An embedding of n points in $\mathbb{R}^d$ corresponds to one point in $\mathbb{R}^{dn}$ (which will be referred to as the *configuration space*) by stacking the coordinates of the n point together.

Every triplet constraint of the form item i is closer to item j than item k can be written as $x^T P_{ijk} x < 0$ where P_{ijk} is a symmetric $nd \times nd$ matrix. Let $T = \{(i,j,k) : d(x_i, x_j) < d(x_i, x_k), i \in [n], j \neq k, j, k \in [n] - \{i\}\}$.

Lemma 0.1 (Perturbing points slightly)**.** *The set of points in the configuration space that satisfies all triplet constraints is open. The analagous statement in $\mathbb{R}^d$ says that given an embedding that satisfies all triplet constraints, each point in the embedding can be perturbed slightly without violating any constraints.*

Proof. Let $x \in \mathbb{R}^{nd}$ be an embedding in the configuration space that does not violate any triplet constraint. Now consider $x + \varepsilon_{ijk} u$ where $||u||_2 = 1$. We will show that there exists ε such that $x + \varepsilon u$ does not violate any triplet constraints, which establishes that the set of points in the configuration space that satifies all triplet constraints is open.
The goal is to show that $(x + \varepsilon_{ijk} u)^T P_{ijk} (x + \varepsilon_{ijk} u) < 0$ for any $(i,j,k) \in T$. Note,

$$\begin{aligned}(x + \varepsilon_{ijk} u)^T P_{ijk} (x + \varepsilon_{ijk} u) &= x^T P_{ijk} x + \varepsilon_{ijk} u^T P_{ijk} x + \varepsilon_{ijk} x^T P_{ijk} u + \varepsilon_{ijk}^2 u^T P_{ijk} u \\ &= x^T P_{ijk} x + 2\varepsilon_{ijk} u^T P_{ijk} x + \varepsilon_{ijk}^2 u^T P_{ijk} u \text{ by symmetry of } P_{ijk}.\end{aligned}$$

Because P_{ijk} is a symmetric real matrix, it is diagonalizable: $P_{ijk} = U \Lambda U^{-1}$. Therefore,

$$\begin{aligned}\max_{||u||_2} u^T P_{ijk} u &= \max_{||u||_2} u^T U \Lambda U^{-1} u \\ &= \max_{||u||_2} u^T \Lambda u \text{ since } U \text{ is orthogonal} \\ &= \sqrt{3}\end{aligned}$$

since the largest eigenvalue of P_{ijk} is $\sqrt{3}$.

Furthermore, for any vector y

$$\begin{aligned}\max_{||u||_2 = 1} u^T y &= \max_{||u||_2 = 1} ||u||_2 ||y||_2 \cos(\theta) \text{ where } \theta \text{ is the angle between } y \text{ and } u \\ &= ||y||_2.\end{aligned}$$

Therefore,

$$2\varepsilon_{ijk} u^T P_{ijk} x + \varepsilon_{ijk}^2 u^T P_{ijk} u < 2\varepsilon_{ijk} ||P_{ijk} x||_2 + \varepsilon_{ijk}^2 \sqrt{3},$$

so let $\epsilon_{ijk} > 0$ be sufficiently small so that

$$2\varepsilon_{ijk} ||P_{ijk} x||_2 + \varepsilon_{ijk}^2 \sqrt{3} < -x^T P_{ijk} x$$

since $-x^T P_{ijk} x > 0$ since x is a valid embedding.
Putting this altogether, we see

$$(x + \varepsilon_{ijk} u)^T P_{ijk} (x + \varepsilon_{ijk} u) = x^T P_{ijk} x + 2\varepsilon_{ijk} u^T P_{ijk} x + \varepsilon_{ijk}^2 u^T P_{ijk} u < x^T P_{ijk} x - x^T P_{ijk} x = 0,$$

so $x + \varepsilon_{ijk} u$ does not violate the (i,j,k) triplet constraint.
Finally, if $\varepsilon = \min_{(i,j,k) \in T} \{\varepsilon_{ijk}\}$, $x + \varepsilon u$ does not violate any triplet constraint for all u such that $||u||_2 = 1$. □

Lemma 0.2 (scaling / cone)**.** *The set of points in the configuration space that satisfies all triplet constraints is a cone. Furthermore, if x satisfies all triplet constraints $-x$ satisfies all triplet constraints.*

Proof. Let x be a point in the configuration space that satisfies all triplet constriants. Let $\alpha \neq 0$, which means for all $(i,j,k) \in T$, $(\alpha x)^T P_{ijk} (\alpha x) = \alpha^2 x^T P_{ijk} x < 0$ since $\alpha^2 > 0$, so αx satisfies all triplet constraints. □

Lemma 0.3 (rotation)**.**

Lemma 0.4 (translation)**.**

Lemma 0.5 (reflection)**.**