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Exponential potentials and cosmological scaling solutions

Edmund J. Copeland
Centre for Theoretical Physics, University of Sussex, Falmer, Brighton BN1 9QH, U. K.

Andrew R. Liddle
Astronomy Centre, University of Sussex, Falmer, Brighton BN1 9QH, U. K.

David Wands
School of Computer Science and Mathematics, University of Portsmouth, Portsmouth PO1 2EG, U. K.

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We present a phase-plane analysis of cosmologies containing a barotropic fluid with equation of state $p_\gamma = (\gamma - 1)\rho_\gamma$, plus a scalar field ϕ with an exponential potential $V \propto \exp(-\lambda\kappa\phi)$ where $\kappa^2 = 8\pi G$. In addition to the well-known inflationary solutions for $\lambda^2 < 2$, there exist scaling solutions when $\lambda^2 > 3\gamma$ in which the scalar field energy density tracks that of the barotropic fluid (which for example might be radiation or dust). We show that the scaling solutions are the unique late-time attractors whenever they exist. The fluid-dominated solutions, where $V(\phi)/\rho_\gamma \to 0$ at late times, are always unstable (except for the cosmological constant case $\gamma = 0$). The relative energy density of the fluid and scalar field depends on the steepness of the exponential potential, which is constrained by nucleosynthesis to $\lambda^2 > 20$. We show that standard inflation models are unable to solve this 'relic density' problem.

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I. INTRODUCTION

Scalar fields have come to play a central role in current models of the early universe. The self-interaction potential energy density of such a field is undiluted by the expansion of the universe and hence can act like an effective cosmological constant driving a period of inflation. The detailed evolution is dependent upon the specific form of the potential V as a function of the scalar field's expectation value ϕ.

A common functional form for the self-interaction potential is an exponential dependence upon the scalar field. It is to be found in higher-order [1] or higher-dimensional gravity theories [2]. In string or Kaluza–Klein type models the moduli fields associated with the geometry of the extra dimensions may have effective exponential potentials due to curvature of the internal spaces, or the interaction of moduli with form fields on the internal spaces. Exponential potentials can also arise due to non-perturbative effects such as gaugino condensation [3].

The possible cosmological roles of exponential potentials have been investigated before, but almost always as a means of driving a period of cosmological inflation [4,5]. This requires potentials that are much flatter than those usually found in particle physics models. The purpose of this paper is to emphasize that scalar fields with exponential potentials may still have important cosmological consequences even if they are too steep to drive a period of inflation [6–8]. We will present a phase-plane analysis to show that scalar fields with exponential potentials contribute a non-negligible energy density at nucleosynthesis unless they are unusually steep. This 'relic density' problem is not alleviated by standard models of inflation.

II. AUTONOMOUS PHASE-PLANE

We will consider a scalar field with an exponential potential energy density $V = V_0 \exp(-\lambda\kappa\phi)$ evolving in a spatially-flat Friedmann–Robertson–Walker (FRW) universe containing a fluid with barotropic equation of state $p_\gamma = (\gamma - 1)\rho_\gamma$, where γ is a constant, $0 \le \gamma \le 2$, such as radiation ($\gamma = 4/3$) or dust ($\gamma = 1$). The evolution equations for a spatially-flat FRW model with Hubble parameter H are

$$\dot{H} = -\frac{\kappa^2}{2}\left(\rho_\gamma + p_\gamma + \dot{\phi}^2\right) , \tag{1}$$

$$\dot{\rho}_\gamma = -3H(\rho_\gamma + p_\gamma) , \tag{2}$$

$$\ddot{\phi} = -3H\dot{\phi} - \frac{dV}{d\phi} , \tag{3}$$

subject to the Friedmann constraint

$$H^2 = \frac{\kappa^2}{3}\left(\rho_\gamma + \frac{1}{2}\dot{\phi}^2 + V\right) , \tag{4}$$

where $\kappa^2 \equiv 8\pi G$. The total energy density of a homogeneous scalar field is $\rho_\phi = \dot{\phi}^2/2 + V(\phi)$.

We define

$$x \equiv \frac{\kappa\dot{\phi}}{\sqrt{6}\,H} \quad ; \quad y \equiv \frac{\kappa\sqrt{V}}{\sqrt{3}\,H} . \tag{5}$$

The evolution equations can then be written as a plane-autonomous system:

x	y	Existence	Stability	Ω_ϕ	γ_ϕ
0	0	All λ and γ	Saddle point for $0<\gamma<2$	0	Undefined
1	0	All λ and γ	Unstable node for $\lambda<\sqrt{6}$ Saddle point for $\lambda>\sqrt{6}$	1	2
-1	0	All λ and γ	Unstable node for $\lambda>-\sqrt{6}$ Saddle point for $\lambda<-\sqrt{6}$	1	2
$\lambda/\sqrt{6}$	$[1-\lambda^2/6]^{1/2}$	$\lambda^2<6$	Stable node for $\lambda^2<3\gamma$ Saddle point for $3\gamma<\lambda^2<6$	1	$\lambda^2/3$
$(3/2)^{1/2}\,\gamma/\lambda$	$[3(2-\gamma)\gamma/2\lambda^2]^{1/2}$	$\lambda^2>3\gamma$	Stable node for $3\gamma<\lambda^2<24\gamma^2/(9\gamma-2)$ Stable spiral for $\lambda^2>24\gamma^2/(9\gamma-2)$	$3\gamma/\lambda^2$	γ

TABLE I. The properties of the critical points.

$$x' = -3x + \lambda\sqrt{\frac{3}{2}}y^2 + \frac{3}{2}x\left[2x^2 + \gamma\left(1 - x^2 - y^2\right)\right] \ , \quad (6)$$

$$y' = -\lambda\sqrt{\frac{3}{2}}xy + \frac{3}{2}y\left[2x^2 + \gamma\left(1 - x^2 - y^2\right)\right] \ , \quad (7)$$

where a prime denotes a derivative with respect to the logarithm of the scale factor, $N \equiv \ln(a)$, and the constraint equation becomes

$$\frac{\kappa^2\rho_\gamma}{3H^2} + x^2 + y^2 = 1 \ . \quad (8)$$

Note that from the constraint equation we have

$$\Omega_\phi \equiv \frac{\kappa^2\rho_\phi}{3H^2} = x^2 + y^2 \ . \quad (9)$$

This is bounded, $0 \leq x^2 + y^2 \leq 1$, for a non-negative fluid density, $\rho_\gamma \geq 0$, and so the evolution of this system is completely described by trajectories within the unit disc. The lower half-disc, $y < 0$, corresponds to contracting universes. As the system is symmetric under the reflection $(x, y) \to (x, -y)$ and time reversal $t \to -t$, we only consider the upper half-disc, $y \geq 0$ in the following discussion.

The effective equation of state for the scalar field at any point is given by

$$\gamma_\phi \equiv \frac{\rho_\phi + p_\phi}{\rho_\phi} = \frac{\dot\phi^2}{V + \dot\phi^2/2} = \frac{2x^2}{x^2 + y^2} \quad (10)$$

Fixed points at finite values of x and y in the phase-plane correspond to solutions where the scalar field has a barotropic equation of state and the scale factor of the universe evolves as $a \propto t^p$ where $p = 2/3\gamma_\phi$.

Depending on the values of γ and λ, we have up to five fixed points (critical points) where $x' = 0$ and $y' = 0$ which are listed in Table I. A full analysis of the stability is given in the Appendix.

Two of the fixed points ($x = \pm 1$, $y = 0$) correspond to solutions where the constraint Eq. (4) is dominated by the kinetic energy of the scalar field with a stiff equation of state, $\gamma_\phi = 2$. As expected these solutions are unstable and are only expected to be relevant at early times.

More surprisingly, however, we find that the barotropic fluid dominated solution ($x = 0$, $y = 0$) where $\Omega_\phi = 0$ is *unstable* for all values of $\gamma > 0$. We will discuss the critical case where $\gamma = 0$ later. But for any $\gamma > 0$, and however steep the potential (i.e. whatever the value of λ), the energy density of the scalar field *never* vanishes with respect to the other matter in the universe.

We are left with only two possible late-time attractor solutions. One of these is the well-known scalar field dominated solution ($\Omega_\phi = 1$) which exists for sufficiently flat potentials, $\lambda^2 < 6$. The scalar field has an effective barotropic index $\gamma_\phi = \lambda^2/3$ giving rise to a power-law inflationary expansion [4] ($\ddot a > 0$) for $\lambda^2 < 2$. Previous phase-plane analyses [5] have shown that a wide class of homogeneous vacuum models approach the spatially-flat FRW model for $\lambda^2 < 2$. We have shown that this scalar field dominated solution is a late-time attractor in the presence of a barotropic fluid when we have $\lambda^2 < 3\gamma$.

However for $\lambda^2 > 3\gamma$ we find a different late-time attractor where neither the scalar-field nor the barotropic fluid entirely dominates the evolution. Instead we have a scaling solution where the energy density of the scalar field remains proportional to that of the barotropic fluid with $\Omega_\phi = 3\gamma/\lambda^2$. This solution was first found by Wetterich [6] and shown to be the global attractor solution for $\lambda^2 > 3\gamma$ in Ref. [7].

The regions of (γ, λ) parameter space leading to different qualitative evolution are indicated in Fig. 1.

I. $\lambda^2 < 3\gamma$. See Fig. 2.
Both kinetic-dominated solutions are unstable nodes. The fluid-dominated solution is a saddle point. The scalar field dominated solution is the late-time attractor, and is inflationary in parameter region Ia and non-inflationary in region Ib.

II. $3\gamma < \lambda^2 < 6$. See Fig. 3.
Both kinetic-dominated solutions are unstable nodes. The fluid-dominated solution is a saddle point. The scalar field dominated solution is

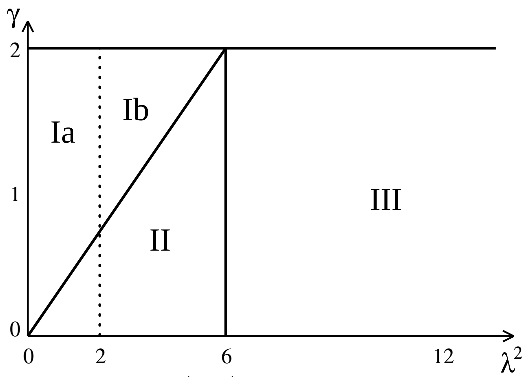

FIG. 1. Regions of (γ, λ) parameter space, as identified in the text. Solutions to the left of the dotted line are inflationary.

a saddle point. The scaling solution is a stable node/spiral.

III. $6 < \lambda^2$. See Fig. 4.
The kinetic-dominated solution with $\lambda x < 0$ is an unstable node. The kinetic-dominated solution with $\lambda x > 0$ is a saddle point. The fluid-dominated solution is a saddle point. The scaling solution is a stable spiral.

III. COSMOLOGICAL CONSEQUENCES

The possible role of scalar fields with exponential potentials with $\lambda^2 < 2$ in driving an inflationary expansion of the early universe has already received considerable attention and so we will not devote much time to discussing this here. However we note that because the scalar field dominated solution is the late-time attractor for $\lambda^2 < 3\gamma$, the existence of such a scalar field today is ruled out unless its energy density has been greatly suppressed relative to the attractor value ($\Omega_\phi = 1$) for most of the "dust-dominated" era where $\gamma = 1$. Nonetheless such models have been considered as possible 'decaying cosmological constant' models [9,8,10].

The peculiar properties of the scaling solution, which we see is the late-time attractor for exponential potentials with $\lambda^2 > 3\gamma$, are sufficiently novel to merit greater investigation. (See also Refs. [11,8].)

The most striking possibility is that a scalar field with an exponential potential could comprise a significant fraction of the energy density of our universe today. Because the effective barotropic index of the homogeneous scalar field would mimic pressureless dust, its dynamical effect would be exactly like cold dark matter. For instance, if $\lambda = 3$ then we expect $\Omega_\phi = 1/3$ today. However the inhomogeneous field can have a different equation of state modifying the evolution of large-scale structure in the universe, as has recently been investigated elsewhere [8,10].

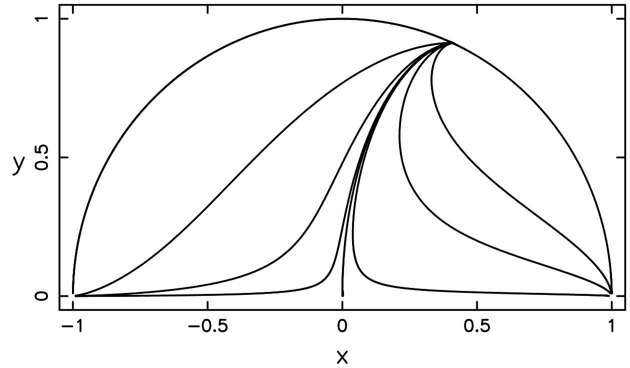

FIG. 2. The phase plane for $\gamma = 1$, $\lambda = 1$. The late-time attractor is the scalar field dominated solution with $x = \sqrt{1/6}$, $y = \sqrt{5/6}$.

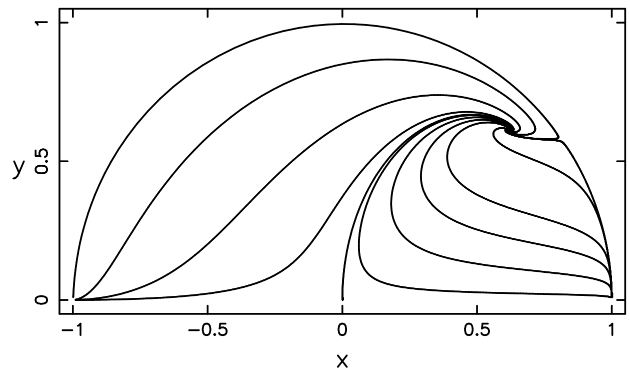

FIG. 3. The phase plane for $\gamma = 1$, $\lambda = 2$. The scalar field dominated solution is a saddle point at $x = \sqrt{2/3}$, $y = \sqrt{1/3}$, and the late-time attractor is the scaling solution with $x = y = \sqrt{3/8}$.

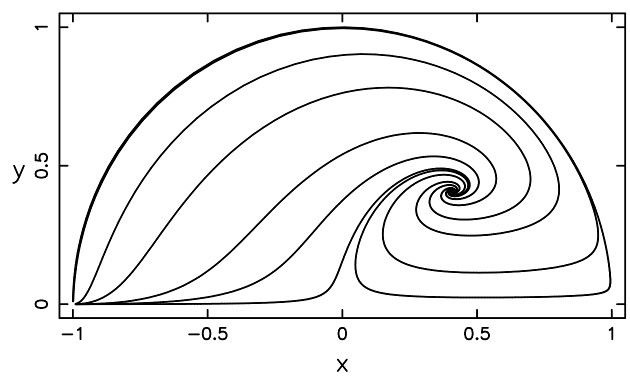

FIG. 4. The phase plane for $\gamma = 1$, $\lambda = 3$. The late-time attractor is the scaling solution with $x = y = \sqrt{1/6}$.

The main problem with this scenario is if the scalar field has a significant contribution to the energy density, throughout the present dust-dominated era, then, unlike conventional cold dark matter, it should also have had a significant effect during the radiation-dominated era. For $\lambda = 3$ we expect $\Omega_\phi = 4/9$ when $p_\gamma = \rho_\gamma/3$.

The tightest constraint on the total energy density of the universe comes from models of nucleosynthesis [6,7]. The primordial abundances of the light elements place tight constraints on the expansion rate, and hence the energy density, at the time of nucleosynthesis, when $T \sim 1$ MeV. If we require $\Omega_\phi < \Omega_\phi^{\rm max}$ at the time of nucleosynthesis, then this implies

$$\lambda^2 > \frac{4}{\Omega_\phi^{\rm max}} \ , \tag{11}$$

The current upper bound on Ω_ϕ at nucleosynthesis is estimated to be in the range 0.13 to 0.2 [8]; we'll adopt the higher value to be conservative. Satisfying the nucleosynthesis bound requires $\lambda^2 > 20$.

Thus there is a relic abundance problem for any particle physics theories that predict the existence of scalar fields with exponential potentials with $\lambda^2 < 20$ at low energies ($T \sim 1$ MeV). Scalar fields with exponential potentials completely dominate the energy density of the universe at nucleosynthesis for $\lambda^2 < 4$, and still have an unacceptably high energy density at nucleosynthesis for $4 < \lambda^2 < 4/\Omega_\phi^{\rm max}$ unless the initial energy density in the field is extraordinarily low.

To quantify exactly how small the initial energy density must be to evade the bound Eq. (11), we expand to first-order about the fluid-dominated solution to find the rate at which Ω_ϕ grows away from zero — see Appendix. We find $\Omega_\phi \propto a^{3\gamma}$, which implies that the scalar field energy density remains essentially constant due to the large friction term in the evolution equation Eq. (3) as the barotropic fluid density redshifts as $\rho_\gamma \propto a^{-3\gamma}$. Thus the scalar field acts like a cosmological constant until Ω_ϕ approaches its attractor value. To have not reached the attractor by some time $t_{\rm f}$ requires an initial value at $t_{\rm i}$ satisfying

$$\Omega_\phi(t_{\rm i}) \lesssim \frac{\rho_\gamma(t_{\rm f})}{\rho_\gamma(t_{\rm i})} \, . \tag{12}$$

IV. THE ROLE OF INFLATION

The usual cosmological solution to relic abundance problems is a period of inflation, during which the unwanted relics have their energy density redshifted to a negligible value relative to the potential energy U of the inflaton field σ.* If inflation ends at some energy density $\rho_\gamma(t_{\rm i}) \sim M^4$, then from Eq. (12) we require

$$\Omega_\phi(t_{\rm i}) \lesssim \left(\frac{1 \ {\rm MeV}}{M}\right)^4 \tag{13}$$

for the scalar field not to have reached the scaling attractor solution by the time of nucleosynthesis. However we will now show that the expected density of the scalar field at nucleosynthesis is not significantly affected by standard models of inflation.

During conventional slow-roll inflation the inflaton field σ has an effective equation of state $\gamma \approx 2\epsilon/3$, where ϵ is the (non-negative) slow-roll parameter controlling the slope of the potential [12]. We can treat γ as effectively constant provided the approach to the scaling attractor (determined by the largest eigenvalue for linear perturbations) is faster than the movement of that attractor. During slow-roll inflation this requires

$$\left|\frac{\epsilon'}{\epsilon}\right| \lesssim 2\epsilon \ , \tag{14}$$

which is valid for most models, including chaotic inflation with $U(\sigma) \propto \sigma^n$ for $n \gtrsim 2$.

The inflaton-dominated solution where $\Omega_\phi = 0$ is not an attractor solution unless $\gamma = 0$ and so for $\lambda^2 > 2\epsilon$ the attractor solution is the scaling solution with

$$\Omega_\phi \approx \frac{2\epsilon}{\lambda^2} \ . \tag{15}$$

Unlike ordinary matter the energy density of a scalar field with an exponential potential *does not vanish* relative to the inflaton energy density during inflation, even though for $\lambda^2 > 2$ the exponential potential would not have an inflationary equation of state in the absence of the inflaton.

To evade the nucleosynthesis bound on λ and satisfy Eq. (13) requires an extremely small value of ϵ. In slow-roll inflation ϵ must be smaller than unity, but is not usually very small. For instance in chaotic inflation with a potential $U(\sigma) = m^2\sigma^2/2$, $\epsilon \simeq 0.01$ when perturbations on the current horizon scale were generated, and increases to unity at the end of inflation before the inflaton starts oscillating about the minimum of its potential. Thus the energy density of a scalar field with an exponential potential is not significantly reduced during chaotic inflation, and can rapidly attain its attractor solution when the radiation-dominated era begins.

Models such as hybrid inflation [13] or thermal inflation [14] are distinctive in that ϵ can remain very small during the final stages of inflation and thus Ω_ϕ may be very small at the attractor scaling solution. However in this case the condition Eq. (14) for quasi-constant γ is violated and the attractor value for Ω_ϕ approaches zero faster than the actual solution. We effectively have $\gamma \to 0$, which corresponds to the critical case where the scaling solution tends to the fluid-dominated solution with $x = y = 0$.

For $\gamma = 0$ the largest eigenvalue for linear perturbations vanishes (see Appendix) and we must consider higher-order perturbations about the critical point to determine its stability. We find that $x = y = 0$ is a stable attractor, but that trajectories only approach this as the

*Note that we are assuming that the field σ is unrelated to the field ϕ which we have been discussing up until now.

logarithm of the scale factor, N. The late-time evolution is given by

$$y^2 = \frac{\sqrt{6}}{\lambda} x \approx \frac{1}{\lambda^2 N} \ . \tag{16}$$

Thus even the extreme case of a cosmological constant (or constant false-vacuum energy density) only dilutes the energy density of the scalar field as the logarithm of the scale factor, $\Omega_\phi \propto 1/N$. Thus a model such as thermal inflation, which is so effective at diluting the abundance of relic moduli particles [14], has a negligible effect on the relic density of the scalar field ϕ as it only lasts for a small number of e-foldings (typically about 15) and the scalar field can rapidly return to its scaling solution after inflation. Even in the case of hybrid inflation the relic density after inflation may be significant unless we have an extremely large number of e-foldings during inflation. To satisfy Eq. (13) requires

$$N \gtrsim \frac{1}{\lambda^2} \left(\frac{M}{1\ \mathrm{MeV}} \right)^4 \tag{17}$$

which could be of order 10^{60} e-foldings for a typical hybrid inflation model. Requiring so much expansion would be a significant constraint on the model.

V. CONCLUSIONS

We have presented a phase-plane analysis of the evolution of a spatially-flat FRW universe containing a barotropic fluid plus a scalar field with an exponential potential $V(\phi) = V_0 \exp(-\lambda\kappa\phi)$. We have shown that the energy density of the scalar field dominates at late times for $\lambda^2 < 3\gamma$. For $\lambda^2 > 3\gamma$ we find that the barotropic fluid does not completely dominate and the energy density of the scalar field remains a fixed fraction of the total density at late times.

This leads to a relic density problem at nucleosynthesis in such models if $\lambda^2 \lesssim 20$. Standard models of inflation do not significantly dilute the initial density of the exponential potential and do not alleviate this bound. Only inflation models with effectively constant energy density and an exponentially large number of e-foldings (such as some models of hybrid inflation) would be able to weaken this bound, unless the radiation-dominated era only begins shortly before nucleosynthesis, e.g. Ref. [15].

We emphasize that we have assumed that there is no direct coupling between the exponential potential and other matter. The only interaction is gravitational.

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APPENDIX: STABILITY OF THE CRITICAL POINTS

In order to study the stability of the critical points (x_c, y_c) we expand about these points

$$x = x_c + u \ , \qquad y = y_c + v \ , \tag{A1}$$

which when substituted into Eqs. (6) and (7) yields, to first-order in the perturbations, equations of motion

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \mathcal{M} \begin{pmatrix} u \\ v \end{pmatrix} \ . \tag{A2}$$

The general solution for the evolution of linear perturbations can be written as

$$u = u_+ \exp(m_+ N) + u_- \exp(m_- N) \ , \tag{A3}$$

$$v = v_+ \exp(m_+ N) + v_- \exp(m_- N) \ , \tag{A4}$$

where $m_\pm$ are the eigenvalues of the matrix $\mathcal{M}$. Thus for stability we require the real part of both eigenvalues to be negative.

For the critical points listed in Table 1 we find:

Fluid-dominated solution:

$$m_- = -\frac{3(2-\gamma)}{2} \ , \qquad m_+ = \frac{3\gamma}{2} \ . \tag{A5}$$

Kinetic-dominated solutions, ($x_c = \pm 1$, $y_c = 0$):

$$m_- = \sqrt{\frac{3}{2}} \left(\sqrt{6} \mp \lambda \right) \ , \qquad m_+ = 3(2-\gamma) \ . \tag{A6}$$

Scalar field dominated solution:

$$m_- = \frac{\lambda^2 - 6}{2} \ , \qquad m_+ = \lambda^2 - 3\gamma \ . \tag{A7}$$

Scaling solution:

$$m_\pm = -\frac{3(2-\gamma)}{4} \left[1 \pm \sqrt{1 - \frac{8\gamma(\lambda^2 - 3\gamma)}{\lambda^2(2-\gamma)}} \right] \ . \tag{A8}$$

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