PROBABILITY
&
QUEUEING THEORY

(As per SRM UNIVERSITY Syllabus)

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UNIT – 5 : MARKOV CHAINS

Syllabus

- Introduction to Stochastic process, Markov process, Markov chain one step & n-step Transition Probability.
- Transition Probability Matrix and Applications
- Chapman Kolmogorov theorem (Statement only) – Applications.
- Classification of states of a Markov chain – Applications

INTRODUCTION

***Random Processes or Stochastic Processes* :** A random process is a collection of random variables $\{X(s,\ t)\}$ which are functions of a real variable t (time). Here $s \in S$ (sample space) and $t \in T$ (index set) and each $\{X(s,\ t)\}$ is a real valued function. The set of possible values of any individual member is called state space.

Classification : Random processes can be classified into 4 types depending on the continuous or discrete nature of the state space S and index set T.

1. Discrete random sequence : If both S and T are discrete
2. Discrete random process : If S is discrete and T is continuous
3. Continuous random sequence : If S is continuous and T is discrete
4. Continuous random process : If both S and T are continuous.

Markov Process : If, for $t_1 < t_2 < t_3 < \cdots < t_n$, we have $P\{X(t) \le x/X(t_1) = x_1, X(t_2) = x_2 \ldots X(t_n) = x_n\} = P\{X(t) \le x/X(t_n) = x_n\}$ then the process $\{X(t)\}$ is called a Markov process. That is, if the future behaviour of the process depends only on the present state and not on the past, then the random process is called a Markov process.

***Markov Chain* :** If, for all n, $P\{X_n = a_n/X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2} \ldots X_0 = a_0\} = P\{X_n = a_n/X_{n-1} = a_{n-1}\}$ then the process $\{X_n; n = 0,1,2 \ldots\}$ is called a Markov chain.

***One Step Transition Probability* :** The conditional probability $P_{ij}(n-1, n) = P\left(X_n = a_j/X_0 = a_i\right)$ is called one step transition probability from state a_i to state a_j in the n^{th} step.

***Homogeneous Markov Chain* :** If the one step transition probability does not depend on the step.
That is, $\boldsymbol{P_{ij}(n-1, n) = P_{ij}(m-1, m)}$ the Markov chain is called a homogeneous markov chain or the chain is said to have stationary transition probabilities**.**

***n - Step Transition Probability* :** $P_{ij}^{(n)} = P\left(X_n = a_j/X_0 = a_i\right)$

***Chapman Kolmogorov Equations* :** If P is the tpm of a homogeneous Markov chain, the n^{th} step tpm $P^{(n)}$ is equal to P^n. That is, $\left[P_{ij}^{(n)}\right] = \left[P_{ij}\right]^n$.

***Regular Matrix* :** A stochastic matrix P is said to be a regular matrix, if all the entries of P^m are positive. A homogeneous Markov chain is said to be regular if its tpm is regular.

Classification of States of a Markov Chain

***Irreducible Chain and Non - Irreducible (or) Reducible*:** If for every i, j we can find some n such that $P_{ij}^{(n)} > 0$, then every state can be reached from every other state, and the Markov chain is said to be irreducible. Otherwise the chain is non – irreducible or reducible.

***Return State* :** State i of a Markov chain is called a return state, if $P_{ij}^{(n)} > 0$ for some $n > 1$.

***Periodic State and Aperiodic State* :** The period d_i of a return state i is the greatest common divisor of all m such that $P_{ij}^{(n)} > 0$. That is, $d_i = GCD\left(m: P_{ii}^{(m)} > 0\right)$. State i is periodic with period d_i if $d_i > 1$ and aperiodic if $d_i = 1$.

***Recurrent (Persistent) State and Transient* :** If $\sum_{n=1}^{\infty} f_{ii}^{(n)} = 1$, the return to state i is certain and the state i is said to be persistent or recurrent. Otherwise, it is said to be transient.

***Null Persistent and Non – null Persistent State* :** $\mu_{ii} = \sum_{n=1}^{\infty} n\, f_{ii}^{(n)}$ is called the mean recurrence time of the state i. If μ_{ii} is finite, the state i is non null persistent. If $\mu_{ii} = \infty$ the state i is null persistent.

***Ergodic State*:** A non null persistent and aperiodic state are called ergodic.

Theorem used to classify states

1. If a Markov chain is irreducible, all its states are of the same type. They are all transient, all null persistent or all non null persistent. All its states are either aperiodic or periodic with the same period.
2. If a Markov chain is finite irreducible, all its states are non null persistent.

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PROBLEMS

6. ***A man either drives a car or catches a train to go to office each day. He never goes 2 days in a row by train but if he drives one day, then the next day he is just as likely to drive again as he is to travel by train. Now suppose that on the first day of the week, the man tossed a fair die and drove to work if and only if a 6 appeared. (i) Find the probability that he takes a train on the 3rd day (ii) Find the probability that he drives to work in the long run.***

Solution : Let T – Train and C - Car. If today he goes by train, next day he will not go by train.

$P(Train \to Train) = 0, P(Train \to Car) = 1, P(Car \to Train) = \frac{1}{2}, P(Car \to Car) = \frac{1}{2}$

$P = \begin{array}{c} \\ T \\ C \end{array}\begin{array}{c} \begin{array}{cc} T & C \end{array} \\ \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array}$. The first day of the week, the man tossed a fair die and drove to work if and only if a 6 appeared.

Initial state probability distribution is obtained by throwing a die.

Probability of going by car = $\frac{1}{6}$ and Probability of going by train = $1 - \frac{1}{6} = \frac{5}{6}$

The 1st day sate distribution is $P^{(1)} = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} \end{bmatrix}$

The 2nd day sate distribution is $P^{(2)} = P^{(1)}P = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{12} & \frac{11}{12} \end{bmatrix}$

The 3rd day sate distribution is $P^{(3)} = P^{(2)}P = \begin{bmatrix} \frac{1}{12} & \frac{11}{12} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{11}{24} & \frac{13}{24} \end{bmatrix}$

(i) *P(*he travels by train on 3rd day*)* = $\frac{11}{24}$

(ii) The limiting form or long run probability distribution. $\boldsymbol{\pi p = \pi}$

$$[\pi_1 \ \ \pi_2]\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = [\pi_1 \ \ \pi_2]$$

$$\frac{\pi_2}{2} = \pi_1 \Rightarrow \pi_2 = 2\,\pi_1 \quad (1)$$

$$\pi_1 + \frac{\pi_2}{2} = \pi_2 \Rightarrow \pi_2 = 2\,\pi_1 \quad (2)$$

$$\pi_1 + \pi_2 = 1 \quad (3)$$

Sub. (1) in (3), $\pi_1 + 2\,\pi_1 = 1 \Rightarrow \boldsymbol{\pi_1} = \frac{\mathbf{1}}{\mathbf{3}}$ (4)

Sub. (4) in (3), $\frac{1}{3} + \pi_2 = 1 \Rightarrow \boldsymbol{\pi_2} = \frac{\mathbf{2}}{\mathbf{3}}$ **,** P(driving in the long run) = $\frac{2}{3}$.

7. ***A college student X has the following study habits. If he studies one night, he is 70% sure not to study the next night. If he does not study one night, he is only 60% sure not to study the next night also. Find (i) the transition probability matrix (ii) how often he studies in the long run.***

Solution : Let S – Studying and N - Not Studying. If he studies one night, next night he is 70% not studying.

$P(Studying \to Not\ Studying) = 0.7,\ P(Studying \to Studying) = 0.3,$

$P(Not\ Studying \to Not\ Studying) = 0.6,\ P(Not\ Studying \to Studying) = 0.4$,

$P = \begin{array}{c} \\ S \\ N \end{array}\begin{array}{c} \begin{array}{cc} S & N \end{array} \\ \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix} \end{array}$. The limiting form or long run probability distribution. $\boldsymbol{\pi p = \pi}$

$$[\pi_1 \ \ \pi_2]\begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix} = [\pi_1 \ \ \pi_2]$$

$$0.3\,\pi_1 + 0.4\,\pi_2 = \pi_1 \Rightarrow 0.4\,\pi_2 = 0.7\,\pi_1 \quad \Rightarrow 4\,\pi_2 = 7\pi_1 \quad (1)$$

$$0.7\,\pi_1 + 0.6\,\pi_2 = \pi_2 \Rightarrow 0.4\,\pi_2 = 0.7\,\pi_1 \quad \Rightarrow 4\,\pi_2 = 7\pi_1 \quad (2)$$

$$\pi_1 + \pi_2 = 1 \quad (3)$$

Sub. (1) in (3), $\frac{4}{7}\,\pi_2 + \pi_2 = 1 \Rightarrow \boldsymbol{\pi_2} = \frac{\mathbf{7}}{\mathbf{11}}$ (4)

Sub. (4) in (3), $\pi_1 + \frac{\mathbf{7}}{\mathbf{11}} = 1 \Rightarrow \boldsymbol{\pi_1} = \frac{\mathbf{4}}{\mathbf{11}}$ **,** P(he studies in the long run) = $\frac{\mathbf{4}}{\mathbf{11}}$.

8. ***Suppose that the probability of a dry day following a rainy day is*** $\frac{\mathbf{1}}{\mathbf{3}}$ ***and that the probability of a rainy day following a dry day is*** $\frac{\mathbf{1}}{\mathbf{2}}$***. Given that May 1 is a dry day, find the prob. that (i) May 3 is also a dry day (ii) May 5 is also a dry day.***

Solution : Let D – Dry day and R - Rainy day.

$$P = \begin{array}{c} \\ D \\ R \end{array}\!\!\begin{array}{c} \begin{array}{cc} D & R \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \end{array}$$

. Initial state probability distribution is the probability distribution on May 1. Since May 1 is a dry day.

P(D) = 1 and P(R) = 0.

$P^{(1)} = [1\ \ 0]$, $P^{(2)} = P^{(1)}\ P = [1\ \ 0]\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, $P^{(3)} = P^{(2)}\ P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{5}{12} & \frac{7}{12} \end{bmatrix}$,

$\mathbf{P(\mathit{May\ 3\ is\ a\ dry\ day}) = \frac{5}{12}}$

$P^{(4)} = P^{(3)}\ P = \begin{bmatrix} \frac{5}{12} & \frac{7}{12} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{29}{72} & \frac{43}{72} \end{bmatrix}$, $P^{(4)}\ P = \begin{bmatrix} \frac{29}{72} & \frac{43}{72} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{173}{432} & \frac{259}{432} \end{bmatrix}$,

$\boldsymbol{P(May\ 5\ is\ a\ dry\ day) = \frac{173}{432}}$

9. ***A salesman territory consists of 3 cities A, B and C. He never sells in the same city on successive days. If he sells in city A, then the next day, he sells in city B. However, if he sells in either B or C, the next day he is twice as likely to sell in city A as in the other city. In the long run, how often does he sell in each of the cities?***

Solution : The tpm of the given problem is $P = \begin{array}{c} \\ A \\ B \\ C \end{array}\!\!\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix} \end{array}$. The limiting form or long run prob. distribution.

$\boldsymbol{\pi p = \pi}$

$$[\pi_1\ \ \pi_2\ \ \pi_3]\begin{bmatrix} 0 & 1 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix} = [\pi_1\ \ \pi_2\ \ \pi_3]$$

$$\frac{2}{3}\ \pi_2 + \frac{2}{3}\ \pi_3 = \pi_1 \Rightarrow 2\,\pi_2 + 2\,\pi_3 = 3\pi_1 \Rightarrow 3\pi_1 - 2\,\pi_2 - 2\,\pi_3 = 0 \qquad (1)$$

$$\pi_1 + \frac{1}{3}\ \pi_3 = \pi_2 \Rightarrow 3\pi_1 + \pi_3 = 3\pi_2 \Rightarrow 3\pi_1 - 3\pi_2 + \pi_3 = 0 \qquad (2)$$

$$\frac{1}{3}\ \pi_2 = \pi_3 \Rightarrow \pi_2 = 3\pi_3 \qquad (3)$$

$$\pi_1 + \pi_2 + \pi_3 = 1 \qquad (4)$$

Sub. (3) in (1), $3\pi_1 - 2\,\pi_2 - 2\,\pi_3 = 0 \Rightarrow 3\pi_1 - 2\,(3\pi_3) - 2\,\pi_3 = 0 \Rightarrow 3\pi_1 = 8\,\pi_3 \Rightarrow \pi_1 = \frac{8}{3}\ \pi_3$ (5)

Sub. (3) and (5) in (4), $\pi_1 + \pi_2 + \pi_3 = 1 \Rightarrow \frac{8}{3}\pi_3 + 3\pi_3 + \pi_3 = 1 \Rightarrow \pi_3 = \frac{3}{20}$ (6)

Sub. (6) in (3), $\pi_2 = \frac{9}{20}$, Sub. (6) in (5), $\pi_1 = \frac{8}{20}$, $\boldsymbol{\pi_1 = 0.40, \pi_2 = 0.45, \pi_3 = 0.15}$

Thus in the long run, he sells 40% of the time in city A, 45% of the time in the city B and 15% of the time in city C.

10. ***A fair die is tossed repeatedly. If X_n denotes the maximum of the numbers occurring in the first n tosses, find the transition probability matrix P of the Markov chain $\{X_n\}$. Find also P^2 and $P(X_2 = 6)$.***

Solution: The state space is $\{1, 2, 3, 4, 5, 6\}$. X_n = maximum of the numbers occurring in the first n trials.

X_{n+1} = maximum of the numbers occurring in the first $(n+1)$ trials = $\max[X_n, number\ in\ the\ (n+1)^{th}\ trial]$.

Let us see how the First Row of the tpm is filled.

$X_n = 1$

$X_{n+1} = 1\ if\ 1\ appears\ in (n+1)^{th}\ trial$
$= 2\ if\ 1\ appears\ in (n+1)^{th}\ trial$
$= 3\ if\ 1\ appears\ in (n+1)^{th}\ trial$
$= 4\ if\ 1\ appears\ in (n+1)^{th}\ trial$
$= 5\ if\ 1\ appears\ in (n+1)^{th}\ trial$
$= 6\ if\ 1\ appears\ in (n+1)^{th}\ trial$

Now, in the $(n+1)^{th}$ trial, each of the numbers 1, 2, 3, 4, 5, 6 occurs with probability $\frac{1}{6}$.

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Let us see how the Second Row of the tpm is filled.

Here $X_n = 2$

$If\ (n+1)^{th}\ trial\ results\ in\ 1\ or\ 2, X_{n+1} = 2$

$If\ (n+1)^{th}\ trial\ results\ in\ 3, X_{n+1} = 3$

$If\ (n+1)^{th}\ trial\ results\ in\ 4, X_{n+1} = 4$

$If\ (n+1)^{th}\ trial\ results\ in\ 5, X_{n+1} = 5$

$If\ (n+1)^{th}\ trial\ results\ in\ 6, X_{n+1} = 6$

If $X_n = 2, P(X_{n+1} = 2) = \frac{2}{6}$ and $P(X_{n+1} = k) = \frac{1}{6},\ k = 3,4,5,6.$

Proceeding similarly, the tpm is

$$\begin{array}{cc} & (n+1)^{th}\ state \\ P = n^{th} & \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{3}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & \frac{4}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \end{bmatrix} \end{array}, P^2 = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{3}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & \frac{4}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \end{bmatrix} \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{3}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & \frac{4}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} \frac{1}{36} & \frac{3}{36} & \frac{5}{36} & \frac{7}{36} & \frac{9}{36} & \frac{11}{36} \\ 0 & \frac{4}{36} & \frac{5}{36} & \frac{7}{36} & \frac{9}{36} & \frac{11}{36} \\ 0 & 0 & \frac{9}{36} & \frac{7}{36} & \frac{9}{36} & \frac{11}{36} \\ 0 & 0 & 0 & \frac{16}{36} & \frac{9}{36} & \frac{11}{36} \\ 0 & 0 & 0 & 0 & \frac{25}{36} & \frac{11}{36} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The initial probability distribution is $P^{(0)} = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}$

$P(X_2 = 6) = \sum_{i=1}^{6} P(X_2 = 6/X_0 = i)P(X_0 = i) = \frac{1}{6}\sum_{i=1}^{6} P_{i6}^2 = \frac{1}{6}[P_{16}^2 + P_{26}^2 + P_{36}^2 + P_{46}^2 + P_{56}^2 + P_{66}^2]$

$P(X_2 = 6) = \frac{1}{6}\left[\frac{11}{36} + \frac{11}{36} + \frac{11}{36} + \frac{11}{36} + \frac{11}{36} + 1\right] = \frac{91}{216}$

11. ***The transition probability matrix of a Markov chain*** $\{X_n\}, n = 1, 2, \ldots$ ***having 3 states 1, 2, 3 is*** $P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$ ***and the initial distribution*** $P^{(0)} = (0.7, 0.2, 0.1)$.

Find (i) $P(X_2 = 3, X_1 = 3, X_0 = 2)$ ***(ii)*** $P(X_2 = 3)$ ***(iii)*** $P(X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2)$.

Solution:

(i) $P(X_2 = 3, X_1 = 3, X_0 = 2) = P(X_2 = 3/ X_1 = 3, X_0 = 2)P(X_1 = 3, X_0 = 2)$

$= P(X_2 = 3/ X_1 = 3)P(X_1 = 3, X_0 = 2) = P_{33}^{(1)}\ P(X_1 = 3, X_0 = 2)$

$= P_{33}^{(1)}\ P(X_1 = 3/X_0 = 2)P(X_0 = 2) = P_{33}^{(1)}\ P_{23}^{(1)}\ P(X_0 = 2)$

$= (0.3)\ (0.2)(0.2) = 0.012$

(ii) $P(X_2 = 3) = P(X_2 = 3, X_0 = 1) + P(X_2 = 3, X_0 = 2) + P(X_2 = 3, X_0 = 3)$

$= P(X_2 = 3/ X_0 = 1)P(X_0 = 1) + P(X_2 = 3/ X_0 = 2)P(X_0 = 2) + P(X_2 = 3/ X_0 = 3)\ P(X_0 = 3)$

$= P_{13}^{(2)}P(X_0 = 1) + P_{23}^{(2)}P(X_0 = 2) + P_{33}^{(2)}\ P(X_0 = 3)$

$= (0.26)(0.7) + (0.34)(0.2) + (0.29)(0.1) = 0.279$

$$P^2 = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{bmatrix}$$

(iii) $P(X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2) = P(X_3 = 2/ X_2 = 3, X_1 = 3, X_0 = 2)P(X_2 = 3, X_1 = 3, X_0 = 2)$

$= P(X_3 = 2/ X_2 = 3)P(X_2 = 3, X_1 = 3, X_0 = 2)$

$= P_{32}^{(1)}P(X_2 = 3/ X_1 = 3, X_0 = 2)P(X_1 = 3, X_0 = 2)$

$= P_{32}^{(1)}P(X_2 = 3/ X_1 = 3)P(X_1 = 3, X_0 = 2)$

$= P_{32}^{(1)}P_{33}^{(1)}P(X_1 = 3/ X_0 = 2)P(X_0 = 2) = P_{32}^{(1)}P_{33}^{(1)}P_{23}^{(1)}P(X_0 = 2)$

$= (0.4)(0.3)(0.2)(0.2) = 0.0048$

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12. **The transition probability matrix of a Markov chain $\{X_n\}, n = 1, 2, \dots$ having 3 states 1, 2, 3 is** $P = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix}$ **& the initial distribution** $P^{(0)} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. **Find (i)** $P(X_3 = 2/X_2 = 1)$ **(ii)** $P(X_2 = 2)$ **(iii)** $P(X_2 = 2,\ X_1 = 1,\ X_0 = 2)$ **(iv)** $P(X_3 = 1,\ X_2 = 2,\ X_1 = 1,\ X_0 = 2)$

Solution:

(i) $P(X_3 = 2/X_2 = 1) = P_{12}^{(1)} = \frac{1}{4}$

(ii) $P(X_2 = 2) = P(X_2 = 2,\ X_0 = 1) + P(X_2 = 2,\ X_0 = 2) + P(X_2 = 2,\ X_0 = 3)$

$= P(X_2 = 2/\ X_0 = 1)P(\ X_0 = 1) + P(X_2 = 2/\ X_0 = 2)P(\ X_0 = 2) + P(X_2 = 2/\ X_0 = 3)\ P(\ X_0 = 3)$

$= P_{12}^{(2)}P(\ X_0 = 1) + P_{22}^{(2)}P(\ X_0 = 2) + P_{32}^{(2)}\ P(\ X_0 = 3) = \left(\frac{5}{16}\right)\left(\frac{1}{3}\right) + \left(\frac{8}{16}\right)\left(\frac{1}{3}\right) + \left(\frac{9}{16}\right)\left(\frac{1}{3}\right) = \frac{1}{6}$

$$P^2 = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix}\begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{10}{16} & \frac{5}{16} & \frac{1}{16} \\ \frac{5}{16} & \frac{8}{16} & \frac{3}{16} \\ \frac{3}{16} & \frac{9}{16} & \frac{4}{16} \end{bmatrix}$$

(iii) $P(X_2 = 2,\ X_1 = 1,\ X_0 = 2) = P(X_2 = 2/\ X_1 = 1,\ X_0 = 2)P(\ X_1 = 1,\ X_0 = 2)$

$= P(X_2 = 2/\ X_1 = 1)P(\ X_1 = 1,\ X_0 = 2) = P_{12}^{(1)}\ P(\ X_1 = 1,\ X_0 = 2)$

$= P_{12}^{(1)}\ P(\ X_1 = 1/X_0 = 2)P(\ X_0 = 2) = P_{12}^{(1)}\ P_{21}^{(1)}\ P(\ X_0 = 2)$

$= \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{3}\right) = 0.0208$

(iv) $P(X_3 = 1,\ X_2 = 2,\ X_1 = 1,\ X_0 = 2) = P(X_3 = 1/\ X_2 = 2,\ X_1 = 1,\ X_0 = 2)P(\ X_2 = 2,\ X_1 = 1,\ X_0 = 2)$

$= P(X_3 = 1/\ X_2 = 2)P(\ X_2 = 2,\ X_1 = 1,\ X_0 = 2)$

$= P_{21}^{(1)}P(\ X_2 = 2/\ X_1 = 1,\ X_0 = 2)P(\ X_1 = 1,\ X_0 = 2)$

$= P_{21}^{(1)}P(\ X_2 = 2/\ X_1 = 1)P(\ X_1 = 1,\ X_0 = 2)$

$= P_{21}^{(1)}P_{12}^{(1)}P(\ X_1 = 1,\ X_0 = 2) = P_{21}^{(1)}P_{12}^{(1)}P(\ X_1 = 1/\ X_0 = 2)P(\ X_0 = 2)$

$= P_{21}^{(1)}P_{12}^{(1)}P_{21}^{(1)}P(\ X_0 = 2) = \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{3}\right) = 0.0052$

13. **Find the nature of the states of the Markov chain with the transition probability matrix** $P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$.

Solution : $P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$, $P^2 = P\ P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$,

$$P^3 = P^2\ P = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} = P,\ \ P^4 = P^3\ P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} = P^2$$

$P_{11}^{(2)} > 0,\ P_{12}^{(1)} > 0,\ P_{13}^{(2)} > 0,$ $\quad P_{21}^{(1)} > 0,\ P_{22}^{(2)} > 0,\ P_{23}^{(3)} > 0\ ,$ $\quad P_{31}^{(2)} > 0,\ P_{32}^{(1)} > 0,\ P_{33}^{(2)} > 0$

The chain is irreducible. Also since there are only 3 states, the chain is finite. i.e., the chain is finite & irreducible.

All the states are non null persistent.

State 1: $P_{11}^{(2)} > 0, P_{11}^{(4)} > 0, P_{11}^{(6)} > 0, \dots\ ,$ $\quad \textit{Period of state } 1 = \ GCD(2, 4, 6 \dots) = 2$

State 2: $P_{22}^{(2)} > 0, P_{22}^{(4)} > 0, P_{22}^{(6)} > 0 \dots\ ,$ $\quad \textit{Period of state } 2 = \ GCD(2, 4, 6 \dots) = 2$

State 3: $P_{33}^{(2)} > 0, P_{33}^{(4)} > 0, P_{33}^{(6)} > 0, \dots$ $\quad \textit{Period of state } 3 = \ GCD(2, 4, 6 \dots) = 2$

All the states 1, 2, 3 have period 2. That is, they are periodic. All the states are non null persistent, but the states are periodic. ***Hence all the states are not ergodic.***

14. Three boys A, B, C are throwing a ball to each other. A always throw the ball to B &B always throws to C but C is just as likely to throw the ball to B as to A. Show that the process is Markovian. Find the transition matrix & classify the states.

Solution:

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \end{array},$$

$$P^2 = P\,P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad P^3 = P^2\,P = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

$$P^4 = P^3\,P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}, \quad P^5 = P^4\,P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & \frac{3}{8} & \frac{1}{2} \end{bmatrix}$$

$P_{11}^{(3)} > 0,\ P_{12}^{(1)} > 0,\ P_{13}^{(2)} > 0\,,\quad P_{21}^{(2)} > 0,\ P_{22}^{(2)} > 0,\ P_{23}^{(1)} > 0\,,\quad P_{31}^{(1)} > 0,\ P_{32}^{(1)} > 0,\ P_{33}^{(2)} > 0$

The chain is irreducible. Also since there are only 3 states, the chain is finite. That is, the chain is finite and irreducible. All the states are non null persistent.

1st state A: $P_{11}^{(3)} > 0, P_{11}^{(5)} > 0, \dots$ $\quad$ $Period\ of\ A = GCD(3,5,\dots) = 1$

2nd state B: $P_{22}^{(2)} > 0, P_{22}^{(3)} > 0, P_{22}^{(4)} > 0 \dots$ $\quad$ $Period\ of\ B = GCD(2,3,4\ \dots) = 1$

3rd state C: $P_{33}^{(2)} > 0, P_{33}^{(3)} > 0, P_{33}^{(4)} > 0, \dots$ $\quad$ $Period\ of\ C = GCD(2,3,4\ \dots) = 1$

All the states A, B, C have period 1. That is, they are aperiodic.

All the states are aperiodic and non null persistent, they are ergodic.

All the Best

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