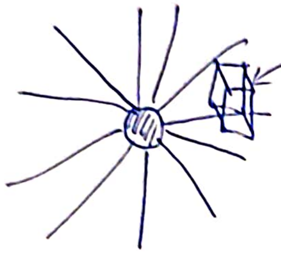
Reissner - Nordström metric ← charged non rotating

Schwarzschild ← uncharged, non-rotating

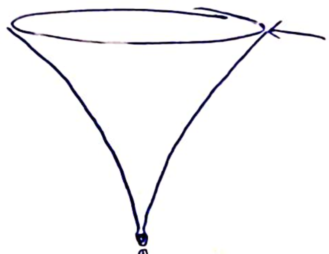
$$ds^2 = e^{\nu} c^2 dt^2 - e^{\lambda} dr^2 - r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$$

Non-zero stress energy tensor due to EM field.

$$T_{\mu\nu} = \frac{1}{4\pi}\left(\frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - F_{\mu\alpha} F^{\alpha}_{\ \nu}\right)$$



This $d\tau$ element will have $\varepsilon = \frac{1}{2}\varepsilon_0 |E_0|^2$ } This will contribute to $T_{\mu\nu}$



r_h

singularity
(All the charge resides here)

* IF photons are mediators of E.M field, then how can any influence get's out of B.H's event horizon?

* Also, wouldn't there be ∞ dilution (sort of red shift) of all photons.

→ Electric field is only in radial direction.

$$A_\mu = \begin{pmatrix} \phi \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

For static charge, only F_{01} component survives

$$F_{01} = \partial_0 A_1 - \partial_1 A_0 = -\frac{\partial A_0}{\partial r}$$

$$T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\alpha}F^{\alpha}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right)$$

energy-momentum density

corrections

energy density T_{00}:

$$T_{00} = \frac{1}{4\pi}\left(F_{00}F^{0}_{0} - \frac{1}{4}g_{00}F_{00}F^{00}\right)$$

We put $T_{\mu\nu}$ in field eq. & get the metric

About $F_{\mu\nu}$

$$F_{\mu\nu} = -F_{\nu\mu}$$

only E.F in x-direction

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & 0 & 0 \\ E_x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

changes signs in spatial direction (for minkowski)

~~$F_{\mu\nu}$~~ $F^{\mu}_{\nu} = g^{\mu\alpha}F_{\alpha\nu}$

$$F^{\mu}_{\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}\begin{pmatrix} 0 & -E_x & 0 & 0 \\ +E_x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -E_x & 0 & 0 \\ -E_x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$F_{0i} = \partial_0 A_i - \partial_i A_0$$

$$= \frac{\partial}{\partial t}(A_r + A_\theta + A_\phi) - \left(\frac{\partial}{\partial r} + \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \phi}\right)(\Phi)$$

Scalar Potential

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1. Homogeneous eq.:

$$\epsilon^{\alpha\beta\mu\nu} \partial_\beta F_{\mu\nu} = 0$$

2. Inhomogeneous eq.

$$\boxed{\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu} \quad \text{4-current}$$

$B \hat{z}$

$$F_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -B & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Properties of $F^{\mu\nu}$

1. $F^{\mu\nu} = -F^{\nu\mu}$

2. 6 independent component (3 for $\vec{E}$ & 3 for $\vec{B}$)

3. Inner Product: $F_{\mu\nu} F^{\mu\nu} = 2\left(B^2 - \frac{E^2}{2}\right)$

   Lorentz invariant

4. determinant: $\det(F) = \frac{1}{c^2} (B.E)^2$

5. Trace $F^\mu{}_\mu = 0$

$A^\mu = (A^0, A^1, A^2, A^3)$ ← using contravariant indices to build a rank 2 tensor.

$\partial^\mu = (-\partial^0, \partial^1, \partial^2, \partial^3)$

$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$

$\vec{B} = \vec{\nabla} \times \vec{A}$

$$\vec{E} = -(\partial^0+\partial^1+\partial^2+\partial^3)A^0 - \underset{\cancel{\delta t}}{\partial^0}(A^1+A^2+A^3)$$

$$E_x = -\partial^1 A^0 + \partial^0 A^1 = \partial^0 A^1 - \partial^1 A^0 = F^{01}$$

$$E_y = -\partial^2 A^0 + \partial^0 A^2 = \partial^0 A^2 - \partial^2 A^0 = F^{02}$$

$$E_z = -\partial^3 A^0 + \partial^0 A^3 = \partial^0 A^3 - \partial^3 A^0 = F^{03}$$

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} i & j & k \\ \partial^1 & \partial^2 & \partial^3 \\ A^1 & A^2 & A^3 \end{vmatrix}$$

$$\vec{\nabla}\times\vec{A} = \hat{i}(\partial^2A^3 - \partial^3A^2) - \hat{j}(\partial^1A^3 - \partial^3A^1) + \hat{k}(\partial^1A^2 - \partial^2A^1)$$

$$B_x = \partial^2A^3 - \partial^3A^2 = F^{23}$$

$$B_y = -(\partial^1A^3 - \partial^3A^1) = F^{13}$$

$$B_z = (\partial^1A^2 - \partial^2A^1) = F^{12}$$

$$F^{\mu\nu} = \begin{array}{c|c} & \xrightarrow{\quad 0 \quad 1 \quad 2 \quad 3 \quad} A^\nu \\ \begin{array}{c}0\\1\\2\\3\end{array}\Big\downarrow \partial^\mu & \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_y & -B_x & & 0 \end{bmatrix}_{4\times4} \end{array}$$

$$\boxed{F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu}$$

$$T_{\mu\nu} = \frac{1}{4\pi}\left(\frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - F_{\mu\alpha} F^{\alpha}_{\ \nu}\right)$$

due to E.M Field

metric on which this E.M Field is sitting.

* But, wouldn't the presence of $T_{\mu\nu}$ change $g_{\mu\nu}$ (or is it small enough to ignore)

If $\parallel A$ $ds^2 = e^{\nu(r)} c^2 dt^2 - e^{\lambda(r)} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$

$g_{00} = e^{\nu}$	$g^{00} = e^{-\nu}$
$g_{11} = -e^{\lambda}$	$g^{11} = -e^{-\lambda}$
$g_{22} = -r^2 \sin^2\theta$	$g^{22} = -\frac{1}{r^2 \sin^2\theta}$

$$F^{\mu\nu} = g^{\mu\beta} g^{\alpha\nu} F_{\alpha\beta}$$

$$g^{\alpha\nu} F_{\alpha\beta} = F^{\nu}_{\ \beta}$$

$$g^{\mu\beta} F^{\nu}_{\ \beta} = F^{\mu\nu}$$

$$F^{01} = g^{00} g^{11} F_{01}$$

$$= (e^{-\nu})(-e^{-\lambda}) \frac{\partial A_0}{\partial r}$$

$$F^{01} = -e^{-(\nu+\lambda)} \frac{\partial A_0}{\partial r}$$

$$F_{\alpha\beta} F^{\alpha\beta} = F_{01} F^{01}$$

$$=$$

$$\boxed{\begin{array}{l}\sqrt{-g} \\ \uparrow \det(g_{\mu\nu})\end{array}}$$

Stress Tensor terms

1. $F_{\alpha\beta} F^{\alpha\beta}$ only F_{01} & F_{10} term are non zero

$$\sum_{\alpha,\beta} F_{\alpha\beta} F^{\alpha\beta}$$

$$F_{01} = \frac{\partial A_0}{\partial r}$$

$$\Rightarrow F_{01} F^{01} + F_{10} F^{10}$$

$$F_{10} = -\frac{\partial A_0}{\partial r}$$

$$\Rightarrow 2\left(e^{-\gamma}\left(-e^{-\lambda}\right)\left(\frac{\partial A_0}{\partial r}\right)^2\right)$$

$$\begin{pmatrix} 0 & F_{01} & 0 & 0 \\ F_{10} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \; \end{pmatrix}$$

$F_{\mu r} \; g$

$$\boxed{F_{\alpha\beta} F^{\alpha\beta} = -2\, e^{-(\gamma+\lambda)}\left(\frac{\partial A_0}{\partial r}\right)^2}$$

(This now became a scalar) which will then be multiplied with the metric $g_{\mu\nu}$

2. <u>Computing $F_{\mu\alpha} F^{\alpha}_{\ \gamma}$</u> :

<u>$\mu=0,\ \gamma=0$</u>

$$\sum_{\alpha=0,1} F_{0\alpha} F^{\alpha}_{\ 0} = F_{00} F^{0}_{\ 1} + F_{01} F^{1}_{\ 0}$$

$$\sum_{\beta} F_{\beta 0}\, g^{\alpha\beta} = F^{\alpha}_{\ 0} \Rightarrow F_{00}\, g^{\alpha 0} + F_{10}\, g^{\alpha 1}$$

$$F^{\alpha}_{\ 0} = F_{10}\, g^{\alpha 1}$$

$$= -\frac{\partial A_0}{\partial r}\, g$$

$$\sum_{\alpha=0,1} F_{\mu\alpha} F^{\alpha}_{\ \gamma} = F_{\mu 1} F^{1}_{\ \gamma} + F_{\mu 0} F^{0}_{\ \gamma}$$

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changing index of a tensor by applying metric tensor

Raising 1st index : $\boxed{T^c_{\ b} = g^{ac}\, T_{ab}}$

$$= \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

$$T^c_{\ b} = \begin{pmatrix} g^{11}T_{11} + g^{12}T_{21} & g^{11}T_{12} + g^{12}T_{22} \\ g^{21}T_{11} \quad g^{22}T_{21} & g^{21}T_{12} \quad g^{22}T_{22} \end{pmatrix}$$

$$\cancel{T_a{}^c} = \cancel{g^{bc}}\, \cancel{T_{ab}}$$

$$= \cancel{\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix}} \cancel{\left(T \right.}$$

Raising 2nd index : $\boxed{T_a{}^c = T_{ab}\, g^{bc}}$

$$= \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix}$$

$$T_a{}^c = \begin{pmatrix} T_{11}g^{11} + T_{12}g^{21} & T_{11}g^{12} + T_{12}g^{22} \\ T_{21}g^{11} + T_{22}g^{21} & T_{21}g^{12} + T_{22}g^{22} \end{pmatrix}$$

$$\cancel{f_{\mu\nu}} \quad f_{ab} \to f_\alpha{}^b$$

$$f_\alpha{}^b = \cancel{f_{ab}\, g^{ab}}\; f_{\alpha c}\, g^{cb}$$

$$= \cancel{\begin{pmatrix} f_{10} & f_{11} & f_{12} & f_{13} \\ 0 & & & \end{pmatrix}}$$

$$= \begin{pmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{pmatrix} \begin{pmatrix} g^{00} & 0 & 0 & 0 \\ 0 & g^{11} & 0 & 0 \\ 0 & 0 & g^{22} & 0 \\ 0 & 0 & 0 & g^{33} \end{pmatrix}$$

$$= \begin{pmatrix} f_{00} g^{00} & f_{01} g^{11} & f_{02} g^{22} & f_{03} g^{33} \\ f_{10} g^{00} & f_{11} g^{11} & f_{12} g^{22} & f_{13} g^{33} \\ f_{20} g^{00} & f_{21} g^{11} & f_{22} g^{22} & f_{23} g^{33} \\ f_{30} g^{00} & f_{31} g^{11} & f_{32} g^{22} & f_{33} g^{33} \end{pmatrix}$$

$$f_{01} = \frac{\partial A_0}{\partial r} \qquad g^{00} = e^{-\nu}$$

$$f_{10} = \frac{-\partial A_0}{\partial r} \qquad g^{11} = -e^{-\lambda}$$

$$\boxed{f_\gamma{}^\alpha = \begin{pmatrix} 0 & f_{01} g^{11} & 0 & 0 \\ f_{10} g^{00} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}$$

$$f_{\mu\alpha} f_\gamma{}^\alpha = \begin{pmatrix} 0 & f_{01} & 0 & 0 \\ f_{10} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & f_{01} g^{11} & 0 & 0 \\ f_{10} g^{00} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$f_{\mu\alpha} f_\gamma{}^\alpha = \begin{pmatrix} f_{01} f_{10} g^{00} & 0 & 0 & 0 \\ 0 & f_{10} f_{01} g^{11} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -\left(\frac{\partial A_0}{\partial r}\right)^2 e^{-\nu} & 0 & 0 & 0 \\ 0 & +\left(\frac{\partial A_0}{\partial r}\right)^2 e^{-\lambda} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Computing $\frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}$

$$F_{\alpha\beta} F^{\alpha\beta} = -2 e^{-(\lambda+\nu)} \left(\frac{\partial A_0}{\partial r}\right)^2$$

$$\frac{1}{4}\begin{pmatrix} e^{\nu} & & & \\ & -e^{\lambda} & & \\ & & -r^2 & \\ & & & -r^2\sin^2\theta \end{pmatrix}\left(-2e^{-(\lambda+\nu)}\left(\frac{\partial A_0}{\partial r}\right)^2\right)$$

$$\cancel{\tfrac{2}{4}}\begin{pmatrix} -\frac{1}{2}e^{-\lambda}\left(\frac{\partial A_0}{\partial r}\right)^2 & 0 & 0 & 0 \\ 0 & +\frac{1}{2}e^{-\nu}\left(\frac{\partial A_0}{\partial r}\right)^2 & 0 & 0 \\ 0 & 0 & +\frac{1}{2}r^2 e^{-(\lambda+\nu)}\left(\frac{\partial A_0}{\partial r}\right)^2 & 0 \\ 0 & 0 & 0 & +\frac{1}{2}r^2\sin^2\theta\, e^{-(\lambda+\nu)}\left(\frac{\partial A_0}{\partial r}\right)^2 \end{pmatrix}$$

Combining everything :

$$T_{\mu\nu} = \frac{1}{4\pi}\left[-F_{\nu\alpha} F_{\nu}{}^{\alpha} + \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}\right]$$

$$= \frac{1}{4\pi}\left[\begin{pmatrix} \left(\frac{\partial A_0}{\partial r}\right)^2 e^{-\nu} & 0 & 0 & 0 \\ 0 & -\left(\frac{\partial A_0}{\partial r}\right)^2 e^{-\lambda} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \left(\frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}\right)\right]$$

$$= \frac{1}{4\pi}\left[\begin{pmatrix} (\partial_r A_0)^2 e^{-\nu} - \frac{1}{2} e^{-\lambda}(\partial_r A_0)^2 & 0 & 0 & 0 \\ 0 & -(\partial_r A_0)^2 e^{-\lambda} + \frac{1}{2} e^{-\nu}(\partial_r A_0)^2 & 0 & 0 \\ 0 & 0 & \frac{1}{2} r^2 e^{-(\lambda+\nu)}(\partial_r A_0)^2 & 0 \\ 0 & 0 & 0 & \frac{1}{2} r^2 \sin^2\theta\, e^{-(\lambda+\nu)}(\partial_r A_0)^2 \end{pmatrix}\right]$$

$$T^{\mu}{}_{\nu} = g^{\mu\alpha} T_{\alpha\nu}$$

$$= \Bigg($$

$$T_\mu{}^{\nu} = \frac{1}{8\pi} e^{-(\nu+\lambda)} (\partial_r A_0)^2 \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$\boxed{\nabla_\beta F^{\alpha\beta} = \frac{4\pi}{c} j^\alpha}$$ ← 4-current density

Covariant derivative

$$\nabla_\beta V^\beta = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\beta} \left(\sqrt{-g}\, V^\beta \right) = \partial_\beta V^\beta + \Gamma^\beta_{\beta\gamma} V^\gamma$$

Where can I find it's derivation?

why only w.r.t β

$$\boxed{\nabla_\beta F^{\alpha\beta} = \frac{-1}{\sqrt{-g}} \frac{\partial}{\partial x^\beta} \left(\sqrt{-g}\, F^{\alpha\beta} \right)}$$

$$\sqrt{-g} = \sqrt{(-1)\, g} = \sqrt{(+1)(+e^\nu)(+e^\lambda)+r^4 \sin^2\theta}$$

$$\sqrt{-g} = r^2 \sin\theta\, e^{\left(\frac{\nu+\lambda}{2}\right)}$$

$\because$ there is no 4-current density, $j^\alpha = 0$

$$\frac{-1}{\sqrt{-g}} \frac{\partial}{\partial x^\beta} \left(\sqrt{-g}\, F^{\alpha\beta} \right) = 0$$

F^{10} & F^{01} are the only surviving components.

So, $$\frac{\partial}{\partial x^\beta} \left(\cancel{\sqrt{}}\, r^2 \sin\theta\, e^{\left(\frac{\nu+\lambda}{2}\right)} F^{\alpha\beta} \right) = 0$$

$$F^{\alpha\beta}\Big|_{\beta\neq 1} = 0$$

$F^{\alpha\beta}$ & ν & λ has no time dependence, so only ∂_r remains.

$$\frac{\partial}{\partial r} \left(r^2 \sin\theta\, e^{\left(\frac{\nu+\lambda}{2}\right)} F^{\alpha 1} \right) = 0$$

he
uld
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for F^{01}

e'd the
eq.

$$\frac{\partial}{\partial r} \left(r^2 e^{\left(\frac{\nu+\lambda}{2}\right)} F^{01} \right) = 0$$

So, $r^2 e^{\left(\frac{\nu+\lambda}{2}\right)} + e^{-(\nu+\lambda)} \partial_r A_0 = \cancel{*}$ const.

$$\boxed{r^2 e^{-\left(\frac{\nu+\lambda}{2}\right)} \partial_r A_0 = \cancel{*}\ \text{Const.}}$$

$$e^{-\lambda}\left(\frac{\lambda'}{R}-\frac{1}{R^2}\right)+\frac{1}{R^2}=\frac{\alpha}{R^4} \quad -(1)$$

$$e^{-\lambda}\left(\frac{\nu'}{R}+\frac{1}{R^2}\right)-\frac{1}{R^2}=-\frac{\alpha}{R^4} \quad -(2)$$

(1) + (2)

$$e^{-\lambda}\frac{\lambda'}{R}+e^{-\lambda}\frac{\nu'}{R}=0$$

$$\nu'=-\lambda'$$

$$\boxed{\nu=-\lambda}$$

$$e^{-\lambda}\left(\frac{\lambda'}{R}-\frac{1}{R^{2}}\right)=\frac{\alpha}{R^{4}}-\frac{1}{R^{2}}$$

$$\xrightarrow{\times R}\quad \lambda'-\frac{1}{R}=e^{\lambda}\left(\frac{\alpha}{R^3}-\frac{1}{R}\right)$$

$$\boxed{\lambda'=e^{\lambda}\left(\frac{\alpha}{R^3}-\frac{1}{R}\right)+\frac{1}{R}}$$

$$e^{-\lambda}\left(\frac{\nu'}{R}+\frac{1}{R^2}\right)=-\frac{\alpha}{R^4}+\frac{1}{R^2}$$

$$\nu'+\frac{1}{R}=e^{\lambda}\left(-\frac{\alpha}{R^3}+\frac{1}{R}\right)$$

$$\boxed{\nu'=e^{\lambda}\left(-\frac{\alpha}{R^3}+\frac{1}{R}\right)-\frac{1}{R}}$$