Project 1: Monte Carlo Methods for Solving PDEs

FUSRP 2022 Midterm Presentations

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Overview

- Explaining the problem
- Issues with popular methods
- Monte Carlo Markov Chains
- Future work and methods

Part 1: Background and Overview of the Problem

An Overview of PDEs

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Answer: $u(x_1, x_2) = x_1 - x_2$

Laplace's Equation on the Unit Disk

- Example: Laplace's Equation in two dimensions:

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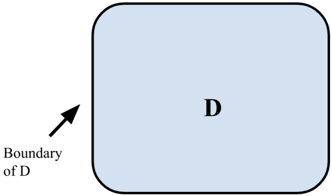
- Can also impose a boundary condition and a particular domain D:

$$\Delta u = 0, \text{ in } D$$
$$u = g \text{ in } \partial D$$



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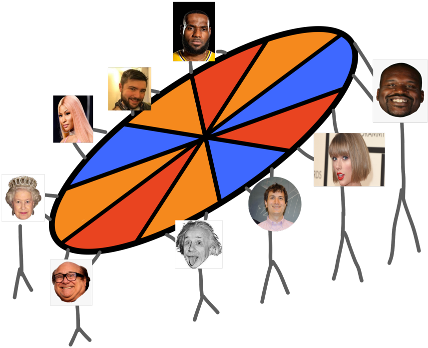

Our Broader Goals Going Forward

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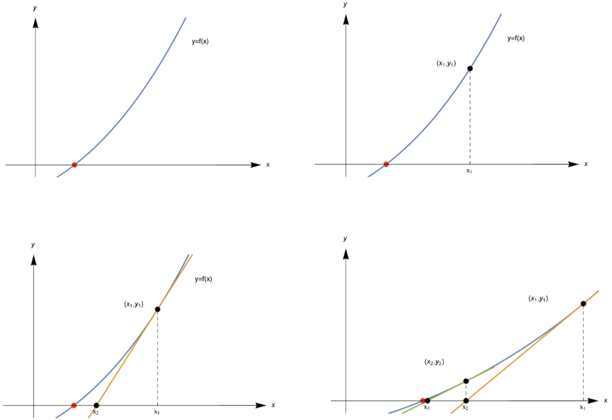

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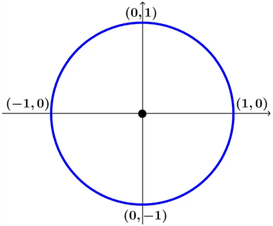

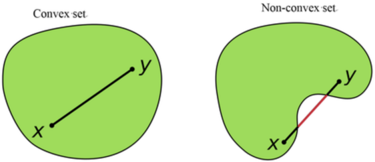

Part 2: Issues With Popular Methods

Popular Numerical Methods

1. Discretize

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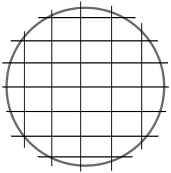
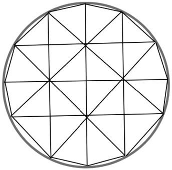
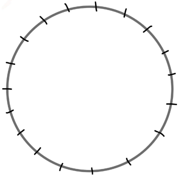
Finite Difference	Finite Element	Boundary Integral Equation
		

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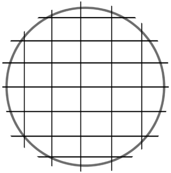	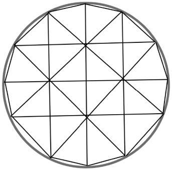	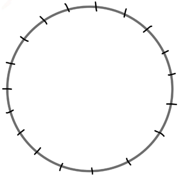

Note:

$$\begin{cases} \Delta u(\mathbf{x}) = 0 & \mathbf{x} \in \Omega \\ u(\mathbf{x}) = g(\mathbf{x}) & \mathbf{x} \in \Gamma \end{cases}$$

$$\Rightarrow u(\mathbf{x}) = \int_\Gamma K(\mathbf{x}, \mathbf{y}) g(\mathbf{y}) dS_{\mathbf{y}} \qquad \mathbf{x} \in \Omega$$

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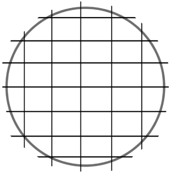	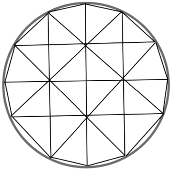	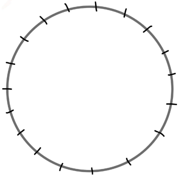

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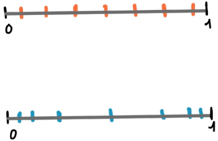
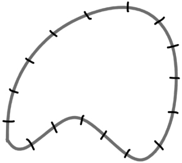
2. Solve the system of equations

Issues

- choosing optimal quadrature for complicated boundaries is non-trivial (e.g. uniform/Chebyshev nodes)





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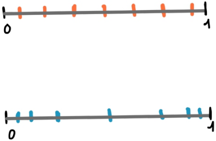
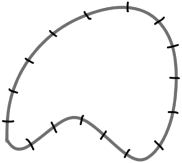
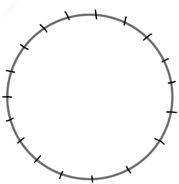
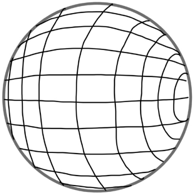




- slow evaluation in high dimensions (e.g. $dim \geq 3$)



dim=2



dim=3

Monte Carlo Methods

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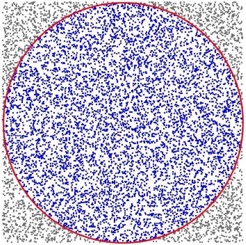

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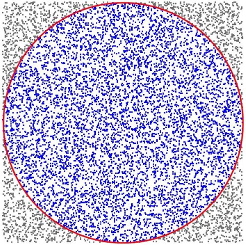

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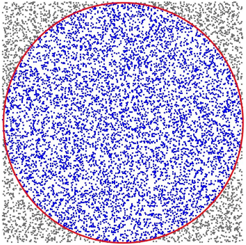

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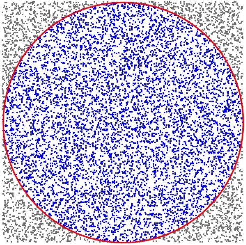

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Note: scales well to higher dimensions

Part 3: Metropolis-Hastings

Introducing Probability to the Equation

Let's try to solve Laplace's PDE on a unit disk with the boundary constraint $g(\theta) = sin(\theta)$. We'll solve at point $(0.5, \frac{\pi}{2})$.

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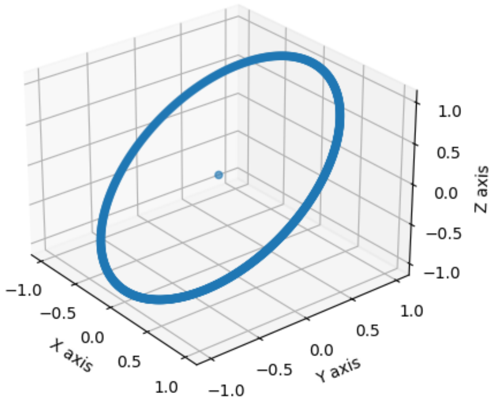
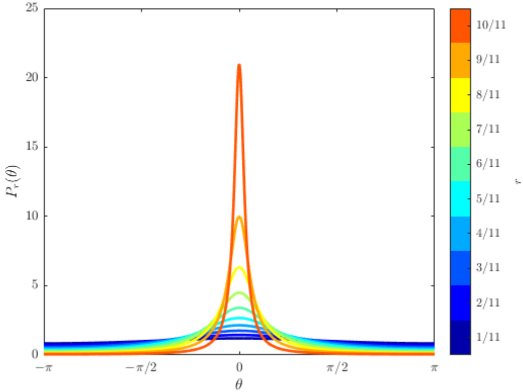


Introducing Probability to the Equation

The analytic solution to Laplace's equation on the unit disc is given by:

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} P(r,\theta-\varphi)g(\varphi)d\varphi$$

where P is the Poisson kernel.



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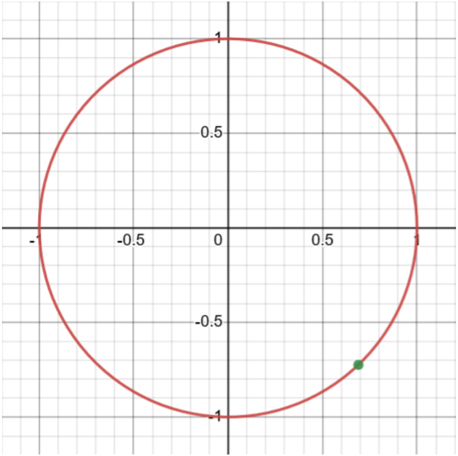
Where $X \sim P(r,\theta - x)$. But how do we sample from the Poisson kernel?

Metropolis-Hastings Algorithm

We can construct a sequence of random variables whose distribution approaches the Poisson kernel!



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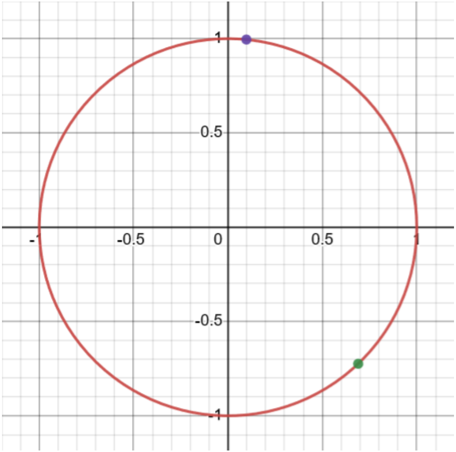

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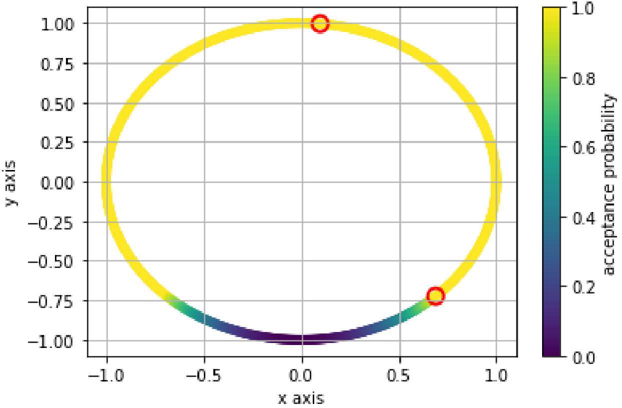

$$A(x'|x) = min(1, \frac{P(r,\theta-x')}{P(r,\theta-x)}) = 1$$

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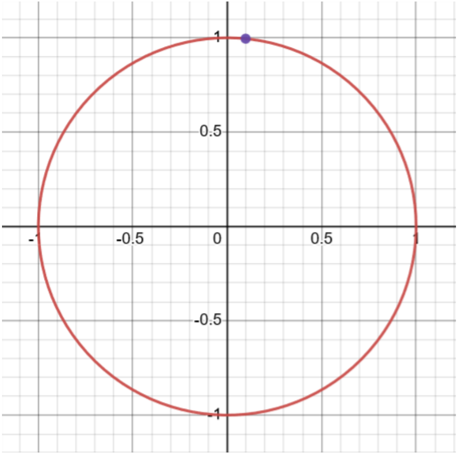

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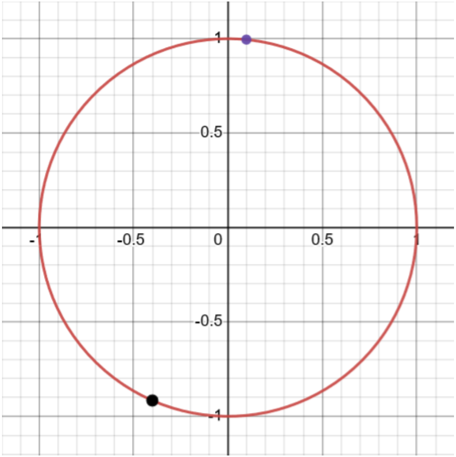

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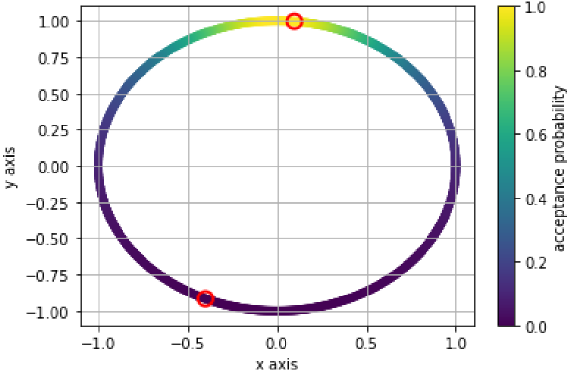

$$A(x'|x) = min(1, \frac{P(r,\theta - x')}{P(r,\theta - x)}) \approx 0.1176$$

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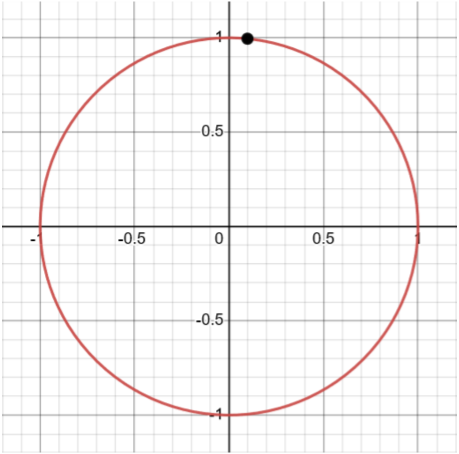
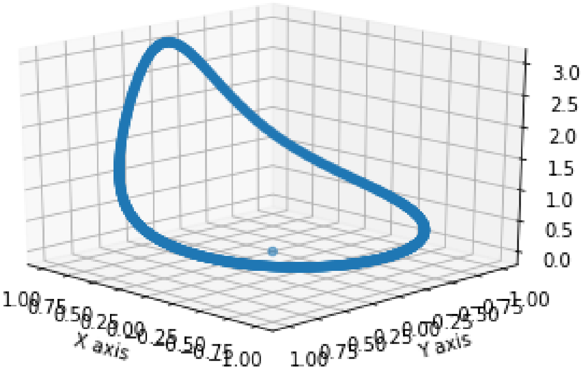


Results

Here's a graph of the Poisson kernel:



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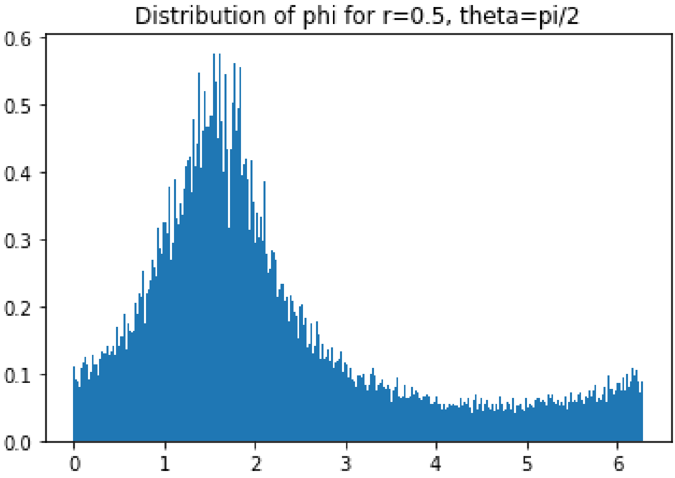
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6

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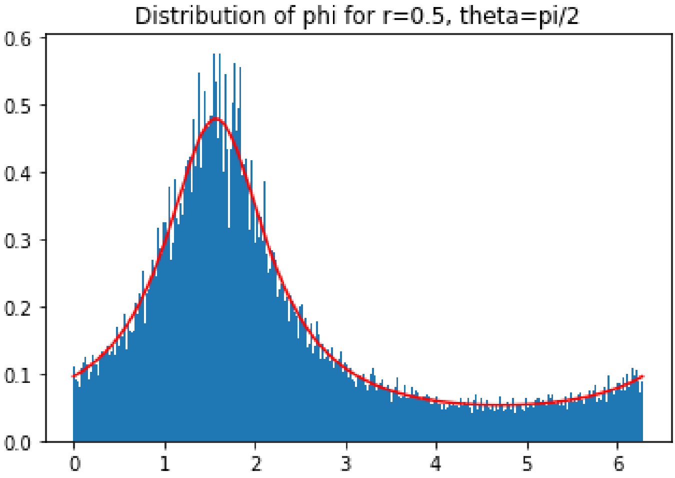

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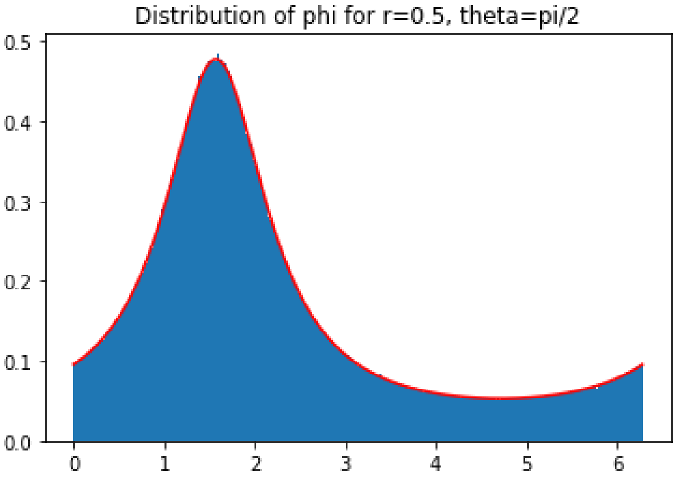

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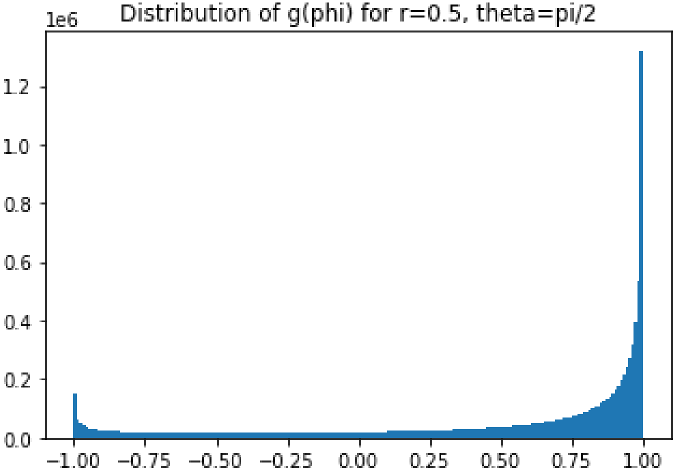
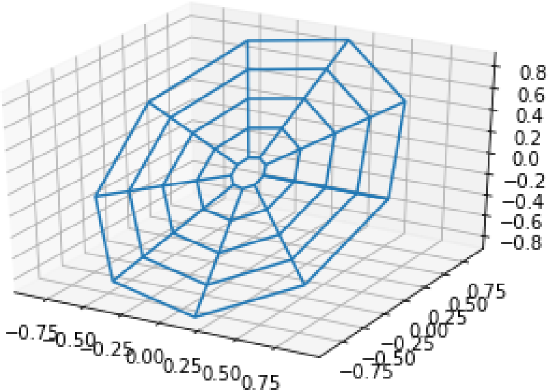
Results

After getting the sequence, we apply g to the sequence and take the mean.



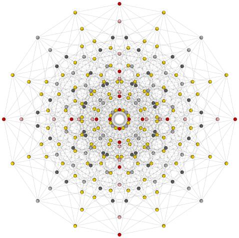
Results

Here's the solution for a bunch of points on the unit disc:



Part 4: Future Work and Methods

Future Directions



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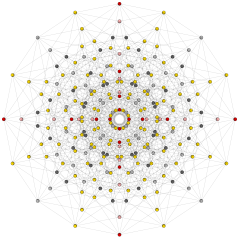

Higher Dimensions

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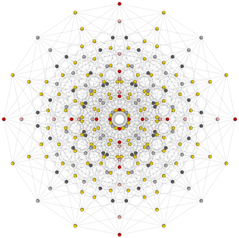

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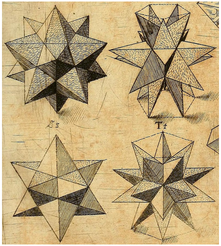

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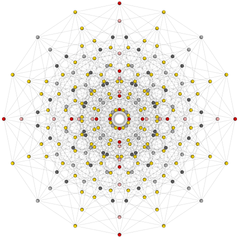

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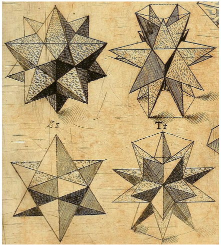

Non-convex Boundary

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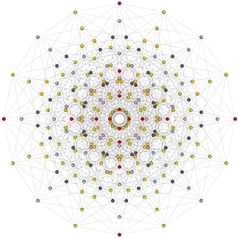

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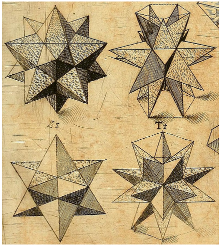

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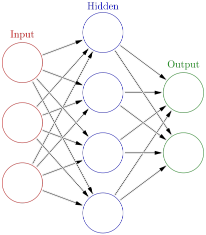

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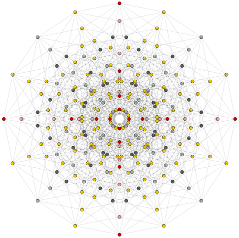

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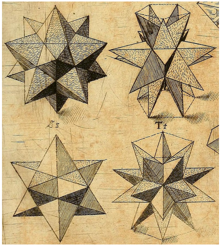

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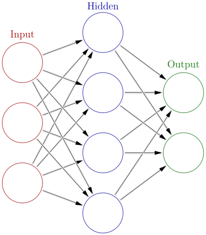
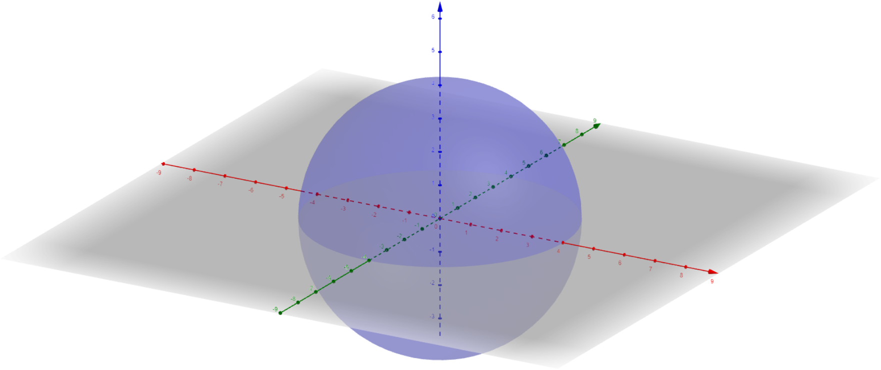


Reinforcement Learning

Unknown Distribution?



Thank you!

References

1. E, W., Li, T. & Vanden-Eijnden, E. (2019). *Applied Stochastic Analysis*. AMS, Providence, R.I. Print.
2. Sabelfeld, K. & Simonov, N. (2016). *Stochastic Methods for Boundary Value Problems: Numerics for High-dimensional PDEs and Applications*. Berlin, Boston: De Gruyter. Link