Assignment 1 Question 3

3. Matrix Rank, Properties, and Eigenspace

Determine the Rank of the Given Matrix.

Find the rank of the matrix A (below). Explain what the rank tells us about the linear independence of the rows and columns of matrix A. Identify if there are any linear dependencies among the rows or columns.

$$A = \begin{bmatrix} 2 & 4 & 1 & 3 \\ -2 & -3 & 4 & 1 \\ 5 & 6 & 2 & 8 \\ -1 & -2 & 3 & 7 \end{bmatrix}$$

In row echelon form, we get:

$$A = \begin{bmatrix} 1 & 2 & 0.5 & 1.5 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 1 & 16.5/19.5 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

The rank is the number of nonzero rows in row echelon form. This 4x4 matrix has rank = 4 or full rank, which means no free variables or linear dependencies. All rows and columns are linearly independent.

In R:

```r
library(Matrix)

A <- matrix(c(2,4,1,3,
              -2,-3,4,1,
              5,6,2,8,
              -1,-2,3,7), byrow = TRUE, nrow = 4)
rank_A <- as.numeric(rankMatrix(A))
rank_A
```

```
[1] 4
```

B. Matrix Rank Boundaries

Given an m x n matrix where m > n, determine the maximum and minimum possible rank, assuming that the matrix is non-zero. Prove that the rank of a matrix equals the dimension of its row space (or column space). Provide an example to illustrate the concept.

1. Maximum rank = n because the number of linearly independent columns can't exceed n
2. Minimum rank = 1 because there must be at least one independent row or column since the matrix is non-zero
3. The rank of a matrix is the same as the dimension of its row space, which is the number of linearly independent rows. Performing row operations on a matrix to get to row echelon form does not change the row space but it allows us to see it clearly by reducing dependent rows to zeroes:

Example matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

Row echelon form with one linearly independent variable (non-zero row) showing the row space and rank of this 3x3 matrix is actually 1:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

In R:

```r
A <- matrix(c(1,2,3,
              2,4,6,
              3,6,9), byrow = TRUE, nrow = 3)
rank_A <- as.numeric(rankMatrix(A))
rank_A
```

```
[1] 1
```

C. Rank and Row Reduction:

Determine the rank of matrix B. Perform a row reduction on matrix B and describe how it helps in finding the rank. Discuss any special properties of matrix B (e.g., is it a rank-deficient matrix?)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 5 & 7 \\ 4 & 10 & 14 \\ 1 & 2.5 & 3.5 \end{bmatrix}$$

Row-reduced to row echelon form:

Swap R3 and R1, subtract 2xR1 from R3 and subtract 2xR1 from R2:

$$B = \begin{bmatrix} 1 & 2.5 & 3.5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The rank is the number of linearly independent rows, which are the non-zero rows in row echelon format, or rank $= 1$.

This is rank deficient because the maximum rank of a 3x3 matrix would be 3, which means this matrix had linearly dependent rows and columns (has free variables). This matrix collapses a 3D space to a 1D vector and is singular or not invertible (can't be reversed).

Check in R:

```r
B <- matrix(c(2,5,7,
              4,10,14,
              1,2.5,3.5), byrow = TRUE, nrow = 3)
rank_B <- as.numeric(rankMatrix(A))
rank_B
```

```
[1] 1
```

D. Compute the eigenvalues and eigenvectors

Find the eigenvalues and eigenvectors of the matrix A. Write out the characteristic polynomial and show your solution step by step. After finding the eigenvalues and eigenvectors, verify that the eigenvectors are linearly independent. If they are not, explain why.

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 5 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

First we find eigenvalues by solving the characteristic equation $det(A - \lambda I)$:

$$det \left(\begin{bmatrix} 3 & 1 & 2 \\ 0 & 5 & 4 \\ 0 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

Multiply by λ:

$$det \left(\begin{bmatrix} 3 & 1 & 2 \\ 0 & 5 & 4 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right)$$

Subtract:

$$det \left(\begin{bmatrix} 3-\lambda & 1 & 2 \\ 0 & 5-\lambda & 4 \\ 0 & 0 & 2-\lambda \end{bmatrix} \right)$$

Find determinant using $det = a(ei - fh) - b(di - fg) + c(dh - eg)$:

From above:
a $= 3 - \lambda$
b $= 1$
c $= 2$
d $= 0$
e $= 5 - \lambda$
f $= 4$
g $= 0$
h $= 0$
i $= 2 - \lambda$

ei $= (5 - \lambda) * (2 - \lambda) = \lambda^2 - 7\lambda + 10$
fh $= 0$
di $= 0$
fg $= 0$
dh $= 0$
eg $= 0$

$det = a(ei - fh) - b(di - fg) + c(dh - eg)$
$det = (3 - \lambda) * (\lambda^2 - 7\lambda + 10)$
$det = -\lambda^3 + 10\lambda^2 - 31\lambda + 30$

Set equal to zero to find null space:
$-\lambda^3 + 10\lambda^2 - 31\lambda + 30 = 0$

Try $\lambda = 2$ (Rational Root Theorem: roots must be factor of 30)
$-8 + 40 - 62 + 30 = 0$ is True

So $\lambda = 2$, therefore $(\lambda - 2)$ is a factor and can be factored out:
$-\lambda^2 - 8\lambda - 15 = 0$

Flip signs to make it easier:
$\lambda^2 + 8\lambda + 15 = 0$
$(\lambda - 3)(\lambda - 5) = 0$

Eigenvalues are {2,3,5}

Now we find eigenvectors by substituting each eigenvalue into the formula $(A - \lambda I)v = 0$:

1. $\lambda = 2$:
   $(A - 2I)v = 0$

$$\left(\begin{bmatrix} 3 & 1 & 2 \\ 0 & 5 & 4 \\ 0 & 0 & 2 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 3 & 1 & 2 \\ 0 & 5 & 4 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right) \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$v_1 + v_2 + 2v_3 = 0$$
$$0 + 3v_2 + 4v_3 = 0$$
$$0 + 0 + 0 = 0$$

So $v_2 = -4/3v_3$ (from second equation) and substituting that into the first equation:

$$v_1 - 4/3v_3 + 2v_3 = 0$$
$$v_1 + 2/3v_3 = 0$$
$$v_1 = -2/3v_3$$

v_3 is free variable; set to t:

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -2/3t \\ -4/3t \\ t \end{bmatrix} = t \begin{bmatrix} -2/3 \\ -4/3 \\ 1 \end{bmatrix} = t \begin{bmatrix} -2 \\ -4 \\ 3 \end{bmatrix}$$

For eigenvalue 2, the eigenvector is $\begin{bmatrix} -2 \\ -4 \\ 3 \end{bmatrix}$

2. $\lambda = 3$:

$$(A - 3I)v = 0$$

$$\left(\begin{bmatrix} 3 & 1 & 2 \\ 0 & 5 & 4 \\ 0 & 0 & 2 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 3 & 1 & 2 \\ 0 & 5 & 4 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \right) \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$v_2 + 2v_3 = 0$$
$$2v_2 + 4v_3 = 0$$
$$-v_3 = 0$$

v_1 does not appear in the equations and is free variable; set to t:

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} t \\ 0t \\ 0t \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda = 3$, the eigenvector is

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

3. $\lambda = 5$:

$$(A - 5I)v = 0$$

$$\left(\begin{bmatrix} 3 & 1 & 2 \\ 0 & 5 & 4 \\ 0 & 0 & 2 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 3 & 1 & 2 \\ 0 & 5 & 4 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \right) \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2v_1 + v_2 + 2v_3 = 0$$

$$4v_3 = 0$$

$$-3v_3 = 0$$

So $v_3 = 0$ and we can substitute:
$-2v_1 + v_2 = 0$
$v_2 = 2v_1$

Because there are more variables than equations, one must be free. We can assume v_1 is free and set it to t:

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} t \\ 2t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

For $\lambda = 5$, the eigenvector is

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

Conclusion: The matrix A is an upper triangular matrix (non-symmetric) with a set of eigenvalues $\{2,3,5\}$. It does have a full set of eigenvectors:

For $\lambda = 2$, the eigenvector is

$$\begin{bmatrix} -2 \\ -4 \\ 3 \end{bmatrix}$$

For $\lambda = 3$, the eigenvector is

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda = 5$, the eigenvector is

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

Validation in R:

```r
A <- matrix(c(3,1,2,
              0,5,4,
              0,0,2),
            nrow=3,
            byrow=TRUE)

A
```

```
     [,1] [,2] [,3]
[1,]    3    1    2
[2,]    0    5    4
[3,]    0    0    2
```

```r
my_eigen <- eigen(A)
my_eigen$values
```

```
[1] 5 3 2
```

```r
my_eigen$vectors
```

```
          [,1] [,2]       [,3]
[1,] 0.4472136    1 -0.3713907
[2,] 0.8944272    0 -0.7427814
[3,] 0.0000000    0  0.5570860
```