Generalized linear models, binary data

Regression models

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Key ideas

- Frequently we care about outcomes that have two values
  - Alive/dead
  - Win/loss
  - Success/Failure
  - etc
- Called binary, Bernoulli or 0/1 outcomes
- Collection of exchangeable binary outcomes for the same covariate data are called binomial outcomes.

Example Baltimore Ravens win/loss

Ravens Data

```
download.file("https://dl.dropboxusercontent.com/u/7710864/data/ravensData.rda"
              , destfile="./data/ravensData.rda",method="curl")
load("./data/ravensData.rda")
head(ravensData)
```

```
  ravenWinNum ravenWin ravenScore opponentScore
1           1        W         24             9
2           1        W         38            35
3           1        W         28            13
4           1        W         34            31
5           1        W         44            13
6           0        L         23            24
```

Linear regression

$$RW_i = b_0 + b_1 RS_i + e_i$$

RW_i - 1 if a Ravens win, 0 if not

RS_i - Number of points Ravens scored

b_0 - probability of a Ravens win if they score 0 points

b_1 - increase in probability of a Ravens win for each additional point

e_i - residual variation due

Linear regression in R

```
lmRavens <- lm(ravensData$ravenWinNum ~ ravensData$ravenScore)
summary(lmRavens)$coef
```

```
                      Estimate Std. Error t value Pr(>|t|)
(Intercept)             0.2850   0.256643   1.111  0.28135
ravensData$ravenScore   0.0159   0.009059   1.755  0.09625
```

Odds

Binary Outcome 0/1

$$\mathrm{RW_i}$$

Probability (0,1)

$$\mathrm{Pr(RW_i|RS_i, b_0, b_1)}$$

Odds $(0, \infty)$

$$\frac{\mathrm{Pr(RW_i|RS_i, b_0, b_1)}}{1 - \mathrm{Pr(RW_i|RS_i, b_0, b_1)}}$$

Log odds $(-\infty, \infty)$

$$\log\left(\frac{\mathrm{Pr(RW_i|RS_i, b_0, b_1)}}{1 - \mathrm{Pr(RW_i|RS_i, b_0, b_1)}}\right)$$

Linear vs. logistic regression

Linear

$$\mathrm{RW_i} = \mathrm{b_0} + \mathrm{b_1 RS_i} + \mathrm{e_i}$$

or

$$\mathrm{E[RW_i|RS_i, b_0, b_1]} = \mathrm{b_0} + \mathrm{b_1 RS_i}$$

Logistic

$$\mathrm{Pr(RW_i|RS_i, b_0, b_1)} = \frac{\exp(\mathrm{b_0} + \mathrm{b_1 RS_i})}{1 + \exp(\mathrm{b_0} + \mathrm{b_1 RS_i})}$$

or

$$\log\left(\frac{\mathrm{Pr(RW_i|RS_i, b_0, b_1)}}{1 - \mathrm{Pr(RW_i|RS_i, b_0, b_1)}}\right) = \mathrm{b_0} + \mathrm{b_1 RS_i}$$

Interpreting Logistic Regression

$$\log\left(\frac{\Pr(RW_i|RS_i, b_0, b_1)}{1 - \Pr(RW_i|RS_i, b_0, b_1)}\right) = b_0 + b_1 RS_i$$

b_0 - Log odds of a Ravens win if they score zero points

b_1 - Log odds ratio of win probability for each point scored (compared to zero points)

$\exp(b_1)$ - Odds ratio of win probability for each point scored (compared to zero points)

Odds

- Imagine that you are playing a game where you flip a coin with success probability p.
- If it comes up heads, you win X. If it comes up tails, you lose Y.
- What should we set X and Y for the game to be fair?

$$E[\mathrm{earnings}] = Xp - Y(1 - p) = 0$$

- Implies

$$\frac{Y}{X} = \frac{p}{1 - p}$$

- The odds can be said as "How much should you be willing to pay for a p probability of winning a dollar?"
  - (If $p > 0.5$ you have to pay more if you lose than you get if you win.)
  - (If $p < 0.5$ you have to pay less if you lose than you get if you win.)

Visualizing fitting logistic regression curves

```
x <- seq(-10, 10, length = 1000)
manipulate(
    plot(x, exp(beta0 + beta1 * x) / (1 + exp(beta0 + beta1 * x)),
         type = "l", lwd = 3, frame = FALSE),
    beta1 = slider(-2, 2, step = .1, initial = 2),
    beta0 = slider(-2, 2, step = .1, initial = 0)
    )
```

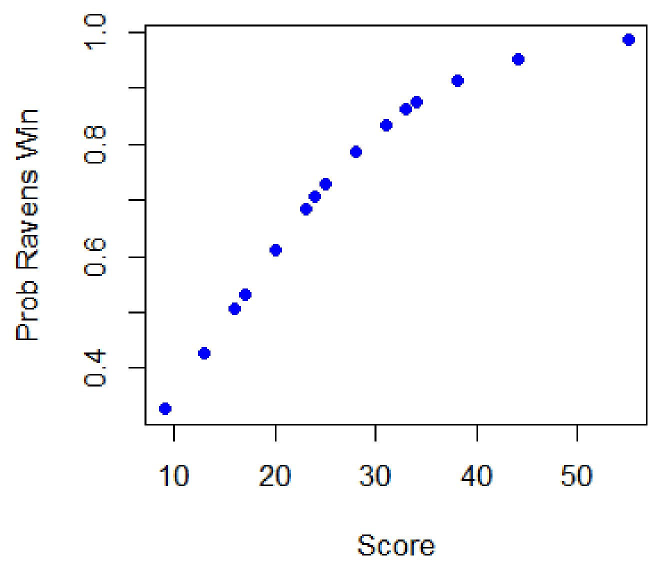
Ravens logistic regression

```
logRegRavens <- glm(ravensData$ravenWinNum ~ ravensData$ravenScore,family="binomial")
summary(logRegRavens)
```

```

Call:
glm(formula = ravensData$ravenWinNum ~ ravensData$ravenScore, 
    family = "binomial")

Deviance Residuals: 
   Min      1Q  Median      3Q     Max  
-1.758  -1.100   0.530   0.806   1.495  

Coefficients:
                      Estimate Std. Error z value Pr(>|z|)
(Intercept)            -1.6800     1.5541   -1.08     0.28
ravensData$ravenScore   0.1066     0.0667    1.60     0.11

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 24.435  on 19  degrees of freedom
Residual deviance: 20.895  on 18  degrees of freedom
AIC: 24.89
```

Ravens fitted values

```
plot(ravensData$ravenScore,logRegRavens$fitted,pch=19,col="blue",xlab="Score",ylab="Prob Ravens Win")
```



Odds ratios and confidence intervals

```
exp(logRegRavens$coeff)
```

```
         (Intercept) ravensData$ravenScore
              0.1864                1.1125
```

```
exp(confint(logRegRavens))
```

```
                          2.5 % 97.5 %
(Intercept)           0.005675  3.106
ravensData$ravenScore 0.996230  1.303
```

ANOVA for logistic regression

```
anova(logRegRavens,test="Chisq")
```

```
Analysis of Deviance Table

Model: binomial, link: logit

Response: ravensData$ravenWinNum

Terms added sequentially (first to last)

                      Df Deviance Resid. Df Resid. Dev Pr(>Chi)
NULL                                     19       24.4         
ravensData$ravenScore  1     3.54        18       20.9     0.06 .
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Interpreting Odds Ratios

- Not probabilities
- Odds ratio of 1 = no difference in odds
- Log odds ratio of 0 = no difference in odds
- Odds ratio < 0.5 or > 2 commonly a "moderate effect"
- Relative risk $\frac{\Pr(RW_i|RS_i=10)}{\Pr(RW_i|RS_i=0)}$ often easier to interpret, harder to estimate
- For small probabilities RR ≈ OR but **they are not the same**!

[Wikipedia on Odds Ratio]

Further resources

- Wikipedia on Logistic Regression
- Logistic regression and glms in R
- Brian Caffo's lecture notes on: Simpson's paradox, Case-control studies
- Open Intro Chapter on Logistic Regression