TRANSFORM AND BOUNDARY VALUE PROBLEMS

MATHEMATICS-III

(18MAB201T)

DEPARTMENT OF MATHEMATICS

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There are several applications of Fourier series in computer science engineering. For example:

- **MP3 Encoding/ Sound Processing:** Fourier series is used in MP3 coding to simplify the MP3 formats file. To compress mp3 or .wmv and simplify it easily to get a fast and more simplified sound by fourier series.
- **Transmissions and processing signals:** Fourier series is used in computer science in transmission and processing of digital signals.Suppose a digital signal may have a frequency of 200Hz. However its rise and fall rates are very much faster than would be expected at 200Hz sine web format. Thats best faster transmission and processing can be got from fourier series.
- **Time Domain conversion:** Fourier series can transform time domain into frequency domain. And this domain is used as a mathematical tool to analyze the signals in that computer uses.

Periodic Function: A function $f(x)$ is said to be periodic if $f(x+p) = f(x)$ for all real x and some positive p, where p is called the period of $f(x)$.
Fourier Series: if $f(x)$ is a periodic function with period 2π, then the Fourier series of the the function $f(x)$ in the interval $[0,\ 2\pi]$ is given by:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \qquad \text{where}$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx \qquad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nxdx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nxdx$$

Note: Here a_0, a_n and b_n are called Euler-Fourier formula.

Dirichlet's Conditions: A periodic function said to be satisfy Dirichlet's Conditions if

- It is a single valued and finite in any interval
- It has at at most a finite number of discontinuities with in the period
- It has finite number of maxima and minima in any one period

- If $m \neq n$, then $\int_0^{2\pi} \sin mx \cos nx dx = 0$
- If $m \neq n$, then $\int_0^{2\pi} \sin mx \sin nx dx = 0$
- If $m \neq n$, then $\int_0^{2\pi} \cos mx \cos nx dx = 0$
- If $n \neq 0$, then $\int_0^{2\pi} \sin^2 nx dx = \pi$
- If $n \neq 0$, then $\int_0^{2\pi} \cos^2 nx dx = \pi$
- $\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2+b^2}\left[a \sin bx - b \cos bx\right]$
- $\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2+b^2}\left[a \cos bx + b \sin bx\right]$

Example 1: Obtain the F.S. of periodicity 2π for $f(x) = e^{-x}$ in the interval $0 < x < 2\pi$ and hence deduce the value of $\sum_{n=2}^{\infty} \frac{(-1)^n}{1+n^2}$.

Solution: Here $f(x) = e^{-x}$, $\quad 0 < x < 2\pi$.

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{2\pi}\int_0^{2\pi} e^{-x}dx \\
&= \frac{1}{\pi}\left[-e^{-x}\right]_0^{2\pi} = \frac{1-e^{-2\pi}}{\pi}
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx dx \\
&= \frac{1}{\pi}\left[\frac{e^{-x}}{(-1)^2+n^2}\left((-1)\cos nx + n\sin nx\right)\right]_0^{2\pi} \\
&= \frac{1}{\pi(n^2+1)}\left[e^{-2\pi}(-1) - (-1)\right] = \frac{1-e^{-2\pi}}{\pi(n^2+1)}.
\end{aligned}
$$

Similarly,

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx dx \\ &= \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}\left((-1)\sin nx - n\cos nx\right)\right]_0^{2\pi} \\ &= \frac{1}{\pi(n^2+1)}\left[e^{-2\pi}(-n)+n\right] = \frac{n(1-e^{-2\pi})}{\pi(n^2+1)}. \end{aligned}$$

$\therefore$ The F.S of e^{-x} is given by

$$\boxed{e^{-x} = \frac{(1-e^{-2\pi})}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{1}{(n^2+1)}\left(\cos nx + n\sin nx\right)\right]}.$$

Next, put $x = \pi$

$$e^{-\pi} = \frac{(1-e^{-2\pi})}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty} \frac{\cos nx}{(n^2+1)}\right] \Rightarrow \frac{\pi e^{-\pi}}{(1-e^{-2\pi})} = \left[\frac{1}{2} + \sum_{n=1}^{\infty} \frac{\cos nx}{(n^2+1)}\right]$$
$$= \frac{\pi}{(e^{\pi} - e^{-\pi})} = \frac{\pi}{2 \sin h\pi}.$$

Putting $n = 1$, we get $\sum_{n=1}^{\infty} \frac{\cos nx}{(n^2+1)} = -1/2$. Therefore,

$$1/2 - 1/2 + \sum_{n=2}^{\infty} \frac{\cos nx}{(n^2+1)} = \frac{\pi cosech\pi}{2}$$

$$\boxed{cosech\pi = \frac{2}{\pi} \sum_{n=2}^{\infty} \frac{(-1)^n}{(n^2+1)}}.$$

Example 2:

$$f(x) = \begin{cases} -k & \text{when} \quad -\pi < x < 0 \\ k & \text{when} \quad 0 < x < \pi \end{cases}$$

and $f(x + 2\pi) = f(x)$ for all x. Derive the F.S. for $f(x)$ and deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots..$

Solution:

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -kdx + \int_{0}^{\pi} kdx\right] \\ &= \frac{1}{\pi}\left[(-kx)_0^{-\pi} + (kx)_\pi^0\right] = 0. \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -k\cos nxdx + \int_{0}^{\pi} k\cos nxdx\right] \\ &= \frac{1}{\pi}\left[\left(\frac{-k\sin nx}{n}\right)_{-\pi}^{0} + \left(\frac{k\sin nx}{n}\right)_{\pi}^{0}\right] = \frac{1}{\pi}[0+0] = 0. \end{aligned}$$

Similarly,

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -k\sin nx dx + \int_{0}^{\pi} k\sin nx dx\right] \\
&= \frac{1}{\pi}\left[\left(\frac{k\cos nx}{n}\right)_{-\pi}^{0} + \left(\frac{-k\cos nx}{n}\right)_{\pi}^{0}\right] \\
&= \frac{k}{n\pi}\left[(1-\cos n\pi) + (-\cos n\pi + 1)\right] \\
&= \frac{2k}{n\pi}\left[(1-\cos n\pi)\right] = \frac{2k}{n\pi}\left[(1-(-1)^n)\right]
\end{aligned}$$

$\therefore$

$$b_n = \begin{cases} \frac{4k}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

i.e. $b_1 = \frac{4k}{\pi}$, $b_0 = 0$, $b_3 = \frac{4k}{3\pi}$........ The F.S. is given by

$$\boxed{f(x) = \frac{4k}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + ...\right]}.$$

Putting $x = \frac{\pi}{2}$ we get $f(\frac{\pi}{2}) = \frac{4k}{\pi}\left[1 + \frac{1}{3} + \frac{1}{5} + ...\right]$. But $f(\frac{\pi}{2}) = k$

$$\therefore \boxed{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...}.$$

Note: If $x = a$ is the point discontinuity of $f(x)$, then the F.S at $x = a$ is $\frac{1}{2}[f(a^+) + f(a^-)]$

Example 3: Obtain the F.S of the periodic function defined by

$$f(x) = \begin{cases} -\pi & \text{if} \quad -\pi < x < 0 \\ x & \text{if} \quad 0 < x < \pi \end{cases}$$

and deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\ldots..$

Solution:

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi dx + \int_{0}^{\pi} x dx\right] = \frac{1}{\pi}\left[-\pi^2 + \frac{\pi^2}{2}\right] = -\frac{\pi}{2}$$

$$\begin{aligned} a_n &= \left[\int_{-\pi}^{0} -\pi \cos nx dx + \int_{0}^{\pi} x \cos nx dx\right] \\ &= \frac{1}{\pi}\left[\left(-\pi\frac{\sin nx}{n}\right)_{-\pi}^{0} + \left(x\frac{\sin nx}{n}\right)_{0}^{\pi} - \left(-\frac{\cos nx}{n}\right)_{0}^{\pi}\right] \end{aligned}$$

$a_n = \frac{1}{\pi}\left[\frac{1}{n^2}(\cos n\pi - 1)\right] = \frac{1}{n^2}[(-1)^n - 1]$.
Similarly,

$$
\begin{aligned}
b_n &= \left[\int_{-\pi}^{0} -\pi \sin nx dx + \int_{0}^{\pi} x \sin nx dx\right] \\
&= \frac{1}{\pi}\left[\left((-\pi)\frac{-\cos nx}{n}\right)_{-\pi}^{0} + \left(-x\frac{\cos nx}{n}\right)_{0}^{\pi} - \left(-\frac{\sin nx}{n}\right)_{0}^{\pi}\right] \\
&= \frac{1}{\pi}\left[\frac{\pi}{n^2}(1 - \cos n\pi) - \frac{\pi}{n}\cos n\pi\right] = \frac{1}{n}[1 - 2\cos n\pi] = \frac{1}{n}[1 - 2(-1)^n]
\end{aligned}
$$

$\therefore$

$$
f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\dots\right)
$$

$$
+3\sin x - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} - \frac{\sin 4x}{4} + \dots\dots
$$

Deduction: Here $f(0^-) = -\pi$ and $f(0^+) = 0$, hence

$$\frac{f(0^-) + f(0^+)}{2} = -\frac{\pi}{2} = f(0).$$

$\therefore$

$$\begin{aligned}
f(0) &= -\frac{\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\right) \\
-\frac{\pi}{2} &= -\frac{\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\right) \\
-\frac{\pi}{4} &= -\frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\right) \\
\frac{\pi^2}{8} &= \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\right).
\end{aligned}$$

Even function: The function $f(x)$ is said to be even function if $f(-x) = f(x)$.

Example: x^2, $\cos x$, and all even degree functions of x.

Odd function: The function $f(x)$ is said to be odd function if $f(-x) = -f(x)$.

Example: x, $\sin x$, and all odd degree functions of x.

Note 1: If a function $f(x)$ is even in $(-a,\ a)$, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$.

Note 2: If a function $f(x)$ is odd in $(-a,\ a)$, then $\int_{-a}^{a} f(x)dx = 0$.

Note 3: If a function $f(x)$ is even in $(-\pi,\ \pi)$, then $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx, \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos x dx \quad b_n = 0.$

Note 4: If a function $f(x)$ is odd in $(-\pi,\ \pi)$, then $a_0 = 0, \quad a_n = 0, \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin x dx.$

Example 4: Find the F.S of $f(x) = x + x^2$ in $(-\pi,\ \pi)$ of periodicity 2π and hence deduce $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$.

Solution:

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)dx = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} xdx + \int_{-\pi}^{\pi} x^2dx\right] \\ &= \frac{1}{\pi}\left[2\int_0^{\pi} x^2dx\right] = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2\pi^2}{3}. \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\cos nxdx = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\cos nxdx + \int_{-\pi}^{\pi} x^2\cos nxdx\right] \\ &= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x^2\cos nxdx\right] = \frac{2}{\pi}\left[\int_0^{\pi} x^2\cos nxdx\right] \\ &= \frac{2}{\pi}\left[x^2\frac{\sin nx}{n} - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi} \\ &= \frac{2}{\pi}\left[\frac{2\pi}{n^2}\cos n\pi\right] = \frac{4}{n^2}(-1)^n. \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx dx = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\sin nx dx + \int_{-\pi}^{\pi} x^2 \sin nx dx\right] \\ &= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\sin nx dx\right] = \frac{2}{\pi}\left[\int_{0}^{\pi} x\sin nx dx\right] \\ &= \frac{2}{\pi}\left[-x\frac{\cos nx}{n} - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi} = \frac{2}{\pi}\left[-\frac{\pi}{n}\cos n\pi\right] = -\frac{2}{n}(-1)^n. \end{aligned}$$

$$\therefore \boxed{f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n\left[\frac{4}{n^2}\cos nx - \frac{2}{n}\sin nx\right].}.$$

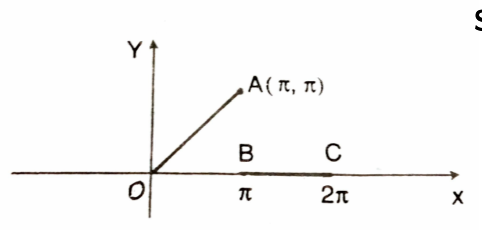
Deduction: As $x = -\pi$ and $x = \pi$ are the end points, therefore F.S at $x = \pi$ is average value of $f(x)$ at $x = -\pi$ and $x = \pi$. i.e

$$f(\pi) = \frac{f(-\pi) + f(\pi)}{2} = \frac{(-\pi + \pi^2) + (\pi + \pi^2)}{2} = \pi^2.$$

Using this in the F.S we get

$$\begin{aligned}
\pi^2 &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n \left[\frac{4}{n^2}\cos n\pi\right] = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^{2n}\frac{4}{n^2} \\
\Rightarrow &\quad \frac{2\pi^2}{3} = \sum_{n=1}^{\infty}(-1)^{2n}\frac{4}{n^2} \\
\Rightarrow &\quad \frac{2\pi^2}{3} = \sum_{n=1}^{\infty}\frac{1}{n^2}.
\end{aligned}$$

Example 5: Write down the analytic expression of the following function given in the graph. Hence find the F.S

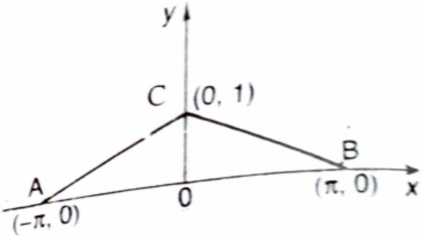


Solution: Here we will have

$$f(x) = \begin{cases} x & \text{for } \ 0 < x < \pi \\ 0 & \text{for } \ \pi < x < 2\pi. \end{cases}$$

Then find the F.S as usual.

Example 6: Write down the analytic expression of the following function given in the graph. Hence find the F.S

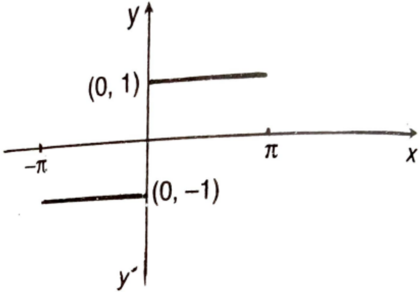


Solution: Equation of BC $y = -\frac{1}{\pi}(x-\pi)$ and equation of AC $y = \frac{1}{\pi}(x+\pi)$. Hence

$$f(x) = \begin{cases} \frac{1}{\pi}(x+\pi) & \text{if } -\pi < x < 0 \\ -\frac{1}{\pi}(x-\pi) & \text{if } 0 < x < \pi. \end{cases}$$

Then find the F.S as usual.

Example 7: Write down the analytic expression of the following function given in the graph. Hence find the F.S

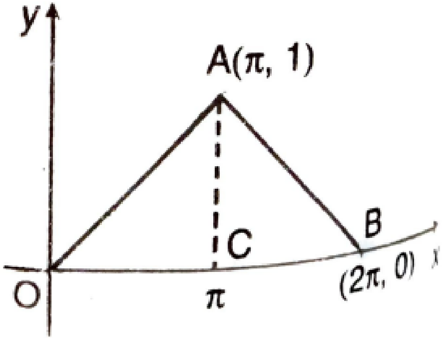


Solution: Here the two segments are parallel to $x-$axis, therefore

$$f(x) = \begin{cases} -1 \ \text{ if } & -\pi < x < 0 \\ 1 \ \text{ for } & 0 < x < \pi. \end{cases}$$

Then find the F.S as usual.

Example 8: Write down the analytic expression of the following function given in the graph. Hence find the F.S



Solution: Equation of AB $y = -\frac{1}{\pi}(x - 2\pi)$ and equation of OA $y = \frac{1}{\pi}x$. Hence

$$f(x) = \begin{cases} \frac{x}{\pi} & \text{if } 0 < x < \pi \\ \frac{1}{\pi}(2\pi - x) & \text{if } \pi < x < 2\pi. \end{cases}$$

Then find the F.S as usual.

Half Range Cosine Series: Let $f(x)$ be an even function in the interval $(-\pi, \pi)$ such that $f(x) = f(-x)$ in $(0, \pi)$, then the Fourier half range cosine series is given by

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.}$$

Where

$$\boxed{a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx \quad \text{and} \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nxdx.}$$

Half Range Sine Series: Let $f(x)$ be an odd function in the interval $(-\pi, \pi)$ and defined in $(0, \pi)$, such that $f(x) = -f(-x)$ for $x \in (-\pi, 0)$, then the Fourier half range sine series is given by

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad \text{where} \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nxdx.}$$

Exampe 1: Determine the half range Fourier sine series for

$$f(x) = \begin{cases} x, & 0 < x < \pi/2 \\ \pi - x, & \pi/2 < x < \pi \end{cases} .$$

Solution: We know the sine series is
$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$, where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx dx$ for $n \geq 1$.

$$\begin{aligned} b_n &= \frac{2}{\pi}\left[\int_0^{\pi/2} x \sin nx dx + \int_{\pi/2}^{\pi} (\pi - x) \sin nx dx\right] \\ &= \frac{2}{\pi}\left[-x\frac{\cos nx}{n} - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi/2} \\ &+ \frac{2}{\pi}\left[-(\pi - x)\frac{\cos nx}{n} - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{\pi/2}^{\pi} \end{aligned}$$

$$\Rightarrow \quad \frac{2}{\pi}\left[\frac{1}{n^2}(\sin\frac{n\pi}{2}-0)-\frac{1}{n^2}(\sin n\pi-\sin\frac{n\pi}{2})\right]=\frac{4}{n^2\pi}\sin\frac{n\pi}{2}.$$

Therefore,

$$f(x) = \sum_{n=1}^{\infty}\frac{4}{n^2\pi}\sin\frac{n\pi}{2}\sin nx=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin nx.$$

Exampe 2: Express $f(x)=x(\pi-x)$, $0<x<\pi$ as a F.S of periodicity 2π containing only
(i) cosine term only
(ii) sine term only.
Hence deduce
$\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}.......=\frac{\pi^2}{12}$
and
$1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}.......=\frac{\pi^3}{32}.$

Solution: We know the cosine series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$, where

$$a_0 = \frac{2}{\pi}\left[\int_0^{\pi} x(\pi - x)dx\right] = \frac{2}{\pi}\left[\frac{\pi x^2}{2} - \frac{x^3}{3}\right] = \frac{2}{\pi}\left[\frac{\pi^3}{2} - \frac{\pi^3}{3}\right] = \frac{\pi^2}{3}$$

$$\begin{aligned}
a_n &= \frac{2}{\pi}\left[\int_0^{\pi} (x\pi - x^2)\cos nx dx\right] \\
&= \frac{2}{\pi}\left[(x\pi - x^2)\frac{\sin nx}{n} - (\pi - 2x)\left(-\frac{\cos nx}{n^2}\right) + (-2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi} \\
&= \frac{2}{\pi}\left[\left(-\frac{\pi\cos n\pi}{n^2}\right) - \left(\frac{\pi\cos 0}{n^2}\right)\right] = -\frac{2}{n^2}\left(1 + (-1)^n\right)
\end{aligned}$$

$$\Rightarrow a_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ -\frac{4}{n^2} & \text{if } n \text{ is even} \end{cases}$$

Deduction:

$$x(\pi - x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} -\frac{4}{(2n)^2}\cos 2nx = \frac{\pi^2}{6} - \sum_{n=1}^{\infty} \frac{1}{n^2}\cos 2nx.$$

Put $x = \frac{\pi}{2}$

$$\frac{\pi^2}{4} = \frac{\pi^2}{6} - \sum_{n=1}^{\infty} \frac{1}{n^2}\cos n\pi = \frac{\pi^2}{6} - \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\frac{\pi^2}{4} - \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \ldots\ldots$$

Sine series:

$$b_n = \frac{2}{\pi}\left[\int_0^{\pi}(x\pi - x^2)\sin nx dx\right]$$

$$= \frac{2}{\pi}\left[(x\pi - x^2)\left(-\frac{\cos nx}{n}\right) - (\pi - 2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\frac{-2(-1)^n}{n^3}\right) + \frac{2}{n^3}\right] = \frac{4}{\pi n^3}\left(1 - (-1)^n\right)$$

$$b_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ -\frac{8}{\pi n^3} & \text{if } n \text{ is odd} \end{cases}$$

$\therefore f(x) = x(\pi - x) = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin(2n-1)x.$
Putting $x = \frac{\pi}{2}$ and simplifying we get

$$\frac{\pi^2}{4} = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3} \Rightarrow \frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3}\ldots\ldots$$

If a function $f(x)$ has period other than 2π, it's F.S can be obtained by making a change of variable so that the new variable has period 2π. Suppose $f(x)$ has period $2L$ i.e. $f(x+2L) = f(x)$ for all x. Assume that x varies from $-L$ to L and t varies from $-\pi$ to π. Then we can write

$$\frac{x}{2L} = \frac{t}{2\pi} \quad \Rightarrow \frac{x}{L} = \frac{t}{\pi} \quad \Rightarrow t = \frac{\pi x}{L}$$

Then the F.S in the interval $(0, 2L)$ will be given as

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{l}\right] \quad \text{where} \\ a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)dx, \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx. \end{aligned}$$

If a function $f(x)$ has period has a $2L$ in $c < x < c + 2L$, then introducing the variable t where $-\pi < t < \pi$ as

$$\frac{x}{2L} = \frac{t}{2\pi} \qquad \Rightarrow \frac{x}{L} = \frac{t}{\pi} \qquad \Rightarrow t = \frac{\pi x}{L},$$

with $d = t = \frac{\pi c}{L}$, we can observe that if $c < x < c + 2L$, then $d < t < d + 2\pi$ i.e. the length of the interval is 2π. As such the F.S will be given as

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{l}\right] \qquad \text{where} \\ a_0 &= \frac{1}{L}\int_c^{c+2L} f(x)dx, \qquad a_n = \frac{1}{L}\int_c^{c+2L} f(x)\cos\frac{n\pi x}{L}dx, \\ b_n &= \frac{1}{L}\int_c^{c+2L} f(x)\sin\frac{n\pi x}{L}dx. \end{aligned}$$

Note 1: The F.S of an even function $f(x)$ defined in the interval $(-L, L)$ will contain cosine term only i.e. $b_n = 0$

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \qquad \text{where} \\ a_0 &= \frac{2}{L}\int_0^L f(x)dx, \qquad a_n = \frac{2}{L}\int_0 f(x)\cos\frac{n\pi x}{L}dx. \end{aligned}$$

Note 2: The F.S for an odd function $f(x)$ defined in the interval $(-L, L)$ will contain sine term only i.e. $a_0 = 0, \quad a_n = 0$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \quad \text{where} \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx.$$

Half Range Cosine Series: Let $f(x)$ be a function defined in the interval $(0, L)$ such that $f(x) = f(-x)$ in $(-L, 0)$, then the Fourier half range cosine series with period $2L$ is given by

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}.}$$

Where

$$\boxed{a_0 = \frac{2}{L} \int_0^L f(x)dx \quad \text{and} \quad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx.}$$

Half Range Sine Series: Suppose $f(x)$ be a function defined in the interval $(0, L)$ such that $f(x) = -f(-x)$ for $x \in (-L, 0)$, then the Fourier half range sine series with period $2L$ is given by

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \quad \text{where} \quad b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin \frac{n\pi x}{L} dx.}$$

Exampe 1: Find the Fourier series for $f(x) = 1 - x^2$, in $-1 < x < 1$.
Solution: We know $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}\right]$ where $L = 1$.

$$
\begin{aligned}
a_0 &= \int_{-1}^{1} (1 - x^2) dx = 2 \int_{0}^{1} (1 - x^2) dx \quad \text{f(x) is even function} \\
&= 2 \left[x - \frac{x^3}{3}\right]_0^1 = \frac{4}{3} \\
a_n &= \int_{-1}^{1} (1 - x^2) \cos n\pi x dx = 2 \int_{0}^{1} (1 - x^2) \cos n\pi x dx, \\
&= \left[(1 - x^2)\frac{\sin n\pi x}{n\pi} - (-2x)\left(-\frac{\cos n\pi x}{(n\pi)^2}\right) + (-x)\left(-\frac{\sin n\pi x}{(n\pi)^3}\right)\right]_0^1 \\
&= \frac{4}{n^2\pi^2}(-1)^{n+1}.
\end{aligned}
$$

As $f(x) = 1 - x^2$ is an even function, so $f(x)\sin n\pi x$ is an odd function and hence $b_n = \int_{-1}^{1}(1-x^2)\sin n\pi x dx = 0$. Finally the F.S is given by

$$\boxed{f(x) = \frac{2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}(-1)^{n+1}\cos n\pi x = \frac{2}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}\cos n\pi x.}$$

Exampe 2: Obtain the Fourier series expansion of $f(x)$ given that

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < 1 \\ 2 & \text{for } 1 < x < 3 \end{cases}$$

and $f(x) = 3/2$ when $x = 0, 1, 3$ and $f(x+3) = f(x)\ \forall x$.
Solution: As $f(x)$ is defined in $(0, 3)$ and again the period is 3
$\therefore 2L = 3 \quad \Rightarrow L = 3/2$. Hence the F.S is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{3/2} + b_n \sin\frac{n\pi x}{3/2}\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi x}{3} + b_n \sin \frac{2n\pi x}{3} \right].$$

Now

$$\begin{aligned}
a_0 &= \frac{2}{3} \int_0^3 f(x)dx = \frac{2}{3} \left[\int_0^1 1dx + \int_1^3 2dx \right] \\
&= \frac{2}{3} \left[x|_0^1 + 2x|_1^3 \right] = \frac{2}{3}(1+4) = \frac{10}{3} \\
a_n &= \frac{2}{3} \left[\int_0^1 \cos \frac{2n\pi x}{3} dx + \int_1^3 \cos \frac{2n\pi x}{3} dx \right], \\
&= \frac{2}{3} \left[\left(\frac{\sin \frac{2n\pi x}{3}}{\frac{2n\pi x}{3}} \right)_0^1 + 2 \left(\frac{\sin \frac{2n\pi x}{3}}{\frac{2n\pi x}{3}} \right)_1^3 \right] = \frac{2}{3} \left[-\frac{3}{2n\pi} \sin \frac{2n\pi}{3} \right] \\
&= -\frac{1}{n\pi} \sin \frac{2n\pi}{3}.
\end{aligned}$$

If we take $n = 3m$ i.e. n is multiple of 3 , then we get $a_3 = a_6 = a_9 ...... = 0$. Hence

$$a_1 = -\frac{1}{\pi}\sin\frac{2\pi}{3} = -\frac{\sqrt{3}}{2\pi}, \qquad a_2 = -\frac{1}{2\pi}\sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2\pi}.\left(\frac{1}{2}\right),$$

$$a_4 = -\frac{1}{4\pi}\sin\frac{8\pi}{3} = -\frac{\sqrt{3}}{2\pi}.\left(\frac{1}{4}\right), \quad a_5 = -\frac{1}{5\pi}\sin\frac{10\pi}{3} = \frac{\sqrt{3}}{2\pi}.\left(\frac{1}{5}\right).$$

$$\begin{aligned} b_n &= \frac{2}{3}\left[\int_0^1 \sin\frac{2n\pi x}{3}dx + \int_1^3 \sin\frac{2n\pi x}{3}dx\right], \\ &= \frac{2}{3}\left[\left(-\frac{\cos\frac{2n\pi x}{3}}{\frac{2n\pi x}{3}}\right)_0^1 + 2\left(-\frac{\cos\frac{2n\pi x}{3}}{\frac{2n\pi x}{3}}\right)_1^3\right] \\ &= -\frac{1}{n\pi}\left(1 - \cos\frac{2n\pi}{3}\right). \end{aligned}$$

When $n = 3m$ i.e. n is multiple of 3 , then we get $b_n = -\frac{1}{3m\pi}(1-1) = 0$.
Hence $b_3 = b_6 = a_9 = ... = 0$.

$$b_1 = -\frac{1}{\pi}\left(1 - \cos\frac{2\pi}{3}\right) = -\frac{3}{2\pi}, \qquad b_2 = -\frac{1}{2\pi}\cos\frac{4\pi}{3} = -\frac{3}{2\pi}.\left(\frac{1}{2}\right),$$

$$a_4 = -\frac{1}{4\pi}\cos\frac{8\pi}{3} = -\frac{3}{2\pi}.\left(\frac{1}{4}\right), \quad a_5 = -\frac{1}{5\pi}\sin\frac{10\pi}{3} = -\frac{3}{2\pi}.\left(\frac{1}{5}\right).$$

$\therefore$ The F.S is given by

$$f(x) = \frac{5}{3} - \frac{\sqrt{3}}{2\pi}\left[\cos\frac{2\pi x}{3} - \frac{1}{2}\cos\frac{4\pi x}{3} + \frac{1}{4}\cos\frac{8\pi x}{3} - \frac{1}{5}\cos\frac{10\pi x}{3} + ...\right]$$
$$-\frac{3}{2\pi}\left[\sin\frac{2\pi x}{3} + \frac{1}{2}\sin\frac{4\pi x}{3} + \frac{1}{4}\sin\frac{8\pi x}{3} + \frac{1}{5}\sin\frac{10\pi x}{3} + ...\right]$$

Exampe 3: Find the half range cosine series for the function $f(x) = (x-1)^2$ in the interval $0 < x < 1$. Hence show that $\pi^2 = 6\{1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ...\}$

Solution: Here $L = 1$ the F.S is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x$.
Hence

$$\begin{aligned}
a_0 &= \frac{2}{1}\int_0^1 f(x)dx = 2\int_0^1 (x-1)^2 dx \left[\frac{(x-1)^3}{3}\right]_0^1 = \frac{3}{3} \\
a_n &= 2\int_0^1 (x-1)^2 \cos n\pi x dx \\
&= 2\left[(x-1)^2\left(\frac{\sin n\pi x}{n\pi}\right) - 2(x-1)\left(-\frac{\cos n\pi x}{(n\pi)^2}\right) + 2\left(-\frac{\sin n\pi x}{(n\pi)^3}\right)\right]_0^1 \\
&= 2\left[\frac{2}{n^2\pi^2}\right] = \frac{4}{n^2\pi^2}.
\end{aligned}$$

Hence

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2} \cos n\pi x.$$

As

$$\frac{f(0^-) + f(0^+)}{2} = \frac{1+1}{2} = 1$$

$\therefore$ The F.S becomes

$$f(0) = 1 = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \Rightarrow \frac{2}{3} = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \pi^2 = 6 \sum_{n=1}^{\infty} \frac{1}{n^2} = 6\{1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots\}.$$

Root Mean Square(RMS) value of a function: The root mean square value of a function $y = f(x)$ over the given interval (a, b) is defined as

$$\bar{y} = \sqrt{\frac{\int_a^b y^2 dx}{b-a}}.$$

If the interval is $(0, 2\pi)$, then it can be written as

$$\bar{y} = \sqrt{\frac{\int_0^{2\pi} y^2 dx}{2\pi}} \qquad \Rightarrow \bar{y}^2 = \frac{1}{2\pi}\int_0^{2\pi} y^2 dx.$$

Note: If $y = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$ is the F.S, then multiplying it with $f(x)$ and integrating term by term in the interval $(0, 2\pi)$, we get

$$\boxed{\bar{y}^2 = \frac{{a_0}^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}\left[{a_n}^2 + {b_n}^2\right].}$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx,\ \ a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nxdx,\ \ b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nxdx$$

$$\Rightarrow a_0\pi = \int_0^{2\pi} f(x)dx,\ \ a_n\pi = \int_0^{2\pi} f(x)\cos nxdx,\ \ b_n\pi = \int_0^{2\pi} f(x)\sin nxdx$$

Example 4: Expand $f(x) = x - x^2$ as F.S $-1 < x < 1$ and using this series find R.M.S values of $f(x)$ in the interval.

Solution: Here $L = 1$

$$\begin{aligned} a_0 &= \frac{1}{1}\int_{-1}^{1}(x - x^2)dx = 2\int_0^1(-x^2)dx = 2\left[-\frac{x^3}{3}\right]_0^1 = -\frac{2}{3} \\ a_n &= \int_{-1}^{1}(x - x^2)\cos n\pi xdx = 2\int_0^1(-x^2)\cos n\pi xdx \\ &= -2\left[x^2\left(\frac{\sin n\pi x}{n\pi}\right) - 2x\left(\frac{-\cos n\pi x}{(n\pi)^2}\right) + 2\left(\frac{-\sin n\pi x}{(n\pi)^3}\right)\right]_0^1 \end{aligned}$$

$$a_n = -2\left(\frac{2\cos n\pi}{n^2\pi^2}\right) = \frac{4}{n^2\pi^2}(-1)^{n+1}$$

$$b_n = \int_{-1}^{1}(x-x^2)\sin n\pi x dx = 2\int_0^1 x\sin n\pi x dx$$

$$= 2\left[x\left(\frac{-\cos n\pi x}{n\pi}\right)-\left(\frac{-\sin n\pi x}{(n\pi)^2}\right)\right]_0^1 = 2\left[\frac{-\cos n\pi}{n\pi}\right] = \frac{2}{n\pi}(-1)^{n+1}$$

$\therefore$ The F.S is

$$f(x) = x - x^2 = -\frac{1}{3} + \sum_{n=1}^{\infty}\left[\frac{4}{n^2\pi^2}(-1)^{n+1}\cos n\pi x + \frac{2}{n\pi}(-1)^{n+1}\sin n\pi x\right].$$

R.M.S value: The R.M.S value of $f(x)$ is given as

$$\bar{y} = \sqrt{\frac{\int_{-1}^{1}(x-x^2)^2 dx}{2}} = \sqrt{\frac{1}{2}\int_{-1}^{1}(x^2+x^4-2x^3)dx}$$

$$= \sqrt{\int_{0}^{1}(x^2+x^4)dx} = \sqrt{\left[\frac{x^3}{3}+\frac{x^5}{5}\right]_0^1} = \sqrt{\frac{1}{3}+\frac{1}{5}} = \sqrt{\frac{8}{15}}.$$

Example 5: Express $f(x) = x$ in half range cosine series and sine series of periodicity $2l$ in the range $0 < x < 1$ and deduce the value $\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+........+\infty$.

Solution: For the cosine series we will get

$$a_0 = \frac{2}{l}\int_0^l x dx = l$$

$$a_n = \frac{2}{l}\int_0^l x \cos\frac{n\pi x}{l} dx = \left[x\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - \left(\frac{\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_0^l$$

$$a_n = \begin{cases} -\frac{4l}{n^2\pi^2} & \text{if n is odd} \\ 0 & \text{if n is even} \end{cases}$$

$$\begin{aligned} f(x) = x &= \frac{l}{2} - \frac{4l}{\pi^2} \sum_{n=1,3,5....}^{\infty} \left[\frac{1}{n^2} \cos \frac{n\pi x}{l} \right] \\ &= \frac{l}{2} - \frac{4l}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{l} \right] \end{aligned}$$

Deduction using Parseval's Theorem:

$$\text{(Range)} \left[\frac{{a_0}^2}{4} + \frac{1}{2} \sum_{n=1,3,5....}^{\infty} {a_n}^2 \right] = \int_0^l [f(x)]^2 dx$$

$$l \left[\frac{l^2}{4} + \frac{1}{2} \sum_{n=1,3,5...}^{\infty} \frac{16l^2}{n^4\pi^4} \right] = \int_0^l x^2 dx = \frac{l^3}{3}$$

$$\Rightarrow \quad \frac{8l^3}{\pi^4} \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^4} = \frac{l^3}{3} - \frac{l^3}{4} = \frac{l^3}{12}$$

$$\Rightarrow \quad \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^4} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\ldots\ldots = \frac{\pi^4}{96}$$

Deduction for sine series: Here $b_n = \frac{2l}{n\pi}$, so

$$\text{(Range)} \left[\frac{1}{2}\sum_{n=1}^{\infty} {b_n}^2\right] = \int_0^l [f(x)]^2 dx \quad \Rightarrow l\left[\frac{1}{2}\sum_{n=1}^{\infty} \frac{4l^2}{n^2\pi^2}\right] = \int_0^l x^2 dx = \frac{l^3}{3}$$

$$\Rightarrow \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{3} \quad \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

Definition: The process of finding the F.S for a function given by numerical values is known as harmonic analysis.
In harmonic the F.S is given by

$$
\begin{aligned}
f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \\
&= \frac{a_0}{2} + [a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \ldots\ldots\ldots\ldots] \\
&\qquad + [b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \ldots\ldots\ldots\ldots],
\end{aligned}
$$

where the first term $a_1 \cos x + b_1 \sin x$ is called the fundamental or first harmonic, $a_2 \cos 2x + b_2 \sin 2x$ is second harmonic and so on so....
Here the co-efficient can be calculated as

$$
\begin{aligned}
a_0 &= 2[\text{Mean values of } \ f(x)] \\
a_n &= 2[\text{Mean values of } \ f(x) \cos nx] \\
b_n &= 2[\text{Mean values of } \ f(x) \sin nx].
\end{aligned}
$$

Example 1: Compute the first 3 harmonic of the F.S for $f(x)$ from the following table:

x	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$f(x)$	2.34	3.01	3.68	4.15	3.69	2.20	0.83	0.51	0.88	1.09	1.19	1.64

Solution: Let the F.S upto 3 harmonic in $(0, 2\pi)$ be
$f(x) = \frac{a_0}{2} + [a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x] + [b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x]$.
Now let us find the value of a_0 as

$$\begin{aligned} a_0 &= 2[\text{Mean values of } f(x)] = \frac{2}{n}\sum f(x) = \frac{2}{12}\sum f(x) \\ &= \frac{1}{6}[2.34 + 3.01 + 3.68 + ... + 1.09 + 1.19 + 1.64] \\ &= \frac{1}{6} \times 25.21 = 4.202. \end{aligned}$$

Now we can calculate the other values using the following table:

x	$f(x)$	$f(x)\cos x$	$f(x)\sin x$	$f(x)\cos 2x$	$f(x)\sin 2x$	$f(x)\cos 3x$	$f(x)\sin 3x$
30°	2.34	2.026	1.17	1.17	2.026	0	2.34
60°	3.01	1.505	2.607	-1.505	2.607	-3.01	0
90°	3.68	0	3.68	-3.65	0	0	-3.68
120°	4.15	-2.075	3.594	-2.075	-3.594	4.15	0
150°	3.69	-3.196	1.845	1.845	-3.195	0	3.69
180°	2.20	-2.20	0	2.20	0	-2.20	0
210°	0.83	-0.719	-0.415	0.415	0.719	0	-0.83
240°	0.51	-0.255	-.442	-0.255	0.442	0.51	0
270°	0.88	0	-0.88	-0.88	0	0	0.88
300°	1.09	0.545	-0.944	-0.545	-0.944	-1.09	0
330°	1.19	1.030	-0.595	0.595	-1.030	0	-1.19
360°	1.64	1.64	0	1.64	0	1.64	0
$\sum$	25.21	-1.699	10.504	-1.125	-2.969	0	1.21

$$\begin{aligned} a_1 &= 2[\text{Mean values of } \; f(x)\cos x] = \frac{2}{n}\sum f(x)\cos x \\ &= \frac{2}{12} \times -1.699 = -0.2832. \end{aligned}$$

$$
\begin{aligned}
a_2 &= \frac{2}{12}\sum f(x)\cos 2x = \frac{1}{6}\times -1.125 = -0.1875\\
a_3 &= \frac{2}{12}\sum f(x)\cos 3x = \frac{1}{6}\times 0 = 0.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
b_1 &= \frac{2}{12}\sum f(x)\sin x = \frac{1}{6}\times 10.504 = 1.7506\\
b_2 &= \frac{2}{12}\sum f(x)\sin 2x = \frac{1}{6}\times -2.969 = -0.495\\
b_3 &= \frac{2}{12}\sum f(x)\sin 3x = \frac{1}{6}\times 1.21 = 0.202.
\end{aligned}
$$

$\therefore$ The F.S upto 3 harmonic in $(0, 2\pi)$ be

$$
\begin{aligned}
f(x) = 2.101 &+ [-0.2832\cos x - 0.1875\cos 2x]\\
&+ [1.7506\sin x - 0.495\sin 2x + 0.202\sin 3x].
\end{aligned}
$$

Example 2: Compute the first 3 harmonic of the F.S of $f(x)$ from the following table:

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
$f(x)$	1	1.4	1.9	1.7	1.5	1.2	1

Solution: Here we need to remember that the value of the function $f(x)$ is known at $x = 0$ and $x = 2\pi$.
Therefore, we exclude the last value $x = 2\pi$.
Let the F.S upto 3 harmonic in $(0, 2\pi)$ be
$f(x) = \frac{a_0}{2} + [a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x] + [b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x]$.

x	$f(x)$	$f(x)\cos x$	$f(x)\sin x$	$f(x)\cos 2x$	$f(x)\sin 2x$	$f(x)\cos 3x$	$f(x)\sin 3x$
0	1	1	0	1	0	1	0
$\pi/3$	1.4	0.7	1.2124	-0.7	1.2124	-1.4	0
$2\pi/3$	1.9	-0.95	1.6454	-0.95	-1.6454	1.9	0
π	1.7	-1.7	0	1.7	0	-1.7	0
$4\pi/3$	1.5	-0.75	-1.299	-0.75	1.299	1.5	0
$5\pi/3$	1.2	0.6	-1.0392	-0.6	-1.0392	-1.2	0
$\sum$	8.7	-1.1	0.5196	-0.3	-0.1732	0.1	0

$$
\begin{aligned}
a_0 &= 2[\text{Mean values of } \ f(x)] = \frac{2}{6}\sum f(x) = \frac{1}{3}\times 8.7 = 2.9 \\
a_1 &= 2[\text{Mean values of } \ f(x)\cos x] = \frac{1}{3}\times -1.1 = -0.367 \\
a_2 &= 2[\text{Mean values of } \ f(x)\cos 2x] = \frac{1}{3}\times -0.3 = -0.1 \\
a_3 &= 2[\text{Mean values of } \ f(x)\cos 3x] = \frac{1}{3}\times 0.1 = 0.033.
\end{aligned}
$$

$$
\begin{aligned}
b_1 &= 2[\text{Mean values of } \; f(x)\sin x] = \frac{1}{3} \times 0.5196 = 0.1732 \\
b_2 &= 2[\text{Mean values of } \; f(x)\sin 2x] = \frac{1}{3} \times -0.1732 = -0.058 \\
b_3 &= 2[\text{Mean values of } \; f(x)\sin 3x] = \frac{1}{3} \times 0 = 0.
\end{aligned}
$$

$\therefore$ The F.S up to 3 harmonic is given by

$$
f(x) = 1.45 + [-0.367\cos x - 0.1\cos 2x + 0.033\cos 3x] \\
+[0.1732\sin x - 0.058\sin 2x].
$$

Example 3: The values of x and the corresponding values of $f(x)$ over a period T are given below. Show that

$$f(x) = 0.75 + 0.37\cos\theta + 1.004\sin\theta, \quad \text{where} \quad \theta = \frac{2\pi x}{T}.$$

x	0	$T/6$	$T/3$	$T/2$	$2T/3$	$5T/6$	T
$f(x)$	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

Solution: Let us change this problem in to the problem in terms of θ with $\theta = \frac{2\pi x}{T}$ as

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
$f(x)$	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

As we know the value of the function $f(x)$ at $x = 0$ and $x = 2\pi$, so we exclude the last value $x = 2\pi$.
Let us consider the F.S as $f(x) = \frac{a_0}{2} + a_1\cos\theta + b_1\sin\theta$. Then construct the following table:

x	$f(x)$	$f(x)\cos x$	$f(x)\sin x$
0	1.98	1.98	0
$\pi/3$	1.30	0.65	1.258
$2\pi/3$	1.05	-0.525	-0.9093
π	1.3	-1.3	0
$4\pi/3$	-0.88	0.44	0.762
$5\pi/3$	-0.25	0.6	-1.0392
$\sum$	4.5	1.12	3.013

$$a_0 = \frac{2}{6}\sum f(x) = \frac{1}{3} \times 4.5 = 1.5$$

$$a_1 = 2[\text{Mean values of } \; f(x)\cos x] = \frac{1}{3} \times 1.12 = 0.373$$

$$b_1 = 2[\text{Mean values of } \; f(x)\sin x] = \frac{1}{3} \times 3.013 = 1.004.$$

$\therefore$ The F.S is given by $f(x) = 0.75 + 0.37\cos\theta + 1.005\sin\theta$.