Transforms and BVP - Unit III
(1D Wave and Heat Equations)

Classification of Second-Order PDEs

Classification of PDEs is an important concept because the general theory and methods of solution usually apply only to a given class of equations. Let us discuss the classification of PDEs involving two independent variables.

Classification with two independent variables

Consider the following general second order linear PDE in two independent variables:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu + G = 0, \qquad (1)$$

where A, B, C, D, E, F and G are functions of the independent variables x and y. The equation (1) may be written in the form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + f(x, y, u_x, u_y, u) = 0, \qquad (2)$$

where

$$u_x = \frac{\partial u}{\partial x}, u_y = \frac{\partial u}{\partial y}, u_{xx} = \frac{\partial^2 u}{\partial x^2}, u_{xy} = \frac{\partial^2 u}{\partial x \partial y}, u_{yy} = \frac{\partial^2 u}{\partial y^2}$$

Assume that A, B and C are continuous functions of x and y possessing continuous partial derivatives of as high order as necessary. The classification of PDE is motivated by the classification of second order algebraic equations in two-variables

$$ax^2 + bxy + cy^2 + dx + ey + f = 0. \tag{3}$$

We know that the nature of the curves will be decided by the principal part $ax^2 + bxy + cy^2$,
i.e., the term containing highest degree. Depending on the sign of the discriminant $b^2 - 4ac$, we classify the curve as follows:

If $b^2 - 4ac > 0$ then the curve traces hyperbola.
If $b^2 - 4ac = 0$ then the curve traces parabola.
If $b^2 - 4ac < 0$ then the curve traces ellipse.

With suitable transformation, we can transform (3) into the following normal form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ (hyperbola)}, \; x^2 = y \text{ (parabola)}, \; \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ (ellipse)}.$$

Linear PDE with constant coefficients.

Let us first consider the following general linear second order PDE in two independent variables x and y with constant coefficients:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0, \tag{4}$$

where the coefficients A, B, C, D, E, F and G are constants. The nature of the equation (4) is determined by the principal part containing highest partial derivatives i.e.,

$$Lu \equiv Au_{xx} + Bu_{xy} + Cu_{yy}. \tag{5}$$

For classification, we attach a symbol to (5) as $P(x, y) = Ax^2 + Bxy + Cy^2$ (as if we have replaced x by $\frac{\partial}{\partial x}$ and y by $\frac{\partial}{\partial y}$). Now depending on the sign of the discriminant $(B^2 - 4AC)$,the classification of (4) is done as follows:

If $B^2 - 4AC > 0$ then Eq.(4) is hyperbolic.
If $B^2 - 4AC = 0$ then Eq.(4) is parabolic.
If $B^2 - 4AC < 0$ then Eq.(4) is elliptic.

Linear PDE with variable coefficients. The above classification of (4) is still valid if the coefficients A, B, C, D, E and F depend on x, y. In this case, the conditions (6), (7) and (8) should be satisfied at each point (x, y) in the region where we want to describe its nature e.g., for elliptic we need to verify

$$B^2(x,y) - 4A(x,y)C < 0,$$

for each (x, y) in the region of interest. Thus, we classify linear PDE with variable coefficients as follows:

If $B^2(x,y) - 4A(x,y)C(x,y) > 0$ at (x, y) then Eq.(4) is hyperbolic.
If $B^2(x,y) - 4A(x,y)C(x,y) = 0$ at (x, y) then Eq.(4) is parabolic.
If $B^2(x,y) - 4A(x,y)C(x,y) < 0$ at (x, y) then Eq.(4) is elliptic.

Note: Eq. (4) is hyperbolic, parabolic, or elliptic depends only on the coefficients of the second derivatives. It has nothing to do with the first-derivative terms, the term in u, or the non-homogeneous term.

Examples

1. $u_{xx} + u_{yy} = 0$ (Laplace equation). Here, $A = 1,\ B = 0,\ C = 1$ and $B^2 - 4AC = -4 < 0$. Therefore, it is an elliptic type.

2. $u_t = u_{xx}$ (Heat equation). Here, $A = -1,\ B = 0,\ C = 0$ and $B^2 - 4AC = 0$. Therefore, it is of parabolic type.

3. $u_{tt} + u_{xx} = 0$ (Wave equation). Here, $A = -1,\ B = 0,\ C = 1$ and $B^2 - 4AC = 4 > 0$.Therefore, it is of hyperbolic type.

4. $u_{xx} + xu_{yy} = 0,\ x \neq 0$ (Tricomi equation) $B^2 - 4AC = -4x$. Given PDE is hyperbolic for $x < 0$ and elliptic for $x > 0$. This example shows that equations with variable coefficients can change form in the different regions of the domain.

5. $y^2u_{xx} - 2xyu_{xy} + x^2u_{yy} + 2u_x - 3u = 0$. Here $A = y^2$, $B = -2xy$ and $C = x^2$. So, $B^2 - 4AC = 0$. Hence the given PDE is a parabolic type PDE.

6. $y^2u_{xx} + u_{yy} + u_x^2 + u_y^2 + 7 = 0$.
Here $A = y^2$, $B = 0$ and $C = 1$. So, $B^2 - 4AC = -4y^2$. Given PDE is parabolic for $y = 0$ and elliptic for $y \neq 0$.

7. $(x+1)z_{xx} + \sqrt{2}(x+y+1)z_{xy} + (y+1)z_{yy} + yz_x - xz_y = 0$.
Here $A = x+1$, $B = \sqrt{2}(x+y+1)$ and $C = y+1$.
So, $B^2 - 4AC = 2(x^2+y^2-1)$.
If $x^2+y^2 = 1$ then $B^2 - 4AC = 0$. It is parabolic.
If $x^2+y^2 < 1$ then $B^2 - 4AC < 0$. It is elliptic.
If $x^2+y^2 > 1$ then $B^2 - 4AC > 0$. It is hyperbolic.

8. $z_{xx} + 4z_{xy} + (x^2+4y^2)z_{yy} = \sin(x+y)$.
Here $A = 1$, $B = 4$ and $C = x^2+4y^2$.
So, $B^2 - 4AC = 16 - (4x^2+16y^2)$.
If $x^2+4y^2 = 4$ then $B^2 - 4AC = 0$. It is parabolic.
If $x^2+4y^2 < 4$ then $B^2 - 4AC > 0$. It is hyperbolic.
If $x^2+4y^2 > 4$ then $B^2 - 4AC < 0$. It is elliptic.

VARIABLE SEPARABLE SOLUTIONS OF THE WAVE EQUATION

The one dimensional wave equation is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

Let

$$y(x,t) = X(x).T(t) \tag{1}$$

be a solution of the equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \tag{2}$$

where $X(x)$ is a function of x alone and $T(t)$ is a function of t alone.

Then $\frac{\partial^2 y}{\partial t^2} = XT''$ and $\frac{\partial^2 y}{\partial x^2} = X''T$, where $T'' = \frac{d^2 T}{dt^2}$ and $X'' = \frac{d^2 X}{dx^2}$ and satisfy equation (2)

i.e. $$XT'' = a^2 X''T$$

i.e.

$$\frac{X''}{X} = \frac{T''}{a^2T} \tag{3}$$

The L.H.S. of (3) is a function of x alone and the R.H.S. is a function of t alone. They are equal for all values of the independent variables x and t. This is possible only if each is a constant.

$\therefore$

$$\frac{X''}{X} = \frac{T''}{a^2T} = k$$

where k is a constant.

$\therefore$

$$X'' - kX = 0 \tag{4}$$

and

$$T'' - ka^2T = 0 \tag{5}$$

The nature of the solutions of (4) and (5) depends on the nature of values of k. Hence the following three cases arise.

Case 1

k is positive. Let $k = p^2$

Then equations (4) and (5) become

$$(D^2 - p^2)X = 0$$

and

$$({D'}^2 - p^2a^2)T = 0$$

where

$$D \equiv \frac{d}{dx} \text{ and } D' \equiv \frac{d}{dt}$$

The solutions of these equations are

$$X = Ae^{px} + Be^{-px}$$

and

$$T = Ce^{pat} + De^{-pat}$$

Case 2

k is negative. Let $k = -p^2$

Then equations (4) and (5) become

$$(D^2 + p^2)X = 0$$

and

$$({D'}^2 + p^2 a^2)T = 0$$

The solutions of these equations are $X = A\cos px + B\sin px$ and $T = C\cos pat + D\sin pat$.

Case 3

$k = 0$. Then equations (4) and (5) become

$$\frac{d^2X}{dx^2} = 0$$

and

$$\frac{d^2T}{dt^2} = 0$$

The solutions of these equations (found by integrating them) are $X = Ax + B$ and $T = Ct + D$. Since $y(x,t) = X \cdot T$ is the solution of the wave equation, the three mathematically possible solutions of the wave equation are

$$y(x,t) = (Ae^{px} + Be^{-px})(Ce^{pat} + De^{-pat}) \tag{6}$$

$$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \tag{7}$$

and

$$y(x,t) = (Ax + B)(Ct + D) \tag{8}$$

Example 1

A uniform string is stretched and fastened to two points l apart. Motion is started by displacing the string into the form of the curve (i) $y = k\sin^3(\frac{\pi x}{l})$ and (ii) $y = kx(l-x)$ and then releasing it from this position at time $t = 0$. Find the displacement of the point of the string at a distance x from one end at time t:

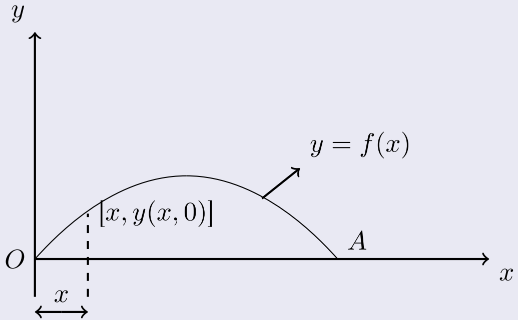

Fig.1

The displacement $y(x,t)$ of the point of the string at a distance x from the left end 0 at time t is given by the equation (Fig.1)

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \tag{1}$$

since the ends of the string $x = 0$ and $x = l$ are fixed, they do not undergo any displacement at any time.
Hence

$$y(0,t) = 0, \quad \text{for} \quad t \geq 0, \tag{2}$$

and

$$y(l,t) = 0, \quad \text{for} \quad t \geq 0. \tag{3}$$

since the string is released from rest initially, that is, at $t = 0$, the initial velocity of every point of the string in the y-direction (the direction of motion of the string) is zero.
Hence

$$\frac{\partial y}{\partial t}(x,0) = 0, \quad \text{for} \quad 0 \leq x \leq l, \tag{4}$$

since the string is initially displaced into the form of the curve $y = f(x)$, the coordinates $[x, y(x,0)]$ satisfy the equation $y = f(x)$, where $y(x,0)$ is the initial displacement of the point 'x' in the y-direction.

Hence

$$y(x,0) = f(x), \quad \text{for} \quad 0 \leq x \leq l, \tag{5}$$

where $f(x) = k\sin^3(\frac{\pi x}{l})$ in (i) and $= kx(l-x)$ in (ii). Conditions (2), (3), (4) and (5) are collectively called ***boundary conditions*** of the problem.We have to get the solution of Eq. (1), that satisfies the boundary conditions. Of the three mathematically possible solutions of Eq. (1), the appropriate solution, consistent with the vibration of the string is

$$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \tag{6}$$

where A,B,C,D and p are arbitrary constants that are to be found out by using the boundary conditions.
Using boundary condition (2) in (6), we have

$$A(C\cos pat + D\sin pat) = 0 \quad \text{for all } \ t \geq 0$$

$$\therefore \qquad A = 0$$

Using boundary condition (3) in (6), we have

$$B\sin pl(C\cos pat + D\sin pat) = 0 \quad \text{for all } t \geq 0$$

$$\therefore \qquad B\sin pl = 0$$

$$\therefore \qquad \text{Either } B = 0 \text{ or } \sin pl = 0.$$

If $B = 0$, the solution becomes $y(x,t) = 0$, which is meaningless.

$$\therefore \qquad \sin pl = 0$$

$$\therefore \qquad pl = n\pi$$

$$\text{or} \qquad p = \tfrac{n\pi}{l}, \quad where \quad n = 0, 1, 2, 3, ...\infty$$

Differentiating both the sides of (6) partially with respect to t, we have

$$\frac{\partial y}{\partial t}(x,t) = B\sin px \cdot pa(-C\sin pat + D\cos pat) \tag{7}$$

where

$$p = \tfrac{n\pi}{l}$$

Using boundary condition (4) in (7), we have

$$B \sin px \cdot pa \cdot D = 0, \quad \text{for} \quad 0 \leq x \leq l$$

As $B \neq 0$ and $p \neq 0$, we get $D = 0$.

Using these values of A, p and D in(6), the solution reduces to

$$y(x,t) = BC \sin \tfrac{n\pi x}{l} \cos \tfrac{n\pi at}{l}$$

where $n = 1, 2, 3, ...\infty$
Taking $BC = k$, Eq.(1) has infinitely many solutions given below.

$$y(x,t) = k \sin \tfrac{\pi x}{l} \cos \tfrac{\pi at}{l}$$

$$y(x,t) = k \sin \tfrac{2\pi x}{l} \cos \tfrac{2\pi at}{l}$$

$$y(x,t) = k \sin \tfrac{3\pi x}{l} \cos \tfrac{3\pi at}{l}, \text{ etc.}$$

Since Eq.(1) is linear, a linear combination of the R.H.S. members of all the above solutions is the general solution of Eq.(1). Thus the most general solution of Eq.(1) is

$$y(x,t) = \textstyle\sum_{n=1}^{\infty}(c_n k)\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

$$\text{or} \quad y(x,t) = \sum_{n=1}^{\infty}\lambda_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \tag{8}$$

where λ_n is yet to be found out.

Using boundary conditions (5) in (8) we have

$$\sum_{n=1}^{\infty}\lambda_n \sin\frac{n\pi x}{l} = f(x), \quad for \;\; 0 \leq x \leq l \tag{9}$$

If we can express $f(x)$ in a series comparable with the L.H.S. series of (9), we can get the values of λ_n.

$(i)\ f(x) = k\sin^3\frac{\pi x}{l}$

$$= \tfrac{k}{4}(3\sin\tfrac{\pi x}{l} - \sin\tfrac{3\pi x}{l})$$

Using this form of $f(x)$ in (9) and comparing like terms, we get

$$\lambda_1 = \tfrac{3k}{4}, \lambda_3 = -\tfrac{k}{4}, \lambda_2 = 0 = \lambda_4 = \lambda_5 = ...$$

Using these values in(8), the required solution is

$$y(x,t) = \tfrac{3k}{4}\sin\tfrac{\pi x}{l}\cos\tfrac{\pi at}{l} - \tfrac{k}{4}\sin\tfrac{3\pi x}{l}\cos\tfrac{3\pi at}{l}$$

$(ii)\ f(x) = kx(l-x).$

If we expand $f(x)$ as Fourier half-range sine series in $(0,l)$, that is, in the form

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

,
it is comparable with the L.H.S. series of (9).

Thus

$\lambda_n = b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$, by Euler's formula

$$= \frac{2k}{l}\int_0^l (lx - x^2)\sin\frac{n\pi x}{l}dx$$

$$= \frac{2k}{l}\left[(lx - x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l-2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$= \frac{4kl^2}{n^3\pi^3}\{1 - (-1)^n\}$$

$$= \begin{cases} 0, & \text{if n is even} \\ \frac{8kl^2}{n^3\pi^3}, & \text{if n is odd} \end{cases}$$

Using this value of λ_n in (8), the required solution is

$$y(x,t) = \frac{8kl^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{l}\cos\frac{(2n-1)\pi at}{l}.$$

Example 2

A tightly stretched string of length $2l$ is fastened at both ends. The midpoint of the string is displaced by a distance b transversely and the string is released from rest in this position. Find the displacement of any point of the string at any subsequent time.

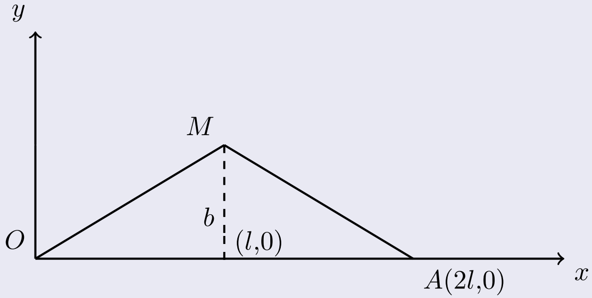

Fig.2

Let us find the analytic equation of the curve which the string assumes initially. The initial curve consists of the two lines OM and MA, where the origin O represents one end of the string and OA is the equilibrium position of the string (Fig.2).

Equation of OM is $y = \frac{b}{l}x$.

Equation of MA is $\frac{y-0}{b-0} = \frac{x-2l}{l-2l}$ or $y = \frac{b}{l}(2l - x)$.

The displacement $y(x,t)$ of any point 'x' of the string at any time 't' is given by

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \tag{1}$$

we have to solve Eq.(1), satisfying the following boundary conditions.

$$y(0,t) = 0, \quad \text{for} \quad t \geq 0, \tag{2}$$

$$y(2l,t) = 0, \quad \text{for} \quad t \geq 0, \tag{3}$$

since the ends of the string are fixed.

$$\frac{\partial y}{\partial t}(x,0) = 0, \quad \text{for} \quad 0 \leq x \leq 2l, \tag{4}$$

since the string is released from rest initially.

$$y(x,0) = \begin{cases} \frac{b}{l}x, & \text{for } 0 \le x \le l \\ \frac{b}{l}(2l - x), & \text{for } l \le x \le 2l \end{cases} \tag{5}$$

since $y(x,0)$ is given by $f(x)$, where $y = f(x)$ is the equation of the initial position curve of the string.

The suitable solution of Eq.(1), consistent with the vibration of the string, is

$$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \tag{6}$$

Using boundary condition (2) in (6), we have

$$A(C\cos pat + D\sin pat) = 0, \quad \text{for all } t \ge 0$$

$$\therefore A = 0$$

Using boundary condition (3) in (6), we have

$$B\sin 2pl(C\cos pat + D\sin pat) = 0, \quad \text{for all } t \ge 0$$

$$\therefore \qquad \text{Either } B = 0 \text{ or } \sin 2pl = 0$$

If we assume that $B = 0$, it results in a trivial solution.

$\therefore \qquad \sin 2pl = 0$

$\therefore \qquad 2pl = n\pi$ or $p = \frac{n\pi}{2l}$, where $n = 0, 1, 2, ...\infty$.

Differentiating both sides of (6) partially with respect to t, we have

$$\frac{\partial y}{\partial t}(x,t) = B \sin px \cdot pa(-C \sin pat + D \cos pat) \tag{7}$$

where $p = \frac{n\pi}{2l}$

Using boundary condition (4) and (7), we have

$$B \sin px \cdot pa \cdot D = 0, \quad \text{for} \quad 0 \leq x \leq 2l$$

As $B \neq 0$ and $p \neq 0$, we get $D = 0$.

Using these values of A,p and D in (6), it reduces to

$$y(x,t) = k \sin \frac{n\pi x}{2l} \cos \frac{n\pi at}{2l} \tag{8}$$

where $n = 1, 2, ...\infty$.

Therefore the most general solution of Eq.(1) [got as a linear combination of the R.H.S.members of solution in(8)] is

$$y(x,t) = \sum_{n=1}^{\infty} \lambda_n \sin\frac{n\pi x}{2l}\cos\frac{n\pi at}{2l} \tag{9}$$

Using boundary condition (5) in (9), we have

$$\sum_{n=1}^{\infty} \lambda_n \sin\frac{n\pi x}{2l} = f(x) \tag{10}$$

where

$$f(x) = \begin{cases} \frac{b}{l}x, & in\,(0,l) \\ \frac{b}{l}(2l-x), & in\,(l,2l) \end{cases}$$

$f(x)$ can be expressed in the form of a series similar to the L.H.S. of (10) by means of Fourier sine series in $(0,2l)$.
Let it be $\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2l}$.

Thus $\lambda_n = b_n = \dfrac{2}{2l}\displaystyle\int_0^{2l} f(x)\sin\frac{n\pi x}{2l}dx$

$$= \frac{1}{l}[\int_0^l \frac{b}{l}x\sin\frac{n\pi x}{2l}dx + \int_l^{2l}\frac{b}{l}(2l-x)\sin\frac{n\pi x}{2l}dx]$$

$$= \frac{b}{l^2}[\{x(\frac{-\cos\frac{n\pi x}{2l}}{\frac{n\pi}{2l}}) - (\frac{-\sin\frac{n\pi x}{2l}}{\frac{n^2\pi^2}{4l^2}})\}_0^l$$

$$+ \{(2l-x)(\frac{-\cos\frac{n\pi x}{2l}}{\frac{n\pi}{2l}}) - (-1)(\frac{-\sin\frac{n\pi x}{2l}}{\frac{n^2\pi^2}{4l^2}})\}_l^{2l}]$$

$$= \frac{b}{l^2}[\{-\frac{2l^2}{n\pi}\cos\frac{n\pi}{2} + \frac{4l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\} + \{\frac{2l^2}{n\pi}\cos\frac{n\pi}{2} + \frac{4l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\}]$$

$$= \frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Using this value of λ_n in (9), the required solution is

$$y(x,t) = \frac{8b}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{2l}\cos\frac{n\pi at}{2l} \qquad (11)$$

Example 3

A taut string of length $2l$, fastened at both ends, is disturbed from its position of equilibrium by imparting to each of its points an initial velocity of magnitude $k(2lx - x^2)$. Find the displacement function $y(x,t)$.

The displacement $y(x,t)$ of any point 'x' of the string at any time 't' is given by

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \tag{1}$$

We have to solve Eq. (1), satisfying the following boundary conditions.

$$y(0,t) = 0, \quad \text{for} \quad t \geq 0 \tag{2}$$

$$y(2l,t) = 0, \quad \text{for} \quad t \geq 0 \tag{3}$$

$$y(x,0) = 0, \quad \text{for} \quad 0 \leq x \leq 2l \tag{4}$$

$$\frac{\partial y}{\partial t}(x,0) = k(2lx - x^2), \quad \text{for} \quad 0 \leq x \leq 2l \tag{5}$$

The appropriate solution of Eq.1, consistent with the vibration of the string, is

$$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \tag{6}$$

Using the boundary condition (2) in (6), we have

$$A(C\cos pat + D\sin pat) = 0, \quad \text{for all} \quad t \geq 0$$

$$\therefore \qquad A = 0$$

Using boundary condition (3) in (6), we have

$$B\sin 2lp(C\cos pat + D\sin pat) = 0, \quad \text{for all} \quad t \geq 0.$$

$$\therefore \qquad \text{Either } B = 0 \text{ or } \sin 2lp = 0.$$

If we assume that $B = 0$, it leads to a trivial solution.

$$\therefore \qquad \sin 2lp = 0$$

$$\therefore \qquad 2pl = n\pi \text{ or } p = \frac{n\pi}{2l}, \quad \text{where} \quad n = 0, 1, 2, ...\infty.$$

Using boundary condition (4) in (6), we have

$$B \sin px \cdot C = 0, \quad \text{for} \quad 0 \leq x \leq 2l, \quad \text{where} \quad p = \frac{n\pi}{2l}$$

As $B \neq 0$, we get $C = 0$.
Using these values of A, p and C in (6), it reduces to

$$y(x,t) = k \sin \frac{n\pi x}{2l} \sin \frac{n\pi at}{2l} \tag{7}$$

where $n = 0, 1, 2, ...\infty$.

Therefore the most general solution of Eq. (1) is

$$y(x,t) = \sum_{n=1}^{\infty} \lambda_n \sin \frac{n\pi x}{2l} \sin \frac{n\pi at}{2l} \tag{8}$$

Differentiating both sides of (8) partially with respect to t, we have

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty} \left(\frac{n\pi a}{2l} \cdot \lambda_n \right) \sin \frac{n\pi x}{2l} \cos \frac{n\pi at}{2l} \tag{9}$$

Using boundary condition (5) in (9), we have

$$\sum_{n=1}^{\infty}\left(\frac{n\pi a}{2l}\cdot\lambda_n\right)\sin\frac{n\pi x}{2l}=k(2lx-x^2),\quad \text{for } 0\le x\le 2l$$
$$=\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{2l}$$

which is Fourier half-range sine series of $k(2lx-x^2)$ in $(0,2l)$.

Comparing the like terms, we get

$$\frac{n\pi a}{2l}\cdot\lambda_n=b_n=\frac{2}{2l}\int_0^{2l}k(2lx-x^2)\sin\frac{n\pi x}{2l}dx$$

$\therefore\quad \lambda_n=$
$$\frac{2k}{n\pi a}\left[(2lx-x^2)\left(\frac{-\cos\frac{n\pi x}{2l}}{\frac{n\pi}{2l}}\right)-(2l-2x)\left(\frac{-\sin\frac{n\pi x}{2l}}{\frac{n^2\pi^2}{4l^2}}\right)+(-2)\left(\frac{\cos\frac{n\pi x}{2l}}{\frac{n^3\pi^3}{8l^3}}\right)\right]_0^{2l}$$

$$=\frac{32kl^3}{n^4\pi^4a}\{1-(-1)^n\}$$

$$=\begin{cases}0, & \text{if n is even}\\ \frac{64kl^3}{n^4\pi^4a}, & \text{if n is odd}\end{cases}$$

Using this values of λ_n in (8), the required solution is

$$y(x,t)=\frac{64kl^3}{\pi^4a}\sum_{n=0}^{\infty}\frac{1}{(2n-1)^4}\sin\frac{(2n-1)\pi x}{2l}\sin\frac{(2n-1)\pi at}{2l}.$$

Example 4

Solve the problem of the vibrating string for the following boundary conditions.

$(i) \quad y(0,t) = 0, \quad (ii) \quad y(l,t) = 0,$

$(iii) \quad \frac{\partial y}{\partial t}(x,0) = v_0 \sin \frac{\pi x}{l}$ and $(iv) \quad y(x,0) = y_0 \sin \frac{2\pi x}{l}$.

The displacement $y(x,t)$ of any point 'x' of the string at any time 't' is given by

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}. \tag{1}$$

We have to solve Eq.(1), satisfying the given boundary conditions.

The proper solution of Eq.(1), consistent with the vibration of the string is

$$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \tag{2}$$

Using the boundary condition (i) in (2), we have

$$A(C\cos pat + D\sin pat) = 0, \quad \text{for all} \quad t \geq 0$$

$\therefore$ $$A = 0$$

Using boundary condition (ii) in (2), we have

$$B\sin pl(C\cos pat + D\sin pat) = 0, \quad \text{for all } t \geq 0.$$

$\therefore$ $$\text{Either } B = 0 \text{ or } \sin pl = 0.$$

If we assume that $B = 0$, we get a trivial solution.

$\therefore$ $$\sin pl = 0$$

$\therefore$ $$pl = n\pi \text{ or } p = \tfrac{n\pi}{l}, \quad \text{where } n = 0, 1, 2, ...\infty.$$

The next two boundary conditions contain non zero values on the R.H.S. Hence we should proceed to use them, only after getting the general solution of Eq.(1). Using the values of A and p in (2), it reduces to

$$y(x,t) = a'\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} + b\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l} \tag{3}$$

where $BC = a'$, $BD = b$ and $n = 0, 1, 2, ...\infty$.

The most general solution of (1) is got by superposing the above infinitely many solutions. That is

$$y(x,t) = \sum_{n=1}^{\infty}[(c_n a') \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l} + (c_n b) \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l}]$$

Taking $c_n a' = \lambda_n$ and $c_n b = \mu_n$, the most general solution of (1) becomes

$$y(x,t) = \sum_{n=1}^{\infty} \lambda_n \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l} + \sum_{n=0}^{\infty} \mu_n \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l} \tag{4}$$

Differentiating both sides of (4) partially with respect to t, we have

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty}(-\frac{n\pi a}{l}\cdot\lambda_n) \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l} + \sum_{n=0}^{\infty}(\frac{n\pi a}{l}\cdot\mu_n) \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l} \tag{5}$$

Using boundary condition (iii) in (5), we have

$$\sum_{n=1}^{\infty}(\frac{n\pi a}{l}\mu_n) \sin\frac{n\pi x}{l} = v_0 \sin\frac{\pi x}{l}$$

Comparing terms, we get

$$\frac{\pi a}{l}\mu_1 = v_0, \quad \frac{n\pi a}{l}\mu_n = 0, \quad \text{for} \quad n = 2, 3, 4, ...\infty$$

$\therefore \qquad \mu_1 = \frac{lv_0}{\pi a}$, and $\mu_2 = 0 = \mu_3 = \mu_4 = ...$

Using boundary condition (iv) in (4), we have

$\sum_{n=1}^{\infty} \lambda_n \sin \frac{n\pi x}{l} = y_0 \sin \frac{2\pi x}{l}$

Comparing the like terms, we get

$\lambda_2 = y_0$ and $\lambda_1 = 0 = \lambda_3 = \lambda_4 = ...$

Using these values of λ_n and μ_n in (4), the required solution is

$y(x, t) = y_0 \sin \frac{2\pi x}{l} \cos \frac{2\pi at}{l} + \frac{lv_0}{\pi a} \sin \frac{\pi x}{l} \sin \frac{\pi at}{l}$.

VARIABLE SEPARABLE SOLUTIONS OF THE HEAT EQUATION

The one dimensional heat flow equation is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

Let

$$u(x,t) = X(x) \cdot T(t) \tag{2}$$

be a solution of Eq.(1), where $X(x)$ is a function of x alone and $T(t)$ is a function of t alone.

Then $\frac{\partial u}{\partial t} = XT'$ and $\frac{\partial^2 u}{\partial x^2} = X''T$, where $T' = \frac{dT}{dt}$ *and* $X'' = \frac{d^2X}{dx^2}$, satisfy Eq.(1)
i.e.

$$XT' = \alpha^2 X''T$$

i.e.

$$\frac{X''}{X} = \frac{T'}{\alpha^2 T} \tag{3}$$

The L.H.S. of (3) is a function of x alone and the R.H.S. is a function of t alone. They are equal for all values of independent variables x and t. This is possible only if each is a constant.
Therefore,

$$\frac{X''}{X} = \frac{T'}{\alpha^2 T} = k, \quad \text{where } k \text{ is a constant.}$$

$\therefore$

$$X'' - kX = 0 \tag{4}$$

and

$$T' - k\alpha^2 T = 0 \tag{5}$$

The nature of the solution of (4) and (5) depends on the nature of the values of k.

Hence we have following three cases:

Case 1: k is positive. Let $k = p^2$.

Then equations (4) and (5) become

$(D^2 - p^2)X = 0$ and $(D' - p^2\alpha^2)T = 0$, where

$$D \equiv \tfrac{d}{dx} \text{ and } D' \equiv \tfrac{d}{dt}.$$

The solutions of these equations are

$$X = C_1 e^{px} + C_2 e^{-px} \text{ and } T = C_3 e^{p^2\alpha^2 t}.$$

Case 2: k is negative. Let $k = -p^2$.

Then equations(4) and (5) become

$(D^2 + p^2)X = 0$ and $(D' + p^2\alpha^2)T = 0$

The solutions of these equations are

$$X = C_1 \cos px + C_2 \sin px \text{ and } T = C_3 e^{-p^2\alpha^2 t}.$$

Case 3: $k = 0$

Then equations (4) and (5) become

$$\frac{d^2X}{dx^2} = 0 \text{ and } \frac{dT}{dt} = 0$$

The solutions of these equations are

$$X = C_1x + C_2 \text{ and } T = C_3.$$

Since $u(x,t) = X \cdot T$ is the solution of Eq.(1), the three mathematically possible solutions of Eq.(1) are

$$u(x,t) = (Ae^{px} + Be^{-px})e^{p^2\alpha^2 t}, \tag{6}$$

$$u(x,t) = (A\cos px + B\sin px)e^{-p^2\alpha^2 t}, \tag{7}$$

and

$$u(x,t) = Ax + B, \tag{8}$$

where C_1C_3 and C_2C_3 have been taken as A and B.

Out of this three mathematically possible solutions derived, we have to choose that solution which is consistent with physical nature of the problem and the given boundary conditions.

(i) As we are dealing with heat condition, $u(x, t)$, representing the temperature at any point at any time t, must decreases. In other words, $u(x, t)$ can not be infinite as $t \to \infty$. Hence solution (7) is the proper solution.

(ii) When the heat flow is under steady-state conditions, the temperature at any point does not vary with time, that is independent with time. Hence the proper solution in steady-state heat flow problems is solution (8).

In problems, we may directly assume that (7) or (8) is the proper solution, according to whether the temperature distribution in the bar is under transient or steady-state conditions.

Example 1

A uniform bar of length l through which heat flows is insulated at its sides.The ends are kept at zero temperature. If the initial temperature at the interior points of the bar is given by $(i)\ k\sin^3\frac{\pi x}{l}$, $(ii)\ k(lx-x^2)$, for $0<x<l$, find the temperature distribution in the bar after time t.

The temperature $u(x,t)$ at a point of the bar, which is at a distance x from one end, at time t is given by the equation

$$\frac{\partial u}{\partial t}=\alpha^2\frac{\partial^2 u}{\partial x^2} \tag{1}$$

Since the ends $x=0$ and $x=l$ are kept at zero temperature, that is, the ends are maintained at zero temperature at all times, we have

$$u(0,t)=0, \quad \text{for } t\geq 0 \tag{2}$$

$$u(l,t)=0, \quad \text{for } t\geq 0 \tag{3}$$

Since the initial temperature at the interior points of the bar is $f(x)$, we have

$$u(x,0) = f(x), \quad \text{for } 0 < x < l \tag{4}$$

where $f(x) = k\sin^3 \frac{\pi x}{l}$ in (i) and $= k(lx - x^2)$ in (ii).
we have to get the solution of Eq.(1) that satisfies the boundary conditions (2), (3) and (4).
Of the three mathematically possible solutions of Eq.(1), the appropriate solution that satisfies the condition $u \neq \infty \;\; as \;\; t \longrightarrow \infty$ is

$$u(x,t) = (A\cos px + B\sin px)e^{-p^2\alpha^2 t} \tag{5}$$

where A, B and p are arbitrary constants that are to be found out by using the boundary conditions.
Using boundary condition(2) in (5), we have

$$\therefore \qquad Ae^{-p^2\alpha^2 t} = 0, \quad \text{for all } t \geq 0$$

$$\therefore \qquad A = 0$$

Using boundary condition (3) in (5), we have

Using boundary condition (3) in (5), we have

$$B \sin pl \, e^{-p^2\alpha^2 t} = 0, \quad \text{for all } t \geq 0$$

$$\therefore \qquad B \sin pl = 0$$

$$\text{i.e.} \qquad \text{either } B = 0 \text{ or } \sin pl = 0$$

If we assume that $B = 0$, the solution becomes $u(x,t) = 0$, which is meaningless.

$$\therefore \qquad \sin pl = 0$$

$$\therefore \qquad pl = n\pi$$

$$\text{or} \qquad p = \tfrac{n\pi}{l}, \text{ where } n = 0, 1, 2, ...\infty.$$

Using these values of A and p in (5), the solution reduces to

$$u(x,t) = B \sin \frac{n\pi x}{l} \cdot e^{-\frac{n^2\pi^2\alpha^2 t}{l^2}} \tag{6}$$

Where $n = 1, 2, ...\infty$.

Superposing the infinitely many solutions contained in Step (6),we get the most general solution of Eq.(1) as

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} e^{-\frac{n^2\pi^2\alpha^2 t}{l^2}} \tag{7}$$

Using the boundary condition(4) in (7), we have

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = f(x), \quad \text{for } 0 < x < l \tag{8}$$

If we can express f(x) in a series comparable with the L.H.S. series of (8), we can get the values of B_n. Since $\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$ is of the Fourier half-range sine series of a function, in most situations we may have to expand $f(x)$ as a Fourier half-range sine series.

(i) $f(x) = k\sin^3(\frac{\pi x}{l})$

$$= \frac{k}{4}(3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l})$$

Using this from of $f(x)$ in (8) and comparing like terms, we get

$$B_1 = \frac{3k}{4}, \quad B_3 = -\frac{k}{4}, \quad B_2 = B_4 = B_5 = ... = 0$$

Using these values in (7), the required solution is

$$u(x,t) = \frac{3k}{4}\sin\frac{\pi x}{l}e^{-\pi^2\alpha^2 t/l^2} - \frac{k}{4}\sin\frac{3\pi x}{l}e^{-9\pi^2\alpha^2 t/l^2}$$

(ii) $f(x) = k(lx - x^2)$ in $0 < x < l$

Let the Fourier half-range sine series of $f(x)$ in $(0,l)$ be

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

Using this form of $f(x)$ in (8) and comparing like terms, we get

$B_n = b_n = \frac{2}{l}\int_0^1 k(lx - x^2)\sin\frac{n\pi x}{l}dx$

$= \frac{2k}{l}[(lx - x^2)(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}) - (l - 2x)(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}) + (-2)(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}})]_0^l$

$= \frac{4kl^2}{n^3\pi^3}\{1 - (-1)^n\}$

$$= \begin{cases} 0, & \text{if n is even} \\ \frac{8kl^2}{n^3\pi^3}, & \text{if n is odd} \end{cases}$$

Using this value of B_n in (7), the required solution is

$$u(x,t) = \frac{8kl^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{l}\cdot e^{\frac{-(2n-1)^2\pi^2\alpha^2 t}{l^2}}$$

Example 2

Find the temperature distribution in a homogeneous bar of length π which is insulated laterally, if the ends are kept at zero temperature and if, initially, the temperature is k at the center of the bar and falls uniformly to zero at its ends.

It is easy to derive the initial condition as

$$u(x,0) = \begin{cases} \frac{2k}{\pi}x, & \text{in } 0 \le x \le \frac{\pi}{2} \\ \frac{2k}{\pi}(\pi - x), & \text{in } \frac{\pi}{2} \le x \le \pi \end{cases}$$

The temperature distribution $u(x,t)$ in the bar is given by

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial t^2}. \tag{1}$$

We have to solve Eq. (1) satisfying the following boundary conditions.

$$u(0,t) = 0, \quad \text{for all } t \ge 0 \tag{2}$$

$$u(\pi,t) = 0, \quad \text{for all } t \ge 0 \tag{3}$$

$$u(x,0)=\begin{cases}\frac{2k}{\pi}x, & \text{in } 0\le x\le \frac{\pi}{2}\\ \frac{2k}{\pi}(\pi-x), & \text{in } \frac{\pi}{2}\le x\le \pi\end{cases} \tag{4}$$

As $u(x,t)$ has to remain finite when $t\to\infty$, the proper solution of Eq.(1) is

$$u(x,t)=(A\cos px+B\sin px)e^{-p^2\alpha^2 t} \tag{5}$$

Using boundary condition (2) in (5), we have

$$A\cdot e^{-p^2\alpha^2 t}=0, \quad \text{for all } t\ge 0$$

$\therefore$ $$A=0$$

Using boundary condition (3) in (5), we have

$$B\sin p\pi\cdot e^{-p^2\alpha^2 t}=0, \quad \text{for all } t\ge 0$$

$\therefore$ $$B=0 \;\text{ or }\; \sin p\pi=0$$

$B=0$ leads to a trivial solution.

$\therefore$ $$\sin p\pi=0$$

$\therefore$ $$p\pi=n\pi \;\text{ or }\; p=n \text{ where } n=0,1,2,...\infty$$

Using these values of A and p in (5), it reduces to

$$u(x,t) = B \sin nx e^{-n^2\alpha^2 t}, \tag{6}$$

where $n = 1, 2, 3, ...\infty$.
Therefore the most general solution of Eq.(1) is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin nx e^{-n^2\alpha^2 t} \tag{7}$$

Using boundary condition (4) in (7), we have

$$\sum_{n=1}^{\infty} B_n \sin nx = f(x) \text{ in } (0,\pi), \text{ where}$$

$$f(x) = \begin{cases} \frac{2k}{\pi} x, & \text{in } 0 \le x \le \frac{\pi}{2} \\ \frac{2k}{\pi}(\pi - x), & \text{in } \frac{\pi}{2} \le x \le \pi \end{cases}$$

If the Fourier half-range sine series of $f(x)$ in $(0,\pi)$ is $\sum_{n=1}^{\infty} b_n \sin nx$, it is comparable with $\sum_{n=1}^{\infty} B_n \sin nx$.

$$B_n = b_n = \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \frac{2k}{\pi} x \sin nx dx + \int_{\frac{\pi}{2}}^{\pi} \frac{2k}{\pi}(\pi - x)\sin nx dx\right]$$

$$= \frac{4k}{\pi^2}\left[\left\{x\left(\frac{-\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right\}_0^{\frac{\pi}{2}} + \left\{(\pi - x)\left(\frac{-\cos nx}{n}\right) + \left(\frac{-\sin nx}{n^2}\right)\right\}_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Using this value of B_n in (7), the required solution is

$$u(x,t) = \frac{8k}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin nx e^{-n^2\alpha^2 t}$$

$$\text{or} \quad u(x,t) = \frac{8k}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin(2n-1)x e^{-(2n-1)^2\alpha^2 t}.$$

Example 3

A rod of length 20cm has its ends A and B kept at 30^oC and 90^oC respectively, until steady-state conditions prevail. If the temperature at each end is then suddenly reduced to $0°C$ and maintained so, find the temperature $u(x,t)$ at a distance x from A at time t.

When steady-state deg conditions prevail, the temperature at any point of the bar does not depend on t, but only on x. Hence when steady-state conditions prevailing the bar, the temperature distribution is given by

$$\frac{d^2u}{dx^2} = 0. \tag{1}$$

$[\because \frac{\partial u}{\partial t} = 0$ and $\frac{\partial^2 u}{\partial x^2}$ becomes $\frac{d^2u}{dx^2}.]$

We have to solve (1) satisfying the following boundary conditions

$$u(0) = 30 \tag{2}$$

and

$$u(20) = 90 \tag{3}$$

Solving Eq.(1),we get

$$u(x) = C_1x + C_2 \tag{4}$$

Using (2) in (4), we get $C_2 = 30$.
Using (3) in (5), we get $C_1 = 3$.
Using these values in (4), the solution of Eq.(1) is

$$u(x) = 3x + 30. \tag{5}$$

That is, as long as the steady-state conditions prevail in the bar, the temperature distribution in it is given by (5).

Once we alter the end temperatures (or the end conditions), the heat flow or the temperature distribution in the bar will not be under steady-state conditions and hence will depend on time also.However the temperature distribution at the interior points on the bar in the steady-state will be the initial temperature distribution in the transient state.

In the transient state, the temperature distribution in the bar is given by

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{6}$$

The corresponding boundary conditions are

$$u(0,t) = 0, \quad \text{for all} \quad t \geq 0 \tag{7}$$

$$u(20,t) = 0, \quad \text{for all} \quad t \geq 0 \tag{8}$$

$$u(x,0) = 3x + 30, \quad \text{for} \quad 0 < x < 20 \tag{9}$$

As $u \neq \infty$ when $t \to \infty$, the proper solution of Eq.(6) is

$$u(x,t) = (A\cos px + B\sin px)e^{-p^2\alpha^2 t} \tag{10}$$

Using boundary condition (7) in (10), we have

$$A \cdot e^{-p^2\alpha^2 t} = 0, \quad \text{for all} \quad t \geq 0$$

$$\therefore \qquad A = 0$$

Using boundary condition (8) in (10), we have

$$B\sin 20p \cdot e^{-p^2\alpha^2 t} = 0, \quad \text{for all} \quad t \geq 0$$

$$\therefore \qquad B = 0 \quad \text{or} \quad \sin 20p = 0$$

$B = 0$ leads to a trivial solution.

$$\therefore \qquad \sin 20p = 0$$

$\therefore \qquad 20p = n\pi \quad \text{or} \quad p = \frac{n\pi}{20}, \text{ where } \quad n = 0, 1, 2, ...\infty$

Using these values of A and p in (10), it reduces to

$$u(x,t) = A \sin \frac{n\pi x}{20} e^{-n^2\pi^2\alpha^2 t/20^2} \tag{11}$$

where $n = 1, 2, 3, ...\infty$
Therefore the most general solution of Eq.(6) is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{20} e^{-n^2\pi^2\alpha^2 t/400} \tag{12}$$

Using boundary condition (9) in (12), we have

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{20} = 3x + 30 \quad \text{in} \quad (0, 20)$$

$$\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20}$$

which is Fourier half-range since series of $(3x + 30)$ in $(0, 20)$.
comparing like terms,

$$B_n = b_n = \frac{2}{20}\int_0^{20} (3x+30)\sin\frac{n\pi x}{20}dx$$

$$= \frac{3}{10}\left[(x+10)\left(\frac{-\cos\frac{n\pi x}{20}}{\frac{n\pi}{20}}\right) - \left(-\frac{\sin\frac{n\pi x}{20}}{\frac{n^2\pi^2}{20^2}}\right)\right]_0^{20}$$

$$= -\frac{6}{n\pi}\{30(-1)^n - 10\} = \frac{60}{n\pi}\{1-3(-1)^n\}$$

Using this value of B_n in (12), the required solution is

$$u(x,t) = \frac{60}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\{1-3(-1)^n\}\sin\frac{n\pi x}{20}e^{-n^2\pi^2\alpha^2 t/400}.$$

Example 4

Solve the one dimensional heat flow equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

satisfying the following boundary conditions

(i) $\frac{\partial u}{\partial x}(0, t) = 0$, for all $t \geq 0$

(ii) $\frac{\partial u}{\partial x}(\pi, t) = 0$, for all $t \geq 0$; and

(iii) $u(x, 0) = \cos^2 x, \quad 0 < x < \pi$

When conditions (i) and (ii) are satisfied, it means that the ends $x = 0$ and $x = \pi$ of the bar are thermally insulated, so that heat cannot flow in or out through these ends.
The appropriate solution of the equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

satisfying the condition that $u \neq \infty$ when $t \to \infty$ is

$$u(x,t) = (A\cos px + B\sin px)e^{-p^2\alpha^2 t} \tag{2}$$

Differentiating (2) partially with respect to x, we have

$$\frac{\partial u}{\partial x}(x,t) = p(-A\sin px + B\cos px)e^{-p^2\alpha^2 t} \tag{3}$$

Using boundary condition (i) in (3), we have

$$p \cdot B \cdot e^{p^2\alpha^2 t} = 0, \quad \text{for all } t \geq 0$$

$$\therefore \qquad B = 0 \; [\because \text{ if } p = 0, u(x,t) = A, \text{ which is meaningless}]$$

When the zero left end temperature condition was used in the proper solution, we got $A = 0$ in all the earlier examples. When the zero left end temperature gradient condition is used, we get $B = 0$.
Using boundary condition (ii) in (3), we have

$$-pA \sin p\pi \cdot e^{-p^2\alpha^2 t} = 0, \quad \text{for all} \ \ t \geq 0$$

$$\therefore \qquad \text{Either } A = 0 \text{ or } \sin p\pi = 0$$

$A = 0$ leads to a trivial solution.

$$\therefore \qquad \sin p\pi = 0$$

$$\therefore \qquad p\pi = n\pi, \text{ where } n = 0, 1, 2, ...\infty$$

Using these values of B and p in (2), it reduces to

$$u(x,t) = A \cos nx \cdot e^{-n^2\alpha^2 t} \tag{4}$$

where $n = 1, 2, ...\infty$.

$n = 0$ gives $u(x,t) = A$, which cannot be omitted. Therefore the most general solution of Eq.(1) is

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos nx e^{-n^2\alpha^2 t} \tag{5}$$

Using the boundary condition (iii) in (5), we have

$$\sum_{n=0}^{\infty} A_n \cos nx = \cos^2 x \quad \text{in} \quad (0,\pi) \tag{6}$$

In general, we have to expand the function in the R.H.S. as a Fourier half-range cosine series in $(0,\pi)$ so that it may be compared with L.H.S. series. In this problem, it is not necessary. We can rewrite $\cos^2 x$ as $\frac{1}{2}(1+\cos 2x)$, so that comparison is possible.
Thus $\sum_{n=0}^{\infty} A_n \cos nx = \frac{1}{2} + \frac{1}{2}\cos 2x.$
Comparing like terms, we have

$$A_0 = \tfrac{1}{2}, \quad A_2 = \tfrac{1}{2}, \quad A_1 = A_3 = A_4 = ... = 0$$

Using these values of A_n's in (5), the required solution is

$$u(x,t) = \tfrac{1}{2} + \tfrac{1}{2}\cos 2x e^{-4\alpha^2 t}.$$

Example 5 (With non-zero boundary conditions)

A bar 10 cm long has originally a temperature of 0^oC throughout its length. At time $t = 0$ sec, the temperature at the end $x = 0$ is raised to 20^oC, while that at the end $x = 10$ is raised to 40^oC. Determine the resulting temperature distribution in the bar.

The temperature distribution $u(x,t)$ in the bar is given by the equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

We have to solve Eq.(1) satisfying the following The corresponding boundary conditions are

$$u(0,t) = 20, \quad \text{for all } t \geq 0 \tag{2}$$

$$u(10,t) = 40, \quad \text{for all } t \geq 0 \tag{3}$$

$$u(x,0) = 0, \quad \text{for } 0 < x < 10 \tag{4}$$

In all the earlier problems, the boundary values in (2) and (3) were zero each and hence we were able to get the values of two of the unknown constants in the proper solution easily. The usual procedure will not give the values of unknown constants in the proper solution in this example, since we have non-zero values in the boundary conditions (2) and (3). Hence we adopt a slightly different procedure.

Let

$$u(x,t) = u_1(x) + u_2(x,t), \tag{5}$$

Using (5) in (1), we get

$$\frac{\partial}{\partial t}(u_1 + u_2) = \alpha^2 \frac{\partial^2}{\partial x^2}(u_1 + u_2),$$

This gives rise to the two equations

$$\frac{d^2 u_1}{dx^2} = 0, \tag{6}$$

and

$$\frac{\partial u_2}{\partial t} = \alpha^2 \frac{\partial^2 u_2}{\partial x^2}, \tag{7}$$

Solving Eq.(6) along with the conditions $u_1(0) = 20$ and $u_1(10) = 40$, we get

$$u_1(x) = 2x + 20, \tag{8}$$

Now we have to solve Eq.(7) along with the boundary and initial conditions given below

$$u_2(0,t) = u(0,t) - u_1(0) = 0, \quad \text{for all } t \geq 0$$

$$u_2(10,t) = u(10,t) - u_1(10) = 0, \quad \text{for all } t \geq 0$$

$$u_2(x,0) = u(x,t) - u_1(x) = -(20x + 20), \quad \text{for } 0 < x < 10$$

Proceeding like previous examples we can find the solution for $u_2(x,t)$ as

$$u_2(x,t) = \frac{40}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \{2(-1)^n - 1\} \sin \frac{n\pi x}{10} e^{-n^2\pi^2\alpha^2 t/100}. \tag{9}$$

From Eqs.(5), (8) and (9) we get the required solution as

$$u(x,t) = (2x + 20) + \frac{40}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \{2(-1)^n - 1\} \sin \frac{n\pi x}{10} e^{-n^2\pi^2\alpha^2 t/100}.$$

Short-Answer questions

Write one dimensional wave equation and its possible solutions

One dimensional wave equation is given by

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}.$$

Its possible solutions are given by

$$y(x,t) = (Ae^{px} + Be^{-px})(Ce^{pat} + De^{-pat}),$$
$$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat),$$
$$y(x,t) = (Ax + B)(Ct + D).$$

where A, B, C and D are arbitrary constants.

In one dimensional wave equation $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$, what does a^2 stands for?

In one dimensional wave equation the value of a^2 is stands for $a^2 = \frac{T}{m}$, where T is the tension and m is mass per unit length.

Write one dimensional heat equation and its possible solutions.

One dimensional wave equation is given by

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}.$$

Its possible solutions are given by

$$\begin{aligned} u(x,t) &= (Ae^{px} + Be^{-px})e^{p^2 a^2 t}, \\ u(x,t) &= (A\cos px + B\sin px)e^{-p^2 a^2 t}, \\ u(x,t) &= Ax + B, \end{aligned}$$

where A and B are arbitrary constants.

In one dimensional heat equation $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$, what does a^2 stands for?

In one dimensional wave equation the value of a^2 is stands for $a^2 = \frac{\kappa}{\rho c}$, where κ is the thermal conductivity, ρ the density of the material and c is specific heat of the material.

What is meant by steady state condition in heat flow?

When steady state conditions prevail in the material, the temperature at any point of the plate does not depend on t, but depends only on x.

The ends of a rod of length 20 cm are maintained at the temperature 10^oC and 20^oC respectively until steady state prevails. Determine the steady state temperature of the rod.

1D heat equation is $u_t = a^2 u_{xx}$. In steady state, $\frac{d^2u}{dx^2} = 0$, that implies $u(x) = Ax + B$.
Applying given conditions, i.e., $u(0) = 10$ and $u(20) = 20$ we can get

$$u(x) = \frac{1}{2}x + 10$$

What is the basic difference between the solutions of the 1D wave and heat equations?

The solution for wave equation is periodic in t, but the solution for heat equation is NOT periodic in t.