Course M1105

Vector functions
and functions of several variables

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Chapter 1

Topology of $\mathbb{R}^n$

1.1 Norms and distances on $\mathbb{R}^n$

1.1.1 The space $\mathbb{R}^n$

Definition *1.1 We define the space* $\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n-times}$ *by*

$$\mathbb{R}^n = \{x = (x_1, \cdots, x_n) : x_1, \cdots, x_n \in \mathbb{R}\},$$

with the addition and the scalar multiplication
$x + y = (x_1, \cdots, x_n) + (y_1, \cdots, y_n) = (x_1 + y_1, \cdots, x_n + y_n), \ \forall x, y \in \mathbb{R}^n$
and $\alpha x = \alpha (x_1, \cdots, x_n) = (\alpha x_1, \cdots, \alpha x_n), \ \forall \alpha \in \mathbb{R}, \forall x \in \mathbb{R}^n$.

Let $e_1 = (1, 0, \cdots, 0)$, $e_2 = (0, 1, 0, \cdots, 0), \cdots, e_n = (0, \cdots, 0, 1)$ be the canonical basis of $\mathbb{R}^n$.
An element $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ is therefore written in the form

$$x = x_1 e_1 + x_2 e_2 + \cdots x_n e_n = \sum_{i=1}^{n} x_i e_i.$$

Matrix notation is often used $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

- For $n = 2$, $\mathbb{R}^2 = \{X = (x, y) : x, y \in \mathbb{R}\}$, representing the $xy-plane$, with the orthonormal system $\left(O, \overrightarrow{i}, \overrightarrow{j}\right)$ and $X = x\overrightarrow{i} + y\overrightarrow{j}$, $\forall X \in \mathbb{R}^2$.
- For $n = 3$, $\mathbb{R}^3 = \{X = (x, y, z) : x, y, z \in \mathbb{R}\}$, representing the $xyz-space$, with the orthonormal system $\left(O, \overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}\right)$ and $X = x\overrightarrow{i} + y\overrightarrow{j} + z\overrightarrow{k}$, $\forall X \in \mathbb{R}^3$.

Definition *1.2* $\mathbb{R}^n$ *is equipped with a scalar product defined, for two vectors* $x = (x_1, \cdots, x_n)$ *and* $y = (y_1, \cdots, y_n)$ *of* $\mathbb{R}^n$, *by*

$$x \cdot y = \langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i.$$

Theorem** 1.1 **(Cauchy-Schwarz Inequality)
$\forall x=(x_1,\cdots,x_n)\in\mathbb{R}^n,\ \forall y=(y_1,\cdots,y_n)\in\mathbb{R}^n,$ we have

$$\left|\sum_{i=1}^{n}x_iy_i\right|\leq\left(\sum_{i=1}^{n}x_i^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n}y_i^2\right)^{\frac{1}{2}}.$$

Proof : Suppose that x and y are not collinear.

We have $\sum\limits_{i=1}^{n}(tx_i+y_i)^2>0$, for all $t\in\mathbb{R}$. Then

$\sum\limits_{i=1}^{n}(tx_i+y_i)^2=\sum\limits_{i=1}^{n}(t^2x_i^2+2tx_iy_i+y_i^2)=\left(\sum\limits_{i=1}^{n}x_i^2\right)t^2+2\left(\sum\limits_{i=1}^{n}x_iy_i\right)t+\left(\sum\limits_{i=1}^{n}y_i^2\right)>0.$

Let $a=\sum\limits_{i=1}^{n}x_i^2,\ b=\sum\limits_{i=1}^{n}x_iy_i$ and $c=\sum\limits_{i=1}^{n}y_i^2\implies at^2+2bt+c>0,$

as $a>0\Longrightarrow\Delta'=b^2-ac<0\Longrightarrow b^2<ac$

$\Longrightarrow\left(\sum\limits_{i=1}^{n}x_iy_i\right)^2<\left(\sum\limits_{i=1}^{n}x_i^2\right)\left(\sum\limits_{i=1}^{n}y_i^2\right)$, hence the inequality.

If x and y are collinear, then $\exists t_0\in\mathbb{R}^*$ such that $y=t_0x$, therefore

$\left(\sum\limits_{i=1}^{n}x_i^2\right)^{\frac{1}{2}}\left(\sum\limits_{i=1}^{n}y_i^2\right)^{\frac{1}{2}}=\left(\sum\limits_{i=1}^{n}x_i^2\right)^{\frac{1}{2}}\left(\sum\limits_{i=1}^{n}t_0^2x_i^2\right)^{\frac{1}{2}}=|t_0|\sum\limits_{i=1}^{n}x_i^2$

and $\left|\sum\limits_{i=1}^{n}x_iy_i\right|=|t_0|\sum\limits_{i=1}^{n}x_i^2$, hence the equality.

1.1.2 Norms and distances

***Definition** 1.3 A norm on $\mathbb{R}^n$ is all mapping*

$$N:\mathbb{R}^n\longrightarrow[0,\infty[,$$

verifying the three properties :
(N_1) $\forall x\in\mathbb{R}^n,\ N(x)=0\Longleftrightarrow x=0_{\mathbb{R}^n}$; *(Positivity)*
(N_2) $\forall\alpha\in\mathbb{R},\ \forall x\in\mathbb{R}^n,\ N(\alpha x)=|\alpha|\,N(x)$; *(Homogeneity)*
(N_2) $\forall x,y\in\mathbb{R}^n,\ N(x+y)\leq N(x)+N(y)$. *(Triangular inequality)*

***Definition** 1.4 A distance on $\mathbb{R}^n$ is all mapping*

$$d:\mathbb{R}^n\times\mathbb{R}^n\longrightarrow[0,\infty[,$$

that satisfies :
(D_1) $\forall x,y\in\mathbb{R}^n,\ d(x,y)=0\Longleftrightarrow x=y$;
(D_2) $\forall x,y\in\mathbb{R}^n,\ d(x,y)=d(y,x)$;
(D_3) $\forall x,y,z\in\mathbb{R}^n,\ d(x,z)\leq d(x,y)+d(y,z)$.

Remark : For all norm $\|\cdot\|$ on $\mathbb{R}^n$, we can associate a distance $d(\cdot,\cdot)$ such that for $x,y\in\mathbb{R}^n$

$$d(x,y)=N(y-x).$$

The converse is not true, i.e., there are distances that are not deduced from a norm.

Note : For $n=1$, the unique usual norm on $\mathbb{R}$ is the absolute value $N(x)=|x|$, and the associated distance is defined by $d(x,y)=|y-x|$.

1.1.3 Usual Norms and associated distances

In what follows we will study the three usual norms of the space $\mathbb{R}^2$.

- **First usual norm on** $\mathbb{R}^2$: Let $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \mathbb{R}^2$.

  The first usual norm on $\mathbb{R}^2$ is defined by

$$\|x\|_1 = |x_1| + |x_2|,$$

  and its associated distance is given by

$$d_1(x, y) = \|y - x\|_1 = |y_1 - x_1| + |y_2 - x_2|.$$

***Proposition** 1.1 $\|\cdot\|_1$ is a norm on $\mathbb{R}^2$ and d_1 is a distance on $\mathbb{R}^2$.*

Proof : Let $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \mathbb{R}^2$.
$(N1)$ $\|x\|_1 = 0 \iff |x_1| + |x_2| = 0 \iff |x_1| = |x_2| = 0 \iff x_1 = x_2 = 0 \iff x = (0, 0)$,
$(N2)$ $\|\alpha x\|_1 = |\alpha x_1| + |\alpha x_2| = |\alpha|\,|x_1| + |\alpha|\,|x_2| = |\alpha|\,(|x_1| + |x_2|) = |\alpha|\,\|x\|_1$, $\forall \alpha \in \mathbb{R}$,
$(N3)$ $\|x + y\|_1 = |x_1 + y_1| + |x_2 + y_2| \leq |x_1| + |y_1| + |x_2| + |y_2| \leq \|x\|_1 + \|y\|_1$.
Let $x, y, z \in \mathbb{R}^2$.
$(D1)$ $d_1(x, y) = 0 \iff \|y - x\|_1 \iff y - x = 0 \iff x = y$,
$(D2)$ $d_1(x, y) = \|y - x\|_1 = \|-(x - y)\|_1 = \|x - y\|_1 = d_1(y, x)$,
$(D3)$ $d_1(x, z) = \|z - x\|_1 = \|z - y + y - x\|_1 \leq \|z - y\|_1 + \|y - x\|_1 \leq d_1(x, y) + d_2(y, z)$.

- **Second usual norm on** $\mathbb{R}^2$: Let $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \mathbb{R}^2$.

  The second usual norm, called euclidean norm, on $\mathbb{R}^2$ is defined by

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2}$$

  and its associated distance is given by

$$d_2(x, y) = \|y - x\|_2 = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}.$$

***Proposition** 1.2 $\|\cdot\|_2$ is a norm on $\mathbb{R}^2$ and d_2 is a distance on $\mathbb{R}^2$.*

Proof : Let $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \mathbb{R}^2$.
$(N1)$ $\|x\|_2 = 0 \iff \sqrt{x_1^2 + x_2^2} = 0 \iff x_1^2 + x_2^2 = 0 \iff x_1 = x_2 = 0 \iff x = (0, 0)$,
$(N2)$ $\|\alpha x\|_2 = \sqrt{(\alpha x_1)^2 + (\alpha x_2)^2} = \sqrt{\alpha^2\,(x_1^2 + x_2^2)} = |\alpha|\,\sqrt{x_1^2 + x_2^2} = |\alpha|\,\|x\|_2$, $\forall \alpha \in \mathbb{R}$,
$(N3)$

$$\begin{aligned} \|x + y\|_2^2 &= (x_1 + y_1)^2 + (x_2 + y_2)^2 = |x_1 + y_1|^2 + |x_2 + y_2|^2 \\ &\leq (|x_1| + |y_1|)^2 + (|x_2| + |y_2|)^2 \\ &\leq x_1^2 + y_1^2 + 2\,|x_1|\,|y_1| + x_2^2 + y_2^2 + 2\,|x_2|\,|y_2| \\ &\leq (x_1^2 + x_2^2) + (y_1^2 + y_2^2) + 2\,(|x_1|\,|y_1| + |x_2|\,|y_2|). \end{aligned}$$

Using Cauchy-Schwarz inequality $|x_1|\,|y_1| + |x_2|\,|y_2| \leq \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}$,
then $\|x + y\|_2^2 \leq \|x\|_2^2 + \|y\|_2^2 + 2\,\|x\|_2\,\|y\|_2 \leq (\|x\|_2 + \|y\|_2)^2$
therefore $\|x + y\|_2 \leq \|x\|_2 + \|y\|_2$.
For the distance, the proof is similar to the one of the previous proposition.

- **Third usual norm on** $\mathbb{R}^2$: Let $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \mathbb{R}^2$.

  The third usual norm, called infinite norm, on $\mathbb{R}^2$ is defined by

$$\|x\|_\infty = \max(|x_1|, |x_2|),$$

  and its associated distance is given by

$$d_\infty(x,y) = \|y - x\|_\infty = \max(|y_1 - x_1|, |y_2 - x_2|).$$

***Proposition** 1.3 $\|\cdot\|_\infty$ is a norm on $\mathbb{R}^2$ and d_∞ is a distance on $\mathbb{R}^2$.*

Proof : Let $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \mathbb{R}^2$.
$(N1)$ $\|x\|_\infty = 0 \iff \max(|x_1|, |x_2|) = 0 \iff |x_1| = |x_2| = 0 \iff x_1 = x_2 = 0 \iff x = (0,0)$,
$(N2)$ $\|\alpha x\|_\infty = \max(|\alpha x_1|, |\alpha x_2|) = \max(|\alpha| |x_1|, |\alpha| |x_2|) = |\alpha| \max(|x_1|, |x_2|) = |\alpha| \|x\|_\infty$, $\forall \alpha \in \mathbb{R}$,
$(N3)$ $\|x + y\|_\infty = \max(|x_1 + y_2|, |x_2 + y_2|)$
we have $|x_1 + y_1| \leq |x_1| + |y_1| \leq \max(|x_1|, |x_2|) + \max(|y_1|, |y_2|) \leq \|x\|_\infty + \|y\|_\infty$,
similarly $|x_2 + y_2| \leq \|x\|_\infty + \|y\|_\infty$, then $\|x + y\|_\infty \leq \|x\|_\infty + \|y\|_\infty$
For the distance, the proof is similar to the one of the previous theorem.

Example : Let $A(2,3)$ and $B(-1,2)$ be two points of $\mathbb{R}^2$. Calculate $d(A,B)$ with respect to the three usual distances.
Solution : $d_1(A,B) = |-1-2| + |2-3| = 4$.
$d_2(A,B) = \sqrt{(-1-2)^2 + (2-3)^2} = \sqrt{10}$,
$d_\infty(A,B) = \max\{|-1-2|, |2-3|\} = 3$.

Remarks : (1) In the same way, we can define the three usual norms on $\mathbb{R}^n$ by

$$\begin{aligned} \|x\|_1 &= |x_1| + \cdots + |x_n| &&= \sum_{i=1}^n |x_i|, \\ \|x\|_2 &= \sqrt{x_1^2 + \cdots + x_n^2} &&= \sqrt{\sum_{i=1}^n x_i^2}, \\ \|x\|_\infty &= \max(|x_1|, \cdots, |x_n|) &&= \max_{1 \leq i \leq n} |x_i|, \end{aligned}$$

$\forall x = (x_1, \cdots, x_n) \in \mathbb{R}^n$.
(2) $\forall i = 1, \cdots, n,\ |x_i| \leq \|x\|$, whatever the norm.
(3) The norm $\|\cdot\|_2$ is associated to the inner product $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$ between the two vectors $x, y \in \mathbb{R}^n$, with $\|x\|_2 = \sqrt{\langle x, x\rangle}$.
(4) From Cauchy-Shwarz inequality, we can deduce that for all $x, y \in \mathbb{R}^n$

$$|\langle x, y\rangle| \leq \|x\|_2 \|y\|_2.$$

***Definition** 1.5 For $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, we define the Hölder's norm of order p $(1 \leq p < \infty)$ by*

$$\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{\frac{1}{p}} = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}.$$

1.2 Neighborhoods on $\mathbb{R}^n$

1.2.1 Open balls, closed balls and spheres in $\mathbb{R}^n$

Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$, d be the associated distance, $r > 0$ and a be a point of $\mathbb{R}^n$.

***Definition** 1.6 We call open ball of $\mathbb{R}^n$ of center a and radius r, associated to $\|\cdot\|$, the set*

$$B(a,r) = \{x \in \mathbb{R}^n : d(a,x) < r\} = \{x \in \mathbb{R}^n : \|x-a\| < r\}.$$

***Definition** 1.7 We call closed ball of $\mathbb{R}^n$ of center a and radius r, associated to $\|\cdot\|$, the set*

$$\overline{B}(a,r) = \{x \in \mathbb{R}^n : d(a,x) \leq r\} = \{x \in \mathbb{R}^n : \|x-a\| \leq r\}$$

***Definition** 1.8 We call sphere of $\mathbb{R}^n$ of center a and radius r, associated to $\|\cdot\|$, the set*

$$S(a,r) = \{x \in \mathbb{R}^n : d(a,x) = r\} = \{x \in \mathbb{R}^n : \|x-a\| = r\}.$$

Remark : If the center is the origin and $r = 1$, the closed balls respectively, spheres are called unit balls respectively, unit spheres.

1.2.2 Balls associated to the usual norms of $\mathbb{R}^2$

- **Associated ball to** $\|\cdot\|_1$: Let $A(a,b) \in \mathbb{R}^2$ and $r > 0$.

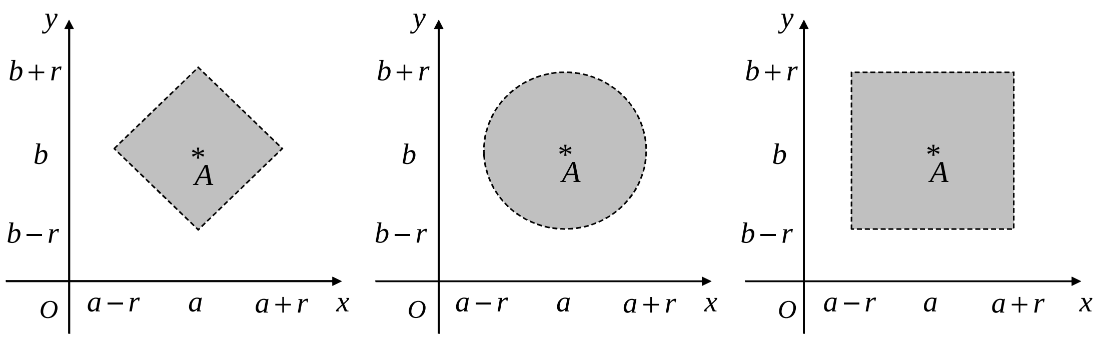
  $B_1(A,r) = \{M \in \mathbb{R}^2 : d_1(A,M) < r\} = \{(x,y) \in \mathbb{R}^2 : \|(x,y)-(a,b)\|_1 < r\}.$
  Let $M(x,y) \in B_1(A,r) \Longrightarrow d_1(A,M) < r \Longrightarrow |x-a| + |y-b| < r.$
  Geometrically, B_1 is the inside of the square of center A and side $\sqrt{2}r$ rotated $\frac{\pi}{4}$ private of its boundary drawn in dotted line.

- **Associated ball to** $\|\cdot\|_2$: Let $A(a,b) \in \mathbb{R}^2$ and $r > 0$.

  $B_2(A,r) = \{M \in \mathbb{R}^2 : d_2(A,M) < r\} = \{(x,y) \in \mathbb{R}^2 : \|(x,y)-(a,b)\|_2 < r\}.$
  Let $M(x,y) \in B_2(A,r) \Longrightarrow d_2(A,M) < r \Longrightarrow \sqrt{(x-a)^2 + (y-b)^2} < r$
  $\Longrightarrow (x-a)^2 + (y-b)^2 < r^2.$
  Geometrically, B_2 is the disk $D(A,r)$ of center A and radius r without the circumference of radius r drawn in dotted line.

- **Associated ball to** $\|\cdot\|_\infty$: Let $A(a,b) \in \mathbb{R}^2$ and $r > 0$.

  $B_\infty(A,r) = \{M \in \mathbb{R}^2 : d_\infty(A,M) < r\} = \{(x,y) \in \mathbb{R}^2 : \|(x,y)-(a,b)\|_\infty < r\}.$
  Let $M(x,y) \in B_\infty(A,r) \Longrightarrow d_\infty(A,M) < r \Longrightarrow \max(|x-a|, |y-b|) < r$
  $\Longrightarrow |x-a| < r$ and $|y-b| < r.$
  Geometrically, B_∞ is the inside of the square of center A and side $2r$ private of its boundary drawn in dotted line.

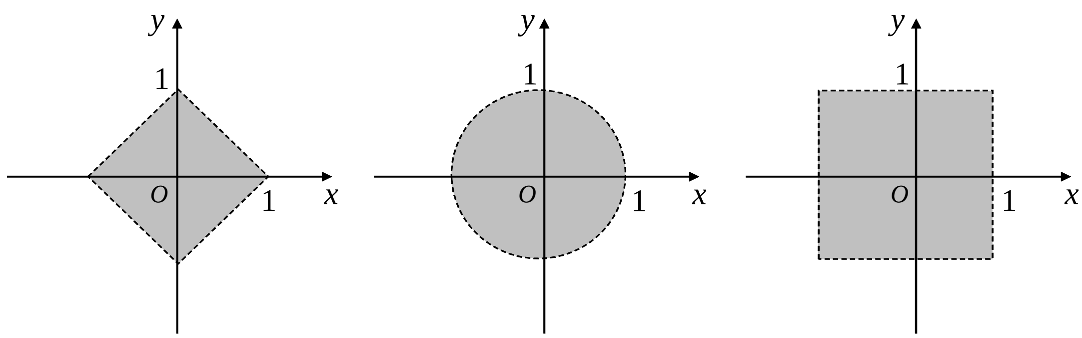


Example : The (open) unit balls associated to the three usual norms are

$$B_1(O,1) = \{(x,y) \in \mathbb{R}^2 : \|(x,y)\|_1 = |x| + |y| < 1\},$$
$$B_2(O,1) = \left\{(x,y) \in \mathbb{R}^2 : \|(x,y)\|_2 = \sqrt{x^2+y^2} < 1\right\}$$

and

$$B_\infty(O,1) = \{(x,y) \in \mathbb{R}^2 : \|(x,y)\|_\infty = \max(|x|,|y|) < 1\}.$$



Remarks : (1) In $\mathbb{R}$, we obtain the interval $]a-r, a+r[$.

(2) In $\mathbb{R}^3$, we well obtain full regular octahedron, full spheres (balls) and bull cubes respectively.

1.2.3 Equivalent norms

Definition *1.9 Two norms N_1 and N_2 on $\mathbb{R}^n$ are said to be equivalent if there exist $\alpha > 0$ and $\beta > 0$ such that*

$$\forall x \in \mathbb{R}^n, \quad \alpha N_2(x) \le N_1(x) \le \beta N_2(x).$$

Proposition *1.4 Let N_1 and N_2 be two norms on $\mathbb{R}^n$.*

The following statements are equivalent

(i) There exist $\alpha > 0$ and $\beta > 0$ such that

$$\forall x \in \mathbb{R}^n, \quad \alpha N_2(x) \le N_1(x) \le \beta N_2(x).$$

(ii) There exist $\alpha > 0$ and $\beta > 0$ such that

$$B_{N_1}(0,\alpha) \subseteq B_{N_2}(0,1) \subseteq B_{N_1}(0,\beta).$$

(iii) There exist $\alpha > 0$ and $\beta > 0$ such that

$$B_{N_2}\left(0, \frac{1}{\beta}\right) \subseteq B_{N_1}(0,1) \subseteq B_{N_2}\left(0, \frac{1}{\alpha}\right).$$

***Proposition** 1.5 For all $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, we have*

$$\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 \leq n \|x\|_\infty .$$

Proof : $\forall i = 1, \cdots, n, \;\; |x_i| \leq \|x\|_2 \Longrightarrow \max\limits_{1 \leq i \leq n} |x_i| \leq \|x\|_2 \Longrightarrow \|x\|_\infty \leq \|x\|_2$,

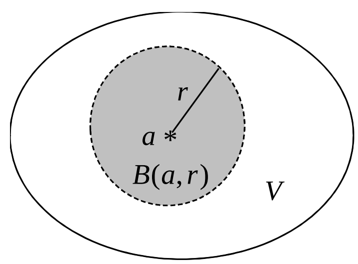
$$\|x\|_2^2 = x_1^2 + \cdots + x_n^2 \leq \sum_{i=1}^{n} x_i^2 + 2 \sum_{i=1;\; i<j}^{n} |x_i|\,|x_j| \leq (|x_1| + \cdots + |x_n|)^2 \leq \|x\|_1^2 \Longrightarrow \|x\|_2 \leq \|x\|_1 ,$$

$$\|x\|_1 = |x_1| + \cdots + |x_n| \leq \|x\|_\infty + \cdots + \|x\|_\infty \leq n \|x\|_\infty .$$

Remark : All the norms on $\mathbb{R}^n$ are equivalent.

1.2.4 Neighborhood

***Definition** 1.10 Let $a \in \mathbb{R}^n$ and $V \subset \mathbb{R}^n$. We say that V is a neighborhood of a, if there exists a real $r > 0$ such that $B(a, r) \subseteq V$.*



***Definition** 1.11 We call pointed neighborhood of a, noted $\widehat{V}$ all neighborhood of a not containing a.*

***Proposition** 1.6 The intersection of two neighborhoods of a is a neighborhood of a.*

Proof : Consider two neighborhoods V and W of a, then

$$\exists r_1 > 0 \;/\; B(a, r_1) \subseteq V \qquad \text{and} \qquad \exists r_2 > 0 \;/\; B_2(a, r_2) \subseteq W.$$

Let $r = \inf(r_1, r_2) \Longrightarrow B(a, r) \subseteq B(a, r_1) \subseteq V$ and $B(a, r) \subseteq B(a, r_2) \subseteq W \Longrightarrow B(a, r) \subseteq V \cap W$, then $V \cap W$ is a neighborhood of a.

1.3 Convergence in $\mathbb{R}^n$

1.3.1 Convergence of a vector sequence

***Definition** 1.12 A vector sequence of $\mathbb{R}^n$ is all sequence $(x_k)_{k \geq 0}$ such that $x_k = (x_k^1, \cdots, x_k^n)$ with $x_k^i \in \mathbb{R}$, $\forall i = 1, \cdots, n$.*

***Definition** 1.13 Let $(x_k)_{k\geq 0}$ be a vector sequence of $\mathbb{R}^n$, $a=(a_1,\cdots,a_n)\in\mathbb{R}^n$ and $\|\cdot\|$ be a norm on $\mathbb{R}^n$. We say that $(x_k)_{k\geq 0}$ converges to a with respect to $\|\cdot\|$ if one of the following properties is verified :*

(i) $(\forall\varepsilon>0)\,(\exists k_0\in\mathbb{N})(\forall k\geq k_0 \Longrightarrow \|x_k-a\|<\varepsilon)$.

(ii) The numerical sequence $(\|x_k-a\|)_{k\geq 0}$ *tends to* 0.

In this case, we denote $x_k \overset{\|\cdot\|}{\longrightarrow} a$ *when* $k\longrightarrow\infty$ *and we say that* a *is the limit of* $(x_k)_k$, *i.e.,* $\lim\limits_{k\longrightarrow\infty} x_k = a$.

Example : Show that $\lim\limits_{n\longrightarrow\infty}\left(\dfrac{n}{n+2}, 2-\dfrac{1}{n^2}\right)=(1,2)$.

Solution : Let the vector sequence $(x_n)_{n\geq 1}$ such that $x_n=\left(\dfrac{n}{n+2}, 2-\dfrac{1}{n^2}\right)$.

$\|x_n-(1,2)\|_\infty = \max\left(\left|\dfrac{n}{n+2}-1\right|, \left|2-\dfrac{1}{n^2}-2\right|\right)=\max\left(\left|\dfrac{1}{n+2}\right|, \left|\dfrac{1}{n^2}\right|\right)$.

We have $\lim\limits_{n\longrightarrow\infty}\left|\dfrac{1}{n+2}\right| = \lim\limits_{n\longrightarrow\infty}\left|\dfrac{1}{n^2}\right|=0 \Longrightarrow \lim\limits_{n\longrightarrow\infty}\|x_n-(1,2)\|_\infty=0$.

***Proposition** 1.7 A vector sequence $(x_k)_k$ is convergent in $\mathbb{R}^n$ if and only if the sequences $(x_k^1)_k,\cdots,(x_k^n)_k$ are convergent in $\mathbb{R}$, and we have*

$$\lim_{k\longrightarrow\infty} x_k = \left(\lim_{k\longrightarrow\infty} x_k^1,\cdots,\lim_{k\longrightarrow\infty} x_k^n\right).$$

Proof : Consider the norm $\|\cdot\|_\infty$ and suppose that $x_k \overset{\|\cdot\|_\infty}{\longrightarrow} a$ when $k\longrightarrow\infty$,
with $a=(a_1,\cdots,a_n)\in\mathbb{R}^n$.
We have $\lim\limits_{k\longrightarrow\infty}\|x_k-a\|_\infty=0 \Longleftrightarrow \lim\limits_{k\longrightarrow\infty}\max\limits_{1\leq i\leq n}|x_k^i-a_i|=0 \Longleftrightarrow \lim\limits_{k\longrightarrow\infty}|x_k^i-a_i|=0$
$\Longleftrightarrow x_k\longrightarrow a_i,\ \forall i=1,\cdots,n$.

***Proposition** 1.8 If a vector sequence $(x_k)_k$ of $\mathbb{R}^n$ has a limit, it is unique.*

Proof : Suppose that $x_k \overset{\|\cdot\|}{\longrightarrow} a$ and $x_k \overset{\|\cdot\|}{\longrightarrow} b$
$\Longrightarrow (\forall\varepsilon>0)\,(\exists k_0\in\mathbb{N})(\forall k\geq k_0 \Longrightarrow \|x_k-a\|<\varepsilon$ and $\|x_k-b\|<\varepsilon)$
$\Longrightarrow \|a-b\|\leq\|a-x_k\|+\|x_k-b\|<2\varepsilon,\ \forall\varepsilon>0$,
then $\|a-b\|=0\Longrightarrow a=b$.

Example : $\lim\limits_{n\longrightarrow\infty}\left(n\sin\dfrac{1}{n},\dfrac{(-1)^n}{n}\right)=\lim\limits_{n\longrightarrow\infty}\left(\lim\limits_{n\longrightarrow\infty} n\sin\dfrac{1}{n},\lim\limits_{n\longrightarrow\infty}\dfrac{(-1)^n}{n}\right)=(1,0)$.

***Definition** 1.14 A vector sequence $(x_k)_k$ of $\mathbb{R}^n$ is said to be divergent if it doesn't admit a limit in $\mathbb{R}^n$.*

Example : Study the convergence of the sequence $(x_n)_{n\geq 1}$ such that $x_n=\left(2^n,\dfrac{1}{n}\right)$.

Solution : $\lim\limits_{n\longrightarrow\infty} x_n = \lim\limits_{n\longrightarrow\infty}\left(2^n,\dfrac{1}{n}\right)=\left(\lim\limits_{n\longrightarrow\infty}2^n,\lim\limits_{n\longrightarrow\infty}\dfrac{1}{n}\right)=(\infty,0)$, therefore the sequence is divergent.

Definition *1.15 We call sub-sequence of the sequence* $(x_k)_k$ *of* $\mathbb{R}^n$*, every sequence of the form* $\left(x_{\sigma(k)}\right)$ *where* $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$ *is a strictly increasing mapping.*

Proposition *1.9 Let* $(x_k)_k$ *be a vector sequence of* $\mathbb{R}^n$*. If* $x_k \overset{\|\cdot\|}{\longrightarrow} a$ *when* $k \longrightarrow \infty$*, then every sub-sequence of* $(x_k)_k$ *converges to a. But the reciprocal is not true.*

Remark : If there exists two sub-sequences of a sequence $(x_k)_k$ of $\mathbb{R}^n$ converging to two different limits, then the sequence $(x_k)_k$ is divergent.

Example : Study the convergence of the sequence $(x_n)_{n\geq 1}$ such that $x_n = \left(\dfrac{(-1)^n n}{n+1}, \dfrac{n+(-1)^n}{n^2}\right)$.

Solution : We have $\lim\limits_{n\longrightarrow\infty} x_{2n} = \lim\limits_{n\longrightarrow\infty} \left(\dfrac{2n}{2n+1}, \dfrac{2n+1}{4n^2}\right) = (1,0)$ and

$\lim\limits_{n\longrightarrow\infty} x_{2n+1} = \lim\limits_{n\longrightarrow\infty} \left(\dfrac{-(2n+1)}{2n+2}, \dfrac{2n}{(2n+1)^2}\right) = (-1,0)$. Therefore $(x_n)_{n\geq 1}$ is divergent.

1.3.2 Theorems on the sequences

Proposition *1.10 Let* $(x_k)_k$ *be a vector sequence of* $\mathbb{R}^n$*. If* $x_k \overset{\|\cdot\|}{\longrightarrow} a$ *when* $k \longrightarrow \infty$*, then the numerical sequence* $(\|x_k\|)_k$ *converges to* $\|a\|$.

Proof : We have $\forall k \geq 0,\ 0 \leq |\|x_k\| - \|a\|| \leq \|x_k - a\|$
$\Longrightarrow 0 \leq \lim\limits_{k\longrightarrow\infty} |\|x_k\| - \|a\|| \leq \lim\limits_{k\longrightarrow\infty} \|x_k - a\| \leq 0$
$\Longrightarrow \lim\limits_{k\longrightarrow\infty} |\|x_k\| - \|a\|| = 0 \Longrightarrow \lim\limits_{k\longrightarrow\infty} \|x_k\| = \|a\|$.

Definition *1.16 Let* $(x_k)_k$ *be a vector sequence of* $\mathbb{R}^n$ *and* $\|\cdot\|$ *be a norm on* $\mathbb{R}^n$*. We say that* $(x_k)_k$ *is bounded in* $\mathbb{R}^n$ *if there exists* $M > 0$ *such that* $\forall k \geq 0,\ \ \|x_k\| \leq M$.

Proposition *1.11 Let* $(x_k)_k$ *be a vector sequence of* $\mathbb{R}^n$*. If* $x_k \overset{\|\cdot\|}{\longrightarrow} a$ *when* $k \longrightarrow \infty$*, then the sequence* $(x_k)_k$ *is bounded in* $\mathbb{R}^n$.

Proof : First, let us recall that we say $(x_k)_k$ is bounded in $\mathbb{R}^n$ iff the sequences $(x_k^1)_k, \cdots, (x_k^n)_k$ are bounded in $\mathbb{R}$.
As $x_k \overset{\|\cdot\|}{\longrightarrow} a$, then $(\forall\varepsilon > 0)\,(\exists k_0 \in \mathbb{N})\,(\forall k \geq k_0 \Longrightarrow \|x_k - a\| < \varepsilon)$.
For $\varepsilon = 1$, $(\exists k_0 \in \mathbb{N})\,(\forall k \geq k_0,\ \ \|x_k\| < 1 + \|a\|)$.
Take $M = \max\{\|x_k\|, \cdots, \|x_{k_0}\|, 1 + \|a\|\} \Longrightarrow \forall k \geq 0,\ \ \|x_k\| \leq M$.

Remarks : (1) For a sequence to be divergent, it is sufficient to show that it is not bounded.
(2) If a sequence is bounded, this does not imply that it is convergent.

Example : Let $x_n = (\cos n, \sin n)$, for $n \geq 0$.
$\|x_n\|_1 = |\cos n| + |\sin n| \leq 2$, $\forall n \geq 0$, but $(x_n)_{n\geq 0}$ is not convergent.

Theorem *1.2 Let $(x_k)_k$ and $(y_k)_k$ be two vector sequences of $\mathbb{R}^n$. If $x_k \xrightarrow{\|\cdot\|} a$ and $y_k \xrightarrow{\|\cdot\|} b$ when $k \longrightarrow \infty$, then the sequence $(\alpha x_k + \beta y_k)_k$ converges to $\alpha a + \beta b$, for $\alpha, \beta \in \mathbb{R}$.*

Proof : We have $\forall k \geq 0,\ \alpha x_k + \beta y_k - \alpha a - \beta b = \alpha(x_k - a) + \beta(y_k - b)$

$\Longrightarrow \forall k \geq 0,\ 0 \leq \|\alpha x_k + \beta y_k - \alpha a - \beta b\| \leq |\alpha| \|x_k - a\| + |\beta| \|y_k - b\|$

$\Longrightarrow 0 \leq \lim\limits_{k \longrightarrow \infty} \|\alpha x_k + \beta y_k - \alpha a - \beta b\| \leq |\alpha| \lim\limits_{k \longrightarrow \infty} \|x_k - a\| + |\beta| \lim\limits_{k \longrightarrow \infty} \|y_k - b\| \leq 0$

$\Longrightarrow \lim\limits_{k \longrightarrow \infty} \|\alpha x_k + \beta y_k - \alpha a - \beta b\| = 0.$

Theorem *1.3 Let $(x_k)_k$ be a vector sequence of $\mathbb{R}^n$ and $(\alpha_k)_k$ be a scalar sequence of $\mathbb{R}$. If $x_k \xrightarrow{\|\cdot\|} a$ and $\alpha_k \longrightarrow \alpha$ when $k \longrightarrow \infty$, then the sequence $(\alpha_k x_k)_k$ converges to αa.*

Proof : We have $\forall k \geq 0,\ \alpha_k x_k - \alpha a = \alpha_k x_k - \alpha_k a + \alpha_k a - \alpha a$

$\Longrightarrow \forall k \geq 0,\ 0 \leq \|\alpha_k x_k - \alpha a\| \leq |\alpha_k| \|x_k - a\| + |\alpha_k - \alpha| \|a\|$

$\Longrightarrow 0 \leq \lim\limits_{k \longrightarrow \infty} \|\alpha_k x_k - \alpha a\| \leq \lim\limits_{k \longrightarrow \infty} |\alpha_k| \|x_k - a\| + \|a\| \lim\limits_{k \longrightarrow \infty} |\alpha_k - \alpha| \leq 0$

$\Longrightarrow \lim\limits_{k \longrightarrow \infty} \|\alpha_k x_k - \alpha a\| = 0.$

1.4 Topological concepts on $\mathbb{R}^n$

1.4.1 Open, closed and bounded set

Definition *1.17 Let E be a subset of $\mathbb{R}^n$.*
E is said to be open on $\mathbb{R}^n \Longleftrightarrow (\forall x \in E)\,(\exists r > 0)\,(B(x, r) \subseteq E)$.

Definition *1.18 Let E be a subset of $\mathbb{R}^n$.*
E is said to be closed on $\mathbb{R}^n \Longleftrightarrow (\forall x \in \mathbb{R}^n)\,(\forall r > 0)\,(B(x, r) \cap E \neq \emptyset \Longrightarrow x \in E)$.

Definition *1.19 Let E be a subset of $\mathbb{R}^n$. We call complementary of E, the set*

$$E^c = \{x \in \mathbb{R}^n : x \notin E\}.$$

Proposition *1.12 Let E be a subset of $\mathbb{R}^n$. E^c is closed if and only if E is open.*

Remarks : (1) The sets $\mathbb{R}^n$ and $\emptyset$ are at the same time open and closed in $\mathbb{R}^n$.
(2) Any singleton $\{a\}$ of $\mathbb{R}^n$ is closed.
(3) Any open set is neighborhood of each of its points.

Proposition *1.13 Let I be a set of $\mathbb{N}$, $(U_i)_{i\in I}$ be a family of open and $(Fi)_{i\in I}$ be a family of closed of $\mathbb{R}^n$.*
(i) $\bigcup\limits_{i\in I} U_i$ is an open and $\bigcap\limits_{i\in I} F_i$ is a closed of $\mathbb{R}^n$.
(ii) If I is finite, then $\bigcap\limits_{i\in I} U_i$ is an open and $\bigcup\limits_{i\in I} F_i$ is a closed of $\mathbb{R}^n$.

Remark : On the other hand if I is not finite $\bigcap\limits_{i\in I} U_i$ is not necessarily open and $\bigcup\limits_{i\in I} F_i$ is not necessarily closed.

Example : $\bigcap\limits_{n\geq 1} B_2\left(0,\dfrac{1}{n}\right) = \{0\}$ not open and $\bigcup\limits_{n\geq 1} \overline{B}_2\left(0,1-\dfrac{1}{n}\right) = B_2\left(0,1\right)$ not closed.

***Definition** 1.20 Let E be a subset of $\mathbb{R}^n$ and $\|\cdot\|$ be a norm on $\mathbb{R}^n$.*
E is said to be bounded with respect to $\|\cdot\|$ on $\mathbb{R}^n$ $\Longleftrightarrow \exists M>0 \;/\; \forall x\in E,\;\; \|x\|\leq M$, i.e., $E\subseteq \overline{B}\left(O,M\right)$.

Properties : (1) All ball of $(\mathbb{R}^n,\|\cdot\|)$ is bounded.
(2) Any subset of a bounded set is bounded.
(3) Any sequence of a bounded set is bounded.

Remark : To show that a set is not bounded, it sufficient to find a sequence in this set which is not bounded.

Examples :
(1) $E=\{(x,y)\in\mathbb{R}^2 : x^2+y^2<1\}=B_2(O,1)$ is open and bounded.
(2) $E=\{(x,y)\in\mathbb{R}^2 : x^2+y^2\leq 1\}=\overline{B}_2(O,1)$ is closed and bounded
(3) $E=\{(x,y)\in\mathbb{R}^2 : x^2+y^2>1\}=\overline{B}_2(O,1)^c$ is open and not bounded.
(4) $E=\{(x,y)\in\mathbb{R}^2 : x^2+y^2\geq 1\}=B_2(O,1)^c$ is closed and not bounded.
(5) $E=\{(x,y)\in\mathbb{R}^2 : 1<x^2+y^2<4\}=B_2(O,2)\cap\overline{B}_2(O,1)^c$ is open and bounded.
(6) $E=\{(x,y)\in\mathbb{R}^2 : 1\leq x^2+y^2<9\}=B_2(O,3)\cap B_2(O,1)^c$ is neither open nor closed, but bounded.

1.4.2 Adherence, Interior and Boundary

***Definition** 1.21 Let E be a subset of $\mathbb{R}^n$. We call interior of E, the open set*

$$\overset{o}{E}=\{x\in\mathbb{R}^n : \exists r>0,\; B(x,r)\subseteq E\}.$$

***Definition** 1.22 Let E be a subset of $\mathbb{R}^n$. We call adherence or closure of E, the closed set*

$$\overline{E}=\{x\in\mathbb{R}^n : \forall r>0,\; E\cap B(x,r)\neq\emptyset\}.$$

Properties : Let E and F be two parts of $\mathbb{R}^n$, then
(1) $\overset{o}{E}\subseteq E\subseteq\overline{E}$, $\left(\overline{E}\right)^c=\overset{o}{E^c}$ and $\left(\overset{o}{E}\right)^c=\overline{E^c}$;
(2) $E\subseteq F\Longrightarrow \overset{o}{E}\subseteq\overset{o}{F}$ and $\overline{E}\subseteq\overline{F}$.
(3) $\overline{E}$ is the smallest closed containing E and $\overset{o}{E}$ is the largest open contained in E.

***Definition** 1.23 Let E be a subset of $\mathbb{R}^n$. An element $a\in\mathbb{R}^n$ is said to be a boundary point of E if*

$$(\forall r>0)\left(E\cap B(a,r)\neq\emptyset \text{ and } E^c\cap B(a,r)\neq\emptyset\right).$$

Definition *1.24 Let E be a subset of $\mathbb{R}^n$. We call boundary of E, the set of all boundary points of E. It is given by*

$$\partial E = \overline{E} \setminus \overset{o}{E}.$$

Proposition *1.14 Let E be a part of $\mathbb{R}^n$, then*

(i) $x \in \overset{o}{E} \Longleftrightarrow (\forall (x_k) \subseteq \mathbb{R}^n \ / \ x_k \longrightarrow x)(\exists k_0 \in \mathbb{N} \ / \ \forall k \geq k_0, \ x_k \in E)$;

(ii) $x \in \overline{E} \Longleftrightarrow (\exists (x_k) \subseteq E \ / \ x_k \longrightarrow x)$.

Proof : (i) $\Longrightarrow$) Let $x \in \overset{o}{E} \Longrightarrow \exists r > 0 \ / \ B(x,r) \subseteq E$. Take $(x_k) \subseteq \mathbb{R}^n \ / \ x_k \longrightarrow x$, then $(\forall \varepsilon > 0)\,(\exists k_0 \in \mathbb{N})\,(\forall k \geq k_0 \Longrightarrow \|x_k - x\| < \varepsilon)$.

Take $\varepsilon = r \Longrightarrow \|x_k - x\| < r, \ \forall k \geq k_0 \Longrightarrow x_k \in B(x,r) \subseteq E, \ \forall k \geq k_0$.

$\Longleftarrow$) Suppose that $x \notin \overset{o}{E} \Longrightarrow \forall r > 0, \ B(x,r) \nsubseteq E \Longrightarrow \forall r > 0, \ \exists y \in B(x,r) \ / \ y \notin E$.

Take $r = \dfrac{1}{k}$, for $k \in \mathbb{N} - \{0\} \Longrightarrow \forall k \geq 1, \ \exists y_k \in B(x,r) \ / \ y_k \notin E$

$\Longrightarrow \forall k \geq 1, \ \|y_k - x\| < \dfrac{1}{k} \Longrightarrow y_k \longrightarrow x \Longrightarrow y_k \in E, \ \forall k \geq 1$, contradiction.

(ii) $\Longrightarrow$) Let $x \in \overline{E} \Longrightarrow (\forall r > 0)(E \cap B(x,r) \neq \emptyset)$.

Take $r = \dfrac{1}{k}$, for $k \in \mathbb{N} - \{0\} \Longrightarrow \forall k \geq 1, \ \exists x_k \in E \ / \ x_k \in B\left(x, \dfrac{1}{k}\right)$

$\Longrightarrow \forall k \geq 1, \ \|x_k - x\| < \dfrac{1}{k} \Longrightarrow \exists (x_k) \subseteq E \ / \ x_k \longrightarrow x$.

$\Longleftarrow$) Suppose that $\exists (x_k) \subseteq E \ / \ x_k \longrightarrow x \Longrightarrow (\forall r > 0)\,(\exists k_0 \in \mathbb{N})(\forall k \geq k_0 \Longrightarrow \|x_k - x\| < r)$
$\Longrightarrow x_k \in \ B(x,r), \ \forall k \geq k_0 \Longrightarrow x_k \in \ E \cap B(x,r), \ \forall k \geq k_0 \Longrightarrow (\forall r > 0)(E \cap B(x,r) \neq \emptyset)$
$\Longrightarrow x \in \overline{E}$.

Corollary *1.1 Let E be a part of $\mathbb{R}^n$, then E is closed if and only if every convergent sequence of elements of E converges in E, i.e., $\forall (x_k) \subseteq E \ / \ x_k \longrightarrow x \Longrightarrow x \in E$.*

Corollary *1.2 Let E be a part of $\mathbb{R}^n$, then*

(i) E is open $\Longleftrightarrow \overset{o}{E} = E$.

(ii) E is closed $\Longleftrightarrow \overline{E} = E$

Example : Show that $E = \{(x,y) \in \mathbb{R}^2 : 2x + 3y = 1 \ \}$ is closed on $\mathbb{R}^2$.

Solution : Prove that $\overline{E} = E$. Since $E \subseteq \overline{E}$, it remains to show that $\overline{E} \subseteq E$.

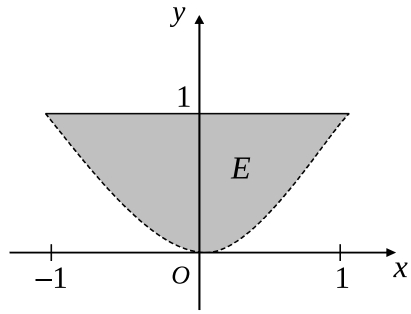
Let $(a,b) \in \overline{E}$, then $\exists\,((x_k, y_k))_{k \geq 0}$ a sequence of E such that $(x_k, y_k) \longrightarrow (a,b)$.

We have $2x_k + 3y_k = 1$ and $x_k \longrightarrow a$, $y_k \longrightarrow b \Longrightarrow \lim\limits_{k \to \infty} (2x_k + 3y_k) = 2a + 3b = 1 \Longrightarrow (a,b) \in E$.

Example : Let $E = \{(x,y) \in \mathbb{R}^2 : x^2 - y < 0 \text{ and } y \leq 1\}$.

1. Sketch E and show that it is bounded.
2. Determine $\overset{o}{E}$, $\overline{E}$, E^c and ∂E.
3. Is E open ? closed ?

Solution : 1.



Let $(x, y) \in E \Longrightarrow x^2 - y < 0$ and $y \leq 1 \Longrightarrow 0 < x^2 < y \leq 1 \Longrightarrow |x| < 1$ and $|y| \leq 1$.
If we consider the norm $\|(x,y)\|_1 = |x| + |y| \Longrightarrow \|(x,y)\|_1 \leq 2$, i.e., $E \subset \overline{B}_1(O, 2)$.
If we consider the norm $\|(x,y)\|_2 = \sqrt{x^2 + y^2} \Longrightarrow \|(x,y)\|_2 \leq \sqrt{2}$, i.e., $E \subset \overline{B}_2\left(O, \sqrt{2}\right)$.
If we consider the norm $\|(x,y)\|_\infty = \max(|x|, |y|) \Longrightarrow \|(x,y)\|_\infty \leq 1$, i.e., $E \subset \overline{B}_\infty(O, 1)$.

2. $\overset{o}{E} = \{(x, y) \in \mathbb{R}^2 : x^2 - y < 0 \text{ and } y < 1\}$.
$\overline{E} = \{(x, y) \in \mathbb{R}^2 : x^2 - y \leq 0 \text{ and } y \leq 1\}$.
$E^c = \{(x, y) \in \mathbb{R}^2 : x^2 - y \geq 0 \text{ or } y > 1\}$.
$$\begin{aligned} \partial E &= \{(x, y) \in \mathbb{R}^2 : (x^2 = y \text{ and } -1 \leq x \leq 1) \text{ or } (y = 1 \text{ and } -1 \leq x \leq 1)\} \\ &= \{(x, y) \in \mathbb{R}^2 : y = x^2 \text{ and } |x| \leq 1\} \cup \{(x, y) \in \mathbb{R}^2 : y = 1) \text{ and } |x| \leq 1\}. \end{aligned}$$

3. Take the point $(0, 1) \in E$ and the sequence $((0, y_k)) \subseteq \mathbb{R}^2$ such that $y_k = 1 + \dfrac{1}{k}$, for $k \geq 1$.
$(0, y_k) \longrightarrow (0, 1)$ but $(0, y_k) \notin E \Longrightarrow (0, 1) \notin \overset{o}{E} \Longrightarrow \overset{o}{E} \neq E$, then E is not open.
Take the sequence $((0, y_k)) \subseteq E$ such that $y_k = \dfrac{1}{k}$, for $k \geq 1$.
$(0, y_k) \longrightarrow (0, 0)$ but $(0, 0) \notin E$, then E is not closed.

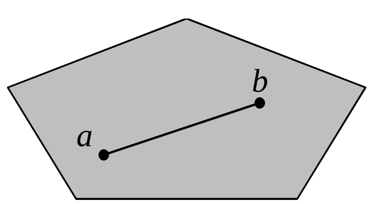
1.4.3 Convex and Connected sets

Definition *1.25 Let $a, b \in \mathbb{R}^n$ we define the segment noted $[a, b]$ by*

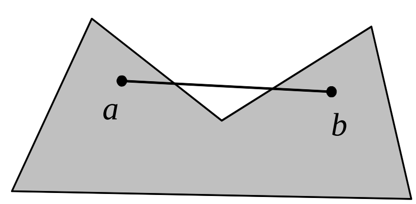
$$\begin{aligned} [a, b] &= \{x \in \mathbb{R}^n : x = \alpha a + \beta b;\ \alpha, \beta \in \mathbb{R}^+ \text{ and } \alpha + \beta = 1\} \\ &= \{x \in \mathbb{R}^n : x = ta + (1-t)b;\ t \in [0, 1]\}. \end{aligned}$$

Definition *1.26 A subset D of $\mathbb{R}^n$ is said to be convex if $\forall (a, b) \in D \times D$, the segment $[a, b] \subset D$.*

Examples : (1) The open and closed balls of $\mathbb{R}^n$ are convex.
Take for example the closed unit ball $\overline{B}(O, 1) = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$.
For $x, y \in \overline{B}(O, 1)$, we have $\|x\| \leq 1$ and $\|y\| \leq 1$
$\Longrightarrow \|tx + (1-t)y\| \leq |t| \|x\| + |1 - t| \|y\| \leq t + 1 - t \leq 1$
$\Longrightarrow tx + (1-t)y \in \overline{B}(O, 1)$.
(2)



Convex



Not convex

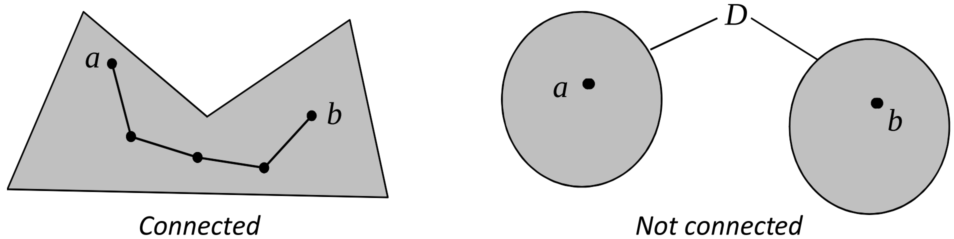
(3) $D = \{(x,y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4\}$ is not convex.

In fact, the points $(1,1)$ and $(-1,-1) \in D$ but $\frac{1}{2}(1,1) + \left(1 - \frac{1}{2}\right)(-1,-1) = (0,0) \notin D$.

Definition *1.27 A subset D of $\mathbb{R}^n$ is said to be connected if $\forall (a,b) \in D \times D$, there is a finite sequence $x_0 = a,\ x_1, \cdots, x_{k-1},\ x_k = b$ of elements of D such that the segments $[x_i, x_{i+1}] \subset D$, $\forall i = 0, \cdots, k-1$.*

Examples : (1) Every convex of $\mathbb{R}^n$ is connected.

(2)



(3) $D = \{(x,y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4\}$ is connected.

(4) $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1 \text{ or } x^2 + y^2 \geq 4\}$ is not connected.

1.5 Exercises

Exercise *1.1 Using Cauchy-Schwarz inequality show that*

$$\forall x \in \mathbb{R}^n, \quad \|x\|_1 \leq \sqrt{n}\, \|x\|_2 .$$

Exercise *1.2 Show that, if $\|\cdot\|$ is a norm on $\mathbb{R}^n$, then $\forall x, y \in \mathbb{R}^n$ we have*

1. $|\|x\| - \|y\|| \leq \|x + y\|$;
2. $\|x\| + \|y\| \leq \|x + y\| + \|x - y\|$;
3. $\dfrac{\|x-y\|}{\|x\|} \leq \rho < 1 \Longrightarrow \dfrac{\|y-x\|}{\|y\|} \leq \dfrac{\rho}{1-\rho}$, *with $x \neq 0$ and $y \neq 0$.*

Exercise *1.3 If $d(\cdot,\cdot)$ is a distance associated to a norm $\|\cdot\|$ on $\mathbb{R}^n$, verify that*

1. $\forall x, y, z \in \mathbb{R}^n,\ d(x,y) = d(x+z, y+z)$;
2. $\forall x, y \in \mathbb{R}^n$ *and* $\forall \alpha \in \mathbb{R},\ d(\alpha x, \alpha y) = |\alpha|\, d(x,y)$.

Exercise *1.4 Check whether each mapping $d : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^+$ is a distance on $\mathbb{R}$:*

1. $d(x,y) = |x^2 - y^2|$
2. $d(x,y) = |x^3 - y^3|$
3. $d(x,y) = |\arctan x - \arctan y|$
4. $d(x,y) = \max\{x, y\}$

Exercise *1.5 Verify if each of the following forms defines a norm on $\mathbb{R}^2$:*

1. $||X|| = |5x + 3y|$, *for* $X = (x,y) \in \mathbb{R}^2$
2. $||X|| = |x + y| + |2x - y|$, *for* $X = (x,y) \in \mathbb{R}^2$
3. $||X|| = \dfrac{|x| + |y|}{1 + |x| + |y|}$, *for* $X = (x,y) \in \mathbb{R}^2$
4. $||X|| = \max(|x+y|, |x-y|)$, *for* $X = (x,y) \in \mathbb{R}^2$
5. $||X|| = |x + y + z| + |x - y + 2z|$, *for* $X = (x,y,z) \in \mathbb{R}^3$

Exercise *1.6 Let N be the mapping defined on $\mathbb{R}^2$ by*

$$N(X) = |x| + |x \ + y|, \ for\ X = (x, y) \in \mathbb{R}^2.$$

1. *Show that N is a norm on $\mathbb{R}^2$.*
2. *Determine the closed unit ball $\overline{B}(0,1)$ associated to N.*
3. *Determine graphically the smallest constant β and the biggest constant α, such that*

$$\overline{B}_1(O, \alpha) \subseteq \overline{B}\,(0,1) \subseteq \overline{B}_1(O, \beta).$$

Exercise *1.7 Let N be the mapping defined on $\mathbb{R}^2$ by*

$$N(X) = \max\left(|x + 4y|, |x - y|\right), \ for\ X = (x, y) \in \mathbb{R}^2.$$

1. *Show that N is a norm on $\mathbb{R}^2$.*
2. *Determine the open unit ball $B\,(0,1)$ associated to N.*
3. *Determine the best constants α and β such that*

$$\forall X \in \mathbb{R}^2, \quad \alpha \left\|X\right\|_2 \leq N(X) \leq \beta \left\|X\right\|_2.$$

Exercise *1.8 Let N be the mapping defined on $\mathbb{R}^2$ by*

$$N(X) = |x| + \sqrt{x^2 + y^2}, \ for\ X = (x, y) \in \mathbb{R}^2.$$

1. *Show that N is a norm on $\mathbb{R}^2$.*
2. *Determine the open unit ball $B\,(0,1)$ in $\mathbb{R}^2$ associated to N.*
3. *Verify that*

$$\forall X \in \mathbb{R}^2, \quad \left\|X\right\|_2 \leq N(X) \leq 2\left\|X\right\|_2.$$

Exercise *1.9 Study the convergence of the sequences $(x_n)_n$ of $\mathbb{R}^2$ defined by*

1. $x_n = \left(\dfrac{1}{n+1}, \left(\dfrac{1}{2}\right)^n\right)$
2. $x_n = \left(\dfrac{n^2+1}{n-1}, \dfrac{n+1}{n-1}\right)$
3. $x_n = \left(1, n \sin\dfrac{1}{n}\right)$
4. $x_n = \left(\dfrac{\sqrt{n}+1}{n+1}, \dfrac{\sqrt{n}+n}{n+1}\right)$
5. $x_n = \left(\dfrac{2^n-1}{3^n-2}, \dfrac{n+2^n}{n2^n}\right)$
6. $x_n = \left(\cos\dfrac{n\pi}{2}, \sin\dfrac{n\pi}{2}\right)$

Exercise *1.10 Study the convergence of the sequence $(x_n)_{n\geq 0}$ of $\mathbb{R}^2$ such that $x_n = \left(\dfrac{\cos\sqrt{n}}{2^n}, \dfrac{\sin\sqrt{n}}{2^n}\right)$, then the sequence $(y_n)_{n\geq 0}$ such that $y_n = \dfrac{1}{\left\|x_n\right\|_2}\left(x_{n+1} - \dfrac{1}{2}x_n\right)$.*

Exercise *1.11 Indicate if the following subsets of $\mathbb{R}^2$ are bounded for the usual norms :*

$A = \{(x, y) \in \mathbb{R}^2 \ : \ x^2 + 5y^2 \leq 2\}$

$B = \{(x, y) \in \mathbb{R}^2 \ : \ |x + y| \leq 1\}$

$C = \{(x, y) \in \mathbb{R}^2 \ : \ \cos x \leq \cos y\}$

$D = \{(x, y) \in \mathbb{R}^2 \ : \ 0 \leq y \leq 1 \text{ and } |x| \leq y\}$

Exercise *1.12 Consider the set $\mathbb{R}^2$ equipped with the usual norms. By writing the following subsets of $\mathbb{R}^2$ as a union or intersection in terms of the balls, say whether they are open or closed :*

$A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1 \text{ and } (x-2)^2 + (y-1)^2 \leq 4\}$
$B = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \cup \{(x,y) \in \mathbb{R}^2 : (x-2)^2 + (y-1)^2 \leq 4\}$
$C = \{(x,y) \in \mathbb{R}^2 : 4 \leq (x-2)^2 + (y-2)^2 \leq 9\}$
$D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1 \text{ and } |x| + |y-1| \leq 1\}$
$E = \{(x,y) \in \mathbb{R}^2 : 2 < \max\{|x-1|, |y|\} < 3\}$
$F = \{(x,y) \in \mathbb{R}^2 : \max\{|x|, |y|\} \leq 2 \text{ and } |x-1| + |y| \leq 3\}$

Exercise *1.13 Determine, with justification, whether the following sets are open or closed for the usual norms :*

$A = \{(x, x^2+1) \in \mathbb{R}^2 : x \in \mathbb{R}\}$
$B = \{(x,y) \in \mathbb{R}^2 : xy + x^2 < 4y^2\}$
$C = \{(x,y) \in \mathbb{R}^2 : e^x \leq x \cos y\}$
$D = \{(x, x^2) \in \mathbb{R}^2 : x > 0\}$
$E = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1 \text{ and } y \in \mathbb{R}\}$
$F = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 > 4 \text{ and } y \geq 1\}$
$G = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1 \text{ and } y \geq x\}$
$H = \{(x,y) \in \mathbb{R}^2 : 1 < |x-y| < x^2 + 1\}$

Exercise *1.14 Sketch E and determine E^c, $\overset{o}{E}$, $\overline{E}$ and ∂E in the following cases :*

1. $E = \{(x,y) \in \mathbb{R}^2 : 0 < x \leq 2 \text{ and } 0 \leq y < 2\}$
2. $E = \{(x,y) \in \mathbb{R}^2 : \ |x+y| \leq 1\}$
3. $E = \{(x,y) \in \mathbb{R}^2 : \ |xy| \leq 1\}$
4. $E = \{(x,y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4,\ y > x,\ y > 3x\}$
5. $E = \{(x,y) \in \mathbb{R}^2 : \ x^2 + y^2 \leq 2x \text{ and } x + y > 2\}$

Exercise *1.15* 1. *Let*

$$A = \{(x,y) \in \mathbb{R}^2 : y \geq x > 0\} \qquad \text{and} \qquad B = \{(x,y) \in \mathbb{R}^2 : y \geq x \geq 0\}$$

and let $\mathbb{R}^2$ be equipped with the usual norms.

a) Show that A is neither open nor closed in $\mathbb{R}^2$.
b) Show that B is closed in $\mathbb{R}^2$.
c) Let $C = \{(x,y) \in B : x = 0\}$. Show that $C \subset \overline{A}$.
d) Deduce that $\overline{A} = B$.

2. *Same question with $A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9 \text{ and } x > 2\}$.*
3. *Same question with $A = \{(x,y) \in \mathbb{R}^2 : x^2 - y < 0 \text{ and } x + y \geq 1\}$.*

Exercise *1.16 Let*

$$A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 2x,\ x + y > 2\} \quad \text{and} \quad B = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 2x,\ x + y > 2\}$$

and let $\mathbb{R}^2$ be equipped with the usual norms.

1. *Show that A is neither open nor closed in $\mathbb{R}^2$.*
2. *Show that B is open in $\mathbb{R}^2$.*
3. *Let $C = \{(x,y) \in A : x^2 + y^2 - 2y = 0\}$. Show that $C \subset (\overset{o}{A})^c$*
4. *Deduce that $\overset{o}{A} = B$.*

Chapter 2

Real-valued functions of several variables - Limits and continuity

2.1 Functions of several real variables

Definition *2.1 We call a real valued function of* n *real variables* $x_1, \cdots, x_n$*, any mapping defined from a subset* D *of* $\mathbb{R}^n$ *into* $\mathbb{R}$*, that for every vector point* $x = (x_1, \cdots, x_n) \in D$ *corresponds a real image* $f(x) = f(x_1, \cdots, x_n) \in \mathbb{R}$*. It is denoted by*

$$f: \begin{array}{ccc} D \subseteq \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ (x_1, \cdots, x_n) & \longmapsto & f(x_1, \cdots, x_n) \end{array}$$

Definition *2.2 The set of* $(x_1, \cdots, x_n) \in \mathbb{R}^n$ *for which* f *is defined is called domain of definition of* f*, noted* D_f*, with*

$$D_f = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : f(x_1, \cdots, x_n) \text{ exists in } \mathbb{R}\}.$$

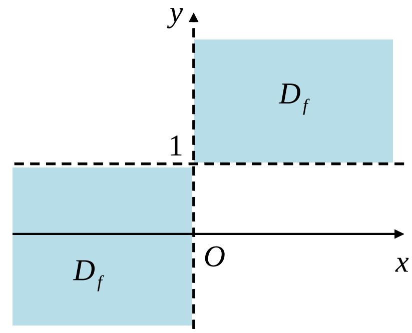
- If $n = 1$, we have a function of one variable : $f: \begin{array}{ccc} D \subseteq \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x) \end{array}$

- If $n = 2$, we have a function of two variables : $f: \begin{array}{ccc} D \subseteq \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x, y) & \longmapsto & f(x, y) \end{array}$

- If $n = 3$, we have a function of three variables : $f: \begin{array}{ccc} D \subseteq \mathbb{R}^3 & \longrightarrow & \mathbb{R} \\ (x, y, z) & \longmapsto & f(x, y, z) \end{array}$

Examples : (1) The domain of definition of the function $f(x, y) = \ln(xy - x)$ is

$$\begin{aligned} D_f &= \{(x, y) \in \mathbb{R}^2 : x(y - 1) > 0\} \\ &= \{(x, y) \in \mathbb{R}^2 : (x > 0 \text{ and } y > 1) \text{ or } (x < 0 \text{ and } y < 1)\} \\ &= \{(x, y) \in \mathbb{R}^2 : x > 0 \text{ and } y > 1\} \cup \{(x, y) \in \mathbb{R}^2 : (x < 0 \text{ and } y < 1\}. \end{aligned}$$



(2) The domain of definition of the function $g(x, y, z) = \dfrac{\ln(1-|xy|)}{z}$ is
$D_g = \{(x, y, z) \in \mathbb{R}^3 : 1 - |xy| > 0 \text{ and } z \neq 0\} = \{(x, y, z) \in \mathbb{R}^3 : -1 < xy < 1 \text{ and } z \neq 0)\}$.
(3) The function $h(x, y) = \sqrt{x^2 + (x-y)^2 + 1}$ is defined $\forall (x, y) \in \mathbb{R}^2$.

***Theorem** 2.1 Let f and g be two functions of n variables defined, respectively, on D_f and D_g, then αf is defined on D_f and the functions $f \pm g$, fg and $\dfrac{f}{g}$ (for $g(x) \neq 0$, $\forall x \in D_g$) are defined on $D_f \cap D_g$ with*
(i) $(\alpha f)(x_1, \cdots, x_n) = \alpha f(x_1, \cdots, x_n)$, $\forall \alpha \in \mathbb{R}$;
(ii) $(f \pm g)(x_1, \cdots, x_n) = f(x_1, \cdots, x_n) \pm g(x_1, \cdots, x_n)$;
(iii) $(fg)(x_1, \cdots, x_n) = f(x_1, \cdots, x_n) g(x_1, \cdots, x_n)$;
(iv) $\left(\dfrac{f}{g}\right)(x_1, \cdots, x_n) = \dfrac{f(x_1, \cdots, x_n)}{g(x_1, \cdots, x_n)}$.

***Definition** 2.3 Given the diagram $D \subseteq \mathbb{R}^n \xrightarrow{f} I \subseteq \mathbb{R} \xrightarrow{g} \mathbb{R}$ such that $f(D) \subseteq I$.*
The function $g \circ f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ is called composite function of f and g with

$$(g \circ f)(x_1, \cdots, x_n) = g(f(x_1, \cdots, x_n)).$$

Example : If $f(x, y) = \dfrac{x+y}{x-y}$ and $g(x) = \dfrac{x+1}{x-1}$,

then $(g \circ f)(x, y) = g(f(x, y)) = \dfrac{\dfrac{x+y}{x-y} + 1}{\dfrac{x+y}{x-y} - 1} = \dfrac{x}{y}$,
with $D_f = \{(x, y) \in \mathbb{R}^2 : x \neq y\}$, $D_g = \mathbb{R} - \{1\}$ and $D_{g \circ f} = \{(x, y) \in \mathbb{R}^2 : x \neq y \text{ and } y \neq 0\}$.

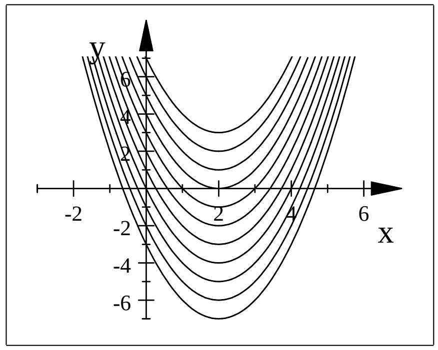
***Definition** 2.4 Let $k \in \mathbb{R}$ and $f : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function of two variables. We call level curve the set*

$$L_k = \{(x, y) \in D : f(x, y) = k\}.$$

Example : The function $f(x, y) = -x^2 + 4x + y$, has level curves given by the equations

$$y = x^2 - 4x + k.$$

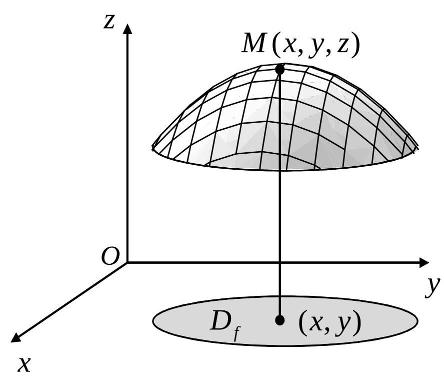
These are parabolas admitting all the same axis of symmetry ($x = 2$).



***Definition** 2.5 Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$. The set*

$$G_f = \left\{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D_f \quad \text{and } z = f(x, y)\right\}$$

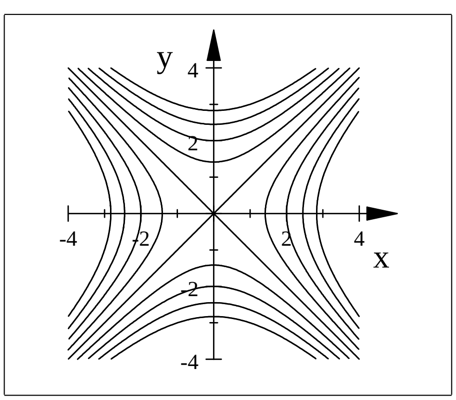
is called the graph of f, representing a surface of the $xyz-$ space whose Cartesian equation is $z = f(x, y)$.

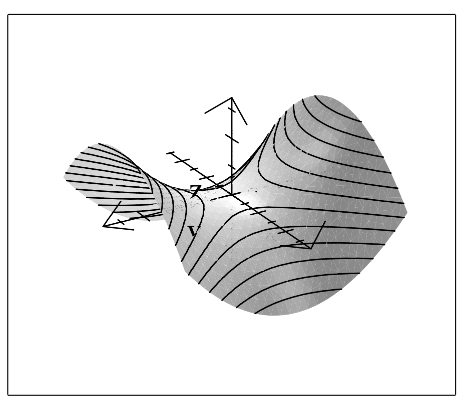


Example : Construct the surface $z = x^2 - y^2$.
Solution : The function $f(x, y) = x^2 - y^2$, has level curves given by the equations

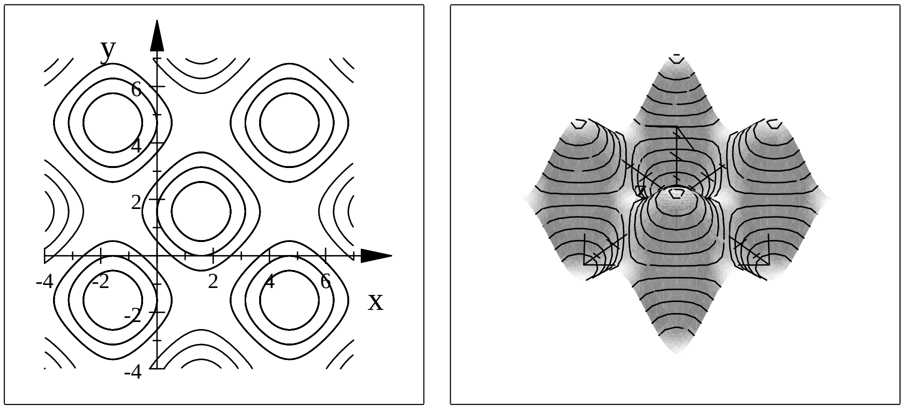
$$x^2 - y^2 = k.$$

For $k = 0$, we have $y = \pm x$, for $k > 0$, $x^2 - y^2 = k$ is a hyperbola of axis $x'x$ and for $k < 0$, $y^2 - x^2 = -k$ is a hyperbola of axis $y'y$.





Example : Level curves and representative surface of the function $z = f(x, y) = \sin x + \sin y$.



***Definition** 2.6 Let $k \in \mathbb{R}$ and $f : D \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a function of three variables. We call level surface the set*

$$S_k = \{(x, y, z) \in D : f(x, y, z) = k\}.$$

Example : The function $f(x, y, z) = x^2 + y^2 + z^2$, has level surfaces given by the equations

$$x^2 + y^2 + z^2 = k.$$

These are spheres admitting all the same center O and of radius $R = \sqrt{k}$ with $k \geq 0$.

2.2 Limits of functions of several variables

Let D be an open of $\mathbb{R}^n$, $f : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ and $a = (a_1, \cdots, a_n) \in D$ or $\overline{D}$.

***Definition** 2.7 We say that $L \in \mathbb{R}$ is the limit of $f(x)$ when $x = (x_1, \cdots, x_n)$ tends to a if and only if*

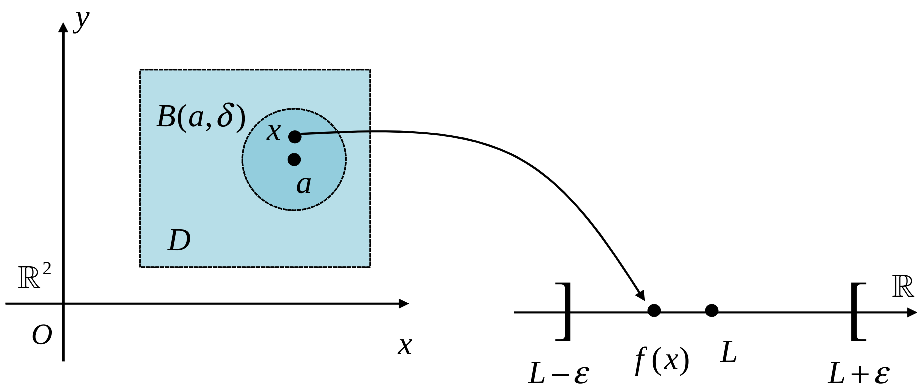
$$(\forall \varepsilon > 0)\,(\exists \delta > 0)\,(\|x - a\| < \delta \Longrightarrow |f(x) - L| < \varepsilon),$$

whatever the norm $\|\cdot\|$, and we write

$$\lim_{x \longrightarrow a} f(x) = L \qquad or \qquad f(x) \longrightarrow L \text{ when } x \longrightarrow a.$$

In other words

$$(\forall \varepsilon > 0)\,(\exists B(a, \delta))\,(\forall x \in B(a, \delta),\; f(x) \in]L - \varepsilon, L + \varepsilon[).$$



Note : The limits at infinity points and the notions of the attached limits are defined in the same manners to those of the functions of one variable.

Example : Let $f(x,y) = x^2y - y$ and $P(2,3)$. Prove that $\lim\limits_{(x,y)\longrightarrow(2,3)} f(x,y) = 9$.

Solution : Let $\varepsilon > 0$, find $\delta > 0$ / $\|(x,y)-(2,3)\| < \delta \Longrightarrow |f(x,y) - 9| < \varepsilon$,

$$\begin{aligned} |f(x,y)-9| &= |x^2y - 12 - y + 3| = |x^2y - 4y + 4y - 12 - y + 3| \\ &= |(x^2-4)y + 4(y-3) - (y-3)| = |(x-2)(x+2)y + 3(y-3)| \\ &\leq |x-2||x+2||y| + 3|y-3| \end{aligned}$$

Consider the norm $\|(x,y)-(2,3)\|_\infty = \max(|x-2|, |y-3|)$, then we have
$|x-2| < \delta$ and $|y-3| < \delta \Longrightarrow |f(x,y) - 9| < \delta(|x+2||y| + 3)$.
Let $\delta < 1$, i.e., we consider that $M(x,y) \in B_\infty(P,1)$

$$\Longrightarrow \begin{cases} |x-2| < 1 \\ |y-3| < 1 \end{cases} \Longrightarrow \begin{cases} -1 < x-2 < 1 \\ -1 < y-3 < 1 \end{cases} \Longrightarrow \begin{cases} 3 < x+2 < 5 \\ 2 < y < 4 \end{cases} \Longrightarrow \begin{cases} |x+2| < 5 \\ |y| < 4 \end{cases}$$

$\Longrightarrow |f(x,y) - 9| < 23\delta < \varepsilon$ if $\delta < \dfrac{\varepsilon}{23}$.

We take $\delta = \inf\left(1, \dfrac{\varepsilon}{23}\right)$, i.e., we consider the neighborhood $V_P = B_\infty(P,1) \cap B_\infty\left(P, \dfrac{\varepsilon}{23}\right)$.

***Theorem** 2.2 If the limit of $f(x)$ exists at a point a this limit is unique.*

Proof : Let L_1 and L_2 be two limits of f at the point a,
then $(\forall\varepsilon > 0)\,(\exists\delta_1 > 0)\,(\|x-a\| < \delta_1 \Longrightarrow |f(x) - L_1| < \varepsilon)$,
and $(\forall\varepsilon > 0)\,(\exists\delta_2 > 0)\,(\|x-a\| < \delta_2 \Longrightarrow |f(x) - L_2| < \varepsilon)$.
Let $\delta = \inf\{\delta_1, \delta_2\}$, then for $\|x-a\| < \delta$,
$|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| \leq |f(x) - L_1| + |f(x) - L_2| < 2\varepsilon, \forall\varepsilon > 0$
$\Longrightarrow |L_1 - L_2| = 0 \Longrightarrow L_1 = L_2$.

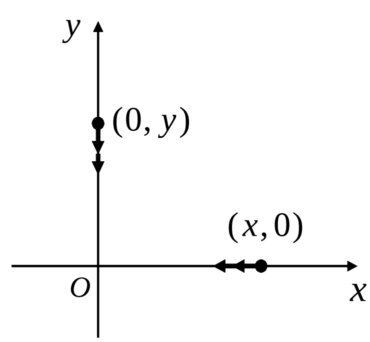
***Corollary** 2.1 If when x approaches a following two different paths (ways) $f(x)$ has two different limits or if when there is no finite limit for at least a path, then $f(x)$ doesn't admit a limit at the point a. This means that if the limit depends on the path followed then the limit does not exist.*

Example : Let $f(x,y) = \begin{cases} \dfrac{x+y}{x-y} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$. Does f have a limit at the origin ?

Solution : Following the path $y = 0$, $\lim\limits_{x\longrightarrow 0} f(x,0) = \lim\limits_{x\longrightarrow 0} \dfrac{x}{x} = 1$,

following the path $x = 0$, $\lim\limits_{y\longrightarrow 0} f(0,y) = \lim\limits_{y\longrightarrow 0} \dfrac{y}{-y} = -1$,

then f doesn't have a limit at the point $(0,0)$.



Note : If we obtain the same limit following at least two different paths, it doesn't mean that f has a limit.

Example : Let $f(x,y) = \begin{cases} \dfrac{y^2}{x} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$. Does f have a limit at the origin ?

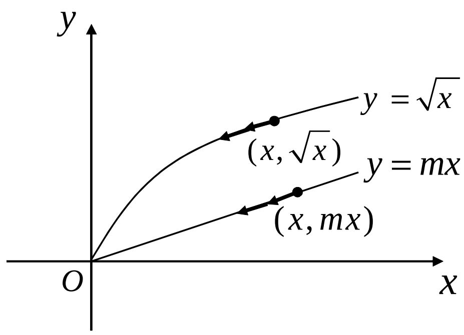
Solution : Following the rectilinear path $y = mx$ with $m \neq 0$, we have

$\lim\limits_{x \longrightarrow 0} f(x, mx) = \lim\limits_{x \longrightarrow 0} \dfrac{m^2x^2}{x} = \lim\limits_{x \longrightarrow 0} m^2 x = 0, \ \forall m \in \mathbb{R}^*,$

therefore, we have the same limit following an infinity of paths.

But following the path $y = \sqrt{x}$, with $x > 0$, $\lim\limits_{x \longrightarrow 0} f(x, \sqrt{x}) = \lim\limits_{x \longrightarrow 0} \dfrac{x}{x} = 1 \neq 0.$

Then f doesn't have a limit at the point $(0,0)$.



***Theorem** 2.3 If $f(x)$ has a limit at a point a, then f is bounded in a neighborhood of a.*

Proof : Let $\lim\limits_{x \longrightarrow a} f(x) = L$, then $(\forall \varepsilon > 0)(\exists \delta > 0)(\|x - a\| < \delta \Longrightarrow |f(x) - L| < \varepsilon)$,

for $\varepsilon = 1$, $|f(x)| - |L| < 1 \Longrightarrow \forall x \in B(a, \delta), \ |f(x)| < |L| + 1.$

Properties : If $\lim\limits_{x \longrightarrow a} f(x) = L$ and $\lim\limits_{x \longrightarrow a} g(x) = L'$, then

(1) $\lim\limits_{x \longrightarrow a} (f \pm g)(x) = L \pm L'$;

(2) $\lim\limits_{x \longrightarrow a} (fg)(x) = LL'$;

(3) $\lim\limits_{x \longrightarrow a} (\alpha f)(x) = \alpha L, \ \forall \alpha \in \mathbb{R}$;

(4) $\lim\limits_{x \longrightarrow a} \left(\dfrac{f}{g}\right)(x) = \dfrac{L}{L'}$ (with $L' \neq 0$).

Proof : Similar to the proof of the functions of one variable.

***Theorem** 2.4 Let $\lim\limits_{x \longrightarrow a} f(x) = L$ and $\lim\limits_{x \longrightarrow a} g(x) = L'$,*

(i) If $f(x) \geq 0, \ \forall x \in V_a$, then $L \geq 0$;
(ii) If $f(x) \leq g(x), \ \forall x \in V_a$, then $L \leq L'$;
(iii) If $\lim\limits_{x \longrightarrow a} |f(x)| = 0$, then $L = 0$;
(iv) If $L = L'$ and $f(x) \leq h(x) \leq g(x), \ \forall x \in V_a$, then $\lim\limits_{x \longrightarrow a} h(x) = L$;
(v) If $L = 0$ and g is bounded in a V_a, then $\lim\limits_{x \longrightarrow a} f(x)g(x) = 0$.

Proof : Similar to the proof of the functions of one variable.

• **Limits in polar coordinates :** Let $x = r\cos\theta$ and $y = r\sin\theta$, for $r > 0$ and $\theta \in [0, 2\pi[$.

As $r^2 = x^2 + y^2$, we observe that $(x, y) \longrightarrow (0, 0) \Longleftrightarrow r \longrightarrow 0$. Hence

$$\lim_{(x,y)\longrightarrow(0,0)} f(x, y) = \lim_{r\longrightarrow 0} f(r\cos\theta, r\sin\theta) = \lim_{r\longrightarrow 0} F(r, \theta).$$

Example : Let $f(x, y) = \dfrac{x^2y^2}{x^2 + y^2}$. Show that $\displaystyle\lim_{(x,y)\longrightarrow(0,0)} f(x, y) = 0$,

(i) by using polar coordinates;
(ii) by Sandwich theorem.

Solution : (i) We have $x^2 + y^2 = r^2$, with $r \longrightarrow 0$ when $(x, y) \longrightarrow (0, 0)$.

$$\lim_{(x,y)\longrightarrow(0,0)} f(x, y) = \lim_{(x,y)\longrightarrow(0,0)} \frac{x^2y^2}{x^2 + y^2} = \lim_{r\longrightarrow 0} \frac{r^4\cos^2\theta\sin^2\theta}{r^2} = \lim_{r\longrightarrow 0} r^2\cos^2\theta\sin^2\theta = 0$$

since $\displaystyle\lim_{r\longrightarrow 0} r^2 = 0$ and $|\cos^2\theta\sin^2\theta| = |\cos\theta|^2 |\sin\theta|^2 \leq 1$.

(ii) We have $0 \leq x^2 \leq x^2 + y^2 \Longrightarrow 0 \leq \dfrac{x^2y^2}{x^2 + y^2} \leq y^2$, $\forall (x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$

$$\Longrightarrow \lim_{(x,y)\longrightarrow(0,0)} |f(x, y)| \leq \lim_{(x,y)\longrightarrow(0,0)} y^2 = 0 \Longrightarrow \lim_{(x,y)\longrightarrow(0,0)} |f(x, y)| = 0$$

$$\Longrightarrow \lim_{(x,y)\longrightarrow(0,0)} f(x, y) = 0.$$

Remark : To find the limit of $f(x, y)$ when $(x, y) \longrightarrow (a, b)$ for $(a, b) \neq (0, 0)$, it is enough to return it to the neighborhood of $(0, 0)$ by using the change of variables $X = x - a$ and $Y = y - b$. Otherwise, we write

$$\lim_{(x,y)\longrightarrow(a,b)} f(x, y) = \lim_{(X,Y)\longrightarrow(0,0)} f(X + a, Y + b) = \lim_{(X,Y)\longrightarrow(0,0)} F(X, Y).$$

Example : Let $f(x, y) = \dfrac{xy + y - 2x - 2}{\sqrt{(x + 1)^2 + (y - 2)^2}}$. Find $\displaystyle\lim_{(x,y)\longrightarrow(-1,2)} f(x, y)$.

Solution : Let $X = x + 1$ and $Y = y - 2$,

$$\Longrightarrow \lim_{(x,y)\longrightarrow(-1,2)} f(x, y) = \lim_{(x,y)\longrightarrow(-1,2)} \frac{(x + 1)(y - 2)}{\sqrt{(x + 1)^2 + (y - 2)^2}} = \lim_{(X,Y)\longrightarrow(0,0)} \frac{XY}{\sqrt{X^2 + Y^2}}.$$

By setting $X = r\cos\theta$ and $Y = r\sin\theta$,

$$\lim_{(x,y)\longrightarrow(-1,2)} f(x, y) = \lim_{r\longrightarrow 0} \frac{r^2\cos\theta\sin\theta}{r} = \lim_{r\longrightarrow 0} r\cos\theta\sin\theta = 0.$$

Example : Find $\displaystyle\lim_{(x,y)\longrightarrow(+\infty,+\infty)} \frac{\ln(x + y)}{x + y}$. Does the limit $\displaystyle\lim_{(x,y)\longrightarrow(+\infty,+\infty)} \frac{\ln(x - y)}{x - y}$ exist ?

Solution : Let $u = x + y \Longrightarrow \displaystyle\lim_{(x,y)\longrightarrow(+\infty,+\infty)} \frac{\ln(x + y)}{x + y} = \lim_{u\longrightarrow+\infty} \frac{\ln u}{u} = 0$.

For the second limit we consider the path $x - y = m$ with $m > 0$

$$\Longrightarrow \lim_{(x,y)\longrightarrow(+\infty,+\infty)} \frac{\ln(x - y)}{x - y} = \frac{\ln m}{m}.$$ Then the limit does not exist.

Proposition *2.1 Let $u : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function, then*

$$\lim_{x \longrightarrow a} u(f(x)) = u\left(\lim_{x \longrightarrow a} f(x)\right).$$

Example : Let $f(x,y) = \dfrac{\arcsin(x^2+y^2)}{x^2+y^2}$ and $u(t) = t^2 - t + 2$. Calculate $\lim_{(x,y) \longrightarrow (0,0)} u(f(x,y))$.

Solution : The function u is continuous on $\mathbb{R}$ and by setting $z = x^2+y^2$, then

$$\lim_{(x,y) \longrightarrow (0,0)} u(f(x,y)) = u\left(\lim_{(x,y) \longrightarrow (0,0)} f(x,y)\right) = u\left(\lim_{z \longrightarrow 0} \frac{\arcsin z}{z}\right) = u(1) = 2.$$

2.3 Continuity of functions of several variables

Let D be an open of $\mathbb{R}^n$, $f : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}$, $x = (x_1, \cdots, x_n) \in D$ and $a = (a_1, \cdots, a_n) \in D$.

Definition *2.8 We say that f is continuous at the point a when $f(x)$ has a finite limit at a and that*

$$\lim_{x \longrightarrow a} f(x) = f(a)$$

i.e.,

$$(\forall \varepsilon > 0)\,(\exists \delta > 0)\,(\|x-a\| < \delta \Longrightarrow |f(x) - f(a)| < \varepsilon).$$

Note : An equivalence of the definition is given by

$$f(x) = f(a) + \varepsilon(x-a)$$

with $\varepsilon(x-a) \longrightarrow 0$ when $x \longrightarrow a$.

Example : Let $f(x,y) = \begin{cases} (x+y)\sin\dfrac{1}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$

Show that f is continuous at the origin.

Solution : Let $\varepsilon > 0$, find $\delta > 0$ / $\|(x,y) - (0,0)\| < \delta \Longrightarrow |f(x,y) - f(0,0)| < \varepsilon$.

$|f(x,y)| = \left|(x+y)\sin\dfrac{1}{x^2+y^2}\right| = |x+y|\left|\sin\dfrac{1}{x^2+y^2}\right| \leq |x+y| \leq |x| + |y|$.

If we consider the norm $\|(x,y)\|_1 = |x| + |y| \Longrightarrow |f(x,y)| < \delta < \varepsilon$, then we take $\delta < \varepsilon$.

If we consider the norm $\|(x,y)\|_2 = \sqrt{x^2+y^2} \Longrightarrow |f(x,y)| < \sqrt{2}\delta < \varepsilon$, then we take $\delta < \varepsilon/\sqrt{2}$.

If we consider the norm $\|(x,y)\|_\infty = \max(|x|,|y|) \Longrightarrow |f(x,y)| < 2\delta < \varepsilon$, then we take $\delta < \varepsilon/2$.

Theorem *2.5 All continuous function at a point a is bounded on a neighborhood of a.*

Proof : If f is continue at the point a, then
for $\varepsilon = \varepsilon_0$ given, $\exists \delta > 0$ / $\|x-a\| < \delta \Longrightarrow |f(x) - f(a)| < \varepsilon_0 \Longrightarrow |f(x)| < |f(a)| + \varepsilon_0$,
therefore f is bounded on $B(a,\delta)$.

Theorem *2.6 If f and g are two continuous functions at the point a, then $f \pm g$, αf, fg and $\dfrac{f}{g}$ ($g(x) \neq 0$ in a neighborhood of a) are continuous at the point a.*

Proof : If f and g are continuous at the point a then
$(\forall \varepsilon > 0)\,(\exists \delta_1 > 0)\,(\|x-a\| < \delta_1 \Longrightarrow |f(x) - f(a)| < \varepsilon)$
and $(\forall \varepsilon > 0)\,(\exists \delta_2 > 0)\,(\|x-a\| < \delta_2 \Longrightarrow |g(x) - g(a)| < \varepsilon)$.

▶ Continuity of $f+g$: for $\delta = \inf(\delta_1, \delta_2)$ we have for $\|x-a\| < \delta$,

$$\begin{aligned}
|(f+g)(x) - (f+g)(a)| &= |[f(x) - f(a)] + [g(x) - g(a)]| \\
&\leq |f(x) - f(a)| + |g(x) - g(a)| \\
&< \varepsilon + \varepsilon \leq 2\varepsilon.
\end{aligned}$$

▶ Continuity of αf : for $\alpha = 0$, nothing to prove. For $\alpha \neq 0$ and $\delta = \delta_1$ we have
$\|x-a\| < \delta \Longrightarrow |(\alpha f)(x) - (\alpha f)(a)| = |\alpha|\, |f(x) - f(a)| < |\alpha| \varepsilon$.

▶ Continuity of fg :

$$\begin{aligned}
|(fg)(x) - (fg)(a)| &= |f(x)g(x) - f(x)g(a) + f(x)g(a) - f(a)g(a)| \\
&\leq |f(x)g(x) - f(x)g(a)| + |f(x)g(a) - f(a)g(a)| \\
&\leq |f(x)|\,|g(x) - g(a)| + |f(x) - f(a)|\,|g(a)|
\end{aligned}$$

f and g being continuous at a, then f and g are bounded in a neighborhood of a, then
$\forall x \in B(a, \delta')$, $|f(x)| \leq K$ and $|g(x)| \leq L$,
then for $\delta = \inf(\delta_1, \delta_2, \delta')$, we have
$\|x-a\| < \delta \Longrightarrow |(fg)(x) - (fg)(a)| < (K+L)\,\varepsilon$.

▶ Continuity of $\dfrac{f}{g}$:

$$\begin{aligned}
\left|\left(\frac{f}{g}\right)(x) - \left(\frac{f}{g}\right)(a)\right| &= \left|\frac{f(x)}{g(x)} - \frac{f(a)}{g(x)} + \frac{f(a)}{g(x)} - \frac{f(a)}{g(a)}\right| \\
&\leq \left|\frac{f(x)}{g(x)} - \frac{f(a)}{g(x)}\right| + \left|\frac{f(a)}{g(x)} - \frac{f(a)}{g(a)}\right| \\
&\leq \frac{1}{|g(x)|}\,|f(x) - f(a)| + \left|\frac{f(a)}{g(a)\,g(x)}\right|\,|g(x) - g(a)|
\end{aligned}$$

f and g being continuous at a, then f and g are bounded in a neighborhood of a (with $g(x) \neq 0$), then $\forall x \in B(a, \delta')$, $\dfrac{1}{|g(x)|} \leq K$ and $\left|\dfrac{f(a)}{g(a)\,g(x)}\right| \leq L$,
then for $\delta = \inf(\delta_1, \delta_2, \delta')$, we have
$\|x-a\| < \delta \Longrightarrow \left|\left(\dfrac{f}{g}\right)(x) - \left(\dfrac{f}{g}\right)(a)\right| < (K+L)\,\varepsilon$.

Example : Given the following polynomial of $\mathbb{R}^2$: $f(x,y) = x^2y + xy^3 - 2x$
Let $(x,y) \in \mathbb{R}^2$. The first and second projection functions

$$\mathrm{Pr}_1(x,y) = x \text{ and } \mathrm{Pr}_2(x,y) = y$$

are continuous on $\mathbb{R}^2$. We write

$$f(x,y) = \mathrm{Pr}_1^2(x,y)\,\mathrm{Pr}_2(x,y) + \mathrm{Pr}_1(x,y)\,\mathrm{Pr}_2^3(x,y) - 2\,\mathrm{Pr}_1(x,y),$$

then using the previous theorem f is continuous on $\mathbb{R}^2$.

Theorem *2.7 Given the composition $D \subseteq \mathbb{R}^n \xrightarrow{f} I \subseteq \mathbb{R} \xrightarrow{g} \mathbb{R}$. If f is continuous at the point $a \in D$ and g is continuous at the point $f(a) \in I$, then $g \circ f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuous at a.*

Proof : f is continuous at the point a then
$(\forall\varepsilon > 0)\,(\exists\delta > 0)\,(\|x - a\| < \delta \Longrightarrow |f(x) - f(a)| < \varepsilon)$.
g is continuous at the point $f(a)$ then
$(\forall\varepsilon' > 0)\,(\exists\delta' > 0)\,(|f(x) - f(a)| < \delta' \Longrightarrow |g(f(x)) - g(f(a))| < \varepsilon')$.
Let $\varepsilon = \varepsilon' = \delta'$, then $\|x - a\| < \delta \Longrightarrow |(g \circ f)(x) - (g \circ f)(a)| < \varepsilon$.

Definition *2.9 We say that $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuous on D if f is continuous at each point of D.*

Example : Prove that the function $f(x, y) = \sin \dfrac{xy}{x^2 + y^2}$ is continuous on $D = \mathbb{R}^2 \setminus \{(0, 0)\}$.

Solution : Let $u(x, y) = \dfrac{xy}{x^2 + y^2}$ and $g(u) = \sin u$.

The functions v / $v(x, y) = xy = \operatorname{Pr}_1(x, y)\operatorname{Pr}_2(x, y)$ and w / $w(x, y) = x^2 + y^2 = \operatorname{Pr}_1^2(x, y) + \operatorname{Pr}_2^2(x, y)$ are continuous on $\mathbb{R}^2$, with $x^2 + y^2 \neq 0$, $\forall(x, y) \in D$, then $u = \dfrac{v}{w}$ is continuous on D, and since g is continuous on $\mathbb{R}$, then $f = g \circ u$ is continuous on D.

- **Extension by continuity :** Let $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be defined and continuous on a domain $D \subset \mathbb{R}^n$ except at a point $a \in D$. If $\lim\limits_{x \longrightarrow a} f(x) = L$ exists and is finite then we can extend f by continuity on D. Its extension g is defined on D by

$$g(x) = \begin{cases} f(x) & \text{if} \quad x \in D \setminus \{a\} \\ L & \text{if} \quad x = a \end{cases}$$

Example : Let $f(x, y) = \dfrac{xy^2}{x^2 + y^2}$, for $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$.
is f extendable by continuity at $(0, 0)$?

Solution : $\lim\limits_{(x,y) \longrightarrow (0,0)} f(x, y) = \lim\limits_{(x,y) \longrightarrow (0,0)} \dfrac{xy^2}{x^2 + y^2} = \lim\limits_{r \longrightarrow 0} \dfrac{r^3 \cos\theta \sin^2\theta}{r^2} = \lim\limits_{r \longrightarrow 0} r \cos\theta \sin^2\theta = 0$
then f is extendable by continuity on $\mathbb{R}^2$ and its extension is given by

$$g(x, y) = \begin{cases} \dfrac{xy^2}{x^2 + y^2} & \text{if} \quad (x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\} \\ 0 & \text{if} \quad (x, y) = (0, 0) \end{cases}$$

Definition *2.10 Let $f : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ and $S \subset \mathbb{R}$. The set*

$$f^{-1}(S) = \{x \in D : f(x) \in S\}$$

is called reciprocal image of S by f.

Theorem *2.8 Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuous function.*
(i) For all open U of $\mathbb{R}$, $f^{-1}(U)$ is an open of $\mathbb{R}^n$.
(ii) For all closed F of $\mathbb{R}$, $f^{-1}(F)$ is a closed of $\mathbb{R}^n$.

Example : Let $f(x, y) = x^2 + y^2$ that is continuous on $\mathbb{R}^n$, then
$f^{-1}([1, 4]) = \{(x, y) \in \mathbb{R}^2 : f(x, y) \in [1, 4]\} = \{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4\}$ is a closed and
$f^{-1}(]1, 4[) = \{(x, y) \in \mathbb{R}^2 : f(x, y) \in]1, 4[\} = \{(x, y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 4\}$ is an pen.

2.4 Partial continuity of functions of several variables

Let D be an open of $\mathbb{R}^n$, $f : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}$, $x = (x_1, \cdots, x_n) \in D$ and $a = (a_1, \cdots, a_n) \in D$.

***Definition** 2.11 For $i = 1, \cdots, n$, the mapping from $\mathbb{R}$ to $\mathbb{R}$ defined by :*

$$f_i : x_i \longmapsto f_i(x_i) = f(a_1, \cdots, a_{i-1}, x_i, a_{i+1}, \cdots, a_n)$$

is called i^{th} partial mapping of f at the point a.

***Definition** 2.12 If the mapping f_i is continuous at a_i, we say that f is continuous with respect to x_i at the point a.*

***Definition** 2.13 If $f_1, \cdots, f_n$ are continuous at $a_1, \cdots, a_n$, respectively, we say that f is partially continuous at a.*

***Proposition** 2.2 If f is continuous at the point a, then $f_1, \cdots, f_n$ are also continuous at $a_1, \cdots, a_n$, respectively.*

Proof : f is continuous at a, then $(\forall \varepsilon > 0)\,(\exists \delta > 0)\,(\|x - a\| < \delta \Longrightarrow |f(x) - f(a)| < \varepsilon)$.
We know that for $i = 1, \cdots, n$, $|x_i - a_i| \leq \|x - a\| < \delta$, whatever the norm,
then $\forall \varepsilon > 0,\ \exists \delta > 0 \ /\ \ |x_i - a_i| < \delta$
$\Longrightarrow |f(a_1, \cdots, a_{i-1}, x_i, a_{i+1}, \cdots, a_n) - f(a_1, \cdots, a_{i-1}, a_i, a_{i+1}, \cdots, a_n)| < \varepsilon$
therefore $(\forall \varepsilon > 0)\,(\exists \delta > 0)\,(|x_i - a_i| < \delta \Longrightarrow |f_i(x_i) - f_i(a_i)| < \varepsilon)$.

Remark : If f is partially continuous at the point a, it is not necessarily continuous at a.

***Definition** 2.14 Let $f : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$. We call the restriction of f on the curve of equation $y = g(x)$ a function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\varphi(x) = f(x, g(x))$. We say that this restriction of f is continuous at a point $P(a, b)$ if*

$$\lim_{x \longrightarrow a} \varphi(x) = \lim_{x \longrightarrow a} f(x, g(x)) = f(a, b).$$

Example : Let $f(x, y) = \begin{cases} \dfrac{xy^2}{x^2 + y^4} & \text{if} \quad (x, y) \neq (0, 0) \\ 0 & \text{if} \quad (x, y) = (0, 0) \end{cases}$

1. Show that f is partially continuous at $(0, 0)$.
2. Show that the restriction of f on the straight line $y = mx$, $\forall m \in \mathbb{R}$, is continuous at $(0, 0)$.
3. Find the limit when $(x, y) \longrightarrow (0, 0)$ of the restriction of f on the parabola $x = y^2$.
4. Is there an equivalence between continuity and partial continuity ?

Solution :

1. $\lim\limits_{x \longrightarrow 0} f_1(x) = \lim\limits_{x \longrightarrow 0} f(x, 0) = 0 = f(0, 0)$ and $\lim\limits_{x \longrightarrow 0} f_2(y) = \lim\limits_{x \longrightarrow 0} f(0, y) = 0 = f(0, 0)$,
then f is partially continuous at $(0, 0)$.
2. For $y = mx$ and $x \neq 0$, $\varphi(x) = f(x, mx) = \dfrac{m^2x^3}{x^2 + m^4x^4} = \dfrac{m^2x}{1 + m^4x^2}$.
$\lim\limits_{x \longrightarrow 0} \varphi(x) = \lim\limits_{x \longrightarrow 0} f(x, mx) = 0 = f(0, 0)$, then f is continuous in any rectilinear direction passing through the origin.
3. For $x = y^2$ and $y \neq 0$, $\lim\limits_{y \longrightarrow 0} \varphi(y) = \lim\limits_{y \longrightarrow 0} f(y^2, y) = \lim\limits_{y \longrightarrow 0} \dfrac{y^4}{y^4 + y^4} = \dfrac{1}{2} \neq 0 = f(0, 0)$.
4. No, f is partially continuous but it is not continuous at $(0, 0)$.

2.5 Exercises

***Exercise** 2.1 Determine the domain of the following functions :*

1. $f(x,y) = \arcsin \dfrac{x}{y}$
2. $f(x,y) = \dfrac{\arcsin x}{\arcsin y}$
3. $f(x,y) = \sqrt{y^2 - 4x^2 - 16}$
4. $f(x,y) = \ln \dfrac{xy}{1 - xy}$
5. $f(x,y) = \sqrt{\dfrac{y^2 - 1}{1 - x^2}}$
6. $f(x,y) = \ln \dfrac{x}{1 - x^2 - y^2}$
7. $f(x,y) = \sqrt{x \cos y}$
8. $f(x,y) = \dfrac{1}{\sqrt{y - \sqrt{x}}}$
9. $f(x,y) = \exp\left(\dfrac{y}{x^2 + y^2 - 1}\right)$

***Exercise** 2.2 Determine the level curves of the following functions :*

1. $f(x,y) = x^2 + y^2 - 4x + 6y + 13,$ *for* $k \in \mathbb{R}$
2. $f(x,y) = \dfrac{x^2 + y}{x + y^2},$ *for* $k \in \{0, -1\}$
3. $f(x,y) = \dfrac{xy - x + y}{xy},$ *for* $k \in \{1, 2\}$
4. $f(x,y) = \dfrac{x^4 + y^4}{8 - x^2 y^2},$ *for* $k = 2$

***Exercise** 2.3 Study the limit when $(x,y) \longrightarrow (0,0)$ of the following functions :*

1. $f(x,y) = \dfrac{\sqrt{1 + x^2} - 1}{\sqrt{1 + y^2} - 1}$
2. $f(x,y) = \dfrac{x^2 - 2xy + 5y^2}{3x^2 + 4y^2}$
3. $f(x,y) = \dfrac{y^2 \ln(x^2 + y^2)}{\sqrt{x^2 + y^2}}$
4. $f(x,y) = \dfrac{\ln(1 + xy^2)}{y^2}$
5. $f(x,y) = \dfrac{y^2 \sin x}{x^2 + y^2 + |x + y|}$
6. $f(x,y) = \dfrac{e^{x^2 y^2} - \cos xy}{\ln(1 + x^2 + y^2)}$
7. $f(x,y) = \dfrac{x^2}{y \ln(y - x^2)}$
8. $f(x,y) = x^y$

***Exercise** 2.4 Let $\alpha > 0$ and f be the function of two variables defined by*

$$f(x,y) = \frac{x^2 \ln(1 + y^2) - y^2 \ln(1 + x^2)}{\sqrt{1 + (x^2 + y^2)^\alpha} - 1}.$$

Discuss according to the parameter α the existence of $\lim\limits_{(x,y) \to (0,0)} f(x,y)$.

***Exercise** 2.5 Find the limits as $\|(x,y)\| \longrightarrow \infty$ of the following functions :*

1. $f(x,y) = \dfrac{x^2 + y^4}{x^4 + y^2}$
2. $f(x,y) = \dfrac{x \arctan y}{1 + x^2 + y^2}$

***Exercise** 2.6 Show, using the definition of the limit at a point, that*

1. $\lim\limits_{(x,y) \to (0,0)} \dfrac{x + y}{1 + x^2 + y^2} = 0$
2. $\lim\limits_{(x,y) \to (0,0)} \dfrac{xy - \sin y}{2 + \cos x} = 0$

***Exercise** 2.7 Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by*

$$f(x,y) = \begin{cases} \dfrac{\sin(xy - y^2)}{\sqrt{x^2 + y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

1. *Verify that* $|f(x,y)| \leq |x - y|$, *for all* $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$.
2. *Deduce that f is continuous at the point $(0,0)$.*

Exercise *2.8 Consider the function* $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *defined by*

$$f(x,y) = \begin{cases} \dfrac{(x+y)\ln(1+|xy|)}{\sin(x^2+y^2)} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

1. *Show that in the neighborhood of the point* $(0,0)$, f *is equivalent to a function* g.
2. *Using polar coordinates, show that* f *is continuous at the point* $(0,0)$.

Exercise *2.9 Study the continuity at the origin* $O(0,0)$ *of the following functions :*

1. $f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$
2. $f(x,y) = \begin{cases} \dfrac{x^3y^3}{x^{12}+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$
3. $f(x,y) = \begin{cases} \dfrac{x}{x+y} & \text{if } x+y \neq 0 \\ 0 & \text{if } x+y = 0 \end{cases}$
4. $f(x,y) = \begin{cases} \dfrac{\sin(x^2y^2)}{x^2y^2+|x-y|} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$
5. $f(x,y) = \begin{cases} \dfrac{y^2}{x-y} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$
6. $f(x,y) = \begin{cases} \dfrac{x^2y^3\arctan\dfrac{y}{x}}{\ln(1+x^4+y^4+2x^2y^2)} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Exercise *2.10 Let* $\alpha, \beta \in \mathbb{R}^+$ *and* $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *be the function defined by*

$$f(x,y) = \begin{cases} \dfrac{|x|^\alpha\,|y|^\beta}{x^2+y^2-xy} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

1. *Verify that* f *is continuous on* $\mathbb{R}^2 \setminus \{(0,0)\}$.
2. *Give a necessary and sufficient condition on* α *and* β *so that* f *is continuous on* $\mathbb{R}^2$.

(Hint : We can verify that $\dfrac{1}{2} \leq 1-\sin\theta\cos\theta \leq \dfrac{3}{2}$*)*

Exercise *2.11 Let the function* $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *defined by*

$$f(x,y) = \begin{cases} \dfrac{y^2-x^2}{|y-x|} & \text{if } y \neq x \\ 0 & \text{if } y = x \end{cases}$$

Study the continuity of f *on the straight line* $y = x$.

Exercise *2.12 Let the function* $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *defined by*

$$f(x,y) = \begin{cases} xy\sin\dfrac{1}{xy} & \text{if } xy \neq 0 \\ 0 & \text{if } xy = 0 \end{cases}$$

Study the continuity of f *at each point of* $\mathbb{R}^2$.

Exercise *2.13 Let the function* $f:\mathbb{R}^2 \longrightarrow \mathbb{R}$ *defined by*

$$f(x,y)=\begin{cases} \dfrac{\sin xy - xy}{\ln\left(1+x^2y^2\right)} & \text{if } \ xy\neq 0 \\ 0 & \text{if } \ xy=0 \end{cases}$$

1. *Show that* f *is continuous on the set* $A=\left\{(x,y)\in\mathbb{R}^2 : xy=0\right\}$.
2. *Deduce that it is continuous on* $\mathbb{R}^2$.

Exercise *2.14 Let the function* $f:\mathbb{R}^2 \longrightarrow \mathbb{R}$ *defined by*

$$f(x,y)=\begin{cases} 10-x^2-y^2 & \text{if } \ x^2+y^2\leq 9 \\ \sqrt{x^2+y^2-9} & \text{if } \ x^2+y^2>9 \end{cases}$$

1. *Study the continuity of* f *on the set* $A=\left\{(x,y)\in\mathbb{R}^2 : x^2+y^2=9\right\}$.
2. *Deduce the domain of continuity of* f.

Exercise *2.15 Study if each of the following functions can be extended by continuity at* $(0,0)$ *and give its extension* g :

1. $f(x,y)=\dfrac{xy}{x^3+3y^2}$
2. $f(x,y)=\dfrac{x^2y}{2x^2+3y^2}$
3. $f(x,y)=\dfrac{x\sin(xy^2)}{\left(x^2+y^2\right)^2}$

Exercise *2.16 Let the function* f *given by*

$$f(x,y)=\frac{x^2+y^2}{|x|+|y|}.$$

1. *Show that* $x^2+y^2\leq\left(|x|+|y|\right)^2$, $\forall\,(x,y)\in\mathbb{R}^2$ *then calculate* $\lim\limits_{(x,y)\to(0,0)} f(x,y)$.
2. *Show that* $|\cos\theta|+|\sin\theta|\geq 1$, $\forall\theta\in\mathbb{R}$ *then find again* $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ *using polar coordinates.*
3. *Deduce that the function* g *given by*

$$g(x,y)=\frac{\sin xy}{|x|+|y|}$$

is extendable by continuity at $(0,0)$.

Exercise *2.17 Extend by continuity the function* f *and give its extension* g *:*

1. $f(x,y)=\dfrac{\cos x-\cos y}{x-y}$ *on the line* $y=x$
2. $f(x,y)=\dfrac{\sin(y^2-x)}{y^2-x}e^{y^2+x}$ *on the parabola* $y^2=x$
3. $f(x,y)=\dfrac{e^{x^2y^2}-\cos(xy)}{y^2}$ *on the line* $y=0$

Chapter 3

Differentiability for real-valued functions of several variables

3.1 Partial derivatives of a function of several variables

***Definition** 3.1 Let* $\begin{array}{rccc} f: & D\subset \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & f(x,y) \end{array}$ *be a function defined in an open D of $\mathbb{R}^n$.*

The first order partial derivative of f with respect to x at a point $P(a,b)\in D$ is defined by

$$\frac{\partial f}{\partial x}(a,b)=\lim_{x\longrightarrow a}\frac{f(x,b)-f(a,b)}{x-a}=\lim_{h\longrightarrow 0}\frac{f(a+h,b)-f(a,b)}{h}.$$

Similarly, the first order partial derivative of f with respect to y at the point $P(a,b)$ is defined by

$$\frac{\partial f}{\partial y}(a,b)=\lim_{y\longrightarrow b}\frac{f(a,y)-f(a,b)}{y-b}=\lim_{k\longrightarrow 0}\frac{f(a,b+k)-f(a,b)}{k}.$$

Note : By fixing $y=b$, the partial derivative of f with respect to x at the point $P(a,b)$ is therefore the derivative at the point $x=a$ of the first partial function $f_1 : x \longrightarrow f(x,b)$ of f with

$$\frac{\partial f}{\partial x}(a,b)=\lim_{x\longrightarrow a}\frac{f_1(x)-f_1(a)}{x-a}=f_1'(a).$$

Similarly

$$\frac{\partial f}{\partial y}(a,b)=\lim_{y\longrightarrow b}\frac{f_2(y)-f_2(b)}{y-b}=f_2'(b).$$

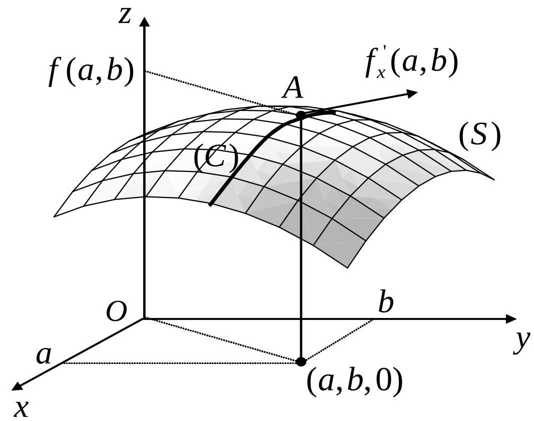
Example : Let $f(x,y)=x^2+x\sqrt{y}$ and $P(1,4)$. Calculate $\frac{\partial f}{\partial x}(1,4)$ and $\frac{\partial f}{\partial y}(1,4)$.

Solution :

$$\frac{\partial f}{\partial x}(1,4)=\lim_{x\longrightarrow 1}\frac{f(x,4)-f(1,4)}{x-1}=\lim_{x\longrightarrow 1}\frac{x^2+2x-3}{x-1}=\lim_{x\longrightarrow 1}\frac{(x-1)(x+3)}{x-1}=\lim_{x\longrightarrow 1}(x+3)=4,$$
$$\frac{\partial f}{\partial y}(1,4)=\lim_{y\longrightarrow 4}\frac{f(1,y)-f(1,4)}{y-4}=\lim_{y\longrightarrow 4}\frac{\sqrt{y}-2}{y-4}=\lim_{y\longrightarrow 4}\frac{y-4}{(y-4)(\sqrt{y}+2)}=\lim_{y\longrightarrow 4}\frac{1}{\sqrt{y}+2}=\frac{1}{4}.$$

- **Geometric interpretation** : Let $f : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ and $P(a,b) \in D$.

Recall that the set $S = \{M(x,y,z) \in \mathbb{R}^3 : z = f(x,y)\}$ is the representative surface of the function f. The first partial function $f_1(x) = f(x,b)$ of f represents a curve (C) on (S) (called line of coordinates) of equation $z = f_1(x)$ and located in the plane $y = b$. In this plane the partial derivative $\dfrac{\partial f}{\partial x}(a,b)$ is the slope of the tangent at the point $A(a,b,f(a,b))$ to the curve (C). Similarly for $\dfrac{\partial f}{\partial y}(a,b)$.



General case : More generally, the partial derivative with respect to the variable x_i for a function of n variables $f : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ at a point $a = (a_1, \cdots, a_n) \in D$, is defined by

$$\frac{\partial f}{\partial x_i}(a) = \lim_{h \longrightarrow 0} \frac{f(a_1, \cdots a_{i-1}, a_i + h, a_{i+1}, \cdots, a_n) - f(a_1, \cdots, a_i, \cdots, a_n)}{h}.$$

Remarks : (1) The partial derivative $\dfrac{\partial f}{\partial x_i}$ may be denoted by f'_{x_i} or $D_i f$.

(2) In practice, to calculate $\dfrac{\partial f}{\partial x_i}(x_1, \cdots, x_n)$ at each point $(x_1, \cdots, x_n)$ of D, it is sufficient to derive f as a function of the single variable x_i, the other variables are considered as constant.

Example : Let $f(x,y,z) = xe^{-y}\cos z$. Calculate by two methods the partial derivatives of f at the point $P(2,0,\pi)$.
Solution : First method (by definition) :

$$\frac{\partial f}{\partial x}(2,0,\pi) = \lim_{x \longrightarrow 2} \frac{f(x,0,\pi) - f(2,0,\pi)}{x-2} = \lim_{x \longrightarrow 2} \frac{-x+2}{x-2} = -1,$$

$$\frac{\partial f}{\partial y}(2,0,\pi) = \lim_{y \longrightarrow 0} \frac{f(2,y,\pi) - f(2,0,\pi)}{y-0} = \lim_{y \longrightarrow 0} \frac{-2e^{-y}+2}{y} \overset{HR}{=} \lim_{y \longrightarrow 0} \frac{2e^{-y}}{1} = 2,$$

$$\frac{\partial f}{\partial z}(2,0,\pi) = \lim_{z \longrightarrow \pi} \frac{f(2,0,z) - f(2,0,\pi)}{z-\pi} = \lim_{z \longrightarrow \pi} \frac{2\cos z + 2}{z-\pi} \overset{HR}{=} \lim_{z \longrightarrow \pi} \frac{-2\sin z}{1} = 0.$$

Second method (by calculation) :

$$\frac{\partial f}{\partial x}(x,y,z) = e^{-y}\cos z \Longrightarrow \frac{\partial f}{\partial x}(2,0,\pi) = -1,$$

$$\frac{\partial f}{\partial y}(x,y,z) = -xe^{-y}\cos z \Longrightarrow \frac{\partial f}{\partial y}(2,0,\pi) = 2,$$

$$\frac{\partial f}{\partial z}(x,y,z) = xe^{-y}\sin z \Longrightarrow \frac{\partial f}{\partial z}(2,0,\pi) = 0.$$

Note : The existence of the partial derivatives of a function of several variables at a given point does not guarantee the continuity at this point.

Example : Given the function $f(x,y) = \begin{cases} \dfrac{x-1}{y-1} & \text{if } \ y \neq 1 \\ 0 & \text{if } \ y = 1 \end{cases}$

1. Find $\dfrac{\partial f}{\partial x}(1,1)$ and $\dfrac{\partial f}{\partial y}(1,1)$.
2. Is f continuous at the point $(1,1)$?

Solution : 1. $\dfrac{\partial f}{\partial x}(1,1) = \lim\limits_{x \longrightarrow 1} \dfrac{f(x,1) - f(1,1)}{x-1} = \lim\limits_{x \longrightarrow 1} \dfrac{0-0}{x-1} = 0$

$\dfrac{\partial f}{\partial y}(1,1) = \lim\limits_{y \longrightarrow 1} \dfrac{f(1,y) - f(1,1)}{y-1} = \lim\limits_{y \longrightarrow 1} \dfrac{0-0}{y-1} = 0$

2. Let the path $y = x$, then $\lim\limits_{x \longrightarrow 1} f(x,x) = \lim\limits_{x \longrightarrow 1} \dfrac{x-1}{x-1} = 1 \neq 0 = f(1,1)$.

Therefore f is not continuous at $(1,1)$, however $\dfrac{\partial f}{\partial x}(1,1)$ and $\dfrac{\partial f}{\partial x}(1,1)$ exist at $(1,1)$.

- **Differentiation rules :** The rules of partial differentiation are the same of the functions of one variable. Consider two functions of n variables $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ and $v : \mathbb{R}^n \longrightarrow \mathbb{R}$, we have for $i = 1, \cdots, n$

$$\frac{\partial}{\partial x_i}(\alpha u) = \alpha \frac{\partial u}{\partial x_i} \qquad \frac{\partial}{\partial x_i}(u+v) = \frac{\partial u}{\partial x_i} + \frac{\partial v}{\partial x_i} \qquad \frac{\partial}{\partial x_i}(u^n) = n \frac{\partial u}{\partial x_i} u^{n-1}$$

$$\frac{\partial}{\partial x_i}(uv) = \frac{\partial u}{\partial x_i} v + u \frac{\partial v}{\partial x_i} \qquad \frac{\partial}{\partial x_i}\left(\frac{u}{v}\right) = \frac{\dfrac{\partial u}{\partial x_i} v - u \dfrac{\partial v}{\partial x_i}}{v^2}$$

If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a differentiable function of one real variable, then for $i = 1, \cdots, n$

$$\frac{\partial}{\partial x_i}(f \circ u) = \frac{df}{du} \frac{\partial u}{\partial x_i}.$$

Example : Let $f(x,y) = \ln(xy + \tan y) = \ln u(x,y)$.

Then $f'_x(x,y) = \dfrac{u'_x(x,y)}{u(x,y)} = \dfrac{y}{xy + \tan y}$ and $f'_y(x,y) = \dfrac{u'_y(x,y)}{u(x,y)} = \dfrac{x + 1 + \tan^2 y}{xy + \tan y}$.

3.2 Higher order partial derivatives

As the functions of one variable, a function of several variables may have second, third, and higher partial derivatives.

- **Second order partial derivatives :** Let a function $\begin{array}{cccc} f: & D \subset \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & f(x,y) \end{array}$ having first order partial derivatives in a certain domain $D \subseteq \mathbb{R}^2$. The functions $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ may each have two partial derivatives. We can therefore define the following second order partial derivatives :

$$
\frac{\partial f}{\partial x}
\begin{array}{l}
\nearrow \quad \dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) = (f'_x)'_x = f''_{xx} \\[2ex]
\searrow \quad \dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = (f'_x)'_y = f''_{xy}
\end{array}
\qquad
\frac{\partial f}{\partial y}
\begin{array}{l}
\nearrow \quad \dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \left(f'_y\right)'_x = f''_{yx} \\[2ex]
\searrow \quad \dfrac{\partial^2 f}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right) = \left(f'_y\right)'_y = f''_{yy}
\end{array}
$$

- ► $\dfrac{\partial^2 f}{\partial x^2}$ and $\dfrac{\partial^2 f}{\partial y^2}$ are the pure second derivatives;
- ► $\dfrac{\partial^2 f}{\partial y \partial x}$ and $\dfrac{\partial^2 f}{\partial x \partial y}$ are the mixed second derivatives.

In general the second order partial derivatives of a function of n variables

$$
\begin{array}{rccc}
f: & D \subset \mathbb{R}^n & \longrightarrow & \mathbb{R} \\
 & (x_1, \cdots, x_n) & \longmapsto & f(x_1, \cdots, x_n)
\end{array}
$$

are given by

$$
\frac{\partial^2 f}{\partial x_i \, \partial x_j}, \forall i, j = 1, \cdots, n.
$$

Remark : Third and higher partial derivatives are defined in like manner.

Example : Let $f(x, y) = x^2 \sin(xy)$. Find all second partial derivatives of f.

Solution : $\dfrac{\partial f}{\partial x}(x, y) = 2x \sin(xy) + x^2 y \cos(xy)$, $\dfrac{\partial f}{\partial y}(x, y) = x^3 \cos(xy)$,

$\dfrac{\partial^2 f}{\partial x^2}(x, y) = 2 \sin(xy) + 4xy \cos(xy) - x^2 y^2 \sin(xy)$, $\dfrac{\partial^2 f}{\partial y^2}(x, y) = -x^4 \sin(xy)$,

$\dfrac{\partial^2 f}{\partial y \partial x}(x, y) = 3x^2 \cos(xy) - x^3 y \sin(xy)$, $\dfrac{\partial^2 f}{\partial x \partial y}(x, y) = 3x^2 \cos(xy) - x^3 y \sin(xy)$.

In this example we note that $\dfrac{\partial^2 f}{\partial y \partial x}(x, y) = \dfrac{\partial^2 f}{\partial x \partial y}(x, y)$, $\forall (x, y) \in \mathbb{R}^2$.

Theorem** 3.1 **(Schwarz theorem) : *Let D be an open of $\mathbb{R}^2$. If $f : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ has continuous partial derivatives $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$, $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$ in a neighborhood of a point $P(a, b) \in D$, then*

$$
\frac{\partial^2 f}{\partial x \partial y}(a, b) = \frac{\partial^2 f}{\partial y \partial x}(a, b).
$$

More generally, if $f : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ is a function having continuous first and second order partial derivatives at a point $x = (x_1, \cdots, x_n) \in D$ then

$$
\frac{\partial^2 f}{\partial x_i \partial x_j}(x) = \frac{\partial^2 f}{\partial x_j \partial x_i}(x), \quad \forall i, j = 1, \cdots, n.
$$

Proof : Set, for $(a+h,b+k)\in D$,

$$E(h,k)=f(a+h,b+k)-f(a+h,b)-f(a,b+k)+f(a,b).$$

Let $\varphi(x)=f(x,b+k)-f(x,b)$, we have $E(h,k)=\varphi(a+h)-\varphi(a)$.
Using the M.V.T. applied to φ, there exists $\alpha\in]0,1[$ such that $\varphi(a+h)-\varphi(a)=h\varphi'(a+\alpha h)$, we therefore obtain

$$E(h,k)=h\left[\frac{\partial f}{\partial x}(a+\alpha h,b+k)-\frac{\partial f}{\partial x}(a+\alpha h,b)\right].$$

Let now, $\psi(y)=\dfrac{\partial f}{\partial x}(a+\alpha h,y)$, we have $E(h,k)=h\left[\psi(b+k)-\psi(k)\right]$.
Using the M.V.T. applied to ψ, there exists $\beta\in]0,1[$ such that $\psi(b+k)-\psi(b)=k\psi'(b+\beta k)$, which gives

$$E(h,k)=hk\frac{\partial^2 f}{\partial y\partial x}(a+\alpha h,b+\beta k).$$

In a similar way, there exists $s,t\in]0,1[$ such that

$$E(h,k)=kh\frac{\partial^2 f}{\partial x\partial y}(a+sh,b+tk).$$

This gives that $\dfrac{\partial^2 f}{\partial y\partial x}(a+\alpha h,b+\beta k)=\dfrac{\partial^2 f}{\partial x\partial y}(a+sh,b+tk)$.
As $\dfrac{\partial^2 f}{\partial x\partial y}$ and $\dfrac{\partial^2 f}{\partial y\partial x}$ are continuous at $P(a,b)$ and making $(h,k)\longrightarrow(0,0)$, we obtain

$$\frac{\partial^2 f}{\partial x\partial y}(a,b)=\frac{\partial^2 f}{\partial y\partial x}(a,b).$$

Definition *3.2 Let D be an open of $\mathbb{R}^n$ and $f:D\subset\mathbb{R}^n\longrightarrow\mathbb{R}$.*
(i) We say that f is of class C^0 on D if it is continuous on D.
(ii) We say that f is of class C^1 on D if, f and all its first-order partial derivatives are continuous on D.
(iii) We say that f is of class C^k on D ($k\in\mathbb{N}$) if, f and all its partial derivatives up to order k are continuous on D.
(iv) We say that f is of class C^∞ on D if it is of class C^k on D, for all $k\in\mathbb{N}$.

Example : Let $f(x,y)=\begin{cases}\dfrac{x^3y-xy^3}{x^2+y^2} & \text{if}\quad (x,y)\neq(0,0)\\ 0 & \text{if}\quad (x,y)=(0,0)\end{cases}$

Calculate $\dfrac{\partial^2 f}{\partial x\partial y}(0,0)$ and $\dfrac{\partial^2 f}{\partial y\partial x}(0,0)$. Conclusion ?

Solution : We have $\dfrac{\partial^2 f}{\partial x\partial y}(0,0)=\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)(0,0)=\lim\limits_{x\longrightarrow0}\dfrac{\dfrac{\partial f}{\partial y}(x,0)-\dfrac{\partial f}{\partial y}(0,0)}{x-0}$.

$\dfrac{\partial f}{\partial y}(0,0)=\lim\limits_{y\longrightarrow0}\dfrac{f(0,y)-f(0,0)}{y-0}=\lim\limits_{y\longrightarrow0}\dfrac{\dfrac{0}{y^2}-0}{y}=\lim\limits_{y\longrightarrow0}\dfrac{0}{y}=0;$
$\dfrac{\partial f}{\partial y}(x,y)=\dfrac{(x^3-3xy^2)(x^2+y^2)-(x^3y-xy^3)(2y)}{(x^2+y^2)^2}=\dfrac{x^5-4x^3y^2-xy^4}{(x^2+y^2)^2}$

$\Longrightarrow \dfrac{\partial f}{\partial y}(x,0) = \dfrac{x^5}{x^4} = x$, for $x \neq 0$.
Or using the definition

$$\frac{\partial f}{\partial y}(x,0) = \lim_{y \longrightarrow 0} \frac{f(x,y)-f(x,0)}{y-0} = \lim_{y \longrightarrow 0} \frac{\dfrac{x^3y-xy^3}{x^2+y^2} - \dfrac{0}{x^2}}{y} = \lim_{y \longrightarrow 0} \frac{x^3-xy^2}{x^2+y^2} = \frac{x^3}{x^2} = x$$

$\Longrightarrow \dfrac{\partial^2 f}{\partial x \partial y}(0,0) = \lim\limits_{x \longrightarrow 0} \dfrac{x-0}{x} = 1.$

Similarly $\dfrac{\partial^2 f}{\partial y \partial x}(0,0) = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)(0,0) = \lim\limits_{y \longrightarrow 0} \dfrac{\dfrac{\partial f}{\partial x}(0,y) - \dfrac{\partial f}{\partial x}(0,0)}{y-0}$,
with $\dfrac{\partial f}{\partial x}(0,0) = 0$ and $\dfrac{\partial f}{\partial x}(x,y) = \dfrac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2}$

$\Longrightarrow \dfrac{\partial^2 f}{\partial y \partial x}(0,0) = \lim\limits_{y \longrightarrow 0} \dfrac{-y-0}{y} = -1.$

We conclude that $\dfrac{\partial^2 f}{\partial x \partial y}(0,0) \neq \dfrac{\partial^2 f}{\partial y \partial x}(0,0)$ and therefore f is not class C^2 at $(0,0)$.

3.3 Derivative of a composite function (The Chain Rule)

***Theorem** 3.2 Let I be an open interval of $\mathbb{R}$, D be an open of $\mathbb{R}^2$ and the following composition :*

$$\begin{array}{ccccc} I \subset \mathbb{R} & \xrightarrow{g} & D \subset \mathbb{R}^2 & \xrightarrow{f} & \mathbb{R} \\ t & \longmapsto & (x(t), y(t)) & \longmapsto & f(x(t), y(t)) = F(t) \end{array}$$

If the functions x and y are differentiable at a point $t_0 \in I$ and if f has continuous first partial derivatives at the point $(x_0, y_0) = (x(t_0), y(t_0)) \in D$, then $F = f \circ g : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ is differentiable at the point t_0, with

$$F'(t_0) = \frac{dF}{dt}(t_0) = \frac{\partial f}{\partial x}(x_0,y_0)x'(t_0) + \frac{\partial f}{\partial y}(x_0,y_0)y'(t_0).$$

Proof : We have

$$F(t) - F(t_0) = f(x,y) - f(x_0,y_0) = f(x,y) - f(x_0,y) + f(x_0,y) - f(x_0,y_0).$$

Let $\varphi(x) = f(x,y)$ and $\psi(y) = f(x_0,y)$. We have $F(t) - F(t_0) = \varphi(x) - \varphi(x_0) + \psi(y) - \psi(y_0)$.
By M.V.T. applied to φ, there exists $\alpha \in]x_0, x[$ or $]x, x_0[$ such that $\varphi(x) - \varphi(x_0) = (x - x_0)\varphi'(\alpha)$ and by M.V.T. applied to ψ, there exists $\beta \in]y_0, y[$ or $]y, y_0[$ such that $\psi(y) - \psi(y_0) = (y - y_0)\psi'(\beta)$, we therefore obtain

$$F(t) - F(t_0) = (x - x_0)\frac{\partial f}{\partial x}(\alpha, y) + (y - y_0)\frac{\partial f}{\partial x}(x_0, \beta).$$

According to the continuity of $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ and the differentiability of x and y, we obtain

$$\begin{aligned} F'(t_0) &= \lim_{t \longrightarrow t_0} \frac{F(t) - F(t_0)}{t - t_0} \\ &= \lim_{t \longrightarrow t_0} \frac{x(t) - x(t_0)}{t - t_0} \lim_{(x,y) \longrightarrow (x_0,y_0)} \frac{\partial f}{\partial x}(\alpha, y) + \lim_{t \longrightarrow t_0} \frac{y(t) - y(t_0)}{t - t_0} \lim_{(x,y) \longrightarrow (x_0,y_0)} \frac{\partial f}{\partial y}(x_0, \beta) \\ &= \frac{\partial f}{\partial x}(x_0,y_0)\, x'(t_0) + \frac{\partial f}{\partial y}(x_0,y_0)\, y'(t_0). \end{aligned}$$

Example : Calculate by two different methods the derivative of the function

$$F(t) = (t^2+5t+6)^{\cos t}, \text{ for } t \notin [-3,-2].$$

Solution :
First method (direct) :

$\ln F(t) = \cos t \ln(t^2+5t+6) \Longrightarrow \dfrac{F'(t)}{F(t)} = -\sin t \ln(t^2+5t+6) + \dfrac{2t+5}{t^2+5t+6}\cos t$

$\Longrightarrow F'(t) = F(t)\left[-\sin t \ln(t^2+5t+6) + \cos t \dfrac{2t+5}{t^2+5t+6}\right].$

Second method (Chain rule) :

Let $x(t) = t^2+5t+6$, $y(t) = \cos t$ and $f(x,y) = x^y = F(t)$.

$\dfrac{dF}{dt}(t) = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt} = \dfrac{y}{x}x^y(2t+5) + x^y(\ln x)(-\sin t)$

$\Longrightarrow F'(t) = F(t)\left[\dfrac{2t+5}{t^2+5t+6}\cos t - \sin t \ln(t^2+5t+6)\right].$

Example : Let $x = x(t)$, $y = y(t)$ and $F(t) = f(x,y)$ where f is of class C^2 and x, y are $2-times$ continuously differentiable. Show that

$$F''(t) = \frac{\partial^2 f}{\partial x^2}[x'(t)]^2 + 2\frac{\partial^2 f}{\partial x \partial y}x'(t)\,y'(t) + \frac{\partial^2 f}{\partial y^2}[y'(t)]^2 + \frac{\partial f}{\partial x}x''(t) + (t)\frac{\partial f}{\partial y}y''(t).$$

Solution : We have

$F'(t) = \dfrac{\partial f}{\partial x}x'(t) + \dfrac{\partial f}{\partial y}y'(t) \Longrightarrow F''(t) = \dfrac{d}{dt}\left(\dfrac{dF}{dt}\right) = \dfrac{d}{dt}\left(\dfrac{\partial f}{\partial x}x'(t)\right) + \dfrac{d}{dt}\left(\dfrac{\partial f}{\partial y}y'(t)\right)$

$$\begin{aligned}
\Longrightarrow F''(t) &= \frac{d}{dt}\left(\frac{\partial f}{\partial x}\right)x'(t) + \frac{\partial f}{\partial x}x''(t) + \frac{d}{dt}\left(\frac{\partial f}{\partial y}\right)y'(t) + \frac{\partial f}{\partial y}y''(t) \\
&= \left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)x'(t) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)y'(t)\right]x'(t) + \frac{\partial f}{\partial x}x''(t) \\
&\quad + \left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)x'(t) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)y'(t)\right]y'(t) + \frac{\partial f}{\partial y}y''(t) \\
&= \frac{\partial^2 f}{\partial x^2}(x'(t))^2 + 2\frac{\partial^2 f}{\partial x \partial y}x'(t)y'(t) + \frac{\partial^2 f}{\partial y^2}(y'(t))^2 + \frac{\partial f}{\partial x}x''(t) + \frac{\partial f}{\partial y}y''(t).
\end{aligned}$$

General case : Let I be an open interval of $\mathbb{R}$, D be an open of $\mathbb{R}^n$ and the following composition

$$\begin{array}{ccccc}
I \subset \mathbb{R} & \xrightarrow{g} & D \subset \mathbb{R}^n & \xrightarrow{f} & \mathbb{R} \\
t & \longmapsto & (x_1(t), \cdots, x_n(t)) & \longmapsto & f(x_1, \cdots, x_n) = F(t)
\end{array}$$

If the functions $x_1, \cdots, x_n$ are differentiable at a point $t_0 \in I$ and if f has continuous first partial derivatives at the point $x_0 = g(t_0) = (x_1(t_0), \cdots, x_n(t_0)) \in D$, then $F = f \circ g : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ is differentiable at the point t_0, with

$$F'(t_0) = \frac{dF}{dt}(t_0) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_0)x_i'(t_0).$$

3.4 Directional derivative

Let $f : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function defined in an open D of $\mathbb{R}^n$, $a = (a_1, \ldots, a_n) \in D$ and $u = (u_1, \cdots, u_n)$ be a unit vector of $\mathbb{R}^n$ ($\|u\| = 1$).

***Definition** 3.3 The directional derivative of f in the direction of the vector $\overrightarrow{u}$, at the point a is defined by*

$$D_u f\,(a) = \frac{\partial f}{\partial u}(a) = \lim_{t \longrightarrow 0} \frac{f(a+tu) - f(a)}{h} = \lim_{t \longrightarrow 0} \frac{f(a_1 + tu_1, \cdots, a_n + tu_n) - f(a_1, \cdots, a_n)}{t}.$$

***Definition** 3.4 We define the gradient of f at a point $x = (x_1, \cdots, x_n)$ by*

$$\overrightarrow{\mathrm{grad}} f(x) = \left(\frac{\partial f}{\partial x_1}(x), \cdots, \frac{\partial f}{\partial x_n}(x) \right) \in \mathbb{R}^n.$$

***Theorem** 3.3 If f is of class C^1 in a neighborhood of a, then*

$$\frac{\partial f}{\partial u}(a) = \overrightarrow{u} \cdot \overrightarrow{\mathrm{grad}} f(a).$$

Proof : Let $F(t) = f(a+tu)$, then $\dfrac{\partial f}{\partial u}(a) = \lim\limits_{t \longrightarrow 0} \dfrac{F(t) - F(0)}{t} = F'(0)$.

Using chain rule $F'(0) = \sum\limits_{i=1}^{n} \dfrac{\partial f}{\partial x_i}(a) x_i'(0) = \sum\limits_{i=1}^{n} \dfrac{\partial f}{\partial x_i}(a) u_i = \overrightarrow{u} \cdot \overrightarrow{\mathrm{grad}} f(a)$,

then $\dfrac{\partial f}{\partial u}(a) = \overrightarrow{u} \cdot \overrightarrow{\mathrm{grad}} f(a)$.

Note : $\dfrac{\partial f}{\partial u}$ can also denoted by $D_u f$.

***Theorem** 3.4 The maximum value of $\dfrac{\partial f}{\partial u}(a)$ occurs in the direction of $\overrightarrow{\mathrm{grad}} f\,(a)$.*
This maximum value equals (in module)

$$\left| \frac{\partial f}{\partial u}(a) \right| = \left\| \overrightarrow{\mathrm{grad}} f(a) \right\|.$$

Proof : $\left| \overrightarrow{u} \cdot \overrightarrow{\mathrm{grad}} f(a) \right|$ is maximal when $\overrightarrow{u}$ and $\overrightarrow{\mathrm{grad}} f(a)$ are collinear, with

$\left| \dfrac{\partial f}{\partial u}(a) \right| = \left| \overrightarrow{u} \cdot \overrightarrow{\mathrm{grad}} f(a) \right| = \|\overrightarrow{u}\| \left\| \overrightarrow{\mathrm{grad}} f(a) \right\| = \left\| \overrightarrow{\mathrm{grad}} f(a) \right\|$.

It is the case where $\overrightarrow{u}$ and $\overrightarrow{\mathrm{grad}} f(a)$ have the same direction.

Example : Let $f(x, y) = x^2 + y^2$, $P(1, 1)$ and $\overrightarrow{u} = \cos\alpha\, \overrightarrow{i} + \sin\alpha\, \overrightarrow{j}$ with $0 \leq \alpha \leq \dfrac{\pi}{2}$.

1. Find the directional derivative of f at the point P, in the direction of $\overrightarrow{u}$.
2. In what direction this directional derivative is maximal ?

Solution : 1. $\overrightarrow{\text{grad}} f(x,y) = 2x\overrightarrow{i} + 2y\overrightarrow{j} \Longrightarrow \overrightarrow{\text{grad}} f(1,1) = 2\overrightarrow{i} + 2\overrightarrow{j}$
$\dfrac{\partial f}{\partial u}(1,1) = \overrightarrow{u} \cdot \overrightarrow{\text{grad}} f(1,1) = \left(\cos\alpha\overrightarrow{i} + \sin\alpha\overrightarrow{j}\right) \cdot \left(2\overrightarrow{i} + 2\overrightarrow{j}\right) = 2(\cos\alpha + \sin\alpha)$.
2. This derivative is maximal when $2|\cos\alpha + \sin\alpha| = \left\|\overrightarrow{\text{grad}} f(1,1)\right\| = 2\sqrt{2}$
$\Longleftrightarrow \cos\alpha + \sin\alpha = \sqrt{2}$, i.e., when $\alpha = \dfrac{\pi}{4}$.

3.5 Differentiability in several variables

Recall that the differentiability of the functions of one variable is the same thing that the existence of the derivative and that we have

$$f'(a) = \lim_{h \longrightarrow 0} \frac{f(a+h) - f(a)}{h}$$

i.e., the exists a mapping $E : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$f(a+h) - f(a) = f'(a)h + |h|E(h) \approx f'(a)h.$$

where $E(h) \longrightarrow 0$ as $h \longrightarrow 0$.

Let $f : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function defined in an open D of $\mathbb{R}^2$.

Definition 3.5 *We say that f is differentiable at $P(a,b) \in D$ if there exists two real α and β and a mapping $E : \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that*

$$f(a+h, b+k) - f(a,b) = \alpha h + \beta k + \|(h,k)\| \, E(h,k)$$

where $E(h,k) \longrightarrow 0$ as $(h,k) \longrightarrow (0,0)$. In other way

$$\lim_{(h,k) \longrightarrow (0,0)} E(h,k) = \lim_{(h,k) \longrightarrow (0,0)} \frac{f(a+h, b+k) - f(a,b) - \alpha h - \beta k}{\|(h,k)\|} = 0.$$

The norm $\|\cdot\|$ is one of the three usual norms of $\mathbb{R}^2$.

We deduce from this definition that if f is differentiable a the point P, then $\dfrac{\partial f}{\partial x}(a,b)$ and $\dfrac{\partial f}{\partial y}(a,b)$ exist; moreover α and β are unique with

$$\alpha = \frac{\partial f}{\partial x}(a,b) \qquad \text{and} \qquad \beta = \frac{\partial f}{\partial y}(a,b).$$

In fact, $\dfrac{\partial f}{\partial x}(a,b) = \lim\limits_{h \longrightarrow 0} \dfrac{f(a+h,b) - f(a,b)}{h} = \lim\limits_{h \longrightarrow 0} \dfrac{\alpha h + |h| \, E(h,0)}{h} = \alpha$. Similarly for $\dfrac{\partial f}{\partial y}(a,b)$.

Example : Let $f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$

1. Find $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$.

2. Show that f is differentiable at the point $(0,0)$.

Solution :

1. $\dfrac{\partial f}{\partial x}(0,0) = \lim\limits_{x\longrightarrow 0} \dfrac{f(x,0)-f(0,0)}{x-0} = \lim\limits_{x\longrightarrow 0} \dfrac{0-0}{x} = 0.$

Similarly $\dfrac{\partial f}{\partial y}(0,0) = 0.$

2. Let $\alpha = \beta = 0$ and $E(h,k)$ such that

$f(h,k) - f(0,0) = 0h + 0k + \|(h,k)\| \, E(h,k) \Longrightarrow E(h,k) = \dfrac{f(h,k)}{\|(h,k)\|}.$

Take $\|(h,k)\|_2 = \sqrt{h^2+k^2}$, then $E(h,k) = \dfrac{h^2k^2}{(h^2+k^2)^{3/2}}.$

$$\lim_{(h,k)\longrightarrow(0,0)} E\left(h,k\right) = \lim_{(h,k)\longrightarrow(0,0)} \frac{h^2k^2}{\left(h^2+k^2\right)^{3/2}} = \lim_{r\longrightarrow 0} \frac{r^4\cos^2\theta\sin^2\theta}{r^3} = \lim_{r\longrightarrow 0} r\cos^2\theta\sin^2\theta = 0.$$

Then f is differentiable at $(0,0)$.

Theorem *3.5 If f is differentiable at a point $P(a,b)\in D$, then f is continuous at this point.*

Proof : If f is differentiable at the point P, then $\exists \alpha,\beta\in\mathbb{R}$ such that

$$f(a+h,b+k) - f(a,b) = \alpha h + \beta k + \|(h,k)\| \, E(h,k).$$

When $(h,k) \longrightarrow (0,0)$, we have $M(x,y) \longrightarrow P(a,b)$ with $x = a+h$ and $y = b+k$ and then $\lim\limits_{(x,y)\longrightarrow(a,b)} \left[f(x,y) - f\left(a,b\right)\right] = 0$, therefore f is continuous at P.

Remark : The reciprocal of this theorem is not true.

Proposition *3.1 If $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are continuous in a neighborhood of a point $P(a,b)\in D$, then f is differentiable at this point.*

Proof : Set, for $(a+h,b+k)\in D$,

$$E\left(h,k\right) = \frac{1}{\|(h,k)\|}\left[f\left(a+h,b+k\right) - f\left(a,b\right) - h\frac{\partial f}{\partial x}\left(a,b\right) - k\frac{\partial f}{\partial y}\left(a,b\right)\right].$$

Using the M.V.T., and based on to the proof of the chain theorem, we therefore obtain

$$\begin{aligned} f\left(a+h,b+k\right) - f\left(a,b\right) &= f\left(a+h,b+k\right) - f\left(a,b+k\right) + f\left(a,b+k\right) - f\left(a,b\right) \\ &= h\frac{\partial f}{\partial x}\left(a+\alpha h,b+k\right) + k\frac{\partial f}{\partial y}\left(a,b+\beta k\right), \end{aligned}$$

which gives

$$E\left(h,k\right) = \frac{1}{\|(h,k)\|}\left[h\left(\frac{\partial f}{\partial x}\left(a+\alpha h,b+k\right) - \frac{\partial f}{\partial x}\left(a,b\right)\right) + k\left(\frac{\partial f}{\partial y}\left(a,b+\beta k\right) - \frac{\partial f}{\partial y}\left(a,b\right)\right)\right].$$

Then

$$\begin{aligned} \left|E\left(h,k\right)\right| &\leq \frac{|h|}{\|(h,k)\|}\left|\frac{\partial f}{\partial x}\left(a+\alpha h,b+k\right) - \frac{\partial f}{\partial x}\left(a,b\right)\right| + \frac{|k|}{\|(h,k)\|}\left|\frac{\partial f}{\partial y}\left(a,b+\beta k\right) - \frac{\partial f}{\partial y}\left(a,b\right)\right| \\ &\leq \left|\frac{\partial f}{\partial x}\left(a+\alpha h,b+k\right) - \frac{\partial f}{\partial x}\left(a,b\right)\right| + \left|\frac{\partial f}{\partial y}\left(a,b+\beta k\right) - \frac{\partial f}{\partial y}\left(a,b\right)\right| \end{aligned}$$

As $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$ are continuous at $P(a,b)$ and making $(h,k) \longrightarrow (0,0)$, we obtain

$$\lim_{(h,k)\longrightarrow(0,0)} |E(h,k)| = 0.$$

Notes : (1) If one of the partial derivatives at P doesn't exist, we can conclude that f is not differentiable at P.

(2) The existence of the partial derivatives at P doesn't provide the differentiability, it is necessary that they are continuous at this point.

Example : Let $f(x,y) = \begin{cases} \dfrac{xy}{|x|+|y|} & \text{if } \ (x,y) \neq (0,0) \\ 0 & \text{if } \ (x,y) = (0,0) \end{cases}$

1. Calculate $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$.
2. Is f differentiable at $(0,0)$?

Solution :

1. $\dfrac{\partial f}{\partial x}(0,0) = \lim\limits_{x\longrightarrow 0} \dfrac{f(x,0)-f(0,0)}{x-0} = \lim\limits_{x\longrightarrow 0} \dfrac{0-0}{x} = 0$; similarly $\dfrac{\partial f}{\partial y}(0,0) = 0$.
2. Suppose that f is differentiable at $(0,0)$, then

$f(h,k) - f(0,0) = 0h + 0k + \|(h,k)\| \, E(h,k) \Longrightarrow E(h,k) = \dfrac{f(h,k)}{\|(h,k)\|}$.

Take $\|(h,k)\|_1 = |h| + |k|$, then $E(h,k) = \dfrac{hk}{(|h|+|k|)^2}$.

Following the path $h = k$, we have $\lim\limits_{h\longrightarrow 0} E(h,h) = \lim\limits_{h\longrightarrow 0} \dfrac{h^2}{(2|h|)^2} = \lim\limits_{h\longrightarrow 0} \dfrac{h^2}{4h^2} = \dfrac{1}{4} \neq 0$.

Then f is not differentiable at $(0,0)$.

General case : For the functions of n variables $f : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}$, we say that f is differentiable at a point $a = (a_1, \cdots, a_n) \in D$ if there exists n real α_i, for $i = 1, \cdots, n$, and a mapping $E : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$f(a+h) - f(a) = \sum_{i=1}^{n} \alpha_i h_i + \|h\| \, E(h)$$

where $E(h) \longrightarrow 0$ as $h = (h_1, \cdots, h_n) \longrightarrow 0_{\mathbb{R}^n}$. In other way

$$\lim_{h\longrightarrow 0} E(h) = \lim_{h\longrightarrow 0} \frac{f(a+h) - f(a) - \displaystyle\sum_{i=1}^{n} \alpha_i h_i}{\|h\|} = 0.$$

We deduce that if f is differentiable a the point a, then $\dfrac{\partial f}{\partial x_1}(a), \cdots, \dfrac{\partial f}{\partial x_n}(a)$ exist; moreover $\alpha_1, \cdots, \alpha_n$ are unique with

$$\alpha_i = \frac{\partial f}{\partial x_i}(a), \ \text{for } i = 1, \cdots, n.$$

3.6 Differentials for functions of several variables

Let D be an open of $\mathbb{R}^n$ and $f : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ be a differentiable function at a point $a = (a_1, \dots, a_n) \in \mathbb{R}^n$, then we have

$$f(a_1 + h_1, \cdots, a_n + h_n) - f(a_1, \cdots, a_n) = \frac{\partial f}{\partial x_1}(a)h_1 + \cdots + \frac{\partial f}{\partial x_n}(a)h_n + \|h\|\, E(h).$$

where $h = (h_1, \cdots, h_n) \in \mathbb{R}^n$ and $E(h) \longrightarrow 0$ as $h \longrightarrow 0_{\mathbb{R}^n}$.

Definition *3.6 We call differential of f at the point a the linear mapping denoted $d_a f$ or $df(a) : \mathbb{R}^n \longrightarrow \mathbb{R}$, that for a vector $h = (h_1, \cdots, h_n) \in \mathbb{R}^n$ associates the linear expression in $h_1, \cdots, h_n$ given by :*

$$d_a f(h) = df\,(a)\,(h) = \frac{\partial f}{\partial x_1}(a)h_1 + \cdots + \frac{\partial f}{\partial x_n}(a)h_n = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)h_i.$$

Theorem *3.6 If f is differentiable at a, then*

$$df(a) = \frac{\partial f}{\partial x_1}(a)dx_1 + \cdots + \frac{\partial f}{\partial x_n}(a)dx_n = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)dx_i.$$

Proof : The variables $x_1, \cdots, x_n$ being independents and for $h = (h_1, \cdots, h_n) \in \mathbb{R}^n$,

we have $d\,(x_i)\,(h) = \displaystyle\sum_{j=1}^{n} \frac{\partial x_i}{\partial x_j}(a)h_j = h_i$, then

$df\,(a)\,(h) = \displaystyle\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)d\,(x_i)\,(h) = \left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)dx_i\right)(h)$, for all $h \in \mathbb{R}^n$,

therefore $df(a) = \displaystyle\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)dx_i$.

Theorem *3.7 If f is differentiable at the point a, then f has a directional derivative in any direction $h \in \mathbb{R}^n$ at a, with*

$$df(a)\,(h) = \overrightarrow{h} \cdot \overrightarrow{\text{grad}} f(a) = \frac{\partial f}{\partial h}(a).$$

Proof : If f is differentiable at the point a, then

$$f(a+h) - f(a) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)h_i + \|h\|\, E(h).$$

with $E(h) \longrightarrow 0$ when $h \longrightarrow 0$. Hence

$$\begin{aligned}
\frac{\partial f}{\partial h}(a) &= \lim_{t \longrightarrow 0} \frac{f(a+th) - f(a)}{t} = \lim_{t \longrightarrow 0} \frac{\displaystyle\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)th_i + \|th\|\, E(th)}{t} \\
&= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)h_i + \lim_{t \longrightarrow 0} \frac{|t|\, \|h\|\, E(th)}{t} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)h_i = df(a)\,(h)\,.
\end{aligned}$$

Remark : The reciprocal of this theorem is not true.

Note : Suppose that f is differentiable at a. From (where $h = x - a$)

$$f(x) - f(a) = df(a)\,(x-a) + \|x-a\|\,E(x-a),$$

for $x = (x_1, \cdots, x_n) \in D$, we can deduce the approximation of f in a neighborhood of a

$$f(x) \approx f(a) + df(a)\,(x-a)\,.$$

Example : Let $f(x,y) = x^y = e^{y \ln x}$. Using the differential, give an approximate value of $f(2.01, 2.97)$ from the point $P(2,3)$.
Solution : We have $f(2.01, 2.97) \approx f(2,3) + df(2,3)\,(2.01-2, 2.97-3) = f(2,3) + df(2,3)\,(0.01, -0.03)\,.$
$\dfrac{\partial f}{\partial x}(x,y) = \dfrac{y}{x}x^y \Longrightarrow \dfrac{\partial f}{\partial x}(2,3) = 12$ and $\dfrac{\partial f}{\partial y}(x,y) = (\ln x)x^y \Longrightarrow \dfrac{\partial f}{\partial y}(2,3) = 8\ln 2$
$\Longrightarrow f(2.01, 2.97) \approx f(2,3) + \dfrac{\partial f}{\partial x}(2,3)0.01 + \dfrac{\partial f}{\partial y}(2,3)\,(-0.03) = 2^3 + (12)(0.01) + (8\ln 2)(-0.03)$
$\Longrightarrow f(2.01, 2.97) \approx 8 + 0.12 - 0.24 \ln 2 \approx 7.953\,645.$
By calculator $f(2.01, 2.97) = 2.01^{2.97} = 7.952\,292.$

***Theorem** 3.8 Let f and g be two functions of n variables that are differentiable at a point a of on open D of $\mathbb{R}^n$, then $\alpha f + \beta g$ (for $\alpha, \beta \in \mathbb{R}$), fg, $\dfrac{f}{g}$ ($g(x) \neq 0$ in a neighborhood of a) and f^n are differentiable at a, and we have*
(i) $d(\alpha f + \beta g)\,(a) = \alpha df\,(a) + \beta dg\,(a)\,;$
(ii) $d(fg)\,(a) = g\,(a)\,df\,(a) + f\,(a)\,dg\,(a)\,;$
(iii) $d\left(\dfrac{f}{g}\right)(a) = \dfrac{g\,(a)\,df\,(a) - f\,(a)\,dg\,(a)}{g^2\,(a)};$
(iv) $d(f^n)\,(a) = nf^{n-1}\,(a)\,df\,(a)\,.$

3.7 Exercises

***Exercise** 3.1 Find the first and second partial derivatives of the following functions :*

1. $f(x,y) = (x^3 - y^2)^5 + \ln(x^2 + y^2)$
2. $f(x,y) = x \cos \dfrac{x}{y}$
3. $f(x,y,z) = xe^z - ye^x + ze^y$
4. $f(x,y,z) = x^2 \arctan(yz)$

***Exercise** 3.2 Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be twice differentiable.*

1. *Find the first and second partial derivatives of the following functions :*

$$g(x,y) = f(x^2 + y^2 + xy) \qquad \text{and} \qquad h(x,y,z) = f(z \sin x + \cos y).$$

2. *If $u\,(x,t) = f(x - at) + f(x + at)$, show that* $\dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2}.$

Exercise *3.3 Study the existence of the partial derivatives at the origin, in the following cases :*

1. $f(x,y)=\max\{x^2,y\}$
2. $f(x,y)=\begin{cases} \dfrac{x^3+y^2}{x^2+y^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}$
3. $f(x,y)=\begin{cases} \dfrac{x^3}{x^4+|y-x^2|} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}$

Exercise *3.4 Let the function $f:\mathbb{R}^3\longrightarrow\mathbb{R}$, given by*

$$f(x,y,z)=\begin{cases} \dfrac{xyz}{x^3+y^3+z^3} & \text{if } x^3+y^3+z^3\neq 0 \\ 0 & \text{if } x^3+y^3+z^3=0 \end{cases}$$

Prove that $\dfrac{\partial f}{\partial x}(0,0,0)$, $\dfrac{\partial f}{\partial y}(0,0,0)$ and $\dfrac{\partial f}{\partial z}(0,0,0)$ exist, but f is not continuous at $(0,0,0)$.

Exercise *3.5 Let $f:\mathbb{R}^2\longrightarrow\mathbb{R}$ defined by*

$$f(x,y)=\begin{cases} (x^2+y^2)^x & \text{if } (x,y)\neq(0,0) \\ 1 & \text{if } (x,y)=(0,0) \end{cases}$$

1. *Is the function f continuous at $(0,0)$?*
2. *Determine $\dfrac{\partial f}{\partial x}(x,y)$ and $\dfrac{\partial f}{\partial y}(x,y)$ at any $(x,y)\in\mathbb{R}^2\smallsetminus\{(0,0)\}$.*
3. *Do the partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ exist at the point $(0,0)$?*

Exercise *3.6 Let $f:\mathbb{R}^2\longrightarrow\mathbb{R}$ such that*

$$f(x,y)=\begin{cases} x^4 y\sin\dfrac{1}{x} & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$$

Calculate $\dfrac{\partial^2 f}{\partial x\partial y}(0,0)$ and $\dfrac{\partial^2 f}{\partial y\partial x}(0,0)$.

Exercise *3.7 Let $f:\mathbb{R}^2\longrightarrow\mathbb{R}$ defined by*

$$f(x,y)=\begin{cases} (x^2+y^2)\sin\dfrac{1}{\sqrt{x^2+y^2}} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}$$

1. *Show that f is continuous at $(0,0)$.*
2. *Calculate $\dfrac{\partial f}{\partial x}(x,y)$ and $\dfrac{\partial f}{\partial y}(x,y)$ at any $(x,y)\in\mathbb{R}^2\smallsetminus\{(0,0)\}$.*
3. *Study the continuity of $\dfrac{\partial f}{\partial x}$ at the point $(0,0)$. Is f of class C^1 at $(0,0)$?*

Exercise *3.8 Let* $f:\mathbb{R}^2\longrightarrow\mathbb{R}$ *defined by*

$$f(x,y)=\begin{cases} xy\ln(|x|+|y|) & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

Show that f *is of class* C^1 *on* $\mathbb{R}^2$.

Exercise *3.9 Let* $f:\mathbb{R}^2\longrightarrow\mathbb{R}$ *defined by*

$$f(x,y)=\begin{cases} \dfrac{(x-y)^2}{xy-1} & \text{if } xy\neq 1\\ 0 & \text{if } xy=1\end{cases}$$

Study the differentiability of f *at* $(1,1)$.

Exercise *3.10 Let the function* $f:\mathbb{R}^2\longrightarrow\mathbb{R}$, *defined by*

$$f(x,y)=\begin{cases} \dfrac{x\sin y-y\sin x}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

1. *Study the differentiability of* f *on* $\mathbb{R}^2$.
2. *Show that* f *is of class* C^1 *on* $\mathbb{R}^2$.
3. *Calculate* $\dfrac{\partial^2 f}{\partial x\partial y}(0,0)$ *and* $\dfrac{\partial^2 f}{\partial y\partial x}(0,0)$. *Deduce that* f *is not of class* C^2.

Exercise *3.11 Let* $n\in\mathbb{N}^*$ *and the function* $f:\mathbb{R}^2\longrightarrow\mathbb{R}$, *defined by*

$$f(x,y)=\begin{cases} \dfrac{x^n-y^n}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

1. *Study, according to the values of* n, *the continuity of* f *at* $(0,0)$.
2. *Determine the first order partial derivatives of* f *at the point* $(0,0)$.
3. *Study, according to the values of* n, *the differentiability of* f *at* $(0,0)$.
4. *Is* f *of class* C^1 *at* $(0,0)$ *for* $n=3$ *? for* $n=4$ *?*

Exercise *3.12 Let the function* $f:\mathbb{R}^2\longrightarrow\mathbb{R}$, *defined by*

$$f(x,y)=\begin{cases} 1-e^{1-(x^2+y^2)} & \text{if } x^2+y^2\geq 1\\ 0 & \text{if } x^2+y^2<1\end{cases}$$

Is this function of class C^1 *on* $\mathbb{R}^2$ *?*

Exercise *3.13 Let the function* $f:\mathbb{R}^2\longrightarrow\mathbb{R}$, *defined by*

$$f(x,y)=\begin{cases} \dfrac{\sin xy-xy}{e^{xy}-1} & \text{if } xy\neq 0\\ 0 & \text{if } xy=0\end{cases}$$

1. *Show that* f *is differentiable in* $A=\{(x,y)\in\mathbb{R}^2 : xy=0\}$.
2. *Deduce that* f *is differentiable in* $\mathbb{R}^2$.

Exercise *3.14 Find the total differentials of the following functions :*

1. $f(x,y) = \arcsin(2x+y)$ 2. $f(x,y) = \ln\sqrt{x^2+4y^2}$ 3. $f(x,y) = x^{\sin y}$
4. $f(x,y) = x^2 e^{xy} + \dfrac{1}{y^2}$ 5. $f(x,y,z) = x^2 \sin z + y \ln z$ 6. $f(x,y,z) = z^{xy}$

Exercise *3.15* 1. *Find* $\dfrac{du}{dt}$ *if* $u = xy + xz + yz$, *with* $x = e^t$, $y = e^{-t}$ *and* $z = e^t + e^{-t}$.

2. *Find* $\dfrac{dz}{dx}$ *and* $\dfrac{dz}{dy}$ *if* $z = f(x,y) = x^2y + xy^2$ *and* $y = \ln x$.

3. *If* $f(x,y) = 0$ *and* $g(x,z) = 0$ *and if* f *and* g *are differentiable, show that*

$$\frac{\partial f}{\partial y} \cdot \frac{\partial g}{\partial x} dy = \frac{\partial f}{\partial x} \cdot \frac{\partial g}{\partial z} dz.$$

Exercise *3.16 Let* $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *defined by*

$$f(x,y) = \begin{cases} \dfrac{x^2}{y} \exp \dfrac{y}{x} & \text{if} \quad xy \neq 0 \\ 0 & \text{if} \quad xy = 0 \end{cases}$$

1. *Calculate* $\dfrac{\partial f}{\partial x}(0,0)$ *and* $\dfrac{\partial f}{\partial y}(0,0)$.
2. *Determine, using the definition, the directional derivative of* f *at the point* $(0,0)$ *in any direction of unit vector* $\vec{u} = \alpha\vec{i} + \beta\vec{j}$ *of* $\mathbb{R}^2$ *such that* $\alpha\beta \neq 0$.
3. *Calculate the limit when* $(x,y) \longrightarrow (0,0)$ *of the restriction of* f *on the parabola* $y = x^2$.
4. *Is* f *continuous at* $(0,0)$ *? Conclusion ?*

Exercise *3.17 Let* $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *defined by*

$$f(x,y) = \begin{cases} \dfrac{\sin(xy^2)}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$$

1. *Determine, using the definition, the directional derivative of* f *at the point* $(0,0)$ *in any direction of unit vector* $\vec{u} = \alpha\vec{i} + \beta\vec{j}$ *of* $\mathbb{R}^2$.
2. *Is* f *differentiable at* $(0,0)$ *? Conclusion ?*
3. *Calculate, if it exists,* $\dfrac{\partial^2 f}{\partial x \partial y}(0,0)$ *and* $\dfrac{\partial^2 f}{\partial y \partial x}(0,0)$.

Exercise *3.18 The temperature at any point of a thin sheet is given by*

$$T(x,y) = \frac{100xy}{x^2+y^2}.$$

1. *Find the directional derivative of* T *at the point* $P(2,1)$, *following the direction that makes an angle of* $60°$ *with the* $x-axis$.
2. *What is the direction of the greatest drop in temperature at* P.

Exercise *3.19 Given* $P(a,b,c) \in S(0,1)$ *and the function* $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ *defined by*

$$f(x,y,z) = e^{x^2+y^2+z^2} - xyz.$$

Determine the directional derivative of f *in the direction of the vector* $\overrightarrow{u} = \overrightarrow{OP}$ *at the point* P.

Chapter 4

Applications of the differential in $\mathbb{R}^n$

4.1 Mean value theorem, Taylor's formula and Finite expansions

Let $P(a,b)$ and $M(x,y)$ be two points of an open and convex domain $D \subseteq \mathbb{R}^2$ such that

$$\begin{cases} x = a + ht \\ y = b + kt \end{cases}, \text{ for } t \in [0,1].$$

Consider a function $f : D \longrightarrow \mathbb{R}$ defined on D and we set

$$f(x,y) = f(a+ht, b+kt) = F(t).$$

4.1.1 Mean value theorem

Theorem** 4.1 **(Mean value theorem)
If f is of class C^1 on D, then $\exists\ \theta \in]0,1[$ such that

$$f(a+h, b+k) - f(a,b) = \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)(a+\theta h, b+\theta k).$$

Proof : f is of class C^1 on D, then F is continuous and differentiable on $[0,1]$.
Therefore, according to the mean value theorem applied to $F : [0,1] \longrightarrow \mathbb{R}$ on $[0,1]$, $\exists \theta \in]0,1[$ such that

$$F(1) - F(0) = (1-0)F'(\theta).$$

As $F(0) = f(a,b),\ F(1) = f(a+h, b+k)$ and

$$F'(t) = \frac{\partial f}{\partial x}(x,y)\,x'(t) + \frac{\partial f}{\partial y}(x,y)\,y'(t) = \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)(a+ht, b+kt),$$

then $f(a+h, b+k) - f(a,b) = \left(h\dfrac{\partial f}{\partial x} + k\dfrac{\partial f}{\partial y}\right)(a+\theta h, b+\theta k)$.

General case : Let D be an open domain of $\mathbb{R}^n$, $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function of n variables and $a = (a_1, \cdots, a_n) \in D$. If f is of class C^1 on D and if $[a, a+h] \subset D$ for $h = (h_1, \cdots, h_n)$, then

$\exists\theta\in]0,1[$ such that

$$f(a_1+h_1,\cdots,a_n+h_n)-f(a_1,\cdots,a_n)=\sum_{i=1}^{n}h_i\frac{\partial f}{\partial x_i}(a_1+\theta h_1,\cdots,a_n+\theta h_n).$$

In other words

$$f(a+h)-f(a)=\sum_{i=1}^{n}h_i\frac{\partial f}{\partial x_i}(a+\theta h)=\left\langle\overrightarrow{\text{grad}}f\,(a+\theta h)\,,h\right\rangle.$$

4.1.2 Taylor's formula

***Theorem** 4.2 **(Taylor's formula of order** 1)*
If f is of class C^2 on D, then $\exists\ \theta\in]0,1[$ such that

$$\begin{aligned} f(a+h,b+k) &= f(a,b) + h\frac{\partial f}{\partial x}(a,b)+k\frac{\partial f}{\partial y}(a,b)\\ &+ \frac{1}{2}\left(h^2\frac{\partial^2 f}{\partial x^2}+2hk\frac{\partial^2 f}{\partial x\partial y}+k^2\frac{\partial^2 f}{\partial y^2}\right)(a+\theta h,b+\theta k).\end{aligned}$$

Proof : f is of class C^2 on D, then F is continuous and twice differentiable on $[0,1]$.
Therefore, according to Taylor's formula applied to F on $[0,1]$, $\exists\ \theta\in]0,1[$ such that

$$F(1)=F(0)+(1-0)F'(0)+\frac{1}{2}F''(\theta).$$

We have $F(0)=f(a,b),\ F(1)=f(a+h,b+k),\ F'(0)=h\frac{\partial f}{\partial x}(a,b)+k\frac{\partial f}{\partial y}(a,b)$ and

$$\begin{aligned} F''(t) &= \frac{\partial^2 f}{\partial x^2}(x,y)\left[x'(t)\right]^2+2\frac{\partial^2 f}{\partial x\partial y}(x,y)\,x'(t)\,y'(t)+\frac{\partial^2 f}{\partial y^2}(x,y)\left[y'(t)\right]^2\\ &= \left(h^2\frac{\partial^2 f}{\partial x^2}+2hk\frac{\partial^2 f}{\partial x\partial y}+k^2\frac{\partial^2 f}{\partial y^2}\right)(a+ht,b+kt),\end{aligned}$$

hence the formula.

General case : Let D be an open domain of $\mathbb{R}^n$, $f:D\subseteq\mathbb{R}^n\longrightarrow\mathbb{R}$ be a function of n variables and $a=(a_1,\cdots,a_n)\in D$. If f is of class C^2 on D and if $[a,a+h]\subset D$ for $h=(h_1,\cdots,h_n)$, then $\exists\theta\in]0,1[$ such that

$$f(a+h)=f(a)+\sum_{i=1}^{n}h_i\frac{\partial f}{\partial x_i}(a)+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}h_ih_j\frac{\partial^2 f}{\partial x_i\partial x_j}(a+\theta h).$$

4.1.3 Finite expansions

According to the previous paragraph, we can define the first and second order finite expansions formulas in a neighborhood of a point $P(a,b)$.

Theorem** 4.3 **(First order finite expansion formula)

If f is of class C^1 on D, then the finite expansion up to order 1 of f in a neighborhood of $P(a,b)$ is given by

$$f(a+h,b+k)=f(a,b)+h\frac{\partial f}{\partial x}(a,b)+k\frac{\partial f}{\partial y}(a,b)+r\varepsilon(h,k)$$

with $r=\|(h,k)\|$ and $\varepsilon(h,k)\longrightarrow 0$ as $(h,k)\longrightarrow(0,0)$.

Proof : Using mean value theorem, we have

$$f(a+h,b+k)-f(a,b)=\left(h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}\right)(a+\theta h,b+\theta k).$$

Since $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous in a neighborhood of P, then

$$\begin{aligned} \frac{\partial f}{\partial x}(a+\theta h,b+\theta k) &= \frac{\partial f}{\partial x}(a,b)+\varepsilon_1(h,k) \\ \text{and}\quad \frac{\partial f}{\partial y}(a+\theta h,b+\theta k) &= \frac{\partial f}{\partial y}(a,b)+\varepsilon_2(h,k), \end{aligned}$$

with $\varepsilon_1(h,k)\longrightarrow 0$ and $\varepsilon_2(h,k)\longrightarrow 0$ as $(h,k)\longrightarrow(0,0)$.
This implies

$$\begin{aligned} f(a+h,b+k) &= f(a,b)+h\frac{\partial f}{\partial x}(a,b)+k\frac{\partial f}{\partial y}(a,b)+h\varepsilon_1(h,k)+k\varepsilon_2(h,k) \\ &= f(a,b)+h\frac{\partial f}{\partial x}(a,b)+k\frac{\partial f}{\partial y}(a,b)+\|(h,k)\|\,\varepsilon(h,k) \end{aligned}$$

with $\varepsilon(h,k)=\dfrac{h\varepsilon_1(h,k)+k\varepsilon_2(h,k)}{\|(h,k)\|}$.
It is easy to show that $\varepsilon(h,k)\longrightarrow 0$ when $(h,k)\longrightarrow(0,0)$.

General case : The finite expansion up to order 1, of a function $f:D\subseteq\mathbb{R}^n\longrightarrow\mathbb{R}$ of class C^1, in a neighborhood of $a=(a_1,\cdots,a_n)\in D$ is given by

$$f(a+h)=f(a)+\sum_{i=1}^{n}h_i\frac{\partial f}{\partial x_i}(a)+r\varepsilon(h)$$

with $r=\|h\|$ and $\varepsilon(h)\longrightarrow 0$ as $h=(h_1,\cdots,h_n)\longrightarrow(0,\cdots,0)$.

Theorem** 4.4 **(Second order finite expansion formula)

If f is of class C^2 on D, then the finite expansion up to order 2 of f in a neighborhood of $P(a,b)$ is given by

$$\begin{aligned} f(a+h,b+k) = f(a,b) &+ h\frac{\partial f}{\partial x}(a,b)+k\frac{\partial f}{\partial y}(a,b) \\ &+ \frac{1}{2}\left[h^2\frac{\partial^2 f}{\partial x^2}(a,b)+2hk\frac{\partial^2 f}{\partial x\partial y}(a,b)+k^2\frac{\partial^2 f}{\partial y^2}(a,b)\right]+r^2\varepsilon(h,k) \end{aligned}$$

with $r=\|(h,k)\|$ and $\varepsilon(h,k)\longrightarrow 0$ as $(h,k)\longrightarrow(0,0)$.

Proof : The proof is analogous to the previous theorem using Taylor's formula of order 1.

General case : The finite expansion up to order 2, of a function $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ of class C^2, in a neighborhood of $a = (a_1, \cdots, a_n) \in D$ is given by

$$f(a+h) = f(a) + \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(a) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}(a) + r^2 \varepsilon(h)$$

with $r = \|h\|$ and $\varepsilon(h) \longrightarrow 0$ as $h = (h_1, \cdots, h_n) \longrightarrow (0, \cdots, 0)$.

Example : Give the finite expansion up to order 2 of $f(x,y) = \dfrac{1}{xy}$ in a neighborhood of $(1,1)$.

Solution : We have

$$\begin{aligned} f(x,y) &= f(1+h, 1+k) \\ &= f(1,1) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} \right)(1,1) + \frac{1}{2}\left(h^2 \frac{\partial^2 f}{\partial x^2} + 2hk \frac{\partial^2 f}{\partial x \partial y} + k^2 \frac{\partial^2 f}{\partial y^2} \right)(1,1) + r^2 \varepsilon(h,k) \end{aligned}$$

with $h = x - 1$ and $k = h - 1$.

$\dfrac{\partial f}{\partial x}(x,y) = -\dfrac{1}{x^2 y}$, $\dfrac{\partial f}{\partial y}(x,y) = -\dfrac{1}{xy^2}$, $\dfrac{\partial^2 f}{\partial x^2}(x,y) = \dfrac{2}{x^3 y}$, $\dfrac{\partial^2 f}{\partial y^2}(x,y) = \dfrac{2}{xy^3}$

and $\dfrac{\partial^2 f}{\partial x \partial y}(x,y) = \dfrac{1}{x^2 y^2}$

$\Longrightarrow f(1,1) = 1$, $\dfrac{\partial f}{\partial x}(1,1) = -1$, $\dfrac{\partial f}{\partial y}(1,1) = -1$, $\dfrac{\partial^2 f}{\partial x^2}(1,1) = 2$, $\dfrac{\partial^2 f}{\partial y^2}(1,1) = 2$ and $\dfrac{\partial^2 f}{\partial x \partial y}(1,1) = 1$

$$\begin{aligned} \Longrightarrow f(x,y) &= 1 - h - k + h^2 + hk + k^2 + r^2 \varepsilon(h,k) \\ &= 1 - (x-1) - (y-1) + (x-1)^2 + (x-1)(y-1) + (y-1)^2 + r^2 \varepsilon(x-1, y-1). \end{aligned}$$

4.2 Extrema of functions of two variables

4.2.1 Necessary condition for a local extremum

Let D be an open of $\mathbb{R}^2$, $f : D \longrightarrow \mathbb{R}$ and $P(a,b) \in D$.

***Definition** 4.1 (i) We say that f has a local minimum (or relative) at (a,b) if there exists a neighborhood $V \subseteq D$ of (a,b) such that*

$$f(a,b) \leq f(x,y), \quad \forall (x,y) \in V.$$

(ii) We say that f has a local maximum (or relative) at (a,b) if there exists a neighborhood $V \subseteq D$ of (a,b) such that

$$f(a,b) \geq f(x,y), \quad \forall (x,y) \in V.$$

***Definition** 4.2 (i) We say that f has a strict local minimum at (a,b) if there exists a neighborhood $V \subseteq D$ of (a,b) such that*

$$f(a,b) < f(x,y), \quad \forall (x,y) \in V \text{ and } (x,y) \neq (a,b).$$

(ii) We say that f has a strict local maximum at (a,b) if there exists a neighborhood $V \subseteq D$ of (a,b) such that

$$f(a,b) > f(x,y), \quad \forall (x,y) \in V \text{ and } (x,y) \neq (a,b).$$

- **Graphic interpretation** : The existence of a local minimum (resp. maximum) at (a,b) signifies that in a neighborhood of (a,b), the position of the surface (S) of equation $z=f(x,y)$ is above (resp. below) the plane of equation $z=f(a,b)$.

Remark : An extremum designate a minimum or a maximum.

- **Critical point** : Suppose that f is of class C^1 on D and that it has a local extremum at $P\,(a,b)$. Then the coordinates of P verify the following first order conditions :

$$(NC)\left\{\begin{array}{l}\dfrac{\partial f}{\partial x}(a,b)=0\\ \dfrac{\partial f}{\partial y}(a,b)=0\end{array}\right.$$

This point is called critical point and the condition (NC) is necessary but is not sufficient for f to have a local extrema at (a,b).

Example : For $f(x,y)=xy$ we have $\dfrac{\partial f}{\partial x}(0,0)=\dfrac{\partial f}{\partial y}(0,0)=0$ but there is neither maximum nor minimum at the point $(0,0)$.

4.2.2 Sufficient condition for a local extremum

In the following, based on the second order finite expansion formula in a neighborhood of a point $P\,(a,b)$, we will extract a sufficient condition for the existence of local extrema.

- **Sufficient condition (Monge's notations) :**

Let D be an open of $\mathbb{R}^n$, $f:D\subseteq\mathbb{R}^2\longrightarrow\mathbb{R}$ be a function of class C^2 and $P(a,b)\in D$ be a critical point of f. We have for $(x,y)\in V_P$

$$\begin{aligned}f(x,y)-f(a,b) &= h\frac{\partial f}{\partial x}(a,b)+k\frac{\partial f}{\partial y}(a,b)+\frac{1}{2}\left[h^2\frac{\partial^2 f}{\partial x^2}(a,b)+2hk\frac{\partial^2 f}{\partial x\partial y}(a,b)+k^2\frac{\partial^2 f}{\partial y^2}(a,b)\right]\\ &\quad +r^2\varepsilon(h,k)\\ &= \frac{1}{2}\left[h^2\frac{\partial^2 f}{\partial x^2}(a,b)+2hk\frac{\partial^2 f}{\partial x\partial y}(a,b)+k^2\frac{\partial^2 f}{\partial y^2}(a,b)\right]+r^2\varepsilon(h,k)\end{aligned}$$

When $(x,y)\longrightarrow(a,b)$ the sign of $\Delta f=f(x,y)-f(a,b)$ becomes the one of the quantity between hooks. To facilitate the discussion put

$$A=\frac{\partial^2 f}{\partial x^2}(a,b),\qquad B=\frac{\partial^2 f}{\partial x\partial y}(a,b),\qquad C=\frac{\partial^2 f}{\partial y^2}(a,b).$$

Then $\Delta f\underset{P}{\approx}\dfrac{1}{2}\left[Ah^2+2Bhk+Ck^2\right].$

Let's study the sign of Δf while considering the term between hooks like polynomial of second degree in h. However $\Delta'=b'^2-ac=B^2k^2-ACk^2=\left(B^2-AC\right)k^2$. We set

$$Q(a,b)=B^2-AC=\left(\frac{\partial^2 f}{\partial x\partial y}(a,b)\right)^2-\left(\frac{\partial^2 f}{\partial x^2}(a,b)\right)\left(\frac{\partial^2 f}{\partial y^2}(a,b)\right).$$

Then the sign of Δf depends on $Q(a,b)$. We have three cases, representing the second order condition :

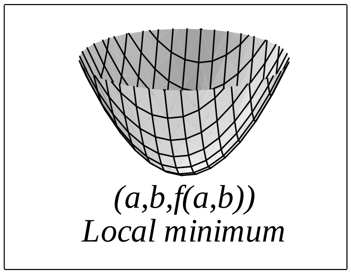
► First case : if $Q(a,b) < 0$, then Δf and A have the same sign and moreover if

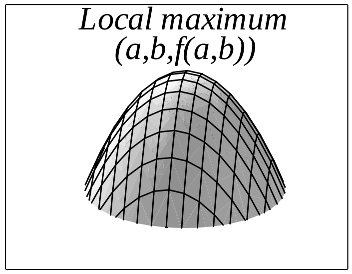
(i) $A > 0 \Longrightarrow f(x,y) > f(a,b),\ \forall\,(x,y) \in V_P$, and f has a local minimum at P;

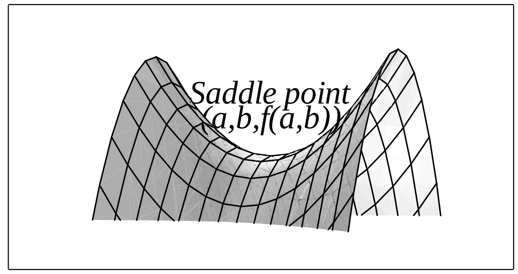
(ii) $A < 0 \Longrightarrow f(x,y) < f(a,b),\ \forall\,(x,y) \in V_P$, and f has a local maximum at P.

► Second case : if $Q(a,b) > 0$, then Δf changes sign and f has a saddle point at P.

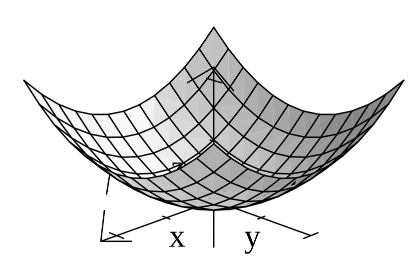
► Third case : if $Q(a,b) = 0$, then $\Delta f \underset{P}{\approx} \dfrac{1}{2A}(Ah + Bk)^2 = 0$ (since $h = -\dfrac{b'}{a} = -\dfrac{Bk}{A}$), the point is therefore degenerate, and we cannot conclude anything.



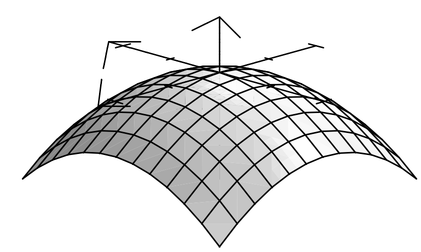




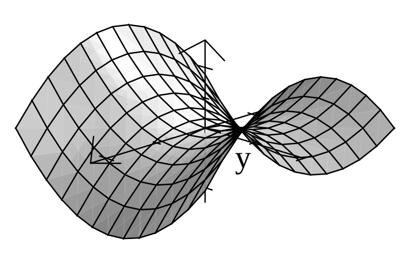
Examples :



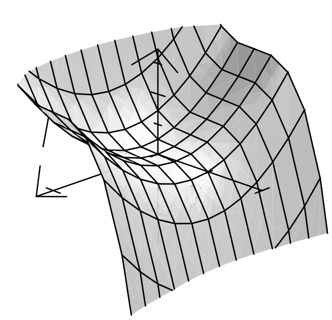
Minimum : $f(x,y) = x^2 + y^2$



Maximum : $f(x,y) = -x^2 - y^2$



Saddle Point : $f(x,y) = x^2 - y^2$



Degenerate point : $f(x,y) = x^3 + y^2$

Example : Study the extrema of the function $z = f(x,y) = x \sin y$.

Solution :

$$\begin{cases} \dfrac{\partial f}{\partial x} = \sin y = 0 \\ \dfrac{\partial f}{\partial y} = x\cos y = 0 \end{cases} \text{ if } \begin{cases} y = k\pi \\ \text{and} \\ x = 0 \text{ or } y = \dfrac{\pi}{2} + k\pi \end{cases} \Longleftrightarrow \begin{cases} x = 0 \text{ and } y = k\pi \\ \text{or} \\ y = k\pi \text{ and } y = \dfrac{\pi}{2} + k\pi \end{cases}$$

The second case is impossible then the critical points are $P_k(0, k\pi)$.

$$Q(x,y) = \left(\frac{\partial^2 f}{\partial x \partial y}(x,y)\right)^2 - \left(\frac{\partial^2 f}{\partial x^2}(x,y)\right)\left(\frac{\partial^2 f}{\partial y^2}(x,y)\right) = \cos^2 y - (0)(-x\sin y) = \cos^2 y$$

$Q(0, k\pi) = \cos^2 k\pi = 1 > 0$.

Therefore f has a saddle point at P_k, $\forall k \in \mathbb{Z}$, with $z_k = f(0, k\pi) = 0$.

4.2.3 Hessian matrix and finding extrema

Definition *4.3 Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function of n variables. We define the Hessian matrix of f at a point $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ by*

$$H_f(x) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right)_{1 \le i,j \le n} = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x_1^2}(x) & \cdots & \dfrac{\partial^2 f}{\partial x_1 \partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f}{\partial x_n \partial x_1}(x) & \cdots & \dfrac{\partial^2 f}{\partial x_n^2}(x) \end{pmatrix}.$$

Its determinant, denoted $\Delta_H(x)$, is called the Hessian of f at x. If f is of class C^2 then this Hessian matrix is symmetric.

Theorem *4.5 Let D be an open of $\mathbb{R}^n$, $f : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function of class C^2 and $P(a,b) \in D$ be a critical point of f. Then*

(i) if $\Delta_H(a,b) > 0$ and $\dfrac{\partial^2 f}{\partial x^2}(a,b) > 0$, f has a local minimum at $P(a,b)$;

(ii) if $\Delta_H(a,b) > 0$ and $\dfrac{\partial^2 f}{\partial x^2}(a,b) < 0$, f has a local maximum at $P(a,b)$;

(iii) if $\Delta_H(a,b) < 0$, f has a saddle point at $P(a,b)$;

(iv) if $\Delta_H(a,b) = 0$, no conclusion can be drawn.

Proof :

$$\Delta_H(x,y) = \begin{vmatrix} \dfrac{\partial^2 f}{\partial x^2}(x,y) & \dfrac{\partial^2 f}{\partial x \partial y}(x,y) \\ \dfrac{\partial^2 f}{\partial y \partial x}(x,y) & \dfrac{\partial^2 f}{\partial y^2}(x,y) \end{vmatrix} = \left(\frac{\partial^2 f}{\partial x^2}(x,y)\right)\left(\frac{\partial^2 f}{\partial y^2}(x,y)\right) - \left(\frac{\partial^2 f}{\partial x \partial y}(x,y)\right)^2$$

$= -Q(x,y)$.

The conclusions then will be obtained from those of $Q(x,y)$.

Example : Study the extrema of the function $z = f(x,y) = x^3 + 3xy^2 - 15x - 12y$.

Solution :

$$\begin{cases} \dfrac{\partial f}{\partial x}(x,y) = 3x^2 + 3y^2 - 15 = 0 \\ \dfrac{\partial f}{\partial y}(x,y) = 6xy - 12 = 0 \end{cases} \text{ if } \begin{cases} x^2 + y^2 = 5 \\ xy = 2 \end{cases} \Longleftrightarrow \begin{cases} (x+y)^2 = 9 \\ xy = 2 \end{cases} \Longleftrightarrow \begin{cases} x + y = \pm 3 \\ xy = 2 \end{cases}$$

The critical points are $P_1(1,2)$, $P_2(2,1)$, $P_3(-1,-2)$ and $P_4(-2,-1)$.

$$\Delta_H(x,y) = \begin{vmatrix} \dfrac{\partial^2 f}{\partial x^2}(x,y) & \dfrac{\partial^2 f}{\partial x \partial y}(x,y) \\ \dfrac{\partial^2 f}{\partial y \partial x}(x,y) & \dfrac{\partial^2 f}{\partial y^2}(x,y) \end{vmatrix} = \begin{vmatrix} 6x & 6y \\ 6y & 6x \end{vmatrix} = 36(x^2 - y^2).$$

$\Delta_H(1,2) = -108 < 0$, then f has a saddle point at P_1, with $z_1 = f(1,2) = -26$;

$\Delta_H(2,1) = 108 > 0$ and $A = \dfrac{\partial^2 f}{\partial x^2}(2,1) = 12 > 0$, then f has a local minimum at P_2, with $z_2 = f(2,1) = -28$;

$\Delta_H(-1,-2) = -108 < 0$, then f has a saddle point at P_3, with $z_3 = f(-1,-2) = 26$;

$\Delta_H(-2,-1) = 108 > 0$ and $A = \dfrac{\partial^2 f}{\partial x^2}(-2,-1) = -12 < 0$, then f has a local maximum at P_4, with $z_3 = f(2,1) = 28$.

4.2.4 Global extremum

***Definition** 4.4 Let D be an open of $\mathbb{R}^2$, $f : D \longrightarrow \mathbb{R}$ and $P(a,b) \in D$.*

(i) We say that f has a global minimum (or absolute) at (a,b) if

$$f(a,b) \leq f(x,y), \quad \forall (x,y) \in D.$$

(ii) We say that f has a global maximum (or absolute) at (a,b) if

$$f(a,b) \geq f(x,y), \quad \forall (x,y) \in D.$$

***Theorem** 4.6 Let D be a closed and bounded domain (compact) of $\mathbb{R}^2$ and $f : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a continuous function. Then f is bounded and attains its bounds. More precisely, there exists $(a,b) \in D$ and $(\alpha,\beta) \in D$ such that*

$$f(a,b) = \inf_{(x,y)\in D} f(x,y) = \min_{(x,y)\in D} f(x,y) \qquad \text{and} \qquad f(\alpha,\beta) = \sup_{(x,y)\in D} f(x,y) = \max_{(x,y)\in D} f(x,y).$$

This theorem shows that f has a global minimum at (a,b) and a global maximum at (α,β).

***Proposition** 4.1 Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuous function such that*

$$\lim_{\|x\| \longrightarrow +\infty} f(x) = +\infty.$$

Then f is bounded below and attains its minimum.

Remark : If $\lim\limits_{\|x\| \longrightarrow +\infty} f(x) = -\infty$, then f is bounded above and attains its maximum.

Example : Study the extrema of the function $z = f(x,y) = x^2 + y^2 + 1$.

Does this function admit a global extrema on $\mathbb{R}^2$? Justify.

Solution : $\begin{cases} \dfrac{\partial f}{\partial x}(x,y) = 2x = 0 \\ \dfrac{\partial f}{\partial y}(x,y) = 2y = 0 \end{cases}$ if $x = y = 0$, then the critical point is $(0,0)$.

$\Delta_H(0,0) = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0$ and $A = \dfrac{\partial^2 f}{\partial x^2}(0,0) = 2 > 0$,

then f has a local minimum at $(0,0)$ with $z_O = f(0,0) = 1$.

$\lim\limits_{\|(x,y)\|_2 \longrightarrow +\infty} f(x,y) = \lim\limits_{\|(x,y)\|_2 \longrightarrow +\infty} \left(\|(x,y)\|_2^2 + 1 \right) = +\infty$,

then f has a global minimum at $(0,0)$.

4.3 Finding extrema with constraints

In this part, we will study the extrema of $f(x, y)$ in the case where the variables x and y are linked by a constraint of the form $g(x, y) = k$. We then consider the problem

$$\begin{cases} \text{Find the extrema of } z = f(x, y), \\ \text{under to the constraint } g(x, y) = k \end{cases}$$

***Definition** 4.5 Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function of class C^2 under the constraint $g(x, y) = k$, with $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ of class C^2. We define the Lagrangian function $L : \mathbb{R}^3 \longrightarrow \mathbb{R}$ associated with the problem by*

$$L(x, y, \lambda) = f(x, y) + \lambda[k - g(x, y)].$$

The parameter λ is called the Lagrange multiplier.

The Hessian of L is given by

$$\Delta_H(x, y, \lambda) = \begin{vmatrix} 0 & \dfrac{\partial g}{\partial x}(x, y) & \dfrac{\partial g}{\partial y}(x, y) \\ \dfrac{\partial g}{\partial x}(x, y) & \dfrac{\partial^2 L}{\partial x^2}(x, y, \lambda) & \dfrac{\partial^2 L}{\partial x \partial y}(x, y, \lambda) \\ \dfrac{\partial g}{\partial y}(x, y) & \dfrac{\partial^2 L}{\partial y \partial x}(x, y, \lambda) & \dfrac{\partial^2 L}{\partial y^2}(x, y, \lambda) \end{vmatrix}.$$

- **Necessary condition for relative extrema :**

The critical values are the solutions of the system

$$\begin{cases} \dfrac{\partial L}{\partial x}(x, y, \lambda) = 0 \\ \dfrac{\partial L}{\partial y}(x, y, \lambda) = 0 \\ \dfrac{\partial L}{\partial \lambda}(x, y, \lambda) = 0 \end{cases}$$

- **Sufficient condition for relative extrema :**

Let a, b and λ_0 be critical values for which

$$\frac{\partial L}{\partial x}(a, b, \lambda_0) = \frac{\partial L}{\partial y}(a, b, \lambda_0) = \frac{\partial L}{\partial \lambda}(a, b, \lambda_0) = 0$$

and the set

$$G = \{(x, y) \in D_f : g\,(x, y) = k\}.$$

The Hessian Δ_H of H is evaluated at the critical values :

- ► if $\Delta_H(a, b, \lambda_0) > 0$, then $f_{/G}$ has a local maximum at (a, b);
- ► if $\Delta_H(a, b, \lambda_0) < 0$, then $f_{/G}$ has a local minimum at (a, b).

Example : Find the extrema of the function $z = f(x,y) = x^2 - xy + 2y$
1. without constraint;
2. with the constraint $x + 2y = 10$;
3. with the constraint $xy = 1$.

Solution :

1. $\begin{cases} \dfrac{\partial f}{\partial x}(x,y) = 2x - y = 0 \\ \dfrac{\partial f}{\partial y}(x,y) = 2 - x = 0 \end{cases} \Longrightarrow \begin{cases} y = 2x = 4 \\ x = 2 \end{cases} \Longrightarrow$ the critical point is $P(2,4)$.

$\Delta_H(x,y) = \begin{vmatrix} 2 & -1 \\ -1 & 0 \end{vmatrix} = -1 < 0$, then f has a saddle point at P, with $z_P = 4$.

2. Let $L(x,y,\lambda) = x^2 - xy + 2y + \lambda[10 - x - 2y]$
and $G = \{(x,y) \in \mathbb{R}^2 : g(x,y) = x + 2y = 10\}$.

$$\begin{cases} \dfrac{\partial L}{\partial x}(x,y,\lambda) = 2x - y - \lambda = 0 \\ \dfrac{\partial L}{\partial y}(x,y,\lambda) = 2 - x - 2\lambda = 0 \\ \dfrac{\partial L}{\partial \lambda}(x,y,\lambda) = 10 - x - 2y = 0 \end{cases} \Longrightarrow \begin{cases} \lambda = 2x - y \\ \lambda = \dfrac{2-x}{2} \\ x + 2y = 10 \end{cases} \Longrightarrow \begin{cases} 5x - 2y = 2 \\ x + 2y = 10 \end{cases}$$

$\Longrightarrow$ the critical values are $x = 2,\ y = 4$ and $\lambda = 0$

$\Delta_H(x,y,\lambda) = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & -1 \\ 2 & -1 & 0 \end{vmatrix} = -12 < 0$, then $f_{/G}$ has a local minimum at P, with $z_P = 4$.

3. Let $L(x,y,\lambda) = x^2 - xy + 2y + \lambda[1 - xy]$
and $G = \{(x,y) \in \mathbb{R}^2 : g(x,y) = xy = 1\}$.

$$\begin{cases} \dfrac{\partial L}{\partial x}(x,y,\lambda) = 2x - y - \lambda y = 0 \\ \dfrac{\partial L}{\partial y}(x,y,\lambda) = 2 - x - \lambda x = 0 \\ \dfrac{\partial L}{\partial \lambda}(x,y,\lambda) = 1 - xy = 0 \end{cases} \Longrightarrow \begin{cases} \lambda = \dfrac{2x-y}{y} \\ \lambda = \dfrac{2-x}{x} \\ xy = 1 \end{cases} \Longrightarrow \begin{cases} y = x^2 \\ xy = 1 \end{cases} \Longrightarrow xy = x^3 = 1$$

$\Longrightarrow$ the critical values are $x = 1,\ y = 1$ and $\lambda = 1$

$\Delta_H(x,y,\lambda) = \begin{vmatrix} 0 & y & x \\ y & 2 & -1-\lambda \\ x & -1-\lambda & 0 \end{vmatrix} \Longrightarrow \Delta_H(1,1,1) = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & -2 \\ 1 & -2 & 0 \end{vmatrix} = -6 < 0$

then $f_{/G}$ has a local minimum at $Q(1,1)$, with $z_Q = 2$.

4.4 Implicit functions

Let D be an open of $\mathbb{R}^2$ and $f : D \longrightarrow \mathbb{R}$ be a continuous function. We say that the relation $f(x,y) = 0$ defines y as an implicit function of x, if there is a continuous function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$, called implicit function, such that $y = \varphi(x)$. This means that when the two variables are linked by an equation, it is locally possible to express one of them as function of the other, but under certain conditions.

Example : Let $f(x,y) = x^2 + y^2 - 1$. We have $\dfrac{\partial f}{\partial x}(x,y) = 2x$ and $\dfrac{\partial f}{\partial y}(x,y) = 2y$.

The curve of equation $f(x,y) = 0$ is obviously the unit circle of $\mathbb{R}^2$.
Let $D = \mathbb{R}\times]0,+\infty[$; at each point $(x_0,y_0) \in D$, we can explicitly determine the function φ in a neighborhood of x_0 by $\varphi(x) = \sqrt{1-x^2} > 0$.

Similarly in $D = \mathbb{R}\times]-\infty, 0[$ with $\varphi(x) = -\sqrt{1-x^2} < 0$.
On the other hand, it is not possible at $(1,0)$ or $(-1,0)$. We notice that in the first two cases $\dfrac{\partial f}{\partial y}(x_0, y_0) \neq 0$ while in the other two cases $\dfrac{\partial f}{\partial y}(\pm 1, 0) = 0$.

Theorem 4.7 (Implicit functions theorem - Case of 2 variables)
Let D be an open of $\mathbb{R}^2$, $f : D \longrightarrow \mathbb{R}$ be a continuous function and $(a,b) \in D$ such that $f(a,b) = 0$. If f is of class C^1 in D and if $\dfrac{\partial f}{\partial y}(a,b) \neq 0$, then there exists a neighborhood $V \subset \mathbb{R}$ of a, a neighborhood $W \subset \mathbb{R}$ of b and a function $\varphi : V \longrightarrow W$ of class C^1 in V, such that we have the equivalence

$$[(x,y) \in V \times W \text{ and } f(x,y) = 0] \Longleftrightarrow [x \in V \text{ and } y = \varphi(x)].$$

This implicit function is determined in a unique way by the relation

$$f(x, \varphi(x)) = 0, \quad \forall x \in V.$$

Moreover, we have for all $x \in V$:

$$\frac{dy}{dx} = \varphi'(x) = -\frac{\dfrac{\partial f}{\partial x}(x, \varphi(x))}{\dfrac{\partial f}{\partial y}(x, \varphi(x))}.$$

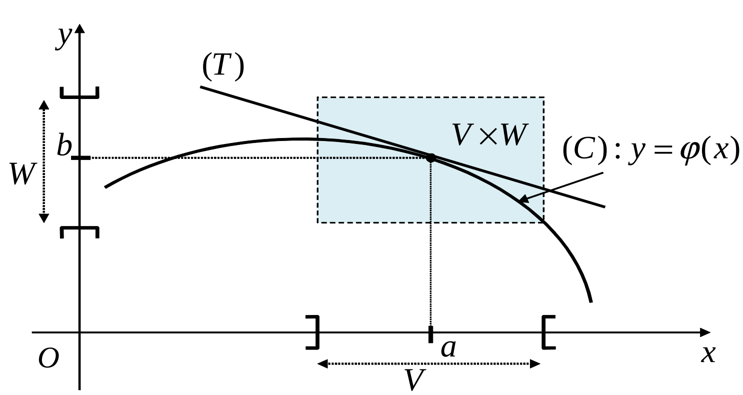
Remark : In particular $\varphi'(a) = -\dfrac{\dfrac{\partial f}{\partial x}(a,b)}{\dfrac{\partial f}{\partial y}(a,b)}$ and the equation of the tangent line (T) to the curve $(C) : y = \varphi(x)$ at the point $A(a,b)$ is

$$(T) : y = \varphi'(a)(x-a) + b,$$

or simply,

$$\frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b) = 0.$$

This situation is schematized by the following graph :



Example : Evaluate y' at the point $A(1,0)$ if $x + y = \cos(xy)$.
Solution : Let $f(x,y) = x + y - \cos(xy)$. We have $f(1,0) = 0$,

$$\frac{\partial f}{\partial x}(x,y) = 1 + y\sin(xy) \Longrightarrow \frac{\partial f}{\partial x}(1,0) = 1 \text{ and } \frac{\partial f}{\partial y}(x,y) = 1 + x\sin(xy) \Longrightarrow \frac{\partial f}{\partial y}(1,0) = 1 \neq 0.$$

Since f, $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are continuous at the point A,
then using the implicit function theorem there exists a neighborhood $V \subset \mathbb{R}$ of 1, a neighborhood $W \subset \mathbb{R}$ of 0 and a function $\varphi : V \longrightarrow W$ of class class C^1 in V, such that

$$f(x, \varphi(x)) = 0, \quad \forall x \in V,$$

moreover

$$\phi'(1) = -\frac{\dfrac{\partial f}{\partial x}(1,0)}{\dfrac{\partial f}{\partial y}(1,0)} = -1 = y'_A.$$

Another method for the determination of $\varphi'(1)$ *:*
Derive $x + y = \cos(xy)$ with respect to x knowing that $y = \varphi(x) : 1 + y' = -(y + xy')\sin(xy)$.
At the point $A(1,0) : 1 + y'_A = -(0 + y'_A)\sin(0) \Longrightarrow y'_A = \varphi'(1) = -1$.

Theorem 4.8 (Implicit functions theorem - Case of 3 variables)
Let D be an open of $\mathbb{R}^3$, $f : D \longrightarrow \mathbb{R}$ be a continuous function and $(a,b,c) \in D$ such that $f(a,b,c) = 0$. If f is of class C^1 in D and if $\dfrac{\partial f}{\partial z}(a,b,c) \neq 0$, then there exists a neighborhood $V \subset \mathbb{R}^2$ of (a,b), a neighborhood $W \subset \mathbb{R}$ of c and a function $\varphi : V \longrightarrow W$ of class C^1 in V, such that we have the equivalence

$$[(x,y,z) \in V \times W \text{ and } f(x,y,z) = 0] \Longleftrightarrow [(x,y) \in V \text{ and } z = \varphi(x,y)].$$

This implicit function is determined in a unique way by the relation

$$f(x, y, \varphi(x,y)) = 0, \quad \forall (x,y) \in V.$$

Moreover, we have for all $(x,y) \in V$:

$$\frac{\partial z}{\partial x} = \varphi'_x(x,y) = -\frac{\dfrac{\partial f}{\partial x}(x,y,\varphi(x,y))}{\dfrac{\partial f}{\partial z}(x,y,\varphi(x,y))} \qquad \text{and} \qquad \frac{\partial z}{\partial y} = \varphi'_y(x,y) = -\frac{\dfrac{\partial f}{\partial y}(x,y,\varphi(x,y))}{\dfrac{\partial f}{\partial z}(x,y,\varphi(x,y))}.$$

Remark : The equation of the tangent plane (P) to the surface $(S) : z = \varphi(x,y)$ at the point $A(a,b,c)$ is given by

$$(P) : z = \varphi'_x(a,b)\,(x-a) + \varphi'_y(a,b)\,(y-b) + c,$$

or simply,

$$\frac{\partial f}{\partial x}(a,b,c)(x-a) + \frac{\partial f}{\partial y}(a,b,c)(y-b) + \frac{\partial f}{\partial z}(a,b,c)(z-c) = 0.$$

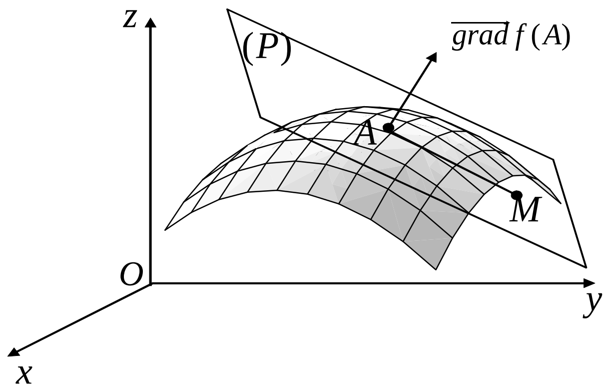
- **Normal vector to a surface** : If (P) is the tangent plane to the surface (S) of equation $f(x,y,z) = 0$ at $A(a,b,c)$ and $M(x,y,z)$ any point of (P) then we have

$$\overrightarrow{AM} \cdot \overrightarrow{\text{grad}} f(A) = 0 \Longleftrightarrow \overrightarrow{AM} \perp \overrightarrow{\text{grad}} f(A).$$

Therefore, $\overrightarrow{N}_A = \overrightarrow{\text{grad}} f(A)$ is an orthogonal vector to (P) at A and the vector

$$\overrightarrow{n} = \frac{\overrightarrow{\text{grad}} f(A)}{\left\| \overrightarrow{\text{grad}} f(A) \right\|}$$

constitutes the unit normal vector to the surface (S) at this point.



Example : Determine $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $x + y + z = xyz$.

Solution : Let $f(x, y, z) = x + y + z - xyz$.

On a $\dfrac{\partial f}{\partial x}(x, y, z) = 1 - yz$, $\dfrac{\partial f}{\partial y}(x, y, z) = 1 - xz$ and $\dfrac{\partial f}{\partial z}(x, y, z) = 1 - xy \neq 0$ if $xy \neq 1$.

Let $D = \{(x, y, z) \in \mathbb{R}^3 : xy \neq 1\}$,

f, $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial z}$ are continuous on D / $f(x_0, y_0, z_0) = 0$ and $\dfrac{\partial f}{\partial z}(x_0, y_0, z_0) \neq 0$ at each point $(x_0, y_0, z_0) \in D$, then, using the implicit function theorem there exists a neighborhood $V \subset \mathbb{R}^2$ of (x_0, y_0), a neighborhood $W \subset \mathbb{R}$ of z_0 and a function $\varphi : V \longrightarrow W$ of class C^1 in V, such that

$$f(x, y, \varphi(x, y)) = 0, \quad \forall\, (x, y) \in V,$$

with $\dfrac{\partial z}{\partial x} = \varphi'_x(x, y) = -\dfrac{1 - yz}{1 - xy}$ and $\dfrac{\partial z}{\partial y} = \varphi'_y(x, y) = -\dfrac{1 - xz}{1 - xy}$.

Another method for the determination of z'_x and z'_y :

Differentiate $x + y + z = xyz$ with respect to x knowing that $z = \varphi(x, y)$:

$1 + z'_x = yz + xyz'_x \Longrightarrow z'_x = \dfrac{yz - 1}{1 - xy}$.

Differentiate $x + y + z = xyz$ with respect to y knowing that $z = \varphi(x, y)$:

$1 + z'_y = xz + xyz'_y \Longrightarrow z'_y = \dfrac{xz - 1}{1 - xy}$.

4.5 Exercises

***Exercise** 4.1 Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by*

$$f(x, y) = x^2 + y^2 + xy.$$

1. *Find the number θ involved in the mean value theorem applied to f in a neighborhood of any point $(a, b) \in \mathbb{R}^2$.*
2. *Develop according to the powers of $(x-1)$ and $(y-2)$ the function f using the finite expansion of order 2 in the neighborhood of $(1, 2)$.*

Exercise *4.2 Let $A(1,1)$ and $M(x,y)$. Using the mean value theorem for functions of two variables on the segment $[AM]$, show that $\forall x,y>0,\ \exists\theta\in]0,1[$ such that*

$$\ln\left(\frac{x+y}{2}\right)=\frac{x+y-2}{2+\theta(x+y-2)}.$$

Exercise *4.3 Let $f:\mathbb{R}^2\longrightarrow\mathbb{R}$ defined by*

$$f(x,y)=xe^y+ye^x.$$

By applying the mean value theorem to f, between the points $O(0,0)$ and $A(a,a)$, show that

$$\forall a\neq 0,\ \exists\theta\in]0,1[\ \textit{such that}\ e^{a(1-\theta)}=1+a\ \theta.$$

Exercise *4.4 Find the finite expansion of order 2, in a neighborhood of the point A, of the following functions :*

1. $f(x,y)=\cos(xy)-y^2$ *with* $A(\pi,1)$
2. $f(x,y)=\arctan\dfrac{x}{y}$ *with* $A(-1,1)$
3. $f(x,y,z)=\ln(1+xyz)$ *with* $A(1,0,2)$
4. $f(x,y,z)=x^2yz+y^2z^3$ *with* $A(1,2,-1)$

Exercise *4.5 Given the function $f:\mathbb{R}^2\longrightarrow\mathbb{R}$ defined by*

$$f(x,y)=\ln(1+x^2+y^2).$$

1. *Give the finite expansion of order 2 of f at $(0,0)$.*
2. *Study the position of the representative surface (S) of the function f, in a neighborhood of $(0,0)$, with respect to its tangent plane at the point $(0,0,0)$.*
3. *Using the mean value theorem to f, show that*

$$|f(x,y)|\leq|x|+|y|,\ \textit{for}\ |x|\leq\frac{1}{2}\ \textit{and}\ |y|\leq\frac{1}{2}.$$

Exercise *4.6 Given the function $f:\mathbb{R}^2\longrightarrow\mathbb{R}$ defined by*

$$f(x,y)=\frac{e^{\sin\left(\sqrt{1+x}-\sqrt{1+y}\right)}}{1+x-y}.$$

1. *Without derivation, give the finite expansion of order 2 of f at $(0,0)$.*
2. *Deduce* $\dfrac{\partial f}{\partial x}(0,0)$, $\dfrac{\partial f}{\partial y}(0,0)$, $\dfrac{\partial^2 f}{\partial x^2}(0,0)$, $\dfrac{\partial^2 f}{\partial y^2}(0,0)$ *and* $\dfrac{\partial^2 f}{\partial x\partial y}(0,0)$.
3. *Determine the values of α for which the limit* $\displaystyle\lim_{(x,y)\longrightarrow(0,0)}\frac{f(x,y)-1}{(x^2+y^2)^\alpha}$ *exists.*

Exercise *4.7 Given the following function $f:\mathbb{R}^2\longrightarrow\mathbb{R}$ defined by*

$$f(x,y)=3xy^2-y^3+x^2+y^2-1.$$

1. *Determine the critical points of f on $\mathbb{R}^2$.*
2. *Study the local extremum of f on $\mathbb{R}^2$.*
3. *Does f admit a global extremum on $\mathbb{R}^2$?*

Exercise *4.8 Given the function f such that*

$$f(x,y) = x(\ln x)^2 + xy^2.$$

1. *Determine the domain of definition of f.*
2. *Find the critical points of f. Determine their nature.*
3. *Show that the obtained local minimum is a global minimum.*

Exercise *4.9 In what follows find the extremes of f :*

1. $f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$
2. $f(x,y) = \sin^2 y + (x - \cos y)^2$
3. $f(x,y) = x^4 + y^4 - 4(x-y)^2$
4. $f(x,y) = e^{5-3xy}$
5. $f(x,y) = x^4 + 14x^2y^2 - 7y^4 - 4x + 6$
6. $f(x,y) = e^x \sin y$

Exercise *4.10 Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by*

$$f(x,y) = x^3 + y^3 - 3axy$$

where a is a real parameter.

1. *Discuss, according to the values of a, the existence of the critical points of f.*
2. *Discuss, according to the values of a, the nature of the critical points of f.*

Exercise *4.11 Let $g :]0,\infty[\longrightarrow \mathbb{R}$ given by*

$$g(x) = x^2 + \ln x.$$

1. *Show that there exists $\alpha \in]0,\infty[$ such that $g(\alpha) = 0$.*
2. *Let $D = \{(x,y) \in \mathbb{R}^2 : x > 0\}$ and $f : D \longrightarrow \mathbb{R}$ defined by*

$$f(x,y) = xe^y + y\ln x.$$

Show that f has a critical point P in D that will be determined as a function of α.

3. *Does f it admit a local extremum at P ?*

Exercise *4.12* 1. *Decompose the number* 1000 *in three numbers whose product is maximum.*
2. *Find the minimum distance between the surface $xyz = 1$ and the origin.*

Exercise *4.13 Determine the bounds of*

$$f(x,y) = 3xy - 3x^2 - y^3$$

on $D = \{(x,y) \in \mathbb{R}^2 : |x| \leq 1 \text{ and } |y| \leq 1\}$.

Exercise *4.14 Let the real function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by*

$$f(x,y) = (5 - 2x + y)e^{x^2 - y}.$$

1. *Find the critical point of f.*
2. *Is this point a global extremum of f ?*
3. *Consider the function $\varphi_a(x) = f(x, x^2 - a)$, $a \in \mathbb{R}$. Verify that φ_a has a minimum value m_a.*
4. *Compare m_a with the value of f on its critical point.*

Exercise *4.15 Using the Lagrangian multiplier, find the maximum and the minimum of the function*

$$f(x,y) = e^{xy-y}$$

such that this function takes their values on the unit circle.

Exercise *4.16 Let the functions f and g defined by*

$$f(x,y) = xy \qquad \text{and} \qquad g(x,y) = \frac{1}{x} + \frac{1}{y}.$$

1. *Determine the extrema of f on $\mathbb{R}^2$ subject to the constraint $g(x,y) = 2$.*
2. *Determine the extrema of g on D_g subject to the constraint $f(x,y) = 1$.*

Exercise *4.17 Let la function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by*

$$f(x,y) = x^2 + y^2 - xy.$$

1. *Determine the critical points of f.*
2. *Determine the critical points of f on the unit circle.*
3. *What are the maxima and the minima of f restricted to the disk*

$$D = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\right\}.$$

Exercise *4.18* 1. *Find the domain D of $\mathbb{R}^2$ in which the relation*

$$f(x,y) = \left(x^2 + y^2 - 1\right)^2 - 4x^2 = 0$$

defines an implicit function $y = \varphi(x)$ and find y'.
2. *Same question for $f(x,y) = x \ln y - y \ln x = 0$.*

Exercise *4.19 Let Γ be the subset of $\mathbb{R}^2$ given by*

$$\Gamma = \left\{(x,y) \in \mathbb{R}^2 : x^3 + y^3 = 3xy + 1\right\}.$$

1. *Using the implicit function theorem, show that in the neighborhood of $(0,1)$, Γ is the graph of a function $y = \varphi(x)$ such that $\varphi(0) = 1$.*
2. *Calculate $\varphi'(0)$, $\varphi''(0)$ and $\varphi'''(0)$.*
3. *Give the finite expansion up to order 3 of φ in a neighborhood of 0.*
4. *Deduce the equation of the tangent line (T) to Γ at the point $(0,1)$. Discuss the relative position of (T) with respect to Γ in a neighborhood of $(0,1)$.*

Exercise *4.20* 1. *Show, that in a neighborhood of the point $(0,0)$, the relation*

$$e^{x-y} = 1 + \sin x + y$$

defines an implicit function $y = \varphi(x)$ verifying $e^{x-\varphi(x)} = 1 + \sin x + \varphi(x)$.
2. *Calculate $\varphi(0)$, $\varphi'(0)$ and $\varphi''(0)$.*
3. *Deduce that $\lim\limits_{x \longrightarrow 0} \dfrac{\varphi(x)}{x^2} = \dfrac{1}{4}$.*
4. *Can we apply the implicit function theorem at $(0,0)$, to define $x = \psi(y)$ in a neighborhood of 0 ?*

Exercise *4.21 Consider the function f defined on $\mathbb{R}^2$ by*

$$f(x,y) = x(x^2+y^2) - a(x^2-y^2);\ a>0$$

and denotes by (C) the curve defined by $f(x,y)=0$.
1. *Can we apply the implicit function theorem at the point $(0,0)$?*
2. *Show that the equation $f(x,y)=0$ can be written in the form $x=\varphi(y)$ in the neighborhood of $(a,0)$.*
3. *Determine the finite expansion up to order 2 of φ in a neighborhood of 0.*
4. *Deduce the equation of the tangent line (T) to (C) at the point $(a,0)$ and the position of (C) with respect to (T).*

Exercise *4.22 Find the domain D of $\mathbb{R}^3$ in which the relation*

$$f(x,y,z) = z^3 - xz - y = 0$$

defines an implicit function $z=\varphi(x,y)$ and calculate $\dfrac{\partial^2 z}{\partial x^2}, \dfrac{\partial^2 z}{\partial y^2}$ and $\dfrac{\partial^2 z}{\partial x \partial y}$.

Exercise *4.23* 1. *Prove that the relation*

$$x^2 + xz + e^{xyz} + \sin\frac{\pi y}{2} = 3$$

defines an implicit function $z=\varphi(x,y)$ in a neighborhood of $(1,1)$ such that $\varphi(1,1)=0$.

2. *Calculate $\dfrac{\partial\varphi}{\partial x}(1,1)$ and $\dfrac{\partial\varphi}{\partial y}(1,1)$.*
3. *Determine the equation of the tangent plane (P) to the surface (S) of equation $z=\varphi(x,y)$ at the point $A(1,1,0)$.*
4. *Can we apply the implicit function theorem at $(1,1,0)$, to define $y=\psi(x,z)$ in a neighborhood of $(1,0)$?*

Exercise *4.24 Let $f:\mathbb{R}\longrightarrow\mathbb{R}$.*
1. *Find sufficient condition in order that, in a neighborhood of a point, the relation*

$$y - xz = f(z)$$

may define z as an implicit function of x and y.

2. *Show that the partial derivatives of this function satisfy $\dfrac{\partial z}{\partial x} + z\dfrac{\partial z}{\partial y} = 0$.*

Exercise *4.25 Given the surface $(S): 2x^2+2yz = z^2+1$ and the point $A(1,0,-1)$.*
1. *Determine a unit normal vector to (S) at the point A.*
2. *Determine the equation of the tangent plane (P) at the point A.*
3. *Determine the equation of the normal line (N) to (S) at the point A.*

Exercise *4.26 Two surfaces are orthogonal at a given point if their two tangent planes at this point are perpendicular. Show that the following surfaces*

$$x^2+y^2+z^2 = 50 \qquad \text{and} \qquad x^2+y^2-10z+25=0$$

are orthogonal at the point $A(3,4,5)$.

Exercise *4.27 We say that a function $f : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ is homogenous of degree m if*

$$\forall \lambda > 0, \quad f(\lambda x, \lambda y) = \lambda^m f(x, y).$$

1. *Show that if f is homogenous of degree m and if it has first order partial derivatives then f satisfies Euler's identity*

$$x\frac{\partial f}{\partial x}(x, y) + y\frac{\partial f}{\partial y}(x, y) = mf(x, y).$$

2. *Given the function $f(x, y) = \dfrac{xy}{x^2 + y^2} \cos \dfrac{x - y}{x + y}$ with $x + y \neq 0$.*

Show without calculate the partial derivatives of f that $x\dfrac{\partial f}{\partial x}(x, y) + y\dfrac{\partial f}{\partial y}(x, y) = 0$.

3. *Let $f(x, y) = \ln u(x, y)$ where $u : D \subset \mathbb{R}^2 \longrightarrow]0, \infty[$ is a differentiable function and homogeneous of degree m. Show that*

$$x\frac{\partial f}{\partial x}(x, y) + y\frac{\partial f}{\partial y}(x, y) = m.$$

Chapter 5

Vector-valued functions

5.1 Vector functions of one real variable

Let D be a non empty set of $\mathbb{R}$.

Definition *5.1 A vector function of one real variable $t \in I$ is a mapping f from D into $\mathbb{R}^n$ $(n \geq 2)$, that for every point t of D associates a vector image $f(t) = (f_1(t), \cdots, f_n(t))$ of $\mathbb{R}^n$, and we write*

$$\begin{aligned} f: \quad D &\longrightarrow \mathbb{R}^n \\ t &\longmapsto f(t) = (f_1(t), \cdots, f_n(t)) = \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix} \end{aligned}$$

The real functions $f_i : D \longrightarrow \mathbb{R}$, for $i = 1, \cdots, n$, are called the components of $f = (f_1, \cdots, f_n)$.

Definition *5.2 Let $f : \mathbb{R} \longrightarrow \mathbb{R}^n$ such that $f(t) = (f_1(t), \cdots, f_n(t))$ be a vector function of one real variable. The set for which f is defined is called domain of definition of f, noted D_f, with*

$$D_f = \{t \in \mathbb{R} : f(t) \text{ exists in } \mathbb{R}^n\}.$$

Example : Let $f : \mathbb{R} \longrightarrow \mathbb{R}^3$ such that

$$f(t) = (f_1(t), f_2(t), f_3(t)) = (1 + t^2, \sqrt{2 - t}, \ln t).$$

Its domain is $D_f =]0, 2]$.

Definition *5.3 A parametric curve (C) of parameter $t \in D$ is the image of a certain vector function $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}^n$, given by par*

$$f(t) = (x_1(t), \cdots, x_n(t)) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}.$$

Such a function is called a parameterization of the curve and the $x_i(t)$, for $i = 1, \cdots, n$, are the parametric equations or the parametric coordinates of (C).

• **Position vector in** $\mathbb{R}^3$: Let $M(x,y,z)$ be a point of $\mathbb{R}^3$ such that

$$\overrightarrow{OM} = \overrightarrow{r}(t) = (x(t), y(t), z(t)) = x(t)\overrightarrow{i} + y(t)\overrightarrow{j} + z(t)\overrightarrow{k} \quad \text{with} \quad t \in D \subseteq \mathbb{R}.$$

This vector is called position vector of which O is the origin and M is its extremity.

Examples : (1) The position vector

$$\overrightarrow{r}(t) = (9-4t)\overrightarrow{i} + (6t-4)\overrightarrow{j} + (3t+3)\overrightarrow{k}, \text{ for } t \in \mathbb{R}$$

represents a parameterization of a straight line (D) of the space $(Oxyz)$ passing through the point $A(9,-4,3)$ with director vector $\overrightarrow{V}(-4,6,3)$.

(2) The function $f : \mathbb{R} \longrightarrow \mathbb{R}^2$ such that

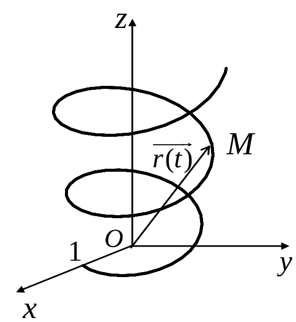
$$f(t) = (x(t), y(t)) = (a + R\cos t, b + R\sin t)$$

is a parameterization of the circle $C(I(a,b), R)$, of the plane (xOy).

(3) The function $f : [0,\infty[\longrightarrow \mathbb{R}^3$ such that

$$f(t) = (x(t), y(t), z(t)) = (\cos t, \sin t, t)$$

represents the following parametric curve (C_f) of the space $(Oxyz)$:



***Definition** 5.4 Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}^n$ such that $f(t) = (f_1(t), \cdots, f_n(t))$ be a vector function of one real variable and $v = (v_1, \cdots, v_n) \in \mathbb{R}^n$. We say that v is a limit of $f(t)$ at the point t_0 if and only if*

$$(\forall \varepsilon > 0)(\exists \delta > 0)(|t - t_0| < \delta \Longrightarrow \|f(t) - v\| < \varepsilon), \quad \forall \text{ the norm } \|\cdot\|.$$

This is equivalent to $\lim\limits_{t \longrightarrow t_0} f_i(t) = v_i,\ \forall i = 1, \cdots, n,$ *and then we write*

$$\lim_{t \longrightarrow t_0} f(t) = \left(\lim_{t \longrightarrow t_0} f_1(t), \cdots, \lim_{t \longrightarrow t_0} f_n(t)\right) = (v_1, \cdots, v_n) = v.$$

Example : Let $f :]0,\infty[\longrightarrow \mathbb{R}^3$ such that

$$f(t) = \left(t\ln t, \frac{\sin t}{t}, \frac{e^t - 1}{t}\right).$$

Then $\lim\limits_{t \longrightarrow 0} f(t) = \left(\lim\limits_{t \longrightarrow 0} t\ln t, \lim\limits_{t \longrightarrow 0} \dfrac{\sin t}{t}, \lim\limits_{t \longrightarrow 0} \dfrac{e^t - 1}{t}\right) = (0,1,1).$

Definition *5.5 Let* $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}^n$ *such that* $f(t) = (f_1(t), \cdots, f_n(t))$ *be a vector function of one real variable and* $t_0 \in D$*. We say that* f *is continuous at the point* t_0 *if and only if*

$$(\forall \varepsilon > 0)\,(\exists \delta > 0)\,(|t - t_0| < \delta \Longrightarrow \|f(t) - f(t_0)\| < \varepsilon)\,, \quad \forall \text{ the norm } \|\cdot\|\,.$$

This is equivalent to saying f_i *is continuous at the point* t_0, $\forall i = 1, \cdots, n$, *with*

$$\lim_{t \longrightarrow t_0} f(t) = \left(\lim_{t \longrightarrow t_0} f_1(t), \cdots, \lim_{t \longrightarrow t_0} f_n(t) \right) = (f_1(t_0), \cdots, f_n(t_0)) = f(t_0).$$

Definition *5.6 Let* $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}^n$ *be a vector function of one real variable. We say that* f *is continuous on* D *if and only if* f *is continuous at each point of* D*.*

5.2 Differentiability of vector functions of one real variable

Let D be a non empty set of $\mathbb{R}$.

Definition *5.7 Let* $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}^n$ *such that* $f(t) = (f_1(t), \cdots, f_n(t))$ *be a vector function of one real variable and* $t_0 \in D$*. We say that* f *is differentiable at the point* t_0 *if and only if* $\lim_{t \longrightarrow t_0} \frac{1}{t - t_0}[f(t) - f(t_0)]$ *exists in* $\mathbb{R}^n$*. This limit, denoted by* $f'(t_0)$*, is called derivative of* f *at* t_0*.*

This is equivalent to saying that f_i *is differentiable at the point* t_0, $\forall i = 1, \cdots, n$, *with*

$$\begin{aligned} f'(t_0) &= \lim_{t \longrightarrow t_0} \frac{1}{t - t_0}[f(t) - f(t_0)] \\ &= \left(\lim_{t \longrightarrow t_0} \frac{f_1(t) - f_1(t_0)}{t - t_0}, \cdots, \lim_{t \longrightarrow t_0} \frac{f_n(t) - f_n(t_0)}{t - t_0} \right) \\ &= (f_1'(t_0), \cdots, f_n'(t_0)). \end{aligned}$$

- **Geometric interpretation of the derivative in $\mathbb{R}^3$:**

  We consider, for $t \in D \subseteq \mathbb{R}$, a parametric curve of the space $(Oxyz)$ given by

$$(C) : \overrightarrow{OM} = \overrightarrow{r}(t) = (x(t), y(t), z(t)) = x(t)\,\overrightarrow{i} + y(t)\,\overrightarrow{j} + z(t)\,\overrightarrow{k}\,.$$

  The vector derivative

$$\frac{d\overrightarrow{r}}{dt}(t_0) = x'(t_0)\,\overrightarrow{i} + y'(t_0)\,\overrightarrow{j} + z'(t_0)\,\overrightarrow{k}$$

  represents a tangent vector to the curve (C_f) at a point $M_0 \in C_f$, of parameter $t_0 \in D$. The vector equation of the tangent line (T) to (C_f) at M_0 is given by

$$(T) : \overrightarrow{OM} = \overrightarrow{r}(s) = \overrightarrow{r}(t_0) + s\frac{d\overrightarrow{r}}{dt}(t_0).$$

Example : Given the parametric curve

$$(C) : \overrightarrow{OM} = \overrightarrow{r}(t) = \left(3t^2 - 7\right)\overrightarrow{i} + \left(t^3 - 3t\right)\overrightarrow{j} + \left(t^3 - 2t\right)\overrightarrow{k}\,, \quad \text{for } t \in \mathbb{R}.$$

Give the parametric equations of the tangent line to (C) at the point $M_0\,(5,2,4)$.

Solution : $t_0 = 2$ and $\dfrac{d\overrightarrow{r}}{dt}(t) = 6t\,\overrightarrow{i} + (3t^2-3)\,\overrightarrow{j} + (3t^2-2)\,\overrightarrow{k} \Longrightarrow \dfrac{d\overrightarrow{r}}{dt}(2) = 12\,\overrightarrow{i} + 9\,\overrightarrow{j} + 10\,\overrightarrow{k}$.

The parametric equations of the tangent line are $\begin{cases} x = 12s+5 \\ y = 9s+2 \\ z = 10s+4 \end{cases}$

***Definition** 5.8 Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}^n$ such that $f(t) = (f_1(t), \cdots, f_n(t))$ be a vector function of one real variable. If $f'(t)$ exists for all t of D, then*

$$\begin{array}{rccl} f' : & D \subset \mathbb{R} & \longrightarrow & \mathbb{R}^n \\ & t & \longmapsto & f'(t) = (f_1'(t), \cdots, f_n'(t)) = \begin{pmatrix} f_1'(t) \\ \vdots \\ f_n'(t) \end{pmatrix} \end{array}$$

defines a vector function. If f' is differentiable at $t_0 \in D$, we say that f is twice differentiable at t_0 and that it has a second order derivative at t_0. It will be noted by

$$f''(t_0) = \lim_{t \longrightarrow t_0} \frac{1}{t-t_0}[f'(t) - f'(t_0)].$$

In the same way we can define derivatives of higher orders.

***Definition** 5.9 Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}^n$ such that $f(t) = (f_1(t), \cdots, f_n(t))$ be a vector function of one real variable and $k \in \mathbb{N}$. We say that f is of class C^k on D if and only if f is $k-$times continuously differentiable on D.*

This is equivalent to saying f_i is $k-$times continuously differentiable (class C^k) on D, $\forall i = 1, \cdots, n$.

Properties : Let $u : D \subset \mathbb{R} \longrightarrow \mathbb{R}^n$ and $v : D \subset \mathbb{R} \longrightarrow \mathbb{R}^n$ be two differentiable vector functions of one real variable on D. Then

(1) $[u(t) + v(t)]' = u'(t) + v'(t), \quad \forall t \in D;$

(2) $[\alpha u(t)]' = \alpha u'(t), \forall t \in D, \ \forall \alpha \in \mathbb{R};$

(3) $[u(t) \cdot v(t)]' = u'(t) \cdot v(t) + u(t) \cdot v'(t), \quad \forall t \in D;$

(4) $[u(t) \wedge v(t)]' = u'(t) \wedge v(t) + u(t) \wedge v'(t), \quad \forall t \in D.$

5.3 Mappings from $\mathbb{R}^n$ into $\mathbb{R}^m$ $(n, m \geq 2)$

***Definition** 5.10 Let $D \subset \mathbb{R}^n$. We say that f is a function from D (domain of f) into $\mathbb{R}^m$ if for every vector point $x = (x_1, \cdots, x_n)$ of D corresponds a vector image*

$$y = (y_1, \cdots, y_m) = f(x) = (f_1(x), \cdots, f_m(x)) = \begin{pmatrix} (f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix} \in \mathbb{R}^m$$

where the components f_j, for $j = 1, \cdots, m$, are functions of n variables from $\mathbb{R}^n$ into $\mathbb{R}$. We denote

$$\begin{array}{rccl} f : & D \subset \mathbb{R}^n & \longrightarrow & \mathbb{R}^m \\ & (x_1, \cdots, x_n) & \longmapsto & f(x_1, \cdots, x_n) = (y_1, \cdots, y_m) \end{array}$$

f is called a vector function of several real variables.

Definition *5.11 Let $f : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$. The set of the images $y = f(x)$ such that $x \in D$ is called the range of f, denoted $f(D) = \{f(x) \in \mathbb{R}^m : x \in D\} \subseteq \mathbb{R}^m$.*

Examples : (1) Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \ / \ f(x, y, z) = (x^2 + y^2 + z^2, x + y + z)$.
We have $D = \mathbb{R}^3$.
For $(x, y, z) \in D$, $f_1(x, y, z) = x^2 + y^2 + z^2 \in \mathbb{R}^+$ and $f_2(x, y, z) = x + y + z \in \mathbb{R}$
then $f(D) = \mathbb{R}^+ \times \mathbb{R}$.
(2) Let $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \ / \ g(x, y) = \left(\dfrac{1}{x+y}, \dfrac{1}{x-y}\right)$.
$D = \{(x, y) \in \mathbb{R}^2 : |x| \neq |y|\}$.
For $(x, y) \in D$, $g_1(x, y) = \dfrac{1}{x+y} \neq 0$ and $g_2(x, y) = \dfrac{1}{x-y} \neq 0$,
then $g(D) = \{(X, Y) \in \mathbb{R}^2 : X \neq 0 \text{ and } Y \neq 0\}$.

Definition *5.12 Let $f : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$. We define the graph of f by*

$$G_f = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : x \in D \text{ and } y = f(x)\}.$$

Definition *5.13 Given the diagram*

$$\begin{array}{ccccc} D \subseteq \mathbb{R}^n & \xrightarrow{f} & D' \subseteq \mathbb{R}^m & \xrightarrow{g} & \mathbb{R}^p \\ x & \longmapsto & y = f(x) & \longmapsto & z = g(y) = g(f(x)) \end{array}$$

where $f(D) \subseteq D'$. We define the composite function of f and g by $g \circ f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^p$ such that

$$(g \circ f)(x_1, \cdots, x_n) = g(f(x_1, \cdots, x_n)) = g(y_1, \cdots, y_m) = (z_1, \cdots, z_p).$$

5.4 Limit and continuity for functions from $\mathbb{R}^n$ into $\mathbb{R}^m$ $(n, m \geq 2)$

Let D be an open of $\mathbb{R}^n$, $a = (a_1, \cdots, a_n) \in D$ or $\overline{D}$ and $f : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that

$$\begin{array}{llcl} f: & D \subset \mathbb{R}^n & \longrightarrow & \mathbb{R}^m \\ & x = (x_1, \cdots, x_n) & \longmapsto & f(x) = (f_1(x), \cdots, f_m(x)) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix} \end{array}$$

Definition *5.14 We say that $L = (L_1, \cdots, L_m) \in \mathbb{R}^m$ is the limit of $f(x)$ when $x = (x_1, \cdots, x_n)$ tends to a if and only if*

$$(\forall \varepsilon > 0)\, (\exists \delta > 0)\, \left(\|x - a\|_{\mathbb{R}^n} < \delta \Longrightarrow \|f(x) - L\|_{\mathbb{R}^m} < \varepsilon\right),$$

whatever the norms $\|\cdot\|_{\mathbb{R}^n}$ and $\|\cdot\|_{\mathbb{R}^m}$.
This is equivalent to $\lim\limits_{x \longrightarrow a} f_i(x) = L_i, \ \forall i = 1, \cdots, m$, and we write

$$\lim_{x \longrightarrow a} f(x) = \left(\lim_{x \longrightarrow a} f_1(x), \cdots, \lim_{x \longrightarrow a} f_m(x)\right) = (L_1, \cdots, L_m) = L.$$

Example : $\lim\limits_{(x,y)\longrightarrow(0,0)}\left(xy\cos\dfrac{1}{xy},\dfrac{\sin xy}{xy}\right)=\left(\lim\limits_{(x,y)\longrightarrow(0,0)}xy\cos\dfrac{1}{xy},\lim\limits_{(x,y)\longrightarrow(0,0)}\dfrac{\sin xy}{xy}\right)=(0,1).$

Definition *5.15 We say that f is continuous at a point $a\in D$ when $f(x)$ has a finite limit at a and that $\lim\limits_{x\longrightarrow a}f(x)=f(a)$, i.e.,*

$$(\forall\varepsilon>0)\,(\exists\delta>0)\,(\|x-a\|_{\mathbb{R}^n}<\delta\Longrightarrow\|f(x)-f(a)\|_{\mathbb{R}^m}<\varepsilon)\,.$$

Proposition *5.1 f is continuous at a point $a\in D$ if and only if $f_1,\cdots,f_m$ are continuous at a.*

Proof : $\Longrightarrow$) f is continuous at the point a then
$(\forall\varepsilon>0)\,(\exists\delta>0)\,(\|x-a\|_{\mathbb{R}^n}<\delta\Longrightarrow\|f(x)-f(a)\|_{\mathbb{R}^m}<\varepsilon)$.
We have $\forall i=1,\cdots,m,\ \ |f_i(x)-f_i(a)|\leq\|f(x)-f(a)\|_{\mathbb{R}^m}<\varepsilon$, $\forall$ the norm $\|\cdot\|_{\mathbb{R}^m}$
$\Longleftarrow$) For all $i=1,\cdots,m$, f_i is continuous at the point a then
$(\forall\varepsilon>0)\,(\exists\delta_i>0)\,(\|x-a\|_{\mathbb{R}^n}<\delta_i\Longrightarrow|f_i(x)-f_i(a)|<\varepsilon)$.
Let $\delta=\inf(\delta_1,\cdots,\delta_m)$, then $\|x-a\|_{\mathbb{R}^n}<\delta\Longrightarrow\|f(x)-f(a)\|_{\mathbb{R}^m}<\varepsilon$.
Therefore f is continuous at the point a.

Proposition *5.2 f is continuous on D if and only if $f_1,\cdots,f_m$ are continuous on D.*

Example : Let $f:\mathbb{R}^2\longrightarrow\mathbb{R}^2$ defined by

$$f(x,y)=(f_1(x,y),f_2(x,y))=(\ln(1+x^2+y^2),x\arctan y).$$

Show that f is continuous at each point of $\mathbb{R}^2$.
Solution : $1+x^2+y^2>0$, $\forall\,(x,y)\in\mathbb{R}^2\Longrightarrow f_1\ /\ f_1(x,y)=\ln(1+x^2+y^2)$ is continuous in $\mathbb{R}^2$
and $f_2\ /\ \ f_2(x,y)=x\arctan y$ it is too, then $f=(f_1,f_2)$ is continuous in $\mathbb{R}^2$.

Theorem *5.1 Given the composition $D\subseteq\mathbb{R}^n\ \xrightarrow{f}\ D'\subseteq\mathbb{R}^m\ \xrightarrow{g}\ \mathbb{R}^p$. If f is continuous at the point $a\in D$ and g is continuous at the point $f(a)\in D'$, then $g\circ f:D\subseteq\mathbb{R}^n\longrightarrow\mathbb{R}^p$ is continuous at a.*

Proof : f is continuous at the point a then
$(\forall\varepsilon>0)\,(\exists\delta>0)\,(\|x-a\|_{\mathbb{R}^n}<\delta\Longrightarrow\|f(x)-f(a)\|_{\mathbb{R}^m}<\varepsilon)$.
g is continuous at the point $f(a)$ then
$(\forall\varepsilon'>0)\,(\exists\delta'>0)\,(\|f(x)-f(a)\|_{\mathbb{R}^n}<\delta'\Longrightarrow\|g(f(x))-g(f(a))\|_{\mathbb{R}^p}<\varepsilon')$.
Let $\varepsilon=\varepsilon'=\delta'$, then $\|x-a\|_{\mathbb{R}^n}<\delta\Longrightarrow\|(g\circ f)(x)-(g\circ f)(a)\|_{\mathbb{R}^p}<\varepsilon$.

5.5 Vector partial derivatives and Jacobian matrix

Consider, for $n,m\geq 2$, the function

$$\begin{array}{llll} f: & D\subset\mathbb{R}^n & \longrightarrow & \mathbb{R}^m \\ & x=(x_1,\cdots,x_n) & \longmapsto & f(x)=(f_1(x),\cdots,f_m(x))=\begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x)\end{pmatrix}\end{array}$$

where $f_1,\cdots,f_m$ are functions of the variables $x_1,\cdots,x_n$.

Definition *5.16 The first vector partial derivative of f with respect to the j^{th} variable x_j at a point $x=(x_1,\cdots,x_n)\in D$ is given by*

$$\frac{\partial f}{\partial x_j}(x)=\left(\frac{\partial f_1}{\partial x_j}(x),\cdots,\frac{\partial f_m}{\partial x_j}(x)\right)=\begin{pmatrix}\dfrac{\partial f_1}{\partial x_j}(x)\\ \vdots\\ \dfrac{\partial f_m}{\partial x_j}(x)\end{pmatrix}\in\mathbb{R}^m.$$

Example : Let $f(x,y)=\begin{pmatrix}x^2y\\ xy^2\\ \ln(x+y)\end{pmatrix}$, with $x+y>0$. Calculate $\dfrac{\partial f}{\partial x}(2,-1)$ and $\dfrac{\partial f}{\partial y}(2,-1)$.

Solution : $\dfrac{\partial f}{\partial x}(2,-1)=\begin{pmatrix}2xy\\ y^2\\ \dfrac{1}{x+y}\end{pmatrix}_{(2,-1)}=\begin{pmatrix}-4\\ 1\\ 1\end{pmatrix}$ and $\dfrac{\partial f}{\partial y}(2,-1)=\begin{pmatrix}x^2\\ 2xy\\ \dfrac{1}{x+y}\end{pmatrix}_{(2,-1)}=\begin{pmatrix}4\\ -4\\ 1\end{pmatrix}$.

Note : In the same way we can define the second vector partial derivatives.

From the vector partial derivatives $\dfrac{\partial f}{\partial x_j}(a)=\begin{pmatrix}\dfrac{\partial f_1}{\partial x_j}(a)\\ \vdots\\ \dfrac{\partial f_m}{\partial x_j}(a)\end{pmatrix}$, for $j=1,\cdots,n$, we can define a matrix called Jacobian matrix of f at a, denoted $M_f(a)$. It is given by

$$M_f(a)=\begin{pmatrix}\dfrac{\partial f_1}{\partial x_1}(a) & \cdots & \dfrac{\partial f_1}{\partial x_n}(a)\\ \vdots & \ddots & \vdots\\ \dfrac{\partial f_m}{\partial x_1}(a) & \cdots & \dfrac{\partial f_m}{\partial x_n}(a)\end{pmatrix}=\left(\frac{\partial f_1}{\partial x_1}(a)\right)_{\substack{i=1,\cdots,m\\ j=1,\cdots,n}}.$$

For $m=n$, the determinant of $M_f(a)$ is called the Jacobian of f at a. It is given by

$$J_f(a)=\det(M_f(a))=\begin{vmatrix}\dfrac{\partial f_1}{\partial x_1}(a) & \cdots & \dfrac{\partial f_1}{\partial x_n}(a)\\ \vdots & \ddots & \vdots\\ \dfrac{\partial f_n}{\partial x_1}(a) & \cdots & \dfrac{\partial f_n}{\partial x_n}(a)\end{vmatrix}=\frac{D(f_1,\cdots,f_n)}{D(x_1,\cdots,x_n)}(a).$$

Example : Let $f:\mathbb{R}^2\longrightarrow\mathbb{R}^2$ defined by $f(x,y)=(u,v)=(xy,x^2+y^2)$. Calculate $J_f(x,y)$.

Solution : $J_f(x,y)=\dfrac{D(u,v)}{D(x,y)}=\begin{vmatrix}\dfrac{\partial u}{\partial x}(x,y) & \dfrac{\partial u}{\partial y}(x,y)\\ \dfrac{\partial v}{\partial x}(x,y) & \dfrac{\partial v}{\partial y}(x,y)\end{vmatrix}=\begin{vmatrix}y & x\\ 2x & 2y\end{vmatrix}=2(y^2-x^2).$

Definition 5.17 *We say that f is of class C^k on D ($k \in \mathbb{N}$) if, f and all its partial derivatives up to order k are continuous on D.*

This is equivalent to say that $f_1, \cdots, f_m$ are of class C^k on D.

5.6 Differentiability of functions from $\mathbb{R}^n$ into $\mathbb{R}^m$

Consider, for $n, m \geq 2$, the function

$$\begin{array}{llll} f: & D \subset \mathbb{R}^n & \longrightarrow & \mathbb{R}^m \\ & x = (x_1, \cdots, x_n) & \longmapsto & f(x) = (f_1(x), \cdots, f_m(x)) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix} \end{array}$$

where $f_1, \cdots, f_m$ are functions of the variables $x_1, \cdots, x_n$.

Definition 5.18 *We say that f is differentiable at a point $a = (a_1, \cdots, a_n) \in D$ if there exists a linear mapping $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and a mapping $E : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that*

$$f(a+h) - f(a) = L(h) + \|h\| E(h)$$

with $E(h) \longrightarrow 0_{\mathbb{R}^m}$ when $h = (h_1, \cdots, h_n) \longrightarrow 0_{\mathbb{R}^n}$. In other way

$$\lim_{h \longrightarrow 0} E(h) = \lim_{h \longrightarrow 0} \frac{f(a+h) - f(a) - L(h)}{\|h\|} = 0.$$

The norm $\|\cdot\|$ is one of the three usual norms of $\mathbb{R}^n$.

We can deduce that if f is differentiable at point a, then $M_f(a)$ exists and the mapping L is unique with

$$L(h) = M_f(a)h.$$

Example : Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $f(x, y) = (x + y, x - y)$.

1. Calculate $M_f(a, b)$, for $(a, b) \in \mathbb{R}^2$.
2. Show that f is differentiable at the point (a, b).

Solution :

1. $M_f(a,b) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(a,b) & \frac{\partial f_1}{\partial y}(a,b) \\ \frac{\partial f_2}{\partial x}(a,b) & \frac{\partial f_2}{\partial y}(a,b) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

2. Let $E : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that
$f(a+h, b+k) - f(a,b) = M_f(a,b) \begin{pmatrix} h \\ k \end{pmatrix} + \|(h,k)\| E(h,k)$
$\Longrightarrow E(h,k) = \dfrac{1}{\|(h,k)\|} \left[\begin{pmatrix} a+h+b+k \\ a+h-b-k \end{pmatrix} - \begin{pmatrix} a+b \\ a-b \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} \right] = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
$\Longrightarrow \lim_{(h,k) \longrightarrow (0,0)} \|E(h,k)\| = 0$. Then f is differentiable at (a, b).

Definition *5.19 We call differential of f at a point a, denoted df_a or $df(a)$, the following linear mapping*

$$\begin{array}{rccl} df(a): & \mathbb{R}^n & \longrightarrow & \mathbb{R}^m \\ & h & \longmapsto & df\,(a)\,(h) = M_f\,(a)\,h \end{array}$$

Example : Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined by $f(x, y) = (x^2 + e^y, x + y \sin z)$. Calculate its differential at the point $P\,(1, 1, \pi)$.

Solution : $M_f\,(1, 1, \pi) = \begin{pmatrix} 2x & e^y & 0 \\ 1 & \sin z & y \cos z \end{pmatrix}_{(1,1,\pi)} = \begin{pmatrix} 2 & e & 0 \\ 1 & 0 & -1 \end{pmatrix}.$

Let $h = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} \in \mathbb{R}^3 \Longrightarrow df\,(1, 1, \pi)\,(h) = \begin{pmatrix} 2 & e & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = \begin{pmatrix} 2h_1 + h_2 e \\ h_1 - h_3 \end{pmatrix}.$

Proposition *5.3 f is differentiable at a point $a = (a_1, \cdots, a_n) \in D$ if and only if $f_1, \cdots, f_m$ are differentiable at a, with $df\,(a) = (df_1\,(a)\,, \cdots, df_n\,(a))$.*

Proof : f is differentiable at $a \Longleftrightarrow \lim\limits_{h \longrightarrow 0} \dfrac{f(a + h) - f(a) - M_f(a)h}{\|h\|} = 0$

$$\Longleftrightarrow \lim_{h \longrightarrow 0} \frac{f_i(a + h) - f_i(a) - \sum\limits_{j=1}^{n} \dfrac{\partial f_i}{\partial x_j}(a)h_j}{\|h\|} = 0,\ \forall i = 1, \cdots, m$$

$\Longleftrightarrow f_i$ is differentiable at a, $\forall i = 1, \cdots, m$.

Proposition *5.4 f is differentiable on D if and only if $f_1, \cdots, f_m$ are differentiable on D.*

5.7 Differential of a composite function

Let the composition $D \subset \mathbb{R}^n \xrightarrow{f} D' \subset \mathbb{R}^m \xrightarrow{g} \mathbb{R}^p$. If f is differentiable at a point $a \in D$ and g is differentiable at the point $b = f\,(a) \in D'$, then $h = g \circ f : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}^p$ is differentiable at a, and we have

$$d\,(g \circ f)\,(a) = d(f\,(a)) \circ dg(a),$$

which is equivalent to

$$M_{g \circ f}(a) = M_g(f\,(a))M_f(a),$$

such that

$$\begin{aligned} M_g(b)M_f(a) &= \begin{pmatrix} \dfrac{\partial g_1}{\partial y_1}(b) & \cdots & \dfrac{\partial g_1}{\partial y_m}(b) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial g_p}{\partial y_1}(b) & \cdots & \dfrac{\partial g_p}{\partial y_m}(b) \end{pmatrix} \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1}(a) & \cdots & \dfrac{\partial f_1}{\partial x_n}(a) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial x_1}(a) & \cdots & \dfrac{\partial f_m}{\partial x_n}(a) \end{pmatrix} \\ &= \left(\frac{\partial g_i}{\partial y_k}(b) \right)_{\substack{1 \le i \le, p \\ 1 \le k \le m}} \left(\frac{\partial f_k}{\partial x_j}(a) \right)_{\substack{1 \le k \le m \\ 1 \le j \le n}} = \left(\frac{\partial h_i}{\partial x_j}(a) \right)_{\substack{1 \le i \le, p \\ 1 \le j \le n}} = M_h(a) \end{aligned}$$

with $\dfrac{\partial h_i}{\partial x_j}(a) = \dfrac{\partial g_i}{\partial y_1}(b) \dfrac{\partial f_1}{\partial x_j}(a) + \cdots + \dfrac{\partial g_i}{\partial y_m}(b) \dfrac{\partial f_m}{\partial x_j}(a) = \sum\limits_{k=1}^{m} \dfrac{\partial g_i}{\partial y_k}(b) \dfrac{\partial f_k}{\partial x_j}(a).$

Proposition *5.5 Given the differentiable transformation $(T(D) \subset D')$*

$$\begin{array}{lcl} D \subset \mathbb{R}^n & \xrightarrow{T} & D' \subset \mathbb{R}^n \\ x = (x_1, \cdots, x_n) & \longmapsto & u = (u_1, \cdots, u_n) \end{array}$$

Let $f : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ and $F : D' \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $f = F \circ T$. If F is differentiable on D' then f is differentiable on D, and we have

$$\frac{\partial f}{\partial x_j} = \frac{\partial F}{\partial u_1}\frac{\partial u_1}{\partial x_j} + \cdots + \frac{\partial F}{\partial u_n}\frac{\partial u_n}{\partial x_j}, \text{ for } j = 1, \cdots, n$$

Proof : From what precedes

$M_f(x) = M_{F\circ T}(x) = M_F(T(x))M_T(x) = M_F(u)M_T(x)$, for $x = (x_1, \cdots, x_n) \in D$. Then

$$\begin{pmatrix} \frac{\partial f}{\partial x_1}(x) & \cdots & \frac{\partial f}{\partial x_n}(x) \end{pmatrix} = \begin{pmatrix} \frac{\partial F}{\partial u_1}(u) & \cdots & \frac{\partial F}{\partial u_n}(u) \end{pmatrix} \begin{pmatrix} \frac{\partial u_1}{\partial x_1}(x) & \cdots & \frac{\partial u_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial u_n}{\partial x_1}(x) & \cdots & \frac{\partial u_n}{\partial x_n}(x) \end{pmatrix}$$

hence

$$\begin{cases} \frac{\partial f}{\partial x_1} = \frac{\partial F}{\partial u_1}\frac{\partial u_1}{\partial x_1} + \cdots + \frac{\partial F}{\partial u_n}\frac{\partial u_n}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} = \frac{\partial F}{\partial u_1}\frac{\partial u_1}{\partial x_n} + \cdots + \frac{\partial F}{\partial u_n}\frac{\partial u_n}{\partial x_n} \end{cases}$$

Proposition *5.6 Let D be an open of $\mathbb{R}^n$ and $f : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a differentiable and invertible mapping in D. If $J_f(x) \neq 0$, $\forall x \in D$, then f^{-1} is differentiable in $f(D)$, with*

$$df^{-1}(f(x)) = [df(x)]^{-1},$$

which is equivalent to

$$M_{f^{-1}}(f(x)) = [M_f(x)]^{-1}.$$

Corollary *5.1 Let D be on open of $\mathbb{R}^n$ and $f : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a differentiable and invertible mapping in D. If $J_f(x) \neq 0$, $\forall x \in D$, then*

$$J_{f^{-1}}(f(x)) = \frac{1}{J_f(x)}.$$

Proof : From what precedes we have

$M_{f^{-1}}(f(x))M_f(x) = M_{f^{-1}\circ f}(x) = M_{Id}(x) = I_n$

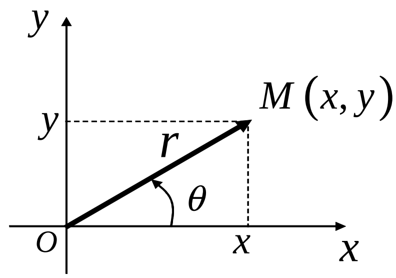
$\Longrightarrow J_{f^{-1}}(f(x))J_f(x) = 1 \Longrightarrow J_{f^{-1}}(f(x)) = \dfrac{1}{J_f(x)}.$

5.8 Coordinate Systems

- **Polar coordinates in $\mathbb{R}^2$:**

In the $xy-plane$: $\mathbb{R}^2$, we consider a point $M(x,y)\neq(0,0)$. Let

$$r=\left\|\overrightarrow{OM}\right\|=\sqrt{x^2+y^2} \qquad \text{and} \qquad \theta=\left(\overrightarrow{Ox},\overrightarrow{OM}\right).$$



The couple $(r,\theta)\in]0,+\infty[\times[0,2\pi[$ defines the polar coordinates of M , given by the following bijective mapping :

$$\begin{array}{rccc}\phi: &]0,+\infty[\times[0,2\pi[& \longrightarrow & \mathbb{R}^2-\{(0,0)\}\\ & (r,\theta) & \longmapsto & (x,y)\end{array}$$

with $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$. The Jacobian of this function is

$$J_\phi(r,\theta)=\frac{D(x,y)}{D(r,\theta)}=\begin{vmatrix}\frac{\partial x}{\partial r}(r,\theta) & \frac{\partial x}{\partial \theta}(r,\theta)\\ \frac{\partial y}{\partial r}(r,\theta) & \frac{\partial y}{\partial \theta}(r,\theta)\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r\cos^2\theta+r\sin^2\theta=r\neq 0.$$

Its reciprocal mapping is given by

$$\begin{array}{rccc}\phi^{-1}: & \mathbb{R}^2-\{(0,0)\} & \longrightarrow &]0,+\infty[\times[0,2\pi[\\ & (x,y) & \longmapsto & (r,\theta)\end{array}$$

with $\begin{cases} r=\sqrt{x^2+y^2} \\ \theta=\arctan\frac{y}{x} \end{cases}$ and $J_{\phi^{-1}}(x,y)=\dfrac{1}{J_\phi(r,\theta)}=\dfrac{1}{r}=\dfrac{1}{\sqrt{x^2+y^2}}$.

Note : The mapping ϕ is called transformation or transition function of the polar coordinates to the Cartesian coordinates.

Example : Give the polar equation of the curve with Cartesian equation :

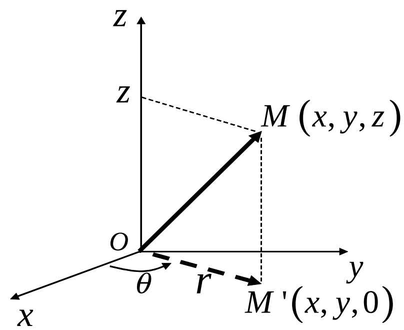
$$x^2+y^2=\sqrt{x^2+y^2}-y.$$

Solution : Let $x=r\cos\theta$ and $y=r\sin\theta$ with $x^2+y^2=r^2\Longrightarrow r^2=r-r\sin\theta\Longrightarrow r=1-\sin\theta$.

- **Cylindrical coordinates in $\mathbb{R}^3$:**

In the $xyz-space:\mathbb{R}^3$, we consider a point $M(x,y,z)\notin z'z$.
Let $M'(x,y,0)=\underset{(xOy)}{\Pr}M\ (x,y,z)\neq(0,0,0)$,

$$r=\left\|\overrightarrow{OM'}\right\|=\sqrt{x^2+y^2}\qquad\text{and}\qquad\theta=\left(\overrightarrow{Ox},\overrightarrow{OM'}\right).$$



The cylindrical coordinates of M are defined by the following bijective mapping :

$$\begin{array}{rccc}\phi: &]0,+\infty[\times[0,2\pi[\times\mathbb{R} & \longrightarrow & \mathbb{R}^3-\{z'z\}\\ & (r,\theta,z) & \longmapsto & (x,y,z)\end{array}$$

with $\begin{cases}x=r\cos\theta\\ y=r\sin\theta\\ z=z\end{cases}$. The Jacobian of this function is

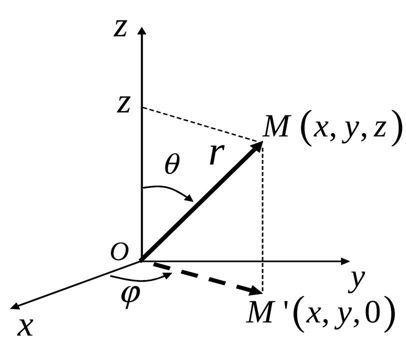
$$J_\phi(r,\theta,z)=\frac{D(x,y,z)}{D(r,\theta,z)}=\begin{vmatrix}\frac{\partial x}{\partial r}(r,\theta,z) & (r,\theta,z) & \frac{\partial x}{\partial z}(r,\theta,z)\\ \frac{\partial y}{\partial r}(r,\theta,z) & \frac{\partial y}{\partial \theta}(r,\theta,z) & \frac{\partial y}{\partial z}(r,\theta,z)\\ \frac{\partial z}{\partial r}(r,\theta,z) & \frac{\partial z}{\partial \theta}(r,\theta,z) & \frac{\partial z}{\partial z}(r,\theta,z)\end{vmatrix}$$

$$=\begin{vmatrix}\cos\theta & -r\sin\theta & 0\\ \sin\theta & r\cos\theta & 0\\ 0 & 0 & 1\end{vmatrix}=r\cos^2\theta+r\sin^2\theta=r\neq0.$$

- **Spherical coordinates in** $\mathbb{R}^3$:

In the $xyz-space:\mathbb{R}^3$, we consider a point $M(x,y,z)\notin z'z$.
Let $M'(x,y,0)=\underset{(xOy)}{\Pr}M\ (x,y,z)\neq(0,0,0)$,

$$r=\left\|\overrightarrow{OM}\right\|=\sqrt{x^2+y^2+z^2},\qquad\varphi=\left(\overrightarrow{Ox},\overrightarrow{OM'}\right)\qquad\text{and}\qquad\theta=\left(\overrightarrow{Oz},\overrightarrow{OM}\right).$$



The spherical coordinates of M are defined by the following bijective mapping :

$$\begin{array}{rcl} \phi : \ ]0,+\infty[\times]0,\pi[\times[0,2\pi[& \longrightarrow & \mathbb{R}^3 - \{z'z\} \\ (r,\theta,\varphi) & \longmapsto & (x,y,z) \end{array}$$

with $\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$. The Jacobian of this function is

$$\begin{aligned} J_\phi(r,\theta,\varphi) = \frac{D(x,y,z)}{D(r,\theta,\varphi)} &= \begin{vmatrix} \frac{\partial x}{\partial r}(r,\theta,\varphi) & \frac{\partial x}{\partial \theta}(r,\theta,\varphi) & \frac{\partial x}{\partial \varphi}(r,\theta,\varphi) \\ \frac{\partial y}{\partial r}(r,\theta,\varphi) & \frac{\partial y}{\partial \theta}(r,\theta,\varphi) & \frac{\partial y}{\partial \varphi}(r,\theta,\varphi) \\ \frac{\partial z}{\partial r}(r,\theta,\varphi) & \frac{\partial z}{\partial \theta}(r,\theta,\varphi) & \frac{\partial z}{\partial \varphi}(r,\theta,\varphi) \end{vmatrix} \\ &= \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} \\ &= r^2\cos^2\theta\cos^2\varphi\sin\theta + r^2\cos^2\theta\sin\theta\sin^2\varphi + r^2\cos^2\varphi\sin^3\theta + r^2\sin^3\theta\sin^2\varphi \\ &= r^2\sin\theta \neq 0. \end{aligned}$$

Example : Give the spherical equation of the sphere of equation :

$$x^2 + y^2 + z^2 = 2z.$$

Solution : Let $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$ and $z = r\cos\theta$ with $x^2+y^2+z^2 = r^2$ $\Longrightarrow r^2 = 2r\cos\theta \Longrightarrow r = 2\cos\theta$.

5.9 Exercises

***Exercise** 5.1 Given the function $f : \mathbb{R} \longrightarrow \mathbb{R}^2$ defined by*

$$f(t) = (x(t), y(t)) = \begin{cases} \left(\dfrac{t-\sqrt{2-t}}{t-1}, \dfrac{t\ln t}{t-1}\right) & \text{if } \ t \neq 1 \\ \left(\dfrac{3}{2}, 1\right) & \text{if } \ t = 1 \end{cases}$$

1. *Find its domain of definition.*
2. *Show that f is continuous at $t = 1$?*
3. *Show that f is differentiable at $t = 1$?*

***Exercise** 5.2 Let the function $f : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}^3$ defined by*

$$f(t) = (x(t), y(t), z(t)) = \left(\frac{\ln(1+t^2)}{t}, \frac{\sqrt{1+t^2}-1}{t}, \frac{\tan t - t}{t^2}\right).$$

1. *Show that the function f is extendible by continuity on $\mathbb{R}$ and give its extension g.*
2. *Is g differentiable at 0 ?*
3. *Is g of class C^1 on $\mathbb{R}$?*

Exercise *5.3 Determine the domain and the range of each of the following functions :*

1. $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *with* $f(x, y) = \left(\sqrt{4 - x^2 - y^2}, x + y\right)$
2. $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ *with* $f(x, y) = \left(\dfrac{x}{\sqrt{x^2 + y^2}}, \dfrac{y}{\sqrt{x^2 + y^2}}, 1\right)$
3. $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ *with* $f(x, y, z) = \left(\sqrt{25 - x^2 - y^2 - z^2}, \sqrt{z - 2}\right)$

Exercise *5.4 Study the existence of the following limits :*

1. $\lim\limits_{(x,y)\longrightarrow(0,0)} \left(\dfrac{xy}{\sqrt{x^2 + y^2}}, \dfrac{x + y}{\sqrt{x^2 + y^2}}\right)$
2. $\lim\limits_{(x,y)\longrightarrow(0,0)} \left(\dfrac{\ln(1 + |xy|)}{|x| + |y|}, \dfrac{xy}{x^2 + y^2}\right)$
3. $\lim\limits_{(x,y)\longrightarrow(0,0)} \left((x + y)\sin\dfrac{1}{x^2 + y^2}, \dfrac{\sin(x^2 + y^2)}{x^2 + y^2}\right)$

Exercise *5.5 Let the function* $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *defined by*

$$f(x, y) = \begin{cases} \left(\dfrac{xy - y^2}{\sqrt{x^2 + y^2}}, \dfrac{\sin x^2}{x^2 + y^2}\right) & \textit{if} \quad (x, y) \neq (0, 0) \\ (0, 0) & \textit{if} \quad (x, y) = (0, 0) \end{cases}$$

Is f *continuous on* $\mathbb{R}^2$ *?*

Exercise *5.6 Find the vector partial derivatives of order* 1 *and* 2 *of* f *at the point* A :

1. *Let* $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *with* $f(x, y) = (x\cos y, x\sin y)$ *at the point* $A(1, \pi)$.
2. *Let* $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ *with* $f(x, y, z) = (x^2 + y^2 + z^2, 2xyz, x + y + z)$ *at the point* $A(1, 2, -1)$.

Exercise *5.7 Find the Jacobian of the transformation* $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *defined by*

$$T(x, y) = (u(x, y), v(x, y)) = \left(\frac{x + y}{1 - xy}, \arctan x + \arctan y\right), \ \textit{for } xy \neq 1.$$

Exercise *5.8 Let* $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *be the function defined by*

$$f(x, y) = (x^2 - y^2 - 2x\ln y, \sin x^2 + 2xy).$$

1. *Show that* f *is of class* C^1 *in* $\mathbb{R}^2$.
2. *Determine the Jacobian matrix* M_f *of* f *at the point* $(0, 1)$.
3. *Deduce the differential* $df(0, 1)$.
4. *The directional derivative of a function* $f = (f_1, \cdots, f_m)$ *at a point* a *in a direction* $u = (u_1, \cdots, u_m)$ *is given by*

$$D_u f(a) = (D_u f_1(a), \cdots, D_u f_m(a)) = M_f(a)u.$$

Calculate $D_u f(0, 1)$ *in the direction of* $u = \left(\dfrac{\sqrt{3}}{2}, -\dfrac{1}{2}\right)$.

Exercise *5.9 Consider the two functions* $f:\mathbb{R}^3 \longrightarrow \mathbb{R}^2$ *and* $g:\mathbb{R}^2 \longrightarrow \mathbb{R}^3$ *defined by*

$$f(x,y,z) = (xyz, x+y+z) \qquad \text{and} \qquad g(u,v) = (u^2+v^2, u^2-v^2, uv).$$

1. *Prove that* f *is differentiable at every point* $(x,y,z) \in \mathbb{R}^3$ *and calculate its Jacobian matrix.*
2. *Prove that* g *is differentiable at every point* $(u,v) \in \mathbb{R}^2$ *and calculate its Jacobian matrix.*
3. *Give the Jacobian matrix of* $g \circ f$.
4. *Deduce the differential* $d\,(g \circ f)\,(1,-1,1)$.

Exercise *5.10 If* $u = f(x,y)$, $x = r\cos\theta$ *and* $y = r\sin\theta$, *show that*

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2.$$

Exercise *5.11 Let* $U = \{(x,y) \in \mathbb{R}^2 : x > 0\}$ *and* $f : U \longrightarrow \mathbb{R}$ *be a differentiable function on* U *verifying*

$$x\frac{\partial f}{\partial x}(x,y) + y\frac{\partial f}{\partial y}(x,y) = 1.$$

Let F *be the function such that* $F(r,\theta) = f(r\cos\theta, r\sin\theta)$, *for* $r > 0$ *and* $\theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$.

1. *Find* $\frac{\partial F}{\partial r}$ *and* $\frac{\partial F}{\partial \theta}$ *in terms of* $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$.
2. *Show that* $r\frac{\partial F}{\partial r}(r,\theta) = 1$ *and deduce that* $F(r,\theta) = \ln r + \varphi(\theta)$.
3. *We define* ψ *by* $\psi(\tan\theta) = \varphi(\theta)$. *Show that* $f(x,y) = \frac{1}{2}\ln(x^2+y^2) + \psi\left(\frac{y}{x}\right)$.

Exercise *5.12 Let* $U = \{(x,y) \in \mathbb{R}^2 : x > 0 \text{ and } y > 0\}$ *and* $f : U \longrightarrow \mathbb{R}$ *be a differentiable function on* U *satisfying the first order differential equation*

$$x\frac{\partial f}{\partial x}(x,y) + y\frac{\partial f}{\partial y}(x,y) = xyf(x,y).$$

Consider the transformation $T : U \longrightarrow U$ *defined by* $T(x,y) = (u = xy, v = \frac{x}{y})$. *Let* F *be the function such that* $f(x,y) = (F \circ T)(x,y) = F(u,v)$, *for* $(u,v) \in U$.

1. *Show that* $J_T(x,y) \neq 0$, *for all* $(x,y) \in U$.
2. *Find* $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$ *in terms of* $\frac{\partial F}{\partial u}$ *and* $\frac{\partial F}{\partial v}$.
3. *Show that* $2\frac{\partial F}{\partial u}(u,v) = F(u,v)$.
4. *Determine* $F(u,v)$ *and deduce* $f(x,y)$ *if* $f(x,1) = e^x$.

Exercise *5.13 Let* $U = \{(x,y) \in \mathbb{R}^2 : x > 0 \text{ and } y > 0\}$ *and* $f : U \longrightarrow \mathbb{R}$ *be a function of class* C^2 *on* U *verifying the second order differential equation*

$$x^2\frac{\partial^2 f}{\partial x^2}(x,y) = y^2\frac{\partial^2 f}{\partial y^2}(x,y).$$

Let F *be the function such that* $f(x,y) = (F \circ T)(x,y) = F(u,v)$ *with* $u = xy$ *and* $v = \frac{x}{y}$.

1. *Find* $\dfrac{\partial^2 f}{\partial x^2}$ *and* $\dfrac{\partial^2 f}{\partial y^2}$ *in terms of* $\dfrac{\partial^2 F}{\partial u^2}$, $\dfrac{\partial^2 F}{\partial v^2}$ *and* $\dfrac{\partial^2 F}{\partial u \partial v}$.
2. *Show that* $2u\dfrac{\partial^2 F}{\partial u \partial v}(u,v) = \dfrac{\partial F}{\partial v}(u,v)$.
3. *Determine* $F(u,v)$ *and deduce* $f(x,y)$.

Chapter 6

Scalar and vector fields

6.1 Recalls

- **Vector :** A vector $\overrightarrow{H}$ of the space $\mathbb{R}^3$ is written as

$$\overrightarrow{H} = X\overrightarrow{i} + Y\overrightarrow{j} + Z\overrightarrow{k} \qquad \text{or} \qquad \overrightarrow{H}(X,Y,Z),$$

where the components $X,\ Y,\ Z$ are the orthogonal projections of $\overrightarrow{H}$ on the coordinates axes of the orthonormal system $\left(O, \overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}\right)$. Its module is

$$\left\|\overrightarrow{H}\right\| = \sqrt{X^2 + Y^2 + Z^2}.$$

Properties : Let $\overrightarrow{H}(X,Y,Z)$ and $\overrightarrow{V}(P,Q,R)$ be two vectors of $\mathbb{R}^3$, then
(1) $\overrightarrow{H} + \overrightarrow{V} = (X+P)\overrightarrow{i} + (Y+Q)\overrightarrow{j} + (Z+R)\overrightarrow{k}$;
(2) $\alpha\overrightarrow{H} = \alpha X\overrightarrow{i} + \alpha Y\overrightarrow{j} + \alpha Z\overrightarrow{k}$, $\forall\alpha \in \mathbb{R}$;
(3) $\overrightarrow{H} + \overrightarrow{V} = \overrightarrow{V} + \overrightarrow{H}$;
(4) $\left(\overrightarrow{H} + \overrightarrow{V}\right) + \overrightarrow{W} = \overrightarrow{H} + \left(\overrightarrow{V} + \overrightarrow{W}\right)$, for all vector $\overrightarrow{W}$ of $\mathbb{R}^3$.

- **Scalar product :** Let $\overrightarrow{H}(X,Y,Z)$ and $\overrightarrow{V}(P,Q,R)$ be two vectors of $\mathbb{R}^3$.

We define the scalar product of $\overrightarrow{H}$ and $\overrightarrow{V}$ by

$$\overrightarrow{H} \cdot \overrightarrow{V} = XP + YQ + ZR.$$

Properties : Let $\overrightarrow{H}$, $\overrightarrow{V}$ and $\overrightarrow{W}$ be three vectors of $\mathbb{R}^3$, then
(1) $\overrightarrow{H} \cdot \overrightarrow{V} = \left\|\overrightarrow{H}\right\|\left\|\overrightarrow{V}\right\| \cos\left(\overrightarrow{H}, \overrightarrow{V}\right)$;
(2) $\overrightarrow{H} \cdot \overrightarrow{V} = \overrightarrow{V} \cdot \overrightarrow{H}$;
(3) $\left(\alpha\overrightarrow{H}\right) \cdot \overrightarrow{V} = \overrightarrow{H} \cdot \left(\alpha\overrightarrow{V}\right) = \alpha\left(\overrightarrow{H} \cdot \overrightarrow{V}\right)$, $\forall\alpha \in \mathbb{R}$;
(4) $\overrightarrow{H} \cdot \left(\overrightarrow{V} + \overrightarrow{W}\right) = \overrightarrow{H} \cdot \overrightarrow{V} + \overrightarrow{H} \cdot \overrightarrow{W}$.

- **Cross product :** Let $\overrightarrow{H}(X,Y,Z)$ and $\overrightarrow{V}(P,Q,R)$ be two vectors of $\mathbb{R}^3$.

  We define the cross product of $\overrightarrow{H}$ and $\overrightarrow{V}$ by

$$\overrightarrow{H}\wedge\overrightarrow{V}=\begin{vmatrix}\overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k}\\ X & Y & Z\\ P & Q & R\end{vmatrix}=(YR-ZQ)\,\overrightarrow{i}-(XR-ZP)\,\overrightarrow{j}+(XQ-YP)\,\overrightarrow{k}.$$

Properties : Let $\overrightarrow{H}$, $\overrightarrow{V}$ and $\overrightarrow{W}$ be three vectors of $\mathbb{R}^3$, then
(1) $\left\|\overrightarrow{H}\wedge\overrightarrow{V}\right\|=\left\|\overrightarrow{H}\right\|\left\|\overrightarrow{V}\right\|\left|\sin\left(\overrightarrow{H},\overrightarrow{V}\right)\right|$;
(2) $\overrightarrow{H}\wedge\overrightarrow{V}=-\,\overrightarrow{V}\wedge\overrightarrow{H}$;
(3) $\left(\alpha\overrightarrow{H}\right)\wedge\overrightarrow{V}=\overrightarrow{H}\wedge\left(\alpha\overrightarrow{V}\right)=\alpha\left(\overrightarrow{H}\wedge\overrightarrow{V}\right)$, $\forall\alpha\in\mathbb{R}$;
(4) $\overrightarrow{H}\wedge\left(\overrightarrow{V}+\overrightarrow{W}\right)=\overrightarrow{H}\wedge\overrightarrow{V}+\overrightarrow{H}\wedge\overrightarrow{W}$.

- **Mixed product :** Let $\overrightarrow{H}(X,Y,Z)$, $\overrightarrow{V}(P,Q,R)$ and $\overrightarrow{W}(L,M,N)$ be three vectors of $\mathbb{R}^3$.

  We define the mixed product of $\overrightarrow{H}$, $\overrightarrow{V}$ and $\overrightarrow{W}$ by

$$\overrightarrow{H}\cdot\left(\overrightarrow{V}\wedge\overrightarrow{W}\right)=\begin{vmatrix}X & Y & Z\\ P & Q & R\\ L & M & N\end{vmatrix}.$$

- **Double cross product :** Let $\overrightarrow{H}$, $\overrightarrow{V}$ and $\overrightarrow{W}$ be three vectors of $\mathbb{R}^3$.

  The double cross product of $\overrightarrow{H}$, $\overrightarrow{V}$ and $\overrightarrow{W}$ is given by

$$\overrightarrow{H}\wedge\left(\overrightarrow{V}\wedge\overrightarrow{W}\right)=\left(\overrightarrow{H}\cdot\overrightarrow{W}\right)\overrightarrow{V}-\left(\overrightarrow{H}\cdot\overrightarrow{V}\right)\overrightarrow{W}.$$

6.2 Scalar field - Vector field

Let $\left(O,\overrightarrow{i},\overrightarrow{j},\overrightarrow{k}\right)$ be an orthonormal system and M be a point of the space $\mathbb{R}^3$, of coordinates (x,y,z) :

$$\overrightarrow{OM}=x\,\overrightarrow{i}+y\,\overrightarrow{j}+z\,\overrightarrow{k}.$$

Definition *6.1 All mapping from a domain $D\subset\mathbb{R}^3$ to $\mathbb{R}$ that to each point $M\in D$ corresponds a scalar $U(M)$ in $\mathbb{R}$ is called scalar field. We denote*

$$U(M)=U(x,y,z).$$

Example : The mass density at a certain point M of a domain D is a scalar field given by

$$\rho(M) = \frac{dm}{dv}$$

where dm is the elementary mass and dv is the elementary volume at M.

Definition *6.2 All mapping from a domain $D \subset \mathbb{R}^3$ to $\mathbb{R}^3$ that to each point $M \in D$ corresponds a vector $\overrightarrow{H}(M)$ of $\mathbb{R}^3$ is called vector field. We denote*

$$\overrightarrow{H}(M) = X(M)\overrightarrow{i} + Y(M)\overrightarrow{j} + Z(M)\overrightarrow{k} = X(x,y,z)\overrightarrow{i} + Y(x,y,z)\overrightarrow{j} + Z(x,y,z)\overrightarrow{k},$$

of which the components $X,\ Y,\ Z$ are scalars fields.

Properties : Let $\overrightarrow{H}$ and $\overrightarrow{V}$ be two vector fields of $\mathbb{R}^3$ which are differentiable in a domain $D \subset \mathbb{R}^3$, then

(1) $\dfrac{\partial}{\partial x}\left(\overrightarrow{H} \cdot \overrightarrow{V}\right) = \dfrac{\partial}{\partial x}\left(\overrightarrow{H}\right) \cdot \overrightarrow{V} + \overrightarrow{H} \cdot \dfrac{\partial}{\partial x}\left(\overrightarrow{V}\right)$, etc...

(2) $\dfrac{\partial}{\partial x}\left(\overrightarrow{H} \wedge \overrightarrow{V}\right) = \dfrac{\partial}{\partial x}\left(\overrightarrow{H}\right) \wedge \overrightarrow{V} + \overrightarrow{H} \wedge \dfrac{\partial}{\partial x}\left(\overrightarrow{V}\right)$, etc...

Example : Let $r = \|\overrightarrow{r}\| = \left\|\overrightarrow{OM}\right\| = \sqrt{x^2+y^2+z^2}$.

We have $r^2 = \overrightarrow{r} \cdot \overrightarrow{r} \Longrightarrow d\left(r^2\right) = d\left(\overrightarrow{r} \cdot \overrightarrow{r}\right) \Longrightarrow 2rdr = 2\overrightarrow{r} \cdot d\overrightarrow{r} \Longrightarrow dr = \dfrac{\overrightarrow{r}}{r} \cdot d\overrightarrow{r} = \overrightarrow{n} \cdot d\overrightarrow{r}$ where $\overrightarrow{n} = \dfrac{\overrightarrow{r}}{r}$ is the unit vector of $\overrightarrow{r} = \overrightarrow{OM}$.

6.3 The Hamiltonian operator

6.3.1 Gradient of a scalar field

Definition *6.3 The Hamiltonian differential operator (of first order) is defined, in a orthonormal system $(Oxyz)$ of unit vectors $\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}$, by*

$$\overrightarrow{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \frac{\partial}{\partial x}\overrightarrow{i} + \frac{\partial}{\partial y}\overrightarrow{j} + \frac{\partial}{\partial z}\overrightarrow{k}.$$

We call it nabla.

Definition *6.4 Consider a scalar field U defined and differentiable at each point M of a domain D of $\mathbb{R}^3$. We call gradient of U at M, the vector field*

$$\overrightarrow{\text{grad}}U(M) = \overrightarrow{\nabla}U(M) = \frac{\partial U}{\partial x}(M)\overrightarrow{i} + \frac{\partial U}{\partial y}(M)\overrightarrow{j} + \frac{\partial U}{\partial z}(M)\overrightarrow{k}.$$

Properties : Let U and V be two differentiable scalar fields in D, then

(1) $\overrightarrow{\text{grad}}\,(U+V) = \overrightarrow{\text{grad}}U + \overrightarrow{\text{grad}}V$;

(2) $\overrightarrow{\text{grad}}\,(\alpha U) = \alpha\overrightarrow{\text{grad}}U$, $\forall \alpha \in \mathbb{R}$;

(3) $\overrightarrow{\text{grad}}\,(UV) = V\ \overrightarrow{\text{grad}}U + U\ \overrightarrow{\text{grad}}V$.

Proof :

$$\begin{aligned}
(1)\ \overrightarrow{\text{grad}}\,(U+V) &= \frac{\partial}{\partial x}(U+V)\overrightarrow{i} + \frac{\partial}{\partial y}(U+V)\overrightarrow{j} + \frac{\partial}{\partial z}(U+V)\overrightarrow{k}\\
&= \left(\frac{\partial U}{\partial x}\overrightarrow{i} + \frac{\partial U}{\partial y}\overrightarrow{j} + \frac{\partial U}{\partial z}\overrightarrow{k}\right) + \left(\frac{\partial V}{\partial x}\overrightarrow{i} + \frac{\partial V}{\partial y}\overrightarrow{j} + \frac{\partial V}{\partial z}\overrightarrow{k}\right)\\
&= \overrightarrow{\text{grad}}U + \overrightarrow{\text{grad}}V;\\
(2)\ \overrightarrow{\text{grad}}\,(\alpha U) &= \frac{\partial}{\partial x}(\alpha U)\overrightarrow{i} + \frac{\partial}{\partial y}(\alpha U)\overrightarrow{j} + \frac{\partial}{\partial z}(\alpha U)\overrightarrow{k}\\
&= \alpha\left(\frac{\partial U}{\partial x}\overrightarrow{i} + \frac{\partial U}{\partial y}\overrightarrow{j} + \frac{\partial U}{\partial z}\overrightarrow{k}\right)\\
&= \alpha\overrightarrow{\text{grad}}U;\\
(3)\ \overrightarrow{\text{grad}}\,(UV) &= \frac{\partial}{\partial x}(UV)\overrightarrow{i} + \frac{\partial}{\partial y}(UV)\overrightarrow{j} + \frac{\partial}{\partial z}(UV)\overrightarrow{k}\\
&= V\frac{\partial U}{\partial x}\overrightarrow{i} + U\frac{\partial V}{\partial x}\overrightarrow{i} + V\frac{\partial U}{\partial y}\overrightarrow{j} + U\frac{\partial V}{\partial y}\overrightarrow{j} + V\frac{\partial U}{\partial z}\overrightarrow{k} + U\frac{\partial V}{\partial z}\overrightarrow{k}\\
&= V\left(\frac{\partial U}{\partial x}\overrightarrow{i} + \frac{\partial U}{\partial y}\overrightarrow{j} + \frac{\partial U}{\partial z}\overrightarrow{k}\right) + U\left(\frac{\partial V}{\partial x}\overrightarrow{i} + \frac{\partial V}{\partial y}\overrightarrow{j} + \frac{\partial V}{\partial z}\overrightarrow{k}\right)\\
&= V\ \overrightarrow{\text{grad}}U + U\ \overrightarrow{\text{grad}}V.
\end{aligned}$$

***Proposition** 6.1 Let U and V be two differentiable scalar fields in an open and convex D, then*

$$\overrightarrow{\text{grad}}U\,(M) = \overrightarrow{\text{grad}}V\,(M) \Longleftrightarrow \exists C \in \mathbb{R} \text{ such that } U\,(M) = V\,(M) + C.$$

6.3.2 Divergence of a vector field

***Definition** 6.5 Let $\overrightarrow{H}(X,Y,Z)$ be a vector field that is defined and differentiable at each point M of a domain D of $\mathbb{R}^3$. We call divergence of $\overrightarrow{H}$ at M, the scalar field*

$$\text{div}\,\overrightarrow{H}\,(M) = \overrightarrow{\nabla}\cdot\overrightarrow{H}\,(M) = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}.$$

Properties : Let $\overrightarrow{H}(X,Y,Z)$ and $\overrightarrow{V}(P,Q,R)$ be two differentiable vector fields and U be a differentiable scalar field in D, then

(1) $\text{div}\left(\overrightarrow{H} + \overrightarrow{V}\right) = \text{div}\,\overrightarrow{H} + \text{div}\,\overrightarrow{V}$;

(2) $\text{div}\left(\alpha\overrightarrow{H}\right) = \alpha\,\text{div}\,\overrightarrow{H}$, $\forall \alpha \in \mathbb{R}$;

(3) $\text{div}\left(U\overrightarrow{H}\right) = U\,\text{div}\,\overrightarrow{H} + \overrightarrow{\text{grad}}U\cdot\overrightarrow{H}$.

Proof :

$$\begin{aligned}
(1)\ \text{div}\left(\overrightarrow{H} + \overrightarrow{V}\right) &= \frac{\partial}{\partial x}(X+P) + \frac{\partial}{\partial y}(Y+Q) + \frac{\partial}{\partial z}(Z+R)\\
&= \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) + \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)\\
&= \text{div}\,\overrightarrow{H} + \text{div}\,\overrightarrow{V};
\end{aligned}$$

$$
\begin{aligned}
(2)\ \operatorname{div}\left(\alpha\overrightarrow{H}\right) &= \frac{\partial}{\partial x}(\alpha X)+\frac{\partial}{\partial y}(\alpha Y)+\frac{\partial}{\partial z}(\alpha Z)\\
&= \alpha\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right)\\
&= \alpha\operatorname{div}\overrightarrow{H};\\
(3)\ \operatorname{div}\left(U\overrightarrow{H}\right) &= \frac{\partial}{\partial x}(UX)+\frac{\partial}{\partial y}(UY)+\frac{\partial}{\partial z}(UZ)\\
&= U\frac{\partial X}{\partial x}+X\frac{\partial U}{\partial x}+U\frac{\partial Y}{\partial y}+Y\frac{\partial U}{\partial y}+U\frac{\partial Z}{\partial z}+Z\frac{\partial U}{\partial z}\\
&= U\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right)+X\frac{\partial U}{\partial x}+Y\frac{\partial U}{\partial y}+Z\frac{\partial U}{\partial z}\\
&= U\operatorname{div}\overrightarrow{H}+\overrightarrow{H}\cdot\overrightarrow{\mathrm{grad}}U.
\end{aligned}
$$

6.3.3 Rotational of a vector field

***Definition** 6.6 Let $\overrightarrow{H}(X,Y,Z)$ be a vector field that is defined and differentiable at each point M of a domain D of $\mathbb{R}^3$. We call curl of $\overrightarrow{H}$ at M, the vector field*

$$
\begin{aligned}
\overrightarrow{\mathrm{curl}}\ \overrightarrow{H}(M) &= \overrightarrow{\nabla}\wedge\overrightarrow{H}(M)=\begin{vmatrix}\overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ X & Y & Z\end{vmatrix}\\
&= \left(\frac{\partial Z}{\partial y}-\frac{\partial Y}{\partial z}\right)\overrightarrow{i}-\left(\frac{\partial Z}{\partial x}-\frac{\partial X}{\partial z}\right)\overrightarrow{j}+\left(\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}\right)\overrightarrow{k}.
\end{aligned}
$$

Properties : Let $\overrightarrow{H}(X,Y,Z)$ and $\overrightarrow{V}(P,Q,R)$ be two differentiable vector fields and U be a differentiable scalar field in D, then

(1) $\overrightarrow{\mathrm{curl}}\left(\overrightarrow{H}+\overrightarrow{V}\right)=\overrightarrow{\mathrm{curl}}\ \overrightarrow{H}+\overrightarrow{\mathrm{curl}}\ \overrightarrow{V}$;

(2) $\overrightarrow{\mathrm{curl}}\left(\alpha\overrightarrow{H}\right)=\alpha\ \overrightarrow{\mathrm{curl}}\ \overrightarrow{H},\ \forall\alpha\in\mathbb{R}$;

(3) $\overrightarrow{\mathrm{curl}}\left(U\overrightarrow{H}\right)=U\ \overrightarrow{\mathrm{curl}}\ \overrightarrow{H}+\overrightarrow{\mathrm{grad}}U\wedge\overrightarrow{H}$.

(4) If moreover U is of class C^2 in D, then $\overrightarrow{\mathrm{curl}}\left(\overrightarrow{\mathrm{grad}}U\right)=\overrightarrow{0}$.

(5) If moreover $\overrightarrow{H}$ is of class C^2 in D, then $\operatorname{div}\left(\overrightarrow{\mathrm{curl}}\ \overrightarrow{H}\right)=0$.

Proof :

$$
\begin{aligned}
(1)\ \overrightarrow{\mathrm{curl}}\left(\overrightarrow{H}+\overrightarrow{V}\right) &= \left(\frac{\partial}{\partial y}(Z+R)-\frac{\partial}{\partial z}(Y+Q)\right)\overrightarrow{i}-\left(\frac{\partial}{\partial x}(Z+R)-\frac{\partial}{\partial z}(X+P)\right)\overrightarrow{j}\\
&\quad+\left(\frac{\partial}{\partial x}(Y+Q)-\frac{\partial}{\partial y}(X+P)\right)\overrightarrow{k}\\
&= \left(\frac{\partial Z}{\partial y}-\frac{\partial Y}{\partial z}\right)\overrightarrow{i}-\left(\frac{\partial Z}{\partial x}-\frac{\partial X}{\partial z}\right)\overrightarrow{j}+\left(\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}\right)\overrightarrow{k}\\
&\quad+\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\overrightarrow{i}-\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)\overrightarrow{j}+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\overrightarrow{k}\\
&= \overrightarrow{\mathrm{curl}}\ \overrightarrow{H}+\overrightarrow{\mathrm{curl}}\ \overrightarrow{V};
\end{aligned}
$$

$$
\begin{aligned}
(2)\ \overrightarrow{\text{curl}}\left(\alpha\overrightarrow{H}\right) &= \left(\frac{\partial}{\partial y}(\alpha Z)-\frac{\partial}{\partial z}(\alpha Y)\right)\overrightarrow{i}-\left(\frac{\partial}{\partial x}(\alpha Z)-\frac{\partial}{\partial z}(\alpha X)\right)\overrightarrow{j}\\
&\quad+\left(\frac{\partial}{\partial x}(\alpha Y)-\frac{\partial}{\partial y}(\alpha X)\right)\overrightarrow{k}\\
&= \alpha\left(\frac{\partial Z}{\partial y}-\frac{\partial Y}{\partial z}\right)\overrightarrow{i}-\alpha\left(\frac{\partial Z}{\partial x}-\frac{\partial X}{\partial z}\right)\overrightarrow{j}+\alpha\left(\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}\right)\overrightarrow{k}\\
&= \alpha\ \overrightarrow{\text{curl}}\ \overrightarrow{H};\\
(3)\ \overrightarrow{\text{curl}}\left(U\overrightarrow{H}\right) &= \left(\frac{\partial}{\partial y}(UZ)-\frac{\partial}{\partial z}(UY)\right)\overrightarrow{i}-\left(\frac{\partial}{\partial x}(UZ)-\frac{\partial}{\partial z}(UX)\right)\overrightarrow{j}\\
&\quad+\left(\frac{\partial}{\partial x}(UY)-\frac{\partial}{\partial y}(UX)\right)\overrightarrow{k}\\
&= \left(U\frac{\partial Z}{\partial y}+Z\frac{\partial U}{\partial y}-U\frac{\partial Y}{\partial z}-Y\frac{\partial U}{\partial z}\right)\overrightarrow{i}-\left(U\frac{\partial Z}{\partial x}+Z\frac{\partial U}{\partial x}-U\frac{\partial X}{\partial z}-X\frac{\partial U}{\partial z}\right)\overrightarrow{j}\\
&\quad+\left(U\frac{\partial Y}{\partial x}+Y\frac{\partial U}{\partial x}-U\frac{\partial X}{\partial y}-X\frac{\partial U}{\partial y}\right)\overrightarrow{k}\\
&= U\left[\left(\frac{\partial Z}{\partial y}-\frac{\partial Y}{\partial z}\right)\overrightarrow{i}-\left(\frac{\partial Z}{\partial x}-\frac{\partial X}{\partial z}\right)\overrightarrow{j}+\left(\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}\right)\overrightarrow{k}\right]\\
&\quad+\left(Z\frac{\partial U}{\partial y}-Y\frac{\partial U}{\partial z}\right)\overrightarrow{i}-\left(Z\frac{\partial U}{\partial x}-X\frac{\partial U}{\partial z}\right)\overrightarrow{j}+\left(Y\frac{\partial U}{\partial x}-X\frac{\partial U}{\partial y}\right)\overrightarrow{k}\\
&= U\ \overrightarrow{\text{curl}}\ \overrightarrow{H}+\overrightarrow{\text{grad}}U\wedge\overrightarrow{H}.\\
(4)\ \overrightarrow{\text{curl}}\ \left(\overrightarrow{\text{grad}}U\right) &= \overrightarrow{\text{curl}}\ \left(\frac{\partial U}{\partial x}\overrightarrow{i}+\frac{\partial U}{\partial y}\overrightarrow{j}+\frac{\partial U}{\partial z}\overrightarrow{k}\right)\\
&= \left(\frac{\partial^2 U}{\partial y\partial z}-\frac{\partial^2 U}{\partial z\partial y}\right)\overrightarrow{i}-\left(\frac{\partial^2 U}{\partial x\partial z}-\frac{\partial^2 U}{\partial z\partial x}\right)\overrightarrow{j}+\left(\frac{\partial^2 U}{\partial x\partial y}-\frac{\partial^2 U}{\partial y\partial x}\right)\overrightarrow{k}=\overrightarrow{0}.\\
(5)\ \text{div}\left(\overrightarrow{\text{curl}}\ \overrightarrow{H}\right) &= \text{div}\left[\left(\frac{\partial Z}{\partial y}-\frac{\partial Y}{\partial z}\right)\overrightarrow{i}-\left(\frac{\partial Z}{\partial x}-\frac{\partial X}{\partial z}\right)\overrightarrow{j}+\left(\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}\right)\overrightarrow{k}\right]\\
&= \frac{\partial}{\partial x}\left(\frac{\partial Z}{\partial y}-\frac{\partial Y}{\partial z}\right)-\frac{\partial}{\partial y}\left(\frac{\partial Z}{\partial x}-\frac{\partial X}{\partial z}\right)+\frac{\partial}{\partial z}\left(\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}\right)\\
&= \frac{\partial^2 Z}{\partial x\partial y}-\frac{\partial^2 Y}{\partial x\partial z}-\frac{\partial^2 Z}{\partial y\partial x}+\frac{\partial^2 X}{\partial y\partial z}+\frac{\partial^2 Y}{\partial z\partial x}-\frac{\partial^2 X}{\partial z\partial y}=0.
\end{aligned}
$$

Example : Verify that $\overrightarrow{\text{curl}}\ \left(U\overrightarrow{\text{grad}}U\right)=\overrightarrow{0}$.

Solution : $\overrightarrow{\text{curl}}\ \left(U\overrightarrow{\text{grad}}U\right)=U\ \overrightarrow{\text{curl}}\ \left(\overrightarrow{\text{grad}}U\right)+\overrightarrow{\text{grad}}U\wedge\overrightarrow{\text{grad}}U=\overrightarrow{0}$.

Example : Let $\overrightarrow{n}=\dfrac{\overrightarrow{r}}{r}=\dfrac{1}{r}\left(x\,\overrightarrow{i}+y\,\overrightarrow{j}+z\,\overrightarrow{k}\right)$ with $r=\sqrt{x^2+y^2+z^2}$.

1. Find $\overrightarrow{\text{grad}}r$, div $\overrightarrow{n}$ and $\overrightarrow{\text{curl}}\ \overrightarrow{n}$.
2. Let $\overrightarrow{H}=\overrightarrow{\omega}\wedge\overrightarrow{r}$ with $\overrightarrow{\omega}(a,b,c)$ (cte), then $\overrightarrow{\omega}=\dfrac{1}{2}\overrightarrow{\text{curl}}\ \overrightarrow{H}$.

Solution :

1. We have $\dfrac{\partial r}{\partial x}=\dfrac{2x}{2\sqrt{x^2+y^2+z^2}}=\dfrac{x}{r}$, $\dfrac{\partial r}{\partial y}=\dfrac{y}{r}$ and $\dfrac{\partial r}{\partial z}=\dfrac{z}{r}\Longrightarrow\overrightarrow{\text{grad}}r=\dfrac{\overrightarrow{r}}{r}=\overrightarrow{n}$

$$
\begin{aligned}
\text{div}\,\overrightarrow{n} &= \frac{\partial}{\partial x}\left(\frac{x}{r}\right)+\frac{\partial}{\partial y}\left(\frac{y}{r}\right)+\frac{\partial}{\partial z}\left(\frac{z}{r}\right)=\left(\frac{1}{r}-\frac{x^2}{r^3}\right)+\left(\frac{1}{r}-\frac{y^2}{r^3}\right)+\left(\frac{1}{r}-\frac{y^2}{r^3}\right)\\
&= \frac{3}{r}-\frac{x^2+y^2+z^2}{r^3}=\frac{3}{r}-\frac{r^2}{r^3}=\frac{3}{r}-\frac{1}{r}=\frac{2}{r}.
\end{aligned}
$$

$$\overrightarrow{\text{curl}}\ \overrightarrow{n} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{x}{r} & \dfrac{y}{r} & \dfrac{z}{r} \end{vmatrix} = \left(-\frac{yz}{r^2} + \frac{yz}{r^2}\right)\overrightarrow{i} - \left(-\frac{xz}{r^2} + \frac{xz}{r^2}\right)\overrightarrow{j} + \left(-\frac{yz}{r^2} + \frac{yz}{r^2}\right)\overrightarrow{k} = \overrightarrow{0}$$

2. $\overrightarrow{H} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ a & b & c \\ x & y & z \end{vmatrix} = (bz - cy)\,\overrightarrow{i} - (az - cx)\,\overrightarrow{j} + (ay - bx)\,\overrightarrow{k}$

$$\Longrightarrow \overrightarrow{\text{curl}}\ \overrightarrow{H} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ bz - cy & cx - az & ay - bx \end{vmatrix} = 2a\overrightarrow{i} + 2b\overrightarrow{j} + 2c\overrightarrow{k} = 2\overrightarrow{\omega}.$$

6.4 Laplace equation

Definition *6.7 We define the differential operator of the second order, called Laplacian by*

$$\Delta = \overrightarrow{\nabla} \cdot \overrightarrow{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Definition *6.8 Consider a scalar field U defined and of class C^2 at each point M of a domain D of $\mathbb{R}^3$. The Laplacian of U at M is the scalar field*

$$\Delta U(M) = \frac{\partial^2 U}{\partial x^2}(M) + \frac{\partial^2 U}{\partial y^2}(M) + \frac{\partial^2 U}{\partial z^2}(M).$$

Properties : Let $\overrightarrow{H}(X, Y, Z)$ be a vector filed and U, V be two scalar fields of class C^2 in D, then
(1) $\Delta\overrightarrow{H} = \Delta X\,\overrightarrow{i} + \Delta Y\,\overrightarrow{j} + \Delta Z\,\overrightarrow{k}$;
(2) $\Delta U = \text{div}\left(\overrightarrow{\text{grad}}U\right)$;
(3) $\Delta(UV) = V\ \Delta U + U\ \Delta V + 2\ \overrightarrow{\text{grad}}U \cdot \overrightarrow{\text{grad}}V.$
Proof :

$$\begin{aligned}
(1)\ \Delta\overrightarrow{H} &= \frac{\partial^2\overrightarrow{H}}{\partial x^2} + \frac{\partial^2\overrightarrow{H}}{\partial y^2} + \frac{\partial^2\overrightarrow{H}}{\partial z^2} \\
&= \left(\frac{\partial^2 X}{\partial x^2}\overrightarrow{i} + \frac{\partial^2 Y}{\partial x^2}\overrightarrow{j} + \frac{\partial^2 Z}{\partial x^2}\overrightarrow{k}\right) + \left(\frac{\partial^2 X}{\partial y^2}\overrightarrow{i} + \frac{\partial^2 Y}{\partial y^2}\overrightarrow{j} + \frac{\partial^2 Z}{\partial y^2}\overrightarrow{k}\right) \\
&\quad + \left(\frac{\partial^2 X}{\partial z^2}\overrightarrow{i} + \frac{\partial^2 Y}{\partial z^2}\overrightarrow{j} + \frac{\partial^2 Z}{\partial z^2}\overrightarrow{k}\right) \\
&= \left(\frac{\partial^2 X}{\partial x^2} + \frac{\partial^2 X}{\partial y^2} + \frac{\partial^2 X}{\partial z^2}\right)\overrightarrow{i} + \left(\frac{\partial^2 Y}{\partial x^2} + \frac{\partial^2 Y}{\partial y^2} + \frac{\partial^2 Y}{\partial z^2}\right)\overrightarrow{j} + \left(\frac{\partial^2 Z}{\partial x^2} + \frac{\partial^2 Z}{\partial y^2} + \frac{\partial^2 Z}{\partial z^2}\right)\overrightarrow{k} \\
&= \Delta X\,\overrightarrow{i} + \Delta Y\,\overrightarrow{j} + \Delta Z\,\overrightarrow{k};
\end{aligned}$$

$$\begin{aligned}
(2)\ \text{div}\left(\overrightarrow{\text{grad}}U\right) &= \text{div}\left(\frac{\partial U}{\partial x}\overrightarrow{i} + \frac{\partial U}{\partial y}\overrightarrow{j} + \frac{\partial U}{\partial z}\overrightarrow{k}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial U}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial U}{\partial z}\right) \\
&= \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = \Delta U;
\end{aligned}$$

$$
\begin{aligned}
(3)\ \Delta(UV) &= \operatorname{div}\ \overrightarrow{\operatorname{grad}}(UV) \\
&= \operatorname{div}\left(U\ \overrightarrow{\operatorname{grad}}V + V\ \overrightarrow{\operatorname{grad}}U\right) \\
&= \operatorname{div}\left(U\ \overrightarrow{\operatorname{grad}}V\right) + \operatorname{div}\left(V\ \overrightarrow{\operatorname{grad}}U\right) \\
&= U \operatorname{div}\left(\ \overrightarrow{\operatorname{grad}}V\right) + \overrightarrow{\operatorname{grad}}U \cdot \overrightarrow{\operatorname{grad}}V + V \operatorname{div}\left(\ \overrightarrow{\operatorname{grad}}U\right) + \overrightarrow{\operatorname{grad}}V \cdot \overrightarrow{\operatorname{grad}}U \\
&= \Delta\left(UV\right) = V\ \Delta U + U\ \Delta V + 2\ \overrightarrow{\operatorname{grad}}U \cdot \overrightarrow{\operatorname{grad}}V.
\end{aligned}
$$

***Definition** 6.9 Let $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function of class C^2 in D. We say that f is harmonic if it verifies, at each point M of D, the equation of Laplace*

$$\Delta f\left(M\right) = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}\left(M\right) = 0.$$

Example : Let $f\left(x,y\right) = \arctan \dfrac{y}{x}$. Show that f is harmonic in $D =\{(x,y) \in \mathbb{R}^2 : x \neq 0\}$.

Solution : $\dfrac{\partial f}{\partial x}\left(x,y\right) = \dfrac{-y}{x^2+y^2}$ and $\dfrac{\partial f}{\partial y}\left(x,y\right) = \dfrac{x}{x^2+y^2}$.

Then

$\Delta f\left(x,y\right) = \dfrac{\partial^2 f}{\partial x^2}\left(x,y\right) + \dfrac{\partial^2 f}{\partial y^2}\left(x,y\right) = \dfrac{2xy}{\left(x^2+y^2\right)^2} - \dfrac{2xy}{\left(x^2+y^2\right)^2} = 0.$

6.5 Total differential forms

6.5.1 Total differential form in $\mathbb{R}^2$ and $\mathbb{R}^3$

Consider the differential form of two variables defined in an open D of $\mathbb{R}^2$:

$$\omega = P(x,y)dx + Q(x,y)dy.$$

If ω is the differential of some differentiable function f of class C^1 in D, i.e.,

$$\omega = df\left(x,y\right) = \frac{\partial f}{\partial x}\left(x,y\right)dx + \frac{\partial f}{\partial y}\left(x,y\right)dy,$$

then we must have

$$\frac{\partial f}{\partial x}\left(x,y\right) = P\left(x,y\right) \qquad \text{and} \qquad \frac{\partial f}{\partial y}\left(x,y\right) = Q\left(x,y\right).$$

If f is of class C^2 on D, then we have

$$\frac{\partial^2 f}{\partial y \partial x}\left(x,y\right) = \frac{\partial^2 f}{\partial x \partial y}\left(x,y\right), \text{ which gives } \frac{\partial P}{\partial y}\left(x,y\right) = \frac{\partial Q}{\partial x}\left(x,y\right).$$

***Definition** 6.10 We say that the differential form*

$$\omega = P(x,y)dx + Q(x,y)dy$$

is a total (or exact) differential form if and only if

$$\frac{\partial P}{\partial y}\left(x,y\right) = \frac{\partial Q}{\partial x}\left(x,y\right).$$

***Theorem** 6.1 If ω is total, then there is a function $f : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that $df(x,y) = \omega$.*

- **Determination de** f : Suppose that ω is total and that there exists f such that $df(x,y) = \omega$, then we must have

$$\frac{\partial f}{\partial x}(x,y)\,dx + \frac{\partial f}{\partial y}(x,y)\,dy = P(x,y)dx + Q(x,y)dy \Longleftrightarrow \begin{cases} \dfrac{\partial f}{\partial x}(x,y) = P(x,y) \\ \dfrac{\partial f}{\partial y}(x,y) = Q(x,y) \end{cases}$$

Take $\dfrac{\partial f}{\partial x}(x,y) = P(x,y)$ and integrate with respect to x considering y as constant, then we have

$$f(x,y) = \int P(x,y)dx + C,$$

where C is an independent constant of x but can depend of y, therefore we consider it as a function of y only : $C = C(y)$.

The equation $\dfrac{\partial f}{\partial y}(x,y) = Q(x,y)$ is used to determine $C'(y)$, and finally we integrated $C'(y)$ to obtain $C(y)$.

It is to note that the calculation can be made by first integrating $\dfrac{\partial f}{\partial y}(x,y) = Q(x,y)$.

Example : Let $\omega = (2xy^2 + y\cos x)dx + (2x^2y + \sin x - 2y)dy$.

1. Show that ω is total.
2. Find $f(x,y)$ such that $df(x,y) = \omega$.

Solution : 1. We have $P(x,y) = 2xy^2 + y\cos x \Longrightarrow \dfrac{\partial P}{\partial y}(x,y) = 4xy + \cos x$

and $Q(x,y) = 2x^2y + \sin x - 2y \Longrightarrow \dfrac{\partial Q}{\partial x}(x,y) = 4xy + \cos x$

$\Longrightarrow \dfrac{\partial P}{\partial y}(x,y) = \dfrac{\partial Q}{\partial x}(x,y)$, then ω is total.

2. Since ω is total, then $\exists\ f(x,y)\ /\ df(x,y) = \omega$.

$\dfrac{\partial f}{\partial x}(x,y) = P(x,y) = 2xy^2 + y\cos x$

$\Longrightarrow f(x,y) = \displaystyle\int P(x,y)\,dx = \int (2xy^2 + y\cos x)dx = x^2y^2 + y\sin x + C(y)$

$\dfrac{\partial f}{\partial y}(x,y) = Q(x,y) = 2x^2y + \sin x - 2y \Longrightarrow 2x^2y + \sin x + C'(y) = 2x^2y + \sin x - 2y$

$\Longrightarrow C'(y) = -2y \Longrightarrow C(y) = -y^2 + K.$

Finally $f(x,y) = x^2y^2 + y\sin x - y^2 + K$.

***Definition** 6.11 The differential form of three variables, defined in an open D of $\mathbb{R}^3$ by*

$$\omega = P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz$$

is total if and only if

$$\frac{\partial P}{\partial y}(x,y,z) = \frac{\partial Q}{\partial x}(x,y,z), \quad \frac{\partial P}{\partial z}(x,y,z) = \frac{\partial R}{\partial x}(x,y,z) \quad \text{and} \quad \frac{\partial Q}{\partial z}(x,y,z) = \frac{\partial R}{\partial y}(x,y,z)$$

if and only if

$$\overrightarrow{\text{curl}}\ \overrightarrow{V} = \overrightarrow{0} \quad \text{with} \quad \overrightarrow{V} = P\,\overrightarrow{i} + Q\,\overrightarrow{j} + R\,\overrightarrow{k}.$$

The determination of a function $f : D \subset \mathbb{R}^3 \longrightarrow \mathbb{R}$ such that $df(x,y,z) = \omega$ is made in an analogous manner as the case of two variables.

Example : Let $\omega = (yz - 2x)dx + (xz + z)dy + (xy + y)dz$.
1. Show that ω is total.
2. Find $f(x,y,z)$ such that $df(x,y,z) = \omega$.

Solution : 1. Take $P(x,y,z) = yz - 2x$, $Q(x,y,z) = xz + z$ and $R(x,y,z) = xy + y$.
Let $\overrightarrow{V} = P\overrightarrow{i} + Q\overrightarrow{j} + R\overrightarrow{k}$.

$$\overrightarrow{\text{curl}}\ \overrightarrow{V} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz - 2x & xz + z & xy + y \end{vmatrix} = (x + 1 - x - 1)\,\overrightarrow{i} - (y - y)\,\overrightarrow{j} + (z - z)\,\overrightarrow{k} = \overrightarrow{0}$$

then ω is total.
2. Since ω is total, then $\exists\ f(x,y,z)\ /\ df(x,y,z) = \omega$.

$\dfrac{\partial f}{\partial x}(x,y,z) = P(x,y,z) = yz - 2x$

$\Longrightarrow f(x,y,z) = \displaystyle\int P(x,y,z)dx = \int (yz - 2x)dx = xyz - x^2 + C(y,z)$

$\dfrac{\partial f}{\partial y}(x,y,z) = Q(x,y,z) = xz + z \Longrightarrow xz + \dfrac{\partial C}{\partial y}(y,z) = xz + z \Longrightarrow \dfrac{\partial C}{\partial y}(y,z) = z$

$\Longrightarrow C(y,z) = \displaystyle\int z dy = yz + K(z) \Longrightarrow f(x,y,z) = xyz - x^2 + yz + K(z)$

$\dfrac{\partial f}{\partial z}(x,y,z) = R(x,y,z) = xy + y \Longrightarrow xy + y + K'(z) = xy + y \Longrightarrow K'(z) = 0 \Longrightarrow K(z) = L$

Finally $f(x,y,z) = xyz - x^2 + yz + L$.

6.5.2 Gradient field

***Definition** 6.12 Let $\overrightarrow{V}$ a vector field defined in an open D of $\mathbb{R}^n$. We say that $\overrightarrow{V}$ is a gradient field if there is a scalar field $f : D \longrightarrow \mathbb{R}$ such that*

$$\overrightarrow{V} = \overrightarrow{\text{grad}} f.$$

f is called scalar potential. We say $\overrightarrow{V}$ derives from a potential.

In $\mathbb{R}^3$ the vector field $\overrightarrow{V}(M) = P(M)\,\overrightarrow{i} + Q(M)\,\overrightarrow{j} + R(M)\,\overrightarrow{k}$, is a gradient field if there is a scalar field $f : D \longrightarrow \mathbb{R}$ such that f is a solution of the system :

$$\begin{cases} \dfrac{\partial f}{\partial x}(x,y,z) = P(x,y,z) \\ \dfrac{\partial f}{\partial y}(x,y,z) = Q(x,y,z) \\ \dfrac{\partial f}{\partial z}(x,y,z) = R(x,y,z) \end{cases}$$

***Theorem** 6.2 Let $\overrightarrow{V}$ be a vector field defined in an open and connected domain D of $\mathbb{R}^3$ by*

$$\overrightarrow{V}(M) = P(M)\,\overrightarrow{i} + Q(M)\,\overrightarrow{j} + R(M)\,\overrightarrow{k}.$$

$\overrightarrow{V}$ is a gradient field if and only if $\overrightarrow{\text{curl}}\ \overrightarrow{V} = \overrightarrow{0}$.

Example : Let $\overrightarrow{V}=\left(x+\dfrac{z}{x^2y}\right)\overrightarrow{i}+\left(y+\dfrac{z}{xy^2}\right)\overrightarrow{j}+\left(z-\dfrac{1}{xy}\right)\overrightarrow{k}$.

1. Show that $\overrightarrow{V}$ is a gradient field.

2. Find f such that $\overrightarrow{V}=\overrightarrow{\text{grad}}f$.

Solution : 1.

$$\overrightarrow{\text{curl}}\ \overrightarrow{V}=\begin{vmatrix}\overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ x+\dfrac{z}{x^2y} & y+\dfrac{z}{xy^2} & z-\dfrac{1}{xy}\end{vmatrix}$$

$$=\left(\frac{1}{xy^2}-\frac{1}{xy^2}\right)\overrightarrow{i}-\left(\frac{1}{x^2y}-\frac{1}{x^2y}\right)\overrightarrow{j}+\left(-\frac{z}{x^2y^2}+\frac{z}{x^2y^2}\right)\overrightarrow{k}=\overrightarrow{0}$$

$\overrightarrow{V}$ is a gradient field, and then $\exists\ f(x,y,z)\ /\ df(x,y,z)=\omega$.

2. $\dfrac{\partial f}{\partial x}(x,y,z)=x+\dfrac{z}{x^2y}\Longrightarrow f(x,y,z)=\displaystyle\int\left(x+\frac{z}{x^2y}\right)dx=\frac{x^2}{2}-\frac{z}{xy}+C(y,z)$

$\dfrac{\partial f}{\partial y}(x,y,z)=y+\dfrac{z}{xy^2}\Longrightarrow\dfrac{z}{xy^2}+\dfrac{\partial C}{\partial y}(y,z)=y+\dfrac{z}{xy^2}\Longrightarrow\dfrac{\partial C}{\partial y}(y,z)=y$

$\Longrightarrow C(y,z)=\displaystyle\int ydy=\frac{y^2}{2}+K(z)\Longrightarrow f(x,y,z)=\frac{x^2}{2}-\frac{z}{xy}+\frac{y^2}{2}+K(z)$

$\dfrac{\partial f}{\partial z}(x,y,z)=z-\dfrac{1}{xy}\Longrightarrow-\dfrac{1}{xy}+K'(z)=z-\dfrac{1}{xy}\Longrightarrow K'(z)=z\Longrightarrow K(z)=\dfrac{z^2}{2}+L$

Finally $f(x,y,z)=\dfrac{x^2}{2}+\dfrac{y^2}{2}+\dfrac{z^2}{2}-\dfrac{z}{xy}+L$.

Remark : In the case of a vector field in $\mathbb{R}^2$:

$$\overrightarrow{V}(M)=P(M)\overrightarrow{i}+Q(M)\overrightarrow{j}$$

$\overrightarrow{V}$ is a gradient field if and only if $\dfrac{\partial P}{\partial y}=\dfrac{\partial Q}{\partial x}$.

6.5.3 Integrating factors

- **Integrating factor in $\mathbb{R}^2$**: Consider a differential form

$$\omega=P(x,y)dx+Q(x,y)dy$$

who is not total. If their exists a scalar function $\mu=\mu(x,y)$ such that

$$\mu\omega=\mu P(x,y)dx+\mu Q(x,y)dy$$

is a total differential of a certain function f, i.e.

$$\frac{\partial}{\partial y}(\mu P)=\frac{\partial}{\partial x}(\mu Q),$$

the function μ is called integrating factor of the differential form ω.

Example : Let $\omega = y(1+xy)dx - xdy$.
1. Find an integrating factor of the form $\mu = \mu(y)$ for $\mu\omega$ to be total.
2. Find $f(x,y)$ such that $df(x,y) = \mu(y)\omega$.

Solution : 1. We have $P(x,y) = y + xy^2 \Longrightarrow \dfrac{\partial P}{\partial y}(x,y) = 1 + 2xy$

and $Q(x,y) = -x \Longrightarrow \dfrac{\partial Q}{\partial x}(x,y) = -1$

$\Longrightarrow \dfrac{\partial P}{\partial y}(x,y) \neq \dfrac{\partial Q}{\partial x}(x,y)$, then ω is not total.

$\mu\omega = \mu(y+xy^2)dx - \mu x dy$ is total if $\dfrac{\partial(\mu P)}{\partial y} = \dfrac{\partial(\mu Q)}{\partial x} \Longrightarrow P\dfrac{\partial \mu}{\partial y} + \mu\dfrac{\partial P}{\partial y} = Q\dfrac{\partial \mu}{\partial x} + \mu\dfrac{\partial Q}{\partial x}$

$\Longrightarrow y\,(1+xy)\,\mu'(y) + 2\,(1+xy)\,\mu = 0 \Longrightarrow y\mu'(y) + 2\mu = 0 \Longrightarrow \dfrac{\mu'(y)}{\mu(y)} = -\dfrac{2}{y}$

$\Longrightarrow \displaystyle\int \frac{\mu'(y)}{\mu(y)}dy = -\int \frac{2}{y}dy \Longrightarrow \ln\mu(y) = -2\ln y + k = \ln\frac{C}{y^2} \Longrightarrow \mu(y) = \frac{C}{y^2}.$

2. $\mu\omega$ is total, then $\exists\, f(x,y)\ /\ df(x,y) = \mu(y)\omega = \dfrac{C}{y^2}\,[y(1+xy)dx - xdy] = C\left[\dfrac{ydx - xdy}{y^2} + xdx\right]$

$\Longrightarrow df = C\left[d\left(\dfrac{x}{y}\right) + d\left(\dfrac{x^2}{2}\right)\right] = Cd\left(\dfrac{x}{y} + \dfrac{x^2}{2}\right)$

Finally $f(x,y) = C\left(\dfrac{x}{y} + \dfrac{x^2}{2}\right) + K.$

• **Integrating factor in** $\mathbb{R}^3$: For a differential form of three variables

$$\omega = P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz$$

who is not total, if their exists a scalar function $\mu = \mu(x,y,z)$ such that

$$\mu\omega = \mu P(x,y,z)dx + \mu Q(x,y,z)dy + \mu R(x,y,z)dz$$

is a total differential of a certain function f, i.e.

$$\overrightarrow{\text{curl}}(\mu\overrightarrow{V}) = \overrightarrow{0} \quad \text{with} \quad \overrightarrow{V} = P\overrightarrow{i} + Q\overrightarrow{j} + R\overrightarrow{k},$$

the function μ is called integrating factor of ω.

Remarks :
(1) In case of the differential forms of two variables, there exists always an integrating factor.
(2) $\overrightarrow{\text{curl}}(\mu\overrightarrow{V}) = \overrightarrow{0}$ is equivalent to $\overrightarrow{V}\cdot\overrightarrow{\text{curl}}\,\overrightarrow{V} = 0$, which is a necessary condition for the existence of an integrating factor in case of the differential forms of three variables.

6.6 Exercises

Exercise *6.1 Consider the vector fields*

$$\overrightarrow{H} = 8t^2\overrightarrow{i} + t\overrightarrow{j} - t^3\overrightarrow{k} \qquad \text{and} \qquad \overrightarrow{V} = \sin t\,\overrightarrow{i} - \cos t\,\overrightarrow{j}.$$

where t is a parameter. Calculate $\dfrac{d}{dt}(\overrightarrow{H}\cdot\overrightarrow{V})$, $\dfrac{d}{dt}(\overrightarrow{H}\wedge\overrightarrow{V})$ and $\dfrac{d}{dt}(\overrightarrow{H}\cdot\overrightarrow{H})$.

***Exercise** 6.2* 1. *Let the scalar field*

$$U(x,y,z)=\begin{cases} x^2\tanh\dfrac{y+z^2}{x} & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$$

Calculate $\overrightarrow{\text{grad}}U(0,1,1)$.
2. *Let the vector field*

$$\overrightarrow{H}(x,y,z)=x^3yz\overrightarrow{i}+xz\overrightarrow{j}+\left(x^2+y^2+z^2\right)\overrightarrow{k}$$

Calculate $\text{div}\,\overrightarrow{H}(1,0,1)$ *and* $\overrightarrow{\text{curl}}\,\overrightarrow{H}(1,0,1)$.

***Exercise** 6.3* 1. *If*

$$\overrightarrow{H}=2yz\overrightarrow{i}-x^2y\overrightarrow{j}+xz^2\overrightarrow{k},\qquad \overrightarrow{V}=x^2\overrightarrow{i}+yz\overrightarrow{j}-xy\overrightarrow{k}\qquad \textit{and}\qquad U=2x^2yz^3.$$

Calculate $\overrightarrow{H}\cdot\overrightarrow{\text{grad}}U$, $\overrightarrow{H}\wedge\overrightarrow{\text{grad}}U$, $\overrightarrow{V}\cdot\overrightarrow{\text{curl}}\,\overrightarrow{H}$, $\overrightarrow{V}\wedge\overrightarrow{\text{curl}}\,\overrightarrow{H}$ *and* $\overrightarrow{\text{curl}}\left(U\overrightarrow{H}\right)$.
2. *Let* $\overrightarrow{H}$ *be a vector field of class* C^1 *on* $\mathbb{R}^3$ *and let* $\overrightarrow{V}(a,b,c)$ *be a fixed vector.*
Show that $\text{div}\left(\overrightarrow{H}\wedge\overrightarrow{V}\right)=\overrightarrow{V}\cdot\overrightarrow{\text{curl}}\,\overrightarrow{H}$.
3. *Let*

$$\overrightarrow{V}=x^2\overrightarrow{i}+\sqrt{x^2+y^2+1}\;\overrightarrow{j}+z\overrightarrow{k}.$$

Calculate $\overrightarrow{\text{grad}}\left(\left\|\overrightarrow{V}\right\|_2^2\right)$ *then deduce* $\text{div}\left(\left\|\overrightarrow{V}\right\|_2^2\overrightarrow{V}\right)$.

***Exercise** 6.4* *Let* $D=\left\{(x,y)\in\mathbb{R}^2: xy>1 \textit{ and } x>0\right\}$ *and*

$$U(x,y)=\arctan x+\arctan y-\arctan\frac{x+y}{1-xy}.$$

1. *Identify* D.
2. *Calculate* $\overrightarrow{\text{grad}}U(x,y)$.
3. *Deduce that* U *is equal to a constant on* D *that will be determined.*
(*<u>hint</u> : we can calculate the limit of* D *when* $x\longrightarrow\infty$ *on the path* $y=x$).

***Exercise** 6.5* *Let* $\overrightarrow{n}=\dfrac{\overrightarrow{r}}{r}=\dfrac{1}{r}\left(x\overrightarrow{i}+y\overrightarrow{j}+z\overrightarrow{k}\right)$ *with* $r=\left\|\overrightarrow{r}\right\|_2=\sqrt{x^2+y^2+z^2}$.

1. *Let* $U=U(r)$ *be a differentiable scalar field. Show that* $\overrightarrow{\text{grad}}U=\dfrac{\partial U}{\partial r}\overrightarrow{n}$.
2. *Calculate* $\overrightarrow{\text{grad}}(r^s)$ *and* $\overrightarrow{\text{grad}}(\ln r)$.
3. *Consider the vector field* $\overrightarrow{F}=\dfrac{\ln r}{r}\overrightarrow{r}$. *Determine, if it exists, a potential* $\varphi(r)$ *such that*

$$\overrightarrow{\text{grad}}\varphi=\overrightarrow{F}\qquad \textit{and}\qquad \lim_{r\longrightarrow 0}\varphi(r)=1.$$

4. *Let* $u:\mathbb{R}\longrightarrow\mathbb{R}$ *be a real function of class* C^2. *Find the function* $f:\mathbb{R}^3\longrightarrow\mathbb{R}$ *such that* $f(x,y,z)=u(r)$ *verifying* $\Delta f=0$.

Exercise *6.6 Let* $U(x,y) = \ln\sqrt{x^2+y^2}$. *Show that* $U(x,y)$ *is solution of the problem*

$$\begin{cases} \Delta U(x,y) = 0 & \text{for} \quad 1 < x^2+y^2 < 4 \\ U(x,y) = 0 & \text{for} \quad x^2+y^2 = 1 \\ U(x,y) = \ln 2 & \text{for} \quad x^2+y^2 = 4 \end{cases}$$

Exercise *6.7* 1. *Let* $U : \mathbb{R}^3 \longrightarrow \mathbb{R}$ *be a scalar function. Show that, if* U *and* U^2 *are harmonic, then* U *is constant on* $\mathbb{R}^3$.

2. *Let* $U = \{(x,y) \in \mathbb{R}^2 : x \neq 0\}$. *Find all the mappings* $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ *of class* C^2 *such that the mapping* $f : U \longrightarrow \mathbb{R}$ *defined by* $f(x,y) = \varphi\left(\frac{y}{x}\right)$ *is harmonic.*

Exercise *6.8 Determine* $f(y)$ *that verifies* $f(0) = 0$ *so that the vector field*

$$\overrightarrow{V} = (1-x^2)\overrightarrow{i} + f(y)\overrightarrow{j} + (2x-y)z\overrightarrow{k}$$

is solenoidal (i.e. $\operatorname{div}\overrightarrow{H} = 0$*).*

Exercise *6.9 Let* $a, b, c \in \mathbb{R}$ *and consider the vector field in* $\mathbb{R}^3$ *defined by*

$$\overrightarrow{V} = (x+2y+az)\overrightarrow{i} + (bx-3y-z)\overrightarrow{j} + (4x+cy+2z)\overrightarrow{k}.$$

1. *Find the constants* a, b, c *so that* $\overrightarrow{V}$ *is a gradient field (i.e.* $\overrightarrow{\operatorname{curl}}\,\overrightarrow{V} = \overrightarrow{0}$*).*
2. *Express* $\overrightarrow{V}$ *as the gradient of a scalar potential.*

Exercise *6.10 Let* $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *be a function of class* C^1 *and consider the vector field in* $\mathbb{R}^3$ *defined by*

$$\overrightarrow{V} = (yz + x^2y^3)\overrightarrow{i} + (xz + x^3y^2)\overrightarrow{j} + g(x,y)\overrightarrow{k}.$$

1. *Find the expression of* $g(x,y)$ *verifying* $g(0,0) = 0$ *for* $\overrightarrow{V}$ *to be a gradient field.*
2. *Find then* f *such that* $\overrightarrow{\operatorname{grad}} f = \overrightarrow{V}$ *verifying* $f(1,0,1) = 0$.

Exercise *6.11 Let* $g : \mathbb{R} \longrightarrow \mathbb{R}$ *be a function of class* C^1 *and consider the vector field in* $\mathbb{R}^3$ *defined by*

$$\overrightarrow{V} = 2xzg(z)\overrightarrow{i} - 2yzg(z)\overrightarrow{j} + \left(y^2 - x^2\right)g(z)\overrightarrow{k}.$$

1. *Determine* $g(z)$ *verifying* $g(0) = 0$ *for* $\overrightarrow{V}$ *to be a gradient field.*
2. *Determine then the potential* f *of* $\overrightarrow{V}$.

Exercise *6.12 In what follows, prove that the differential form is total and determine* $f(x,y)$ *such that* $df(x,y) = \omega$:

1. $\omega = (\sin y - y\cos x)dx + (x\cos y - \sin x)dy,$ *with* $f(0,0) = 0$
2. $\omega = (2x-y)e^{\frac{y}{x}}dx + xe^{\frac{y}{x}}dy,$ *for* $x > 0$

Exercise *6.13 In what follows, prove that the differential form is total and determine* $f(x,y,z)$ *such that* $df(x,y,z) = \omega$:

1. $\omega = 6xzdx + 6yzdy + 3(x^2+y^2-2z^2)dz,$ *with* $f(0,0,0) = 0$
2. $\omega = (-2\arctan x + y\ln z)dx + x\ln z dy + \dfrac{xy}{z}dz,$ *for* $z > 0$

Exercise *6.14 Consider the following differential form*

$$\omega = \frac{x-y}{x}dx + dy, \text{ for } x > 0.$$

1. *Show that ω is not total.*
2. *Find an integrating factor $\mu = \mu(x)$ verifying $\mu(1) = 1$ such that $\mu\omega$ is total.*
3. *Integrate $\mu\omega$.*

Exercise *6.15 Consider the differential form*

$$\omega = \frac{1}{\sqrt{x^2+y^2}}dx + \frac{\sqrt{x^2+y^2}-x}{y\sqrt{x^2+y^2}}dy.$$

1. *Set $x = r\cos\theta$ and $y = r\sin\theta$. Express ω in terms of r, θ, dr and $d\theta$.*
2. *Find a function $F(r,\theta)$ such that $dF(r,\theta) = \omega$ and deduce the solution of the differential equation $\omega = 0$.*

Exercise *6.16 Consider the following differential form*

$$\omega = y(y-x-1)dx + xdy.$$

1. *Show that ω is not total.*
2. *Show that ω has an integrating factor of the form $\mu(x,y) = \dfrac{f(x)}{y^2}$.*
3. *Integrate the differential equation $y(y-x-1) + xy' = 0$.*

Exercise *6.17 Consider the following differential form*

$$\omega = -dx - xdy + 2ze^{-y}dz.$$

1. *Show that ω is not total.*
2. *Show that ω has an integrating factor.*
3. *Find an integrating factor $\mu = \mu(y)$ such that $\mu(0) = 1$.*
4. *Deduce the solutions of the differential equation $\omega = 0$.*

Exercise *6.18 Consider the following differential form*

$$\omega = yzdx - xzdy + (x^2+xy)dz.$$

1. *Show that ω is not total.*
2. *Show that ω has an integrating factor.*
3. *Find the constant α so that $\mu(x,y) = (x+y)^\alpha$ is an integrating factor of ω.*
4. *Deduce the solutions of the differential equation $\omega = 0$.*

Exercise *6.19 Consider the following differential form*

$$\omega = ydx + 2xdy + 3xydz.$$

1. *Show that ω is not total.*
2. *Show that ω has an integrating factor.*
3. *Find the constant m so that $\mu(y,z) = ye^{mz}$ is an integrating factor of ω.*
4. *Find the function $f(x,y,z)$ that verifies $df(x,y,z) = \mu(y,z)\,\omega$ with $f(0,0,0) = 0$.*