NONLINEAR ABSORPTIONS ON QUANTUM WELL CONFIGURATIONS *

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ABSTRACT

Here, we present the analysis of half-parabolic, asymmetric, finite, additive quantum well with the presence of a uniform electric field of perpendicular sectioning in the z-axis.

1 Introduction

Our quantum system is concerned of the quantum well:

$$V(z) = \begin{cases} V_0 & x \leq L_1 \\ \frac{1}{2}m_e\omega_z^2 z^2 + \frac{\hbar^2\beta_z}{2m_e z^2} + V_{ext}(z) & L_1 < x < L_2 \\ V_0 & x \geq L_2 \end{cases} \tag{1}$$

Where ω_z is the angular momenta of the electron in the z-direction, m_e the rest mass, $\hbar$ as Planck reduced constant, β_z is the intrinsic constant of the quantum well configuration, and $V_{ext}(z)$ the external electric field of uniform polarized strength in the same dimension. The goal of the main research is then to analyse the nonlinear absorption coefficient of strong electromagnetic waves for electrons confined in an asymmetric finite semi-parabolic quantum well with resonance in the presence of electric fields. Of such, successive operations and formulation will be extensively expressed.

2 Quantum well

A **quantum well**, is a type of constrain that reduce a system of three-dimensional free charge into a two-dimension free-motion system. This is the act of dimensional reduction of specific microscopic matter, which is then called *low-dimensional system*. This is ultimately used for modelling structures of quantum scale, for example, a system of semiconductor (`GaAs` quantum semiconductor system). The quantum well in equation 1 is one of such quantum well, restricting on the z-axis only, leaving the other two dimension free of particle.

In our above well, we notice that it models the factors:

$$V_1(z) = \frac{1}{2}m_e\omega_z^2 z^2, \quad V_2(z) = \frac{\hbar^2\beta_z}{2m_e z^2}, \quad V_3(z) = V_{ext}(z) \tag{2}$$

where V_1 is the half-parabolic (by the coordinate of the range $[L_1, L_2]$), antisymmetric potential, which model a saddle point of potential (simulating some imperfect electron-pulling region), V_2 simulates the smoothing version of the normal square well, by adding reduction on the scale of inverse-square law (factor $1/z^2$), and V_3 for external field potential.

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3 Quantum well without external field

For equation 1, its energy spectrum can be calculated for the external electrical field. If such external field is ruled out, so $V_{\text{ext}}(z) = 0$, then

$$V(z) = \frac{1}{2} m_e \omega_z^2 z^2 + \frac{\hbar^2 \beta_z}{2 m_e z^2} \tag{3}$$

The Schrödinger equation for time-independent, assuming separable solution as always is then:

$$-\frac{\hbar^2}{2m_e} \frac{\partial^2}{\partial z^2} \phi_n(z) + V(z)\phi_n(z) = E\phi_n(z) \tag{4}$$

which is written as we change to the final state of $V(z)$:

$$-\frac{\hbar^2}{2m_e} \frac{\partial^2}{\partial z^2} \phi_n(z) + \left[\frac{1}{2} m_e \omega_z^2 z^2 + \frac{\hbar^2 \beta_z}{2 m_e z^2} \right] \phi_n(z) = E\phi_n(z) \tag{5}$$

This equation is reserved for the z-axis, every other axis can be resolved using the free-particle case. The solution to such problem gives the wavefunction:

$$\phi_n(z) = A_n z^{2s} \exp\left(-\frac{z^2}{2\alpha_z^2} \right) L_n^\delta \left(\frac{z^2}{\alpha_z^2} \right) \tag{6}$$

where L_n^δ are Legendre polynomials, with the following constants:

$$s = \frac{1}{4}\left(1 + \sqrt{1 + 4\beta_z}\right), \quad \delta = 2s - \frac{1}{2}, \quad \alpha_z = \sqrt{\frac{\hbar}{m_e \omega_z}} \tag{7}$$

Here, the constant A_0 is just the normalization constant. Hence, it must satisfy:

$$A_n^2 \int_0^L \left[z^{2s} \exp\left(-\frac{z^2}{2\alpha_z^2} \right) L_n^\delta \left(\frac{z^2}{\alpha_z^2} \right) \right]^2 dz = 1 \tag{8}$$

The ground and first initial state of this wavefunction is

$$\phi_0(z) = A_0 z^{2s} \exp\left(-\frac{z^2}{2\alpha_z^2} \right) \tag{9}$$

$$\phi_1(z) = A_1 z^{2s} \exp\left(-\frac{z^2}{2\alpha_z^2} \right) \left(-\frac{z^2}{\alpha_z^2} + \gamma + 1 \right) \tag{10}$$

which has normalization constant using such calculation being

$$A_0 = \sqrt{2} \alpha_z^{-1/2-2s} \left[\Gamma\left(\frac{1}{2} + 2s \right), \Gamma\left(\frac{1}{2} + 2s, \frac{L^2}{\alpha_z^2} \right) \right]^{-1/2} \tag{11}$$

$$A_1 = 2\sqrt{2} \alpha_z^{-1/2-2s} \left\{ \left[3 + 4\gamma^2 + \gamma(4 - 16s) + 16s^2 \right] \Gamma\left(\frac{1}{2} + 2s \right) - T_1 - T_2 \right\} \tag{12}$$

for the two terms T_1, T_2 being

$$T_1 = 4(1+\gamma)^2 \Gamma\left(\frac{1}{2} + 2s, \frac{L^2}{\alpha_z^2} \right) + \gamma(1+\gamma)\Gamma\left(\frac{3}{2} + 2s, \frac{L^2}{\alpha_z^2} \right) \tag{13}$$

and

$$T_2 = 4\Gamma\left(\frac{3}{2} + 2s, \frac{L^2}{\alpha_z^2} \right) \tag{14}$$

The energy spectrum of the quantum well is then calculated as

$$\mathcal{E} = \hbar\omega_z \left(2n + 1 + \frac{1}{2}\sqrt{1 + 4\beta_z} \right) \tag{15}$$

which reduces to $\hbar\omega_z(2n + 3/2)$ when $\beta_z \to 0$. In general, the form constant is calculated of

$$I_z = 2 \int \phi_0^*(z) \cos(q_z) z \phi_1 \, dz, \quad I_{q_z} = \int_{-\infty}^{+\infty} |I_z|^2 \, dq_z \tag{16}$$

3.1 With electric and magnetic field

In the case of perpendicular within radial axis of electrical field $\vec{B}(0,0,B)$ and magnetic field $\vec{E}(E,0,0)$, we can solve it 'totally similar' to the case of semi-parabolic, asymmetric infinite well, hence the form of the wavefunction is still

$$\phi(\vec{r}) = \phi_0 \Phi(x - x_0) \exp{(ik_y y)} \phi_n(z) \tag{17}$$

where $\Phi(x - x_0)$ is the harmonic wavefunction centres at x_0, $\phi_n(x)$ the calculated step wavefunction, for

$$x_0 = -\frac{\hbar}{eB}\left(k_y + \frac{m_e E}{\hbar B}\right) = \ell^2 \left(k_y - \frac{m_e v_d}{\hbar}\right) \tag{18}$$

The energy spectrum does not change much,

$$\mathcal{E} = \hbar\omega_z \left(2n + 1 + \frac{1}{2}\sqrt{1 + 4\beta_z}\right) + \left(N + \frac{1}{2}\right)\hbar\omega_c + \hbar v_d k_y - \frac{1}{2} m_e v_d^2 \tag{19}$$

4 Absorption of electromagnetic waves

4.1 Hamiltonian of an electromagnetic - phonon interaction in quantum well

To start with the analysis, we need some formulations of the interaction of electron, phonon and external electrical field within the confinement of the quantum well. This can be separated to three operators interactions: external electric field interaction, non-interacting phonon interaction, and electron-phonon interaction.

Call z the space spatial axis being quantized, the Hamiltonian for the electromagnetic-phonon interaction in the quantum well when there exists external electric field $\vec{E}(t) = \vec{E}_0 \sin{(\Omega t)}$ as

$$\mathcal{H} = \underbrace{\sum_{n,\vec{k}_\perp} \mathcal{E}_n \left(\vec{k}_\perp - \frac{e}{\hbar c}\vec{A}(t)\right) a^+_{n,\vec{k}_\perp} a_{n,\vec{k}_\perp}}_{\text{external field}} \tag{20}$$

$$\underbrace{+ \sum_{\vec{q}} \hbar\omega_0 b^+_{\vec{q}} b_{\vec{q}}}_{\text{non-interacting phonons}} \tag{21}$$

$$\underbrace{+ \sum_{\vec{q}} \sum_{n,n',\vec{k}_\perp} C_{\vec{q}} I_{n,n'}(q_z) a^+_{n',\vec{k}_\perp + \vec{q}} a_{n,\vec{k}_\perp} \left(b_{\vec{q}} + b^+_{-\vec{q}}\right)}_{\text{electron-phonon}} \tag{22}$$

with $\mathcal{E}_n$ is the energy of the electron, $\vec{k}_\perp, \vec{q}$ are the vectors of electron and phonon in the system respectively, $|n; \vec{k}_\perp\rangle$, $\langle n; \vec{k}_\perp + \vec{q}|$ being the system's electron state before and after the scattering interaction of the system. For the external field, which is considered orthogonal, $\vec{A}(t)$ is expressed as the electrical potential

$$-\frac{1}{c}\frac{d}{dt}\vec{A}(t) = \vec{E}_0 \sin{(\Omega t)}, \quad \vec{A}(t) = \frac{c}{r}\vec{E}_0 \cos{(\omega t)} \tag{23}$$

$I_{n,n'}(q_z)$ is the electronic form factor of the heterogenous, constituent superlattice - in our case of quantum well is a special case. This is usually expressed as

$$I_{n,n'}(q_z) \int_0^{N_d} \Psi_n^*(z) \Psi_{n'}(z) \exp{(iq_z z)}\, dz = \frac{2}{L}\int_0^L \Psi_n^*(z) \Psi_{n'}(z) \exp{(iq_z z)}\, dz \tag{24}$$

in which $2/L$ is the intrinsic normalization factor for the special case of a quantum well. Energy spectrum $\mathcal{E}_n(\vec{k}_\perp)$ here is expressed by

$$\mathcal{E}_n(\vec{k}_\perp) = \frac{\hbar^2}{2m^*}\left(k_\perp^2 + k_z^{n^2}\right) = \hbar\Omega_B \left(N + \frac{1}{2}\right) + \left(\frac{\hbar^2}{2m^*}\frac{\pi^2 n^2}{L^2}\right) \tag{25}$$

where $\Omega_B = (eB)/(m^* c)$ is the cyclotron frequency of the external field.

To proceed with investigating the quantum effect and motions of the system given the above Hamiltonian, we will use the following commutative operator between the observable operators in the Hamiltonian, such as the following:

$$\{a_i; a_k^+\} = a_i a_k^+ + a_k^+ a_i = \delta_{ik} \tag{26}$$

$$\{a_i^+; a_k^+\} = \{a_i; a_k\} = 0 \tag{27}$$
$$\{b_i, b_k^+\} = b_i b_k^+ - b_k^+ b_i = \delta_{ik} \tag{28}$$
$$\{b_i^+; b_k^+\} = \{b_i; b_k\} = 0 \tag{29}$$

4.2 Momentum operator solution

To investigate further of the interaction in the quantum well as well as constructing quantum-mechanical expression, we use the momentum operator expression for **particle number operator**. We have the following:

$$i\hbar \frac{\partial f_{n,k_\perp}(t)}{\partial t} = \left\langle \left[a^+_{n,\vec{k}_\perp} a_{n,\vec{k}_\perp}; H \right] \right\rangle, \qquad f_{n,k_\perp} = \left\langle a^+_{n,\vec{k}_\perp}; a_{n,\vec{k}_\perp} \right\rangle \tag{30}$$

Applying this for the first term (external force interaction):

$$\left\langle \left[a^+_{n,\vec{k}_\perp} a_{n,\vec{k}_\perp}; H_1 \right] \right\rangle = \left[a^+_{n,\vec{k}_\perp} a_{n,\vec{k}_\perp}; \sum_{n,\vec{k}_\perp} \mathcal{E}_n \left(\vec{k}_\perp - \frac{e}{\hbar c} \vec{A}(t) \right) a^+_{n,\vec{k}_\perp} a_{n,\vec{k}_\perp} \right] \tag{31}$$
$$= \sum_{n,\vec{k}_\perp} \mathcal{E}_n \left(\vec{k}_\perp - \frac{e}{\hbar c} \vec{A}(t) \right) \left[a^+_{n,\vec{k}_\perp} a_{n,\vec{k}_\perp}; a^+_{n,\vec{k}_\perp} a_{n,\vec{k}_\perp} \right] \tag{32}$$
$$= S_1 \tag{33}$$

Consider the commutator $\left[a^+_{n,\vec{k}_\perp} a_{n,\vec{k}_\perp}; a^+_{n,\vec{k}_\perp} a_{n,\vec{k}_\perp} \right]$, we have

$$\left[a^+_{n,\vec{k}_\perp} a_{n,\vec{k}_\perp}; a^+_{n,\vec{k}_\perp} a_{n,\vec{k}_\perp} \right] = a^+_{n,\vec{k}_\perp} a_{n,\vec{k}'_\perp} \delta_{\vec{k}_\perp,\vec{k}'_\perp} - a^+_{n,\vec{k}'_\perp} a_{n,\vec{k}_\perp} \delta_{\vec{k}'_\perp,\vec{k}_\perp} \tag{34}$$

Thereby, for the Kronecker delta being

$$S_1 = \sum_{n,\vec{k}_\perp} \mathcal{E}_n \left(\vec{k}_\perp - \frac{e}{\hbar c} \vec{A}(t) \right) \left(a^+_{n,\vec{k}_\perp} a_{n,\vec{k}'_\perp} \delta_{\vec{k}_\perp,\vec{k}'_\perp} - a^+_{n,\vec{k}'_\perp} a_{n,\vec{k}_\perp} \delta_{\vec{k}'_\perp,\vec{k}_\perp} \right) \tag{35}$$
$$= \sum_{n,\vec{k}_\perp} \mathcal{E}_n \left(\vec{k}_\perp - \frac{e}{\hbar c} \vec{A}(t) \right) a^+_{n,\vec{k}_\perp} a_{n,\vec{k}'_\perp} \delta_{\vec{k}_\perp,\vec{k}'_\perp} - \sum_{n,\vec{k}_\perp} \mathcal{E}_n \left(\vec{k}_\perp - \frac{e}{\hbar c} \vec{A}(t) \right) a^+_{n,\vec{k}'_\perp} a_{n,\vec{k}_\perp} \delta_{\vec{k}'_\perp,\vec{k}_\perp} \tag{36}$$
$$= \mathcal{E}_n \left(\vec{k}_\perp - \frac{e}{\hbar c} \vec{A}(t) \right) a^+_{n,\vec{k}_\perp} a_{n,\vec{k}'_\perp} - \mathcal{E}_n \left(\vec{k}_\perp - \frac{e}{\hbar c} \vec{A}(t) \right) a^+_{n,\vec{k}'_\perp} a_{n,\vec{k}_\perp} \tag{37}$$
$$= 0 \tag{38}$$

For the second term, we do the same operation:

$$S_2 = \left[a^+_{n,\vec{k}_\perp} a_{n,\vec{k}_\perp}; \sum_{\vec{q}} \hbar \omega_{\vec{q}} b^+_{\vec{q}} b_{\vec{q}} \right] \tag{39}$$
$$= \hbar \sum_{\vec{q}} \omega_{\vec{q}} \left[a^+_{n,\vec{k}_\perp} a_{n,\vec{k}_\perp}; b^+_{\vec{q}} b_{\vec{q}} \right] \tag{40}$$
$$= 0 \tag{41}$$

The third term can be calculated as followed.

$$S_2 = \left[a^+_{n,\vec{k}_\perp} a_{n,\vec{k}_\perp}; \sum_{\vec{q}} \sum_{n,n',\vec{k}_\perp} C_{\vec{q}} I_{n,n'}(q_z) a^+_{n',\vec{k}_\perp+\vec{q}} a_{n,\vec{k}_\perp} \left(b_{\vec{q}} + b^+_{-\vec{q}} \right) \right] \tag{42}$$
$$= \sum_{\vec{q}} \sum_{n'n,\vec{k}_\perp} C_{\vec{q}} I_{n,n'}(q_z) \left[a^+_{n,\vec{k}_\perp} a_{n,\vec{k}_\perp}; a^+_{n',\vec{k}'_\perp+\vec{q}} a_{n,\vec{k}'_\perp} \right] \left(b^+_{\vec{q}} + b_{\vec{q}} \right) \tag{43}$$

The expression of the commutator now can be simplified further down, using our identities. For now, we have, Using the same operator, we can simply calculate the third term in the function as follows:

$$S_3 = \left[a^+_{n,\vec{k_\perp}} a_{n,\vec{k_\perp}} \sum_{n,n'} \sum_{k'_\perp,\vec{q}} C(\vec{q})\, I_{nn'}(q_z) a^+_{n',\vec{k'_\perp}+\vec{q}} a_{n,\vec{k'_\perp}} \left(b^+_{-\vec{q}} + b_{\vec{q}} \right) \right]$$
$$= \sum_{n,n'} \sum_{\vec{k'_\perp},\vec{q}} C(\vec{q})\, I_{nn'}(q_z) \left[a^+_{n,\vec{k_\perp}} a_{n,\vec{k_\perp}} a^+_{n',\vec{k'_\perp}+q} a_{n,\vec{k'_\perp}} \right] \left(b^+_{\vec{q}} + b_{\vec{q}} \right)$$

The expression of the commutator can now be simplified further down using our identities. For now,we have:

$$\left[a^+_{n,\vec{k_\perp}} a_{n,\vec{k_\perp}}, a^+_{n',\vec{k'_\perp}+\vec{q}\perp} a_{n,\vec{k'_\perp}} \right]$$
$$= a^+_{n,\vec{k_\perp}} a_{n,\vec{k_\perp}} a^+_{n',\vec{k'_\perp}+\vec{q_1}} a_{n,\vec{k'_\perp}} - a^+_{n',\vec{k'_\perp}+\vec{q_1}} a_{n,\vec{k'_\perp}} a^+_{n,\vec{k_\perp}} a_{n,\vec{k_\perp}}$$
$$= a^+_{n,\vec{k_\perp}} \left(\delta_{n,n',\vec{k_\perp},\vec{k'_\perp}+\vec{q_1}} - a^+_{n',\vec{k'_\perp}+\vec{q}_\perp} a_{n,\vec{k_\perp}} \right) a_{n,\vec{k'_\perp}} - a^+_{n',\vec{k'_\perp}+\vec{q_1}} \left(\delta_{\vec{k'_\perp},\vec{k_\perp}} - a^+_{n,\vec{k_\perp}} a_{n,\vec{k'_\perp}} \right) a_{n,k_\perp}$$
$$= a^+_{n,\vec{k_\perp}} a_{n,\vec{k'_\perp}} \delta_{n,n',\vec{k_\perp},\vec{k'_\perp}+\vec{q}_\perp} - a^+_{n,\vec{k_\perp}} a^+_{n',\vec{k'_\perp}+\vec{q}} a_{n,\vec{k_\perp}} a_{n,\vec{k'_\perp}} - a^+_{n',\vec{k'_\perp}+\vec{q}_\perp} \delta_{\vec{k'_\perp},\vec{k_\perp}} a_{n,\vec{k_\perp}}$$
$$- a^+_{n',\vec{k'_\perp}+\vec{q_1}} a^+_{n,\vec{k_\perp}} a_{n,\vec{k'_\perp}} a_{n,\vec{k_\perp}}$$
$$= a^+_{n,\vec{k_\perp}} a_{n,\vec{k'_\perp}} \delta_{n,n',\vec{k_\perp},\vec{k'_\perp}+\vec{q_1}} - a^+_{n',\vec{k'_\perp}+\vec{q_1}} a_{n,\vec{k_\perp}} \delta_{\vec{k'_\perp},\vec{k_\perp}}$$
$$\to S_3 = \sum_{n,n'} \sum_{\vec{k'_\perp},\vec{q}} C(\vec{q})\, I_{nn'}(q_z) \left(a^+_{n,\vec{k_\perp}} a_n, \vec{k'_\perp} \delta_{n,n',\vec{k_\perp},\vec{k'_\perp}+\vec{q_1}} - a^+_{n',\vec{k'_\perp}+\vec{q_\perp}} a_{n,\vec{k_\perp}} \delta_{\vec{k'_\perp},\vec{k_\perp}} \right) \left(b^+_{\vec{q}} + b_{\vec{q}} \right)$$

$$\delta_{n,n',\vec{k_\perp},\vec{k'_\perp}+\vec{q_1}} = 0 \leftrightarrow \begin{cases} n = n' \\ \vec{k_\perp} = \vec{k'_\perp} + \vec{q} \leftrightarrow \vec{k'_\perp} = \vec{k_\perp} - \vec{q} \end{cases}$$
$$\delta_{\vec{k'_\perp},\vec{k_\perp}} = 1 \leftrightarrow \vec{k'_\perp} = \vec{k_\perp}$$

$$\to S_3 = \sum_{\vec{k'_\perp},\vec{q}} C(\vec{q})\, I_{n,n'}(q_z) \left(a^+_{n,\vec{k_\perp}} a_{n',\vec{k_\perp}-\vec{q_1}} - a^+_{n',\vec{k_\perp}+\vec{q_1}} a_{n,\vec{k_\perp}} \right) \left(b^+_{\vec{q}} + b_{\vec{q}} \right)$$
$$= \sum_{\vec{k'_\perp},\vec{q}} C(\vec{q})\, I_{n,n'}(q_z) \left(a^+_{n,\vec{k_\perp}} a_{n',\vec{k_\perp}-\vec{q_1}} b^+_{-\vec{q}} + a^+_{n,\vec{k_\perp}} a_{n',\vec{k_\perp}-\vec{q_1}} b_{\vec{q}} \right)$$
$$+ \left(-a^+_{n',\vec{k_\perp}+\vec{q_1}} a_{n,\vec{k_\perp}} b^+_{-\vec{q}} - a^+_{n,\vec{k_\perp}} a_{n,\vec{k_\perp}} b_{\vec{q}} \right)$$
$$\to S_3 = \sum_{q_\perp} \sum_{\vec{k'_\perp},\vec{q}} C(\vec{q})\, I_{n,n'}(q_z) \left[\left(a^+_{n,\vec{k_\perp}} a_{n',\vec{k_\perp}-\vec{q_1}} b_{\vec{q}} \right) + \left(a^+_{n',\vec{k_\perp}-\vec{q_1}} a_{n,\vec{k_\perp}} b_{-\vec{q}} \right)^* \right]$$
$$- \left[\left(a^+_{n',\vec{k_\perp}+\vec{q}} a_{n,\vec{k_\perp}} b_{\vec{q}} \right) - \left(a^+_{n,\vec{k_\perp}} a_{n',\vec{k_\perp}+\vec{q_1}} b_{-\vec{q}} \right)^* \right]$$

Set $F_{n,\vec{k_1}_\perp,n',\vec{k_2}_\perp,q'(t)} = \langle a^+_{n,\vec{k_1}_\perp} a_{n',\vec{k_2}_\perp} b_{\vec{q'}} \rangle$:

$$\to \left\langle \left[a^+_{n,\vec{k_\perp}} a_{n,\vec{k_\perp}} \sum_{n,n'} \sum_{k'_\perp,\vec{q'}} C(\vec{q})\, I_{nn'}(q_z)\, a^+_{n',\vec{k'_\perp}+\vec{q},\vec{k'_\perp}} \left(b^+_{-\vec{q}} + b_{\vec{q}} \right) \right] \right\rangle$$
$$= -\sum_{q'} C(\vec{q})\, I_{nn'}(q_z) \left[F_{n',\vec{k_\perp}+\vec{q_1},n,\vec{k_\perp},\vec{q}} + F^*_{n,\vec{k_\perp},n',\vec{k_\perp}-\vec{q},\vec{q}} - F^*_{n',\vec{k_\perp}-\vec{q},n,\vec{k_\perp},-\vec{q}} \right]$$

Getting the sum of 3 terms:

$$\to i\hbar \frac{\partial f_{n,\vec{k_\perp}}(t)}{\partial t} = -\sum_{q'} C(\vec{q})\, I_{nn'}(q_z) \left[F_{n',\vec{k_\perp}+\vec{q_1},n,\vec{k_\perp},\vec{q}} + F^*_{n,\vec{k_\perp},n',\vec{k_\perp}+\vec{q},-\vec{q}} \right]$$

$$
\begin{aligned}
&+\left[-F_{n,\vec{k_\perp},n',\vec{k_\perp}-\vec{q},\vec{q}}-F^*_{n',\vec{k_\perp}-\vec{q},n,\vec{k_\perp},-\vec{q}}\right]\\
\rightarrow \frac{\partial f_{n,\vec{k_\perp}}(t)}{\partial t} =&\frac{i}{\hbar}\sum_{\vec{q}} C(\vec{q})\, I_{nn'}(q_z)\left[F_{n',\vec{k_\perp}+\vec{q},n,\vec{k_\perp},\vec{q}}+F^*_{n,\vec{k_\perp},n',\vec{k_\perp}+\vec{q},-\vec{q}}\right]\\
&+\left[-F_{n,\vec{k_\perp},n',\vec{k_\perp}-\vec{q},\vec{q}}-F^*_{n,\vec{k_\perp}-\vec{q},n,\vec{k_\perp},-\vec{q}}\right]
\end{aligned}
$$

To proceed, we define the time-dependent correlation function $F_{n,\vec{k_{1\perp}},n',\vec{k_{2\perp}},\vec{q'}}(t)$, which describes the expectation value of creating a particle in state $n, k_{1\perp}$, annihilating one in state $n', k_{2\perp}$, and absorbing a phonon with momentum q'. This quantity will allow us to rewrite the equation of motion in a more compact form:

$$
F_{n,\vec{k_{1\perp}},n',\vec{k_{2\perp}},\vec{q'}}(t)=\left\langle a^+_{n,\vec{k_{1\perp}}}a_{n',\vec{k_{2\perp}}}b_{\vec{q}}\right\rangle t
$$

Taking the time derivative of F and applying the Heisenberg equation of motion yields the following commutator with the Hamiltonian H

$$
\begin{aligned}
i\frac{\partial F}{\partial t} =&i\frac{\partial\left(a^+_{n,\vec{k_\perp}},a_{n',\vec{k_{2\perp}}}b_{\vec{q}}\right)}{\partial t}\\
=&\langle[F,H]\rangle t=\left\langle\left[a^+_{n,\vec{k_{1\perp}}},a_{n',\vec{k_{2\perp}}}b_{\vec{q}},H\right]\right\rangle t
\end{aligned}
$$

The first contribution to the commutator, denoted S_1, arises from the single particle part of the Hamiltonian. Explicitly, it can be written as:

$$
\begin{aligned}
S_1=&\left\langle\left[a^+_{n,\vec{k_{1\perp}}}a_{n',\vec{k_{2\perp}}}b_{\vec{q'}},\sum_{n,\vec{k_{3\perp}}}\mathcal{E}_n\left(\vec{k_{3\perp}}-\frac{e}{\hbar c}A(t)\right)a^+_{n,\vec{k_{3\perp}}}a_{n,\vec{k_{3\perp}}}\right]\right\rangle\\
=&\sum_{n,\vec{k_{3\perp}}}\mathcal{E}_n\left(\vec{k_{3\perp}}-\frac{e}{\hbar c}A(t)\right)\left[a^+_{n,\vec{k_{1\perp}}}a_{n',\vec{k_{2\perp}}}b_{\vec{q'}},a^+_{n,\vec{k_{3\perp}}}a_{n,\vec{k_{3\perp}}}\right]
\end{aligned}
$$

Computing the commutator of the operator, this gives:

$$
\begin{aligned}
&\left[a^+_{n,\vec{k_{1\perp}}}a_{n',\vec{k_{2\perp}}}b_{\vec{q'}},a^+_{n,\vec{k_{3\perp}}}a_{n,\vec{k_{3\perp}}}\right]\\
=&a^+_{n,\vec{k_{1\perp}}}a_{n',\vec{k_{2\perp}}}a^+_{n,\vec{k_{3\perp}}}a_{n,\vec{k_{3\perp}}}b_{\vec{q'}}-a^+_{n,\vec{k_{3\perp}}}a_{n,\vec{k_{3\perp}}}a_n a^+_{n,\vec{k_{1\perp}}}a_{n',\vec{k_{2\perp}}}b_{q'}\\
=&a^+_{n,\vec{k_{1\perp}}}\left(\delta_{n',n,\vec{k_{2\perp}},\vec{k_{3\perp}}}-a^+_{n,\vec{k_{3\perp}}}a_{n',\vec{k_{3\perp}}k\vec{q}}-a^+_{n,\vec{k_{3\perp}},\vec{k_{1\perp}}}-a^+_{n,\vec{k_{1\perp}}}a_{n,\vec{k_{3\perp}}}\right)\times a_{n',\vec{k_{2\perp}}}b_{\vec{q}}\\
=&a^+_{n,\vec{k_{1\perp}}}a_{n,\vec{k_{3\perp}}}b_{\vec{q'}}\delta_{n',n,\vec{k_{2\perp}},\vec{k_{3\perp}}}-a^+_{n,\vec{k_{3\perp}}}a_{n',\vec{k_{2\perp}}}b_{\vec{q'}}\delta_{n,\vec{k_{3\perp}},\vec{k_{1\perp}}}
\end{aligned}
$$

After evaluating the commutator, we substitute the result back into the expression for S_1, obtaining:

$$
\begin{aligned}
\rightarrow S_1=&\left\langle\sum_{n,\vec{k_{3\perp}}}\mathcal{E}_n\left(\vec{k_{3\perp}}-\frac{e}{\hbar c}A(t)\right)\left(a^+_{n,\vec{k_{1\perp}}}a_{n,\vec{k_{3\perp}}}b_{\vec{q'}}\delta_{n',n,\vec{k_{2\perp}},\vec{k_{3\perp}}}-a^+_{n,\vec{k_{3\perp}}}a_{n',\vec{k_{2\perp}}}b_{\vec{q'}}\delta_{n,\vec{k_{3\perp}},\vec{k_{1\perp}}}\right)\right\rangle\\
&\delta\delta_{n,n',\vec{k_{2\perp}},\vec{k'_{3\perp}}}=1\leftrightarrow\begin{cases}n=n'\\ \vec{k_{2\perp}}=\vec{k_{3\perp}}\end{cases}\\
&\delta_{n,\vec{k_{3\perp}},\vec{k_{1\perp}}}=1\leftrightarrow\vec{k_{3\perp}}=\vec{k_{1\perp}}\\
\rightarrow S_1=&\mathcal{E}_{n',\vec{k_{2\perp}}}\left(\vec{k_{2\perp}}-\frac{e}{\hbar c}A(t)\right)-\mathcal{E}_{n,\vec{k_{1\perp}}}\left(\vec{k_{1\perp}}-\frac{e}{\hbar c}A(t)\right)\left\langle a^+_{n,\vec{k_{1\perp}}}a_{n',\vec{k_{2\perp}}k\vec{q'}}\right\rangle t
\end{aligned}
$$

The second contribution, S_2, arises from the phonon Hamiltonian. Using the commutation relations for the phonon operators, we find:

$$
S_2=\left\langle\left[a^+_{n,\vec{k_{1\perp}}}a_{n',\vec{k_{2\perp}}}b_{\vec{q'}},\sum_{q_\perp}\hbar\omega_{\vec{q_1}}b^+_{\vec{q_1}}b_{\vec{q_1}}\right]\right\rangle
$$

To compute this term, we evaluate and commutator between $a^{+}_{n,\vec{k}_{1\perp}} a_{n',\vec{k}_{2\perp}} b_{\vec{q}'}$ and $b^{+}_{\vec{q}\perp} b_{\vec{q}_\perp}$:

$$\begin{aligned}
&\left[a^{+}_{n,\vec{k}_{1\perp}} a_{n',\vec{k}_{2\perp}} b_{\vec{q}'}, b^{+}_{\vec{q}\perp} b_{\vec{q}_\perp}\right] \\
=&a^{+}_{n,\vec{k}_{1\perp}} a_{n',\vec{k}_{2\perp}} b_{\vec{q}'} b^{+}_{\vec{q}_\perp} b_{\vec{q}_1} - b^{+}_{\vec{q}_1} b_{\vec{q}_1} a^{+}_{n,\vec{k}_{1\perp}} a_{n',\vec{k}_{2\perp}} b_{\vec{q}'} \\
=&a^{+}_{n,\vec{k}_{1\perp}} a_{n',\vec{k}_{2\perp}} \left(\delta_{\vec{q}',\vec{q}_1} + b^{+}_{\vec{q}_1} b_{\vec{q}'}\right) - b^{+}_{\vec{q}_1} b_{\vec{q}_\perp} a^{+}_{n,\vec{k}_{1\perp}} a'_{n,\vec{k}_{2\perp}} b_{\vec{q}'} \\
=&a^{+}_{n,\vec{k}_{1\perp}} a_{n',\vec{k}_{2\perp}} b_{\vec{q}_1} \delta_{\vec{q}',\vec{q}_1} \\
&\delta_{\vec{q}',\vec{q}_1} = 1 \leftrightarrow \vec{q}' = \vec{q}_1 \\
\rightarrow S_2 =&\hbar\omega q' \left\langle a^{+}_{n,\vec{k}_{1\perp}} a_{n',\vec{k}_{2\perp}} b_{\vec{q}'} \right\rangle
\end{aligned}$$

Finally, the third contribution, S_3, corresponds to the electron-phonon interaction. Its evaluation involves a double summation over intermediate states and phonon momenta. We can write it as:

$$S_3 = \left\langle \left[a^{+}_{n,\vec{k}_{1\perp}} a_{n'}, b_{\vec{q}'} \sum_{n,n'} \sum_{\vec{k}_{3\perp},\vec{q}_1} C\left(\vec{q}_1\right) I_{n'n}\left(q_z\right) a^{+}_{n',\vec{k}_{3\perp}+\vec{q}_1} a_{n,\vec{k}_{3\perp}} \left(b^{+}_{-\vec{q}_1} + b^{+}_{\vec{q}_1}\right) \right] \right\rangle$$

This expression for S_3 contains two terms of opposite sign, the corresponding o phonon absorption and emission process.

$$\begin{aligned}
&\left[a^{+}_{n,\vec{k}_{1\perp}} a_{n',k_{\vec{k}_{2\perp}}} b_{\vec{q}'}, a^{+}_{n',\vec{k}_{3\perp}+\vec{q}_1} a_{n,\vec{k}_{3\perp}} \left(b^{+}_{-\vec{q}_1} + b^{+}_{\vec{q}_1}\right)\right] \\
=&a^{+}_{n,\vec{k}_{1\perp}} a_{n',\vec{k}_{2\perp}} b_{q'} a^{+}_{n',\vec{k}_{3\perp}+\vec{q}_1} a_{n,\vec{k}_{3\perp}} \left(b^{+}_{\vec{q}_1} b_{\vec{q}_1}\right) - a^{+}_{n',\vec{k}_{3\perp}+\vec{q}_1} a_{n,\vec{k}_{3\perp}} \left(b^{+}_{-\vec{q}_1} + b_{\vec{q}_\perp}\right) \\
&\times a^{+}_{n,\vec{k}_{1\perp}} a_{n',\vec{k}_{2\perp} k\vec{q}} \\
=&a^{+}_{n,\vec{k}_{1\perp}} a_{n',\vec{k}_{2\perp}} b_{q'} a^{+}_{n',\vec{k}_{3\perp}} b_{\vec{q}_1} + a^{+}_{n,\vec{k}_{1\perp}} a^{+}_{n',\vec{k}_{3\perp}+\vec{q}_1} a_{n,\vec{k}_{3\perp}} b_{\vec{q}'} b^{+}_{-\vec{q}_1} \\
&- a^{+}_{n',\vec{k}_{3\perp}+\vec{q}_1} a_{n,\vec{k}_{3\perp}} \left(b^{+}_{-\vec{q}_1} + b_{\vec{q}_1}\right) a^{+}_{n,\vec{k}_{1\perp}} a_{n',\vec{k}_{2\perp}} b_{\vec{q}'} \\
=&a^{+}_{n,\vec{k}_{1\perp}} a_{n',\vec{k}_{2\perp}} b_{q'} a^{+}_{n',\vec{k}_{3\perp}} b_{\vec{q}_1} + a^{+}_{n,\vec{k}_{1\perp}} a^{+}_{n',\vec{k}_{3\perp}+\vec{q}_1} a_{n,\vec{k}_{3\perp}} \left(\delta_{\vec{q}',-\vec{q}_1} + b^{+}_{-\vec{q}_1} b_{\vec{q}'}\right) \\
&- a^{+}_{n',\vec{k}_{3\perp}+\vec{q}_1} a_{n,\vec{k}_{3\perp}} \left(b^{+}_{-\vec{q}_1} + b_{\vec{q}_1}\right) a^{+}_{n,\vec{k}_{1\perp}} a_{n',\vec{k}_{2\perp}} b_{\vec{q}'} \\
=&a^{+}_{n,\vec{k}_{1\perp}} \left(\delta_{n',\vec{k}_{2\perp},\vec{k}_{3\perp}+\vec{q}_{1\perp}} - a^{+}_{n',\vec{k}_{3\perp}+\vec{q}_1} a_{n',\vec{k}_{2\perp}}\right) a_{n,\vec{k}_{3\perp}} b_{\vec{q}_1} b_{\vec{q}'} \\
&+ a^{+}_{n,\vec{k}_{1\perp}} \left(\delta_{n',\vec{k}_{2\perp},\vec{k}_{3\perp}+\vec{q}_{1\perp}} - a^{+}_{n',\vec{k}_{3\perp}+\vec{q}_{1\perp}} a_{n',\vec{k}_{2\perp}}\right) a_{n,\vec{k}_{3\perp}} \left(\delta_{\vec{q}',-\vec{q}_1} + b^{+}_{-\vec{q}_1} b_{\vec{q}'}\right) \\
&- a^{+}_{n',\vec{k}_{3\perp}+\vec{q}_1} a_{n,\vec{k}_{3\perp}} \left(b^{+}_{-\vec{q}_1} + b_{\vec{q}_1}\right) a^{+}_{n,\vec{k}_{1\perp}} a_{n',\vec{k}_{2\perp}} b_{\vec{q}'} \\
=&a^{+}_{n,\vec{k}_{1\perp}} a_{n,\vec{k}_{3\perp}} b_{\vec{q}_1} b_{\vec{q}'} \delta_{n',\vec{k}_{2\perp},\vec{k}_{3\perp}+\vec{q}_1} - a^{+}_{n,\vec{k}_{1\perp}} a^{+}_{n',\vec{k}_{3\perp}+\vec{q}_{1\perp}} - a^{+}_{n',\vec{k}_{3\perp}+\vec{q}_1} a_{n',\vec{k}_{2\perp}} a_{n,\vec{k}_{3\perp}} b_{\vec{q}_1} b_{\vec{q}'} \\
&+ a^{+}_{n,\vec{k}_{1\perp}} a_{n,\vec{k}_{3\perp}} \delta_{n',\vec{k}_{2\perp},\vec{k}_{3\perp}+\vec{q}_{1\perp}} \delta_{\vec{q}',-\vec{q}_1} + a^{+}_{n,\vec{k}_{1\perp}} a_{n,\vec{k}_{3\perp}} b^{+}_{-\vec{q}_{1\perp}} b_{\vec{q}'} \delta_{n',\vec{k}_{2\perp},\vec{k}_{3\perp}+\vec{q}_{1\perp}} \\
&- a^{+}_{n,\vec{k}_{1\perp}} a_{n,\vec{k}_{3\perp}} \delta_{n',\vec{k}_{2\perp},\vec{k}_{3\perp}+\vec{q}_{1\perp}} a_{n',\vec{k}_{2\perp} a_{n,\vec{k}_{3\perp}}} \delta_{\vec{q}',-\vec{q}_1} - a^{+}_{n,\vec{k}_{1\perp}} a^{+}_{n',\vec{k}_{3\perp}+\vec{q}_{1\perp}} a_{n',\vec{k}_{2\perp}} a_{n,\vec{k}_{3\perp}} b^{+}_{-\vec{q}_{1\perp}} b_{\vec{q}'} \\
&- a^{+}_{n',\vec{k}_{3\perp}+\vec{q}_{1\perp}} a_{n,\vec{k}_{3\perp}} \left(b^{+}_{-\vec{q}_1} + b_{\vec{q}_{1\perp}}\right) a^{+}_{n,\vec{k}_{1\perp}} a_{n',\vec{k}_{2\perp}\vec{q}'} \\
=&a^{+}_{n,\vec{k}_{1\perp}} a_{n,\vec{k}_{3\perp}} b_{\vec{q}_1} b_{\vec{q}'} \delta_{n',\vec{k}_{2\perp},\vec{k}_{3\perp}+\vec{q}_1} - a^{+}_{n,\vec{k}_{1\perp}} a^{+}_{n',\vec{k}_{3\perp}+\vec{q}_{1\perp}} \left(\delta_{n,\vec{k}_\perp,\vec{k}_{3\perp}} - a_{n,\vec{k}_{3\perp}} a^{+}_{n,\vec{k}_{1\perp}}\right) a_{n',\vec{k}_{2\perp}} b_{\vec{q}_1} b_{\vec{q}'} \\
&- a^{+}_{n',\vec{k}_{3\perp}+\vec{q}_{1\perp}} \left(\delta_{n,\vec{k}_{1\perp},\vec{k}_{3\perp}} - a_{n,\vec{k}_{3\perp}} a^{+}_{n,\vec{k}_{1\perp}}\right) a_{n',\vec{k}_{2\perp}} b^{+}_{-\vec{q}_1} b_{\vec{q}'} + a^{+}_{n,\vec{k}_{1\perp}} a^{+}_{n',\vec{k}_{3\perp}} \delta_{n',\vec{k}_{3\perp}+\vec{q}_{1\perp}} \delta_{\vec{q}',-\vec{q}_1} \\
&- a_{n',\vec{k}_{3\perp}+\vec{q}_1} a_{n,\vec{k}_{3\perp}} \left(b^{+}_{-\vec{q}_1} + b_{\vec{q}_1}\right) a^{+}_{n,\vec{k}_{1\perp}} a_{n',\vec{k}_{2\perp}} b_{\vec{q}'} \\
=&a^{+}_{n,\vec{k}_{1\perp}} a_{n,\vec{k}_{3\perp}} b_{\vec{q}_1} b_{\vec{q}'} \delta_{n',\vec{k}_{2\perp},\vec{k}_{3\perp}+\vec{q}_{1\perp}} - a^{+}_{n',\vec{k}_{3\perp}+\vec{q}_{1\perp}} a_{n',\vec{k}_{2\perp}} b^{+}_{\vec{q}_{1\perp}} b_{\vec{q}'} \delta_{n'\vec{k}_\perp,\vec{k}_{3\perp}}
\end{aligned}$$

$$
\begin{aligned}
&+ a^{+}_{n,\vec{k_{1\perp}}} a^{+}_{n,\vec{k_{3\perp}}} \delta_{n',\vec{k_{2\perp}},\vec{k_{3\perp}}+\vec{q_{1\perp}}} \delta_{\vec{q'},-\vec{q_1}} + a^{+}_{n,\vec{k_{1\perp}}} a_{n,\vec{k_{3\perp}}} b^{+}_{-\vec{q_1}} b_{\vec{q'}} \delta_{n',\vec{k_{2\perp}},\vec{k_{3\perp}}+\vec{q_{1\perp}}} \\
&- a^{+}_{n\vec{k_{1\perp}},\vec{k_{3\perp}}+\vec{q_{1\perp}}} a_{n'+\vec{k_{2\perp}}} a_{n,\vec{k_{3\perp}}} \delta_{\vec{q'},-\vec{q_1}} - a^{+}_{n',\vec{k_{3\perp}}+\vec{q_{1\perp}}} a_{n',\vec{k_{2\perp}}} b^{+}_{-\vec{q_1}} b_{\vec{q'}} \delta_{n,\vec{k_{1\perp}},\vec{k_{3\perp}}}
\end{aligned}
$$

$$
\begin{aligned}
\to S_3 = \Big\langle & \sum_{n,n'} \sum_{\vec{k_{3\perp}},\vec{q_1}} C(\vec{q_1}) I_{nn'}(q_z) a^{+}_{n,\vec{k_{1\perp}}} a_{n,\vec{k_{3\perp}}} b_{\vec{q_1}} b_{\vec{q'}} \delta_{n',\vec{k_{2\perp}},\vec{k_{3\perp}}+\vec{q_{1\perp}}} \\
&- a^{+}_{n',\vec{k_{3\perp}}+\vec{q_{1\perp}}} a_{n',\vec{k_{2\perp}}} b^{+}_{\vec{q_1}} b_{\vec{q'}} \delta_{n'\vec{k_\perp},\vec{k_{3\perp}}} + a^{+}_{n,\vec{k_{1\perp}}} a^{+}_{n,\vec{k_{3\perp}}} \delta_{n',\vec{k_{2\perp}},\vec{k_{3\perp}}+\vec{q_{1\perp}}} \delta_{\vec{q'},-\vec{q_1}} \\
&+ a^{+}_{n,\vec{k_{1\perp}}} a_{n,\vec{k_{3\perp}}} b^{+}_{-\vec{q_1}} b_{\vec{q'}} - a^{+}_{n,\vec{k_{1\perp}},\vec{k_{3\perp}}+\vec{q_{1\perp}}} a_{n'+\vec{k_{2\perp}}} a_{n,\vec{k_{3\perp}}} \delta_{\vec{q'},-\vec{q_1}} - a^{+}_{n',\vec{k_{3\perp}}+\vec{q_{1\perp}}} \\
&- a^{+}_{n',\vec{k_{2\perp}}} b^{+}_{-\vec{q_1}} b_{\vec{q'}} \delta_{n,\vec{k_{1\perp}},\vec{k_{3\perp}}} \Big\rangle t
\end{aligned}
$$

Note:

$$
\begin{aligned}
\delta_{\vec{q'},-\vec{q_1}} &= 1 \leftrightarrow \vec{q'} = -\vec{q_1} \\
\delta_{n,\vec{k_{1\perp}},\vec{k_{3\perp}}} &= 1 \leftrightarrow \vec{k_{1\perp}} = \vec{k_{3\perp}} \\
\delta_{n,\vec{k_{2\perp}},\vec{k_{3\perp}}+\vec{q_1}} &= 1 \leftrightarrow \vec{k_{2\perp}} = \vec{k_{3\perp}} + \vec{q_1}
\end{aligned}
$$

It is important to notice that the conditions: $\vec{q'} = -\vec{q_1}, \vec{k_{2\perp}} = \vec{k_{3\perp}} + \vec{q_{1\perp}} \leftrightarrow \vec{q_{1\perp}} = \vec{k_{2\perp}} - \vec{k_{3\perp}}$ cannot be satisfied simultaneously, which can give a result: $\delta_{\vec{q'},-\vec{q_1}} = \delta_{n',\vec{k_{2\perp}},\vec{k_{3\perp}}+\vec{q_1}} = 1$

$$
\begin{aligned}
\to \sum_{n,n'} \sum_{\vec{k_{3\perp}},q_{1\perp}} a^{+}_{n,\vec{k_{1\perp}}} a_{n,\vec{k_{3\perp}}} \delta_{n',\vec{k_{2\perp}},\vec{k_{3\perp}}+\vec{q_1}} \delta_{\vec{q'},-\vec{q_1}} &= \sum_{n,n'} \sum_{\vec{q_1}} a^{+}_{n,\vec{k_{1\perp}}} a_{n,\vec{k_{2\perp}}-\vec{q_{1\perp}}} \delta_{\vec{q'},-\vec{q_1}} \\
&= \sum_{n,n'} a^{+}_{n,\vec{k_{1\perp}}} a_{n,\vec{k_{2\perp}}+\vec{q'_\perp}}
\end{aligned}
$$

$$
\begin{aligned}
\to S_3 = &- \sum_{n,n'} \sum_{k_{3\perp}} C\left(-\vec{q'}\right) I_{n,n'}(q_z) \left\langle a^{+}_{n,\vec{k_{1\perp}}} a^{+}_{n',\vec{k_{3\perp}}-\vec{q'_\perp}} a_{n',\vec{k_{2\perp}}} a_{n,\vec{k_{3\perp}}} \right\rangle t \\
&- \sum_{n,n'} \sum_{\vec{q_1}} C(q_\perp) I_{nn'}(q_z) \left\langle a^{+}_{n',\vec{k_{1\perp}}+\vec{q_{1\perp}}} a_{n',\vec{k_{2\perp}}} \left(b_{\vec{q_1}} + b^{+}_{-q_1}\right) b_{\vec{q'}} \right\rangle t \\
&+ \sum_{n,n'} \sum_{q_\perp} C(\vec{q_1}) I_{nn'}(q_z) \left\langle a^{+}_{n,\vec{k_{1\perp}}} a_{n,\vec{k_{2\perp}}-\vec{q_{1\perp}}} b_{\vec{q'}} \left(b_{\vec{q_1}} + b^{+}_{-\vec{q_1}}\right) \right\rangle t
\end{aligned}
$$

We now collect the three remaining contributions to the equation of motion. These terms correspond to (i) the free-particle energy difference, (ii) the interaction with the phonon field of momentum $-q$, and (iii) the interaction with the phonon field at momentum $q+$. Combining them yields:

$$
\begin{aligned}
\to i\frac{\partial F}{\partial t} = &- \left[\mathcal{E}_n\left(\vec{k_{1\perp}} - \frac{e}{C}\vec{A}(t)\right) - \mathcal{E}_n\left(\vec{k_{2\perp}} - \frac{e}{c}\vec{A}(t)\right) - \hbar\omega q'\right] F(t) \\
&- \sum_{n,n'} \sum_{k_{3\perp}} C\left(-\vec{q'}\right) I_{n,n'}(q_z) \left\langle a^{+}_{n,\vec{k_{1\perp}}} a^{+}_{n',\vec{k_{3\perp}}-\vec{q'_\perp}} a_{n',\vec{k_{2\perp}}} a_{n,\vec{k_{3\perp}}} \right\rangle t \\
&- \sum_{n,n'} \sum_{\vec{q_{1\perp}}} C(\vec{q_1}) I_{nn'}(q_z) \left\langle a^{+}_{n',\vec{k_{1\perp}}+\vec{q_{1\perp}}} a_{n',\vec{k_{2\perp}}} \left(b_{\vec{q_1}} + b^{+}_{-q_1}\right) b_{\vec{q'}} \right\rangle t \\
&+ \sum_{n,n'} \sum_{\vec{q_{1\perp}}} C(\vec{q_1}) I_{nn'}(q_z) \left\langle a^{+}_{n,\vec{k_{1\perp}}} a_{n,\vec{k_{2\perp}}-\vec{q_{1\perp}}} b_{\vec{q'}} \left(b_{\vec{q_1}} + b^{+}_{-\vec{q_1}}\right) \right\rangle t \\
\to \frac{\partial F}{\partial t} = & i\left[\mathcal{E}_n\left(\vec{k_{1\perp}} - \frac{e}{C}\vec{A}(t)\right) - \mathcal{E}_{n'}\left(\vec{k_{2\perp}} - \frac{e}{c}\vec{A}(t)\right) - \hbar\omega q'\right] F(t) \\
&+ i \sum_{n,n'} \sum_{\vec{k_{3\perp}}} C\left(-\vec{q'}\right) I_{n'm}(\vec{q_z}) \left\langle a^{+}_{n,\vec{k_{1\perp}}} a^{+}_{n,\vec{k_{3\perp}}-\vec{q'_\perp}} a_{n',\vec{k_{2\perp}}} a_{n,\vec{k_{3\perp}}} \right\rangle t \\
&+ i \sum_{n,n'} \sum_{\vec{k_{3\perp}}} C\left(-\vec{q'}\right) I_{n'm}(\vec{q_z}) \left\{ \left\langle a^{+}_{n,\vec{k_{1\perp}}+\vec{q_{1\perp}}} a_{n',\vec{k_{2\perp}}} \left(b_{\vec{q_1}} + b^{+}_{-\vec{q_1}}\right) \right\rangle t \right.
\end{aligned}
$$

$$-\left\langle a^{+}_{n,\vec{k_{1\perp}}} a_{n',\vec{k_{2\perp}}-q_{1\perp}} b_{\vec{q'}} \left(b_{\vec{q_1}} + b^{+}_{-\vec{q_1}}\right)\right\rangle t\}$$

Since S_2 is approximately equal to $f_n^2, \vec{k_\perp}$ and the electron occupation in the semiconductor is much smaller than unity $f_n, \vec{k_\perp} \ll 1$, this term is negligible. Consequently, the time-evolution equation reduces to:

$$\begin{aligned}
\rightarrow i\frac{\partial F}{\partial t} = &- \left[\mathcal{E}_n\left(\vec{k_{1\perp}} - \frac{e}{C}\vec{A}(t)\right) - \mathcal{E}_n\left(\vec{k_{2\perp}} - \frac{e}{c}\vec{A}(t)\right) - \hbar\omega q'\right] F(t) \\
&+ i\sum_{n,n'}\sum_{\vec{k_{3\perp}}} C\left(-\vec{q'}\right) I_{n'm}\left(\vec{q_z}\right)\left\{\left\langle a^{+}_{n,\vec{k_{1\perp}}+\vec{q_1}} a_{n',\vec{k_{2\perp}}} \left(b_{\vec{q_1}} + b^{+}_{-\vec{q_1}}\right)\right\rangle t\right. \\
&- \left\langle a^{+}_{n,\vec{k_{1\perp}}} a_{n',\vec{k_{2\perp}}-q_{1\perp}} b_{\vec{q'}} \left(b_{\vec{q_1}} + b^{+}_{-\vec{q_1}}\right)\right\rangle t\}
\end{aligned}$$

To fully specify the problem, we impose the boundary conditions and define the auxiliary quantities. In particular, at $t \rightarrow \infty$ the correlation function vanishes:

$$F_{n,\vec{k_{1\perp}},n',\vec{k_{2\perp}},q'(t=\infty)} = 0$$

Furthermore, the electron and phonon distributions are defined as

$$\begin{aligned}
f_n, \vec{k_1}(t) &= \left\langle a^{+}_{n,\vec{k_\perp}} a_{n,\vec{k_\perp}}\right\rangle t \\
N_{\vec{q}}(t) &= \left\langle b^{+}_{\vec{q}} b_{\vec{q}}\right\rangle t
\end{aligned}$$

With these definitions, the correlations function can be written as a product.

$$\rightarrow F_{n,\vec{k_{1\perp}},n',\vec{k_{2\perp}},\vec{q'}}(t) = P(t)\cdot F(t)$$

where $P(t)$ is a known prefactor and $F(t)$ satisfies a nonhomogeneous differential equation.

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