B.Tech. Degree Examination, November 2018
Third to Seventh Semester
15MA201-Transforms and Boundary Value Problems

Time: Three hours **Max. Marks: 100**

Part - A (20 $\times$ 1=20 Marks)
Answer ALL Questions

1. The partial differential equation formed by eliminating arbitrary constant a, b is $z = (x+a)(y+b)$ is
   (A) $z = p+q$ (B) $z = p-q$ (C) $z = p/q$ (D) $z = pq$
   Sol: Given $z = p+q$ (1)
   Equation (1) partially differentiating w.r.to x and y, we get
   $\dfrac{\partial z}{\partial x} = (y+b)$ and $\dfrac{\partial z}{\partial y} = (x+a)$
   Therefore $p = (y+b)$ where $p = \dfrac{\partial z}{\partial x}$ (2)
   and $q = (x+a)$ where $q = \dfrac{\partial z}{\partial y}$. (3)
   Substituting equations (2) and (3), in (1) we get $z = pq$.
   Ans. D

2. The complementary function of $(D^2 + 2DD' + D'^2)z = 0$ is
   (A) $\phi(y-x) + \phi_2(y-x)$ (B) $\phi(y-x) + x\phi_2(y-x)$
   (C) $\phi(y-x) + \phi_2(y+x)$ (D) $\phi(y-x) + x\phi_2(y+x)$
   Sol: The auxiliary equation is $m^2 + 2m + 1 = 0$ where $D = m, D' = 1$.
   $(m+1)^2 = 0 \Rightarrow m = -1, -1$. C.F. is $z = \phi_1(y-x) + x\phi_2(y-x)$.
   Ans. B

3. The particular integral of $(D^2 - 2DD')z = e^{2x}$ is
   (A) $\dfrac{e^{2x}}{4}$ (B) $\dfrac{e^{2x+y}}{4}$ (C) e^{2x} (D) $\dfrac{e^{2x}}{2}$
   Sol: P.I.$= \dfrac{1}{D^2 - 2DD'}e^{2x} = \dfrac{1}{4}e^{2x}$ where $D = 2, D' = 0$.
   Ans. A

4. The complete solution of $z = px + qy + p^2q^2$ is
   (A) $z = ax + by^2 + ab^2$ (B) $z = ax^2 + by + ab^2$
   (C) $z = ax + by + a^2b^2$ (D) $z = ax + by + ab$
   Sol: Given $z = px + qy + p^2q^2$. This is clairaut's form.
   Hence the complete integral is $z = ax + by + a^2b^2$.
   Ans. C

5. $\sin x$ is a periodic function with period
   (A) π (B) $\dfrac{\pi}{2}$ (C) 2π (D) 4π
   Ans. C

6. The constant a_0 of the Fourier series for the function $f(x) = k$ in $0 \leq x \leq 2\pi$

   (A) k (B) $2k$ (C) 0 (D) $\dfrac{k}{2}$

   Sol: $a_0 = \dfrac{1}{\pi}\int\limits_0^{2\pi} f(x)dx = \dfrac{1}{\pi}\int\limits_0^{2\pi} kdx = \dfrac{k}{\pi}\left(x\right)_0^{2\pi} = 2k.$

   Ans. B

7. The RMS value of $f(x) = x$ in $-1 \leq x \leq 1$ is

   (A) 1 (B) 0 (C) $\dfrac{1}{\sqrt{3}}$ (D) -1

   Sol: RMS value of $f(x) = \sqrt{\dfrac{\int\limits_a^b (f(x))^2 dx}{b-a}} = \sqrt{\dfrac{\int\limits_{-1}^{1} x^2 dx}{2}} = \dfrac{1}{\sqrt{3}}.$

   Ans. C

8. Half range cosine series for $f(x)$ is $(0, \pi)$ is

   (A) $\sum\limits_{n=1}^{\infty} a_n \cos nx$ (B) $\dfrac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos nx$

   (C) $\sum\limits_{n=1}^{\infty} b_n \sin nx$ (D) $\dfrac{a_0}{2} - \sum a_n \cos nx$

   Ans. B

9. The proper solution of the problems of vibration of string is

   (A) $y(x,t) = (Ae^{\lambda x} + Be^{-\lambda x})(Ce^{\lambda at} + De^{-\lambda at})$ (B) $y(x,t) = (Ax+B)(ct+1)$

   (C) $y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at)$ (D) $y(x,t) = Ax + B$

   Ans. C

10. The one dimensional wave equation is

    (A) $\dfrac{\partial u}{\partial t} = a^2\dfrac{\partial^2 u}{\partial x^2}$ (B) $\dfrac{\partial^2 y}{\partial t^2} = a^2\dfrac{\partial^2 y}{\partial x^2}$

    (C) $\dfrac{\partial y}{\partial t} = a\dfrac{\partial^2 y}{\partial x^2}$ (D) $\dfrac{\partial^2 y}{\partial x^2} = a^2\dfrac{\partial^2 y}{\partial t^2}$

    Ans. B

11. One dimensional heat equation is used to find

    (A) Density (B) Temperature Distribution (C) Time (D) Displacement

    Ans. B

12. A rod of length l has its ends A and B kept at $0°C$ and $100°C$ respectively, until steady state conditions prevail. Then the initial condition is given by

    (A) $u(x,0) = ax + b + 100l$ (B) $u(x,0) = \dfrac{100x}{l}$

    (C) $u(x,0) = 100xl$ (D) $u(x,0) = (x+l)100$

    Sol: In steady state, the P.D.E. becomes $\dfrac{d^2u}{dx^2} = 0$

    Therefore the solution is $u(x) = ax + b$ (1)

    The initial conditions are $u(0) = 0$ and $u(l) = 100$.

    Using these conditions in (1), we obtain

    $u(0) = 0 + b \Rightarrow b = 0$ and $u(l) = la \Rightarrow 100 = la \Rightarrow a = \dfrac{100}{l}.$

    Therefore $u(x) = \dfrac{100x}{l}$. **Ans. B**

13. $F[e^{iax}f(x)]$ is

(A) $F(s+a)$ (B) $F(s-a)$ (C) $F(as)$ (D) $F\left(\dfrac{s}{a}\right)$

Sol: $F[e^{iax}f(x)] = \dfrac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} e^{iax}f(x)e^{isx}dx = \dfrac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} e^{i(s+a)x}f(x)dx = F(s+a).$

Ans. A

14. $F[xf(x)] =$

(A) $\dfrac{dF(s)}{ds}$ (B) $i\dfrac{dF(s)}{ds}$ (C) $-i\dfrac{dF(s)}{ds}$ (D) $-\dfrac{dF(s)}{ds}$

Sol: We know that $F\left\{x^n f(x)\right\} = (-i)^n \dfrac{d^n}{ds^n}F(s).$

Put $n=1,$ we get $F[xf(x)] = -i\dfrac{dF(s)}{ds}.$

Ans. C

15. The Fourier cosine transform of $F_c[e^{-4x}]$

(A) $\sqrt{\dfrac{2}{\pi}}\left(\dfrac{4}{16+s^2}\right)$ (B) $\sqrt{\dfrac{2}{\pi}}\left(\dfrac{4}{4+s^2}\right)$ (C) $\sqrt{\dfrac{\pi}{2}}\left(\dfrac{4}{16+s^2}\right)$ (D) $\sqrt{\dfrac{\pi}{2}}\left(\dfrac{4}{4+s^2}\right)$

Sol:

$$\begin{aligned} F_c(e^{-4x}) &= \sqrt{\frac{2}{\pi}}\int\limits_0^{\infty} e^{-4x}\cos sx dx = \sqrt{\frac{2}{\pi}}\left[\frac{e^{-4x}}{4^2+s^2}\left(-4\cos sx + s\sin sx\right)\right]_0^{\infty} \\ &= \sqrt{\frac{2}{\pi}}\frac{4}{4^2+s^2}. \end{aligned}$$

Ans. A

16. $F(f(x)*g(x))$ is

(A) $F(s)+G(s)$ (B) $F(s)-G(s)$ (C) $F(s)G(s)$ (D) $F(s)/G(s)$

Ans. C

17. $Z(7)$ is

(A) $\dfrac{z}{z-1}$ (B) $7.\dfrac{z}{z-1}$ (C) $\dfrac{1}{7}.\dfrac{z}{z-1}$ (D) $\dfrac{z-1}{z}$

Sol: We know that $Z(k) = \dfrac{kz}{z-1} \Rightarrow Z(7) = \dfrac{7z}{z-1}.$

Ans. B

18. $Z[na^n] =$

(A) $\dfrac{az}{(z-a)^2}$ (B) $\dfrac{z}{(z-a)^2}$ (C) $\dfrac{a}{(z-a)^2}$ (D) $\dfrac{z}{(z-a)^3}$

Sol: We know that $Z[nf(t)] = -z\dfrac{dF(z)}{dz} \Rightarrow Z[na^n] = -z\dfrac{d}{dz}\left[\dfrac{z}{z-a}\right] = \dfrac{az}{(z-a)^2}.$

Ans. A

19. If $Z[f(t)] = F(z)$ then $\lim\limits_{z\to\infty} F(z) =$

(A) $f(0)$ (B) $f(1)$ (C) $\lim\limits_{x\to\infty} f(t)$ (D) $f(\infty)$

Ans. A

20. $\phi(z) = \dfrac{z^n(2z+4)}{(z-2)^3}$ has a pole of order
(A) 2 (B) 1 (C) 3 (D) 4
Ans. C

Part - B (5 × 4=20 Marks)
Answer ANY FIVE Questions

21. Form the partial differential equation by eliminating f from $z = xy + f(x^2+y^2+z^2)$.
Sol: Given $z = xy + f(x^2+y^2+z^2)$.
Rewrite the given equation $z - xy = f(x^2+y^2+z^2)$.
Partially differentiate with respect to x and y, we get

$$\frac{\partial z}{\partial x} - y = f'(x^2+y^2+z^2)\left(2x + 2z\frac{\partial z}{\partial x}\right) \quad (1)$$

$$\text{and } \frac{\partial z}{\partial y} - x = f'(x^2+y^2+z^2)\left(2y + 2z\frac{\partial z}{\partial y}\right) \quad (2)$$

$$(1) \Rightarrow \frac{p-y}{2(x+zp)} = f'(x^2+y^2+z^2), \text{ where } p = \frac{\partial z}{\partial x} \quad (3)$$

$$(2) \Rightarrow \frac{q-x}{2(y+zq)} = f'(x^2+y^2+z^2), \text{ where } q = \frac{\partial z}{\partial y} \quad (4)$$

From (3) and (4), we get $\dfrac{p-y}{(x+zp)} = \dfrac{q-x}{(y+zq)}$
$\Rightarrow (p-y)(y+zq) = (q-x)(x+zp)$
$\Rightarrow py + pqz - y^2 - qyz = qx + pqz - x^2 - pxz$
Hence $(y+xz)p - (yz+x)q = y^2 - x^2$.

22. Find the half range Fourier sine series for $f(x) = x$ in $0 < x < \pi$.
Sol: Let $f(x) = \sum\limits_{n=1}^{\infty} b_n \sin nx$ where $b_n = \dfrac{2}{\pi}\int\limits_0^{\pi} f(x)\sin nx dx$.
Now $b_n = \dfrac{2}{\pi}\int\limits_0^{\pi} x\sin nx dx = \dfrac{2}{\pi}\left[x\left(\dfrac{-\cos nx}{n}\right) - 1\left(\dfrac{-\sin nx}{n^2}\right)\right]_0^{\pi} = \dfrac{2}{\pi}\left[\dfrac{-\pi}{n}(-1)^n\right]$.
Therefore $x = \sum\limits_{n=1}^{\infty} \dfrac{2}{n}(-1)^{n-1}\sin nx$.

23. Write the one dimensional heat flow equation and all the possible solutions.
Sol: one dimensional heat flow equation: $\dfrac{\partial u}{\partial t} = \alpha^2\dfrac{\partial^2 u}{\partial x^2}$ and possible solutions are

$$u(x,t) = \begin{cases} (A_1e^{\lambda x} + B_1e^{-\lambda x})C_1e^{\alpha^2\lambda^2 t} \\ (A_2\cos\lambda x + B_2\sin\lambda x)C_2e^{-\alpha^2\lambda^2 t} \\ (A_3x + B_3)C_3 \end{cases}$$

24. Find the Fourier sine transform of $f(x) = e^{-ax}, a > 0$.
Sol:

$$F_s(e^{-ax}) = \sqrt{\frac{2}{\pi}}\int\limits_0^{\infty} e^{-ax}\sin sx dx = \sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{a^2+s^2}(-a\sin sx - s\cos sx)\right]_0^{\infty}$$

$$= \sqrt{\frac{2}{\pi}}\frac{s}{a^2+s^2}.$$

25. Find $Z-$ transform of $r^n \cos n\theta$.
Proof: We know that $Z\{a^n\} = \dfrac{z}{z-a}$ if $|z| > |a|$. Taking $a = re^{i\theta}$

$$Z\left\{r^n e^{in\theta}\right\} = \frac{z}{z-re^{i\theta}} = \frac{z}{z-r(\cos\theta + i\sin\theta)} = \frac{z}{(z-r\cos\theta) - ir\sin\theta}$$

$$\Rightarrow Z\left[r^n(\cos n\theta + i\sin n\theta)\right] = \frac{z[(z-r\cos\theta)+ir\sin\theta]}{[(z-r\cos\theta)-ir\sin\theta][(z-r\cos\theta)+ir\sin\theta]}$$

$$= \frac{z[(z-r\cos\theta)+ir\sin\theta]}{[(z-r\cos\theta)^2+\sin^2\theta]}.$$

Equating real parts, we get $z(r^n\cos n\theta) = \dfrac{z(z-r\cos\theta)}{z^2-2zr\cos\theta+r^2}$ if $|z| > |r|$.

26. Find $Z^{-1}\left(\dfrac{1}{(z-1)(z-2)}\right)$ by convolution theorem.
Sol:

$$Z^{-1}\left[\frac{z^2}{(z-1)(z-2)}\right] = Z^{-1}\left[\frac{1}{z-1}.\frac{1}{z-2}\right]$$

$$= Z^{-1}\left(\frac{1}{z-1}\right).Z^{-1}\left(\frac{1}{z-2}\right)$$

$$= u(n-1) * 2^{n-1}u(n-1) = \sum_{m=1}^{n} 1^{n-m}.2^{m-1}$$

$$= 1 + 2 + 2^2 + \ldots + 2^n$$

$$= \frac{1-2^{n+1}}{1-2} = 2^{n+1} - 1$$

27. Solve $p^2 + q^2 = x + y$.
Sol: Given equation is separable type.
Therefore $p^2 - x = y - q^2 = a$ (say)
$\Rightarrow p = \sqrt{x+a}$ and $q = \sqrt{y-a}$.
We know that $dz = pdx + qdy$.
$dz = \sqrt{x+a}dx + \sqrt{y-a}dy$
Integrating, we have $z = \dfrac{2}{3}(x+a)^{3/2} + \dfrac{2}{3}(y-a)^{3/2} + c$. (1)
This is the complete integral.
Differentiating partially with respect to c in (1), we get 0=1 which is absurd.
Hence, there is no singular integral.
To find general integral, put $c = f(a)$ in (1)
$z = \dfrac{2}{3}(x+a)^{3/2} + \dfrac{2}{3}(y-a)^{3/2} + f(a)$. (2)
Differentiating partially with respect to a, we get
$0 = \sqrt{x+a} - \sqrt{y-a} + f'(a)$. (3)
Eliminating a between (2) and (3), we get the general integral.

Part - C (5 $\times$ 12=60 Marks)
Answer ALL Questions

28. a. i. Solve $z = px + qy + \sqrt{1+p^2+q^2}$.
Sol: The given equation is Clairaut's form.
Therefore the complete integral is $z = ax + by + \sqrt{1+a^2+b^2}$. (1)

Differentiating (1) w. r. to a and b, we get

$$0 = x + \frac{1}{2}(1+a^2+b^2)^{1/2-1}.2a \Rightarrow x = \frac{-a}{\sqrt{1+a^2+b^2}} \qquad (2)$$

and $0 = y + \frac{1}{2}(1+a^2+b^2)^{1/2-1}.2b \Rightarrow y = \frac{-b}{\sqrt{1+a^2+b^2}}$ (3)

Now $1-x^2-y^2 = 1 - \frac{a^2+b^2}{1+a^2+b^2} \Rightarrow 1-x^2-y^2 = \frac{1}{1+a^2+b^2}$

$\Rightarrow \sqrt{1+a^2+b^2} = \frac{1}{\sqrt{1-x^2-y^2}}$

$(2) \Rightarrow a = -x\sqrt{1+a^2+b^2} = \frac{-x}{\sqrt{1-x^2-y^2}}$

$(3) \Rightarrow b = -y\sqrt{1+a^2+b^2} = \frac{-y}{\sqrt{1-x^2-y^2}}$

Substituting in (1), we get

$$z = -\frac{x^2}{\sqrt{1-x^2-y^2}} - \frac{y^2}{\sqrt{1-x^2-y^2}} + \frac{1}{\sqrt{1-x^2-y^2}} = \frac{1-x^2-y^2}{\sqrt{1-x^2-y^2}}$$

$z = \sqrt{1-x^2-y^2} \Rightarrow z^2 = 1-x^2-y^2$. Hence $x^2+y^2+z^2 = 1$.

Put $b = f(a)$ in (1), we get $z = ax + f(a)y + \sqrt{1+a^2+(f(a))^2}$ (4)

Differentiate (4) with respect to a to get the general solution.

ii. Solve $x(z^2-y^2)p + y(x^2-z^2)q = z(y^2-x^2)$.

Sol: The auxiliary equations are $\frac{dx}{x(z^2-y^2)} = \frac{dy}{y(x^2-z^2)} = \frac{dz}{z(y^2-x^2)}$.

Taking the Lagrange's multipliers x, y, z, we get

$$\frac{xdx}{x^2(z^2-y^2)} = \frac{ydy}{y^2(x^2-z^2)} = \frac{zdz}{z^2(y^2-x^2)}$$

Each is equal to $\frac{xdx+ydy+zdz}{\sum x^2(z^2-y^2)} = \frac{xdx+ydy+zdz}{0}$

Hence $xdx+ydy+zdz = 0$.

Integrating, we get $\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2} = a \Rightarrow x^2+y^2+z^2 = 2a = a_1$

Also, taking the Lagrang's multipliers $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$, we get

$$\frac{\frac{dx}{x}}{z^2-y^2} = \frac{\frac{dy}{y}}{x^2-z^2} = \frac{\frac{dz}{z}}{y^2-x^2}$$

Each is equal to $\frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{\sum(z^2-y^2)} = \frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{0}$

Hence $\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z} = 0$

Integrating, we get $\log x + \log y + \log z = \log b \Rightarrow xyz = b$.

Therefore, the general solution is $\phi(x^2+y^2+z^2, xyz) = 0$.

(OR)

b. i. Solve $(D^2 - 2DD' + D'^2) = \cos(x-3y)$.

Sol: The auxiliary equation is $m^2 - 2m + 1 = 0$ where $D = m, D' = 1$.

$\Rightarrow (m-1)^2 = 0 \Rightarrow m = 1, 1$.

The complementary function (C.F.) is $z = f_1(y+x) + xf_2(y+x)$.

To find Particular intergral (P.I.):

$$\begin{aligned}\text{P.I.} &= \frac{1}{D^2 - 2DD' + D'^2)} \cos(x-3y)\\ &= \frac{\cos(x-3y)}{-1-2(3)-9} \quad \text{repalce by } D^2 = -1, DD^{'} = -(-3), D^{'2} = -(-3^2)\\ &= -\frac{1}{16}\cos(x-3y)\end{aligned}$$

The complete solution is $z = C.F. + P.I.$

$z = f_1(y+x) + xf_2(y+x) - \dfrac{1}{16}\cos(x-3y).$

ii. Solve $(D^2 - DD^{'2}) = e^{x+2y}.$

Sol: The auxiliary equation is $m^2 - 1 = 0$ where $D = m, D^{'} = 1.$

$\Rightarrow m^2 = 1 \Rightarrow m = -1, 1.$

The complementary function (C.F.) is $z = f_1(y-x) + f_2(y+x).$

To find Particular intergral (P.I.):

$$\begin{aligned}\text{P.I.} &= \frac{1}{D^2 - DD'^2} e^{x+2y}\\ &= \frac{e^{x+2y}}{1-(1)(4)} \quad \text{repalce by } D = 1, D^{'} = 2\\ &= -\frac{1}{3}e^{x+2y}\end{aligned}$$

The complete solution is $z = C.F. + P.I.$

$z = f_1(y-x) + f_2(y+x) - \dfrac{1}{3}e^{x+2y}.$

29. a. Find the Fourier series of $f(x) = x + x^2$ in $(-\pi, \pi)$ of periodicity 2π. Hence deduce that $\sum \dfrac{1}{n^2} = \dfrac{\pi^2}{6}.$

Sol: Given the function $f(x)$ is neither even nor odd.

Let $f(x) = \dfrac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos nx + \sum\limits_{n=1}^{\infty} b_n \sin nx$ (1)

where $a_0 = \dfrac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)dx,$ $a_n = \dfrac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)\cos nx dx$ and $b_n = \dfrac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)\sin nx dx.$

To find $a_0, a_n b_n$:

$$\begin{aligned}a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} xdx + \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx\\ &= 0 + \frac{2}{\pi}\int_{0}^{\pi} x^2 dx, \text{ since } x \text{ is odd and } x^2 \text{ is even.}\\ &= \frac{2\pi^2}{3}\end{aligned}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx dx + \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx dx$$
$$= 0 + \frac{2}{\pi}\int_{0}^{\pi} x^2\cos nx dx, \text{ since } x\cos nx \text{ is odd}$$
$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$
$$= \frac{2}{\pi}\left[2\pi\frac{\cos n\pi}{n}\right], \text{ since } \sin 0 = \sin n\pi = 0$$
$$= \frac{4(-1)^n}{n^2}$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx dx + \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin nx dx$$
$$= \frac{2}{\pi}\int_{0}^{\pi} x\sin nx dx + 0, \text{ since } x^2\sin nx \text{ is odd}$$
$$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - 1\left(\frac{-\sin nx}{n}\right)\right]_0^{\pi} = \frac{2}{\pi}\left[\pi\left(\frac{-\cos n\pi}{n}\right)\right]$$
$$= \frac{-2(-1)^n}{n}$$

Substituting the values of a_0, a_n, b_n in (1), we get
$x + x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n\left[\frac{4}{n^2}\cos nx - \frac{2}{n}\sin nx\right]$

Deduction: $x = \pi$ is an end point in the range. Hence the value of the Fourier series at $x = \pi$ is equal to $\frac{1}{2}[f(\pi) + f(-\pi)] = \frac{1}{2}[(\pi + \pi^2) + (-\pi + \pi^2)] = \pi^2$.
Hence $\frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos n\pi = \pi^2 \Rightarrow 4\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{2}{3}\pi^2$. Therefore $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$.

(OR)

b. Find the Fourier series upto second harmonic from the following data:

x	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π
y	1	1.4	1.9	1.7	1.5	1.2	1

Sol: Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{2} a_n\cos nx + \sum_{n=1}^{2} b_n\sin nx$ where $a_0 = \frac{2}{m}\sum f(x)$,
$a_n = \frac{2}{m}\sum f(x)\cos nx$ and $b_n = \frac{2}{m}\sum f(x)\sin nx$.

x	$f(x)$	$\cos x$	$\sin x$	$\cos 2x$	$\sin 2x$
0	1	1	0	1	0
$\frac{\pi}{3}$	1.4	0.5	0.866	-0.5	0.866
$\frac{2\pi}{3}$	1.9	-0.5	0.866	-0.5	-0.866
π	1.7	-1	0	1	0
$\frac{4\pi}{3}$	1.5	-0.5	-0.866	0.5	0.866
$\frac{5\pi}{3}$	1.2	0.5	-0.866	0.5	-0.866

Now $a_0 = \frac{1}{3}[1 + 1.4 + 1.9 + 1.7 + 1.5 + 1.2] = 2.9$
$a_1 = \frac{2}{6}\sum f(x)\cos x = \frac{1}{3}[1 + 0.7 - 0.95 - 1.7 - 0.75 + 0.6] = -0.3667$
$a_2 = \frac{2}{6}\sum f(x)\cos 2x = \frac{1}{3}[1 - 0.7 - 0.95 + 1.7 - 0.75 - 0.6] = -0.1$
$b_1 = \frac{2}{6}\sum f(x)\sin x = \frac{1}{3}[0 + 1.2124 + 1.6454 + 0 - 1.299 - 1.0392] = 0.1732$
$b_2 = \frac{2}{6}\sum f(x)\sin 2x = \frac{1}{3}[0 + 1.2124 - 1.6454 + 1.299 - 1.0392] = -0.0577$
Hence $f(x) = 1.45 - 0.3667\cos x - 0.1\cos 2x + 0.1732\sin x - 0.0577\sin 2x$.

30. a. A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in its equilibrium position. If it is set vibrating giving each point a velocity $3x(l - x)$, find the displacement.

Sol. The displacement of the string $y(x,t)$ is governed by $\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}$.
The boundary conditions are
(i) $y(0,t) = 0, t \geq 0$ (ii) $y(l,t) = 0, t \geq 0$.
The initial conditions are
(iii) $y(x,0) = 0, 0 \leq x \leq l$ (iv) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 3x(l-x), 0 \leq x \leq l$.
The proper solution is $y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at)$. (1)
Using boundary condition (i) in (1), $A(C\cos\lambda at + D\sin\lambda at) = 0 \Rightarrow A = 0$.
$A = 0$ in (1), we get $y(x,t) = B\sin\lambda x(C\cos\lambda at + D\sin\lambda at)$. (2)
Applying the boundary condition (ii) in (2), $B\sin\lambda l(C\cos\lambda at + D\sin\lambda at) = 0$.
Since $B \neq 0$ and $\sin nl = 0 \Rightarrow \lambda l = n\pi \Rightarrow \lambda = \frac{n\pi}{l}$.
$\lambda = \frac{n\pi}{l}$ in (2), we get

$$y(x,t) = B\sin\frac{n\pi x}{l}\left(C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right). \quad (3)$$

Using the initial condition (iii) in (3), we get $B\sin\frac{n\pi x}{l}.C = 0$
Since $B \neq 0, C = 0$.
Therefore $y(x,t) = B\sin\frac{n\pi x}{l}.D\sin\frac{n\pi at}{l}$.
The most general solution is $y(x,t) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$ (4)
Using initial condition (iv) in (4), we get

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}.\frac{n\pi a}{l}$$

$$\Rightarrow \left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty} B_n.\frac{n\pi a}{l}\sin\frac{n\pi x}{l} = 3x(l-x)$$

This is half-range Fourier sine series. Therefore

$$\begin{aligned}
B_n.\frac{n\pi a}{l} &= \frac{2}{l}\int_0^l 3(lx-x^2)\sin\frac{n\pi x}{l}dx \\
&= \frac{6}{l}\left[(lx-x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l-2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l \\
&= \frac{6}{l}\left[-2\cos n\pi.\frac{l^3}{n^3\pi^3} + 2\frac{l^3}{n^3\pi^3}\right], \text{ since } \sin 0 = \sin n\pi = 0 \\
&= \frac{6}{l}.\frac{2l^3}{n^3\pi^3}[-(-1)^n + 1] \\
&= \frac{12l^2}{n^3\pi^3}[1-(-1)^n] = \begin{cases} \dfrac{24l^2}{n^3\pi^3} \text{ if } n \text{ is odd} \\ 0 \text{ if } n \text{ is even} \end{cases}
\end{aligned}$$

Hence $B_n = \dfrac{24l^3}{n^4\pi^4 a}$ if n is odd.

Substituting the value of B_n in (4), we get $y(x,t) = \sum_{n=odd} \dfrac{24l^3}{n^4\pi^4 a}\sin\dfrac{n\pi x}{l}\sin\dfrac{n\pi at}{l}$.

(OR)

b. A rod of length l has its end A and B kept at $0°C$ and $100°C$ respectively until steady state conditions prevail. If the temperature at B is reduced suddenly to $0°C$ and kept so, while that of A is maintained, find the temperature $u(x,t)$.

Sol: The P.D.E. of one dimensional heat flow is $\dfrac{\partial u}{\partial t} = \alpha^2\dfrac{\partial^2 u}{\partial x^2}$. (1)

In steady state, the P.D.E. becomes $\dfrac{d^2u}{dx^2} = 0$. (2)

In steady state, the solution is $u(x) = ax + b$. (3)

The initial conditions are $u(0) = 0$ and $u(l) = 100$.

Using these conditions in (3), we obtain $u(0) = 0 + b \Rightarrow b = 0$ and
$u(l) = la + b \Rightarrow 100 = la + \Rightarrow a = \dfrac{100}{l}$. Therefore $u(x) = \dfrac{100}{l}$.

If the temperature at B is reduced to $0°C$, then the temperature distribution changes from steady state to unsteady state (transient state).

In transient state, the boundary conditions are

(i) $u(0,t) = 0$ for all $t \geq 0$ (ii) $u(l,t) = 0$ for all $t \geq 0$.

The initial condition is (iii) $u(x,0) = \dfrac{100}{l}$ for $0 < x < l$.

In transient state, the proper solution is $u(x,t) = (A\cos\lambda x + B\sin\lambda x)e^{-\alpha^2\lambda^2 t}$. (4)

Using (i) in (4), we get $u(0,t) = 0 = Ae^{-\alpha^2\lambda^2 t} \Rightarrow A = 0$.

$A = 0$ in (4), $u(x,t) = B\sin\lambda x e^{-\alpha^2\lambda^2 t}$. (5)

Using (ii) in (5), we get $u(l,t) = 0 = B\sin lx e^{-\alpha^2\lambda^2 t}$.

Since $B \neq 0, \sin l\lambda = 0 \Rightarrow l\lambda = n\pi \Rightarrow \lambda = \dfrac{n\pi}{l}$.
$\lambda = \dfrac{n\pi}{l}$ in (5), we get $u(x,t) = B\sin\dfrac{n\pi x}{l}e^{-\alpha^2n^2\pi^2t/l^2}$.
The most general solution is $u(x,t) = \sum\limits_{n=1}^{\infty} B_n \sin\dfrac{n\pi x}{l}e^{-\alpha^2n^2\pi^2t/l^2}$. (6)
Using (iii) in (6), we get $u(x,0) = \sum\limits_{n=1}^{\infty} B_n \sin\dfrac{n\pi x}{l} = \dfrac{100}{l}$.
This is a half range sine series. Therefore

$$\begin{aligned}
B_n &= \frac{2}{l}\int_0^l \frac{100}{l}\sin\frac{n\pi x}{l}dx \\
&= \frac{200}{l^2}\left[x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 1\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l \\
&= \frac{200}{l^2}\left[-\frac{l^2}{n\pi}\cos n\pi\right], \text{ since } \sin 0 = \sin n\pi = 0 \\
&= \frac{200}{n\pi}(-1)^{n+1}
\end{aligned}$$

Substituting the value of B_n in (6), we get
$u(x,t) = \sum\limits_{n=1}^{\infty} \dfrac{200}{n\pi}(-1)^{n+1}\sin\dfrac{n\pi x}{l}e^{-\alpha^2n^2\pi^2t/l^2}$.

31. a. Find the Fourier transform of $f(x) = \begin{cases} 1-|x|, & \text{for } |x|<1 \\ 0, & \text{for } |x|>1 \end{cases}$ and hence prove that $\int\limits_0^{\infty}\left(\dfrac{\sin x}{x}\right)^4 dx = \dfrac{\pi}{3}$.

Sol:

$$\begin{aligned}
F\{f(x)\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-|x|)e^{isx}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-|x|)(\cos sx + i\sin sx)dx \\
&= \frac{2}{\sqrt{2\pi}}\int_0^1 (1-x)\cos sx dx + i.0, \quad \text{since } (1-|x|)\sin sx \text{ is odd}
\end{aligned}$$

$$\begin{aligned}
F\{f(x)\} &= \sqrt{\frac{2}{\pi}}\left[(1-x)\left(\frac{\sin sx}{s}\right) - (-1)\left(\frac{-\cos sx}{s^2}\right)\right]_0^1 \\
&= \sqrt{\frac{2}{\pi}}\left[\left(-\frac{\cos s}{s^2}\right) - \left(\frac{-1}{s^2}\right)\right] \\
&= \sqrt{\frac{2}{\pi}}\left(\frac{1-\cos s}{s^2}\right) = \sqrt{\frac{2}{\pi}}.\frac{2\sin^2(s/2)}{s^2}
\end{aligned}$$

By Parseval's identity $\int\limits_{-\infty}^{\infty} |F(s)|^2 ds = \int\limits_{-\infty}^{\infty} |f(x)|^2 dx$

$\Rightarrow \int\limits_{-\infty}^{\infty} \frac{2}{\pi}.\frac{4\sin^4(s/2)}{s^4} ds = \int\limits_{-1}^{1} (1-|x|)^2 dx$

$\Rightarrow \frac{8}{\pi}.2\int\limits_{0}^{\infty} \frac{\sin^4(s/2)}{s^4} ds = 2\int\limits_{0}^{1} (1-x)^2 dx$

Put $t = \frac{s}{2} \Rightarrow 2t = s$. Therefore $2\,dt = ds$ and $t = 0$ to $t = \infty$

$\Rightarrow \frac{8}{\pi}\int\limits_{0}^{\infty} \frac{\sin^4 t}{(2t)^4}.2dt = \left[\frac{(1-x)^3}{-3}\right]_0^1 \Rightarrow \int\limits_{0}^{\infty} \frac{\sin^4 t}{t^4} dt = \frac{\pi}{3}.$

(OR)

b. Using transform method to evaluate $\int\limits_{0}^{\infty} \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)}$.

Sol: Consider $f(x) = e^{-ax}$ and $g(x) = e^{-bx}$.

$$F_c(e^{-ax}) = \sqrt{\frac{2}{\pi}} \int\limits_{0}^{\infty} e^{-ax}\cos sx dx = \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+s^2}\left(-a\cos sx + s\sin sx\right)\right]_0^{\infty}$$
$$= \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2}.$$

Similarly $G_c(s) = \sqrt{\frac{2}{\pi}}.\frac{b}{b^2+s^2}$

Using Parseval's identity $\int\limits_{0}^{\infty} F_c(s)G_c(s) ds = \int\limits_{0}^{\infty} f(x)g(x) dx$

$\frac{2}{\pi}\int\limits_{0}^{\infty} \frac{ab}{(a^2+s^2)(b^2+s^2)} ds = \int\limits_{0}^{\infty} e^{-(a+b)x} dx \Rightarrow \frac{2}{\pi}\int\limits_{0}^{\infty} \frac{ab}{(a^2+s^2)(b^2+s^2)} ds = \frac{1}{a+b}$

$\Rightarrow \int\limits_{0}^{\infty} \frac{dx}{(a^2+x^2)(b^2+x^2)} = \frac{\pi}{2ab(a+b)}.$

32. a. i. Find $Z(a^n)$ and $Z(n^2)$.

Sol:

$$\begin{aligned} Z(a^n) &= \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n \\ &= 1 + az^{-1} + (az^{-1})^2 + (az^{-1})^3 + \dots \\ &= (1-az^{-1})^{-1} \text{ if } \left|\frac{a}{z}\right| < 1 \\ &= \left(\frac{z-a}{z}\right)^{-1} = \frac{z}{z-a} \text{ if } |z| > |a| \end{aligned}$$

To find $Z\left\{n^2\right\}$:

$Z\{n\} = -z\frac{d}{dz}Z(1)$ by property

$Z\{n\} = -z\frac{d}{dz}\left(\frac{z}{z-1}\right) = -z\left[\frac{(z-1).1 - z.1}{(z-1)^2}\right] = \frac{z}{(z-1)^2}.$

$Z\left\{n^2\right\} = Z\left\{n.n\right\} = -z\dfrac{d}{dz}\left(\dfrac{z}{(z-1)^2}\right) = -z\left[\dfrac{(z-1)^2.1 - z.2(z-1)}{(z-1)^4}\right] = \dfrac{z(z+1)}{(z-1)^3}.$

ii. Using residues find the inverse $Z-$ transform of $\dfrac{z}{(z-1)(z-2)}$.

Sol: Given $f(z) = \dfrac{z}{(z-1)(z-2)}$.

$f(z)z^{n-1} = \dfrac{z^n}{(z-1)(z-2)}$ has simple poles at $z=1$ and $z=2$.

Therefore $f(n) = \sum R$ where $\sum R$ is the sum of the residue of $f(z)z^{n-1}$.

$R_1 = \{\text{Residue}\}_{z=1} = \lim\limits_{z\to 1}(z-1)\dfrac{z^n}{(z-1)(z-2)} = -1$ and

$R_2 = \{\text{Residue}\}_{z=2} = \lim\limits_{z\to 2}(z-2)\dfrac{z^n}{(z-1)(z-2)} = 2^n.$

Therefore $f(n) = R_1 + R_2 = 2^n - 1$.

(OR)

b. Solve the equation $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$, given $y_0 = y_1 = 0$ by using $Z-$ transform.

Sol: Taking Z-transform on both sides of the equation, we get

$Z(y_{n+2}) + 6Z(y_{n+1}) + 9Z(y_n) = Z(2^n)$

$z^2\left[Y(z) - y_0 - \dfrac{y_1}{z}\right] + 6\left[z(Y(z) - y_0)\right] + 9Y(z) = \dfrac{z}{z-2}$

$\Rightarrow (z^2+6z+9)Y(z) = \dfrac{z}{z-2} \Rightarrow Y(z) = \dfrac{z}{(z-2)(z^2+6z+9)}$

$\Rightarrow Y(z) = \dfrac{z}{(z-2)(z+3)^2}.$

$Y(z)$ has simple pole at $z=2$ and pole of order 2 at $z=-3$.

Therefore $y(n) = \sum R$ where $\sum R$ is the sum of the residue of $Y(z)z^{n-1}$.

$R_1 = \{\text{Residue}\}_{z=2} = \lim\limits_{z\to 2}(z-2)\dfrac{z^n}{(z-2)(z+3)^2} = \dfrac{2^n}{25}$ and

$R_2 = \{\text{Residue}\}_{z=-3} = \lim\limits_{z\to -3}\dfrac{d}{dz}(z+3)^2\dfrac{z^n}{(z-2)(z+3)^2}$

$= \lim\limits_{z\to -3}\left[\dfrac{(z-2).nz^{n-1} - z^n.1}{(z-2)^2}\right] = \dfrac{(-3)^n}{25}\left[\dfrac{5n}{3} - 1\right].$

Hence $y(n) = \dfrac{2^n}{25} + \dfrac{(-3)^n}{25}\left[\dfrac{5n}{3} - 1\right].$