Thursday, October 28, 2021 4:12 PM

Taylor's theorem – Problems

$a = -2 \quad b = 1$

① Expand $xy^2 + 2x - 3y$ in powers of $(x+2)$ and $(y-1)$ upto second degree terms

② Expand $e^x \cos y$ in the powers of x and y upto third degree terms

③ Expand $\tan^{-1}(y/x)$ at $(1,1)$ upto quadratic terms

Solution

① $f(x,y) = xy^2 + 2x - 3y$

at $(-2,1)$

$f(-2,1) = -9$

$$f(x,y) = f(a,b) + \left[(x+2) f_x(-2,1) + (y-1) f_y(-2,1)\right] + \frac{1}{2!}\left[(x+2)^2 f_{xx}(-2,1)\right.$$
$$\left. + 2(x+2)(y-1) f_{xy}(-2,1) + (y-1)^2 f_{yy}(-2,1)\right] + \cdots$$

$$f_x = \frac{\partial f}{\partial x} = y^2 + 2 \text{ at } (-2,1) = 3$$

$$f_y = \frac{\partial f}{\partial y} = 2xy - 3 \text{ at } (-2,1) = -7$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 0$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 2x \text{ at } (-2,1) = -4$$

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 2y \text{ at } (-2,1) = 2$$

$$f(x,y) = -9 + \left[(x+2)3 + (y-1)(-7)\right] + \frac{1}{\cancel{2}}\left[(x+2)^2(0)\right.$$
$$\left. + 2(x+2)(y-1)\cdot\cancel{2} + (y-1)^2(-\cancel{4}^{\,2})\right] + \cdots$$

$$\boxed{f(x,y) = -9 + \left[3(x+2) - 7(y-1)\right] + \left[2(x+2)(y-1) - 2(y-1)^2\right] + \cdots}$$

② $f(x,y) = e^x \cos y$

$f(0,0) = e^0 \cos 0 = 1 \cdot 1 = 1$

$$f_x = \frac{\partial f}{\partial x} = e^x \cos y \text{ at } (0,0) = 1$$

$$f_y = \frac{\partial f}{\partial y} = -e^x \sin y \text{ at } (0,0) = 0$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = e^x \cos y \text{ at } (0,0) = 1$$

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -e^x \sin y \text{ at } (0,0) = 0$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -e^x \cos y \text{ at } (0,0) = -1$$

$$f_{xxx} = \frac{\partial^3 f}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial}{\partial x}\left(e^x \cos y\right) = e^x \cos y \text{ at } (0,0) = 1$$

$$f_{xxy} = \frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x \partial y}\right) = \frac{\partial}{\partial x}\left(-e^x \sin y\right) = -e^x \sin y \text{ at } (0,0) = 0$$

$$f_{xyy} = \frac{\partial^3 f}{\partial x \partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y^2}\right) = \frac{\partial}{\partial x}\left(-e^x \cos y\right) = -e^x \cos y \text{ at } (0,0) = -1$$

$$f_{yyy} = \frac{\partial^3 f}{\partial y^3} = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y^2}\right) = \frac{\partial}{\partial y}\left(-e^x \cos y\right) = e^x \sin y \text{ at } (0,0) = 0$$

$$f(x,y) = 1 + \left[x \cdot 1 + y \cdot 0\right] + \frac{1}{2}\left[x^2 \cdot 1 + \cancel{2xy \cdot 0} + y^2(-1)\right]$$
$$+ \frac{1}{6}\left[x^3 \cdot 1 + \cancel{3x^2y \cdot 0} + 3xy^2(-1) + \cancel{y^3 \cdot 0}\right] + \cdots$$

$$\boxed{f(x,y) = 1 + x + \frac{x^2}{2} - \frac{y^2}{2} + \frac{x^3}{6} - \frac{xy^2}{2} + \cdots}$$

③ $f(x,y) = \tan^{-1}(y/x)$

$f(1,1) = \tan^{-1}(1/1) = \pi/4$

$$\frac{d}{dx}\tan^{-1}x = \left(\frac{1}{1+x^2}\right)$$

$$\frac{d}{dx}\tan^{-1}(y/x) = \frac{1}{1+(y/x)^2}\left[y\left(\frac{-1}{x^2}\right)\right]$$

$$\frac{d}{dy}\tan^{-1}(y/x) = \frac{1}{1+(y/x)^2}\left(\frac{1}{x}\right)\cdot 1$$

$$f_x = \frac{\partial f}{\partial x} = \frac{1}{1+(y/x)^2}\left(\frac{-y}{x^2}\right)$$

$$\boxed{f_x = \frac{-y}{x^2+y^2}}$$

$$f_y = \frac{\partial f}{\partial y} = \frac{1}{1+(y/x)^2}\left(\frac{1}{x}\right)$$

$$\boxed{f_y = \frac{x}{x^2+y^2}}$$

$$f_{xx} = \frac{\partial}{\partial x}\left(-y(x^2+y^2)^{-1}\right)$$
$$= -y(-1)(x^2+y^2)^{-2}(2x)$$

$$\boxed{f_{xx} = \frac{2xy}{(x^2+y^2)^2}}$$

$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{2xy}{(x^2+y^2)^2}$$

$$f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2}$$

$$\boxed{f_{xy} = \frac{y^2-x^2}{(x^2+y^2)^2}}$$

$$\boxed{f_{yy} = \frac{2xy}{(x^2+y^2)^2}}$$