CALCULUS AND LINEAR ALGEBRA

UNIT-IV

(Applications of Differential Calculus)

DEPARTMENT OF MATHEMATICS

SRM Institute of Science and Technology

Introduction

The rate of change of the direction of tangent with respect to arc lenngth as the point p moves along the curve is called curvature vector of the curve whose magnitude is called the curvature at p. Radius of curvature. The reciprocal of the curvature of a curve at any point P is called the radius of curvature at P and is denoted by rho.

- Radius of curvature for Cartesian Curve $y = f(x)$, is given by $\rho = \dfrac{[1 + {y_1}^2]^{3/2}}{y_2}$, where $y_1 = dy/dx$ and $y_2 = d^2y/dx^2$
- Radius of curvature for parametric equations $x = f(t), y = g(t)$ is given by $\rho = \dfrac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}$ where $z' = dz/dt$ and $z' = d^2y/dx^2$
- Radius of curvature for polar curve $r = f(\theta)$ is given by

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 - rr_2 + 2r_1^2}$$

Problem-1

Find the radius of curvature at the point (3a/2, 3a/2) of the Folium $x^3 + y^3 = 3axy$.

Soln: Differentiating with respect to x, we get

$3x^2 + 3y^2\frac{dy}{dx} = 3a\left(y + x\frac{dy}{dx}\right)$

$$(y^2 - ax)\frac{dy}{dx} = ay - x^2 \qquad (1)$$

$\therefore \dfrac{dy}{dx}$ at $(3a/2, 3a/2) = -1$

Differentiating (1),

$\left(2y\frac{dy}{dx} - a\right)\dfrac{dy}{dx} + (y^2 - ax)\dfrac{d^2y}{dx^2} = a\dfrac{dy}{dx} - 2x$

$\therefore \dfrac{d^2y}{dx^2}$ at $(3a/2, 3a/2) = 32/3a$

Hence $\rho\ at(3a/2, 3a/2) = \dfrac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} = \dfrac{[1 + (-1)^2]^{3/2}}{-32/3a}$

$= \dfrac{3a}{8\sqrt{2}}$ (in magnitude).

Problem-2

Show that the radius of curvature at any point of the cycloid $x = a(\theta + sin\theta), y = a(1 - cos\theta)$ is $4a\ cos\theta/2$.

Soln: We have $\dfrac{dy}{dx} = a(1 + cos\theta), \dfrac{dy}{d\theta} = asin\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} + \frac{dx}{d\theta} = \frac{asin\theta}{a(1+cos\theta)} = \frac{2sin\theta/2\ cos\theta/2}{2cos^2\theta/2} = tan\theta/2$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\cdot\frac{d\theta}{dx} = \frac{1}{2}sec^2\frac{\theta}{2}\cdot\frac{1}{a(1+cos\theta)}$$

$$= \frac{1}{2}sec^2\frac{\theta}{2}\cdot\frac{1}{2acos^2\theta/2} = \frac{1}{4a}sec^4\theta/2.$$

$$\rho = \frac{[1+(dy/dx)^2]^{3/2}}{d^2y/dx^2} = \frac{4a(1+tan^2\theta/2)^{3/2}}{sec^4\theta/2}$$

$$= 4a\cdot(sec^2\theta/2)^{3/2}\cdot cos^4\theta/2 = 4a\ cos\theta/2.$$

Problem-3

Prove that the radius of curvature at any point of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$, is three times the length of the perpendicular from the origin to the tangent at that point.

Soln: The parametric equation of the curve is

$$
\begin{aligned}
x = acos^3 t, y &= asin^3 t \\
x'(= dx/dt) = -3acos^2 t \sin t,\ \ y' &= 3asin^2 t\ \ cost. \\
x'' = -3a(cos^3 t - 2cost\ \ sin^2 t) &= 3a \cos t(2sin^2 t - cos^2 t) \\
y'' = 3a(2sint\ \ cost - sin^3 t) &= 3a\ \ sin\ \ t(2cos^2 t - sin^2 t) \\
x'^2 + y'^2 = 9a^2(cos^4 tsin^2 t + sin^4 tcos^2 t) &= 9a^2 sin^2 t\ \ cos^2 t \\
x'y'' - y'x'' &= -9a^2 cos^2 tsin^2 t(2cos^2 t - sin^2 t) \\
-9a^2 cos^2 tsin^2 t(2sin^2 t - cos^2 t) &= -9a^2 sin^2 t\ \ cos^2 t \\
\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''} = \frac{27a^3 sin^3 t\ \ cos^3 t}{-9a^2 sin^2 t\ \ cos^2 t} &= -3asint\ \ cost.
\end{aligned}
$$

Problem-3 Cont...

Since $\dfrac{dy}{dx} = \dfrac{y'}{x'} = -tant$,
$\therefore$ Equation of the tangent at $(acos^3t, asin^3t)$ is
$y - asin^3t = -tan\ t(x - acos^3t)$
$x \tan t + y - a \sin t = 0$
length of $\perp$ from (0,0) on (2)$=\dfrac{0+0-asint}{\sqrt{(tan^2t+1)}} = -asint\ cost$.
Thus $\rho = 3p$.

Problem-4

Show that the radius of curvature at any point of the cardioid $r = a(1 - cos\theta)$ varies as $\sqrt{r}$.

Soln: Differentiating w.r.t. θ, we get

$r_1 = asin\theta, r_2 = acos\theta$

$\therefore (r^2 + r_1^2)^{3/2} = [a^2(1 - cos\theta)^2 + a^2 sin^2\theta]^{3/2} = a^3[2(1 - cos\theta)]^{3/2}$

$r^2 - rr_2 + 2r_1^2 = a^2(1 - cos\theta)^2 - a^2(1 - cos\theta)cos\theta + 2a^2 sin^2\theta = 3a^2(1 - cos\theta)$

Thus $\rho = \dfrac{(r^2 + r_1^2)^{3/2}}{r^2 - rr_2 + 2r_1^2} = \dfrac{a^3 2\sqrt{2}(1 - cos\theta)^{3/2}}{3a^2(1 - cos\theta)}$

$= \dfrac{2\sqrt{2}}{3} a(1 - cos\theta)^{1/2} = \dfrac{2\sqrt{2a}}{3} \left(\dfrac{r}{a}\right)^{1/2} \propto \sqrt{r}.$

CENTRE OF CURVATURE

Let Γ be a simple curve having tangent at each point. At any point P on this curve we can draw a circle having the same curvature at P as the curve Γ.

This circle is called the circle of curvature and its centre is called the centre of curvature and its radius is the radius of curvature of Γ at P.

Centre of curvature at any point $P(x, y)$ on the curve $y = f(x)$ is given by

$$\overline{x} = x - \frac{y_1(1 + y_1^2)}{y_2}$$
$$\overline{y} = y + \frac{1 + y_1^2}{y_2}$$

Equation of the circle of curvature at P is $(x - \overline{x})^2 + (y - \overline{y})^2 = \rho^2$.

EVOLUTE

The locus of centre of curvature of a given curve Γ is called the evolute of the curve.

The given curve Γ is called an involute of the evolute. In fact, for the evolute there are many involutes.

Procedure to Find the Evolute

Let $y = f(x)$ (1) be the equation of the given curve. If $(\overline{x}, \overline{y})$ is the centre of curvature at any point $P(x, y)$ on (1), then

$$\overline{x} = x - \frac{y_1(1 + y_1^2)}{y_2} \tag{2}$$

$$\overline{y} = y + \frac{1 + y_1^2}{y_2} \tag{3}$$

Eliminating x, y using (1), (2) and (3), we get a relation in $\overline{x}, \overline{y}$. Replacing $\overline{x}$ by x and $\overline{y}$ by y, we get the equation of locus of $(\overline{x}, \overline{y})$, which is the evolute of the given curve.

Problem-1

Find the coordinates of the center of curvature at any point of the parabola $y^2 = 4ax$. Hence show that its evolute is $27ay^2 = 4(x-2a)^3$

Soln: We have $2yy_1 = 4a$ i.e. $y_1 = 2a/y$ and

$y_2 = -\dfrac{2a}{y^2} \cdot y_1 = -\dfrac{4a^2}{y^3}$

If $(\overline{x}, \overline{y})$ be the center of curvature, then

$$\overline{x} = x - \frac{y_1(1+y_1^2)}{y_2} = x - \frac{2a/y(1+4a^2/y^2)}{-4a^2/y^3}$$
$$= x + \frac{y^2+4a^2}{2a} = x + \frac{4ax+4a^2}{2a} = 3x+2a \tag{4}$$

and

$$\overline{y} = y + \frac{1+y_1^2}{y_2} = y + \frac{1+4a^2/y^2}{-4a^2/y^2}$$
$$= y - \frac{y(y^2+4a^2)}{4a^2} = \frac{-y^3}{4a^2} = -\frac{2x^{3/2}}{\sqrt{a}} \tag{5}$$

Problem-1 Cont...

To find the evolute, we have to eliminate x from (4) and (5)

$\therefore (\overline{y})^2 = \dfrac{4x^3}{a} = \dfrac{4}{a}\left(\dfrac{\overline{x}-2a}{3}\right)^3$

or $27a(\overline{y})^2 = 4(\overline{x}-2a)^3$.

Thus the locus of $(\overline{x},\overline{y})$ i.e., evolute is $27ay^2 = 4(x-2a)^3$.

Problem-2

Show that the evolute of the cycloid
$x = a(\theta - sin\theta), y = a(1 - cos\theta)$ is another equal cycloid.

Soln: We have $y_1 = \dfrac{dy}{d\theta} + \dfrac{dx}{d\theta} = \dfrac{asin\theta}{a(1 - cos\theta)} = cot\frac{\theta}{2}.$

$$y_2 = \frac{d}{dx}(y_1) = \frac{d}{d\theta}(cot\theta/2) \cdot \frac{d\theta}{dx}$$

$$= -cosec^2\theta/2 \cdot 1/2 \cdot \frac{1}{a(1 - cos\theta)} = -\frac{1}{4asin^4\theta/2}$$

If$(\overline{x}, \overline{y})$ be the center of curvature, then

$$\overline{x} = x - \frac{y_1(1 + y_1^2)}{y_2} =$$

$$a(\theta - sin\theta) + cot\theta/2\left(-4asin^4\theta/2\right)\left(1 + cot^2\theta/2\right)$$

$$= a(\theta - sin\theta) + \frac{cos\theta/2}{sin\theta/2} \cdot 4asin^4\theta/2 \cdot cosec^2\theta/2$$

$$= a(\theta - sin\theta) + 4asin\theta/2cos\theta/2 = a(\theta - sin\theta) + 2asin\theta = a(\theta + sin\theta)$$

Problem-2 Cont...

$$\overline{y} = y + \frac{1+y_1^2}{y_2} = a(1-cos\theta) + \left(1+cot^2\theta/2\right)\left(-4asin^4\theta/2\right)$$
$$= a(1-cos\theta) - 4asin^4\theta/2 \cdot cosec^2\theta/2$$
$$a(1-cos\theta) - 4asin^2\theta/2$$
$$a(1-cos\theta) - 2a(1-cos\theta) = -a(1-cos\theta)$$

Hence the locus of $(\overline{x}, \overline{y})$ i.e., the evolute, is given by
$x = a(\theta + sin\theta), y = -a(1-cos\theta)$ which is another equal cycloid.

ENVELOPE

Consider the system of straight lines $y = mx + \frac{1}{m}$ (1) where m is a parameter. For different values of m, we have different straight lines and so (1) represents a family of straight lines. Each member of this family touches the curve $y^2 = 4x$. So, these lines cover the curve $y^2 = 4x$. This curve is called the envelope of the family of lines. We shall now define envelope.

Definition: Let $f(x, y, \alpha) = 0$ be a single parameter family of curves, where α is the parameter. The envelope of this family of curves is a curve which touches every member of the family.

Problem-1

Find the envelope of the family of lines $y = mx + \sqrt{(1 + m^2)}$, m being the parameter.
Soln: We have

$$(y - mx)^2 = 1 + m^2 \tag{6}$$

Differentiating (6) partially with respect to m,

$$2(y - mx)(-x) = 2m \ or \ m = xy/(x^2 - 1) \tag{7}$$

Now eliminate m from (6) and (7).
Substituting the values of m in (6), we get
$\left(y - \dfrac{x^2y}{x^2 - 1}\right)^2 = 1 + \left(\dfrac{xy}{x^2 - 1}\right)^2$ or
$y^2 = (x^2 - 1)^2 + x^2y^2$
$x^2 + y^2 = 1$ which is the required equation of the envelope

Problem-2

Find the envelope of a system of concentric and coaxial ellipses of constant area.

Soln: Taking the common axes of the system of ellipses as the coordinate axes, the equation to an ellipse of the family is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{8}$$

where a and b are the parameters.

The area of the ellipse $=\pi ab$ which is given to be constant, say $= \pi c^2$

$$ab = c^2 \ or \ b = c^2/2 \tag{9}$$

Substituting in (8),

$$\frac{x^2}{a^2} + \frac{y^2}{(c^2/a)^2} = 1 \ or \ x^2 a^{-2} + (y^2/c^4)a^2 = 0 \tag{10}$$

which is given family of ellipses with a as the only parameter.

Problem-2 Cont...

Differentiating partially (10) with respect to a,

$$-2x^2a^{-3} + 2(y^2/c^4)a = 0 \text{ or } a^2 = c^2x/y \tag{11}$$

Eliminate a from (10) and (11)
Substituting the values of a^2 in (10), we get

$$x^2(y/c^2x) + (y^2/c^4)(c^2x/y) = 1 \text{ or } 2xy = c^2$$

which is the required equation of the envelope.

Problem-3

Find the evolute of the parabola $y^2 = 4ax$.
Soln: Any normal to the parabola is

$$y = mx - 2am - am^3 \tag{12}$$

Differentiating it with respect to m partially,
$0 = x - 2a - 3am^2$ or $m = [(x - 2a)/3a]^{1/2}$
Substituting this value of m in (12),
$$y = \left(\frac{x-2a}{3a}\right)^{1/2}\left[x - 2a - a \cdot \frac{x-2a}{3a}\right]$$
Squaring both sides, we have
$27ay^2 = 4(x - 2a)^3$
Which is the evolute of the parabola.

Beta, Gamma Functions

The beta function is defined as

$$\beta(p,q) = \int\limits_0^1 x^{p-1}(1-x)^{q-1}dx$$

where $m, n > 0$. Note that $\beta(p,q) = \beta(q,p)$.
The gamma function is defined as

$$\Gamma(n) = \int\limits_0^\infty x^{n-1}e^{-x}dx$$

where n > 0.
Note: (1) $\Gamma(1) = 1$
(2) $\Gamma(n+1) = n\Gamma(n) = n!$
(3) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$
The relation between Beta and Gamma functions is

$$\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

Problems Beta, Gamma Functions

Show that $\Gamma(n) = \int\limits_0^1 \left(log\frac{1}{y}\right)^{n-1} dy, (n > 0).$

Soln: $\Gamma(n) = \int\limits_0^\infty x^{n-1}e^{-x}dx (n > 0)$

$= \int\limits_0^1 \left(log\frac{1}{y}\right)^{n-1} y\left(-\frac{1}{y}dy\right)$

put $y = e^{-x}$

$i.e., x = log(1/y)$

so that $dx = -(1/y)dy$

$= \int\limits_0^1 \left(log\frac{1}{y}\right)^{n-1} dy.$

Problems based on Beta, Gamma Functions

Show that $\beta(p, q) = \int\limits_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}} dy = \int\limits_0^1 \frac{x^{p-1} + x^{q-1}}{(1+x)^{p+q}} dx$

Soln: $\beta(p, q) = \int\limits_0^1 x^{p-1}(1-x)^{q-1} dx$

$$= \int\limits_\infty^0 \frac{1}{(1+y)^{p-1}} \left(\frac{y}{1+y}\right)^{q-1} \frac{-1}{(1+y)^2} dy$$

put $x = \dfrac{1}{1+y}$ i.e., $y = \frac{1}{x} - 1$

so that $dx = \dfrac{-1}{(1+y)^2} dy$

$$= \int\limits_0^\infty \frac{y^{q-1}}{(1+Y)^{p+q}} dy = \int\limits_0^1 \frac{y^{q-1}}{(1+y)^{p+q}} dy + \int\limits_1^\infty \frac{y^{q-1}}{(1+y)^{p+q}} dy$$

Now substituting $y = 1/z$ in the second integral, we get

$$\int\limits_1^\infty \frac{y^{q-1}}{(1+y)^{p+q}} dy = \int\limits_1^0 \frac{1}{z^q - 1} \cdot \frac{1}{(1+1/z)^{p+q}} \left(-\frac{1}{z^2}\right) dz$$

$$= \int\limits_0^1 \frac{z^{p-1}}{(1+z)^{p+q}} dz$$

Problems based on Beta, Gamma Functions

Express the following integral in terms of gamma function

$$\int_0^1 \frac{dx}{\sqrt{(1-x^4)}}$$

Soln: $\displaystyle\int_0^1 \frac{dx}{\sqrt{(1-x)^4}}$

Put $x^2 = sin\theta, i.e., x = sin^{1/2}\theta$

so that $dx = 1/2 sin^{-1/2}\theta \ cos\ \theta\ d\theta$

$$= \int_0^{\pi/2} \frac{1}{2} \frac{sin^{-1/2}\theta\ cos\theta\ d\theta}{\sqrt{(1-sin^2\theta)}}$$

$$= 1/2 \int_0^{\pi/2} sin^{-1/2}\theta\ d\theta$$

$$= \frac{1}{2} \cdot \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma\left(\frac{-1/2+1}{2}\right)}{\Gamma\left(\frac{-1/2+2}{2}\right)}$$

$$= \frac{\sqrt{\pi}}{4} \frac{\Gamma(1/4)}{\Gamma(3/4)}$$

Problems based on Beta, Gamma Functions

Express the following integral in terms of gamma function

$$\int_0^{\pi/2} \sqrt{(tan\ \theta)\ d\theta}$$

Soln: $\int_0^{\pi/2} \sqrt{(tan\theta)d\theta} = \int_0^{\pi/2} sin^{1/2}\theta\ cos^{-1/2}\theta\ d\theta$

$$= \frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{\frac{-1}{2}+1}{2}\right)}{2\Gamma\left(\frac{\frac{1}{2}-\frac{1}{2}+2}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{2\Gamma(1)}$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right).$$

Problems based on Beta, Gamma Functions

Evaluate $\int\limits_0^\infty e^{-ax}x^{m-1} sin\ bx\ dx$ in terms of Gamma functions.

Soln: We have $\Gamma(m) = \int\limits_0^\infty e^{-x}x^{m-1}dx$

$= \int\limits_0^\infty e^{-ay}a^m y^{m-1}dy$

$$\int\limits_0^\infty e^{-ay}y^{m-1}dy = \Gamma(m)/a^m \qquad (13)$$

Then $I = \int\limits_0^\infty e^{-ax}x^{m-1} sin\ bx\ dx$

$= \int\limits_0^\infty e^{-ax}x^{m-1}$ (Imaginary part of e^{ibx})dx

=I.P. of $\int\limits_0^\infty e^{-(a-ib)x}x^{m-1}dx$

=I.P. of $\{\Gamma(m)/(a-ib)^m$ by (1)

Problems based on Beta, Gamma Functions

$=$I.P. of $\{\Gamma(m)/[r^m(cos\theta - i\ sin\ \theta)^m]$ where $a = rcos\theta, b = r\ sin\theta$
$=$I.P. of $\Gamma(m)/[r^m(cos\ m\theta - i\ sin\ m\theta)]$ (Using Demoivre's theorem)

$=$I.P. of $\left\{\dfrac{\Gamma(m).(cos\ m\theta + i\ sin\ m\theta)}{r^m(cos\ m\theta + i\ sin\ m\theta)(cos\ m\theta - i\ sin\ m\theta)}\right\}$

$=\dfrac{\Gamma(m)}{r^m} sin\ m\theta$ Where $r = \sqrt{(a^2 + b^2)}, \theta = tan^{-1}b/a.$