac{1}{k+1} = \frac{1}{4}$, $\lambda' = \mu(1 - P_0) = 6\left(1 - \frac{1}{4}\right) = 4.5$, $L_s = \frac{k}{2} = \frac{3}{2} = 1.5$ train, $W_s = \frac{1.5}{4.5} = 0.3333$ hours

3. **A petrol pump with only one pump can accommodate 5 cars. The arrival of cars is Poisson with a mean rate of 10 per hour. The service time is exponentially distributed with a mean 2 minutes. How many cars are in the petrol pump on an average? What is the probability of a newly arriving customer finding the system full and leaving without availing service?**

Solution: $\lambda = 10/$hr, $\mu = 30/$hr, $k = 5$, $P_0 = \frac{\left(1 - \frac{\lambda}{\mu}\right)}{\left[1 - \left(\frac{\lambda}{\mu}\right)^{k+1}\right]} = \frac{\left(1 - \frac{10}{30}\right)}{\left[1 - \left(\frac{10}{30}\right)^{5+1}\right]} = 0.667$, $\lambda \neq \mu$

$L_s = \left(\frac{\lambda}{\mu - \lambda}\right) - \frac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1 - \left(\frac{\lambda}{\mu}\right)^{k+1}} = \left(\frac{10}{30 - 10}\right) - \frac{(5+1)\left(\frac{10}{30}\right)^{5+1}}{1 - \left(\frac{10}{30}\right)^{5+1}} = 0.492$

$P(\text{System full}) = P(5 \text{ cars in the system}) = \left(\frac{\lambda}{\mu}\right)^5 P_0 = \left(\frac{10}{30}\right)^5 (0.667) = 0.00274.$

4. **A one person barber shop has 6 chairs to accommodate people waiting for a hair cut. Assume that customers who arrive when all the 6 chairs are full leave without entering the barber shop. Customers arrive at the average rate of 3 per hour and spend an average of 15 minutes in the barber chair.**
   (i) What is the probability that a customer can get directly into the barber chair upon arrival?
   (ii) What is the expected number of customers waiting for a hair cut?
   (iii) How much time can a customer expect to spend in the barber shop?
   (iv) What fraction of potential customers are turned away?
   Solution: $\lambda = 3/$hr, $\mu = 4/$hr, $k = 6 + 1 = 7$

$P_0 = \frac{\left(1 - \frac{\lambda}{\mu}\right)}{\left[1 - \left(\frac{\lambda}{\mu}\right)^{k+1}\right]} = \frac{\left(1 - \frac{3}{4}\right)}{\left[1 - \left(\frac{3}{4}\right)^{7+1}\right]} = 0.2778$, $\lambda \neq \mu$,

Effective arrival rate $\lambda' = \mu(1 - P_0) = 4(1 - 0.2778) = 2.89/hour$

(i) Probability of empty system $= P_0 = 0.2778$

(ii) $L_s = \left(\frac{\lambda}{\mu - \lambda}\right) - \frac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1 - \left(\frac{\lambda}{\mu}\right)^{k+1}} = \left(\frac{3}{4-3}\right) - \frac{(7+1)\left(\frac{3}{4}\right)^{7+1}}{1 - \left(\frac{3}{4}\right)^{7+1}} = 2.11$, $L_q = L_s - \frac{\lambda'}{\mu} = 2.11 - \frac{2.89}{4} = 1.3875$

(iv) $W_s = \frac{L_s}{\lambda'} = \frac{2.11}{2.89} = 0.7301/hour$

(v) $P(a \text{ customer is turned away}) = P(\text{system is full}) = P_7 = \left(\frac{\lambda}{\mu}\right)^7 P_0 = \left(\frac{3}{4}\right)^7 (0.2778) = 0.037$

   Hence 3.7% of potential customers are turned away.

All the Best

Regards!
Dr. P. GodhandaRaman
Assistant Professor (Senior Grade)
Department of Mathematics
SRM IST, Kattankulathur – 603 203
Email : godhanda@gmail.com, Mobile : 9941740168