Random Matrix & Spectral Analysis

1

Foundations

Motivation:
In complex systems (finance, neuroscience, climate, genetics), we often work with **large datasets**. A key object is the **correlation (or covariance) matrix**. But when data are noisy and sample sizes are limited, **separating meaningful signals from random fluctuations is challenging.**

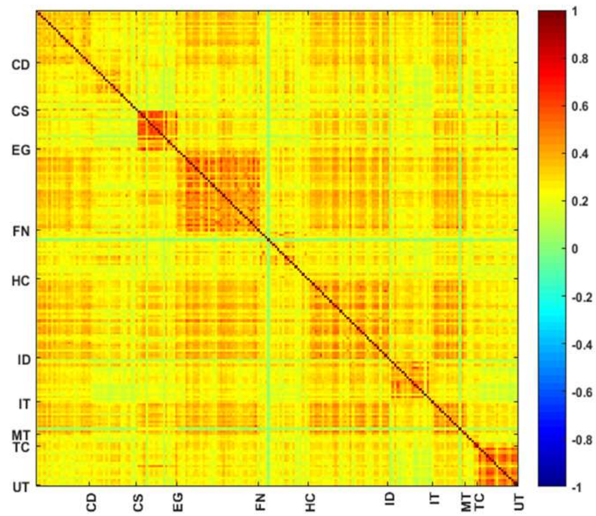
Random Matrix Theory (RMT) provides statistical laws for eigenvalues/eigenvectors of random matrices and helps us distinguish signal from noise.

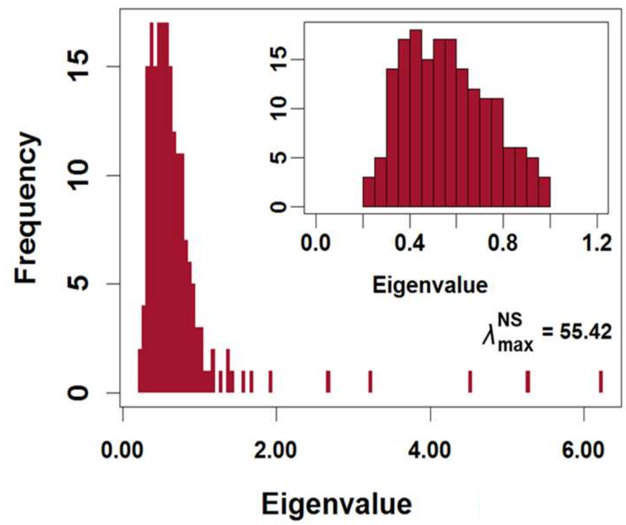
Linear Algebra Basics:
- **Eigenvalues (λ) and Eigenvectors (v)**: For a matrix A, solving $Av = \lambda v$.
- **Spectral Decomposition**: Any symmetric matrix can be decomposed into eigenvectors and eigenvalues.
- Eigenvalues capture the **variance explained** along directions defined by eigenvectors.

Long vs. Short Time Series:
- **Long time series**: $T \gg N$(samples >> variables). Covariance estimates are reliable.
- **Short time series**: $T \sim N$or $T < N$. Estimates are noisy, eigenvalue distribution is distorted, and spurious correlations appear.
- This motivates using RMT to filter noise.

2

Matrix and its Eigen spectra





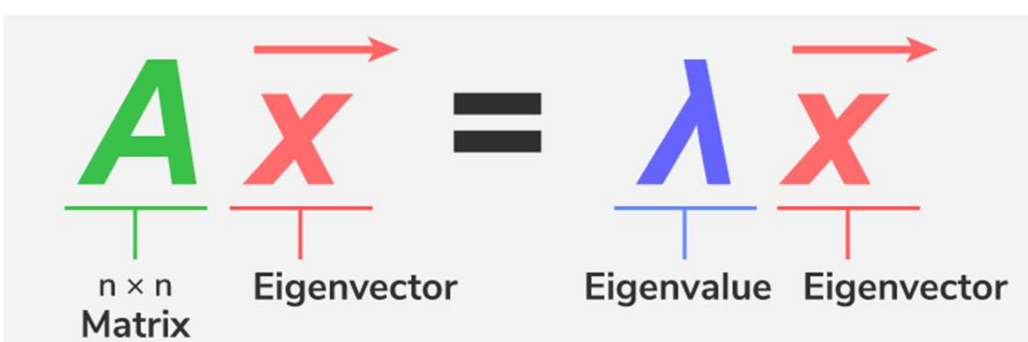
3

Linear Algebra Basics

Eigenvalues and Eigenvectors

The word 'Eigen' is of German Origin which means 'characteristic'.

Eigenvector Equation



https://www.geeksforgeeks.org/eigen-values/

4

Eigenvalue and Eigenvector

- Let A be an nxn matrix.
- A scalar λ is called an Eigenvalue of A if there is a non-zero vector X such that AX = λX. Such a vector X is called an Eigenvector of A corresponding to λ.
- Example: $\begin{bmatrix}2\\1\end{bmatrix}$ is an Eigenvector of A = $\begin{bmatrix}3&2\\3&-2\end{bmatrix}$ for λ = 4

$$\begin{pmatrix}3&2\\3&-2\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} \overset{?}{=} 4\begin{pmatrix}2\\1\end{pmatrix}$$

$$\begin{pmatrix}3\cdot2+2\cdot1\\3\cdot2+(-2)\cdot1\end{pmatrix} \overset{?}{=} \begin{pmatrix}8\\4\end{pmatrix}$$

$$\begin{pmatrix}8\\4\end{pmatrix} \overset{\checkmark}{=} \begin{pmatrix}8\\4\end{pmatrix}$$

5

Finding Eigenvalues and Eigenvectors

$$A = \begin{bmatrix}7&3\\3&-1\end{bmatrix}$$

① $\lambda I = \lambda\begin{bmatrix}1&0\\0&1\end{bmatrix} = \begin{bmatrix}\lambda&0\\0&\lambda\end{bmatrix}$

② $A - \lambda I = \begin{bmatrix}7&3\\3&-1\end{bmatrix} - \begin{bmatrix}\lambda&0\\0&\lambda\end{bmatrix}$

$= \begin{bmatrix}7-\lambda&3\\3&-1-\lambda\end{bmatrix}$

③ $\det\begin{bmatrix}7-\lambda&3\\3&-1-\lambda\end{bmatrix}$

$= (7-\lambda)(-1-\lambda) - (3)(3)$

$= -7-7\lambda+\lambda+\lambda^2-9$

$= \lambda^2-6\lambda-16$

(4) Solving for λ:
(λ – 8) (λ + 2) = 0
λ = 8 and λ = -2 are the Eigen values

(5) Consider A – λ I

⑤ $\begin{bmatrix}7-\lambda&3\\3&-1-\lambda\end{bmatrix}$

$\lambda = 8$:

$\begin{bmatrix}7-8&3\\3&-1-8\end{bmatrix} = \begin{bmatrix}-1&3\\3&-9\end{bmatrix}$ = B

Solve B X = 0

$$\begin{bmatrix}-1&3\\3&-9\end{bmatrix}\begin{bmatrix}X1\\X2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

-X1 + 3X2 = 0 ⟶ X1 = 3X2

3X1 – 9X2 = 0 ⟶ 3X1 = 9X2 ➔ X1 = 3X2

If X2 = 1;
X1 = 3

$\begin{bmatrix}3\\1\end{bmatrix}$ is an eigenvector for **λ = 8**

6

Finding Eigenvalues and Eigenvectors

For λ = -2

$$\begin{bmatrix} 7-(-2) & 3 \\ 3 & -1-(-2) \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$$

Solve B X = 0

$$\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} X1 \\ X2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$9X1 + 3X2 = 0 \longrightarrow X2 = -3X1$
$3X1 + X2 = 0 \longrightarrow X2 = -3X1$

If X1 = 1;
X2 = -3 $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ is an eigenvector for **λ = -2**

Verification $A = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$

$AX = \lambda X$

For λ = 8 and $X = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 24 \\ 8 \end{bmatrix} = 8\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

An *nxn* matrix has *n* eigenvalues and the corresponding eigenvectors.

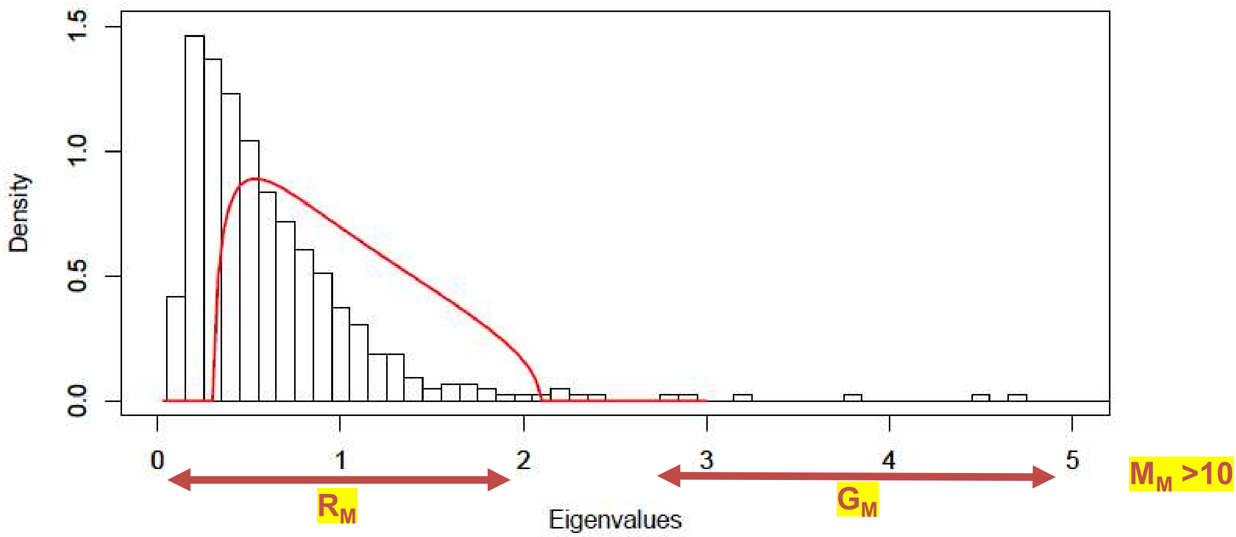
Spectral Analysis Basics

Spectrum of a Matrix (Eigen spectra): The set of eigenvalues of a matrix.
For a correlation/covariance matrix, it represents the **spread of variance** across different modes.

Spectral Analysis in Time Series:
By analyzing the spectrum of the covariance/correlation matrix of multivariate time series, we can uncover:
- Dominant modes (large eigenvalues → meaningful factors).
- Noise modes (small eigenvalues → random fluctuations).

Role of Eigenvalue Distribution:
- If eigenvalues follow RMT predictions (random noise), they may not carry useful information.
- Deviations (outliers beyond theoretical limits) often indicate **true correlations or structure**.

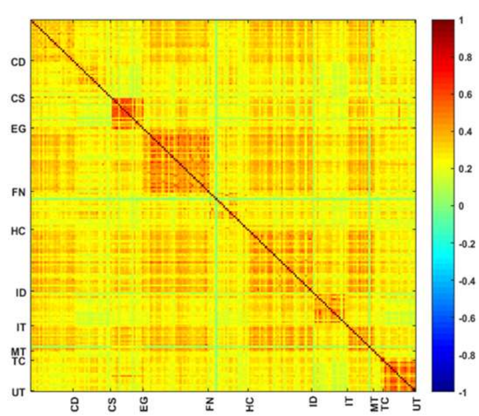
Eigenvalue spectra

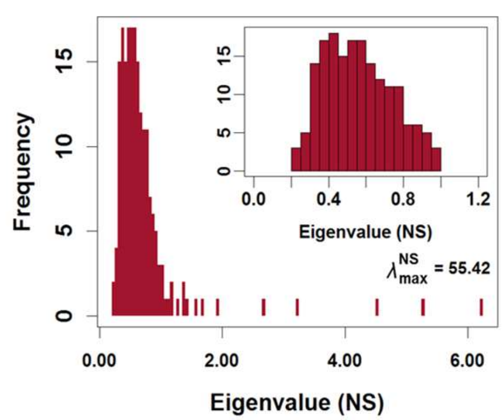


9

Eigenvalue Decomposition:

$$C = C^M + C^G + C^R,$$
$$= \lambda_1 a_1 a_1' + \sum_{i=2}^{N_G} \lambda_i a_i a_i' + \sum_{i=N_G+1}^{N} \lambda_i a_i a_i',$$





10

Random Matrices & Ensembles

Introduction to RMT:
Studies statistical properties of eigenvalues/eigenvectors when matrix entries are **random**.

Wishart Ensemble:
Constructed as $W = \frac{1}{T} X X^{\top}$,where X is an $N \times T$matrix with Gaussian entries. This models empirical covariance matrices.

Correlated Wishart Ensemble:

- Real-world data are not purely random.
- Extension to include **true correlations** between variables, embedded within random fluctuations.

11

Wishart Ensembles

Let us construct a large random matrix $\boldsymbol{B}$ arising from N random time series each of length T, where the entries of a time series are real independent random variables

drawn from a standard Gaussian distribution with zero mean and variance σ^2, such that the resulting matrix $\boldsymbol{B}$ is $N \times T$. Then the Wishart matrix can be constructed as

$$\boldsymbol{W} = \frac{1}{T} \boldsymbol{B} \boldsymbol{B}'.$$

Correlated Wishart ensemble:

$$\boldsymbol{W} = \frac{1}{T} \boldsymbol{G} \boldsymbol{G}',$$

where $\boldsymbol{G} = \zeta^{1/2} \boldsymbol{B}$, is a $N \times T$ matrix; $\boldsymbol{G}'$ is the $T \times N$ transpose matrix of $\boldsymbol{G}$, and the $N \times N$ positive definite symmetric matrix ζ controls the actual correlations and a fixed positive definite matrix.

12

Limiting Distributions:
Marchenko–Pastur (MP) Law:

The spectrum of eigenvalues for the Wishart orthogonal ensemble can be calculated analytically. For the limit $N \to \infty$ and $T \to \infty$, with $Q = T/N$ fixed (and bigger than unity), the probability density function of the eigenvalues is given by the Marčenko-Pastur distribution:

$$\bar{\rho}(\lambda) = \frac{Q}{2\pi\sigma^2} \frac{\sqrt{(\lambda_{max} - \lambda)(\lambda - \lambda_{min})}}{\lambda}, \qquad \text{(A)}$$

where σ^2 is the variance of the elements of $\boldsymbol{G}$, while λ_{min} and λ_{max} satisfy the relation:

$$\lambda_{min}^{max} = \sigma^2 \left(1 \pm \frac{1}{\sqrt{Q}}\right)^2. \qquad \text{(B)}$$

For $Q \leq 1$, positive semi-definite matrices $\boldsymbol{W}$, the density $\bar{\rho}(\lambda)$ in the above Eq. (A) normalized to Q and not to unity. Therefore, taking into account the $(N - T)$ zeros, we have

13

Role of $Q = T/N$:

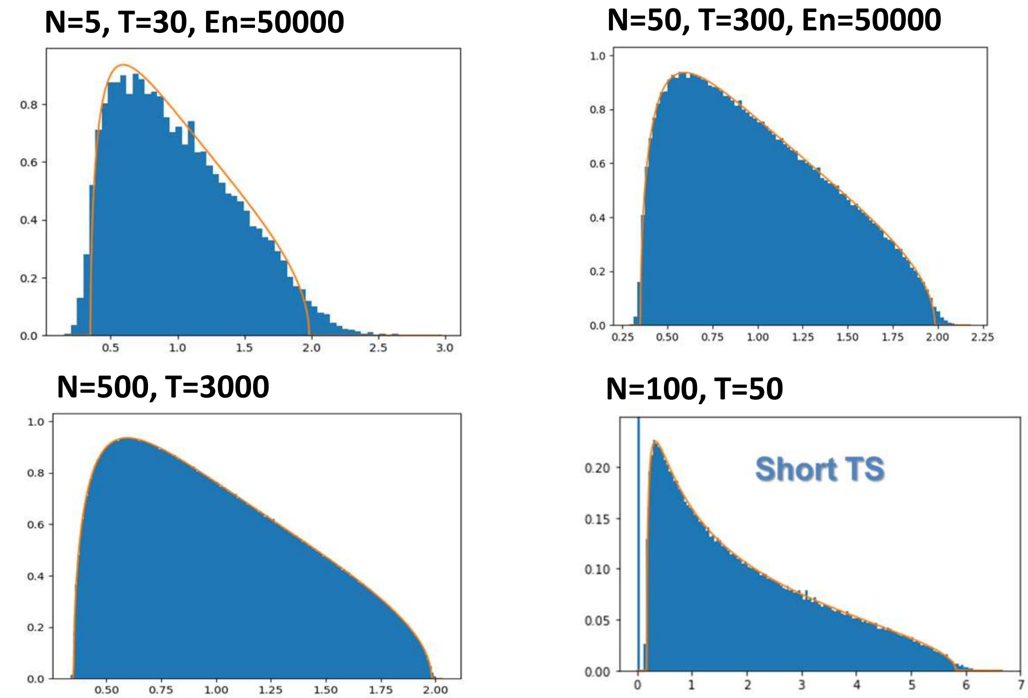
- If $Q \gg 1$: eigenvalue spectrum shrinks, estimates are stable and spectrum is **Marchenko–Pastur (MP).** **Long TS**
- If $Q \approx 1$: strong noise, many spurious correlations.
- If $Q < 1$: covariance matrix becomes singular (rank-deficient). **Short TS**

Effect of Finite Sample Length:

- Real datasets are finite, so the eigenvalue distribution deviates from the MP law.
- Larger deviations indicate possible true signals, but also finite-sample bias.

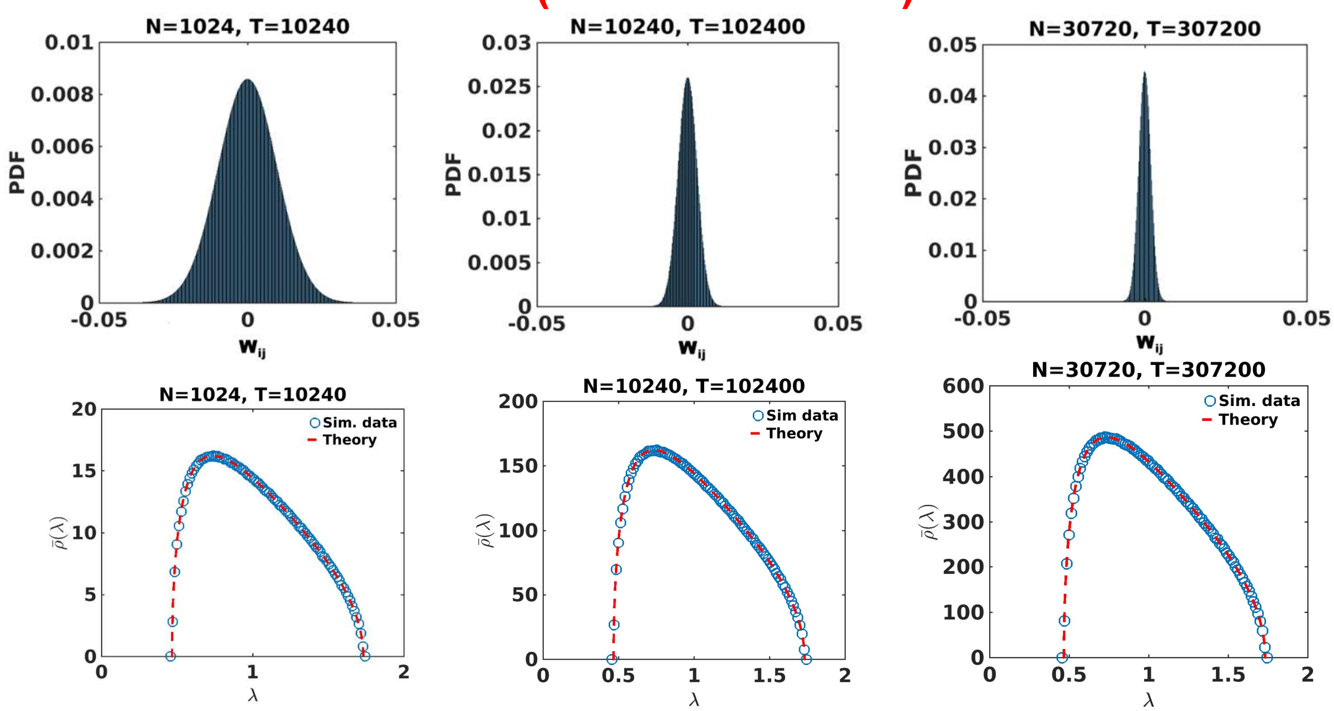
14

Distribution of finite-sized Wishart Matrix Ensemble



15

Effect of finite size (N ≠ Inf & T ≠ Inf)



16

Practical Issues in Time Series

- **Noise Effects:**

Empirical correlation matrices contain both genuine correlations and spurious ones due to **finite sample effects**.

  - Small eigenvalues may be dominated by noise.
  - Large eigenvalues need to be checked against RMT predictions.

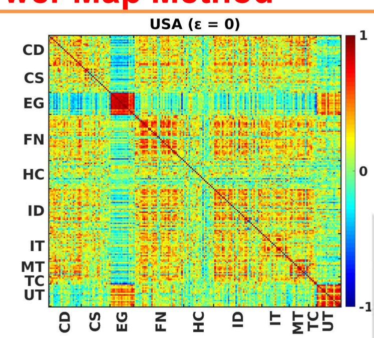
- **Noise Suppression Techniques:**
  - **Eigenvalue Filtering**: Keep only eigenvalues outside the MP bulk.
  - **Nonlinear element-wise transforms (power map / power-filter):** transform raw correlations elementwise

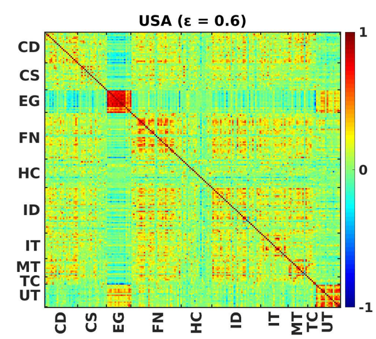
- **Signal vs. Noise Detection:**
  - Outlier eigenvalues beyond MP support often correspond to **meaningful latent factors** (e.g., market mode in finance).

17

Power Map Method

Noise Suppression Technique





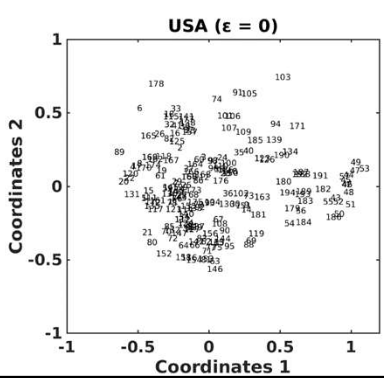
$$C_{kl}^{(q)} = \mathrm{sgn}\, C_{kl} \left|C_{kl}\right|^{q}$$

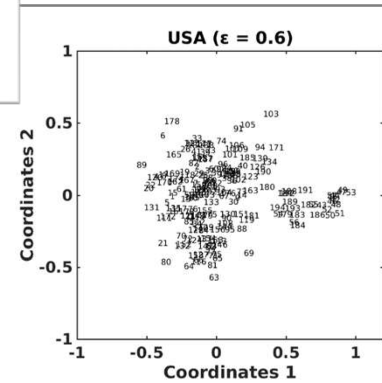
e.g.: $q=1+\varepsilon$, with $\varepsilon=0.01$

Transformation:

$(-0.2)^{1.01} = (-)(0.2)^{1.01}$

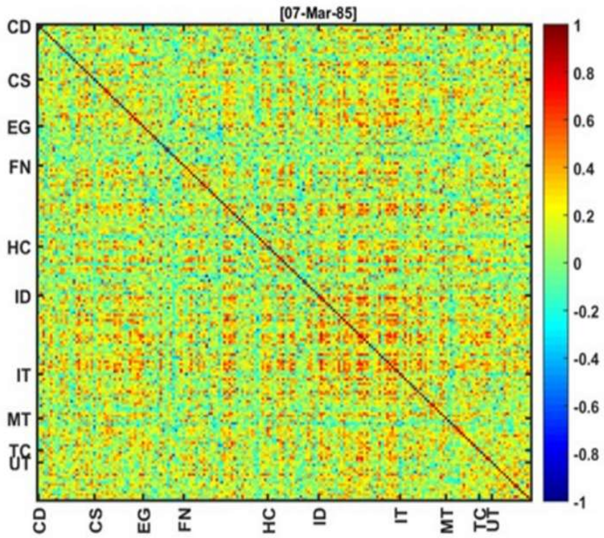
$= -0.1968$

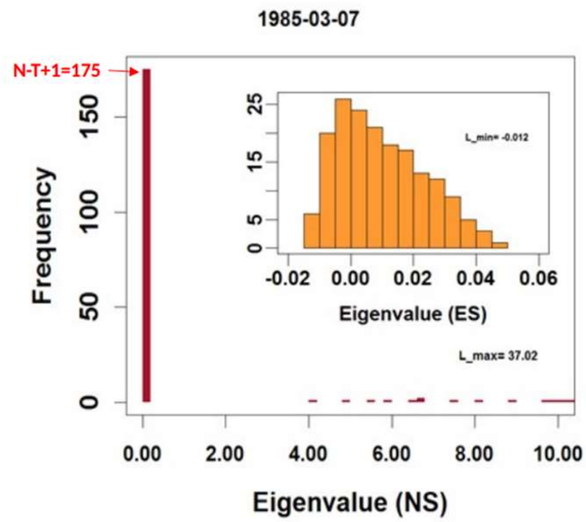




18

Non-critical Period

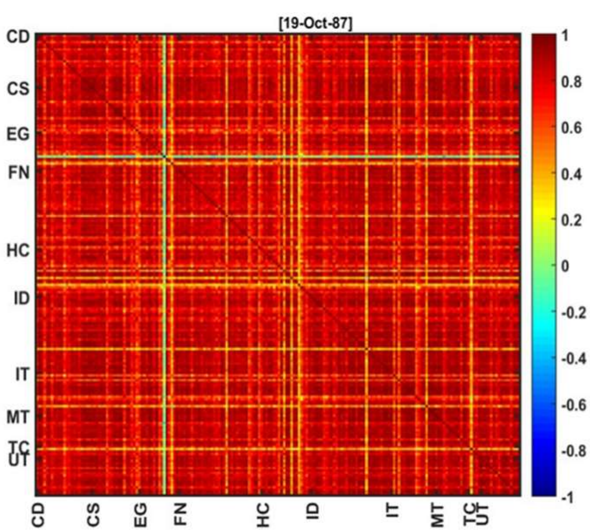


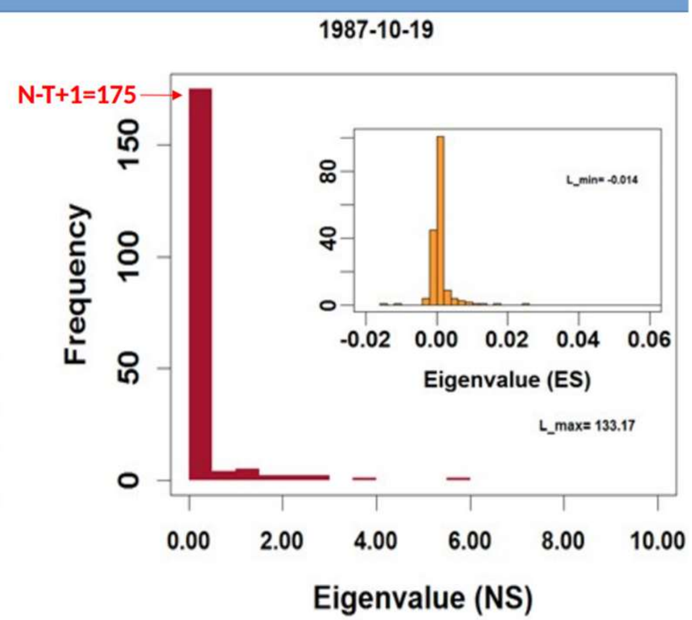


N= 194, T=20

19

Critical Period

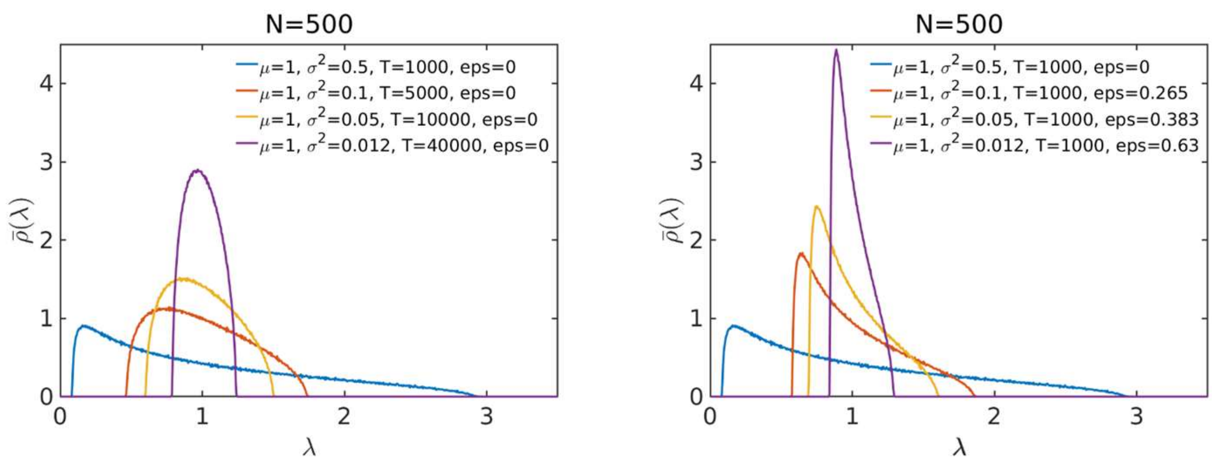




N = 194, T =20

20

Effect of Noise suppression



21

Advanced Applications

- **Powermap Transformation:**
  - A nonlinear transformation applied to eigenvalue spectra to **suppress noise.**
  - Useful in short time series where signal/noise separation is difficult.

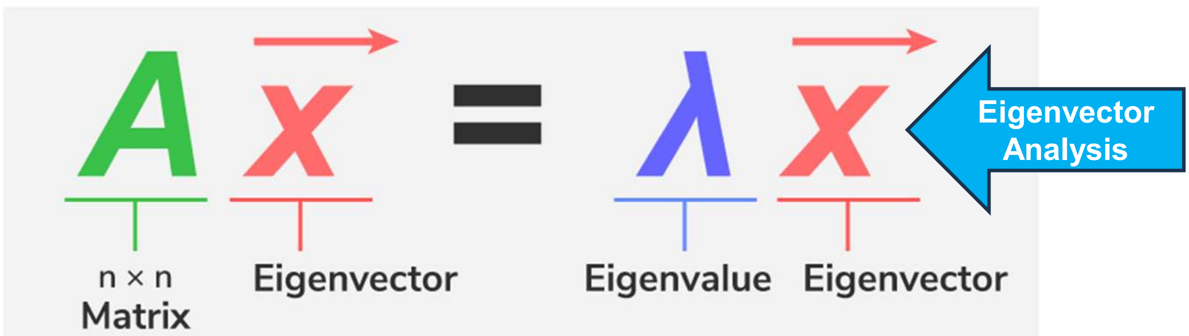
- **Applications:**
  - **Finance:** Distinguishing **market-wide factors** from random stock correlations.
  - **Climate Science:** **Identifying** significant modes in climate variability.
  - **Neuroscience:** **Detecting brain network connectivity** patterns from noisy fMRI/EEG data.

- **Empirical Spectra vs. RMT Predictions:**
  - Real-world eigenvalue distributions are compared to the MP law.
  - Deviations signal true correlations or model inadequacy.

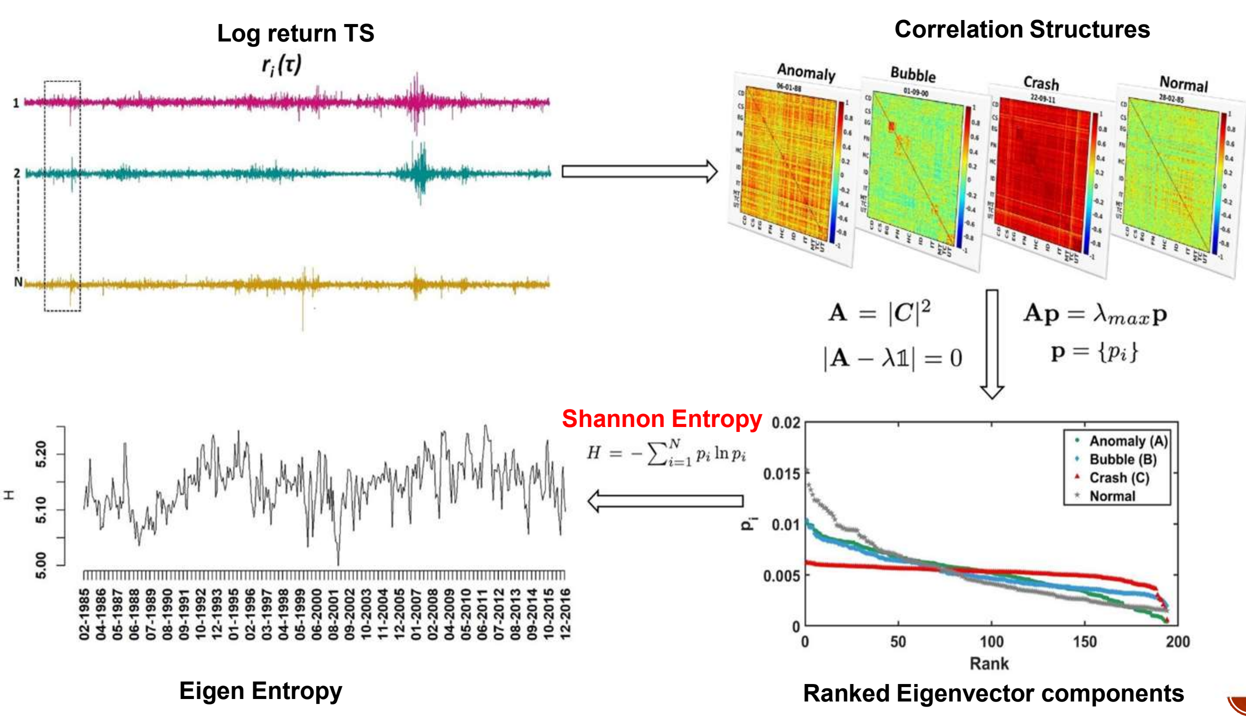
22

Eigenvalues and Eigenvectors

Eigenvector Equation



23



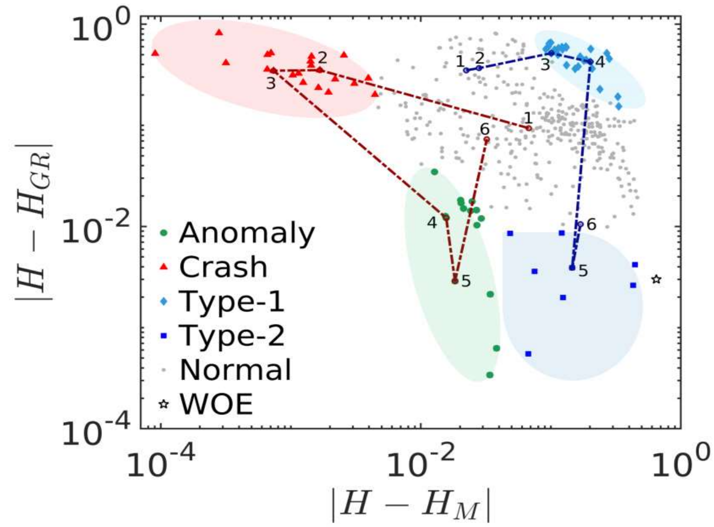
24

- Eigenvector centrality (p_i) is a measure of the **influence of a node** in a network. eg. Google's Page Rank.
- Measure: Normalized non-negative eigenvector components (by the Perron–Frobenius theorem), *i.e.,* corresponds to maximum eigenvalue.

Eigenvalue decomposition

C, C_M, C_G, C_{GR}

H, H_M, H_G, H_{GR}



25

Thank you

26