18MAB101T- CALCULUS AND LINEAR ALGEBRA

Unit II - Function of several variables

In this ppt, we are going to see,

1. Variables
2. Function of several variables
3. Partial derivatives
4. Chain rule
5. Differentiation of Implict functions
6. Total differentiation
7. Total differential
8. Taylor's series

Function of several variables

INTRODUCTION

Definition 1: Independent variable

In a function, the values for the variable which are free to assign is called independent variable.

Definition 2: Dependent variable

In a function, the values for the variable which depends on the value of independent variable is called dependent variable.

Example

$z = x^2 + y^2$

Here x and y are independent variable and z is a dependent variable.

Note: In a function, you have only one dependent variable and the other variables are called independent variable.

Definition 3: Function of several variables

A function which has more than one independent variable is called function of several variables.

Example: $u(x, y, z) = x^2 + y^2 + 2xy - z^2 + xz$

Definition 4: Partial derivative

The derivative of function of several variable with respect to independent variable is called partial derivative and it is denoted by ∂

Example :

$Z = x^3 - y^3 + 3x^2y + 3xy^2$

$\frac{\partial z}{\partial x}$ is called partial derivative with respect to independent variable x

$$\frac{\partial z}{\partial x} = 3x^2 + 6xy + 3y^2$$

In $\frac{\partial z}{\partial x}$, differentiating z with respect to independent variable x and treating the other independent variable as constants.

Example:

Find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$ for $U = e^x sinycosz$.

Solution:

$$\frac{\partial u}{\partial x} = e^x sinycosz$$

$$\frac{\partial u}{\partial y} = e^x cosycosz$$

$$\frac{\partial u}{\partial z} = -e^{x} sinysinz$$

Definition: Chain rule

If $z = f(x,y)$ and x and y are function on t then,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

Example: Find $\frac{dz}{dt}$ where $z = xy^2 + x^2y, x = at^2$ and $y = 2at$

Solution:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$\frac{\partial z}{\partial x} = y^2 + 2xy \qquad \frac{\partial z}{\partial x} = 2xy + x^2$$

$$\frac{dx}{dt} = 2at \qquad \frac{dy}{dt} = 2a$$

$$\frac{dz}{dt} = (y^2 + 2xy)(2at) + (2xy + x^2)(2a)$$

Substituting $x = at^2$ and $y = 2at$ we get

$$\frac{dz}{dt} = 16a^3t^3 + 10a^3t^4$$

If $u = sin\left(\frac{x}{y}\right)$, $x = e^t, y = t^2$ Find $\frac{du}{dt}$

Solution:

$$\frac{du}{dt} = \frac{e^t}{t^2} cos\left(\frac{e^t}{t^2}\right)\left(1 - \frac{2}{t}\right)$$

Differentiation of Implict Function

Consider the implict function $f(x, y) = 0$ then $\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$.

Example: Find $\frac{dy}{dx}$ if $xe^{-y} - 2ye^{x} = 1$

Solution:
Given $f(x,y) = xe^{-y} - 2ye^{x} - 1 = 0$

$$\frac{\partial f}{\partial x} = e^{-y} - 2ye^{x} \quad \frac{\partial f}{\partial y} = e^{-y} - 2ye^{x}$$

$$\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y} = -\frac{e^{-y} - 2ye^{x}}{-xe^{-y} - 2e^{x}}$$

$$= \frac{e^{-y} - 2ye^{x}}{xe^{-y} + 2e^{x}}$$

Find $\frac{dy}{dx}$ if $(\cos x)^y = (\sin y)^x$

Solution:

$$\frac{dy}{dx} = \frac{y\tan x + \log(\sin y)}{\log(\cos x) - x\cot y}.$$

Total differentiation: If $z = f(x_1, x_2, \ldots, x_n)$ where $x_1, x_2, \ldots, x_n$ are all functions on 't' then,

$$\frac{dz}{dt} = \frac{\partial f}{\partial x_1}\cdot\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\cdot\frac{dx_2}{dt} + \cdots + \frac{\partial f}{\partial x_n}\cdot\frac{dx_n}{dt}$$

Example:For $z = f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$ where $x_1(t) = t^2$, $x_2(t) = 2t$ and $x_3(t) = 3t^3$ then find $\frac{dz}{dt}$

Solution:

$$\frac{\partial f}{\partial x_1} = 2x_1 \quad \frac{\partial f}{\partial x_2} = 2x_2 \quad \frac{\partial f}{\partial x_3} = 2x_3$$

$$\frac{dx_1}{dt} = 2t \quad \frac{dx_2}{dt} = 2 \quad \frac{dx_3}{dt} = 9t^2$$

$$\begin{aligned}\frac{dz}{dt} &= 2(t^2)(2t) + 2(2t)(2) + 2(3t^2)(qt^2) \\ &= 4t^3 + 8t + 54t^5 \\ &= 54t^5 + 4t^3 + 8t.\end{aligned}$$

Total differential:

If $u = f(x_1, x_2, \cdots, x_n)$ then the total differential of u is given by

$$du = \frac{\partial f}{\partial x_1} \cdot dx_1 + \frac{\partial f}{\partial x_2} \cdot dx_2 + \cdots + \frac{\partial f}{\partial x_n} \cdot dx_n.$$

Example: A metal box without a top has inside dimensions 6ft, 4ft and 2ft. If the metal is 0.1ft thick. Find the approximate volume by using the differential.

Solution: Let x, y, z be the dimensions of a metal box. Then its volume is $V = xyz$ From total differential we have

$$\begin{aligned} dV &= \frac{\partial v}{\partial x} \cdot dx + \frac{\partial V}{\partial y} \cdot dy + \frac{\partial v}{\partial z} \cdot dz \\ &= yzdx + xzdy + xydz \\ &= 8(0.2)+12(0.2)+24(0.1) \\ &= 6.4 \text{ cu.ft} \end{aligned}$$

TAYLOR SERIES

The Taylor series expansions of $f(x,y)$ in powers of $(x-a)$ and $(y-b)$ is given by

$$f(x,y) = f(a,b) + [(x-a)f_x(a,b) + (y-b)f_y(a,b)]$$
$$+\frac{1}{2!}[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)]$$
$$+\frac{1}{3!}[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b)f_{xxy}(a,b)$$
$$+3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b) + \dots$$

Where $f_x = \frac{\partial f}{\partial x}$ $\quad f_y = \frac{\partial f}{\partial y}$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} \quad f_{xy} = \frac{\partial^2 f}{\partial x \partial y} \quad f_{yy} = \frac{\partial^2 f}{\partial y^2}$$

$f_{xxx} = \frac{\partial^3 f}{\partial x^3}$ $\quad f_{xxy} = \frac{\partial^3 f}{\partial x^2 \partial y}$ $\quad f_{xyy} = \frac{\partial^3 f}{\partial x \partial y^2}$ and $f_{yyy} = \frac{\partial^3 f}{\partial y^3}$ and so on.

Note: If $a=0$ and $b=0$ then the Taylor's series is reduce to Macularian's series in two variables

$$f(x,y)=f(0,0)+[xf_x(0,0)+yf_y(0,0)]+$$
$$\frac{1}{2!}(x^2f_{xx}(0,0)+2xyf_{xy}(0,0)+y^2f_{yy}(0,0)]+$$
$$\frac{1}{3!}(x^3f_{xxx}(0,0)+3x^2yf_{xxy}(0,0)++3xy^2f_{xyy}(0,0)+y^3f_{yyy}(0,0)]+\dots$$

Problems on Taylor's series

Expand $x^2y+3y-2$ in power of $(x-1)$ and $(y+2)$ using Taylor series upto terms of third degree.

Solution: The Taylor series expansion of $f(x,y)$ in power of $(x-a)$ and $(y-b)$ is given by

$$f(x,y)=f(a,b)+[(x-a)f_x(a,b)+(y-b)f_y(a,b)]$$
$$+\frac{1}{2!}[(x-a)^2f_{xx}(a,b)+2(x-a)(y-b)f_{xy}(a,b)+(y-b)^2f_{yy}(a,b)]+\dots$$

Here $a = 1$ and $b = -2$

$f(x,y) = x^2y + 3y - 2 \quad f(1,-2) = -10$

$f_x = 2xy \quad f_x(1,-2) = -4$

$f_y = x^2 + 3 \quad f_y(1,-2) = 4$

$f_{xx} = 2y \quad f_{xx}(1,-2) = -4$

$f_{xy} = 2x \quad f_{xy}(1,-2) = 2$

$f_{yy} = 0 \quad f_{yy}(1,-2) = 0$

$f_{xxx} = 0 \quad f_{xxx}(1,-2) = 0$

$f_{xxy} = 2 \quad f_{xxy}(1,-2) = 2$

$f_{xyy} = 0 \quad f_{xyy}(1,-2) = 0$

$f_{yyy} = 0 \quad f_{yyy}(1,-2) = 0$

Substituting the values we get

$$f(x,y) = -10 + \frac{1}{1!}((x-1)(-4) + (y+2)(4))$$

$$+\frac{1}{2!}((x-1)^2(-4) + 2(x-1)(y+2)(2)] + \frac{1}{3!}(3(x-1)^2(y+2)(2)] + \cdots$$

$$= -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + (x-y)^2(y+2) + \ldots$$

Expand $e^x \cos y$ in power of x and y as for as the term of the third degree

Solution: $f(x,y) = e^x \cos y \quad a = 0$ and $b = 0$

$f(x,y) = e^x \cos y \quad f(0,0) = 1$

$f_x = e^x \cos y \quad f_x(0,0) = 1$

$f_y = -e^x \sin y \quad f_y(0,0) = 0$

$f_{xx} = e^x \cos y \quad f_{xx}(0,0) = 1$

$f_{xy} = -e^x \sin y \quad f_{xy}(0,0) = 0$

$f_{yy} = -e^x \cos y \quad f_{yy}(0,0) = -1$

$f_{xxx} = e^x \cos y \quad f_{xxx}(0,0) = 1$

$f_{xxy} = -e^x \sin y \quad f_{xxy}(0,0) = 0$

$f_{xyy} = -e^x \cos y \quad f_{xyy}(0,0) = -1$

$f_{yyy} = e^x \sin y \quad f_{yyy}(0,0) = 0$

Substituting these values in the Taylor series we get,

$$f(x,y) = 1 + \frac{x}{1!} + \frac{x^2 - y^2}{2!} + \frac{x^3 - 3xy^2}{3!} + \ldots$$

Using Taylor series verify that $cos(x+y) = 1 - \frac{(x+y)^2}{2!} + \frac{(x+y)^4}{4!} - \ldots$

$f(x,y) = cos(x+y) \quad f(0,0) = 1$

$f_x = f_y = -sin(x+y) \Longrightarrow f_x(0,0) = f_y(0,0) = 0$

$f_{xx} = f_{xy} = f_{yy} = -cos(x+y) \Longrightarrow f_{xx}(0,0) = f_{xy}(0,0) = f_{yy}(0,0) = -1$

$f_{xxx} = f_{xxy} = f_{xyy} = f_{yyy} = sin(x+y) \Longrightarrow f_{xxx}(0,0) = f_{xxy}(0,0) = f_{xyy}(0,0) = f_{yyy}(0,0) = 0$

$f_{xxxx} = f_{xxxy} = f_{xxyy} = f_{xyyy} = f_{yyyy} = cos(x+y) \Longrightarrow f_{xxxx}(0,0) = f_{xxxy}(0,0) = f_{xxyy}(0,0) = f_{xyyy}(0,0) = f_{yyyy}(0,0) = 1$

Substituting these values we get

$$cos(x+y) = 1 - \frac{(x+y)^2}{2!} + \frac{(x+y)^4}{4!} - \ldots$$

Expand the function sin xy in powers of $x-1$ and $y-\frac{\pi}{2}$ up to second degree terms.

$$f(x,y)=\sin xy$$

$$f_x=y\cos(xy)$$

$$f_y=x\cos(xy)$$

$$f_{xx}=-y^2\sin(xy)$$

$$f_{xy}=-xy\sin(xy)+\cos(xy)$$

$$f_{yy}=-x^2\sin(xy)$$

Value of the function at $(1,\pi/2)$

$$f=1$$

$$f_x=0$$

$$f_y=0$$

$$f_{xx}=-\frac{\pi^2}{4}$$

$$f_{xy}=-\frac{\pi}{2}$$

$$f_{yy}=-1$$

Taylor's series expansion is

$$f(1,\pi/2)=1+\left[(x-1)0+\left(y-\pi/2\right)0\right]$$

$$+\frac{1}{2!}\left[(x-1)^2(-\pi^2/4)+2(x-1)\left(y-\pi/2\right)(-\pi/2)+\left(y-\pi/2\right)^2(-1)\right]+\ldots$$

$$=1+\frac{1}{2}\left[-\frac{\pi^2}{4}(x-1)^2-\pi(x-1)\left(y-\frac{\pi}{2}\right)-\left(y-\frac{\pi}{2}\right)^2\right]+\ldots$$

Problems for practice

1. Using Taylor's series expand $e^x log(1+y)$ upto term of the third degree about (0,0)
2. Find the Taylor series expansion of e^{xy} at (1,1) upto third degree terms.
3. Find the expansions for $cosxsiny$ on powers of x and y upto terms of third degree.