Landen's Transformation can also be applied to Elliptic Integrals of this class.

$$E(k,\phi)=\int_0^\phi\sqrt{1-k^2\sin^2\phi}\,d\phi=\int_0^\phi\Delta\phi d\phi. \tag{1}$$

After a series of trivial steps, we arrive at:

$$Ek\phi=\frac{Ek_0\phi_1}{1+k_0}+\frac{k_0\sin\phi_1}{1+k_0}-\tfrac{1}{2}(1-k_0)Fk_0\phi_1. \tag{2}$$

By successive applications of 2, $Ek\phi$ may be made to depend ultimately upon $Ek_{0n}\phi_n$, where k_{0n} approximates to zero and $Ek_{0n}\phi_n$ to ϕ_n.

Or, by reversing, it may be made to depend upon $Ek_n\phi_{0n}$, where k_n approximates to unity and $Ek_n\phi_{0n}$ to $-\cos\phi_{0n}$.

To facilitate this, assume

$$Gk\phi=Ek\phi-Fk\phi,$$

Subtracting from the equation

$$Fk\phi=\frac{1+k_0}{2}Fk_0\phi_1 \text{ (see Chap. IV)},$$

we have

$$Gk\phi=\frac{1}{1+k_0}(Gk_0\phi_1+k_0\sin\phi_1-k_0\,Fk_0\,\phi_1).$$

Repeated applications of this give

$$\begin{aligned}Gk_0\phi_1&=\frac{1}{1+k_{00}}(Gk_{00}\phi_2+k_{00}\sin\phi_2-k_{00}\,Fk_{00}\,\phi_2),\\ &\dots\\ Gk_{0(n-1)}\phi_{n-1}&=\frac{1}{1+k_{0n}}(Gk_{0n}\phi_n+k_{0n}\sin\phi_n-k_{0n}\,Fk_{0n}\,\phi_n).\end{aligned} \tag{3}$$

Whence

$$Gk\phi=\sum_n^1\left\{\frac{k_{0n}(\sin\phi_n-Fk_{0n}\phi_n)}{[\prod_n^{(1+k_{0n})}]}\right\}+\frac{Gk_{0n}\,\phi_n}{[\prod_n^{(1+k_{0n})}]}. \tag{4}$$

But since

$$Fk\phi = \frac{Fk_{0n}\,\phi_n\left[\prod_n^{(1+k_{0n})}\right]}{2^n},$$

or

$$\frac{Fk_{0n}\,\phi_n}{\left[\prod_n^{(1+k_{0n})}\right]} = \frac{2^n Fk\,\phi}{\left[\prod_n^{(1+k_{0n}^2)}\right]};$$

and since, also,

$$\begin{aligned}
\frac{2^n k_{0n}}{\left[\prod_n^1\right](1+k_{0n})^2} &= \frac{k_{0n}}{2^n}\Big[\prod_n^1\Big]\frac{k_{0(n-1)}^2}{k_{0n}} \\
&= \frac{k_{0n}}{2^n}\Big[\prod_n^1\Big]\frac{k_{0(n-1)}}{k_{0n}}\Big[\prod_n^1\Big]k_{0(n-1)} \\
&= \frac{k_{0n}}{2^n}\cdot\frac{k}{k_0}\cdot\frac{k_0}{k_{00}}\cdots\frac{k_{0(n-1)}}{k_{0n}}\cdot k\Big[\prod_n^2\Big]k_{0(n-1)} \\
&= \frac{k^2}{2^n}\Big[\prod_n^2\Big]k_{0(n-1)}.
\end{aligned} \tag{5}$$

we have

$$\begin{aligned}
Gk_{0n}\phi_n &= Ek_{0n}\phi_n - Fk_{0n}\phi_n \\
&= \phi_n - \phi_n = 0,
\end{aligned}$$

$$\begin{aligned}
Gk\phi &= \sum_n^1\left\{\frac{k\sqrt{k_{0n}}\sin\phi_n\left[\prod_n^2\right]\sqrt{k_{0(n-1)}} - k^2\left[\prod_n^2\right]k_{0(n-1)}}{2^n}\right\} \\
&= k\left[\frac{\sqrt{k_0}}{2}\sin\phi_1 + \frac{\sqrt{k_0k_{00}}}{2^2}\sin\phi_2 + \frac{\sqrt{k_0k_{00}k_{03}}}{2^3}\sin\phi_3 + \cdots\right] \\
&\qquad - \frac{k^2}{2}\left[1 + \frac{k_0}{2} + \frac{k_0k_{00}}{2^2} + \frac{k_0k_{00}k_{03}}{2^3} + \cdots\right];
\end{aligned} \tag{6}$$

With substitution of identities into this form, we can arrive at our answer.