18MAB102T
ADVANCED CALCULUS AND COMPLEX ANALYSIS

UNIT-I
MULTIPLE INTEGRALS

Dr. Meena Parvathy Sankar
& Dr. V. Vidhya

TOPICS DISCUSSED

- INTRODUCTION
- REGION OF INTEGRATION
- CHANGING THE ORDER OF INTEGRATION
- PLANE AREA USING DOUBLE INTEGRATION
  - CARTESIAN FORM
  - POLAR FORM
- CHANGE OF VARIABLE FROM CARTESIAN TO POLAR COORDINATES
- VOLUME AS A TRIPLE INTEGRAL

INTRODUCTION

- When a function $f(x)$ is integrated with respect to x between the limits a and b, we get the double integral $\int_a^b f(x)dx$.
- If the integrand is a function $f(x,y)$and if it is integrated with respect to x and y repeatedly between the limits x_0 and x_1 (for x) and between the limits y_0 and y_1 (for y) we get a **double integral** that is denoted by the symbol $\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x,y)dxdy$.
- Extending the concept of double integral one step further, we get the **triple integral,** denoted by $\int_{z_0}^{z_1}\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x,y,z)dxdydz$.

EVALUATION OF DOUBLE AND TRIPLE INTEGRALS

- To evaluate $\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x,y)\,dx\,dy$ first integrate $f(x,y)$ with respect to x partially, treating y as constant temporarily, between the limits x_0 and x_1.
- Then integrate the resulting function of y with respect to y between the limits y_0 and y_1 as usual.
- In notation $\int_{y_0}^{y_1}\left[\int_{x_0}^{x_1} f(x,y)\,dx\right]dy$ (for double integral)

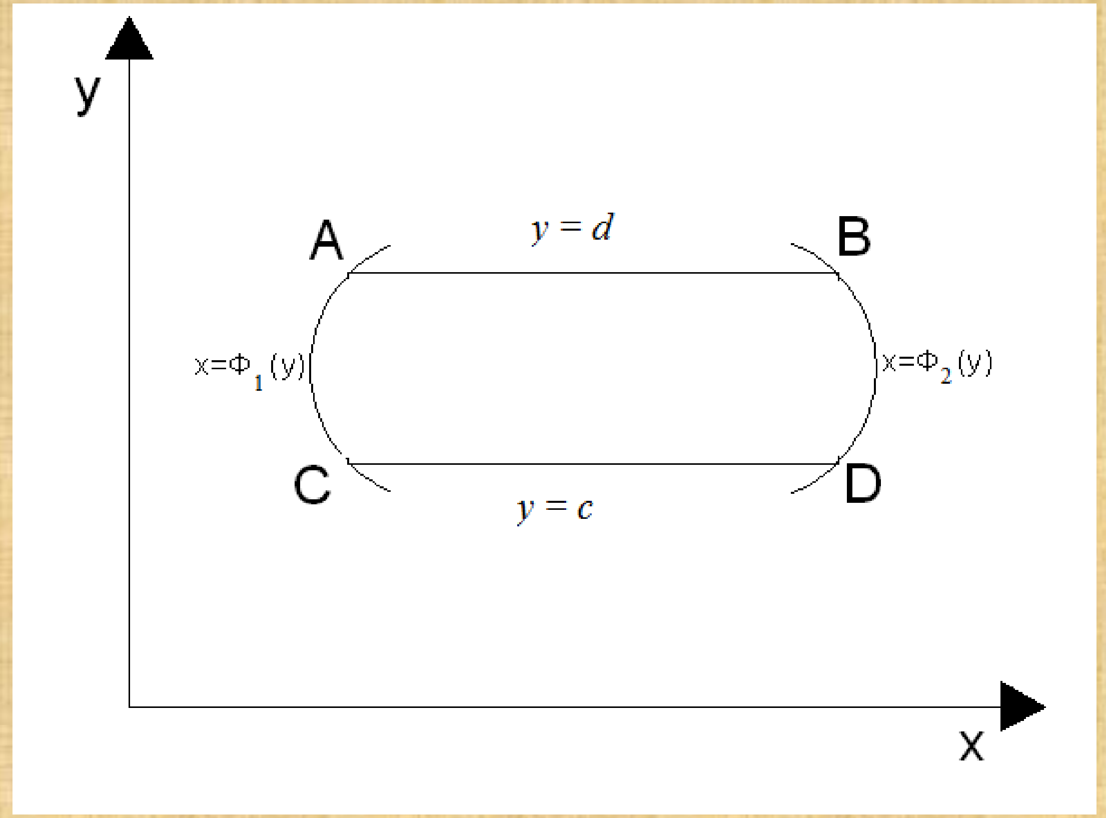
  $\int_{z_0}^{z_1}\left\{\int_{y_0}^{y_1}\left[\int_{x_0}^{x_1} f(x,y,z)\,dx\right]dy\right\}dz$ (for triple integral).

Note:

- Integral with variable limits should be the innermost integral and it should be integrated first and then the constant limits.

REGION OF INTEGRATION

Consider the double integral $\int_c^d \int_{\varphi_1(y)}^{\varphi_2(y)} f(x, y)dxdy$, x varies from $\varphi_1(y)$ to $\varphi_2(y)$ and y varies from c to d. (i.e) $\varphi_1(y) \leq x \leq \varphi_2(y)$and $c \leq y \leq d$. These inequalities determine a region in the $xy - plane$, which is shown in the following figure.This region ABCD is known as the region of integration



EXAMPLE :1

Evaluate $\int_0^1 \int_0^2 y^2 x \, dy \, dx$

Solution:

$$\int_0^1 \int_0^2 y^2 x \, dy \, dx = \int_0^1 x[y^3/3]_0^2 \, dx$$

$$= \frac{8}{3} \int_0^1 x \, dx$$

$$= \frac{8}{3} \left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{4}{3}$$

EXAMPLE :2

Evaluate $\int_2^3 \int_1^2 \frac{1}{xy}\, dy\, dx$

Solution:

$$\int_2^3 \int_1^2 \frac{1}{xy}\, dy\, dx = \int_2^3 [\log x]_1^2 \frac{1}{y}\, dy$$

$$=(log2 - log1) \int_2^3 \frac{1}{y}\, dy$$

$$=log2\ [logy]_2^3$$

$$=log2(log3 - log2)$$

$$=log2. \log\ (3/2)$$

EXAMPLE :3

Evaluate $\int_0^2 \int_1^3 \int_1^2 xy^2 z\,dz\,dy\,dx$

Solution:

$$\int_0^2 \int_1^3 \int_1^2 xy^2 zdzdydx = \int_0^2 \int_1^3 \left[\frac{z^2}{2}\right]_1^2 xy^2 dydx$$

$$= \int_0^2 \int_1^3 \frac{3}{2} xy^2 dydx$$

$$= \frac{3}{2} \int_0^2 \left[\frac{y^3}{3}\right]_1^3 xdx$$

$$= \frac{26}{2} \left[\frac{x^2}{2}\right]_0^2 = 26$$

EXAMPLE :4

Evaluate $\int_0^1 dx \int_0^2 dy \int_1^2 yx^2 z dz$

Solution:

$$\int_0^1 dx \int_0^2 dy \int_1^2 yx^2 z dz = \int_0^1 dx \int_0^2 dy \left[\frac{z^2}{2}\right]_1^2 \; yx^2$$

$$= \frac{3}{2}\int_0^1 \left[\frac{y^2}{2}\right]_0^2 x^2 dx$$

$$= \frac{3}{2}\int_0^1 2\, x^2 dx$$

$$= \left[\frac{x^3}{3}\right]_0^2 \; = 1$$

EXAMPLE :5

Evaluate $\int_0^{\pi}\int_0^{\frac{\pi}{2}}\int_0^1 r^2 \sin\theta \, drd\theta d\emptyset$

Solution:

$$\int_0^{\pi}\int_0^{\frac{\pi}{2}}\int_0^1 r^2 \sin\theta \, drd\theta d\emptyset = \int_0^{\pi}\int_0^{\frac{\pi}{2}} \sin\theta \left[\frac{r^3}{3}\right]_0^1 d\theta d\emptyset$$

$$= \frac{1}{3}\int_0^{\pi}\int_0^{\frac{\pi}{2}} \sin\theta d\theta d\emptyset$$

$$=\frac{1}{3}\int_0^{\pi}[-cos\theta]_0^{\frac{\pi}{2}} d\emptyset$$

$$= \frac{1}{3}\int_0^{\pi} d\emptyset$$

$$= \frac{\pi}{3}$$

EXAMPLE :6

Evaluate $\int_0^1 \int_0^x dx\, dy$

Solution:

$$\int_0^1 \int_0^x dy\, dx = \int_0^1 [y]_0^x\, dx$$

$$= \int_0^1 x\, dx$$

$$= \left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{2}$$

EXAMPLE :7

Evaluate $\int_0^a \int_0^x \int_0^y xyzdxdydz$

Solution:

$$I = \int_0^a \int_0^x [\int_0^y zdz]xydydx$$

$$=\int_0^a \int_0^x \left[\frac{z^2}{2}\right]_0^y xydydx$$

$$= \int_0^a \int_0^x \left[\frac{y^2}{2}\right] xydydx$$

$$=\int_0^a \int_0^x \left[\frac{y^3}{2}\right] dy\, xdx \;\; = \int_0^a \left[\frac{y^4}{8}\right]_0^x xdx$$

$$= \left[\frac{x^6}{48}\right]_0^a = \frac{a^6}{48}$$

EXAMPLE :8

Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dxdydz}{\sqrt{1-x^2-y^2-z^2}}$

Solution:

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1}\left(\frac{z}{\sqrt{1-x^2-y^2}}\right)\right]_0^{\sqrt{1-x^2-y^2}} dxdy$$

$$=\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{\pi}{2} dxdy = \frac{\pi}{2} \int_0^1 [y]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{\pi}{2} \int_0^1 \sqrt{1-x^2} dx$$

$$=\frac{\pi}{2}\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}(x)\right]_0^1$$

$$= \frac{\pi^2}{8}$$

EXAMPLE :9

Evaluate $\int_0^{\pi}\int_0^{a\sin\theta} r dr d\theta$

Solution:

$$I= \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a\sin\theta} d\theta$$

$$=\frac{1}{2}\int_0^{\pi} a^2 sin^2\theta\, d\theta$$

$$=\frac{a^2}{2}\int_0^{\pi}\left[\frac{1-cos2\theta}{2}\right] d\theta$$

$$=\frac{a^2}{2} X \frac{1}{2}\left[\theta-\frac{sin2\theta}{2}\right]_0^{\pi}=\frac{\pi a^2}{4}$$

Example :10

Evaluate $\int_{-\pi/2}^{\pi/2}\int_{0}^{2\cos\theta} r^2 dr d\theta$

Solution:

$$Let \quad I=\int_{-\pi/2}^{\pi/2}\int_{0}^{2\cos\theta} r^2 dr d\theta = \int_{-\pi/2}^{\pi/2}\left[\frac{r^3}{3}\right]_0^{2\cos\theta} d\theta$$

$$=\frac{1}{3}\int_{-\pi/2}^{\pi/2} 8\cos^3\theta d\theta$$

$$=\frac{8}{3}\int_{-\pi/2}^{\pi/2} \cos^3\theta d\theta$$

$$=\frac{8}{3}.2\int_{0}^{\pi/2} \cos^3\theta d\theta = \frac{16}{3}.\frac{2}{3}.1 = \frac{32}{9}$$

EXAMPLE :11

Evaluate $\int_0^1\int_0^1 \frac{dxdy}{\sqrt{1-x^2}\sqrt{1-y^2}}$

Solution:

$$Let\ \ I=\int_0^1\int_0^1 \frac{dxdy}{\sqrt{1-x^2}\sqrt{1-y^2}} = \int_0^1 \frac{dx}{\sqrt{1-x^2}}\int_0^1 \frac{dy}{\sqrt{1-y^2}}$$

$$=[\sin^{-1}x]_0^1[\sin^{-1}y]_0^1$$

$$=[\sin^{-1}1-\sin^{-1}0][\sin^{-1}1-\sin^{-1}0]$$

$$=\frac{\pi}{2}.\frac{\pi}{2}=\frac{\pi}{4}$$

EXAMPLE :12

Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} x^2 y\,dx\,dy$

Solution:

$$Let\ \ I = \int_0^a \int_0^{\sqrt{a^2-x^2}} x^2 y\,dx\,dy = \int_0^a \int_0^{\sqrt{a^2-x^2}} x^2 y\,dy\,dx$$

$$= \int_0^a x^2 \left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx = \frac{1}{2}\int_0^a x^2 (a^2 - x^2)\,dx$$

$$= \frac{1}{2}\int_0^a x^2 (a^2 x^2 - x^4)\,dx = \frac{1}{2}\left[a^2 \frac{x^3}{3} - \frac{x^5}{5}\right]_0^a$$

$$= \frac{1}{2}\left[a^2 \frac{a^3}{3} - \frac{a^5}{5}\right] = \frac{1}{2}.\frac{2a^5}{5} = \frac{a^5}{5}$$

PROBLEMS FOR PRACTICE

Evaluate the following

1. $\int_0^2 \int_0^1 4xy\, dxdy$ — Ans: 4

2. $\int_1^b \int_1^a \frac{1}{xy}\, dxdy$ — Ans: loga.logb

3. $\int_0^1 \int_0^x\, dxdy$ — Ans: 1/2

4. $\int_0^\pi \int_0^{\sin\theta} r\, drd\theta$ — Ans: π/4

5. $\int_0^1 \int_0^2 \int_0^3 xyzdxdydz$ — Ans: 9/2

6. $\int_0^1 \int_0^z \int_0^{y+z} dzdydx$ — Ans: ½

REGION OF INTEGRATION

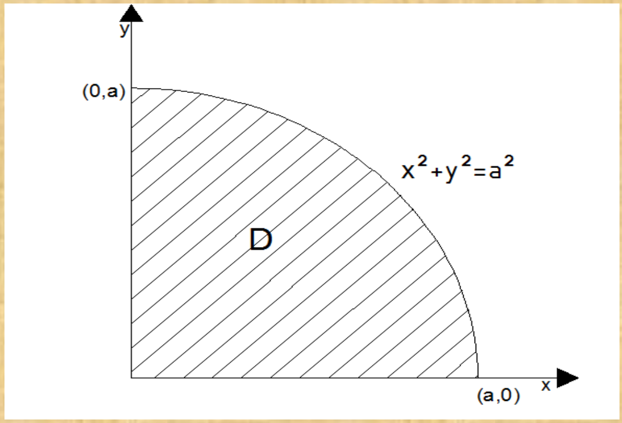
EXAMPLE :1

Sketch the region of integration for $\int_0^a \int_0^{\sqrt{a^2-x^2}} f(x,y)\,dydx.$

Solution:

Given $x = 0\ and\ x = a$; $y = 0\ \ and\ \ y^2 = a^2 - x^2$

$y = 0$ and $x^2 + y^2 = a^2$

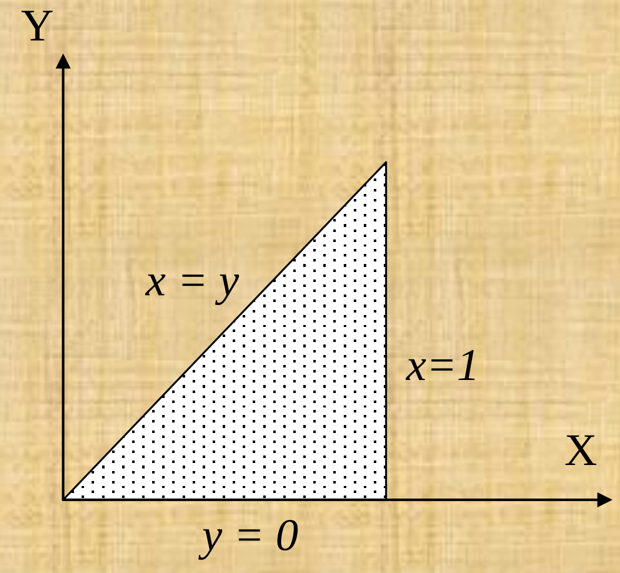


EXAMPLE :2

Sketch the region of integration for $\int_0^1 \int_0^x f(x,y)\, dydx$.

Solution:

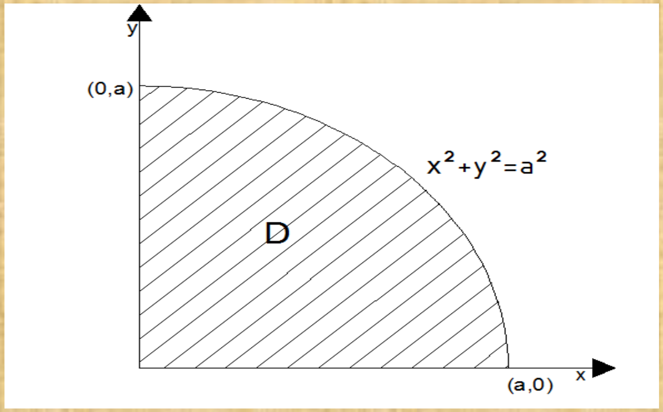
Given $x = 0$; $x = 1$ and $y = 0$; $y = x$.



EXAMPLE :3

Evaluate $\iiint_D \; xyz\, dxdydz$ where D is the region bounded by the positive octant of the sphere $x^2 + y^2 + z^2 = a^2$

Solution:



$$I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyz\, dzdydx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} xy \left[\frac{z^2}{2}\right]_0^{\sqrt{a^2-x^2-y^2}} dydx$$

$$=\frac{1}{2}\int_0^a\int_0^{\sqrt{a^2-x^2}} xy\,(a^2-x^2-y^2)dydx$$

$$=\frac{1}{2}\int_0^a\int_0^{\sqrt{a^2-x^2}} x\,(a^2y-yx^2-y^3)dydx$$

$$=\frac{1}{2}\int_0^a\left[a^2\frac{y^2}{2}-x^2\frac{y^2}{2}-\frac{y^4}{4}\right]_0^{\sqrt{a^2-x^2}} xdx$$

$$=\frac{1}{8}\int_0^a x\,(a^2-x^2)^2\,dx$$

$$=\frac{1}{8}\int_0^a(a^4x-2a^2x^3+x^5)\ dx$$

$$=\frac{1}{8}\left[a^4\frac{x^2}{2}-2a^2\frac{x^4}{4}-\frac{x^6}{6}\right]_0^a=\ \frac{a^6}{48}.$$

PROBLEMS FOR PRACTICE

1.Sketch the region of integration for the following

(i) $\int_0^4 \int_{\frac{y^2}{4}}^{y} \frac{ydxdy}{x^2+y^2}$

(ii) $\int_0^a \int_{a-x}^{\sqrt{a^2-x^2}} y\, dydx$

(iii) $\int_0^1 \int_x^1 \frac{ydxdy}{x^2+y^2}$

2.Evaluate $\iiint_V (xy + yz + zx)dxdydz$, where V is the region of space bounded by *x=0,x=1,y=0,y=2,z=0 and z=3.*

Ans: 33/2

3. Evaluate $\iiint_V \frac{dxdydz}{(1+x+y+z)^3}$, where V is the region of space bounded by *x=0,y=0,z=0 and x+y+z=1*

Ans: $\frac{1}{16}(8log2 - 5)$

4. Evaluate $\iiint_V dxdydz$, where V is the region of space bounded by *x=0,,y=0,,z=0 and 2x+3y+4z=12.*

Ans: 12

CHANGING THE ORDER OF INTEGRATION

CHANGE OF ORDER OF INTEGRATION

- If the limits of integration in a double integral are constants, then the order of integration can be changed, provided the relevant limits are taken for the concerned variables.
- When the limits for inner integration are functions of a variable, the change in the order of integration will result in changes in the limits of integration.

i.e. $\int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y)\,dxdy$ will take the form

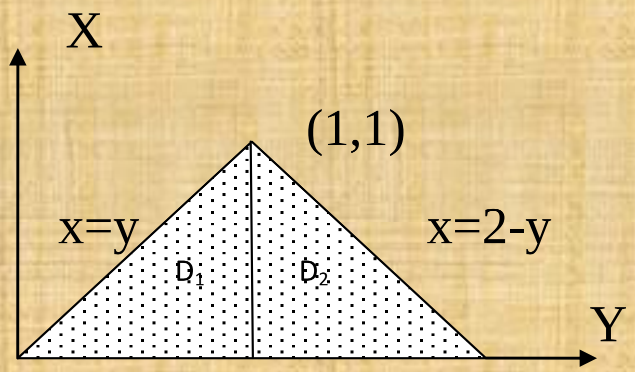
$$\int_a^b \int_{h_1(x)}^{h_2(x)} f(x,y)\,dydx$$

- This process of converting a given double integral into its equivalent double integral by changing the order of integration is called the **change of order of integration**.

EXAMPLE :1

Evaluate $\int_0^1 \int_y^{2-y} xy\,dx\,dy$ by changing the order of integration.

Solution:



Given y : 0 to 1 and x : y to 2-y

By changing the order of integration,

In Region D_1 x : 0 to 1 and y : 0 to x.

In Region D_2 x : 1 to 2 and y : 0 to 2-x.

$$\int_0^1 \int_y^{2-y} xydx\, dy = \int_0^1 \int_0^x xydy\, dx + \int_1^2 \int_0^{2-x} xydy\, dx$$

$$= \int_0^1 x \left[\frac{y^2}{2}\right]_0^x dx + \int_1^2 x \left[\frac{y^2}{2}\right]_0^{2-x} dx$$

$$= \frac{1}{2}\int_0^1 x^3\, dx + \frac{1}{2} \int_1^2 [4x - 4x^2 + x^3] dx$$

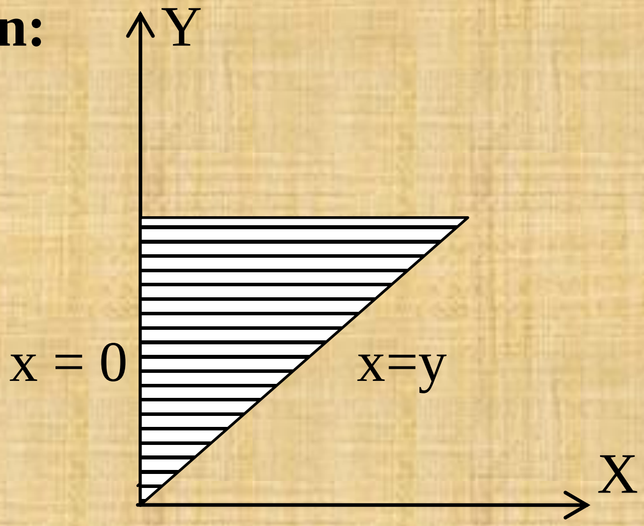
$$= \frac{1}{2}\left[\frac{x^4}{4}\right]_0^1 + \frac{1}{2}\left[2x^2 - \frac{4x^3}{3} + \frac{x^4}{4}\right]_1^2$$

$$= \frac{1}{8} + \frac{5}{24} = \frac{1}{3}$$

EXAMPLE :2

Evaluate $\int_0^{\infty}\int_0^{y} ye^{-\frac{y^2}{x}}dxdy$ by changing the order of integration.

Solution:



Given x=0, x = y, y = 0, y = ∞ .

By changing the order of integration y: x to ∞, x : 0 to ∞

$$\int_0^\infty \int_0^y ye^{-\frac{y^2}{x}} dxdy = \int_0^\infty \int_x^\infty ye^{-\frac{y^2}{x}} dydx$$

$$= \int_0^\infty \int_x^\infty ye^{-\frac{y^2}{x}} d\left(\frac{y^2}{2}\right) dx$$

$$= \frac{1}{2}\int_0^\infty \left[\frac{e^{-\frac{y^2}{x}}}{-1/x}\right]_x^\infty dx \ = \frac{1}{2}\int_0^\infty xe^{-x} dx$$

$Take\ u = x, dv = e^{-x}dx \quad implies\ du = dx, v = -e^{-x},$

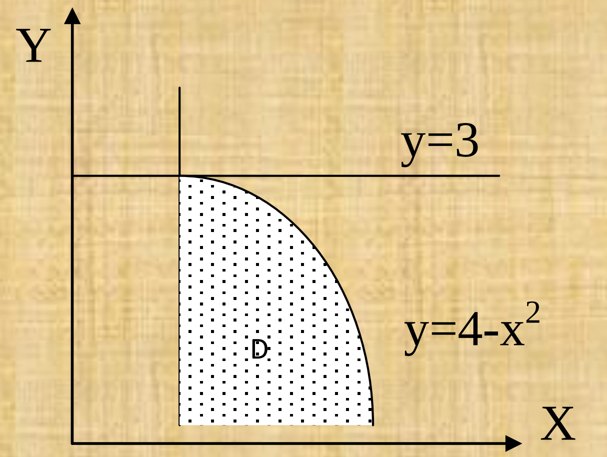
by integration by parts,

$$= \frac{1}{2}\left[x\left(\frac{e^{-x}}{-1}\right) - e^{-x}\right]_0^\infty = \frac{1}{2}$$

EXAMPLE :3

Evaluate $\int_0^3 \int_1^{\sqrt{4-y}} (x+y)dxdy$ by changing the order of integration.

Solution:



Given y=0,y=3 and x=1, x= $\sqrt{4-y}$

By changing the order of integration,

In region D, x : 1 to 2 and y : 0 to 4-x^2

$$\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\,dx\,dy = \int_1^2 \int_0^{4-\mathrm{x}^2} (x+y)\,dy\,dx$$

$$= \int_1^2 \left[xy + \frac{y^2}{2}\right]_0^{4-\mathrm{x}^2} dx$$

$$= \int_1^2 \left[x(4-\mathrm{x}^2) + \frac{(4-\mathrm{x}^2)^2}{2}\right] dx$$

$$= \int_1^2 \left[\frac{x^4}{4} - x^3 - 4x^2 + 4x + 8\right] dx$$

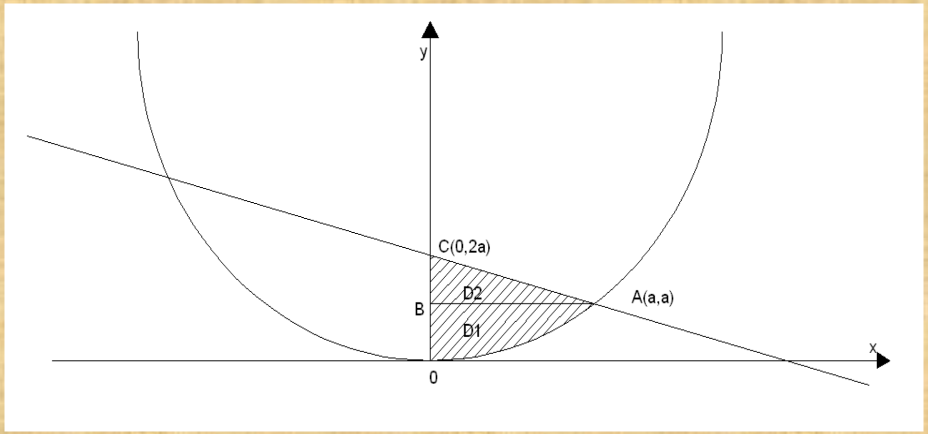
$$= \left[\frac{x^5}{10} - \frac{x^4}{4} - 4\frac{x^3}{3} + 2x^2 + 8x\right]_1^2$$

$$= \frac{241}{8}$$

EXAMPLE :4

Evaluate $\int_0^a \int_{x^2/a}^{2a-x} xy\, dy\, dx$ by changing the order of integration.

Solution:



Given y : x^2/a to $2a - x$ and x : 0 to a

By changing the order of integration,

In Region D_1 x : 0 to $\sqrt{ay}$ and y : 0 to a.

In Region D_2 x : 0 to $2a - y$ and y : a to 2a.

$$\int_0^a \int_{x^2/a}^{2a-x} xy\,dy\,dx = \int_0^a \int_0^{\sqrt{ay}} xydy\,dx + \int_a^{2a} \int_0^{2a-y} xydy\,dx$$

$$= \int_0^a y\left[\frac{x^2}{2}\right]_0^{\sqrt{ay}} dy + \int_0^1 y\left[\frac{x^2}{2}\right]_0^{2a-y} dy$$

$$= \frac{a}{2}\int_0^a y^2\,dy + \frac{1}{2}\int_a^{2a} [4a^2y - 4ay^2 + y^3]dy$$

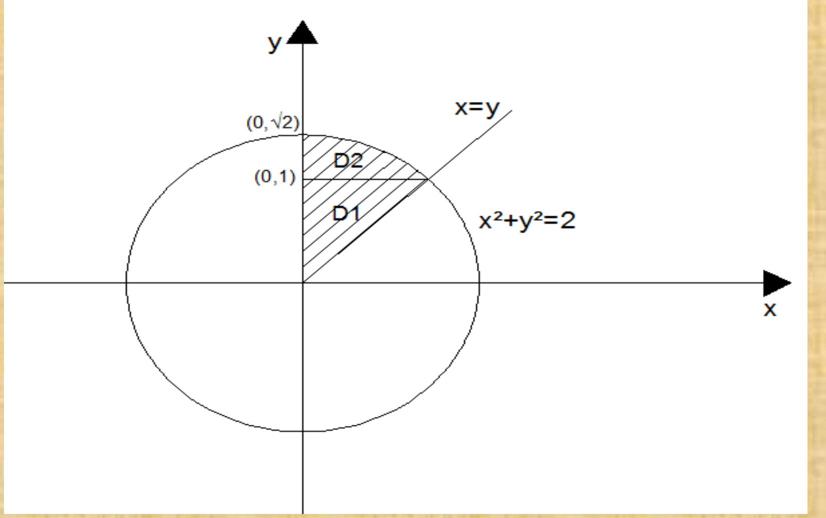
$$= \frac{a}{2}\left[\frac{y^3}{3}\right]_0^a + \frac{1}{2}\left[2a^2y^2 - \frac{4ay^3}{3} + \frac{y^4}{4}\right]_a^{2a}$$

$$= \frac{a^4}{6} + \frac{5a^4}{24} = \frac{3a^4}{8}.$$

EXAMPLE :5

Evaluate $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}} \, dy \, dx$ by changing the order of integration.

Solution:



Given x = 0, x = 1 and y = x, $y^2 = 2-x^2$

By changing the order of integration

In Region D_1, y : 0 to 1,x : 0 to y

In Region D_2, y : 1 to $\sqrt{2}$, x : 0 to $\sqrt{2-y^2}$

$$I = \int_0^1 \int_0^y \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy + \int_1^{\sqrt{2}} \int_0^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy$$

$$= \int_0^1 \left[\sqrt{x^2+y^2}\right]_0^{\sqrt{2}} dy + \int_1^{\sqrt{2}} \left[\sqrt{x^2+y^2}\right]_0^{\sqrt{2-y^2}} dy$$

$$= \int_0^1 (\sqrt{2}y - y)\,dy + \int_1^{\sqrt{2}} (\sqrt{2} - y)\,dy$$

$$= \left((\sqrt{2}-1)\frac{y^2}{2}\right)_0^1 + \left(\sqrt{2}y - \frac{y^2}{2}\right)_1^{\sqrt{2}}$$

$$= 1 - \frac{1}{\sqrt{2}}$$

EXAMPLE:6

INSTITUTE OF SCIENCE & TECHNOLOGY
(Deemed to be University u/s 3 of UGC Act, 1956)

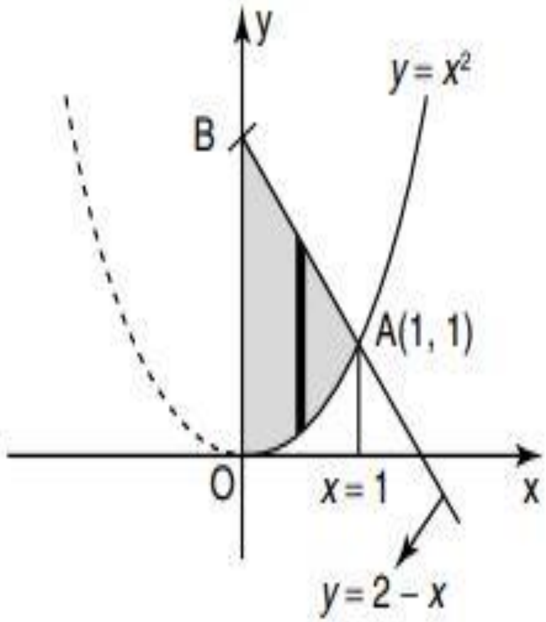
Change the order of integration in $\int_0^1 \int_{x^2}^{2-x} xy\, dydx$ and hence evaluate.

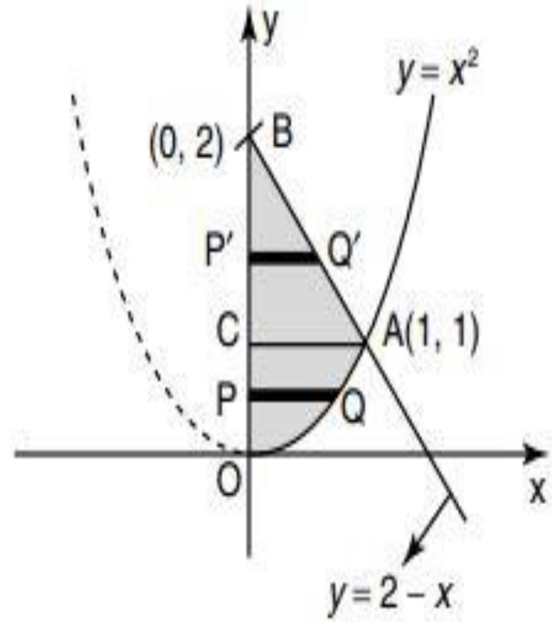
Solution.

Let I = $\int_0^1 \int_{x^2}^{2-x} xy\, dydx$

The region of integration is bounded by $x = 0, x = 1, y = x^2, y = 2 - x$.

In the given integral, first integrate with respect to y and then w.r.to x. After changing the order we have to first integrate w.r.to x, then w.r.to y.


y
$y = x^2$
B
A(1, 1)
O
$x = 1$
x
$y = 2 - x$


y
$y = x^2$
(0, 2)
B
P′
Q′
C
A(1, 1)
P
Q
O
x
$y = 2 - x$

To find A, solve $y = x^2, \quad y = 2 - x$

$$\Rightarrow \qquad x^2 = 2 - x \quad \Rightarrow \quad x^2 + x - 2 = 0 \quad \Rightarrow \quad (x+2)(x-1) = 0 \quad \Rightarrow \quad x = -2, 1$$

Since the region of integration is OAB, $x = 1 \Rightarrow y = 1$

$\therefore$ A is (1, 1) and B is (0, 2), which is the point of intersection of y-axis $x = 0$ and $y = 2 - x$

Now to find the x limits, take a strip parallel to the x-axis. We see there are two types of strips PQ and P′Q′ after the change of order of integration (see Fig. 8.16) with right end points Q and Q′ are respectively on the parabola $y = x^2$ and the line $y = 2 - x$. So, the region OAB splits into two regions OAC and CAB

Hence, the given integral I is written as the sum of two integrals

In the region OAC, x varies from 0 to $\sqrt{y}$ and y varies from 0 to 1

In the region CAB, x varies from 0 to $2-y$ and y varies from 1 to 2

$$\therefore \quad I = \iint_{OAB} xy\,dxdy = \iint_{OAC} xy\,dxdy + \iint_{CAB} xy\,dxdy$$

$$= \int_0^1 \int_0^{\sqrt{y}} x\,y\,dx\,dy + \int_1^2 \int_0^{2-y} xy\,dx\,dy$$

$$= \int_0^1 y \cdot \left[\frac{x^2}{2}\right]_0^{\sqrt{y}} dy + \int_1^2 y\left[\frac{x^2}{2}\right]_0^{2-y} dy$$

$$= \frac{1}{2}\int_0^1 y\,y\,dy + \frac{1}{2}\int_1^2 y\cdot(2-y)^2\,dy$$

$$= \frac{1}{2}\int_0^1 y^2 dy + \frac{1}{2}\int_1^2 y(4-4y+y^2)dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 + \frac{1}{2}\int_1^2 (4y - 4y^2 + y^3)dy$$

$$= \frac{1}{6} + \frac{1}{2}\left[4\frac{y^2}{2} - 4\frac{y^3}{3} + \frac{y^4}{4}\right]_1^2$$

$$= \frac{1}{6} + \frac{1}{2}\left[2(2^2 - 1^2) - \frac{4}{3}(2^3 - 1^3) + \frac{1}{4}(2^4 - 1^4)\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left[6 - \frac{4}{3}\times 7 + \frac{1}{4}\times 15\right] = \frac{1}{6} + \frac{1}{2}.\frac{[72 - 112 + 45]}{12} = \frac{1}{6} + \frac{5}{24} = \frac{9}{24} = \frac{3}{8}$$

PROBLEMS FOR PRACTICE

Evaluate the following by changing the order of integration

1. $\int_0^a \int_x^a (x^2 + y^2)\, dydx$ Ans: $\frac{a^4}{3}$

2. $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy\, dydx$ Ans: $\frac{3a^4}{8}$

3. $\int_0^a \int_{a-y}^{\sqrt{a^2-y^2}} y\, dxdy$ Ans: $\frac{a^3}{6}$

4. $\int_0^1 \int_y^{2-y} xy\, dxdy$ Ans: $\frac{1}{3}$