

# Beta, gamma Functions

## Problems

① Evaluate  $\int_0^1 \frac{dx}{\sqrt{1-x^4}}$

Sol

$$x^2 = \sin^2 \theta \Rightarrow x = \sin^{1/2} \theta$$

$$2x dx = \cos \theta d\theta$$

$$dx = \frac{\cos \theta}{2x} d\theta$$

$x=0$	$x=1$
$\theta=0$	$\theta=\pi/2$

$$dx = \frac{\cos \theta}{2 \sin^{1/2} \theta} d\theta$$

$$dx = \frac{1}{2} \cos \theta \sin^{-1/2} \theta d\theta$$

$$\Rightarrow \int_0^{\pi/2} \frac{1}{2} \frac{\cos \theta \sin^{-1/2} \theta d\theta}{\sqrt{1-\sin^2 \theta}} \left| \begin{array}{l} \sqrt{1-\sin^2 \theta} \\ \sqrt{\cos^2 \theta} \\ \cos \theta \end{array} \right.$$

$$\frac{1}{2} \int_0^{\pi/2} \sin^{-1/2} \theta d\theta \quad (n = -1/2)$$

$$\frac{1}{2} \left( \frac{\sqrt{\pi}}{2} \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n+2}{2})} \right)$$

$$\Gamma(n+1) = n \Gamma(n)$$

$$\frac{\sqrt{\pi}}{4} \frac{\Gamma(1/4)}{\Gamma(3/4)}$$

$$\Gamma(3/4) = \Gamma(1-1/4) = \frac{1}{4} \Gamma(1/4) \quad h=1/4$$

$$\frac{\sqrt{\pi}}{4} \frac{\Gamma(1/4)}{\frac{1}{4} \Gamma(1/4)}$$

$$\boxed{\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \sqrt{\pi}}$$

② Evaluate  $\int_0^{\pi/2} \sqrt{\tan \theta} d\theta$

Sol

$$\sqrt{\tan \theta} = \sqrt{\frac{\sin \theta}{\cos \theta}}$$

$$\int_0^{\pi/2} \sin^{1/2} \theta \cos^{-1/2} \theta d\theta$$

$$p=1/2, q=-1/2$$

$$\Rightarrow \frac{1}{2} \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$\Rightarrow \frac{1}{2} \beta\left(\frac{3}{4}, \frac{1}{4}\right)$$

$$\Rightarrow \frac{1}{2} \frac{\Gamma(3/4) \Gamma(1/4)}{\Gamma(1)}$$

$$= \frac{1}{2} \Gamma(3/4) \Gamma(1/4)$$

$$= \frac{1}{2} \Gamma(1/4) \Gamma\left(\frac{1}{4} + \frac{1}{2}\right)$$

$$= \frac{1}{2} \pi \sqrt{2}$$

$$= \frac{\pi}{\sqrt{2}}$$

$$\beta(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}$$

$$\Gamma(1) = 1$$

$$\Gamma(m) \Gamma(m+1/2) = \frac{\sqrt{\pi}}{2^{2m-1}} \Gamma(2m)$$

$$\Gamma(1/4) \Gamma(1/4 + 1/2) = \frac{\sqrt{\pi}}{2^{-1/2}} \Gamma(2 \cdot 1/4)$$

$$= \frac{\sqrt{\pi}}{2^{-1/2}} \Gamma(1/2)$$

$$= \pi \sqrt{2}$$

$$\boxed{\int_0^{\pi/2} \sqrt{\tan \theta} d\theta = \pi/\sqrt{2}}$$

③ Evaluate  $\int_0^1 (x \log x)^4 dx$

Sol

$$\int_0^1 x^4 (\log x)^4 dx$$

$$\text{Put } x = e^{-t}$$

$$dx = -e^{-t} dt$$

$x=0$	$x=1$
$t=\infty$	$t=0$

$$\Rightarrow - \int_{\infty}^0 e^{-4t} t^4 e^{-t} dt \Rightarrow - \int_{\infty}^0 \Rightarrow \int_0^{\infty}$$

$$\int_0^{\infty} e^{-5t} t^4 dt$$

$$\int_0^{\infty} e^{-5t} t^{5-1} dt \quad a=5$$

$$\Rightarrow \frac{\Gamma(5)}{5^5} = \frac{4!}{3125} = \frac{24}{3125}$$