Z - Transform

* The Z - transform is an Important tool in the Study of discrete time Systems.

* The Z-transform of discrete time sequence $x(n)$, is defined as

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}} \quad \text{--- (1)}$$

where z is a complex variable.

In polar form, z can be expressed as

$$\boxed{z = re^{j\omega}} \quad \text{--- (2)}$$

where r is radius of circle.

* If the sequence $x(n)$ exists for n in the range $-\infty$ to ∞. the equation (1) Represents two sided or the bilateral Z-transform. On the otherhand, if the sequence exists only for $n \geq 0$ the equation (1) changes to

$$\boxed{X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n}}$$

which is called as one sided z-transform.

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1. Find the Z-transform and ROC for the signal. $x(n) = a^n u(n)$.

Formula. $X(z) = \sum_{n=-\alpha}^{\alpha} x(n) z^{-(n)}$ - its complex variable

$z = re^{j\omega}$ - complex, $r = |z|$

r is radius of circle.

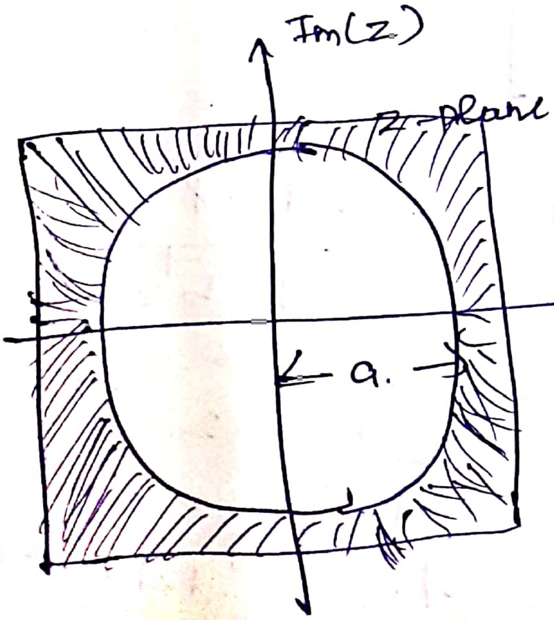
$= \sum_{n=-\alpha}^{\alpha} a^n u(n) z^{-n}$.

$= \sum_{n=0}^{\alpha} a^n z^{-n}$.

$= \sum_{n=0}^{\alpha} (a z^{-1})^n$.

$X(z) = \frac{1}{1-az^{-1}}$; ROC $|z| > |a|$

$= \frac{z}{z-a}$



ROC.

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(2) Determine the z-transform of the signal:

$$x(n) = -b^n u(-n-1)$$ Find ROC.

Solution:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= -\sum_{n=-\infty}^{\infty} b^n u(-n-1) z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} b^n z^{-n}$$

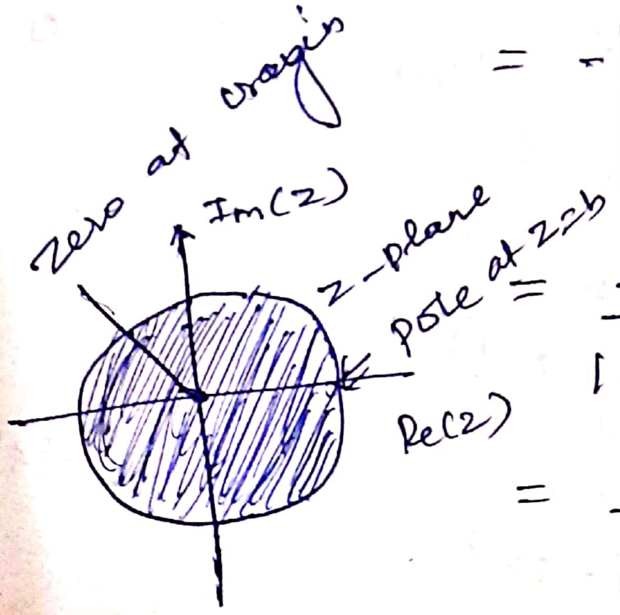
$$= -\sum_{n=1}^{\infty} b^{-n} z^{n}$$

$$= -\sum_{n=1}^{\infty} (b^{-1}z)^n$$

$$= -\sum_{n=0}^{\infty} (b^{-1}z)^n - 1$$

[We know the formula: $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$, $|a|<1$]

$$= -\left[\frac{1}{1-b^{-1}z} - 1\right]$$



$$= \frac{-1}{1-b^{-1}z} + 1 = \frac{\cancel{-1} + \cancel{1} - b^{-1}z}{1-b^{-1}z}$$

$$= \frac{-b^{-1}z}{1-b^{-1}z} = \frac{-\frac{z}{b}}{1-\frac{z}{b}} = \frac{-z}{b-z}$$

$$= \frac{z}{z-b} = \frac{1}{1-bz^{-1}}$$; $|z| < |b|$

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(3). Determine the z-transform of

$x(n) = a^n u(n) - b^n u(-n-1)$ and find ROC.

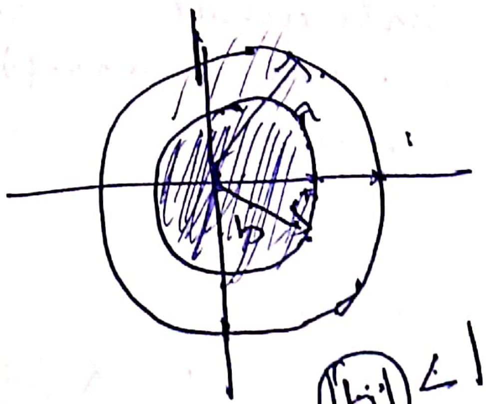
The given sequence is two sided infinite duration sequence having values of n: from $-\alpha$ to α.

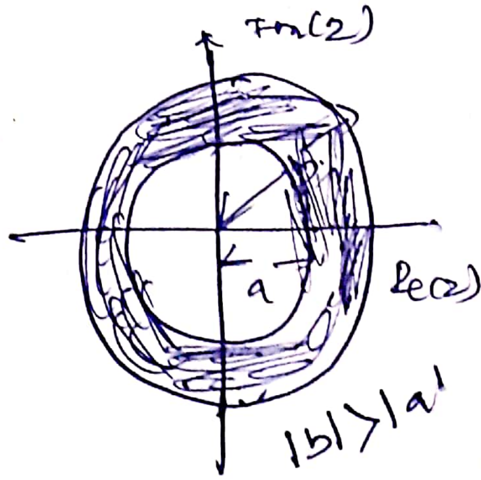
$$\boxed{X(z) = \sum_{n=-\alpha}^{\alpha} x(n) z^{-n}.}$$

$$= \sum_{n=0}^{\alpha} a^n z^{-n} + \sum_{n=-1}^{-\alpha} b^n z^{-n}.$$

$$= \sum_{n=0}^{\alpha} (az^{-1})^n + \sum_{n=1}^{\alpha} (b^{-1}z)^n.$$

$$X(z) = \frac{z}{z-a} + \frac{z}{z-b}$$ ROC $|a| < |z| < |b|$





ROC of a two sided sequence for $|b| < |a|$.

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Z - Transform and ROC Of Finite Duration Sequences.

Right hand Sequence:

* If Sequence is purely right sided sequence. A Right hand sequence is one for which $x(n)=0$ for all $n < n_0$ where n_0 is positive (or) negative but finite.

(3) Find Z-transform and ROC of the causal sequence? $x(n) = \{\underset{\uparrow}{2}, -1, 3, 2, 0, 1\}$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-1} \quad -①$$

One Expanding eq ① we get

$$X(z) = \cdots x(-1)z + x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + x(4)z^{-4} + x(5)z^{-5} \cdots ②$$

The given sequence values are

$x(0)=2$; $x(1)=-1$; $x(2)=3$; $x(3)=2$; $x(4)=0$; $x(5)=1$;

On Substituting the values in eq ②

we have: $X(z) = 2 - z^{-1} + 3z^{-2} + 2z^{-3} + z^{-5}$

The $X(z)$ Converges for all values of z except at $z=0$

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Left - hand Sequences:

* If sequence is purely left-sided sequence. A left hand sequence is one for which $x(n)=0$ for all $n \geq n_0$, where n_0 is positive or negative but finite. If $n_0 \leq 0$ the resulting sequence is anticausal sequence. the ROC is entire z-plane except at $z=\infty$

(4) Find the z-transform and ROC of the anticausal sequence.

$$x(n) = \{3, 2, -1, -4, \underset{\uparrow}{1}\}$$

Solution:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \quad -(1)$$

the given sequence values are

$x(-4)=3$ $x(-3)=2$, $x(-2)=-1$ $x(-1)=-4$,

$x(0)=1$

on substituting the eq (1) we get

$$X(z) = 3z^4 + 2z^3 - z^2 - 4z + 1$$

the $X(z)$ converges for all values of z except at $z=\infty$.

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Two sided sequence.
ROC: entire z-plane except at $z=0$, $z=\infty$.

Find the z-transform of the sequence.

$$x(n) = \{1, 2, 0, -4, \underset{\uparrow}{3}, 2, 1, 6, 5\}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \text{ we get.}$$

$$X(z) = z^4 + 2z^3 - 4z + 3 + 2z^{-1} + z^{-2} + 6z^{-3} + 5z^{-4}$$

The $X(z)$ converges for all values except at $z=0$ and $z=\infty$.

(5) Find z-transform of the following sequences.

i) $x(n) = u(n) - u(n-3)$.

ii) $x(n) = \{1, 2, 3, 2\}$.

iii) $x(n) = \{1, 2, \underset{\uparrow}{-1}, 2, 3\}$

i) $x(n) = u(n) - u(n-3)$

$$X(z) = 1 + z^{-1} + z^{-2}.$$

ROC : Entire z-plane except at $z=0$.

ii) $x(n) = \{1, 2, 3, 2\}$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$X(z) = 1 + 2z^{-1} + 3z^{-2} + 2z^{-3}$$

ROC: Entire z-plane except at $z=0$

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(iii) $x(n) = \{1, 2, -1, 2, 3\}$

$X(z) = z^2 + 2z - 1 + 2z^{-1} + 3z^{-2}$

ROC: Entire z-plane except at $z=0$ and $z=\infty$.

Properties of Region of Convergence

1. The ROC is a ring (or) disk in the z-plane centered at the origin.

2. The ROC cannot contain any poles.

3. If $x(n)$ is a right sided sequence & if $|z| = r$ is the ROC then all finite values of z for which $|z| > r$, will also be in ROC

4. If $x(n)$ is left sided sequences & if $|z| = r$ is the ROC, Then all finite values of z for which $|z| < r$ will also be in ROC.

5. If $x(n)$ is two sided signal. & If $|z| = r$ circle is in ROC, then the ROC will contain a ring in z-plane that include $|z| = r$

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Find the z-transform of the signal $x(n) = [\sin\omega_0 n]\,u(n)$.

$$x(n) = (\sin\omega_0 n)\,u(n)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}.$$

$$= \sum_{n=-\infty}^{\infty} (\sin\omega_0 n)\,u(n) z^{-n} = \sum_{n=0}^{\infty} (\sin\omega_0 n)\, z^{-n}.$$

$$= \sum_{n=0}^{\infty} \left[\frac{[e^{j\omega_0 n} - e^{-j\omega_0 n}]}{2j} \right] z^{-n}$$

$$= \frac{1}{2j} \sum_{n=0}^{\infty} (e^{j\omega_0 n} z^{-n} - \sum_{n=0}^{\infty} e^{-j\omega_0 n} z^{-n}.$$

$$= \frac{1}{2j} \left[\sum_{n=0}^{\infty} (e^{j\omega_0} z^{-1})^{n} - \sum_{n=0}^{\infty} (e^{-j\omega_0} z^{-1})^{n}. \right.$$

$$X(z) = \frac{1}{2j} \left[\frac{1}{1 - e^{j\omega_0} z^{-1}} - \frac{1}{1 - e^{-j\omega_0 n} z^{-1}} \right]$$

$$= \frac{1}{2j} \left[\frac{\cancel{1} - e^{-j\omega_0} z^{-1} - \cancel{1} + e^{j\omega_0} z^{-1}}{(1 - e^{j\omega_0} z^{-1})\,(1 - e^{-j\omega_0 n} z^{-1})} \right]$$

$$= \frac{1}{2j} \left[\frac{e^{j\omega_0} z^{-1} - e^{-j\omega_0 n} z^{-1}}{1 - e^{-j\omega_0 n} z^{-1} - e^{j\omega_0} z^{-1} + z^{-2}} \right]$$

$$X(z) = \left[\frac{(\sin\omega_0)\, z^{-1}}{1 - 2\cos\omega_0 z^{-1} + z^{-2}} \right]$$

ROC: $|z| > 1$

(2). Determine the z-transform, ROC and pole zero locations.

$$x(n) = \left(\frac{2}{3}\right)^n u(n) + \left(\frac{-1}{2}\right)^n u(n).$$

Solution

$$x(n) = \left(\frac{2}{3}\right)^n u(n) + \left(\frac{-1}{2}\right)^n u(n)$$

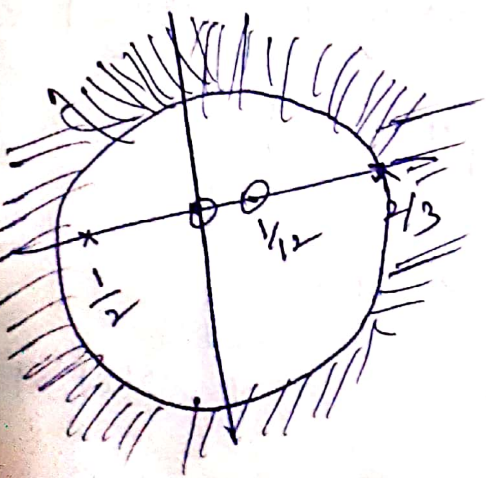
$$= \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n z^{-n} + \sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n z^{-n}.$$

$$= \sum_{n=0}^{\infty} \left(\left(\frac{2}{3}\right) z^{-1}\right)^n + \sum_{n=0}^{\infty} \left(\left(\frac{-1}{2}\right) z^{-1}\right)^n.$$

$$X(z) = \frac{1}{1 - \frac{2}{3} z^{-1}} + \frac{1}{1 + 1/2\, z^{-1}}$$

$$= \frac{z}{z - 2/3} + \frac{z}{z + 1/2}$$

$$= \frac{z^2 + \frac{1}{2} z + z^2 - \frac{2}{3} z}{(z - 2/3)(z + 1/2)}$$



$$= \frac{2z^2 - \frac{1}{6} z}{(z - 2/3)(z + 1/2)}.$$

$$= \frac{z\left(2z - \frac{1}{6}\right)}{(z - 2/3)(z + 1/2)}$$

ROC

$|z| > 2/3$

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