Tuesday, October 5, 2021 3:30 PM

Cayley Hamilton theorem

Every square matrix satisfies its own characteristic equations

$$|\overset{\downarrow}{A}-\lambda I|=0$$

eg) $\underline{\lambda^3-3\lambda^2+4\lambda-5}=0$

$$\boxed{\lambda = A}$$

By cayley Ham the, $A^3-3A^2+4A-5=0$

① Powers of A

② A^{-1}

③ $\underline{8A^9+4A^7+7A^2+4}$

eg) 3×3 $\Rightarrow$ A given, A^2, A^3 cal | 4×4 $\Rightarrow$ A given, A^2, A^3, A^4

Problems

① Verify Cayley Hamilton
② A^{-1}
③ A^4
④ Any Combination of Powers

} 12 marks

① If the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

a) Verify Cayley Hamilton theorem

b) Find A^{-1} and A^4

c) Find the Value of $A^8-5A^7+7A^6-3A^5+A^4-5A^3-8A^2+2A-I$

Sol

Characteristic eqn $|A-\lambda I|=0$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$\underline{\lambda^3-5\lambda^2+7\lambda-3}=0$$

By Cayley Hamilton theorem,

✓ $\boxed{A^3-5A^2+7A-3=0}$

a) $$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 3 & 4 \\ 0 & 1 & 0 \\ 4 & 1 & 5 \end{bmatrix}$$

$$A^3 = A^2 . A = \begin{bmatrix} 5 & 3 & 4 \\ 0 & 1 & 0 \\ 4 & 1 & 5 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 8 & 13 \\ 0 & 1 & 0 \\ 13 & 5 & 14 \end{bmatrix}$$

$$\begin{bmatrix} 14 & 8 & 13 \\ 0 & 1 & 0 \\ 13 & 5 & 14 \end{bmatrix} - \begin{bmatrix} 25 & 15 & 20 \\ 0 & 5 & 0 \\ 20 & 5 & 25 \end{bmatrix} + \begin{bmatrix} 14 & 7 & 7 \\ 0 & 7 & 0 \\ 7 & 0 & 14 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence Verified

b) $A^3-5A^2+7A-3=0$

multiply A^{-1} $\Rightarrow$ $A^2-5A+7-3A^{-1}=0$ $\Rightarrow A^{-1} = \frac{1}{3}\left(A^2-5A+7\right)$

multiply A $\Rightarrow$ $A^4-5A^3+7A^2-3A = 0 \Rightarrow A^4 = 5A^3-7A^2+3A$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -2 & -1 \\ 0 & 3 & 0 \\ -1 & 1 & 2 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 38 & 19 & 37 \\ 0 & 1 & 0 \\ 37 & 18 & 38 \end{bmatrix}$$

c) $A^8-5A^7+7A^6-3A^5+A^4-5A^3-8A^2+2A-I$

$\boxed{A^3-5A^2+7A-3I=0}$

$$A^5(\underbrace{A^3-5A^2+7A-3}_{0})+A\left(A^3-5A^2-8A+2\right)-I$$

$$A\left(A^3-5A^2\underline{-8A+2}\right)+7A^2-7A^2+3A-3A-I$$

$$A(\underbrace{A^3-5A^2+7A-3}_{0})-8A^2-7A^2+2A+3A-I$$

$$\Rightarrow \underline{-15A^2+5A-I}$$

c) $$\begin{bmatrix} -66 & -40 & -55 \\ 0 & -11 & 0 \\ -55 & -15 & -66 \end{bmatrix}$$

Home work

① If $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

Find the Value of $A^5+5A^4-6A^3+2A^2-4A+7$