Solved Probl... described by dif-
ference equation

$$y(n) - 1.5y(n-1) + 0.5y(n-2) = x(n)$$
$$y(-1) = 1; y(-2) = 0$$

$$\frac{d}{dt} = y(n-1)$$

Solution:

Given

$$y(n) - 1.5y(n-1) + 0.5y(n-2) = x(n) \tag{3.15}$$

The homogeneous equation can be obtained by equating the input terms to zero. That is

$$y(n) - 1.5y(n-1) + 0.5y(n-2) = 0 \tag{3.16}$$

The homogeneous solution

$$y_h(n) = \lambda^n. \tag{3.17}$$

Substituting this solution in Eq. (3.16) we get

$$\lambda^n - 1.5\lambda^{n-1} + 0.5\lambda^{n-2} = 0$$
$$\Rightarrow \lambda^{n-2}[\lambda^2 - 1.5\lambda + 0.5] = 0$$
$$\lambda^2 - 1.5\lambda + 0.5 = 0$$
$$(\lambda - 1)(\lambda - 0.5) = 0.$$
$$\lambda_1 = 1; \lambda_2 = 0.5$$

The general form of homogeneous solution is

$$y_h(n) = c_1(1)^n + c_2(0.5)^n \tag{3.18}$$

$c_1 e^{at} + c_2 e^{at}$

From Eq. (3.18)

$$\begin{aligned} y(0) &= c_1 + c_2 \\ y(1) &= c_1 + 0.5c_2 \end{aligned} \tag{3.19}$$

From the homogeneous equation

$$y(0) - 1.5y(-1) + 0.5y(-2) = 0.$$

Given $y(-1) = 1$ and $y(-2) = 0$.
Therefore

$$\begin{aligned} y(0) - 1.5(1) &= 0 \\ \Rightarrow \quad y(0) &= 1.5 \end{aligned}$$

Similarly

$$y(1) - 1.5y(0) + 0.5y(-1) = 0$$

$$y(1) - 1.5(1.5) + 0.5(1) = 0$$

$$\Rightarrow \quad y(1) = 1.75 \tag{3.20a}$$

$$y(0) = 1.5 \tag{3.20b}$$

Comparing Eq. (3.19) and Eq. (3.20) we get

$$\begin{aligned} c_1 + c_2 &= 1.5 \\ c_1 + 0.5c_2 &= 1.75 \\ \Rightarrow \quad c_2 &= -0.5 \\ c_1 &= 2 \end{aligned}$$

The natural response

$$\begin{aligned} y_n(n) &= 2(1)^n - 0.5(0.5)^n \quad \text{for} \quad n \geq 0 \\ &= 2u(n) - 0.5(0.5)^n u(n) \end{aligned} \tag{3.21}$$

3.4 Forced Response (zero state response)

The forced response is the solution of the difference equation for the given input when the initial conditions are zero. That is forced response is the response of a relaxed system for any input $x(n)$. It consists of two parts, homogeneous solution and particular solution. The homogeneous solution can be obtained from the roots of characteristic equation. The particular solution $y_p(n)$ is to satisfy the difference equation for the specific input signal $x(n), n \geq 0$. In otherwords, $y_p(n)$ is a solution satisfying

$$1 + \sum_{k=1}^{N} a_k y_p(n-k) = \sum_{k=0}^{M} b_k x(n-k) \tag{3.22}$$

The general form of the particular solution for several inputs are given in Table (3.1). From the Table (3.1) we can find that, if the input $x(n) = A\cos\omega n$, then $y_p(n) = c_1 \cos\omega n + c_2 \sin\omega n$, where c_1 and c_2 are obtained by substituting $y_p(n)$ and $x(n)$ in the difference equation.

Table 3.1 General form of particular solution for several types of inputs

$x(n)$ input signal	$y_p(n)$ particular solution
A (Step input)	k
AM^n	kM^n
An^M	$k_0 n^M + k_1 n^{M-1} + \ldots k_M$
$A^n N^M$	$A^n [k_0 n^M + k_1 n^{M-1} + \ldots k_M]$
$A\cos\omega n$ / $A\sin\omega n$	$c_1 \cos\omega n + c_2 \sin\omega n$

Note: Here A, k, M, k_i, c_1 and c_2 are constants.

If the input applied to the system and one of the components of the homogeneous solution are equal, then multiply the particular solution by the lower power of n that will give a response components not included in the homogeneous solution. For example, if the homogeneous solution contain the term $c_1(\lambda_1)^n$ and the input is $x(n) = (\lambda_1)^n$ then we assume a particular solution of the form $y_p(n) = c_2 n(\lambda_1)^n$.

The forced response of the system is obtained by summing the particular solution and homogeneous solution and finding the coefficients in the homogeneous solution so that the combined response $y_h(n) + y_p(n)$, satisfies the zero initial conditions.

Solved Problem 3.2 Find the forced response of the system described by the difference equation

$$y(n)-1.5y(n-1)+0.5y(n-2)=x(n)$$

for an input $x(n)=2^n u(n)$

Solution:

From solved problem (3.1) we can find the homogeneous solution of the system is of the form

$$y_h(n)=c_1(1)^n+c_2(0.5)^n. \tag{3.23}$$

For input $x(n) = 2^n u(n)$ the particular solution is of the form (see table 3.1) $y_p(n)=k2^n u(n)$

Substituting $y_p(n)$ and $x(n)$ in the difference equation we get

$$k2^n u(n)-1.5k2^{n-1}u(n-1)+0.5k2^{n-2}u(n-2)=2^n u(n)$$

$y = Kx^n$

$y = K2^n u(n)$

For $n=2$ where none of the terms vanish

$$k(2)^2-1.5k(2)+0.5(k)=2^2$$

$$\Rightarrow \quad 4k-3k+0.5k=4$$

$$1.5k=4$$

$$\Rightarrow \quad k=\frac{8}{3}$$

Select the value of n such that no term vanishes

$0.5z^2 - 1.5z + 1 = 0$

$5z^2 - 15z + 10 = 0$

Therefore the particular solution

$$y_p(n)=\frac{8}{3}2^n u(n)$$

The forced response

$$y_f(n)=y_h(n)+y_p(n)$$

$$=c_1(1)^n+c_2(0.5)^n+\frac{8}{3}2^n u(n) \tag{3.24}$$

From Eq. (3.24)

$$y(0)=c_1+c_2+\frac{8}{3}$$

$$y(1)=c_1+0.5c_2+\frac{16}{3} \tag{3.25}$$

From the difference equation

$$y(0)-1.5y(-1)+0.5(y)(-2)=x(0)$$

$$\Rightarrow \quad y(0)=1$$

$$\boxed{\therefore \quad y(-1)=y(-2)=0}$$

Similarly

$$y(1) - 1.5y(0) + 0.5y(-1) = x(1)$$
$$y(1) - 1.5 = 2$$
$$y(1) = 3.5 \tag{3.26}$$

Comparing Eq. (3.25) and Eq. (3.26) we get

$$c_1 + c_2 + \frac{8}{3} = 1 \quad \Rightarrow \quad c_1 + c_2 = \frac{-5}{3}$$

and

$$c_1 + 0.5c_2 + \frac{16}{3} = \frac{7}{2} \quad \Rightarrow \quad c_1 + 0.5c_2 = \frac{-11}{6}$$

That is

$$c_1 + c_2 = \frac{-5}{3}$$
$$c_1 + 0.5c_2 = \frac{-11}{6}$$

Solving for c_1 and c_2 we get

$$c_1 = -2$$
$$c_2 = \frac{1}{3}$$

The forced response

$$y_f(n) = -2(1)^n + \frac{1}{3}(0.5)^n + \frac{8}{3}(2)^n \quad \text{for} \quad n \geq 0$$
$$= -2u(n) + \frac{1}{3}(0.5)^n u(n) + \frac{8}{3}(2)^n u(n) \tag{3.27}$$