

Landen's Transformation can also be applied to Elliptic Integrals of this class.

$$E(k, \phi) = \int_0^\phi \sqrt{1 - k^2 \sin^2 \phi} d\phi = \int_0^\phi \Delta \phi d\phi. \quad (1)$$

After a series of trivial steps, we arrive at:

$$Ek\phi = \frac{Ek_0\phi_1}{1+k_0} + \frac{k_0 \sin \phi_1}{1+k_0} - \frac{1}{2}(1-k_0)Fk_0\phi_1. \quad (2)$$

By successive applications of 2,  $Ek\phi$  may be made to depend ultimately upon  $Ek_{0n}\phi_n$ , where  $k_{0n}$  approximates to zero and  $Ek_{0n}\phi_n$  to  $\phi_n$ .

Or, by reversing, it may be made to depend upon  $Ek_n\phi_{0n}$ , where  $k_n$  approximates to unity and  $Ek_n\phi_{0n}$  to  $-\cos \phi_{0n}$ .

To facilitate this, assume

$$Gk\phi = Ek\phi - Fk\phi,$$

Subtracting from the equation

$$Fk\phi = \frac{1+k_0}{2}Fk_0\phi_1 \text{ (see Chap. IV),}$$

we have

$$Gk\phi = \frac{1}{1+k_0}(Gk_0\phi_1 + k_0 \sin \phi_1 - k_0 Fk_0\phi_1).$$

Repeated applications of this give

$$\begin{aligned} Gk_0\phi_1 &= \frac{1}{1+k_{00}}(Gk_{00}\phi_2 + k_{00} \sin \phi_2 - k_{00} Fk_{00}\phi_2), \\ &\dots \\ Gk_{0(n-1)}\phi_{n-1} &= \frac{1}{1+k_{0n}}(Gk_{0n}\phi_n + k_{0n} \sin \phi_n - k_{0n} Fk_{0n}\phi_n). \end{aligned} \quad (3)$$

Whence

$$Gk\phi = \sum_n \left\{ \frac{k_{0n}(\sin \phi_n - Fk_{0n}\phi_n)}{[\Pi_n^{(1+k_{0n})}]} \right\} + \frac{Gk_{0n}\phi_n}{[\Pi_n^{(1+k_{0n})}]} \quad (4)$$

But since

$$Fk\phi = \frac{Fk_{0n} \phi_n [\prod_n^{(1+k_{0n})}]}{2^n},$$

or

$$\frac{Fk_{0n} \phi_n}{[\prod_n^{(1+k_{0n})}]} = \frac{2^n Fk \phi}{[\prod_n^{(1+k_{0n}^2)}]};$$

and since, also,

$$\begin{aligned} \frac{2^n k_{0n}}{[\prod_n^1] (1 + k_{0n})^2} &= \frac{k_{0n}}{2^n} [\prod_n^1] \frac{k_{0(n-1)}^2}{k_{0n}} \\ &= \frac{k_{0n}}{2^n} [\prod_n^1] \frac{k_{0(n-1)}}{k_{0n}} [\prod_n^1] k_{0(n-1)} \\ &= \frac{k_{0n}}{2^n} \cdot \frac{k}{k_0} \cdot \frac{k_0}{k_{00}} \cdots \frac{k_{0(n-1)}}{k_{0n}} \cdot k [\prod_n^2] k_{0(n-1)} \\ &= \frac{k^2}{2^n} [\prod_n^2] k_{0(n-1)}. \end{aligned} \tag{5}$$

we have

$$\begin{aligned} Gk_{0n}\phi_n &= Ek_{0n}\phi_n - Fk_{0n}\phi_n \\ &= \phi_n - \phi_n = 0, \end{aligned}$$

$$\begin{aligned} Gk\phi &= \sum_n^1 \left\{ \frac{k\sqrt{k_{0n}} \sin \phi_n [\prod_n^2] \sqrt{k_{0(n-1)}} - k^2 [\prod_n^2] k_{0(n-1)}}{2^n} \right\} \\ &= k \left[ \frac{\sqrt{k_0}}{2} \sin \phi_1 + \frac{\sqrt{k_0 k_{00}}}{2^2} \sin \phi_2 + \frac{\sqrt{k_0 k_{00} k_{03}}}{2^3} \sin \phi_3 + \cdots \right] \\ &\quad - \frac{k^2}{2} \left[ 1 + \frac{k_0}{2} + \frac{k_0 k_{00}}{2^2} + \frac{k_0 k_{00} k_{03}}{2^3} + \cdots \right]; \end{aligned} \tag{6}$$