

Convolution Integral

Find the convolution of $x_1(t)$ and $x_2(t)$ for the following signals

(i) $x_1(t) = u(t)$ $x_2(t) = u(t)$

(ii) $x_1(t) = e^{-at} u(t)$ $x_2(t) = e^{-bt} u(t)$

Solution:

(i) $x_1(t) = u(t)$ $x_2(t) = u(t)$

Convolution formula is

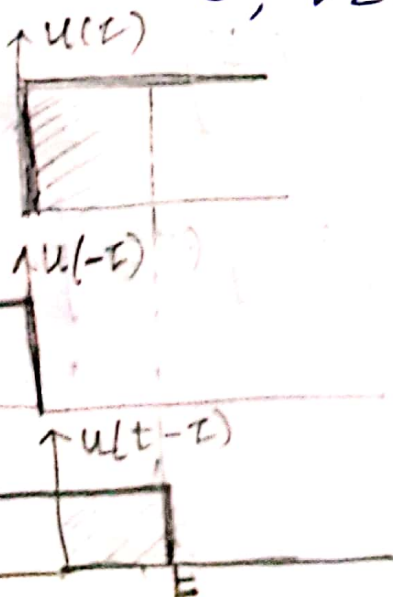
$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) d\tau$$

$$y(t) = x_1(\tau) = u(\tau), \quad x_2(t-\tau) = u(t-\tau)$$

$$= \int_{-\infty}^{\infty} u(\tau) u(t-\tau) d\tau$$

$$u(\tau) = 1; \tau \geq 0 \\ 0; \tau < 0$$

$$u(t-\tau) = 1; \tau \leq t \\ 0; \tau > t$$



$$x_1(t) * x_2(t) = \int_0^t u(\tau) u(t-\tau) d\tau \\ = \int_0^t d\tau$$

$$x_1(t) * x_2(t) = t u(t)$$

Solution:

$$(ii) x_1(t) = e^{-at} u(t) ; x_2(t) = e^{-bt} u(t)$$

$$x_1(\tau) = e^{-a\tau} u(\tau)$$

$$x_2(\tau) = e^{-b\tau} u(\tau)$$

$$x_2(t-\tau) = e^{-b(t-\tau)} u(t-\tau)$$

$$x_1(t) * x_2(t) = \int_0^t x_1(\tau) x_2(t-\tau) d\tau$$

$$= \int_0^t e^{-a\tau} \cdot e^{-b(t-\tau)} d\tau$$

$$= \int_0^t e^{-a\tau} e^{-bt} e^{b\tau} d\tau$$

$$= e^{-bt} \int_0^t e^{-(a-b)\tau} d\tau$$

$$= e^{-bt} \left[\frac{e^{-(a-b)\tau}}{-(a-b)} \right]_0^t$$

$$= \frac{e^{-bt}}{(a-b)} \left[1 - e^{-(a-b)t} \right]$$

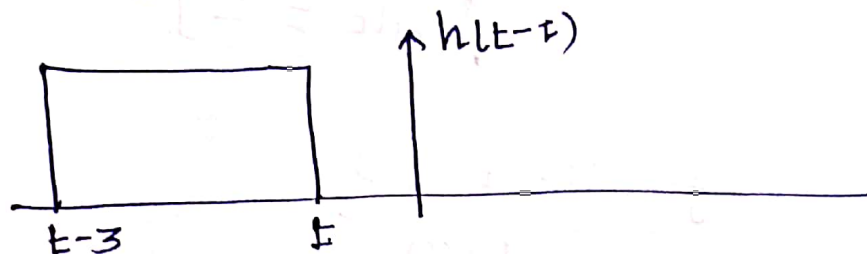
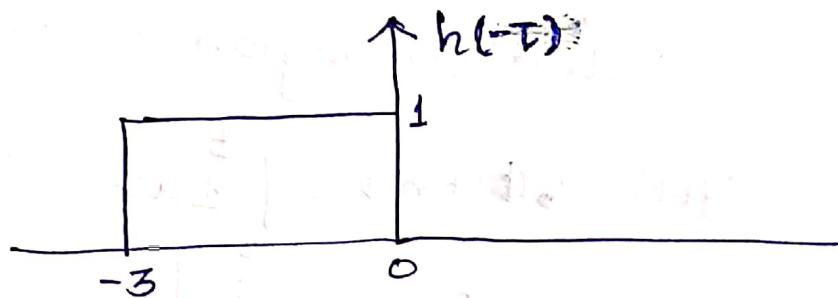
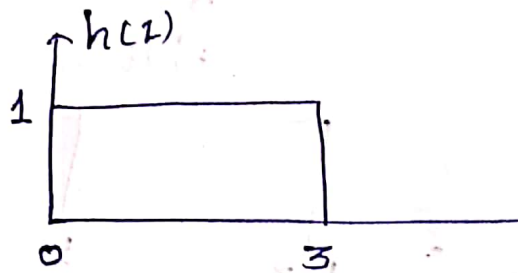
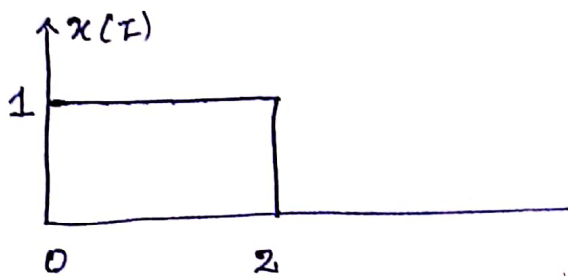
$$x_1(t) * x_2(t) = \frac{1}{(a-b)} \left[e^{-bt} - e^{-at} \right] u(t)$$

Find convolution using Graphical method

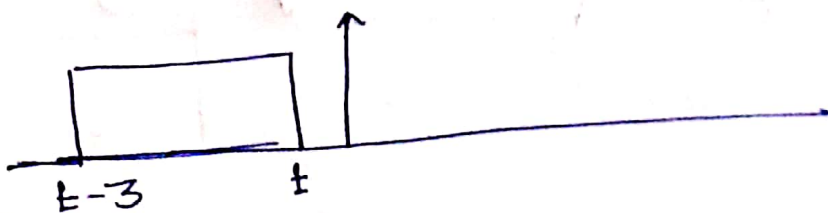
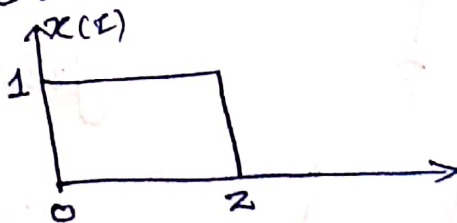
$$(i) x(t) = \begin{cases} 1 & ; 0 \leq t \leq 2 \\ 0 & ; \text{otherwise} \end{cases}$$

$$h(t) = \begin{cases} 1 & ; 0 \leq t \leq 3 \\ 0 & ; \text{otherwise} \end{cases}$$

Solution:



If $t < 0$

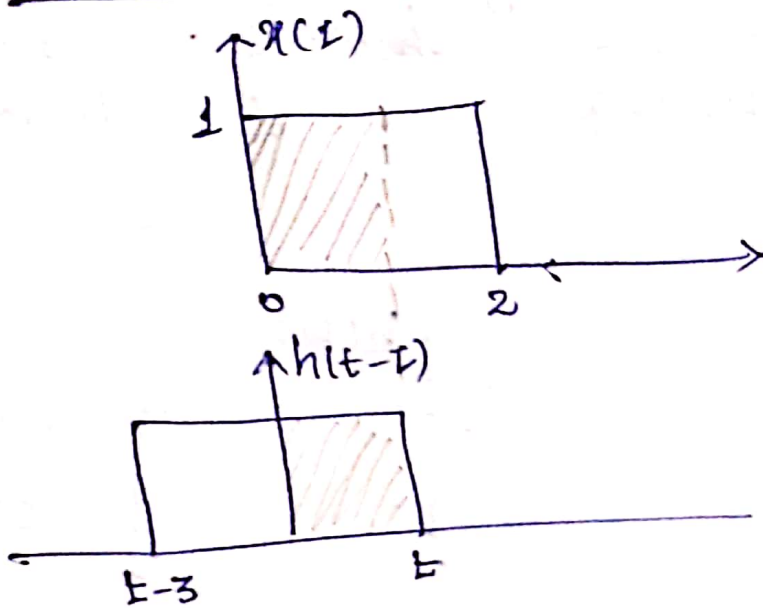


No overlapping b/w

$x(t)$ and $h(t-t)$

then $x_1(t) * x_2(t)$ is
Zero.

If $0 \leq t \leq 2$

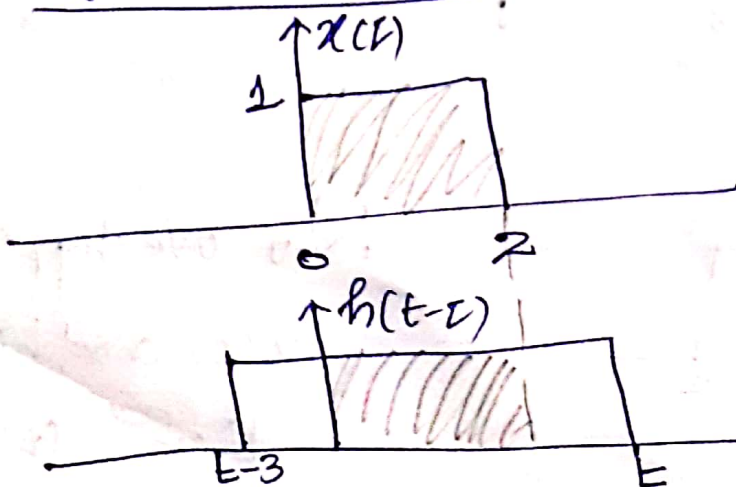


Intersection region is 0 to t

$$y(t) = x(t) * h(t) = \int_0^t 1 \, d\tau$$

$$\boxed{y(t) = t}$$

If $2 \leq t \leq 3$

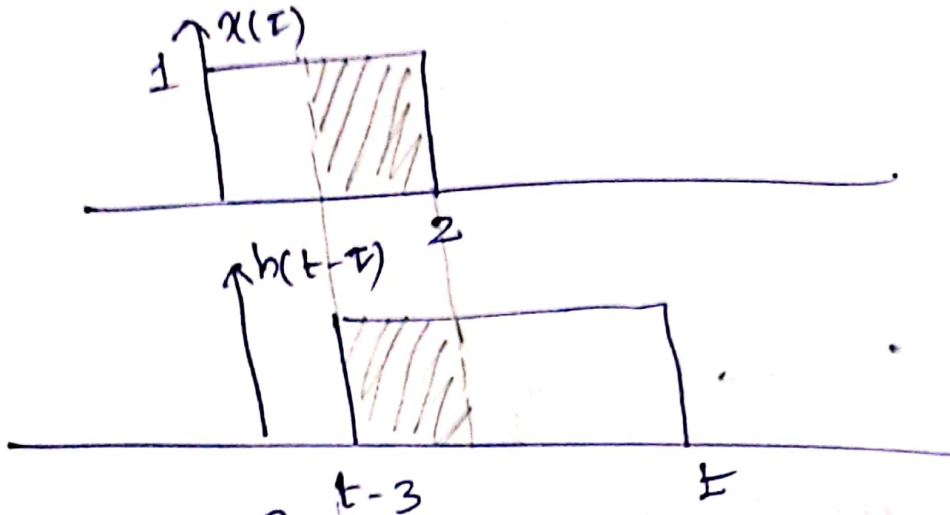


$$y(t) = \int_0^2 1 \, d\tau$$

$$\boxed{y(t) = 2}$$

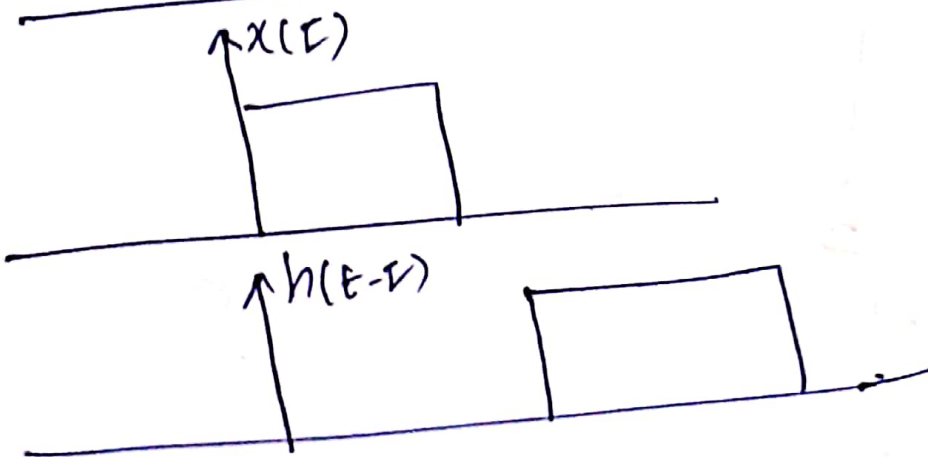
Intersection region is

If $3 \leq t \leq 5$



$$y(t) = \int_{t-3}^2 dz = 2 - (t-3) \\ = 5-t$$

If $t > 5$



No overlapping

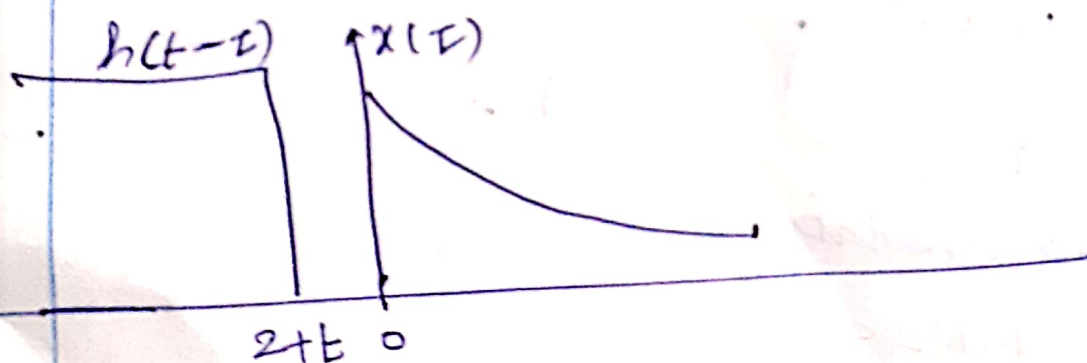
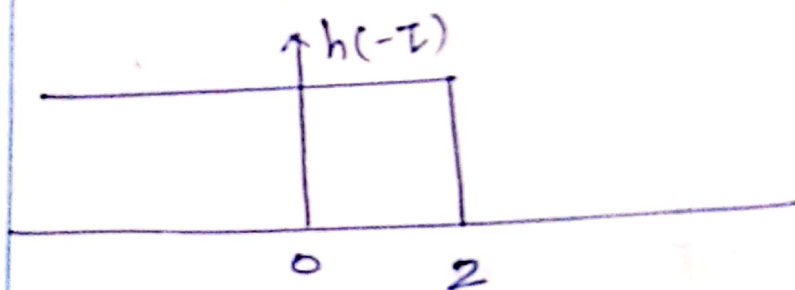
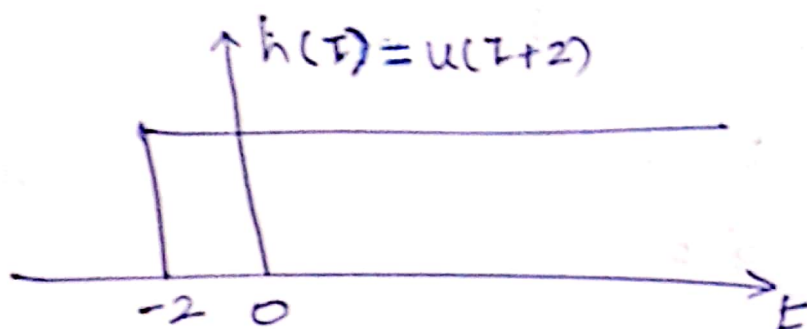
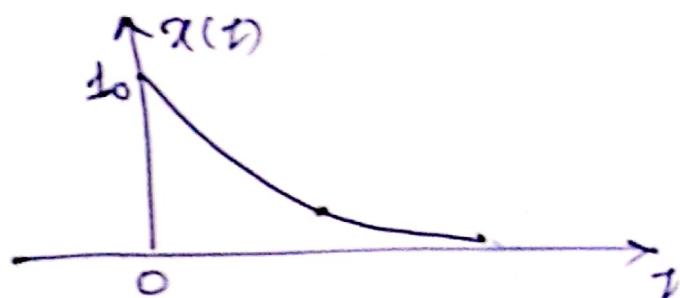
then $y(t) = 0$.

Conclusion:

$$y(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ t & \text{for } 1 \leq t \leq 2 \\ 2 & \text{for } 2 \leq t \leq 3 \\ 5-t & \text{for } 3 \leq t \leq 5 \\ 0 & \text{for } t > 5 \end{cases}$$

(ii) $x(t) = e^{-2t} u(t)$ $h(t) = u(t+2)$

Solution:



If $t < -2$

No overlapping

$$y(t) = 0.$$

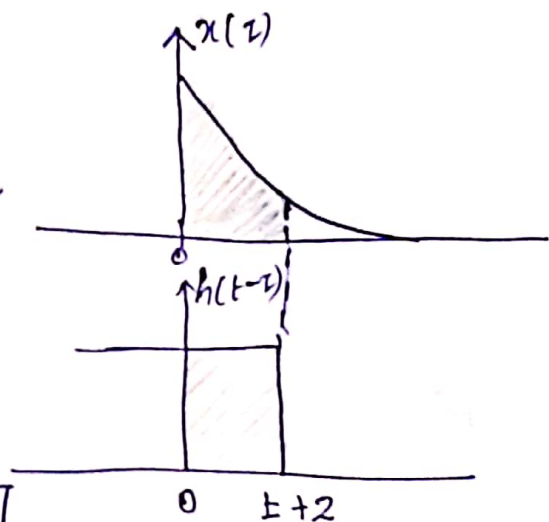
If $t > -2$

$$y(t) = \int_0^{t+2} x(\tau) h(t-\tau) d\tau$$

$$= \int_0^{t+2} e^{-2\tau} d\tau$$

$$= -\frac{1}{2} \left[e^{-2(t+2)} - 1 \right]$$

$$= \frac{1 - e^{-2(t+2)}}{2}$$



Conclusion

$$y(t) = \frac{1 - e^{-2(t+2)}}{2} ; t > -2$$
$$0 ; t < -2$$