MATHS ASSIGNMENT - 1

Q1.

$$A = \begin{bmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{bmatrix}$$

characteristic eqn. of A: $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = 3+(-2)+3 = 4$

S_2: minor of $3 = \begin{vmatrix} -2 & 4 \\ -1 & 3 \end{vmatrix} = -6-(-4) = -2$

minor of $-2 = \begin{vmatrix} 3 & 4 \\ 1 & 3 \end{vmatrix} = 9-4 = 5$

minor of $3 = \begin{vmatrix} 3 & -4 \\ 1 & -2 \end{vmatrix} = -6-(-4) = -2$

So, $S_2 = -2+5-2 = 1$

$S_3 = |A| = \begin{vmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{vmatrix} \Rightarrow 3\begin{vmatrix} -2 & 4 \\ -1 & 3 \end{vmatrix} - (-4)\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} + 4\begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix}$

$\Rightarrow S_3 = 3(-6+4)+4(3-4)+4(-1+2)$

$= -6-4+4 = -6$

Thus, $\lambda^3 - 4\lambda^2 + \lambda - (-6) = 0$

$\Rightarrow \lambda^3 - 4\lambda^2 + \lambda + 6 = 0$

$$\begin{array}{r|l}
\lambda+1 & \lambda^3 - 4\lambda^2 + \lambda + 6 \quad (\lambda^2 - 5\lambda + 6 \\
 & \underset{(-)}{\lambda^3} \underset{(-)}{+} \lambda^2 \\ \hline
 & -5\lambda^2 - \lambda + 6 \\
 & \underset{(+)}{-5\lambda^2} \underset{(+)}{-} 5\lambda \\ \hline
 & 6\lambda + 6 \\
 & \underset{(-)}{6\lambda} \underset{(-)}{+} 6 \\ \hline
 & 0
\end{array}$$

$\Rightarrow (\lambda+1)(\lambda^2-5\lambda+6)=0$

$\Rightarrow (\lambda+1)[\lambda^2-3\lambda-2\lambda+6]=0 \Rightarrow (\lambda+1)[\lambda(\lambda-3)-2(\lambda-3)]=0$

$\Rightarrow (\lambda+1)(\lambda-2)(\lambda-3)=0$

So, $\boxed{\lambda=-1,2,3}$ [Eigenvalues] //

Now, Eigenvector of A : $[A-\lambda I]X=0$

$$\Rightarrow \left[\begin{vmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} (3-\lambda) & -4 & 4 \\ 1 & (-2-\lambda) & 4 \\ 1 & -1 & (3-\lambda) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow (3-\lambda)x_1 - 4x_2 + 4x_3 = 0$$
$$x_1 + (-2-\lambda)x_2 + 4x_3 = 0$$
$$x_1 - x_2 + (3-\lambda)x_3 = 0$$

Since, it is a symmetric matrix

Case 1: $4x_1 - 4x_2 + 4x_3 = 0$ — ①

$x_1 - x_2 + 4x_3 = 0$ — ②

$x_1 - x_2 + 4x_3 = 0$ — ③

from ① & ②,
$$\frac{x_1}{\begin{vmatrix} -4 & 4 \\ -1 & 4 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 4 & 4 \\ 4 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 4 & -4 \\ 1 & -1 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{-12} = \frac{x_2}{-12} = \frac{x_3}{0}$$

$$x_1 = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} //$$

Case 2: $\lambda = 2,$

$$x_1 - 4x_2 + 4x_3 = 0 \quad -(1)$$
$$x_1 - 4x_2 + 4x_3 = 0 \quad -(2)$$
$$x_1 - x_2 + x_3 = 0 \quad -(3)$$

from (2) & (3), $\dfrac{x_1}{\begin{vmatrix} -4 & 4 \\ -1 & 1 \end{vmatrix}} = \dfrac{x_2}{\begin{vmatrix} 4 & 1 \\ 1 & 1 \end{vmatrix}} = \dfrac{x_3}{\begin{vmatrix} 1 & -4 \\ 1 & -1 \end{vmatrix}}$

$$\Rightarrow \quad \frac{x_1}{0} = \frac{x_2}{3} = \frac{x_3}{3}$$

$$\Rightarrow \quad x_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Case 3: $\lambda = 3,$

$$0x_1 - 4x_2 + 4x_3 = 0 \quad -(1)$$
$$x_1 - 5x_2 + 4x_3 = 0 \quad -(2)$$
$$x_1 - x_2 + 0x_3 = 0 \quad -(3)$$

from (1) & (2), $\dfrac{x_1}{\begin{vmatrix} -4 & 4 \\ -5 & 4 \end{vmatrix}} = \dfrac{x_2}{\begin{vmatrix} 4 & 0 \\ 4 & 1 \end{vmatrix}} = \dfrac{x_3}{\begin{vmatrix} 0 & -4 \\ 1 & -5 \end{vmatrix}}$

$$\Rightarrow \quad \frac{x_1}{4} = \frac{x_2}{4} = \frac{x_3}{4}$$

$$\Rightarrow \quad x_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Q2. $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Characteristic Eqtn of A: $|A - \lambda I| = 0$

$\therefore \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = 2+2+2 = 6$

S_2: minor of 2 $(a_{11}) = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4-1 = 3$

minor of 2 $(a_{22}) = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4-1 = 3$

minor of 2 $(a_{33}) = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4-1 = 3$

So, $S_2 = 3+3+3 = 9$

$S_3 = |A| = \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix}$

$\Rightarrow 2\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix}$

$\Rightarrow 2(4-1) + 1(-2+1) + 1(1-2) \Rightarrow 6 - 1 - 1 \Rightarrow 4$

$\therefore \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

<u>To verify</u>: $\lambda = A$ in eqtn;

<u>LHS</u>:

$A^3 - 6A^2 + 9\lambda - 4I = 0$

$$A^3 = \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix} \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix} \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{vmatrix} \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{vmatrix}$$

$$6A^2 = 6 \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix} \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix} \Rightarrow 6 \begin{vmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{vmatrix}$$

$$9A = 9 \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{vmatrix}$$

$$4I = \begin{vmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{vmatrix}$$

So, $\begin{vmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{vmatrix} - \begin{vmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{vmatrix} + \begin{vmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{vmatrix} - \begin{vmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{vmatrix}$

$$\Rightarrow \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} = RHS$$

Hence, verified.

Now,

A^{-1} using Cayley Hamilton Theoram:

Put $\lambda' = A$ in eqtn: $A^3 - 6A^2 + 9A - 4I = 0$

$$\Rightarrow A^3 - 6A^2 + 9A - 4I = 0$$

Multiplying A^{-1} on both sides,

$$A^{-1} \cdot A^3 - 6A^{-1} \cdot A^2 + 9A^{-1} \cdot A - 4A^{-1} \cdot I = 0$$

$$\Rightarrow IA^2 - 6IA + 9I - 4A^{-1} = 0$$

$$\Rightarrow A^2 - 6A + 9I - 4A^{-1} = 0$$

$$\Rightarrow 4A^{-1} = A^2 - 6A + 9I$$

$$\Rightarrow A^{-1} = \frac{1}{4}(A^2 - 5A + 9I)$$

$$\Rightarrow A^{-1} = \frac{1}{4}\left[\begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix}\begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix} - 6\begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix} + 9\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}\right]$$

$$\Rightarrow \frac{1}{4}\left[\begin{vmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{vmatrix} - \begin{vmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ -6 & -6 & 12 \end{vmatrix} + \begin{vmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{vmatrix}\right]$$

$$\Rightarrow A^{-1} = \frac{1}{4}\begin{vmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{vmatrix}$$

//

Now, A^4 using Cayley Hamilton Theoram

Since, $A^3 - 6A^2 + 9A - 4I = 0$

$$\Rightarrow A^3 = 6A^2 - 9A + 4I \quad - (1)$$

$$A^4 = A^3 . A$$

using (1), $\Rightarrow A^4 = A(6A^2 - 9A + 4I)$

$$= 6A^3 - 9A^2 + 4A = 6(6A^2 - 9A + 4I) - 9A^2 + 4A$$

$$= 36A^2 - 54A + 24I - 9A^2 + 4A$$

$$= 27A^2 - 50A + 24I$$

$$\Rightarrow 27\left[\begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix}\begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix}\right] - 50\begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix} + 24\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\Rightarrow A^4 = \begin{bmatrix} 86 & -85 & 85 \\ -85 & 86 & -85 \\ 85 & -85 & 86 \end{bmatrix} //$$

Q3. $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

Characteristic Eqtn: $|A - \lambda I) = 0$

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0$$

$S_1 = 3$

S_2: minor of $(a_{11}) = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = 1 - 1 = 0$

minor of $(a_{22}) = \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} = 1 - 3 = -2$

minor of $(a_{33}) = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1 - 0 = 1$

So, $S_2 = 0 - 2 + 1 = -1$

$$S_3 = |A| = \begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} \Rightarrow 1\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} - 0 + 3\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}$$

$$\Rightarrow \quad 1(1-1) + 3(-2-1) \Rightarrow -9$$

$$\therefore \lambda^3 - 3\lambda^2 + (-1)\lambda - (-9) = 0$$

$$\Rightarrow \lambda^3 - 3\lambda^2 - \lambda + 9 = 0$$

using Cayley Hamilton Theoram,

Put $\lambda = A$

$$\Rightarrow A^3 - 3A^2 - A + 9I = 0$$

$$\Rightarrow A^3 = 3A^2 + A - 9I \quad - (1)$$

using (1),

$$A^4 = A \cdot A^3 = A(3A^2 + A - 9I) = 3A^3 + A^2 - 9A$$

$$\Rightarrow 3(3A^2 + A - 9I) + A^2 - 9A$$

$$\Rightarrow 9A^2 + 3A - 27I + A^2 - 9A$$

$$\Rightarrow 10A^2 - 6A - 27I \quad - (2)$$

$$A^5 = A \cdot A^4$$

using (2), $\Rightarrow A^5 = A(10A^2 - 6A - 27I)$

$$\Rightarrow 10A^3 - 6A^2 - 27A.$$

using (1), $\Rightarrow A^5 = 10(3A^2 + A - 9I) - 6A^2 - 27A$

$$\Rightarrow 30A^2 + 10A - 90I - 6A^2 - 27A$$

$$\Rightarrow 24A^2 - 17A - 90I \quad - (3)$$

using (3),

$$A^6 = A(24A^2 - 17A - 90I) \Rightarrow 24A^3 - 17A^2 - 90A$$

using (1),

$$A^6 = 24(3A^2 + A - 9I) - 17A^2 - 90A \Rightarrow 72A^2 + 24A - 216I - 17A^2 - 90A$$

$$\Rightarrow 55A^2 - 66A - 216I \quad - (4)$$

Now, $A^6 - 5A^5 + 8A^4 - 2A^3 - 9A^2 + 31A + 36I$

using ①, ②, ③ & ④,

$\Rightarrow 55A^2 - 66A - 216I - 5(24A^2 - 17A - 90I) + 8(10A^2 - 6A - 27I)$
$\quad - 2(3A^2 + A - 9I) - 9A^2 + 31A + 36I$

$\Rightarrow 55A^2 - 66A - 216I - 120A^2 + 85A + 450I + 80A^2 - 48A$
$\quad - 216I - 6A^2 - 2A + 18I - 9A^2 + 31A + 36I$

$\Rightarrow 0A^2 + 0A + 72I$

$$\Rightarrow \begin{bmatrix} 72 & 0 & 0 \\ 0 & 72 & 0 \\ 0 & 0 & 72 \end{bmatrix} //$$

Q4. $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

characteristic Eqtn: $|A - \lambda I| = 0$

$\therefore \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = 2+2+2 = 6$

$S_2 =$ minor of $2\ (a_{11}) = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4-1 = 3$

minor of $2\ (a_{22}) = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4-1 = 3$

minor of $2\ (a_{33}) = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4-1 = 3$

So, $S_2 = 3+3+3 = 9$

$S_3 = |A| = \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix} \Rightarrow 2\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix}$

$\Rightarrow 6 - 1 - 1 \Rightarrow 4.$

$\therefore \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

$$
\begin{array}{r|l}
(\lambda - 1) & \lambda^3 - 6\lambda^2 + 9\lambda - 4 \quad (\lambda^2 - 5\lambda + 4 \\
 & \cancel{\lambda^3} - \lambda^2 \\
 & (-) \quad (+) \\
\hline
 & -5\lambda^2 + 9\lambda - 4 \\
 & \cancel{-5\lambda^2} + 5\lambda \\
 & (+) \quad (-) \\
\hline
 & 4\lambda - 4 \\
 & \cancel{4\lambda - 4} \\
 & (-) \quad (+) \\
\hline
 & 0
\end{array}
$$

So, $(\lambda - 1)[\lambda^2 - 5\lambda + 4] = 0$

$\Rightarrow (\lambda - 1)(\lambda - 1)(\lambda - 4) = 0$

Thus, $\boxed{\lambda = 1, 1, 4}$ //

Eigen vectors of A are :

$$A^T = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Since, $A = A^T$
$\Rightarrow$ symmetric matrix

Now, $[A - \lambda I][X] = 0$

$$\Rightarrow \left[\begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} (2-\lambda) & -1 & 1 \\ -1 & (2-\lambda) & -1 \\ 1 & -1 & (2-\lambda) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So, $(2-\lambda)x_1 - x_2 + x_3 = 0$
$-x_1 + (2-\lambda)x_2 - x_3 = 0$
$x_1 - x_2 + (2-\lambda)x_3 = 0$

Case 1: $\lambda = 4$;

$\rightarrow$ Eqtns:

$-2x_1 - x_2 + x_3 = 0$ — ①
$-x_1 - 2x_2 - x_3 = 0$ — ②
$x_1 - x_2 - 2x_3 = 0$ — ③

from ① & ②,

$$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ -2 & -1 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 1 & -2 \\ -1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & -1 \\ -1 & 2 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{3} = \frac{x_2}{-3} = \frac{x_3}{3}$$

$$\Rightarrow x_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Case 2: $\lambda = 1$,

$x_1 - x_2 + x_3 = 0$

let $x_3 = 0 \Rightarrow x_1 - x_2 = 0$
$\Rightarrow x_1 = x_2$

let $x_2 = 1$

$$\Rightarrow x_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Case 3: $\lambda = 1$,

$$x_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

So, $x_1^T x_3$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \Rightarrow a - b + c = 0 \quad \text{— ①}$$

$\therefore x_2^T x_3$

$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \Rightarrow a+b=0 \quad -②$

from ① & ②,

$$\frac{a}{\begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix}} = \frac{b}{\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}} = \frac{c}{\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}}$$

$$\Rightarrow \frac{a}{-1} = \frac{b}{1} = \frac{c}{2}$$

$$\therefore x_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$ //

Now for P;

① $x_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, eigen vectors are $\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

② $x_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, eigenvectors are $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0$

③ $x_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$, eigenvectors are $\frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$

So, $P = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}$

Now, diagonal matrix: $D = P^T \cdot A \cdot P$

$$\Rightarrow \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4/\sqrt{3} & -4/\sqrt{3} & 4/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}$$

$$\Rightarrow D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} //$$

Q5. $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Characteristic eqn for a matrix: $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -2 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (6-\lambda)\left[(3-\lambda)(3-\lambda)-1\right] + 2\left[-2(3-\lambda)+2\right] + 2\left[2-2(3-\lambda)\right] = 0$$

$$\Rightarrow (6-\lambda)\left[9-3\lambda-3\lambda+\lambda^2-1\right] + 2(-6+2\lambda+2) + 2(2-6+2\lambda) = 0$$

$$\Rightarrow (6-\lambda)\left[\lambda^2-6\lambda+8\right] + 2(2\lambda-4) + 2(2\lambda-4) = 0$$

$$\Rightarrow 6\lambda^2 - 36\lambda + 48 + \lambda^3 + 6\lambda^2 - 16 = 0$$

$$\Rightarrow -\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0$$

$$\Rightarrow \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0 \quad \text{— (1)}$$

Now,

Putting 2 as λ in (1),

$$\Rightarrow 8 - 48 + 72 - 32 = 0$$

$$\Rightarrow 0 = 0$$

i.e. $(\lambda - 2) = 0$

$\therefore$

$$
\begin{array}{r l}
(\lambda-2) \overline{\smash{\big)}\, \lambda^3-12\lambda^2+36\lambda-32} & (\lambda^2-10\lambda+16 \\
\underset{(-)}{\cancel{\lambda^3}} \underset{(+)}{-} 2\lambda^2 & \\
\hline
-\cancel{10\lambda^2}+36\lambda-32 & \\
\underset{(+)}{-}\cancel{10\lambda^2} \underset{(-)}{+} 20\lambda & \\
\hline
16\lambda-\cancel{32} & \\
16\cancel{\lambda}-32 & \\
\hline
0 &
\end{array}
$$

$\Rightarrow \lambda^2-10\lambda+16=0$

$\Rightarrow \lambda^2-8\lambda-2\lambda+16=0$

$\Rightarrow \lambda(\lambda-8)-2(\lambda-8)=0$

$\Rightarrow (\lambda-8)(\lambda-2)=0$

$\therefore \lambda=8,2$

eigenvalues $= 8, 2, 2$

Eigenvectors are: $[A-\lambda I]X=0$

If $\lambda=8$,

$$\Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore -x_1-x_2+x_3=0 \quad -(1)$

$2x_1+5x_2+x_3=0 \quad -(2)$

$2x_1-x_2-5x_3=0 \quad -(3)$

from (1) & (2),

$$\frac{x_1}{-1-5} = \frac{x_2}{2+1} = \frac{x_3}{-5+2}$$

$$\Rightarrow \frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1} \quad \Rightarrow \quad x_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$ //

If $\lambda = 2$,

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 4x_1 - 2x_2 + 2x_3 = 0$

$2x_1 - x_2 + x_3 = 0$

let $x_3 = 0$

$2x_1 - x_2 = 0$

$2x_1 = x_2$

$\Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{0} \Rightarrow x_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

let $x_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ which is orthogonal to x_1 & x_2

that is $x_1^T x_3 = 0$ & $x_2^T x_3 = 0$

$\Rightarrow \begin{bmatrix} 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$, $\begin{bmatrix} 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$

$\Rightarrow 2a - b + c = 0$ — (1) & $a + 2b = 0$ — (2)

from (1) & (2),

$\frac{a}{-2} = \frac{b}{1} = \frac{c}{5} \Rightarrow x_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$

$\therefore$ eigenvectors are $= \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

$$N = \begin{bmatrix} 2/\sqrt{6} & 1/\sqrt{5} & -2/\sqrt{30} \\ -1/\sqrt{6} & 2/\sqrt{5} & 1/\sqrt{30} \\ 1/\sqrt{6} & 0/\sqrt{5} & 5/\sqrt{30} \end{bmatrix}$$

$\therefore D = NAN^T$

$$\Rightarrow \begin{bmatrix} 2/\sqrt{6} & 1/\sqrt{5} & -2/\sqrt{30} \\ -1/\sqrt{6} & 2/\sqrt{5} & 1/\sqrt{30} \\ 1/\sqrt{6} & 0/\sqrt{5} & 5/\sqrt{30} \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{5} & 2/\sqrt{5} & 0/\sqrt{5} \\ -2/\sqrt{30} & 1/\sqrt{30} & 5/\sqrt{30} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

let $X = Ny$ be the orthogonal transformation.

Quadratic form $\rightarrow$ $A = X^TAX = (NY)^T A (NY)$

$\Rightarrow Y^TN^TANY$

$A = Y^TDY$ where $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$\therefore A = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$\Rightarrow 8y_1^2 + 2y_2^2 + 2y_3^2$ (conical form).

Hence, rank = 3

index = 3

signature = 3

Nature = positive definite.