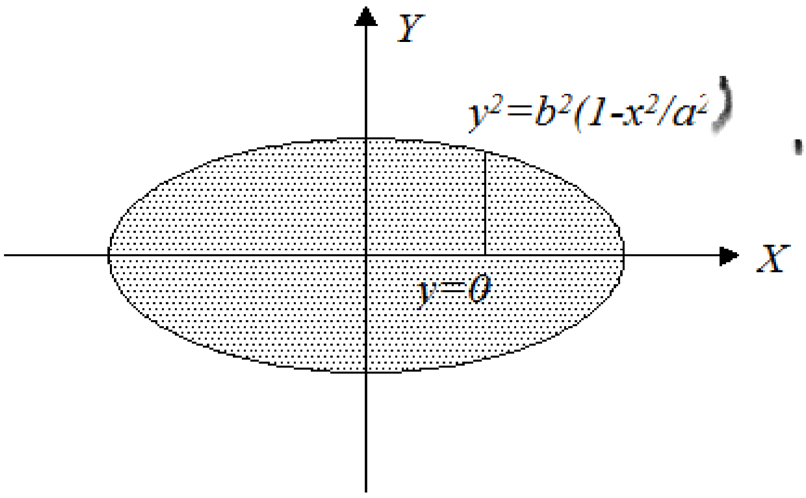
PLANE AREA USING DOUBLE INTEGRAL

CARTESIAN FORM

EXAMPLE :1

Find by double integration, the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Solution:



$$A = 4\iint dydx = 4\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} dydx$$

$$= 4\int_0^a [y]_0^{b\sqrt{1-\frac{x^2}{a^2}}} dx$$

$$= \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$$

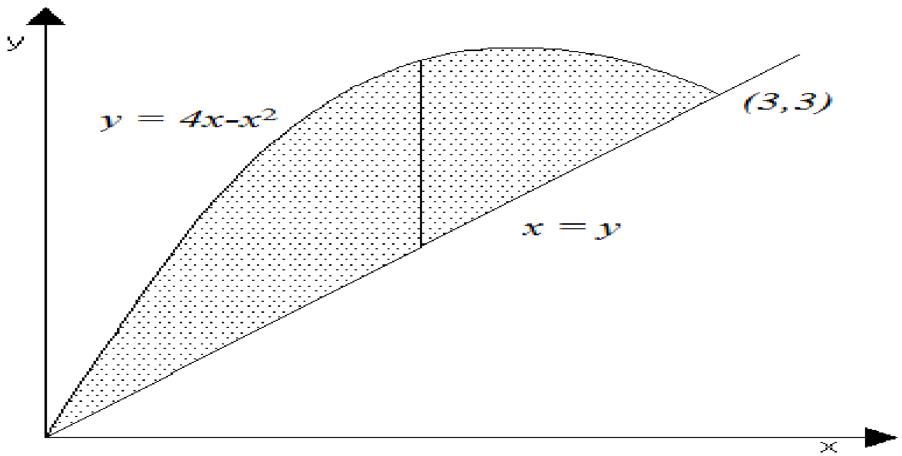
$$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{4b}{a}\text{x}\frac{a^2}{2}\text{x}\frac{\pi}{2} = \pi ab \text{ sq.units.}$$

EXAMPLE :2

Find the area between the parabola $y = 4x - x^2$ and the line $y = x$.

Solution:



Given $y = 4x - x^2$ and $y = x$, solving for x,

$$x = 4x - x^2 \;=>\; 0 = 3x - x^2 \;=>\; 0 = (3 - x)x \;=>\; x = 0{,}3$$

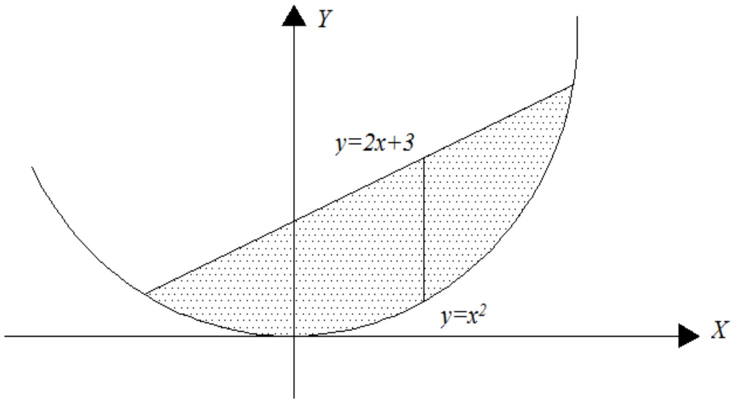
A = $\int_0^3 \int_x^{4x-x^2} dydx = \int_0^3 [y]_x^{4x-x^2} dx$

$= \int_0^3 (3x - x^2) dx$

$= \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3 = \frac{9}{2}$

EXAMPLE :3

Find the area between the parabola $y = x^2$ and the line $y = 2x + 3$.



Solution:

Given $y = x^2$ and $y = 2x + 3$.

solving for x, $x^2 = 2x + 3 \Rightarrow x = -1,3$

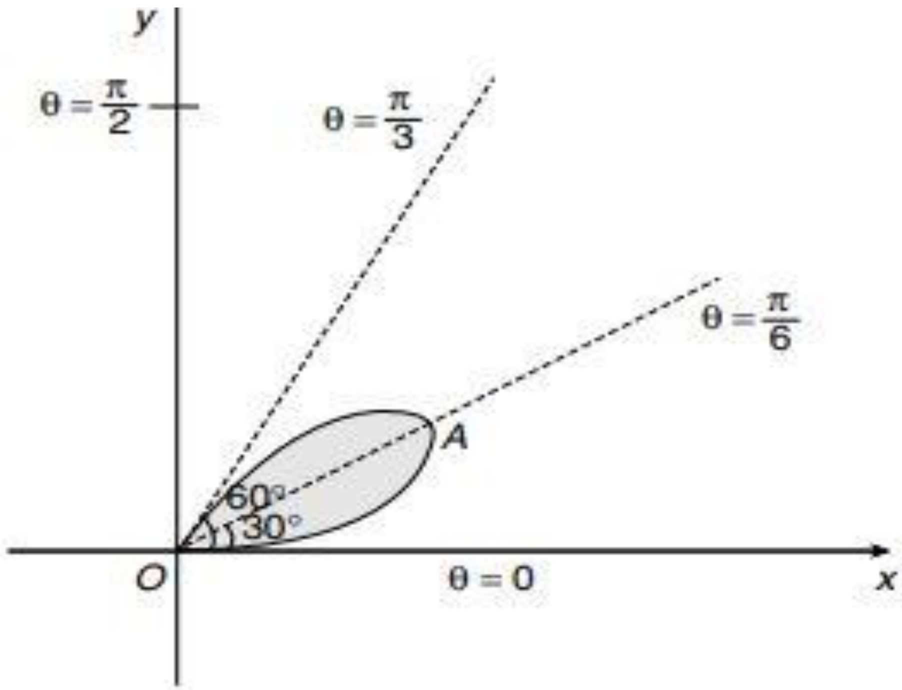
$$A = \int_{-1}^{3}\int_{x^2}^{2x+3} dy\,dx = \int_{-1}^{3}[y]_{x^2}^{2x+3}dx$$

$$= \int_{-1}^{3}(2x + 3 - x^2)dx$$

$$= \left[\frac{2x^2}{2} + 3x - \frac{x^3}{3}\right]_{-1}^{3} = \frac{32}{3}$$

EXAMPLE:4

Find the area of a loop of the curve $r = a\sin 3\theta$.



Solution.

Given the curve is $r = a\sin 3\theta$

When $\theta = 0$, $\quad r = 0$

When $\theta = \frac{\pi}{6}$, $r = a\sin\frac{\pi}{2} = a$, which is the maximum value of r.

When $\theta = \frac{\pi}{3}$, $\quad r = a\sin\pi = 0$

So, as θ varies from 0 to $\frac{\pi}{6}$, x goes from 0 to A

and as θ varies from $\frac{\pi}{6}$ to $\frac{\pi}{3}$, x comes from A

to 0.

So, as θ varies from 0 to $\frac{\pi}{3}$, we get a loop as in **Fig. 6.16**.

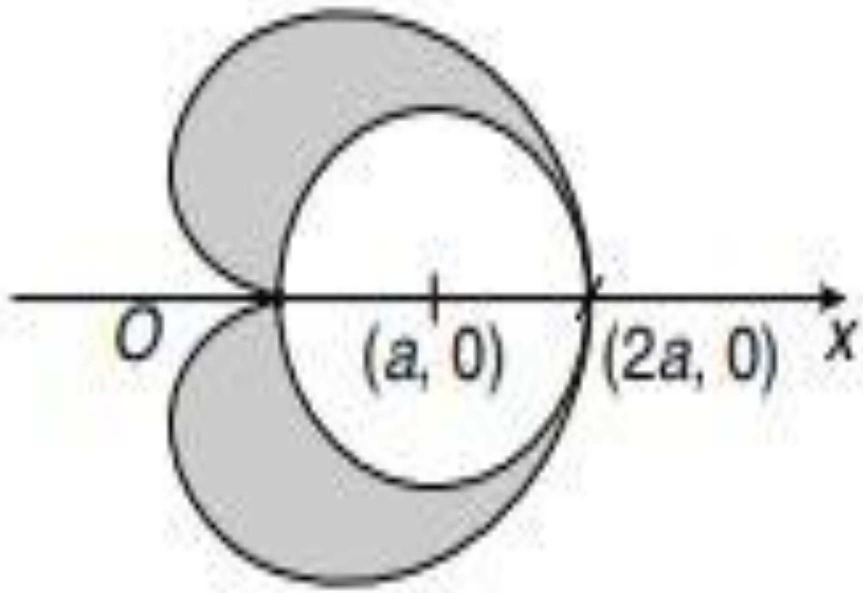
Area of the loop $= \frac{1}{2}\int_0^{\frac{\pi}{3}} r^2 \, d\theta$

$$= \frac{1}{2}\int_0^{\frac{\pi}{3}} a^2 \sin^2 3\theta \, d\theta = \frac{a^2}{2}\int_0^{\frac{\pi}{3}} \left[\frac{1-\cos 6\theta}{2}\right] d\theta$$

$$= \frac{a^2}{4}\left[\theta - \frac{\sin 6\theta}{6}\right]_0^{\frac{\pi}{3}} = \frac{a^2}{4}\left[\frac{\pi}{3} - \frac{\sin 2\pi}{6} - 0\right] = \frac{\pi a^2}{12}$$

EXAMPLE:5

Find the area outside the circle $r = 2a\cos\theta$ and inside the cardioid $r = a(1+\cos\theta)$.



Solution.

Given the circle $r = 2a\cos\theta$ (1)

and the cardioid $r = a(1+\cos\theta)$ (2)

The required area is as shown in the **Fig 6.15**, since the circle lies inside the cardioid.

From (1), when $\theta = 0$, $r = 2a$

and when $\theta = \frac{\pi}{2}$, $r = 0$

To find the point of intersection, solve (1) and (2)

$$\therefore \qquad a(1+\cos\theta) = 2a\cos\theta \quad \Rightarrow \quad \cos\theta = 1 \quad \Rightarrow \quad \theta = 0 \text{ or } 2\pi$$

When $\theta = \frac{\pi}{3}$, $\quad r = 2a \cdot \frac{1}{2} = a$

That is the circle lies inside the cardioid.

Required area A = Area of the cardioid − Area of the circle

$$\text{Area of the cardioid} = \frac{3\pi a^2}{2}$$

Area of the circle = πa^2, since radius is a.

$$\therefore \qquad \text{required area} = \frac{3\pi a^2}{2} - \pi a^2 = \frac{\pi a^2}{2}.$$

PLANE AREA USING DOUBLE INTEGRAL

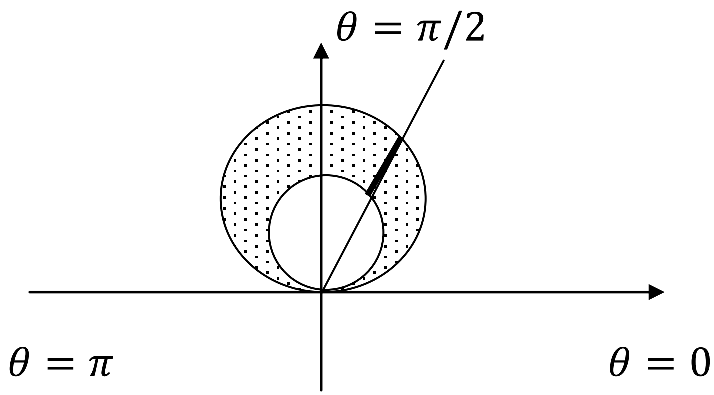
POLAR FORM

EXAMPLE :1

Find the area bounded by the circle

$r = 2\sin\theta \ \ and \ \ r = 4\sin\theta.$

Solution:



$$A = \int_0^{\pi} \int_{2\sin\theta}^{4\sin\theta} r\,dr\,d\theta = \int_0^{\pi} \left[\frac{r^2}{2}\right]_{2\sin\theta}^{4\sin\theta} d\theta$$

$$= 6\int_0^{\pi} (\sin\theta)^2 d\theta$$

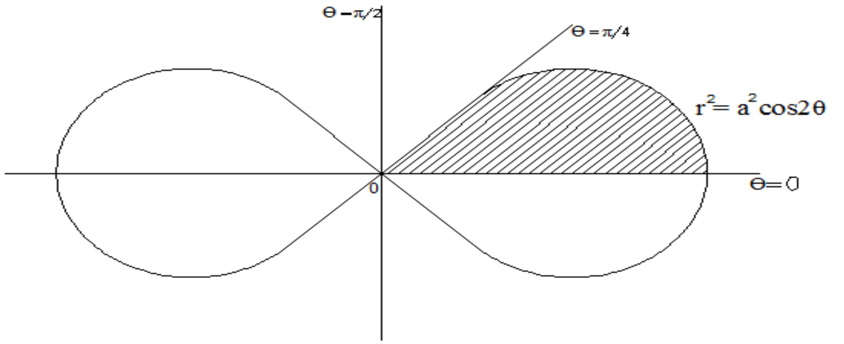
$$= 3\int_0^{\pi} (1 - \cos 2\theta)\,d\theta$$

$$= 3\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi} = 3\pi\,.$$

EXAMPLE :2

Find the area enclosed by the leminiscate $r^2 = a^2 \cos 2\theta$ by double integration.

Solution:



If r = 0 then $\cos 2\theta = 0$ implies $\theta = \frac{\pi}{4}$.

$$A = 4\int_0^{\frac{\pi}{4}} \int_0^{\sqrt{a^2 \cos 2\theta}} r\, dr\, d\theta$$

$$= 4\int_0^{\frac{\pi}{4}} \left[\frac{r^2}{2}\right]_0^{\sqrt{a^2 \cos 2\theta}} d\theta$$

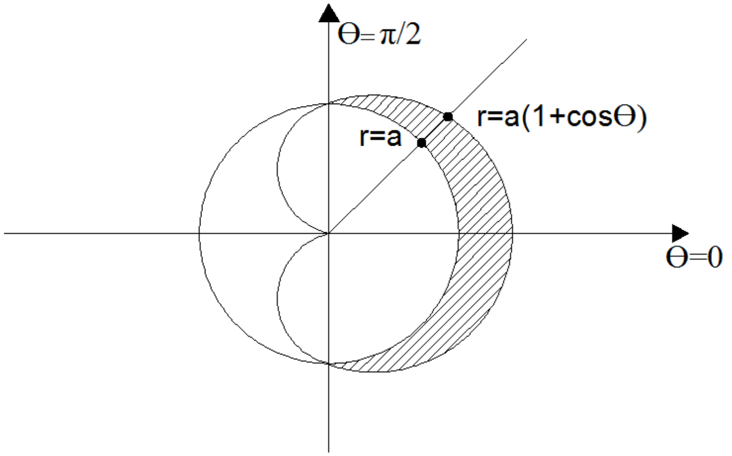
$$= 4a^2 \int_0^{\frac{\pi}{4}} \frac{\cos 2\theta}{2}\, d\theta$$

$$= 4\left[\frac{a^2 \sin 2\theta}{4}\right]_0^{\frac{\pi}{4}} = a^2.$$

EXAMPLE :3

Find the area that lies inside the cardioid $r = a(1 + \cos\theta)$ and outside the circle $r = a,$ by double integration.

Solution:



Solving $r = a(1 + \cos\theta)$ $and\ r = a$

$=> a(1 + \cos\theta) = a$

$=> \cos\theta = 0$

$=> \theta = \frac{\pi}{2}.$

$$A = 2\int_0^{\frac{\pi}{2}}\int_a^{a(1+\cos\theta)} r\,dr\,d\theta = 2\int_0^{\frac{\pi}{2}}\left[\frac{r^2}{2}\right]_a^{a(1+\cos\theta)} d\theta$$

$$= \int_0^{\frac{\pi}{2}}[(a(1+\cos\theta))^2 - a^2]d\theta$$

$$= a^2\int_0^{\frac{\pi}{2}}[2\cos\theta + (\cos\theta)^2]d\theta$$

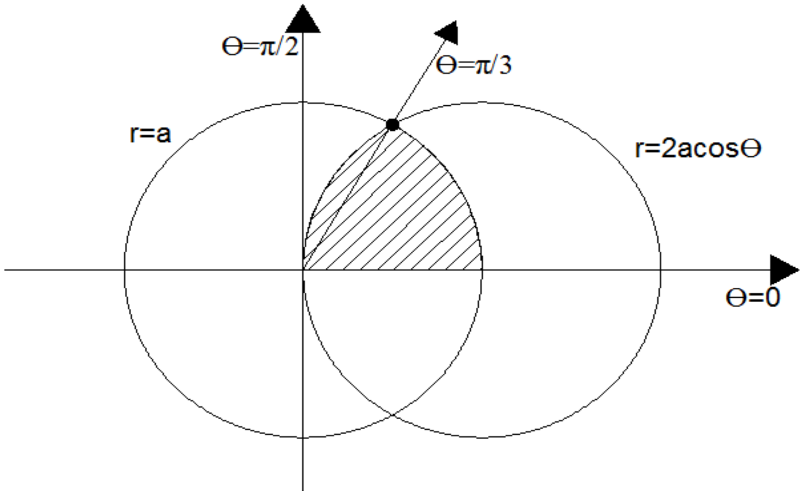
$$= \frac{a^2}{2}\int_0^{\frac{\pi}{2}}[4\cos\theta + 1 + \cos 2\theta]\ d\theta$$

$$= \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2} + 4\sin\theta\right]_0^{\frac{\pi}{2}} = \frac{a^2}{2}(\pi + 8)\ .$$

EXAMPLE :4

Find the common area to the circles $r = a\,, r = 2a\cos\theta$.

Solution:



Given $r = a\,, r = 2a\cos\theta$, solving

- ⇨ $a = 2a\cos\theta$
- ⇨ $\cos\theta = \frac{1}{2}$
- ⇨ $\theta=\pi/3$

when $r = 0 => \cos\theta = 0 => \theta = \pi/2$

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$$A = 2\iint r\,dr\,d\theta$$

$$= 2\int_0^{\frac{\pi}{3}}\int_0^{a} r\,dr\,d\theta + 2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\int_0^{2a\cos\theta} r\,dr\,d\theta$$

$$= 2\int_0^{\frac{\pi}{3}}\left[\frac{r^2}{2}\right]_0^{a} d\theta + 2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\left[\frac{r^2}{2}\right]_0^{2a\cos\theta} d\theta$$

$$= a^2\int_0^{\frac{\pi}{3}} d\theta + 2a^2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}(\cos\theta)^2 d\theta$$

$$= a^2[\theta]_0^{\frac{\pi}{3}} + 2a^2\left[\theta + \frac{\sin 2\theta}{2}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$= a^2\,\frac{\pi}{3} + 2a^2\left(\frac{\pi}{2} - \frac{\pi}{3}\right) - a^2\,\frac{\sqrt{3}}{2}$$

$$= a^2\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$$

PROBLEMS FOR PRACTICE

1.Find by double integration, the area bounded by the parabolas $x^2 = 4ay$ and $y^2 = 4ax$.

Ans: $\frac{16a^2}{3}\ sq.units.$

2.Find by double integration, the smallest area bounded by the circle $x^2 + y^2 = 9$ and the line $x + y = 3$.

Ans: $\frac{9}{4}\ (\pi - 2) sq.units.$

3.Find by double integration, the area common to the parabola $y^2 = x$ and the circle $x^2 + y^2 = 2$.

Ans: $\left(\frac{1}{3} + \frac{\pi}{2}\right) sq\ units.$

4.Find by double integration, the area lying inside the circle $r = a \sin \theta$ and outside the coordinate $r = a(1 - cos\theta)$.

Ans: $a^2 \left(1 - \frac{\pi}{4}\right) sq.units.$

Change of Variables from Cartesian to Polar Coordinates

Change of variables from cartesian to polar coordinates

Let $\iint_R f(x,y)dxdy$ be the double integral.

$$x = r\cos\theta \qquad y = r\sin\theta$$

is the transformation from cartesian to polar coordinates.

Then $dxdy = \left|J\right| drd\theta$

where= $\frac{\partial(x,y)}{\partial(r,\theta)}$ is the Jacobian of the transformation and

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$= r(\cos^2\theta + \sin^2\theta) = r$

$$\iint_R f(x,y)dxdy = \iint_R f(r,\theta) r\, drd\theta$$

Example 1:
Express the following integral in polar coordinates and evaluate

$$\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{dxdy}{\sqrt{a^2-x^2-y^2}}$$

The limits of y are $\sqrt{ax-x^2}$ and $\sqrt{a^2-x^2}$

Upper half of the circles $x^2+y^2-ax=0$
$x^2+y^2=a^2$

To change the given integral to polar coordinates we put

$$x = r\cos\theta, y = r\sin\theta \ \ and \ \ dxdy = rdrd\theta$$

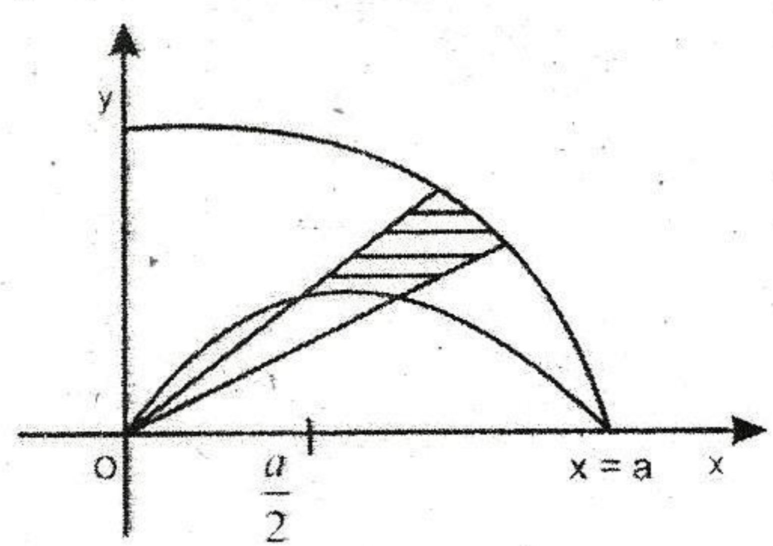
The equations of the circles become

$$i) r^2 - ar\cos\theta = 0 \ (i.e) \ r = a\cos\theta$$

$$ii) r^2 = a^2 \ (i.e) \ r = a$$

$$hence \ r \ changes \ from \ r = a\cos\theta \ to \ a$$

$$and \ \theta \ changes \ from \ 0 \ to \ \pi/2$$



$$I = \int_{0}^{\pi/2} \int_{a\cos\theta}^{a} \frac{r\,dr\,d\theta}{\sqrt{a^2 - r^2}}$$

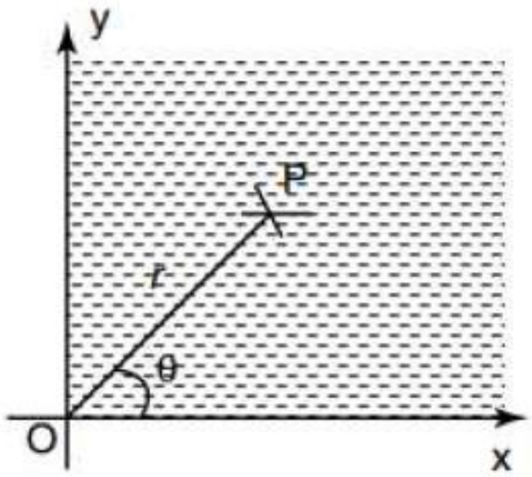
$$= \int_{0}^{\pi/2} a \sin\theta \, d\theta = a$$

Example 2:

Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dxdy$ by changing to polar coordinates and hence evaluate $\int_0^\infty e^{-x^2} dx$

Let $I = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dxdy$

Since x varies from 0 to ∞ and y varies from 0 to ∞, it is clear that the region of integration is the first quadrant as in given figure.



To change to polar coordinates, put $x = r\cos\theta, y = r\sin\theta$.

$\therefore\ dxdy = rdrd\theta$

and $x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta + \sin^2\theta) = r^2$

$\therefore$ r varies from 0 to ∞ and θ varies from 0 to $\dfrac{\pi}{2}$.

$$\therefore I = \int_0^{\pi/2}\int_0^{\infty} e^{-r^2} rdrd\theta$$

Put $r^2 = t \Rightarrow 2rdr = dt \Rightarrow rdr = \dfrac{dt}{2}$

When $r = 0, t = 0$ and when $r = \infty, t = \infty$

$$\therefore I = \int_0^{\pi/2}\left[\frac{1}{2}\int_0^{\infty} e^{-t}dt\right]d\theta = \frac{1}{2}\int_0^{\pi/2}\left[\frac{e^{-t}}{-1}\right]_0^{\infty} d\theta = -\frac{1}{2}\int_0^{\pi/2}(e^{-\infty} - e^0)d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2}(0-1)d\theta = \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{1}{2}[\theta]_0^{\pi/2} = \frac{\pi}{4}$$

$$\therefore \int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}dxdy = \frac{\pi}{4}$$

To find $\int e^{-x^2}dx$.

Now $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dxdy = \int_0^\infty e^{-x^2}dx\int_0^\infty e^{-y^2}dy \Rightarrow \frac{\pi}{4} = \left[\int_0^\infty e^{-x^2}dx\right]^2$

$$\left[\because \int_0^\infty e^{-x^2}dx = \int_0^\infty e^{-y^2}dy\right]$$

$$\therefore \int_0^\infty e^{-x^2}dx = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}$$

Example 3:

Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dydx$ by changing into polar coordinates.

Let $I = \int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dydx$

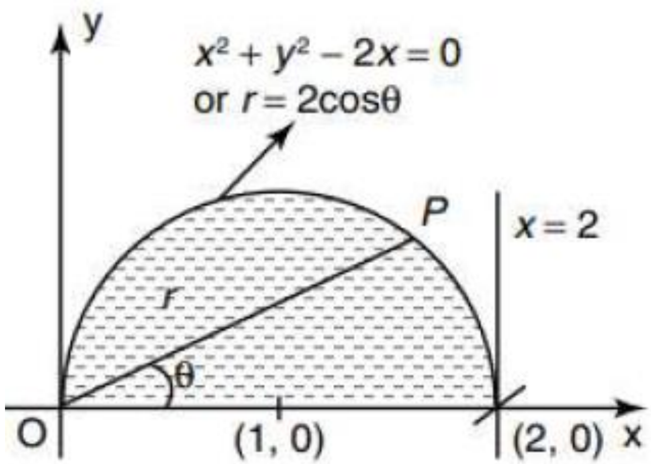
the limits for y are $y=0$ and $y=\sqrt{2x-x^2}$.

Now $y=\sqrt{2x-x^2} \Rightarrow y^2 = 2x-x^2 \Rightarrow x^2+y^2-2x=0 \Rightarrow (x-1)^2+y^2=1.$

Which is a circle with given centre $(1,0)$ and radius $r=1$ and x varies from

0 to 2.

$\therefore$ the region of integration is the upper semi-circle as in below figure.



To change to polar coordinates

Put $x = r\cos\theta, y = r\sin\theta$

$\therefore\ dxdy = rdrd\theta$

$\therefore\ x^2 + y^2 - 2x = 0$

$\Rightarrow r^2\cos^2\theta + r^2\sin^2\theta - 2r\cos\theta = 0$

$\Rightarrow r^2 - 2r\cos\theta = 0 \Rightarrow r(r - 2\cos\theta) = 0 \Rightarrow r = 0, \cos\theta$

Limits of r are $r=0$ and $r=2\cos\theta$ and limits of θ are $\theta=0$ and $\theta=\frac{\pi}{2}$

$$\therefore\ I=\int_0^{\frac{\pi}{2}}\int_0^{2\cos\theta}\frac{r\cos\theta}{r}rdrd\theta=\int_0^{\frac{\pi}{2}}\int_0^{2\cos\theta}r\cos\theta drd\theta$$

$$=\int_0^{\frac{\pi}{2}}\cos\theta\left[\int_0^{2\cos\theta}rdr\right]d\theta=\int_0^{\frac{\pi}{2}}\cos\theta\left[\frac{r^2}{2}\right]_0^{2\cos\theta}d\theta$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\cos\theta.4\cos^2\theta d\theta=2\int_0^{\frac{\pi}{2}}3\cos^3\theta d\theta=2.\frac{3-1}{3}.1=\frac{4}{3}$$

Example 4:

Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)dydx$ by changing into polar coordinates.

Let $I = \int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)dydx$

Limits for x are $x=0$ and $x=\sqrt{a^2-y^2}$.

Now $x=\sqrt{a^2-y^2} \Rightarrow x^2 = a^2-y^2 \Rightarrow x^2+y^2=a^2$ which is circle with centre (0,0) and radius a.

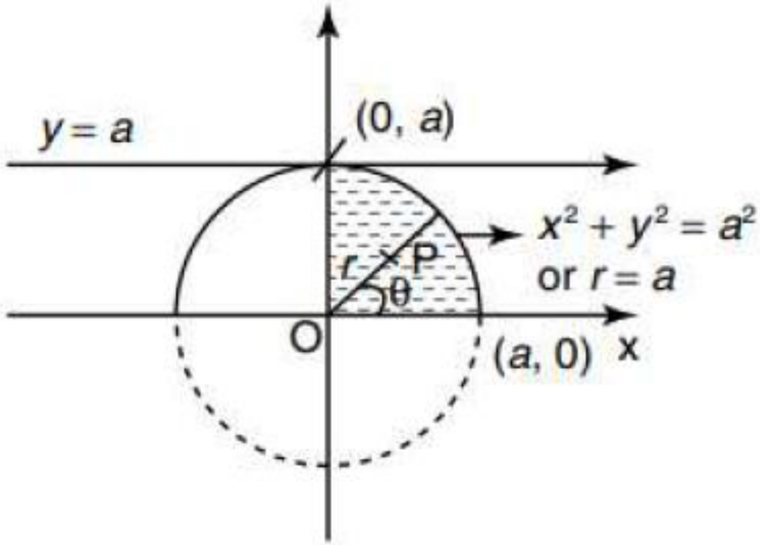
Limits for y are $y=0$ and $y=a$.

$\therefore$ The region of integration is an in below figure bounded $y=0, y=a$ and $x=0, x=\sqrt{a^2-y^2}$.

To change to polar coordinates, put $x=r\cos\theta, y=r\sin\theta$.

$\therefore dxdy = rdrd\theta$ and $x^2+y^2=a^2$

$\therefore x^2+y^2=a^2 \Rightarrow r^2=a^2 \Rightarrow r=\pm a$



$\therefore$ In the given region r varies from 0 to a and θ varies from 0 to $\dfrac{\pi}{2}$.

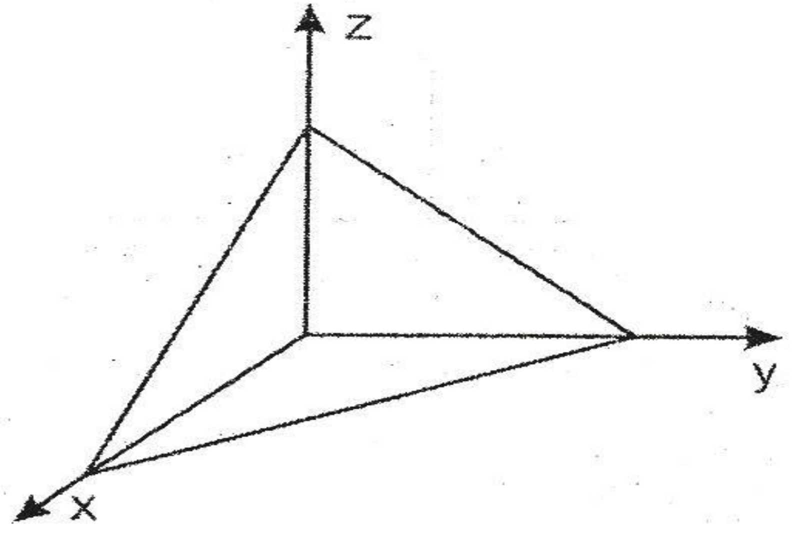
$$\therefore I = \int_0^{\pi/2}\int_0^{a} r^2 r\,dr\,d\theta = \int_0^{\pi/2}\left[\int_0^{a} r^3 dr\right]d\theta = \int_0^{\pi/4}\left[\frac{r^4}{4}\right]_0^{a} d\theta = \int_0^{\pi/4}\left[\frac{a^4}{4}\right]d\theta = \frac{a^4}{4}[\theta]_0^{\pi/2} = \frac{\pi a^4}{8}$$

VOLUME AS A TRIPLE INTEGRAL

Example 1 :

Find the volume of the tetrahedron bounded by the coordinate planes and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

Solution:



$$Volume\ required = \iiint dx\,dy\,dz$$

$$= \int_0^a \int_0^{b(1-\frac{x}{a})} \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} dz\,dy\,dx$$

$$= c\int_0^a \int_0^{b(1-\frac{x}{a})} (1-\frac{x}{a}-\frac{y}{b})$$

$$= \frac{bc}{2}\int_0^a (1-\frac{x}{a})^2\,dx$$

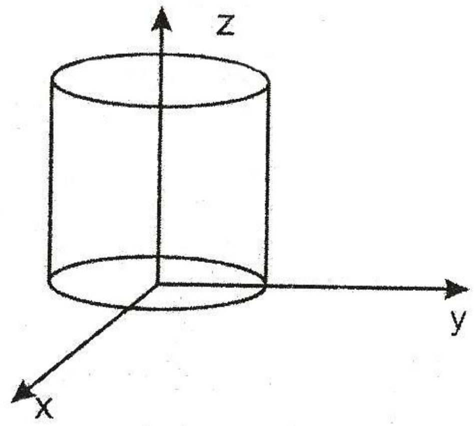
$$= \frac{abc}{6}$$

Example 2:

Find the volume bounded by the cylinder $x^2 + y^2 = 4$

and the planes y+z=4 and z=0

Solution:



Solution:

Z varies from z =0 to z= 4-y and x,y varies all over the points of the circle $x^2 + y^2 = 4$

Volume V = $$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{4-y} dzdydx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-y)dydx$$

$$= 8x2\int_{0}^{2}\sqrt{4-x^2}\,dx$$

$$V = 16[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}] = 16\pi$$

Example 3:

Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$

Solution:

Since the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is symmetric about the cordinate planes,

the volume of the ellipsoid $=8\times$ volume in the first octant.

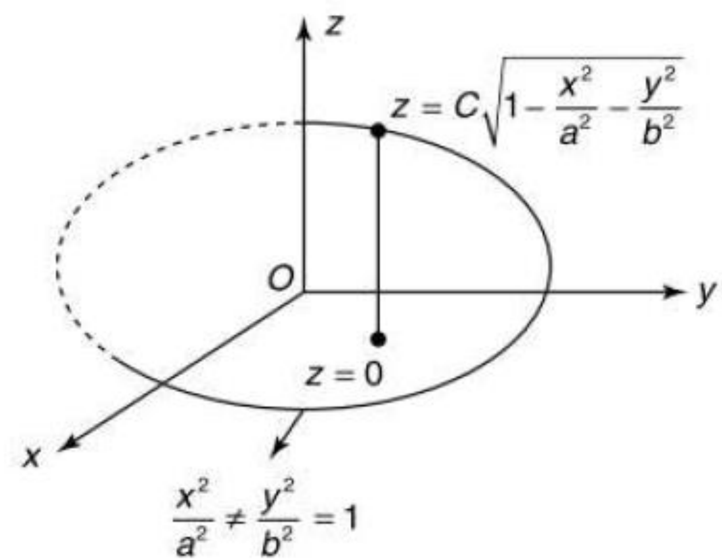
Volume of ellipsoid in the first octant is bounded by the planes,$x=0, y=0, z=0$

and the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

$$\Rightarrow \frac{z^2}{c^2}=1-\frac{x^2}{a^2}-\frac{y^2}{b^2}$$

$$\Rightarrow z^2=c^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)$$

$$\Rightarrow z=\pm c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$$



In the first octant z varies from $z=0$ to $z=c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$

The section of the ellipsoid by the xy plane $z=0$ is the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\Rightarrow y^2=b^2\left(1-\frac{x^2}{a^2}\right)\Rightarrow y=\pm b\sqrt{1-\frac{x^2}{a^2}}$$

y varies from 0 to $b\sqrt{1-\dfrac{x^2}{a^2}}$ and x varies from 0 to a.

$$\text{Volume } V = \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dzdydx = 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} [z]_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dydx$$

$$= 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \left[c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\right] dydx$$

$$= 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \left[\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}\right] dydx$$

$$= \frac{8c}{b}\int_0^a \left[\frac{y}{2}\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)-y^2} + \frac{b^2\left(1-\frac{x^2}{a^2}\right)}{2}\sin^{-1}\frac{y}{b\sqrt{1-\frac{x^2}{a^2}}}\right]_0^{b\sqrt{1-\frac{x^2}{a^2}}} dx$$

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$$= \frac{4c}{b}\int_0^a \left[0 + b^2\left(1 - \frac{x^2}{a^2}\right)\{\sin^{-1}1 - \sin^{-1}0\}\right]dx$$

$$= \frac{4c}{b}\int_0^a b^2\left(1 - \frac{x^2}{a^2}\right).\frac{\pi}{2}\,dx$$

$$= 2\pi bc\int_0^a \left(1 - \frac{x^2}{a^2}\right)dx$$

$$= 2\pi bc\left[x - \frac{1}{a^2}.\frac{x^3}{3}\right]_0^a$$

$$= 2\pi bc\left[a - \frac{1}{3a^2}.a^3 - 0\right]$$

$$= 2\pi bc\left[\frac{2a}{3}\right] = \frac{4\pi abc}{3}$$

Example 4:

Find the volume of the sphere $x^2 + y^2 + z^2 = a^2$ using the triple integrals.

Solution:

Since the sphere $x^2 + y^2 + z^2 = a^2$ is symmetric about the coordinate planes,

the volume of the sphere $= 8 \times$ volume of the first octant.

$$= 8 \times \iiint_V dxdydz$$

$$= 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dzdydx$$

$$= 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} [z]_0^{\sqrt{a^2-x^2-y^2}} dydx$$

$$= 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\, dydx \quad \left[\because \int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{2} \right]$$

$$= 8 \int_0^a \left[\frac{y}{2}\sqrt{(a^2 - x^2) - y^2} + \frac{a^2 - x^2}{2}\sin^{-1}\left(\frac{y}{\sqrt{a^2 - x^2}} \right) \right]_0^{\sqrt{a^2-x^2}} dx$$

$$= 2\pi \int_0^a (a^2 - x^2)dx = 2\pi \left[a^2 x - \frac{x^3}{3} \right] = \frac{4\pi}{3} a^3$$

PROBLEMS FOR PRACTICE

1. Find the volume bounded by $xy-$plane, the cylinder $x^2+y^2=1$ and the plane $x+y+z=3$.
2. Find the volume of the parabolid $x^2+y^2=4z$ cut off by $z=4$.
3. By changing into polar coordinates, evaluate the integral

$$\int_0^{2a}\int_0^{\sqrt{2ax-x^2}}(x^2+y^2)dydx$$

4. Evaluate $\int_0^{4a}\int_{y^2/4a}^{y}\frac{x^2-y^2}{x^2+y^2}dx\,dy$ by changing to polar coordinates.
5. Evaluate $\int_0^{a}\int_y^{a}\frac{x^2dxdy}{\sqrt{x^2+y^2}}$ by changing to polar coordinates.

THE END