18MAB101T- CALCULUS AND LINEAR ALGEBRA

Unit III - Ordinary Differential Equations

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A differential equation is a mathematical equation involving an unknown function and its derivatives.

For example:

1. $$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{-2x}$$
2. $$\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^2 + 2y = \sin 3x$$
3. $$\left(\frac{d^2y}{dx^2}\right)^2 + 2\frac{dy}{dx} + y = 5x$$
4. $$\frac{dy}{dx} + 3y = 5x$$

The **order** of a differential equation is the order of the highest derivative of the unknown function involved in the equation.

The **degree** of a differential equation is the degree of the highest derivative of the unknown function involved in the equation.

The order of the differential equations $(1), (2)$ and (3) is two where as the order of the differential equation (4) is one.

The degree of the differential equations $(1), (2)$ and (4) is one where as the order of the differential equation (3) is two and degree is also two.

LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

An equation of the form

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + ... + a_n y = F(x)$$

where $a_0, a_1, ..., a_n$ are constants, is called a Linear differential equation of order n with constant coefficients.

Let $\frac{d}{dx} = D, \frac{d^2}{dx^2} = D^2$,etc. Then the above equation can be written as

$$\left(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + ... + a_n\right) y = F(x)$$

$$\text{i.e. } [\phi(D)]y = F(x) \text{ ———————— } (1)$$

The general or complete solution of (1) consists of two parts namely

(i) Complementary function (CF) and the (ii) Particular integral (PI)

That is, y=CF+PI

To find the Complementary function:

Form the auxiliary equation by putting $D = m$ in $\phi(D) = 0$. Therefore the auxiliary equation of (1) is $\phi(m) = 0$ which will be a polynomial equation of degree n. By solving this equation we get n roots say $m_1, m_2, ..., m_n$.

Case (i):

If all the roots are real and unequal, i.e. if
$m_1 \neq m_2 \neq ... \neq m_n$, then

$$\text{CF} = c_1 e^{m_1 x} + c_2 e^{m_2 x} + ... + c_n e^{m_n x}$$

Case (ii):

If $m_1 = m_2 = m$ and the remaining be real and unequal, then

$$\text{CF} = (c_1 + c_2 x)e^{mx} + c_3 e^{m_3 x} + ... + c_n e^{m_n x}$$

Case (iii):

If $m_1 = m_2 = m_3 = m$ and the remaining be real and unequal, then

$$\text{CF} = (c_1 + c_2 x + c_3 x^2)e^{mx} + c_4 e^{m_4 x} + ... + c_n e^{m_n x}$$

Case (iv):

If roots are imaginary, i.e. if $m = \alpha \pm i\beta$, then

$$\text{CF} = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

To find the Particular integra1:

Let the given differential equations be $\phi(D)y = F(x)$. If the RHS is zero, ie. if $F(x) = 0$, then there is no particular integral. In this case the complementary function alone constitute the complete solution of the given differential equation.

On the other hand if $F(x) \neq 0$, then we have PI also. The PI is given by

$$\text{PI} = \frac{1}{\phi(D)} F(x)$$

Example 1: Solve $\dfrac{d^2y}{dx^2} - 7\dfrac{dy}{dx} + 12y = 0$

Solution:

Given $\dfrac{d^2y}{dx^2} - 7\dfrac{dy}{dx} + 12y = 0$

(i.e.)$(D^2 - 7D + 12)y = 0$

(i.e.)$\phi(D)y = 0$

To find CF:

Put $D = m$ in $\phi(D) = 0$, the auxiliary equation is $m^2 - 7m + 12 = 0$

$\Rightarrow (m-3)(m-4) = 0 \Rightarrow m = 3, 4$

$\Rightarrow m_1 = 3, m_2 = 4$ are real and unequal.

$\Rightarrow \text{CF} = c_1 e^{m_1 x} + c_2 e^{m_2 x} = c_1 e^{3x} + c_2 e^{4x}$

Since $F(x) = 0$, there is no PI. The complete solution is $y = \text{CF}$

The complete solution is $y = c_1 e^{3x} + c_2 e^{4x}$

Type 1: If $F(x) = e^{ax}$, then PI $= \dfrac{1}{\phi(D)} F(x)$

$$= \frac{1}{\phi(D)} e^{ax}$$

$$[\text{in} \quad \phi(D) \quad \text{replace} \quad D \quad \text{by} \quad a]$$

$$= \frac{1}{\phi(a)} e^{ax}, \text{ provided } \phi(a) \neq 0$$

If $\phi(a) = 0$, then PI $= \dfrac{1}{\phi(D)} e^{ax}$

$$= x.\frac{1}{\phi'(D)} e^{ax}$$

$$\left[\text{in} \quad \phi'(D) \quad \text{replace} \quad D \quad \text{by} \quad a\right]$$

$$= x.\frac{1}{\phi'(a)} e^{ax}, \text{ provided } \phi'(a) \neq 0$$

If $\phi^{'}(a) = 0$, then PI $= x^2.\dfrac{1}{\phi^{''}(D)}e^{ax}$

$$\left[\text{in} \quad \phi^{''}(D) \quad \text{replace} \quad D \quad \text{by} \quad a\right]$$

$$= x^2.\frac{1}{\phi^{''}(a)}e^{ax}, \text{ provided } \phi^{''}(a) \neq 0$$

and so on.

Example 2: Solve $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=e^{-2x}$

Solution:

Given $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=e^{-2x}$

(i.e.)$(D^2+3D+2)y=e^{-2x}$

(i.e.)$\phi(D)y=F(x)$

To find CF :

Put $D=m$ in $\phi(D)=0$, the auxiliary equation is $m^2+3m+2=0$

$\Rightarrow (m+1)(m+2)=0 \Rightarrow m=-1,-2$

$\Rightarrow m_1=-1, m_2=-2$ are real and unequal.

$\Rightarrow \text{CF}=c_1e^{m_1x}+c_2e^{m_2x}=c_1e^{-x}+c_2e^{-2x}$

To find PI :

$$\text{PI} = \frac{1}{\phi(D)}F(x) = \frac{1}{D^2+3D+2}e^{-2x}$$

$$[\text{in} \quad \phi(D) \quad \text{replace} \quad D \quad \text{by} \quad -2]$$

$$= \frac{1}{4-6+2}e^{-2x}, \text{ here } \phi(-2) = 0$$

$$= x.\frac{1}{2D+3}e^{-2x} = x.\frac{1}{-4+3}e^{-2x} = -xe^{-2x}$$

$$\left[\text{in} \quad \phi^{'}(D) \quad \text{replace} \quad D \quad \text{by} \quad -2\right]$$

The complete solution is $y = CF + PI$

$$y = c_1e^{-x} + c_2e^{-2x} - xe^{-2x}$$

Example 3: Solve $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+9y=3e^{4x}$

Solution:

Given $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+9y=3e^{4x}$

(i.e.)$(D^2+6D+9)y=3e^{4x}$

(i.e.)$\phi(D)y=F(x)$

To find CF :

Put $D=m$ in $\phi(D)=0$, the auxiliary equation is $m^2+6m+9=0$

$\Rightarrow (m+3)(m+3)=0 \Rightarrow m=-3,-3$

$\Rightarrow m_1=-3, m_2=-3$ are real and equal.

$\Rightarrow \mathrm{CF}=(c_1+c_2x)e^{mx}=(c_1+c_2x)e^{-3x}$

To find PI :

$$\text{PI} = \frac{1}{\phi(D)}F(x) = \frac{1}{D^2+6D+9}3e^{4x}$$

$$[\text{in} \quad \phi(D) \quad \text{replace} \quad D \quad \text{by} \quad 4]$$

$$= 3.\frac{1}{16+24+9}e^{4x}$$

$$= \frac{3}{49}e^{4x}$$

The complete solution is $y = CF + PI$

$$y = (c_1 + c_2 x)e^{-3x} + \frac{3}{49}e^{4x}$$

Example 4: Solve $\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}+y=e^{-x}+3$

Solution:

Given $\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}+y=e^{-x}+3$

(i.e.)$(D^2+2D+1)y=e^{-x}+3$

(i.e.)$\phi(D)y=F(x)$

To find CF :

Put $D=m$ in $\phi(D)=0$, the auxiliary equation is $m^2+2m+1=0$

$\Rightarrow (m+1)(m+1)=0 \Rightarrow m=-1,-1$

$\Rightarrow m_1=-1, m_2=-1$ are real and equal.

$\Rightarrow \text{CF}=(c_1+c_2x)e^{mx}=(c_1+c_2x)e^{-x}$

To find PI :

$$\text{PI} = \frac{1}{\phi(D)} F(x) = \frac{1}{D^2+2D+1} e^{-x} + 3$$

$$= \frac{1}{D^2+2D+1} e^{-x} + 3 \frac{1}{D^2+2D+1} e^{0x}$$

[in $\phi(D)$ replace D by -1] and
[in $\phi(D)$ replace D by 0]

$$= \frac{1}{(-1)^2+2(-1)+1} e^{-x} + 3e^{0x}$$

[here $\phi(-1) = 0$]

$$= x.\frac{1}{2D+2}e^{-x} + 3$$

$$\left[\text{in} \quad \phi^{'}(D) \quad \text{replace} \quad D \quad \text{by} \quad -1\right],$$

$$= x.\frac{1}{2(-1)+2}e^{-x} + 3$$

$$\left[\text{here} \quad \phi^{'}(-1) = 0\right]$$

$$= \frac{x^2}{2}e^{-x} + 3$$

The complete solution is $y = CF + PI$

$$y = (c_1 + c_2 x)e^{-x} + \frac{x^2}{2}e^{-x} + 3$$

Example 5: Solve $(D^2-1)y = \sin h2x$

Solution:

Given $(D^2-1)y = \sin h2x$

(i.e.)$\phi(D)y = F(x)$

To find CF :

Put $D = m$ in $\phi(D) = 0$, the auxiliary equation is $m^2 - 1 = 0$

$\Rightarrow m = \pm i = 0 \pm i \quad$ (here, $\quad \alpha = 0, \beta = 1$)

The roots are imaginary.

$\Rightarrow \text{CF} = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) = (c_1 \cos x + c_2 \sin x)$

To find PI :

$$\text{PI} = \frac{1}{\phi(D)} F(x) = \frac{1}{D^2-1} \sin h2x$$

$$= \frac{1}{D^2-1}\left[\frac{e^{2x}-e^{-2x}}{2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2-1}e^{2x} - \frac{1}{D^2-1}e^{-2x}\right]$$

[in $\phi(D)$ replace D by 2] and

[in $\phi(D)$ replace D by -2]

$$= \frac{1}{2}\left[\frac{1}{2^2-1}e^{2x} - \frac{1}{(-2)^2-1}e^{-2x}\right]$$

$$= \frac{1}{2}\left[\frac{1}{4-1}e^{2x} - \frac{1}{4-1}e^{-2x}\right]$$

$$= \frac{1}{2}\left[\frac{e^{2x}}{3} - \frac{e^{-2x}}{3}\right]$$

$$= \frac{1}{3}\left[\frac{e^{2x}-e^{-2x}}{2}\right] = \frac{1}{3}\sin h2x$$

The complete solution is $y = CF + PI$

$$y = (c_1 \cos x + c_2 \sin x) + \frac{1}{3}\sin h2x$$

Type 2: If $F(x) = \sin ax \quad \text{or} \quad \cos ax$, then

$$\text{PI} = \frac{1}{\phi(D)} F(x) = \frac{1}{\phi(D)} \sin ax \quad \text{or} \quad \cos ax$$

$$\left[\text{in} \quad \phi(D) \quad \text{replace} \quad D^2 \quad \text{by} \quad -a^2\right]$$

$$= \frac{1}{\phi(-a^2)} \sin ax \quad \text{or} \quad \cos ax, \ \text{provided} \quad \phi(-a^2) \neq 0$$

If $\phi(-a^2) = 0$, then $\text{PI} = x.\dfrac{1}{\phi^{'}(D)} \sin ax \quad \text{or} \quad \cos ax$

$$\left[\text{in} \quad \phi^{'}(D) \quad \text{replace} \quad D^2 \quad \text{by} \quad -a^2\right]$$

$$= x.\frac{1}{\phi^{'}(-a^2)} \sin ax \quad \text{or} \quad \cos ax, \ \text{provided} \quad \phi^{'}(-a^2) \neq 0$$

If $\phi^{'}(-a^2) = 0$, then PI $= x^2.\dfrac{1}{\phi^{''}(D)} \sin ax \quad \text{or} \quad \cos ax$

$$\left[\text{in} \quad \phi^{''}(D) \quad \text{replace} \quad D^2 \quad \text{by} \quad -a^2\right]$$

$$= x^2.\frac{1}{\phi^{''}(-a^2)} \sin ax \quad \text{or} \quad \cos ax, \quad \text{provided} \quad \phi^{''}(-a^2) \neq 0$$

This process may be repeated till the denominator

becoming nonzero when replace D^2 by $-a^2$.

Example 6: Solve $(D^2+3D+2)y = \sin x$

Solution:

Given $(D^2+3D+2)y = \sin x$

(i.e.)$\phi(D)y = F(x)$

To find CF:

Put $D = m$ in $\phi(D) = 0$, the auxiliary equation is $m^2+3m+2 = 0$

$\Rightarrow (m+1)(m+2) = 0 \Rightarrow m = -1, -2$

$\Rightarrow m_1 = -1, m_2 = -2$ are real and unequal.

$\Rightarrow \text{CF} = c_1 e^{-x} + c_2 e^{-2x}$

To find PI :

$$\text{PI} = \frac{1}{\phi(D)} F(x) = \frac{1}{D^2+3D+2} \sin x$$

$$\left[\text{in} \quad \phi(D) \quad \text{replace} \quad D^2 \quad \text{by} \quad -1^2 = -1\right]$$

$$= \frac{1}{-1+3D+2}\sin x$$

$$= \frac{1}{3D+1}\sin x$$

$$= \frac{(3D-1)}{(3D-1)(3D+1)}\sin x$$

$$= \frac{3D-1}{(9D^2-1)}\sin x$$

$$\left[\text{replace} \quad D^2 \quad \text{by} \quad -1^2 = -1\right]$$

$$= \frac{1}{(-9-1)}(3D\sin x - \sin x)$$

$$= -\frac{1}{10}(3\cos x - \sin x)$$

The complete solution is $y = CF + PI$

$$y = c_1 e^{-x} + c_2 e^{-2x} - \frac{1}{10}(3\cos x - \sin x)$$

Example 7: Solve $(D^2 + 6D + 8)y = e^{-2x} + \cos^2 x$

Solution:

Given $(D^2 + 6D + 8)y = e^{-2x} + \cos^2 x$

(i.e.)$\phi(D) = F(x)$

To find CF :

Put $D = m$ in $\phi(D) = 0$, the auxiliary equation is $m^2 + 6m + 8 = 0$

$\Rightarrow (m+2)(m+4) = 0 \Rightarrow m = -2, -4 \Rightarrow m_1 = -2, m_2 = -4$ are real and unequal.

$\Rightarrow \text{CF} = c_1 e^{-2x} + c_2 e^{-4x}$

To find PI :

$$\text{PI} = \frac{1}{\phi(D)} F(x) = \frac{1}{D^2+6D+8}(e^{-2x} + \cos^2 x)$$

$$= \frac{1}{D^2+6D+8} e^{-2x} + \frac{1}{D^2+6D+8} \cos^2 x$$

$$= \quad \text{PI}_1 \quad + \quad \text{PI}_2$$

$$\text{PI}_1 = \frac{1}{D^2+6D+8} e^{-2x}$$

$$[\text{in} \quad \phi(D) \quad \text{replace} \quad D \quad \text{by} \quad -2]$$

$$= \frac{1}{4+6(-2)+8} e^{-2x}, \text{ here } \phi(-2) = 0$$

$$= x.\frac{1}{2D+6} e^{-2x} = x.\frac{1}{2(-2)+6} e^{-2x} = \frac{x}{2} e^{-2x}$$

$$\mathrm{PI}_2 = \frac{1}{D^2+6D+8}\cos^2 x$$

$$= \frac{1}{D^2+6D+8}\left(\frac{1+\cos 2x}{2}\right)$$

$$= \frac{1}{2}\frac{1}{D^2+6D+8}e^{0x} + \frac{1}{2}\frac{1}{D^2+6D+8}\cos 2x$$

[in $\phi(D)$ replace D by 0] and

$\left[\text{in} \quad \phi(D) \quad \text{replace} \quad D^2 \quad \text{by} \quad -2^2 = -4\right]$

$$= \frac{1}{16} + \frac{1}{2}\frac{1}{-4+6D+8}\cos 2x$$

$$= \frac{1}{16} + \frac{1}{2}\frac{1}{6D+4}\cos 2x$$

$$= \frac{1}{16} + \frac{1}{2}\frac{(6D-4)}{(6D+4)(6D-4)}\cos 2x$$

$$= \frac{1}{16} + \frac{1}{2}\frac{6D-4}{(36D^2-16)}\cos 2x$$

$$\left[\text{replace} \quad D^2 \quad \text{by} \quad -2^2 = -4\right]$$

$$= \frac{1}{16} + \frac{1}{2}\frac{6D-4}{(36(-4)-16)}\cos 2x$$

$$= \frac{1}{16} + \frac{1}{2}\frac{6D-4}{-160}\cos 2x$$

$$= \frac{1}{16} + \frac{1}{2}\frac{1}{-160}(6D\cos 2x - 4\cos 2x)$$

$$= \frac{1}{16} - \frac{1}{320}(-12\sin 2x - 4\cos 2x)$$

$$= \frac{1}{16} + \frac{1}{80}(3\sin 2x + \cos 2x)$$

$$\mathrm{PI} = \mathrm{PI}_1 + \mathrm{PI}_2 = \frac{x}{2}e^{-2x} + \frac{1}{16} + \frac{1}{80}(3\sin 2x + \cos 2x)$$

The complete solution is $y = CF + PI$

$$y = c_1 e^{-2x} + c_2 e^{-4x} + \frac{x}{2}e^{-2x} + \frac{1}{16} + \frac{1}{80}(3\sin 2x + \cos 2x)$$

Example 8: Solve $(D^2 + 4)y = \sin x$

Solution:

Given $(D^2 + 4)y = \sin x$

(i.e.)$\phi(D)y = F(x)$

To find CF :

Put $D = m$ in $\phi(D) = 0$, the auxiliary equation is $m^2 + 4 = 0$

$\Rightarrow m = \pm 2i = 0 \pm 2i$ (here, $\alpha = 0, \beta = 2$)

The roots are imaginary.

$\Rightarrow \text{CF} = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) = (c_1 \cos 2x + c_2 \sin 2x)$

To find P.I. :

$$\text{PI} = \frac{1}{\phi(D)} F(x) = \frac{1}{D^2 + 4} \sin 2x$$

$$\left[\text{in} \quad \phi(D) \quad \text{replace} \quad D^2 \quad \text{by} \quad -2^2 = -4\right]$$

$$= \frac{1}{-4+4} \sin 2x$$

$$= \frac{1}{0} \sin 2x \qquad \text{here } \phi(-2) = 0$$

$$= x.\frac{1}{2D} \sin 2x$$

$$= \frac{x}{2}.\frac{D}{D^2} \sin 2x$$

$$\left[\text{replace} \quad D^2 \quad \text{by} \quad -2^2 = -4\right]$$

$$= \frac{x}{2} \cdot \frac{1}{-4} D(\sin 2x)$$

$$= \frac{x}{2} \cdot \frac{1}{-4} (2\cos 2x)$$

$$= -\frac{x}{4} \cos 2x$$

The complete solution is $y = CF + PI$

$$y = (c_1 \cos 2x + c_2 \sin 2x) - \frac{x}{4} \cos 2x$$

Type 3: If $F(x) = x^n$ where n is a constant (+ve integer), then

$$\text{PI} = \frac{1}{\phi(D)} F(x) = \frac{1}{\phi(D)} x^n = \frac{1}{[1 \pm f(D)]} x^n = [1 \pm f(D)]^{-1} x^n$$

[*Express* $\phi(D)$ as $1 \pm f(D)$, *bring it to the Numerator expand* $[1 \pm f(D)]^{-1}$ *as a Binomial series. Operate* x^n *on each term of this expansion*]

Note: Binomial expansions

(i) $(1 + x)^{-1} = 1 - x + x^2 - x^3 + ...$

(ii) $(1 - x)^{-1} = 1 + x + x^2 + x^3 + ...$

Example 9: Solve $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = x^2 + 3x - 1$

Solution:

Given $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = x^2 + 3x - 1$

(i.e.)$(D^2 - 5D + 6)y = x^2 + 3x - 1$

(i.e.)$\phi(D)y = F(x)$

To find CF:

Put $D = m$ in $\phi(D) = 0$, the auxiliary equation is $m^2 - 5m + 6 = 0$

$\Rightarrow (m-2)(m-3) = 0 \Rightarrow m = 2, 3$

$\Rightarrow m_1 = 2, m_2 = 3$ are real and unequal.

$\Rightarrow \text{CF} = c_1 e^{m_1 x} + c_2 e^{m_2 x} = c_1 e^{2x} + c_2 e^{3x}$

To find PI :

$$\text{PI} = \frac{1}{\phi(D)}F(x) = \frac{1}{D^2-5D+6}(x^2+3x-1)$$

$$= \frac{1}{6\left[1+\left(\frac{D^2-5D}{6}\right)\right]}(x^2+3x-1)$$

$$= \frac{1}{6}\left[1+\left(\frac{D^2-5D}{6}\right)\right]^{-1}(x^2+3x-1)$$

$$= \frac{1}{6}\left[1-\left(\frac{D^2-5D}{6}\right)+\left(\frac{D^2-5D}{6}\right)^2-\ldots\right](x^2+3x-1)$$

$$= \frac{1}{6}\left[1-\frac{D^2}{6}+\frac{5D}{6}+\frac{D^4}{36}-\frac{10D^3}{36}+\frac{25D^2}{36}\right](x^2+3x-1)$$

$$= \frac{1}{6}\Big[(x^2+3x-1) - \frac{1}{6}D^2(x^2+3x-1) + \frac{5}{6}D(x^2+3x-1) + 0 - 0 + \frac{25}{36}D^2(x^2+3x-1)\Big]$$

$$= \frac{1}{6}\left[x^2+3x-1-\frac{1}{6}(2)+\frac{5}{6}(2x+3)+\frac{25}{36}(2)\right]$$

$$= \frac{1}{6}\left[x^2+3x-1-\frac{1}{3}+\frac{5}{6}(2x)+\frac{5}{6}(3)+\frac{25}{36}(2)\right]$$

$$= \frac{1}{6}\left[x^2+3x-1-\frac{1}{3}+\frac{5x}{3}+\frac{5}{2}+\frac{25}{18}\right] \quad = \frac{1}{6}\left[x^2+\frac{14x}{3}+\frac{23}{9}\right]$$

The complete solution is $y = CF + PI$

$$y = c_1 e^{2x} + c_2 e^{3x} + \frac{1}{6}\left[x^2 + \frac{14x}{3} + \frac{23}{9}\right]$$

Example 10: Solve $(D^2 + 3D + 2)y = x^2$

Solution:

Given $(D^2 + 3D + 2)y = x^2$

(i.e.)$\phi(D) = F(x)$

To find CF :

Put $D = m$ in $\phi(D) = 0$, the auxiliary equation is $m^2 + 3m + 2 = 0$

$\Rightarrow (m+1)(m+2) = 0 \Rightarrow m = -1, -2$

$\Rightarrow m_1 = -1, m_2 = -2$ are real and unequal.

$\Rightarrow \text{CF} = c_1 e^{-x} + c_2 e^{-2x}$

To find P.I. :

$$\mathrm{PI} = \frac{1}{\phi(D)}F(x) = \frac{1}{D^2+3D+2}x^2$$

$$= \frac{1}{2\left[1+\left(\frac{D^2+3D}{2}\right)\right]}x^2$$

$$= \frac{1}{2}\left[1+\left(\frac{D^2+3D}{2}\right)\right]^{-1}x^2$$

$$= \frac{1}{2}\left[1-\left(\frac{D^2+3D}{2}\right)+\left(\frac{D^2+3D}{2}\right)^2-\dots\right]x^2$$

$$= \frac{1}{2}\left[1-\frac{D^2}{2}-\frac{3D}{2}+\frac{D^4}{4}+\frac{6D^3}{4}+\frac{9D^2}{4}\right]x^2$$

$$= \frac{1}{2}\left(x^2-\frac{1}{2}D^2(x^2)-\frac{3}{2}D(x^2)+0+0+\frac{9}{4}D^2(x^2)\right]$$

$$= \frac{1}{2}\left[x^2 - \frac{2}{2} - \frac{3}{2}(2x) + \frac{9}{4}(2)\right]$$

$$= \frac{1}{2}\left[x^2 - 1 - 3x + \frac{9}{2}\right]$$

$$= \frac{1}{2}\left[x^2 - 3x + \frac{7}{2}\right]$$

The complete solution is $y = CF + PI$

$$y = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{2}\left[x^2 - 3x + \frac{7}{2}\right]$$

Example 11: Find the particular integral of $(D^2 + 2D + 1)y = 1 + x$

Solution:

$$\text{PI} = \frac{1}{\phi(D)} F(x) = \frac{1}{D^2 + 2D + 1}(1 + x)$$

$$
\begin{aligned}
&= \frac{1}{1+(D^2+2D)}(1+x) \\
&= \left[1+(D^2+2D)\right]^{-1}(1+x) \\
&= \left[1-(D^2+2D)+\dots\right](1+x) \\
&= \left[1-D^2-2D\right](1+x) \\
&= \left[1+x-D^2(1+x)-2D(1+x)\right] \\
&= \left[1+x-D^2(1)-D^2(x)-2D(1)-2D(x)\right] \\
&= (1+x-0-0-0-2) = x-1
\end{aligned}
$$

Type 4:

If $F(x) = e^{ax} f(x)$, where $f(x) = x^n$ or $\sin ax$ or $\cos ax$

$$\text{PI} = \frac{1}{\phi(D)} F(x) = \frac{1}{\phi(D)} e^{ax} f(x)$$

$$= e^{ax} \frac{1}{\phi(D+a)} f(x)$$

[ie. replace D by $D+a$]

Note that $\dfrac{1}{\phi(D+a)} f(x)$ will be in any one of the previous

known forms.

Example 12: Solve $(D^2+D+1)y = x^2e^{-x}$

Solution:

Given

$(D^2+D+1)y = x^2e^{-x}$

(i.e.)$\phi(D)y = F(x)$

To find CF :

Put $D = m$ in $\phi(D) = 0$, the auxiliary equation is $m^2+m+1 = 0$

$$\Rightarrow m = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2} = \frac{-1}{2} \pm \frac{i\sqrt{3}}{2}$$

$$\left(\text{here,} \quad \alpha = \frac{-1}{2}, \beta = \frac{\sqrt{3}}{2}\right)$$

The roots are imaginary.

$$\Rightarrow \text{CF} = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) = e^{\frac{-1}{2}x}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right)$$

To find PI:

$$\text{PI} = \frac{1}{\phi(D)}F(x) = \frac{1}{D^2+D+1}e^{-x}x^2$$

$$=e^{-x}\frac{1}{[(D-1)^2+(D-1)+1]}x^2$$

$$=e^{-x}\frac{1}{[D^2-2D+1+D-1+1]}x^2$$

$$=e^{-x}\frac{1}{[D^2-D+1]}x^2$$

$$=e^{-x}\frac{1}{[1+(D^2-D)]}x^2$$

$$= e^{-x}\left[1+(D^2-D)\right]^{-1}x^2$$

$$= e^{-x}\left[1-(D^2-D)+(D^2-D)^2-\dots\right]x^2$$
$$= e^{-x}\left[1-D^2+D+D^4-2D^3+D^2\right]x^2$$

$$= e^{-x}\left[x^2-D^2(x^2)+D(x^2)+D^4(x^2)-2D^3(x^2)+D^2(x^2)\right]$$

$$= e^{-x}\left[x^2-2+2x+0-0+2\right]$$
$$= e^{-x}\left(x^2+2x\right)$$

The complete solution is $y = CF + PI$

$$y = e^{\frac{-1}{2}x}\left(c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right) + e^{-x}\left(x^2+2x\right)$$

Example 13: Solve $\dfrac{d^2y}{dx^2}+4\dfrac{dy}{dx}+4y=e^{3x}\sin x$

Solution:

Given $\dfrac{d^2y}{dx^2}+4\dfrac{dy}{dx}+4y=e^{3x}\sin x$

(i.e.)$(D^2+4D+4)y=e^{3x}\sin x$

(i.e.)$\phi(D)y=F(x)$

To find CF:

Put $D=m$ in $\phi(D)=0$, the auxiliary equation is $m^2+4m+4=0$

$\Rightarrow (m+2)(m+2)=0 \Rightarrow m=-2,-2$

$\Rightarrow m_1=-2, m_2=-2$ are real and equal.

$\Rightarrow \text{CF}=(c_1+c_2x)e^{mx}=(c_1+c_2x)e^{-2x}$

To find PI :

$$\mathrm{PI} = \frac{1}{\phi(D)}F(x) = \frac{1}{D^2+4D+4}e^{3x}\sin x$$

$$= e^{3x}\frac{1}{[(D+3)^2+4(D+3)+4]}\sin x$$

$$= e^{3x}\frac{1}{[D^2+6D+9+4D+12+4]}\sin x$$

$$= e^{3x}\frac{1}{[D^2+10D+25]}\sin x$$

$$\left[\text{in} \quad \phi(D) \quad \text{replace} \quad D^2 \quad \text{by} \quad -1^2=-1\right]$$

$$= e^{3x}\frac{1}{-1+10D+25}\sin x$$

$$= e^{3x}\frac{1}{24+10D}\sin x$$

$$=e^{3x}\frac{1}{2(12+5D)}\sin x$$

$$=\frac{e^{3x}}{2}\frac{(12-5D)}{(12+5D)(12-5D)}\sin x$$

$$=\frac{e^{3x}}{2}\frac{12-5D}{(144-25D^2)}\sin x$$

$$\left[\text{replace} \quad D^2 \quad \text{by} \quad -1^2=-1\right]$$

$$=\frac{e^{3x}}{2}\frac{12-5D}{(144-25(-1))}\sin x$$

$$=\frac{e^{3x}}{2}\frac{1}{169}(12\sin x-5D(\sin x))$$

$$=\frac{e^{3x}}{338}(12\sin x-5\cos x)$$

The complete solution is $y=CF+PI$

$$y=(c_1+c_2x)e^{-2x}+\frac{e^{3x}}{338}(12\sin x-5\cos x)$$

Type 5: If $F(x) = x^n \sin ax \quad \text{or} \quad x^n \cos ax$, then

$$\text{PI} = \frac{1}{\phi(D)} F(x) = \frac{1}{\phi(D)} x^n \sin ax \quad \text{or} \quad x^n \cos ax$$

Now,

$$= \frac{1}{\phi(D)} x^n \cos ax + i \frac{1}{\phi(D)} x^n \sin ax$$

$$= \frac{1}{\phi(D)} x^n (\cos ax + i \sin ax) = \frac{1}{\phi(D)} x^n e^{iax}$$

$$= e^{iax} \frac{1}{\phi(D + ia)} x^n$$

(ie).

$$\frac{1}{\phi(D)} x^n \cos ax + i \frac{1}{\phi(D)} x^n \sin ax = e^{iax} \frac{1}{\phi(D + ia)} x^n$$

$$\Rightarrow \frac{1}{\phi(D)} x^n \cos ax = \text{ Real part of } e^{iax} \frac{1}{\phi(D+ia)} x^n \text{ and}$$

$$\Rightarrow \frac{1}{\phi(D)} x^n \sin ax = \text{ Imaginary part of } e^{iax} \frac{1}{\phi(D+ia)} x^n.$$

$$(\because e^{iax} = \cos ax + i \sin ax)$$

Example 12: Solve $(D^2 - 2D + 1)y = x \sin x$

Solution:

Given $(D^2 - 2D + 1)y = x \sin x$

(i.e.)$\phi(D)y = F(x)$

To find CF:

Put $D = m$ in $\phi(D) = 0$, the auxiliary equation is $m^2 - 2m + 1 = 0$

$\Rightarrow (m-1)(m-1) = 0 \Rightarrow m = 1, 1$

$\Rightarrow m_1 = 1, m_2 = 1$ are real and equal.

$\Rightarrow \text{CF} = (c_1 + c_2 x)e^{mx} = (c_1 + c_2 x)e^{x}$

To find PI:

$$PI = \frac{1}{\phi(D)}F(x) = \frac{1}{D^2-2D+1}x\sin x$$

$$=\text{Imaginary part of } \frac{1}{(D^2-2D+1)}x(\cos x + i\sin x)$$

$$=\text{I.P of } \left\{\frac{1}{[D^2-2D+1]}xe^{ix}\right\} \qquad (\because e^{ix} = \cos x + \sin x)$$

$$=\text{I.P of } \left\{e^{ix}\frac{1}{[(D+i)^2-2(D+i)+1]}x\right\}$$

$$=\text{I.P of } \left\{e^{ix}\frac{1}{[D^2-1+2Di-2D-2i+1]}\right\}x$$

$$=\text{I.P of } \left\{e^{ix}\frac{1}{[D^2+2D(i-1)-2i]}x\right\}$$

$$=\text{I.P of }\left\{e^{ix}\frac{1}{-2i\left[1+\left(\frac{D^2+2D(i-1)}{-2i}\right)\right]}x\right\}$$

$$=\text{I.P of }\left\{e^{ix}\frac{1}{-2i\left[1-\left(\frac{D^2+2D(i-1)}{2i}\right)\right]}x\right\}$$

$$=\text{I.P of }\left\{e^{ix}\frac{1}{-2i}\left[1-\left(\frac{D^2+2D(i-1)}{2i}\right)\right]^{-1}x\right\}$$

$$=\text{I.P of }\left\{e^{ix}\frac{1}{-2i}\left[1+\left(\frac{D^2+2D(i-1)}{2i}\right)-\dots\right]x\right\}$$

$$=\text{I.P of }\left\{e^{-ix}\frac{1}{-2i}*\frac{i}{i}\left[1+\frac{D^2}{2i}+\frac{2Di}{2i}-\frac{2D}{2i}\right]x\right\}$$

$$=\text{I.P of }\left\{e^{ix}\frac{i}{2}\left[x+\frac{D^2(x)}{2i}+D(x)-\frac{D(x)}{i}\right]\right\}$$

$$=\text{I.P of }\left\{e^{ix}\frac{i}{2}\left[x+0+1-\frac{1}{i}*\frac{i}{i}\right]\right\}$$

$$=\text{I.P of }\left\{e^{ix}\frac{i}{2}[x+0+1+i]\right\}$$

$$=\text{I.P of }\left\{\frac{i}{2}(\cos x+i\sin x)(x+i+1)\right\}$$

$$=\text{I.P of }\left\{\frac{i}{2}(x\cos x+ix\sin x+i\cos x-\sin x+\cos x+i\sin x)\right\}$$

$$=\text{I.P of }\left\{\frac{1}{2}(ix\cos x-x\sin x-\cos x-i\sin x+i\cos x-\sin x)\right\}$$

$$=\text{I.P of }\left\{\frac{1}{2}[(-x\sin x-\cos x-\sin x)+i(x\cos x-\sin x+\ cosx)]\right\}$$

$$= \frac{1}{2}(x\cos x - \sin x + \cos x)$$

The complete solution is $y = CF + PI$

$$y = (c_1 + c_2 x)e^x + \frac{1}{2}(x\cos x - \sin x + \cos x)$$

Exercise Problems:

1. $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 2y = 0$
2. $(D^2 + 16)y = e^{-4x}$
3. $(D^2 + 4D + 4)y = e^{-2x}$
4. $(D^2 + 3D + 2)y = \sin 3x$
5. $(D^2 - 4D + 4)y = \cos 2x$
6. $(D^2 - 5D + 6)y = x^2 + 3$
7. Find the particular integral of $(D^2 + 2)y = x^2$
8. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{-x} \sin 2x$
9. $(D^2 + 4D + 3)y = e^x \cos 2x$
10. $(D^2 + 4)y = x \cos x$