Homework 4
Amanda Di Nitto – 260689696
PHYS 512
Due: October 23rd 2020

Problem 1:

The plot below [figure 1] has two gaussians; the blue (i.e. original Guassian) and the orange (i.e. the shifted Gaussian). The original Gaussian was convolved with another Gaussian shifted by -10, meaning the gaussian peak was at -10. This resulted in a convolution of the gaussian with a peak at 10.

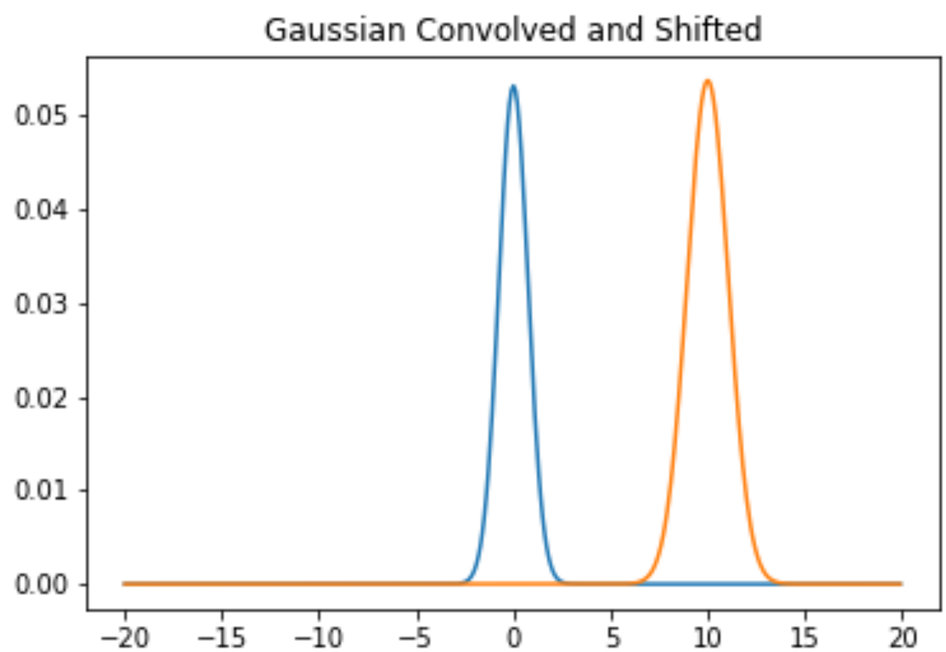


Figure 1: Shifted Gaussian via a Convolution

Problem 2 & 3:

The correlation of a gaussian was taken with itself for problem 2. The result is given in figure 2. For problem 3 the correlation from 2 was compared to a correlation for a shifted gaussian [figure 3]. As seen in the plot, the correlation operation is shift invariant. No matter how large or small the shift the graph looked very similar. It is not very surprising since the gaussian itself did not change only had a shift in the exponential value so taking the correlation should result in a very similar result.

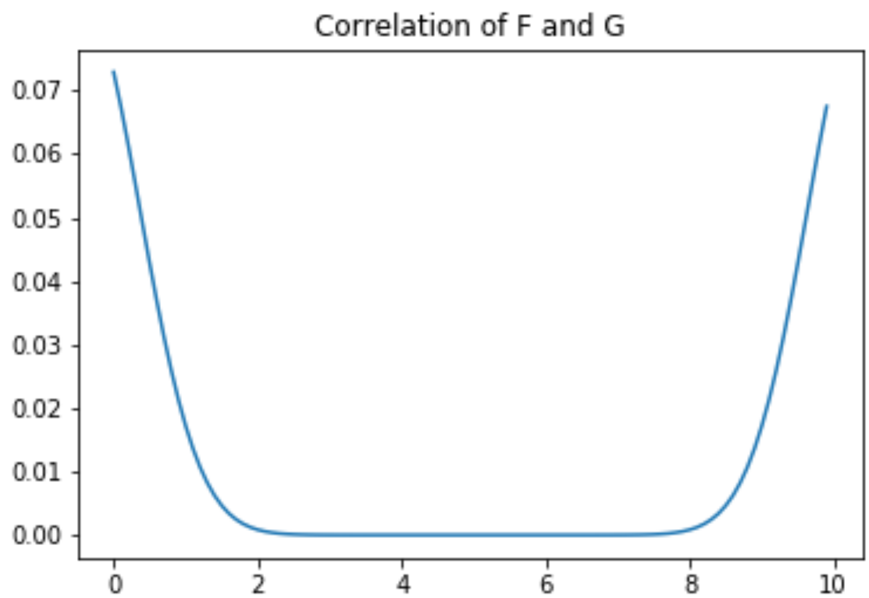


Figure 2(Above): Correlation of Gaussian with itself.

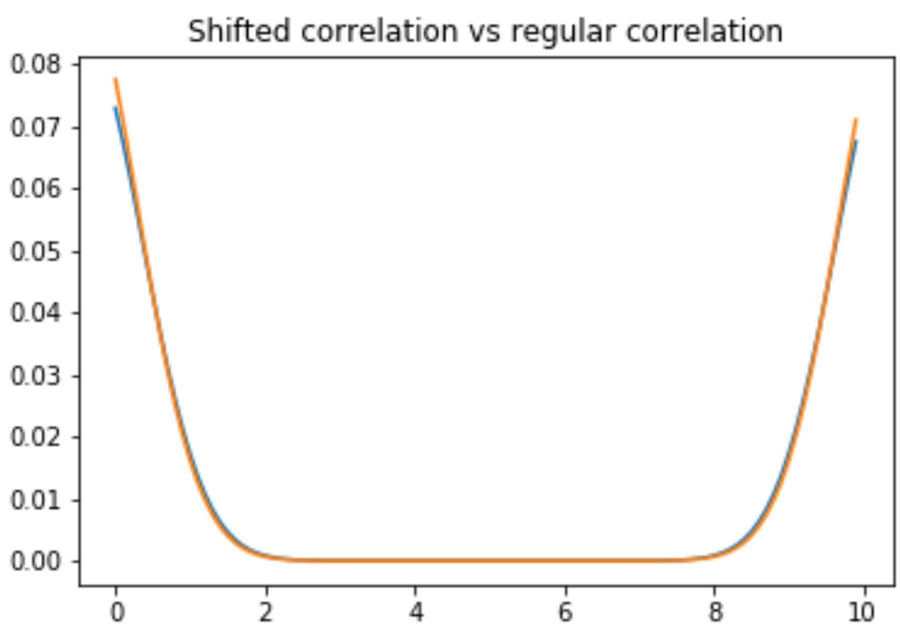


Figure 3(Below): Shifter Correlation Plotted along with Original Correlation

Problem 4:

The wrap around effect occurs due to the circular matrix in the DFT operation. In order to attempt to break the cyclical nature, zeros were substituted into the array for the final values. This will not change the nature of the function it will only change the end point values if the function itself is affected by this wrap around effect. The function chosen for the problem was a gaussian again [figure 4]. The correction was attempted in two different ways: by substituting zeros (orange dots) in at the end of the array, therefore same number of array points as original gaussian, and by adding zeros to the end of the array (black line). When substituting in zeros the plots for the original and the adjusted were identical. When adding the zeros, the array length changed therefore the guassian was shifted slightly compared to what was expected. This changes the peak even meaning it is not the gaussian that was meant to be plotted. Based on this, the way to adjust the function to avoid wrap around is via substitution rather than adding.

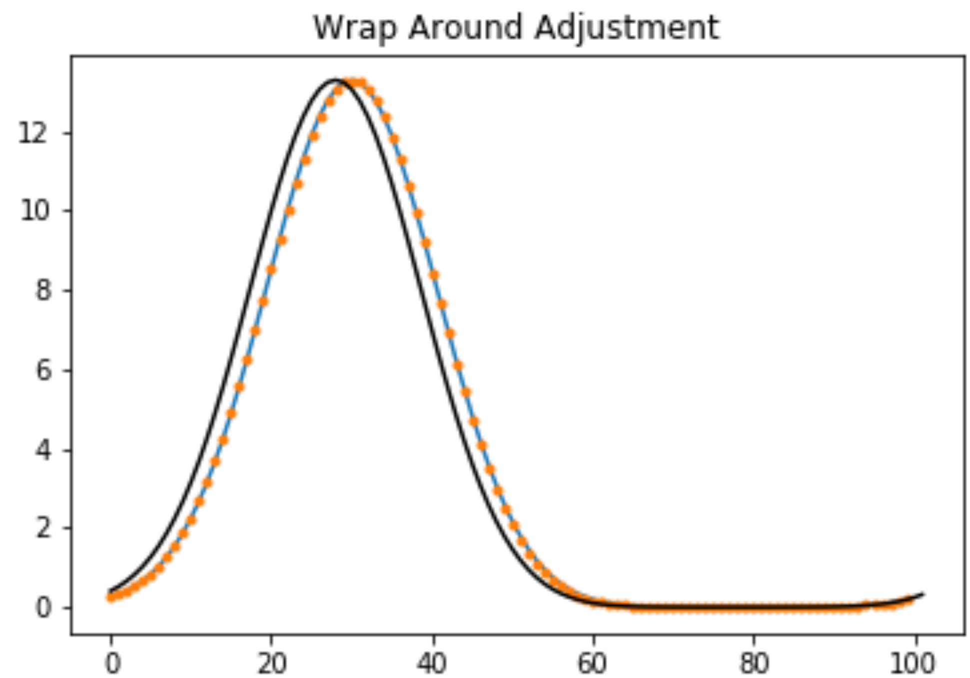


Figure 4: Wrap Around adjusted via substitution (orange) compared to attempted adjustment via addition (black)

Problem 5:

A) By using a geometric series expansion, the summation can be converted like so;

$$\sum \alpha^x = \frac{1-\alpha^N}{1-\alpha}$$

Therefore, by substituting the values of α back in the series becomes:

$$\frac{1-\exp(-2\pi ik)}{1-\exp\left(-\frac{2\pi ik}{N}\right)}$$

B) When *k* approaches zero this summation will go to *N*. The reason is based on the fact that:

$$\sum_{x=0}^{N-1} e^{-\frac{2\pi ikx}{N}} = \sum_{x=0}^{N-1} e^{0} = \sum_{x=0}^{N-1} 1 = N$$

If *k* is not an integer of *N* this will lead to zero. This is due to the fact that the numerator of the geometric expansion does not include an *N* value. This means that as long as *k* is an integer, the numerator becomes zero since the exponential terms will always be a multiple of 2π:

$$numerator = 1-\exp(-2\pi ik) = 1-\mathrm{e}^{0} = 1-1 = 0$$

Therefore, even though the exponential on the denominator will not go to zero due to the fractional value of π, the numerator will drive the whole thing towards zero.

C) In order to use the DFT equation from A we will need to solve for the transform of a sine function:

$$\sum_{x=0}^{N-1} \sin(ax)\, e^{-\frac{2\pi ikx}{N}} = \sum_{x=0}^{N-1} \frac{e^{iax}-e^{-iax}}{2i} e^{-\frac{2\pi ikx}{N}}$$

Using the geometric expansion, we get a final DFT equation of:

$$\frac{1}{2i}\left[\frac{1-e^{-i(2\pi k-aN)}}{1-e^{-i\left(\frac{2\pi k}{N}-a\right)}} - \frac{1-e^{-i(2\pi k+aN)}}{1-e^{-i\left(\frac{2\pi k}{N}+a\right)}}\right]$$

Using $a = 2\pi/T$ where *T* is the period chosen to make k non-integer, the final plot [figure 5] ends up with a delta function that is clearly not perfect. There is an obvious curve at the bottom of the spike which would not be present in a regular delta function. The DFT was taken using numpy as well and the difference between the two is $6.410861857473332x10^{-14}$ which means it is accurate to withing machine precision. In the plot the blue points is the DFT solution and the orange line is the numpy FFT. Since they were identical as both line plots, the points highlights the difference seen between the DFT solution and a real delta function. There are points off of the horizontal axis which is not expected for a delta function.

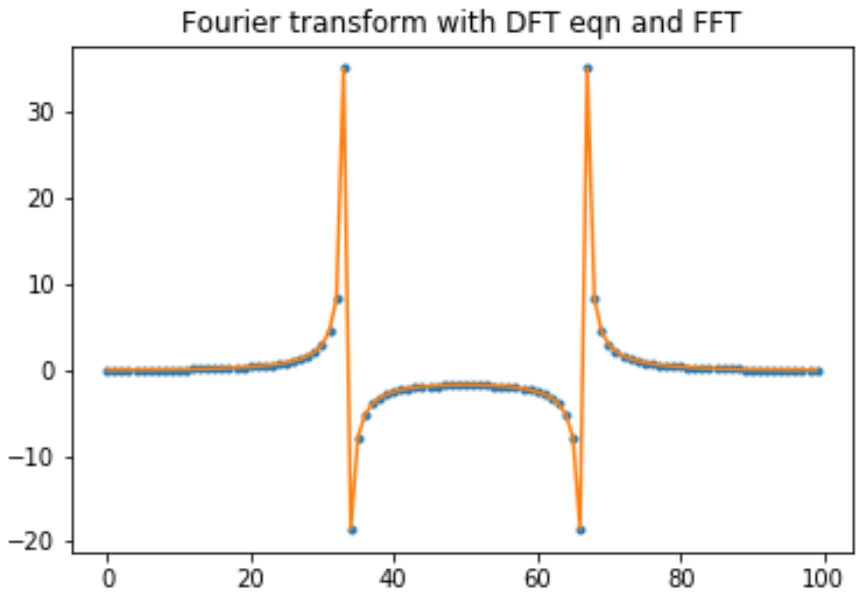


Figure 5: The DFT using analytic equation compared to numpy's FFT

D) In figure 6, the blue represents the un-windowed transform and the orange the windowed. As seen with the point plot again, the blue exhibits a bit of a curve when starting its cruve (due the the spectral leakage) but the orange points have a much cleaner peak, a lot more similar to that of a delta function; There are very few points that stray from the zero x-axis.

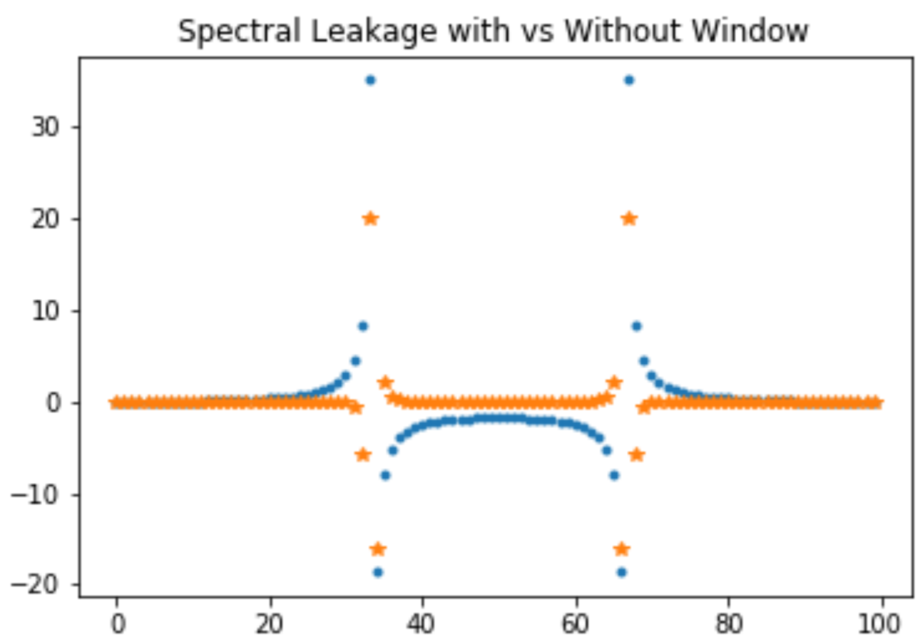
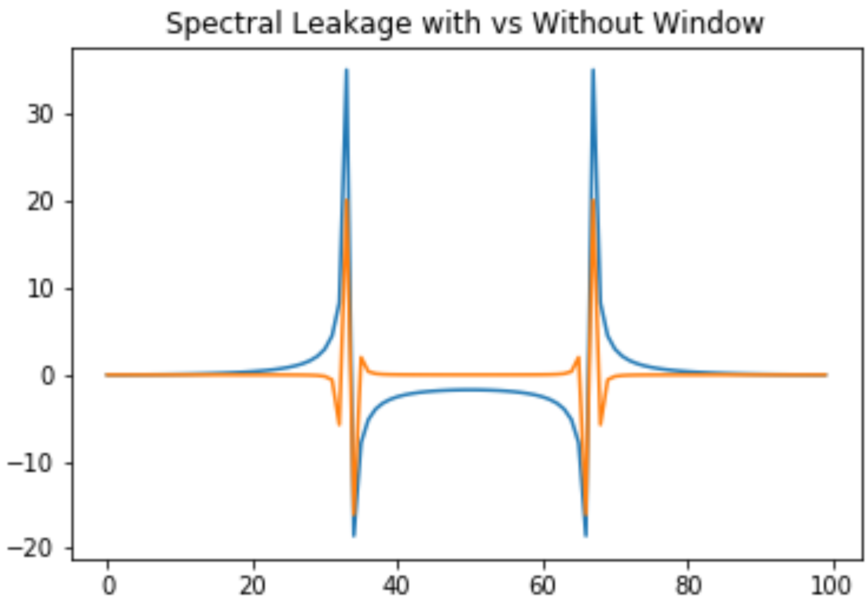


Figure 6: Spectral leakage improvement from un-windowed (blue) to windowed (orange) transform. (Above) in points, (Below) as a line.



E) For this problem I chose to do it numerically. The Fourier transform of the window was taken for $N = 10$ so as to see the relationship for a much shorter set of data. The transform results were: 5, -2.5, 0, 0, 0, -9.02056208e-17, 0, -3.12250226e-16, 0, -2.5. This is exactly N/2, -N/4, 0,…, 0, -N/4. The values for the un-windowed transform are below:

[9.99200722e-16, 6.31757018e-02, 3.69837000e-01, 2.96791321, -2.53490051, -1.73205081, -2.53490051, 2.96791321, 3.69837000e-01, 6.31757018e-02]

By quite a bit of testing the terms were obtained relatively close. It was clear the the N/4 terms were already present as -2.53490051 so now all that was required was the N/2 and zeros. The first term is already small enough to considered zero leaving the rest to create the combination for N/2. The signs, luckily were all the right ones, and by summing the remaining terms N/2 was calculated as 5.069801013599999 leaving zeros for the other terms. Obviously these are slightly off still, but there were only 10 terms calculated so it is highly possible for a higher N that a more precise array of the windowed terms could have been obtained.