Wednesday, December 15, 2021 1:15 PM

Beta, Gamma Function

$\beta(m,n)$ $\quad$ $\Gamma(n)$

Beta function

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad \text{where } m,n>0$$

put $x = 1-y$

$$\beta(n,m) = \int_0^1 y^{n-1}(1-y)^{m-1}\,dy$$

Properties

1. $\beta(m,n) = \beta(n,m)$
2. $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$

   Euler's integral of the first kind.

3. $\beta(m,n) = \int_0^{\infty} \frac{y^{m-1}}{(1+y)^{m+n}}\,dy$
4. $\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$

Gamma Function

$$\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1}\,dx \quad (n>0)$$

Properties

1. $\int_0^{\infty} e^{-ay} y^{n-1}\,dy = \frac{\Gamma(n)}{a^n}$
2. $\Gamma(n+1) = n!$ ✓

   $= n\,\Gamma n$

   eg) $\Gamma(1) = 1$

   $\Gamma(2) = 1$

   $\Gamma(3) = 2$

   $\Gamma\left(\frac{1}{2}+1\right) = \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$

3. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} = 1.772$
4. $\int_0^{\pi/2} \sin^p x \cos^q x\,dx = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$
5. $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}$
6. $\Gamma m\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)$
7. $\Gamma m\,\Gamma(m-1) = \frac{\pi}{\sin m\pi}$