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# MATHS ASSIGNMENT

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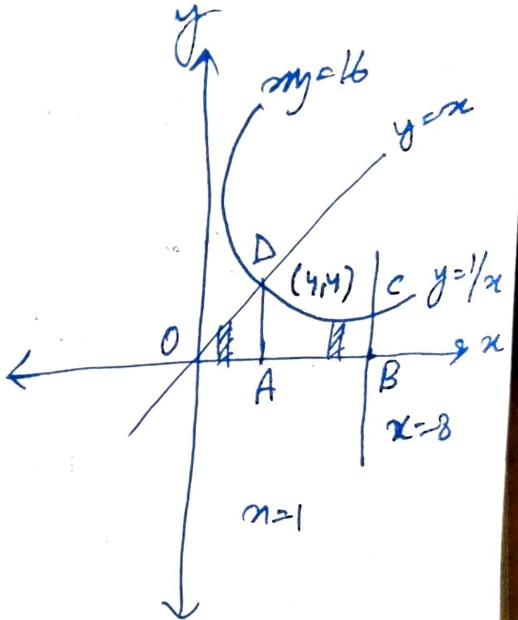
Q1. Evaluate $\iint_A x^2 \, dx \, dy$, where A is the region in the first quadrant bounded by the hyperbola $xy=16$ and the lines $y=x$, $y=0$ and $x=8$.




Ans: $x=y$, $xy=16$

So, $x^2=16$

$\Rightarrow x=4$

When $x=4$ then $y=4$ [$\because y=x$]

So, Point of intersection is $(4,4)$

At Region 1,

in $\Delta$ AOD, $x=0$ to $x=4$

At Region 2,

in $\square$BCD, $x=4$ to $x=8$

Thus, $\int_{x=0}^{x=4}\int_{y=0}^{y=x} x^2 \, dy \, dx + \int_{x=4}^{x=8}\int_{y=0}^{y=16/x} x^2 \, dy \, dx$

$\Rightarrow I = \int_0^4 x^2 [y]_0^x \, dx + \int_4^8 x^2 [y]_0^{16/x} \, dx$

$\Rightarrow \int_0^4 x^3 \, dx + \int_4^8 x^2 \left(\frac{16}{x}\right) dx \Rightarrow \left[\frac{x^4}{4}\right]_0^4 + 16\int_4^8 x \, dx$

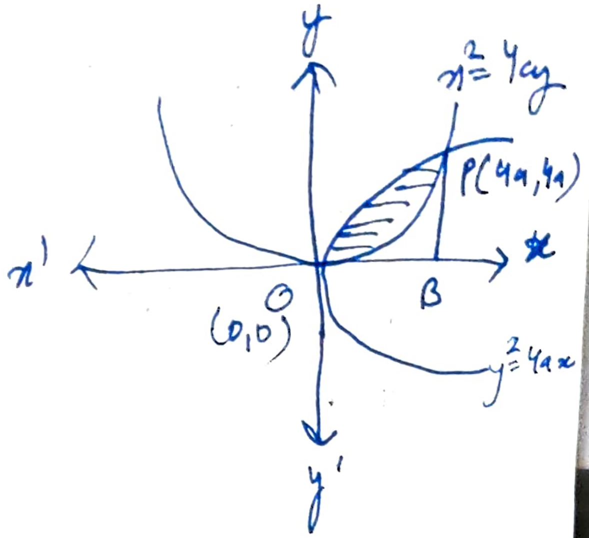
$\Rightarrow \frac{x^4}{4} + 16\left[\frac{x^2}{2}\right]_4^8 \Rightarrow (4)^3 + \frac{16}{2}[8^2-4^2]$

$\Rightarrow 64+8[48] \Rightarrow \boxed{448}$

Q2. Find the area lying between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.

Ans: $y^2 = 4ax$ — (1)

$x^2 = 4ay$ — (2)




$\therefore \left[\frac{x^2}{4a}\right]^2 \Rightarrow 4ax$ by (2)

$\Rightarrow x^4 = 64a^3x$

$\Rightarrow x[x^3 - (4a)^3] = 0$

$\therefore y = 0$ and $y = 4a$

$x = 0$ and $x = 4a$

So, points of intersection of curves are $O(0,0)$ & $P(4a, 4a)$

Required Area $\Rightarrow A =$ (Area under $y^2 = 4ax$) − (Area under $x^2 = 4ay$)

$$\Rightarrow \int_0^{4a} \sqrt{4ax}\,dx - \int_0^{4a} \frac{x^2}{4a}\,dx$$

$$\Rightarrow \sqrt{4a}\cdot\left(\frac{2}{3}\right)\left[x^{3/2}\right]_0^{4a} - \frac{1}{4a}\left(\frac{1}{3}\right)\left[x^3\right]_0^{4a}$$

$$\Rightarrow \left(\frac{4\sqrt{a}}{3} \times 4a\sqrt{4a}\right) - \left(\frac{1}{12a} \times 64a^3\right)$$

$$\Rightarrow \frac{32}{3}a^2 - \frac{16}{3}a^2 \Rightarrow \boxed{\frac{16}{3}a^2 \text{ sq. units}}$$

Q3. Evaluate $\int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a} r^2 \cdot dr.d\theta$

$$\Rightarrow I = \int_0^{\pi/2}\left[\frac{r^3}{3}\right]_{a(1-\cos\theta)}^{a} d\theta$$

$$\Rightarrow \frac{1}{3}\int_0^{\pi/2}(a^3 - a^3(1-\cos\theta)^3)\,d\theta$$

$$\Rightarrow \frac{1}{3}\int_0^{\pi/2}\left[a^3 - a^3(1-\cos^3\theta - 3\cos\theta + 3\cos^2\theta)\right]d\theta$$

$$\Rightarrow \frac{a^3}{3}\int_0^{\pi/2}(\cos^3\theta + 3\cos\theta - 3\cos^2\theta)\,d\theta$$

$$\Rightarrow \frac{a^3}{3}\left[\underset{1}{\int_0^{\pi/2}\cos^3\theta\,d\theta} + \underset{2}{\int_0^{\pi/2}3\cos\theta\,d\theta} - \underset{3}{\int_0^{\pi/2}3\cos^2\theta\,d\theta}\right]$$

1 $\rightarrow \int_0^{\pi/2}\cos\theta(1-\sin^2\theta)\,d\theta$.

let $\sin\theta . t \rightarrow \int_0^1$

$\Rightarrow \cos\theta\,d\theta \cdot dt$

$$\therefore \int_0^1 (1-t^2)\,dt \Rightarrow \left[t - \frac{t^3}{3}\right]_0^1 \Rightarrow 1 - \frac{1}{3} \Rightarrow \frac{2}{3}$$

2 $\rightarrow 3\int_0^{\pi/2}\cos\theta\,d\theta \Rightarrow 3[\sin\theta]_0^{\pi/2} \Rightarrow 3$

3 $\rightarrow 3\int_0^{\pi/2}\cos^2\theta\,d\theta \Rightarrow 3\int_0^{\pi/2}\frac{\cos 2\theta + 1}{2}\,d\theta$

$$\Rightarrow \frac{3}{2}\left[\left(\frac{\sin 2\theta}{2}\right)_0^{\pi/2} + (\theta)_0^{\pi/2}\right]^2 \Rightarrow \frac{3}{2}\left(\frac{\pi}{2}\right) \Rightarrow \frac{3\pi}{4}$$

Hence, $I = \frac{a^3}{3}\left[\frac{2}{3} + 3 - \frac{3\pi}{4}\right] \Rightarrow \frac{a^3}{3}\left[\frac{8+36+9\pi}{12}\right]$

$$\Rightarrow \boxed{\frac{a^3(44-9\pi)}{36}}$$

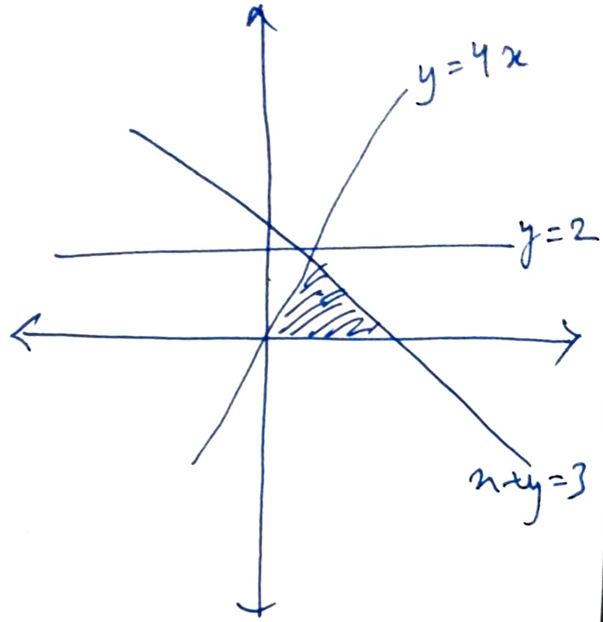
Q4. Evaluate $\iint_A (x^2+y^2)\,dx\,dy$, where $A$ is the region enclosed by the curves $y=4x$, $x+y=3$, $y=2$ and $y=0$.

Ans = $y = 4x$

| y | x |
|---|---|
| 0 | 0 |
| 4 | 1 |
| 8 | 2 |

$x+y=3$

| y | x |
|---|---|
| 0 | 3 |
| 3 | 0 |
| 1 | 2 |




$$\therefore \int_0^2 \int_{y/4}^{(3-y)} (x^2+y^2)\,dx\,dy$$

$$\Rightarrow I = \int_0^2 \left[\frac{x^3}{3} + xy^2\right]_{y/4}^{(3-y)} dy$$

$$\Rightarrow I = \int_0^2 \left[\frac{(3-y)^3}{3} + y^2(3-y) - \frac{y^3}{3\times 64} - \frac{y^3}{4}\right] dy$$

$$\Rightarrow \int_0^2 \left[\frac{27-y^3-27y+9y^2}{3} + 3y^2 - y^3 - \frac{y^3}{192} - \frac{y^3}{4}\right] dy$$

$$\Rightarrow \int_0^2 \left[\frac{27-y^3-27y+9y^2+9y^2-3y^3}{3} - y^3\left(\frac{1}{192}+\frac{1}{4}\right)\right] dy$$

$$\Rightarrow \int_0^2 \left[\frac{-4y^3+18y^2-27y+27}{3} - y^3\left(\frac{49}{192}\right)\right] dy$$

$$\Rightarrow \frac{1}{3}\left[-4\left(\frac{y^4}{4}\right) + 18\left(\frac{y^3}{3}\right) - 27\left(\frac{y^2}{2}\right) + 27y\right]_0^2 - \frac{49}{192}\left[\frac{y^4}{4}\right]_0^2$$

$$\Rightarrow \frac{1}{3}\left[-16+48-\cancel{54}+\cancel{54}\right] - \frac{49}{48}$$

$$\Rightarrow \frac{32}{3} - \frac{49}{48} \Rightarrow \boxed{\frac{463}{48}}$$

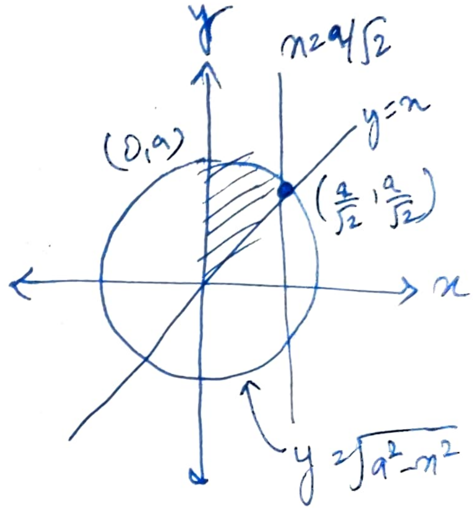
Q5. Change the order of integration and evaluate

$$\int_0^{a/\sqrt{2}} \int_x^{\sqrt{a^2-x^2}} y^2 \, dy \, dx$$

Ans = The region of integration bounded by $y = x$, $y = \sqrt{a^2-x^2}$, $x = 0$, $x = \frac{a}{\sqrt{2}}$

$\Rightarrow y^2 = a^2 - x^2$

$\Rightarrow x^2 + y^2 = a^2$




$$\therefore I = \int_{a/\sqrt{2}}^{a} \int_0^{\sqrt{a^2-y^2}} y^2 \, dx \, dy + \int_0^{a/\sqrt{2}} \int_0^{y} y^2 \, dx \, dy$$

$$\Rightarrow I = \int_{a/\sqrt{2}}^{a} y^2 [x]_0^{\sqrt{a^2-y^2}} dy + \int_0^{a/\sqrt{2}} y^2 [x]_0^{y} dy$$

$$\Rightarrow I = \int_{a/\sqrt{2}}^{a} y^2 \sqrt{a^2-y^2} \, dy + \int_0^{a/\sqrt{2}} y^3 \, dy$$

Put $y = a \sin\theta$

$dy = a \cos\theta \, d\theta$

$$\left[\frac{y^4}{4}\right]_0^{a/\sqrt{2}} = \frac{a^4}{4 \times 4}$$

$$\Rightarrow \frac{a^4}{16}$$

$$\Rightarrow \int_{\pi/4}^{\pi/2} a^2 \sin^2\theta \left(\sqrt{a^2 - a^2\sin^2\theta}\right) . a\cos\theta \, d\theta$$

$$\Rightarrow \int_{\pi/4}^{\pi/2} a^3 \sin^2\theta \cos\theta \times a\cos\theta \, d\theta \Rightarrow \int_{\pi/4}^{\pi/2} a^4 \sin^2\theta \cdot \cos^2\theta \cdot d\theta$$

$$\Rightarrow \int_{\pi/4}^{\pi/2} a^4 (\sin\theta\cos\theta)^2 \Rightarrow a^4 \int_{\pi/4}^{\pi/2} \left[\frac{\sin 2\theta}{2}\right]^2 d\theta \Rightarrow \frac{a^4}{4} \int_{\pi/4}^{\pi/2} \sin^2(2\theta) \, d\theta$$

$$\Rightarrow \frac{a^4}{4} \int_{\pi/4}^{\pi/2} \frac{1-\cos(4\theta)}{2} d\theta \Rightarrow \frac{a^4}{8}\left[[\theta]_{\pi/4}^{\pi/2} - [\sin 4\theta]_{\pi/4}^{\pi/2}\right]$$

$$\Rightarrow \frac{a^4}{8}\left[\left(\frac{\pi}{2} - \frac{\pi}{4}\right) - \left(\sin 2\pi - \sin\pi\right)\right] \Rightarrow \frac{\pi a^4}{32}$$

Hence, $I = \frac{\pi a^4}{32} + \frac{a^4}{16} \Rightarrow \boxed{\frac{a^4}{32}(\pi+2)}$

5

Q6. Evaluate $\iiint x^2yz\,dx\,dy\,dz$ throughout the volume bounded by the planes $x=0$, $y=0$, $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

Ans: $\iiint x^2yz\,dz\,dy\,dx$

$$\Rightarrow \int_0^a \int_0^{b(1-\frac{x}{a})} \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} x^2yz\,dz\,dy\,dx$$

$$\Rightarrow \int_0^a \int_0^{b(1-\frac{x}{a})} x^2y\left[\frac{z^2}{2}\right]_0^{c(1-\frac{x}{a}-\frac{y}{b})} dy\,dx$$

$$\Rightarrow \int_0^a \int_0^{b(1-\frac{x}{a})} \frac{x^2y}{2}\times c^2\left[\left(1-\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2-2\left(1-\frac{x}{a}\right)\left(\frac{y}{b}\right)\right]dy\,dx$$

$$\Rightarrow \int_0^a \int_0^{b(1-\frac{x}{a})} \frac{x^2c^2}{2}\left[y\left(1-\frac{x}{a}\right)^2+\frac{y^3}{b^2}-2\left(1-\frac{x}{a}\right)\left(\frac{y^2}{b}\right)\right]dy\,dx$$

$$\Rightarrow \int_0^a \frac{x^2c^2}{2}\left[\left(1-\frac{x}{a}\right)^2\left(\frac{y^2}{2}\right)+\frac{y^4}{4b^2}-2\left(1-\frac{x}{a}\right)\left(\frac{y^3}{3b}\right)\right]_0^{b(1-\frac{x}{a})} dx$$

$$\Rightarrow \int_0^a \frac{x^2c^2}{2}\left[\left(1-\frac{x}{a}\right)^2\frac{b^2}{2}\left(1-\frac{x}{a}\right)^2+b^4\left(1-\frac{x}{a}\right)^4\times\frac{1}{4b^2}-\frac{2}{3b}\left(1-\frac{x}{a}\right)\left(b^3\left(1-\frac{x}{a}\right)^3\right)\right]$$

$$\Rightarrow \int_0^a \frac{x^2c^2}{2}\left[(a-x)^4\frac{b^2}{2a^4}+(a-x)^4\frac{b^2}{4a^4}-\frac{2b^2}{3a^4}(a-x)^4\right]dx$$

$$\Rightarrow \int_0^a \frac{x^2c^2}{2}\left[\left(\frac{b^2}{2a^4}+\frac{b^2}{4a^4}-\frac{2b^2}{3a^4}\right)(a-x)^4\right]dx$$

$$\Rightarrow \int_0^a \frac{x^2c^2}{2}\left(\frac{b^2}{a^4}\right)\left[\left(\frac{1}{2}+\frac{1}{4}-\frac{2}{3}\right)\right](a-x)^4\,dx$$

$$\Rightarrow \int_0^a \frac{x^2c^2b^2}{2a^4}(a-x)^4\times\left(\frac{6+3-8}{12}\right)dx$$

$$\Rightarrow \int_0^a \frac{c^2b^2}{24a^4}(a-x)^4x^2\,dx \Rightarrow \frac{c^2b^2}{24a^4}\int_0^a (a^2+x^2-2ax)^2x^2\,dx$$

$$\Rightarrow \frac{c^2b^2}{24a^4}\int_0^a \left[(a^2+x^2)^2+4a^2x^2-2(a^2+x^2)(2ax)\right]x^2\,dx$$

$$\Rightarrow \frac{c^2b^2}{24a^4}\int_0^a \left[(a^4+x^4+2a^2x^2+4a^2x^2-4a^3x-4ax^3\right]x^2\,dx$$

$$\Rightarrow \frac{c^2b^2}{24a^4}\int_0^a (a^4x^2+x^6+6a^2x^4-6a^2x^4-4a^3x^3-4ax^5)\,dx$$

$$\Rightarrow \frac{c^2b^2}{24a^4}\left[a^4\left(\frac{x^3}{3}\right)+\frac{x^7}{7}+6a^2\left(\frac{x^5}{5}\right)-4a^3\left(\frac{x^4}{4}\right)-4a\frac{x^6}{6}\right]_0^a$$

$$\Rightarrow \frac{c^2b^2}{24a^4}\left[\frac{a^7}{3}+\frac{a^7}{7}+\frac{6a^7}{5}-\frac{2a^7}{3}\right]$$

$$\Rightarrow \frac{a^3b^2c^2}{24}\left[\frac{1}{3}+\frac{1}{7}+\frac{1}{5}-\frac{2}{3}\right] \Rightarrow \frac{a^3b^2c^2}{24}\times\left(\frac{35+15+21-70}{105}\right)$$

$$\Rightarrow \frac{a^3b^2c^2}{24\times 105} \Rightarrow \boxed{\frac{a^3b^2c^2}{2520}}$$