Homework 1

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Matrix multiplication (5 points): Practice matrix multiplication by hand for the following problems:

P1: $\begin{bmatrix} -3 & 5 \\ 7 & -10 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, P2: $\begin{bmatrix} 4 & 5 & 1 \\ 3 & 7 & 10 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Problem 1

$$\begin{bmatrix} -3 & 5 \\ 7 & -10 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} (-3 * -1) + (5 * 3) \\ (7 * -1) + (-10 * 3) \end{bmatrix} = \begin{bmatrix} 18 \\ -37 \end{bmatrix}$$

Problem 2

$$\begin{bmatrix} 4 & 5 & 1 \\ 3 & 7 & 10 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} (4 * 1) + (5 * 2) + (1 * 3) \\ (3 * 1) + (7 * 2) + (10 * 3) \\ (1 * 1) + (0 * 2) + (1 * 3) \end{bmatrix} = \begin{bmatrix} 17 \\ 47 \\ 4 \end{bmatrix}$$

Orthogonality (10 points): Consider the vectors $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $y = \begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix}$, $z = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

P3: Are any of these vectors orthogonal to each other?

P4: Find a non-zero orthogonal vector for each of these vectors without using any of the other vectors.

Problem 3

Note: If two vectors are orthogonal, then the inner product is equal to zero

We know that you can only take the inner product of two vectors that have the same dimension. Therefore, only the first and third vectors have the possibility of being orthogonal.

$$< x, z >= x^T z = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = (1 * 1) + (1 * -1) = 1 - 1 = 0$$

$\boxed{\text{Therefore, the vectors } x \text{ and } z \text{ are orthogonal.}}$

Problem 4

Vector x

Find an orthogonal vector for x:

$$x^T\omega = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \omega_1 + \omega_2 = 0$$

$$\omega_1 = -\omega_2$$

$$\boxed{x_{orthogonal} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}}$$

Vector y

Find and orthogonal vector for y:

$$y^T\omega = \begin{bmatrix} -3 & 2 & 4 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = -3\omega_1 + 2\omega_2 + 4\omega_3 = 0$$

Pick random values for ω_1 and ω_2 and solve for ω_3

$$-3 * (1) + 2 * (1) + 4 * \omega_3 = -1 + 4\omega_3$$

$$\omega_3 = 1/4$$

$$\boxed{y_{orthogonal} = \begin{bmatrix} 1 \\ 1 \\ 1/4 \end{bmatrix}}$$

Vector z

Find an orthogonal vector for z:

$$z^T\omega = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \omega_1 - \omega_2 = 0$$

$$\omega_1 = \omega_2$$

$$\boxed{z_{orthogonal} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}}$$

Span and basis set (10 points): Consider the vectors $x_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, x_2 = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}, x_3 = \begin{bmatrix} 2 \\ 0 \\ 9 \end{bmatrix}$

P5: Express a basis set of unit vectors for the subspace defined by $span\{x_1, x_2, x_3\}$

Problem 5

Note: In order to be a basis, the vectors need to be independent and cover the entire span of the set.

We can show that all three vectors do not form a basis because they are not independent. There are non-zero real numbers that you can multiply by each vector to obtain the zero vector when all added together. For example:

$$-9\begin{bmatrix}1\\0\\1\end{bmatrix}+\frac{7}{4}\begin{bmatrix}4\\0\\0\end{bmatrix}+1\begin{bmatrix}2\\0\\9\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

This also makes sense intuitively because each vector has a zero for the second element. Therefore, you can think of them all as being 2 dimensional which means the basis cannot have more than 2 vectors. By looking at a set of just 2 vectors, we can see that a set of any two vectors are independent. For example, no multiple of x_1 will equal x_2 or x_3. The same is true for the other vectors. Therefore, we know that choosing any 2 vectors will satisfy the requirement for each vector in the basis to be independent. We also know that we will need at least 2 vectors to completely define a basis.

From inspection, a good choice of vector would be x_2 because any multiple of it will only effect the first term when added. The vector x_1 is also convenient because it contains only ones. We know that they are independent, so we just need to check that they have the same rank as the full set. You can do this by combining the vectors into a matrix and putting it into reduced row-echelon form.

$$S_1 = \{x_1, x_2\}$$

$$S_2 = \{x_1, x_2, x_3\}$$

$$S_1 = \begin{bmatrix}1 & 4\\0 & 0\\1 & 0\end{bmatrix},\ \mathrm{rref}(S_1) = \begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}$$

$$S_2 = \begin{bmatrix}1 & 4 & 2\\0 & 0 & 0\\1 & 0 & 9\end{bmatrix},\ \mathrm{rref}(S_2) = \begin{bmatrix}1 & 0 & 9\\0 & 1 & -1.75\\0 & 0 & 0\end{bmatrix}$$

We can see both sets S_1 and S_2 have the rank of 2. Because both sets contain the same vectors, we can conclude that x_1 and x_2 form a valid basis for the set.

$$\boxed{B = \{x_1, x_2\} \subseteq \{x_1, x_2, x_3\}}$$

Null and Image subspaces (20 points): Consider the matrices $A_1 = \begin{bmatrix}2 & 3 & 5\\-4 & 2 & 3\end{bmatrix}$, $A_2 = \begin{bmatrix}1 & 0 & 1\\5 & 2 & 1\\1 & 2 & 2\end{bmatrix}$, $A_3 = \begin{bmatrix}2 & 1 & 1\\1 & 1 & 0\\1 & 0 & 1\end{bmatrix}$

P6: Find the null space for each of the matrices

P7: Find the Image of each of the matrices

P8: For A_2 and A_3, show that the image space is the orthogonal complement of the null space

Problem 6

Matrix A1

$$A_1 x = \begin{bmatrix} 2 & 3 & 5 \\ -4 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 + 5x_3 \\ -4x_1 + 2x_2 + 3x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2x_1 + 3x_2 + 5x_3 = 0$$

$$-4x_1 + 2x_2 + 3x_3 = 0$$

$$x_1 = \frac{2x_2 + 3x_3}{4}$$

$$2\left(\frac{2x_2 + 3x_3}{4}\right) + 3x_2 + 5x_3 = 0$$

$$x_2 + \frac{3}{2}x_3 + 3x_2 + \frac{10}{2}x_3 = 0$$

$$4x_2 + \frac{13}{2}x_3 = 0$$

$$\underline{x_2 = \frac{-13}{8}x_3}$$

$$x_1 = \frac{\left(2 * \frac{-13}{8}x_3 + 3x_3\right)}{4}$$

$$\underline{x_1 = \frac{-1}{16}x_3}$$

$$\boxed{\text{null}(A_1) = \text{span}\left\{\begin{bmatrix} -1/16 \\ -13/8 \\ 1 \end{bmatrix}\right\} = \left\{\begin{bmatrix} \frac{-1}{16}\beta \\ \frac{-13}{8}\beta \\ \beta \end{bmatrix}\right\}}$$

Matrix A2

$$A_2 x = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_3 \\ 5x_1 + 2x_2 + x_3 \\ x_1 + 2x_2 + 2x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 = -x_3$$

$$5(-x_3) + 2x_2 + x_3 = -4x_3 + 2x_2 = 0$$

$$x_2 = 2x_3$$

$$-x_3 + 2(2x_3) + 2x_3 = 5x_3 = 0$$

$$x_3 = 0, x_2 = 0, x_1 = 0$$

$$\boxed{\text{null}(A_2) = \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\}}$$

Note: the zero vector is not included in the rank of the null space, so rank(null(A_2)) =0

Matrix A3

$$A_3x = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2 + x_3 \\ x_1 + x_2 \\ x_1 + x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 = x_2 = x_3$$

$$2x_1 - x_1 - x_1 = 0$$

$$\boxed{\text{null}(A_3) = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} -\beta \\ \beta \\ \beta \end{bmatrix} \right\}}$$

Problem 7

Matrix A1

$$\text{rref}(A_1) = \begin{bmatrix} 1 & 3/2 & 5/2 \\ -4 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 3/2 & 5/2 \\ 0 & 1 & 13/8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1/16 \\ 0 & 1 & 13/8 \end{bmatrix}$$

$$\boxed{\text{Im}(A_1) = \text{span}\left\{ \begin{bmatrix} 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}}$$

Or, you can do something like this:

$$A_1x = \begin{bmatrix} 2 & 3 & 5 \\ -4 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 + 5x_3 \\ -4x_1 + 2x_2 + 3x_3 \end{bmatrix}$$

$$\alpha_1 \begin{bmatrix} 2 \\ -4 \end{bmatrix} + \alpha_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\boxed{\text{Im}(A_1) = \left\{ \begin{bmatrix} 2\alpha_1 + 3\alpha_2 + 5\alpha_3 \\ -4\alpha_1 + 2\alpha_2 + 3\alpha_3 \end{bmatrix} \right\}}$$

Matrix A2

$$\text{rref}(A_2) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -4 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{\text{Im}(A_2) = \text{span}\left\{ \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} \alpha_1 + \alpha_3 \\ 5\alpha_1 + 2\alpha_2 + \alpha_3 \\ \alpha_1 + 2\alpha_2 + 2\alpha_3 \end{bmatrix} \right\}}$$

Matrix A3

$$\text{rref}(A_3) = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{\text{Im}(A_3) = \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 2\alpha_1 + \alpha_2 + \alpha_3 \\ \alpha_1 + \alpha_2 \\ \alpha_1 + \alpha_3 \end{bmatrix} \right\}}$$

Problem 8

The orthogonal complement is a matrix in which you can multiply every column in the orthogonal complement by that in the original matrix and get the zero vector. Essentially, every vector in the orthogonal complement is perpendicular to every vector in the original matrix and vice versa. I am going to prove this for A_2 and A_3 by showing that you can take the inner product of every vector in the nullspace with every vector in the image space and it will result in the zero vector.

Matrix A2

$$G = \text{Im}(A_2) = \left\{ \begin{bmatrix} \alpha_1 + \alpha_3 \\ 5\alpha_1 + 2\alpha_2 + \alpha_3 \\ \alpha_1 + 2\alpha_2 + 2\alpha_3 \end{bmatrix} \right\}, \quad H = \text{null}(A_2) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$\boxed{G^T H = \begin{bmatrix} \alpha_1 + \alpha_3 & 5\alpha_1 + 2\alpha_2 + \alpha_3 & \alpha_1 + 2\alpha_2 + 2\alpha_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$

Because the null space is only the zero vector, anything multiplied by it in the image space will always equal zero. Therefore, they are orthogonal complements.

Matrix A3

$$G = \text{Im}(A_3) = \left\{ \begin{bmatrix} 2\alpha_1 + \alpha_2 + \alpha_3 \\ \alpha_1 + \alpha_2 \\ \alpha_1 + \alpha_3 \end{bmatrix} \right\}, \quad H = \text{null}(A_3) = \left\{ \begin{bmatrix} -\beta \\ \beta \\ \beta \end{bmatrix} \right\}$$

$$G^T H = \begin{bmatrix} 2\alpha_1 + \alpha_2 + \alpha_3 & \alpha_1 + \alpha_2 & \alpha_1 + \alpha_3 \end{bmatrix} \begin{bmatrix} -\beta \\ \beta \\ \beta \end{bmatrix} = -\beta(2\alpha_1 + \alpha_2 + \alpha_3) + \beta(\alpha_1 + \alpha_2) + \beta(\alpha_1 + \alpha_3)$$

$$\boxed{-\beta(2\alpha_1 + \alpha_2 + \alpha_3) + \beta(\alpha_1 + \alpha_2) + \beta(\alpha_1 + \alpha_3) = -2\beta\alpha_1 - \beta\alpha_2 - \beta\alpha_3 + \beta\alpha_1 + \beta\alpha_2 + \beta\alpha_1 + \beta\alpha_3 = 0}$$

Every term cancels out, which means no matter what values of β or α are used, the image space and the null space of A_3 will always be orthogonal complements.

Eigen vectors and values (10 points): Consider A_2 thruough A_6 below

$$A_2 = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}, A_3 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A_4 = \begin{bmatrix} 4 & 5 & 1 \\ 3 & 7 & 10 \\ 1 & 0 & 1 \end{bmatrix} A_5 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix} A_6 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 5 & 10 \\ 0 & 0 & 8 \end{bmatrix}$$

P9: Find the eigenvalues and eigenvectors

P10: Which (if any) of the matrices are full rank?

P11: What pattern do you notice for matrices A_5 and A_6?

Problem 9

Matrix A2

Using a calculator/Matlab:

$$\boxed{\lambda_1 = 4.3797, v_1 = \begin{bmatrix} -0.2007 \\ -0.7068 \\ -0.6784 \end{bmatrix}}$$

$$\boxed{\lambda_2 = 0.3102 + 1.4789i, v_2 = \begin{bmatrix} 0.1786 - 0.3109i \\ -0.7275 \\ 0.3366 + 0.4785i \end{bmatrix}}$$

$$\boxed{\lambda_3 = 0.3102 - 1.4789i, v_3 = \begin{bmatrix} 0.1786 + 0.3109i \\ -0.7275 \\ 0.3366 - 0.4785i \end{bmatrix}}$$

Matrix A3

$$\det(\lambda I - A_3) = \det\left(\begin{bmatrix} \lambda - 2 & 1 & 1 \\ 1 & \lambda - 1 & 0 \\ 1 & 0 & \lambda - 1 \end{bmatrix}\right) = (\lambda - 1)^2(\lambda - 2) - (\lambda - 1) + (0 - (\lambda - 1)) = 0$$

$$(\lambda - 1)^2(\lambda - 2) - 2(\lambda - 1) = (\lambda - 1)\left[(\lambda - 1)(\lambda - 2) - 2\right]$$

$$(\lambda - 1)\left[\lambda^2 - 3\lambda\right] = \lambda(\lambda - 1)(\lambda - 3)$$

$$\underline{\lambda_1 = 0,\ \lambda_2 = 1,\ \lambda_3 = 3}$$

$$\begin{bmatrix} 2 - \lambda & 1 & 1 \\ 1 & 1 - \lambda & 0 \\ 1 & 0 & 1 - \lambda \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2v_{11} + v_{12} + v_{13} \\ v_{11} + v_{12} \\ v_{11} + v_{13} \end{bmatrix}$$

$$\underline{v_{11} = -v_{12} = -v_{13}}$$

$$\begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} = \begin{bmatrix} v_{21}+v_{22}+v_{23} \\ v_{21} \\ v_{21} \end{bmatrix}$$

$$\underline{v_{21} = 0, \quad v_{22} = -v_{23}}$$

$$\begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} v_{31} \\ v_{32} \\ v_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} v_{31} \\ v_{32} \\ v_{33} \end{bmatrix} = \begin{bmatrix} -v_{31}+v_{32}+v_{33} \\ v_{31}-2v_{32} \\ v_{31}-2v_{33} \end{bmatrix}$$

$$\underline{v_{11} = 2v_{12} = 2v_{13}}$$

$$\lambda_1 = 0, v_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}; \quad \lambda_2 = 1, v_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}; \quad \lambda_3 = 3, v_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

normalized eigenvectors:

$$\boxed{\lambda_1 = 0, v_1 = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}; \quad \lambda_2 = 1, v_2 = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}; \quad \lambda_3 = 3, v_3 = \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}}$$

Matrix A4

Using a calculator/matlab:

$$\boxed{\lambda_1 = 10.2942, v_1 = \begin{bmatrix} 0.6271 \\ 0.7760 \\ 0.0675 \end{bmatrix}}$$

$$\boxed{\lambda_2 = 0.8529 + 2.1708i, v_2 = \begin{bmatrix} -0.7301 \\ 0.4550 - 0.3839i \\ 0.0227 + 0.3348i \end{bmatrix}}$$

$$\boxed{\lambda_3 = 0.8529 - 2.1708i, v_3 = \begin{bmatrix} -0.7301 \\ 0.4550 + 0.3839i \\ 0.0227 - 0.3348i \end{bmatrix}}$$

Matrix A5

$$\det(\lambda I - A_5) = \det\left(\begin{bmatrix} \lambda-4 & 0 & 0 \\ 0 & \lambda-7 & 0 \\ 0 & 0 & \lambda-1 \end{bmatrix}\right) = (\lambda-7)(\lambda-1)(\lambda-4) = 0$$

$$\underline{\lambda_1 = 1, \ \lambda_2 = 4, \ \lambda_3 = 7}$$

$$\begin{bmatrix} 4-\lambda_1 & 0 & 0 \\ 0 & 7-\lambda_1 & 0 \\ 0 & 0 & 1-\lambda_1 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{v_{11} = 0, \ v_{12} = 0, v_{13} = \text{anything}}$$

$$\begin{bmatrix} 4-\lambda_2 & 0 & 0 \\ 0 & 7-\lambda_2 & 0 \\ 0 & 0 & 1-\lambda_2 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{v_{21} = \text{anything},\ \ v_{22} = 0, v_{23} = 0}$$

$$\begin{bmatrix} 4-\lambda_3 & 0 & 0 \\ 0 & 7-\lambda_3 & 0 \\ 0 & 0 & 1-\lambda_3 \end{bmatrix} \begin{bmatrix} v_{31} \\ v_{32} \\ v_{33} \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -6 \end{bmatrix} \begin{bmatrix} v_{31} \\ v_{32} \\ v_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{v_{31} = 0,\ \ v_{32} = \text{anything}, v_{33} = 0}$$

$$\boxed{\lambda_1 = 1, v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix};\quad \lambda_2 = 4, v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix};\quad \lambda_3 = 7, v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$$

Matrix A6

$$\det(\lambda I - A_6) = \det\left(\begin{bmatrix} \lambda-2 & 1 & 1 \\ 0 & \lambda-5 & 10 \\ 0 & 0 & \lambda-8 \end{bmatrix}\right) = (\lambda-5)(\lambda-8)(\lambda-2) = 0$$

$$\underline{\lambda_1 = 2,\ \lambda_2 = 5,\ \lambda_3 = 8}$$

$$\begin{bmatrix} 2-\lambda_1 & 1 & 1 \\ 0 & 5-\lambda_1 & 10 \\ 0 & 0 & 8-\lambda_1 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 3 & 10 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{v_{13} = v_{12} = 0,\ \ v_{11} = \text{anything}}$$

$$\begin{bmatrix} 2-\lambda_2 & 1 & 1 \\ 0 & 5-\lambda_2 & 10 \\ 0 & 0 & 8-\lambda_2 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} = \begin{bmatrix} -3 & 1 & 1 \\ 0 & 0 & 10 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{v_{21} = v_{22} = v_{23} = 0}$$

The system is homogeneous so we need to perform row operations to find the eigenvector

$$\begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{v_{23} = 0, v_{21} = \frac{1}{3} v_{22}}$$

$$\begin{bmatrix} 2-\lambda_3 & 1 & 1 \\ 0 & 5-\lambda_3 & 10 \\ 0 & 0 & 8-\lambda_3 \end{bmatrix} \begin{bmatrix} v_{31} \\ v_{32} \\ v_{33} \end{bmatrix} = \begin{bmatrix} -6 & 1 & 1 \\ 0 & -3 & 10 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_{31} \\ v_{32} \\ v_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{v_{33} = \frac{3}{10} v_2,\ \ v_{31} = \frac{13}{60} v_{32}}$$

$$\boxed{\lambda_1 = 2, v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix};\quad \lambda_2 = 5, v_2 = \begin{bmatrix} 1/(3\sqrt{10/9}) \\ 1/(\sqrt{10/9}) \\ 0 \end{bmatrix};\quad \lambda_3 = 8, v_3 = \begin{bmatrix} 0.2032 \\ 0.9378 \\ 0.2814 \end{bmatrix}}$$

Problem 10

$$\mathrm{rref}(A_2) = \begin{bmatrix}1&0&1\\5&2&1\\1&2&2\end{bmatrix} = \begin{bmatrix}1&0&1\\0&2&-4\\1&2&2\end{bmatrix} = \begin{bmatrix}1&0&1\\0&2&-4\\0&2&1\end{bmatrix} = \begin{bmatrix}1&0&1\\0&2&-4\\0&0&5\end{bmatrix} = \begin{bmatrix}1&0&1\\0&1&-2\\0&0&1\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$\mathrm{rref}(A_3) = \begin{bmatrix}2&1&1\\1&1&0\\1&0&1\end{bmatrix} = \begin{bmatrix}1&1/2&1/2\\1&1&0\\1&0&1\end{bmatrix} = \begin{bmatrix}1&1/2&1/2\\0&1&-1\\1&0&1\end{bmatrix} = \begin{bmatrix}1&1/2&1/2\\0&1&-1\\0&-1/2&1/2\end{bmatrix} = \begin{bmatrix}1&0&1\\0&1&-1\\0&-1/2&1/2\end{bmatrix} = \begin{bmatrix}1&0&1\\0&1&-1\\0&0&0\end{bmatrix}$$

$$\mathrm{rref}(A_4) = \begin{bmatrix}4&5&1\\3&7&10\\1&0&1\end{bmatrix} = \begin{bmatrix}1&-2&-9\\3&7&10\\1&0&1\end{bmatrix} = \begin{bmatrix}1&-2&-9\\0&13&37\\1&0&1\end{bmatrix} = \begin{bmatrix}1&-2&-9\\0&13&37\\0&0&1\end{bmatrix} = \begin{bmatrix}1&-2&-9\\0&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$\mathrm{rref}(A_5) = \begin{bmatrix}4&0&0\\0&7&0\\0&0&1\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$\mathrm{rref}(A_6) = \begin{bmatrix}2&1&1\\0&5&10\\0&0&8\end{bmatrix} = \begin{bmatrix}1&1/2&1/2\\0&1&2\\0&0&1\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$\boxed{\text{All are full rank except for } A_3 \text{ which is rank 2.}}$

Problem 11

For both A_5 and A_6, the eigenvalues are the same as what is on the diagonals. For A_5, the eigenvectors only have a nonzero term in one location. For A_6, the eigenvectors would form an upper triangular matrix when combined together.

Fundamental theorem of linear equations (10 points):

Prove Lemma 11.1 (i.e. show that $Im(V) = N(V^T)^{\perp}$)

- Recall that kernel and null space are synonymous
- The proof is in the book, just explain it in your own words

Problem 12

First, we can define a vector x which is in the image space of V. This can be written as

$$x \in \mathrm{Im}(V)$$

Because we have defined that x is in the image space of V, we know that there must be some vector which you can multiply by V in order to get x. This can be written mathematically as

$$\exists \eta : x = V\eta \tag{1}$$

which means that there exists some η such that x can be formed when multiplied by V. Next, we can define a vector that is within the null space of the tranpose of V, written as

$$z \in \text{null}(V)$$

When a vector, z,is within the null space of a matrix, the result will always be equal to zero when it is multiplied by the original matrix.

$$V^T z = 0 \tag{2}$$

The definition of the orthogonal complement is that the vectors in one set are perpendicular to all of the vectors in another set. This means that their inner product will be zero. Therefore, if we can prove that $z^T x$ is always equal to zero, then every vector z will be orthogonal to every vector x. By using Equations (1) and (2), we get

$$z^T x = z^T V \eta = 0 \tag{3}$$

Equation (3) can be rearranged by using the relationship

$$A^T B = (B^T A)^T \tag{4}$$

therefore,

$$(V^T z)^T \eta = 0$$

We know from Equation (2) that the term $V^T z$ will always be zero. Therefore, η can be anything and still result in a zero value. This proves that x which is contained in the image space of V, is always contained within the orthogonal complement of the null space of V^T. This can be written as

$$x \in (\text{null}(V^T))^\perp$$

Therefore, because we know that x is in both the image space of V and in the orthogonal complement of the null space of V^T, we can say that

$$\text{Im}(V) \subset (\text{null}(V^T))^\perp$$

Now that we have shown that the image space is a subset of the orthogonal complement of the null space, we need to show that the image space is not a strict subset and they are in fact equal. This can be done by comparing the dimensions of each space. If the dimensions of $\text{Im}(V)$ and $\text{null}(V^T)^\perp$ are the same, then that implies that the image space is not a strict subset and must be equal. First, we can define the matrix V as

$$V \in \mathbb{R}^{mxn}$$

From the textbook, we know the relations

$$\dim(\text{null}(V^T)) + \dim(\text{null}(V^T)^\perp) = m$$

$$\dim(\text{null}(V^T)) + \dim(\text{Im}(V^T)) = m$$

We can see that the first term in each equation is the same, therefore, the terms $\dim(\text{null}(V^T)^\perp)$ and $\dim(\text{Im}(V^T))$ must be equal. We also know that the dimension of the image space of a matrix will be the same dimension as the image space of the transpose of a matrix. This proves that $\dim(\text{Im}(V))$ and $\dim(\text{null}(V^T)^\perp)$ are equal. Because the dimensions are equal, this means that $\text{Im}(V)$ cannot be a strict subset of $\text{null}(V^T)^\perp$ and instead must be equal.

State Representation (10 points)

Consider the systems:

1. $$\begin{aligned} y_1^{(3)} + 2\ddot{y}_1 + 3\dot{y}_1 - 4\dot{y}_3 + 5\dot{y}_2 &= 4u_1 - u_3 \\ \ddot{y}_2 + 4y_1 - 3\dot{y}_3 &= u_2 \\ \ddot{y}_3 + \ddot{y}_1 &= 2u_1 + 4u_3 \end{aligned}$$

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2. $$\begin{aligned} 10\ddot{y}_1 + 4\dot{y}_1 - 5\dot{y}_3 &= 4u_1 \\ \ddot{y}_2 + 9y_1 - 3\dot{y}_3 &= u_1 + 7u_2 \\ \ddot{y}_3 + 8\dot{y}_1 + 5\dot{y}_2 &= u_1 + 3u_2 \end{aligned}$$

Write each system in state space form as $\dot{x} = Ax + Bu$

Problem 1

$x_1 = y_1 \quad \dot{x}_1 = \dot{y}_1 = x_2$

$x_2 = \dot{y}_1 \quad \dot{x}_2 = \ddot{y}_1 = x_3$

$x_3 = \ddot{y}_1 \quad \dot{x}_3 = y_1^{(3)} = \cancel{-2\ddot{y}_1 - 3\dot{y}_1 + 4\dot{y}_3 - 5\dot{y}_2 + 4u_1 - u_3}$

$= -2x_3 - 3x_2 + 4x_7 - 5x_5 + 4u_1 - u_3$

$x_4 = y_2 \quad \dot{x}_4 = \dot{y}_2 = x_5$

$x_5 = \dot{y}_2 \quad \dot{x}_5 = \ddot{y}_2 = \cancel{-4y_1 + 3\dot{y}_3 + u_2}$

$-4x_1 + 3x_7 + u_2$

$x_6 = y_3 \quad \dot{x}_6 = \dot{y}_3 = x_7$

$x_7 = \dot{y}_3 \quad \dot{x}_7 = \ddot{y}_3 = \cancel{-\ddot{y}_1 + 2u_1 + 4u_3}$

$= -x_3 + 2u_1 + 4u_3$

$\dot{x} = Ax + Bu$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \\ \dot{x}_7 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -3 & -2 & 0 & -5 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -4 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$y = Cx$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix}$$

Problem 2

$$x_1 = y_1 \quad \dot{x}_1 = \dot{y}_1 = x_2$$

$$x_2 = \dot{y}_1 \quad \dot{x}_2 = \ddot{y}_1 \cancel{= (-4\dot{y}_1 + 5\dot{y}_3 + 4u_1)\frac{1}{10}}$$

$$= -\frac{2}{5}x_2 + \frac{1}{2}x_6 + \frac{2}{5}u_1$$

$$x_3 = y_2 \quad \dot{x}_3 = \dot{y}_2 = x_4$$

$$x_4 = \dot{y}_2 \quad \dot{x}_4 = \ddot{y}_2 = \cancel{-9y_1 + 3\dot{y}_3 + u_1 + 7u_2}$$

$$= -9x_1 + 3x_6 + u_1 + 7u_2$$

$$x_5 = y_3 \quad \dot{x}_5 = \dot{y}_3 = x_6$$

$$x_6 = \dot{y}_3 \quad \dot{x}_6 = \ddot{y}_3 = \cancel{-8\dot{y}_1 - 5\dot{y}_2 + u_1 + 3u_2}$$

$$= -8x_2 - 5x_4 + u_1 + 3u_2$$

$\dot{x} = Ax + Bu$

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \\ \dot{x_5} \\ \dot{x_6} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2/5 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -9 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -8 & 0 & -5 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 2/5 & 0 \\ 0 & 0 \\ 1 & 7 \\ 0 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$y = Cx$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$$

Simulation of unstable System (25 points)

Use SimulationScript.m to simulate an unstable system. Do the following:

- Finish implementing the "eulerIntegration" function
  - Hint: 2017 - Mayne - MPC Theory Computation and Design.pdf ch 8.2 will be very useful
- Change the initial state on line 11 and show the state plot for each of the following (using eigenvectors and eigenvalues from the matrix "A" *defined in the f(t,z,u) function*):
  - Start on an eigenvector of magnitude 5 corresponding to a negative eigenvalue with zero control input (u = 0)
  - Start on an eigenvector of magnitude 5 corresponding to a negative eigenvalue and command a constant control (u = 1)
  - Start on an eigenvector of magnitude 5 corresponding to a positive eigenvalue with zero control input (u = 0)
  - Start on a random initial state with zero control input (u = 0)
- Answer the following
  - What is the behavior of the system for each of the described initial conditions?
  - How does Matlab's ode45 solution compare with the Euler integration for each of the described initial conditions?
  - Which solution do you trust more (ode45 or Euler) and why?
- Turn in .m files

Bullet point 1

Start on an eigenvector of magnitude 5 corresponding to a negative eigenvalue with zero control input (u=0).

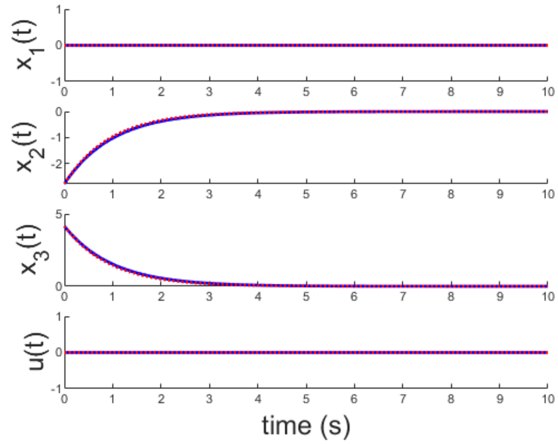


Fig. 1 Negative eigenvalue corresponding to eigenvector $v = [0, -0.5547, 0.8321]$

Answer Questions about the system

For the negative eigenvalue case, ODE45 and the EulerIntegration are fairly close in what they predict. It appears that the state starts out at an initial value but soon converges to zero over time. In this case, either integration method is probably acceptable but ODE45 is likely slightly more accurate because it is a higher order integrator.

Bullet point 2

Start on an eigenvector of magnitude 5 corresponding to a negative eigenvalue and command a constant control (u=1).

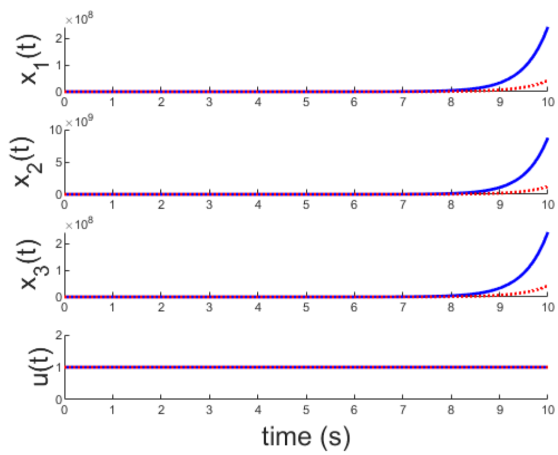


Fig. 2 Negative eigenvalue corresponding to eigenvector $v = [0, -0.5547, 0.8321]$

Answer Questions about the system

For this case, all three components of the state eventually blow up to a very large number, even though the eigenvector only has 2 nonzero components. This could be because machine zero is not exactly zero which is affecting the dynamics. Or, it could be that the other 2 states are influencing state 1. Overall, it seems like a constant control input results in values for the state that diverge.

In this case, the euler integration function and ODE45 are significantly different as the values for the state get lareger. I would likely trust ODE45 more because the Euler method would require very small step sizes in order to accurately capture the large change in state variable. Without sufficiently small step sizes, the Euler method can diverge significantly from the true value.

Bullet point 3

Start on an eigenvector of magnitude 5 corresponding to a positive eigenvalue with zero control input (u=0).

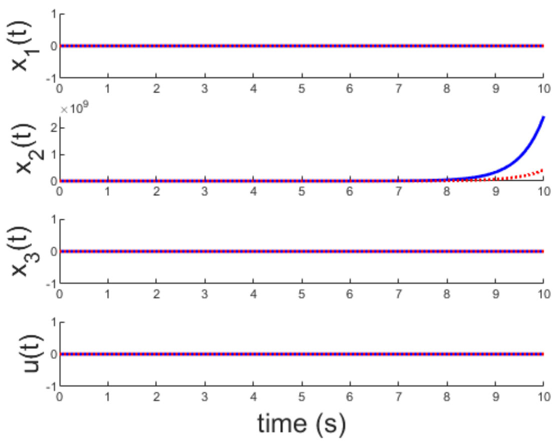


Fig. 3 Positive eigenvalue corresponding to eigenvector $v = [0, 1, 0]$

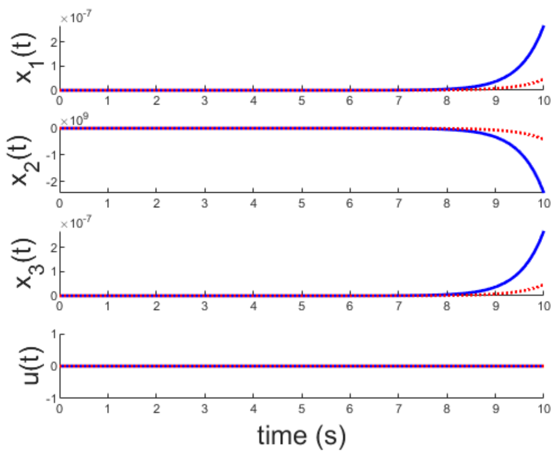


Fig. 4 Positive eigenvalue corresponding to eigenvector $v = [0, -1, 0]$

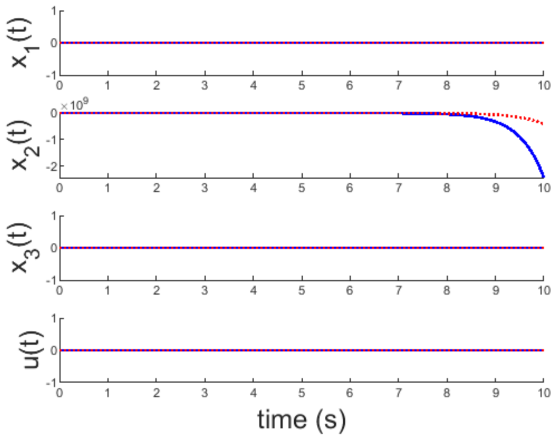


Fig. 5 Positive eigenvalue corresponding to eigenvector $v = [0, -1, 0]$ when not using the calculated eigenvector in MatLab

Answer Questions about the system

For this case, I looked at both eigenvectors corresponding to positive eigenvalues. The only difference in these eigenvectors was the direction. It appears that the state diverges in a different direction depending on the eigenvector used. It also appears that the solution is highly sensitive to numerical error. For example, Figure 4 shows the resulting states when using the eigenvector calculated in MatLab which do not have exact zero terms for the first and third initial state variables. Figure 5 shows the same simulation when explicitly stating that the eigenvector is [0, -1, 0]. With no initial input for the first and third states, you would likely expect something shown in Figure 5. However, the machine precision adds numerical error as shown in Figure 4.

Similar to the previous case, when the values blow up, the Euler integration and ODE45 diverge significantly. For similar reasons as stated before, I would still likely trust ODE45 over an Euler integrator. It is also interesting to note that even though this case used no control input like in case 1, the positive eigenvalue still results in a diverging system.

Bullet point 4

Start on a random initial state with zero control input (u=0).

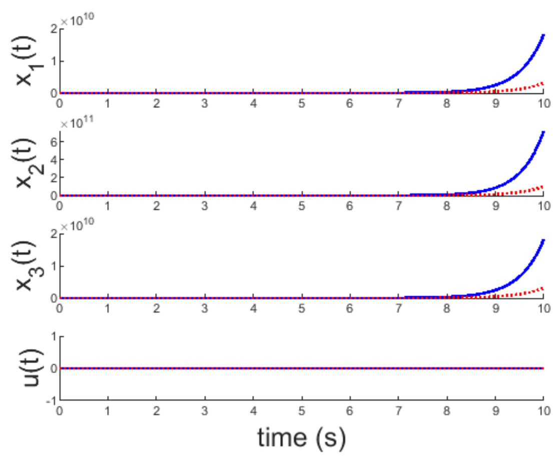


Fig. 6 Random initial state of $x_0 = 5 * [7.5, -1.2, 3]$

Answer Questions about the system

The random initial state also blows up like case 2 and 3. The Euler Integrator and ODE45 are very different so I would trust ODE45 more because it it higher order. It seems that there is likely some criteria that the initial state must meet in order to converge rather than diverge, which is likely linked to the eigenvalues and eigenvectors. This is potentially similar to what we learned in my Nonlinear Dynamics class.

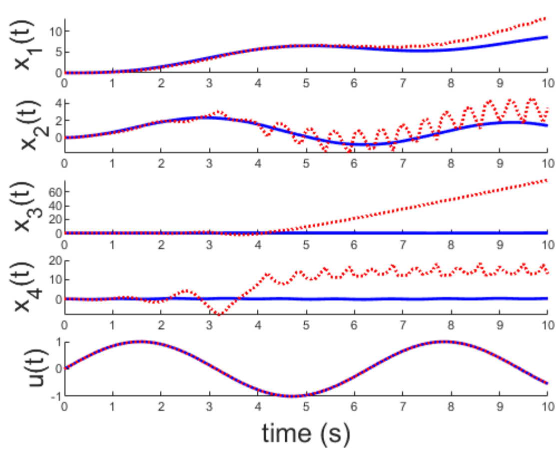
Extra credit: Simulation of a nonlinear system (20 points)

Create a simple simulation for the nonlinear inverted pendulum described here.

- Use the states $[x \quad \dot{x} \quad \theta \quad \dot{\theta}]$
- Plot the state and input vs time with input u = sin(t).
- Have the simulation start at time zero with zero initial state.
- Turn in .m files

Hints:

- You can use the SimulationScript.m as a shell and update the following:
    - The dynamics function
    - The plotting (to plot all four states)
    - The input function
- To get values for $\ddot{x}$ and $\ddot{\theta}$ run PendulumCart.m



I also played with the step size a little bit and saw that the euler integrator (red line) matches much closer. However, this is the result when using the step size given in the file.