Solved Numericals.

1. Find the impulse and step response of the following system:

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n)$$

Soln:-

Given

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n).$$

Taking z-transforms on both sides, we get

$$Y(z) - \frac{3}{4}\left[z^{-1}Y(z) + y(-1)\right] + \frac{1}{8}\left[z^{-2}Y(z) + z^{-1}y(-1) + y(-2)\right] = X(z)$$

On substituting initial conditions, we have

$y(-1) = y(-2) = 0$ that yields

$$Y(z) - \frac{3}{4}z^{-1}Y(z) + \frac{1}{8}z^{-2}Y(z) = X(z)$$

$$\Rightarrow \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$$

Impulse Response.

For $x(n) = \delta(n)$

$X(z) = 1$

$$\Rightarrow Y(z) = \frac{1}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} = \frac{z^2}{z^2 - \frac{3}{4}z + \frac{1}{8}}$$

$$\frac{Y(z)}{z} = \frac{z}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}$$

$$= \frac{A}{z-\frac{1}{2}} + \frac{B}{z-\frac{1}{4}}$$

$$A = \cancel{\left(z-\frac{1}{2}\right)} \left.\frac{z}{\cancel{\left(z-\frac{1}{2}\right)}\left(z-\frac{1}{4}\right)}\right|_{z=\frac{1}{2}}$$

$$= \frac{\left(\frac{1}{2}\right)}{\frac{1}{2}-\frac{1}{4}} = 2$$

$$B = \left(z-\frac{1}{4}\right) \left.\frac{z}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}\right|_{z=\frac{1}{4}}$$

$$= \frac{\frac{1}{4}}{\left(\frac{1}{4}-\frac{1}{2}\right)} = -1$$

$$\Rightarrow \frac{Y(z)}{z} = \frac{2}{z-\frac{1}{2}} - \frac{1}{z-\frac{1}{4}}$$

$$Y(z) = 2\frac{z}{z-\frac{1}{2}} - \frac{z}{z-\frac{1}{4}}$$

On taking inverse z-transform, we get

$$y(n) = 2\left(\frac{1}{2}\right)^n u(n) - \left(\frac{1}{4}\right)^n u(n)$$

Step Response.

For a unit step input,

$$x(n) = u(n)$$

$$X(z) = \frac{z}{z-1}$$

$$\frac{Y(z)}{X(z)} = \frac{1}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}}$$

$$Y(z) = \frac{z}{z-1} \quad \frac{z^2}{z^2-\frac{3}{4}z+\frac{1}{8}}$$

$$\frac{Y(z)}{z} = \frac{z^2}{(z-1)\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}$$

$$= \frac{A}{z-1} + \frac{B}{z-\frac{1}{2}} + \frac{C}{z-\frac{1}{4}}$$

$$A = \cancel{(z-1)} \left.\frac{z^2}{\cancel{(z-1)}\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}\right|_{z=1}$$

$$= \frac{(1)^2}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)} = \frac{1}{\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)} = \frac{8}{3}$$

$$B = \cancel{\left(z-\frac{1}{2}\right)} \left.\frac{z^2}{(z-1)\cancel{\left(z-\frac{1}{2}\right)}\left(z-\frac{1}{4}\right)}\right|_{z=\frac{1}{2}}$$

$$= \frac{\left(\frac{1}{2}\right)^2}{\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-\frac{1}{4}\right)} = \frac{\frac{1}{4}}{\left(\frac{-1}{2}\right)\left(\frac{1}{4}\right)} = -2$$

$$C = \cancel{\left(z-\frac{1}{4}\right)} \left.\frac{z^2}{(z-1)\left(z-\frac{1}{2}\right)\cancel{\left(z-\frac{1}{4}\right)}}\right|_{z=\frac{1}{4}} = \frac{\left(\frac{1}{4}\right)^2}{\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-\frac{1}{2}\right)} = \frac{\frac{1}{16}}{\left(\frac{-3}{4}\right)\left(\frac{-1}{4}\right)} = \frac{1}{3}$$

$$\frac{Y(z)}{z} = \frac{8}{3(z-1)} - \frac{2}{z-\frac{1}{2}} + \frac{1}{3\left(z-\frac{1}{4}\right)}$$

Taking inverse z-transform yields $\underline{\underline{y(n) = \frac{8}{3}u(n) - 2\left(\frac{1}{2}\right)^n u(n) + \frac{1}{3}\left(\frac{1}{4}\right)^n u(n)}}$

② Find the output $y(n)$ of a linear time invariant discrete time system specified by the equation.

$$y(n) - \frac{3}{2}y(n-1) + \frac{1}{2}y(n-2) = 2x(n) + \frac{3}{2}x(n-1)$$

with initial conditions $y(-1) = 0$, $y(-2) = 1$ and input

$$x(n) = \left(\frac{1}{4}\right)^n u(n)$$

Soln:

Given

$$y(n) - \frac{3}{2}y(n-1) + \frac{1}{2}y(n-2) = 2x(n) + \frac{3}{2}x(n-1) \quad \text{— (a)}$$

Taking z-transform on both sides, we get

$$Y(z) - \frac{3}{2}\left[z^{-1}Y(z) + y(-1)\right] + \frac{1}{2}\left[z^{-2}Y(z) + z^{-1}y(-1) + y(-2)\right]$$

$$= 2X(z) + \frac{3}{2}\left[z^{-1}X(z) + x(-1)\right] \quad \text{— (b)}$$

We have,

$y(-1) = 0$; $y(-2) = 1$ and $x(-1) = 0$

On substituting above initial values in eqn — (b), we get

$$Y(z) - \frac{3}{2}z^{-1}Y(z) + \frac{1}{2}\left[z^{-2}Y(z) + 1\right] = X(z)\left[2 + \frac{3}{2}z^{-1}\right]$$

For an input $x(n) = \left(\frac{1}{4}\right)^n u(n)$

We know $X(z) = \dfrac{1}{1 - \frac{1}{4}z^{-1}}$

$$\Rightarrow Y(z)\left[1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}\right] = -\frac{1}{2} + \frac{1}{1 - \frac{1}{4}z^{-1}}\left(2 + \frac{3}{2}z^{-1}\right)$$

$$Y(z) = \frac{-1}{2\left(1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}\right)} + \frac{\left(2 + \frac{3}{2}z^{-1}\right)}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}\right)}$$

$$= \frac{-z^2}{2\left(z^2 - \frac{3}{2}z + \frac{1}{2}\right)} + \frac{z^2\left(2z + \frac{3}{2}\right)}{\left(z - \frac{1}{4}\right)\left(z^2 - \frac{3}{2}z + \frac{1}{2}\right)}$$

$$= \frac{-z^2}{2(z-1)\left(z - \frac{1}{2}\right)} + \frac{z^2\left(2z + \frac{3}{2}\right)}{\left(z - \frac{1}{4}\right)(z-1)\left(z - \frac{1}{2}\right)} = Y_1(z) + Y_2(z)$$

$$Y_1(z) = \frac{-z^2}{2(z-1)\left(z - \frac{1}{2}\right)}$$

$$\frac{Y_1(z)}{z} = \frac{-z}{2(z-1)\left(z - \frac{1}{2}\right)} = \frac{A}{z-1} + \frac{B}{z - \frac{1}{2}} = \frac{-1}{z-1} + \frac{1}{2\left(z - \frac{1}{2}\right)}$$

$$Y_1(z) = \frac{-z}{z-1} + \frac{z}{2\left(z - \frac{1}{2}\right)}$$

$$y_1(n) = -u(n) + 0.5\left(\frac{1}{2}\right)^n u(n)$$

$$Y_2(z) = \frac{z^2\left(2z + \frac{3}{2}\right)}{\left(z - \frac{1}{4}\right)(z-1)\left(z - \frac{1}{2}\right)}$$

$$\frac{Y_2(z)}{z} = \frac{z\left(2z + \frac{3}{2}\right)}{\left(z - \frac{1}{4}\right)(z-1)\left(z - \frac{1}{2}\right)} = \frac{A_1}{z - \frac{1}{4}} + \frac{B_1}{z-1} + \frac{C_1}{z - \frac{1}{2}}$$

$$= \frac{8}{3\left(z - \frac{1}{4}\right)} + \frac{28}{3(z-1)} - \frac{10}{z - \frac{1}{2}}$$

$$Y_2(z) = \frac{8}{3}\frac{z}{z - \frac{1}{4}} + \frac{28}{3}\frac{z}{z-1} - 10\frac{z}{z - \frac{1}{2}}$$

$$A = \left.\cancel{(z-1)}\frac{-z}{2\cancel{(z-1)}\left(z - \frac{1}{2}\right)}\right|_{z=1} = \frac{-1}{2\left(1 - \frac{1}{2}\right)} = -1$$

$$B = \left.\cancel{\left(z - \frac{1}{2}\right)}\frac{-z}{2(z-1)\cancel{\left(z - \frac{1}{2}\right)}}\right|_{z=\frac{1}{2}} = \frac{-\frac{1}{2}}{2\left(\frac{1}{2} - 1\right)} = \frac{1}{2}$$

$$A_1 = \left.\cancel{\left(z - \frac{1}{4}\right)}\frac{z\left(2z + \frac{3}{2}\right)}{\cancel{\left(z - \frac{1}{4}\right)}(z-1)\left(z - \frac{1}{2}\right)}\right|_{z=\frac{1}{4}} = \frac{1}{2} \cdot \frac{16}{3} = \frac{8}{3}$$

$$B_1 = \left.\cancel{(z-1)}\frac{z\left(2z + \frac{3}{2}\right)}{\cancel{(z-1)}\left(z - \frac{1}{4}\right)\left(z - \frac{1}{2}\right)}\right|_{z=1} = \frac{7}{2} \cdot \frac{8}{3} = \frac{28}{3}$$

$$C_1 = \left.\cancel{\left(z - \frac{1}{2}\right)}\frac{z(2z+3)}{\cancel{\left(z - \frac{1}{2}\right)}(z-1)\left(z - \frac{1}{4}\right)}\right|_{z=\frac{1}{2}} = \frac{5}{4}\left(\frac{-8}{1}\right) = -10$$

$$y_2(n) = \frac{8}{3}\left(\frac{1}{4}\right)^n u(n) + \frac{28}{3}u(n) - 10\left(\frac{1}{2}\right)^n u(n)$$

$$y(n) = \frac{25}{3}u(n) + \frac{8}{3}\left(\frac{1}{4}\right)^n u(n) - \frac{19}{2}\left(\frac{1}{2}\right)^n u(n)$$

$y_1(n) + y_2(n)$