Lebanese University

Faculty of Sciences

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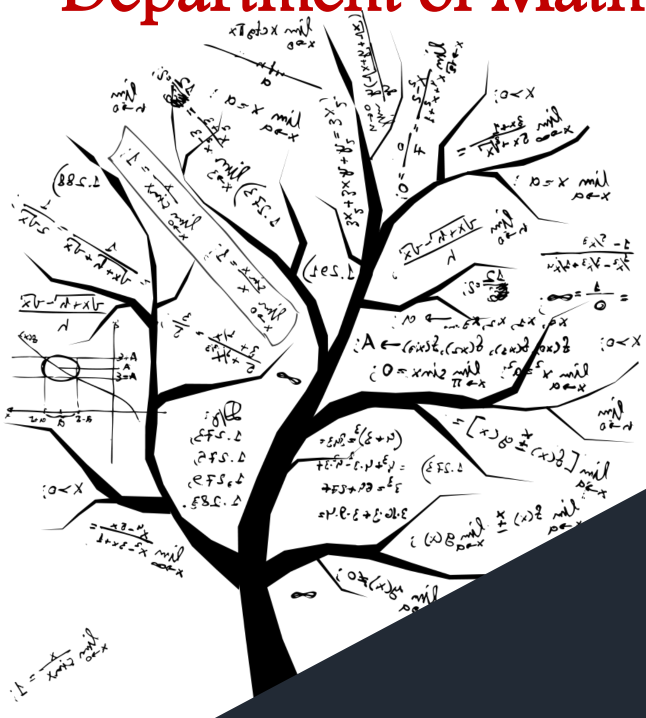
M1103
Algebra III

Department of Mathematics



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PREFACE

This book is intended to serve as lecture notes in linear algebra for the students of the Faculty of Sciences who are in the first year of Mathematics, Statistics and Computer sciences.

This course is divided into eight chapters and one appendix. In chapter I we study the matrices and in chapter II we study the echelon form of a matrix. Chapter III is devoted to the study of determinants and chapter IV is devoted to study of the systems of linear equations. Chapters V and VI deal with the vector spaces and their bases, while chapters VII and VIII are reserved for the study of the linear mappings and their matrix representations. As the proofs of some theorems are hard for the students, and thus should not be done in the classroom, then we give these proofs in the appendix at the end of the course.

Finally a word about the notation: 1.2.3 will denote the third theorem of the 2nd section of chapter one and the first corollary of this theorem is denoted corollary 1.2.3.1. The symbol ■ will indicate the end of a definition, theorem and a corollary.

Contents

CHAPTER I

MATRICES

Throughout this chapter, and unless mentioned otherwise, the letter K denotes a field. The symbols 0_K and 1_K will be denoted 0 and 1 respectively.

§ 1.1. INTRODUCTION.

In this section K can be any non-empty set.

Definition 1. We call (m×n) **matrix** over K, every rectangular table A with m rows and n columns of the form:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

where $a_{ij} \in K$, $\forall 1 \le i \le m$ and $\forall 1 \le j \le n$. ■

If $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ is a (m×n) matrix over K, then A will be written $A=(a_{ij})$. The elements $a_{11},...,a_{1n}$, $a_{21},...,a_{2n}$, ..., $a_{m1},...,a_{mn}$ are called the **entries** (or **coefficients**) of A and the entry a_{ij} is called the **main entry** (or the **(i,j)th entry**) of A.

For each $1 \le i \le m$, $(a_{i1},...,a_{in})$ is called the **ith row** and for each $1 \le j \le n$, $\begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix}$ is called the **jth column** of A.

If m=n, then A is called a **square matrix of order n**.

We denote by $M_{m,n}(K)$ the set of all (m×n) matrices over K. The set of the square matrices of order n will be denoted $M_n(K)$ instead of $M_{n,n}(K)$.

Definition 2. Two matrices $A=(a_{ij}) \in M_{m,n}(K)$ and $B=(b_{ij}) \in M_{s,t}(K)$ are said to be **equal** and we write A = B if m = s, n = t and $a_{ij} = b_{ij}$, for all $1 \le i \le m$ and $1 \le j \le n$. ■

Thus two (m×n) matrices $A=(a_{ij})$ and $B=(b_{ij})$ are equal if and only if $a_{ij} = b_{ij}$, for all $1 \le i \le m$ and $1 \le j \le n$.

Example: If $A = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$, then

$$A = B \Leftrightarrow x = 0,\ y = 1,\ z = 2 \text{ and } t = 3.$$

1.1.1. Let A and B be two (m×n) matrices over K, then

(i) If F_i denotes the ith row of A and E_i that of B, then

$$A=B \Leftrightarrow F_i = E_i \text{ , for all } 1\le i\le m.$$

(ii) If C_i denotes the ith column of A and D_i that of B, then

$$A=B \Leftrightarrow C_i = D_i \text{ , for all } 1\le i\le n.$$

Proof: The proof follows easily from definition 2. ■

If $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ is a (n×n) matrix over K, then the encircled part of A is called the **main diagonal** of A and the entries a_{11}, a_{22},..., a_{nn} are called the **diagonal entries** of A.

§ 1.2. REMARKABLE MATRICES.

Definition 3. A matrix A is said to be a **row** (resp. **column**) **matrix** if A is a (1×n) (resp. (m×1)) matrix over K. ■

Thus A is a row (resp. column) matrix if A has just one row (resp. column).

Examples: 1) A=(0 1 0) is a (1×3) matrix. It is a row matrix.

2) $A = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$ is a (3×1) matrix. It is a column matrix.

3) $A = \begin{pmatrix} 1 & i \\ -1 & 0 \end{pmatrix}$ is a (2×2) matrix. It is a square matrix of order 2.

4) $A = \begin{pmatrix} 1 & -1 & 3 \\ i & 0 & 1 \end{pmatrix}$ is a (2×3) matrix.

Definition 4. Let $A = (a_{ij}) \in M_n(K)$. We say that

(i) A is **lower triangular** if $a_{ij} = 0$, $\forall i<j$.

(ii) A is **upper triangular** if $a_{ij} = 0$, $\forall i>j$.

(iii) A is a **diagonal matrix** if $a_{ij} = 0$, $\forall i\neq j$. ■

A diagonal matrix $A=(a_{ij})$ of order n is usually denoted diag(a_{11},..., a_{nn}).

Examples: 1) The matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 1 & 0 \end{pmatrix}$ is lower triangular, $\begin{pmatrix} 2 & 3 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ is upper triangular and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ are diagonal matrices.

The diagonal matrix of order n over K whose diagonal entries are all equal to 1 is called the **unit matrix** of order n over K and is denoted I_n. Thus if $A=(a_{ij})\in M_n(K)$, then

$$A = I_n \Leftrightarrow a_{ij} = 0 \text{ if } i\neq j \text{ and } a_{ij} = 1 \text{ if } i = j.$$

We have

$$I_1 = (1)\,,\ I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The $(m\times n)$ matrix over K whose entries are all zero is called the **zero matrix** and denoted 0. Thus if $A=(a_{ij})$ is a $(m\times n)$ matrix over K, then

$$A = 0 \Leftrightarrow a_{ij} = 0,\ \forall i,j.$$

§ 1.3. ADDITION OF MATRICES.

Definition 5. We define an addition on $M_{m,n}(K)$ by: if $A=(a_{ij})$ and $B=(b_{ij})$ are two elements of $M_{m,n}(K)$, then we define A+B to be the $(m\times n)$ matrix $A+B=(c_{ij})$, such that $c_{ij}=a_{ij}+b_{ij}$, for all $1\leq i\leq m$ and $1\leq j\leq n$. ■

If $A=(a_{ij})$ and $B=(b_{ij})$ are two elements of $M_{m,n}(K)$, then we have

$$(a_{ij})+(b_{ij})=(a_{ij}+b_{ij}).$$

Example: If $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 & 2 \\ -1 & 4 & -4 \end{pmatrix}$, then

$$A+B = \begin{pmatrix} 1+1 & 2+3 & -1+2 \\ 0+(-1) & 1+4 & 5+(-4) \end{pmatrix} = \begin{pmatrix} 2 & 5 & 1 \\ -1 & 5 & 1 \end{pmatrix}.$$

1.3.1. The following hold for all $A,B,C\in M_{m,n}(K)$:

(i) (A+B)+C = A+(B+C),
(ii) A+B = B+A,
(iii) A+0 = 0+A = A,
(iv) If $A=(a_{ij})$, then the $(m\times n)$ matrix over K, $X=(-a_{ij})$, satisfies A+X=X+A=0.

In particular $M_{m,n}(K)$ is an additive abelian group.

Proof: Put $A=(a_{ij})$, $B=(b_{ij})$ and $C=(c_{ij})$.

(i) Let $A+B=(d_{ij})$, $B+C=(e_{ij})$, $A+(B+C)=(f_{ij})$ and $(A+B)+C=(h_{ij})$. We have

$$h_{ij} = d_{ij}+c_{ij} = (a_{ij}+b_{ij})+c_{ij} = a_{ij}+(b_{ij}+c_{ij}) = a_{ij}+e_{ij} = f_{ij}$$

hence (A+B)+C=A+(B+C).

(ii) Let $A+B=(d_{ij})$ and $B+A=(x_{ij})$. Then $d_{ij} = a_{ij}+b_{ij} = b_{ij}+a_{ij} = x_{ij}$, and so A+B =B+A.

(iii) Let $0=(y_{ij})$ and $A+0=(z_{ij})$. We have $z_{ij} = a_{ij}+y_{ij} = a_{ij}+0 = a_{ij}$, hence A+0=A, and so A+0 = 0+A = A.

(iv) Let $A+X=(b_{ij})$, then $b_{ij}=a_{ij}-a_{ij}=0$, hence A+X=0, and so A+X=X+A=0. ■

If $A=(a_{ij})$ is $(m\times n)$ matrix over K, then the matrix $X=(-a_{ij})$ is called **the opposite** of A and denoted $-A$. Thus if $A=(a_{ij})$, then $-A=(-a_{ij})$.

Example: If $A=\begin{pmatrix} 1 & 3 & i & -1 \\ 5 & i & 2 & -i \end{pmatrix}$, then $-A=\begin{pmatrix} -1 & -3 & -i & 1 \\ -5 & -i & -2 & i \end{pmatrix}$.

§ 1.4. MULTIPLICATION OF A MATRIX BY A SCALAR.

We call **scalar** every element of K.

For each $A=(a_{ij})\in M_{m,n}(K)$ and each $\alpha\in K$, we define αA to be the $(m\times n)$ matrix over K obtained from A by multiplying each entry of A by the scalar α. Thus if $A=(a_{ij})$, then $\alpha A=(\alpha a_{ij})$.

Example: If $\alpha\in\mathbb{R}$ and $A=\begin{pmatrix} 5 & -2 & 3 \\ 2 & -3 & 4 \end{pmatrix}$, then $\alpha A=\begin{pmatrix} 5\alpha & -2\alpha & 3\alpha \\ 2\alpha & -3\alpha & 4\alpha \end{pmatrix}$.

1.4.1. If A and B are $(m\times n)$ matrices over K and $\alpha,\beta\in K$, then
(i) $\alpha(A+B) = \alpha A+\alpha B$.
(ii) $(\alpha+\beta)A = \alpha A+\beta A$.
(iii) $\alpha(\beta A) = (\alpha\beta)A$.
(iv) $1A = A$.

Proof: Easy enough. ■

§ 1.5. PRODUCT OF MATRICES.

Definition 6. Let $A=(a_{ij})$ be $(m\times n)$ matrix and $B=(b_{ij})$ be $(n\times p)$ matrix over K. We say that a matrix $C=(c_{ij})$ is the **product** of A and B and we write C=AB if C is a $(m\times p)$ matrix over K and $c_{ij} = \sum_{t=1}^{n} a_{it}b_{tj}$, $\forall 1\le i\le m$ and $1\le j\le p$. ■

Remark: Let $A=(a_{ij})$ be $(m\times n)$ matrix and $B=(b_{ij})$ be $(n\times p)$ matrix over K. Let $AB=(c_{ij})$. Then

$$c_{ij} = a_{i1}b_{1j}+a_{i2}b_{2j}+\dots+a_{in}b_{nj}.$$

We notice that the ith row of A is $(a_{i1}\ a_{i2}\ \dots\ a_{in})$ and the jth column of B is $\begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix}$, hence c_{ij} is obtained by multiplying the ith row of A and the jth column of B as shown above.

Example: 1) If $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, then

$$AB = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1\times2+2\times1+3\times3 \\ 0\times2+1\times1+2\times3 \end{pmatrix} = \begin{pmatrix} 13 \\ 7 \end{pmatrix}.$$

BA is not defined because the number of columns of B is not equal to that of rows of A.

2) If $A = \begin{pmatrix} 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, then

$$AB=\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix}=(1\times3+2\times4)=(11) \text{ and } BA=\begin{pmatrix} 3 \\ 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \end{pmatrix}=\begin{pmatrix} 3\times1 & 3\times2 \\ 4\times1 & 4\times2 \end{pmatrix}=\begin{pmatrix} 3 & 6 \\ 4 & 8 \end{pmatrix}.$$

We notice that $AB\neq BA$.

Remark: If $AB=BA$, then the binomial formula $(A+B)^m = \sum_{t=0}^{m} C_m^t A^{m-t}B^t$, holds for the matrices in this case.

1.5.1. The following hold

(i) $A(BC)=(AB)C$, $\forall A\in M_{m,n}(K)$, $B\in M_{n,p}(K)$ and $C\in M_{p,r}(K)$.

(ii) $A(B+C)=AB+AC$, $\forall A\in M_{m,n}(K)$ and $B,C\in M_{n,p}(K)$.

(iii) $(A+B)C=AC+BC$, $\forall A,B\in M_{m,n}(K)$ and $C\in M_{n,p}(K)$.

(iv) $AI_n = I_m A = A$, $\forall A\in M_{m,n}(K)$. In particular $AI_n = I_n A = A$, $\forall A\in M_n(K)$.

(v) $0A = A0 = 0$, $\forall A\in M_{m,n}(K)$.

(vi) $(\alpha A)B=A(\alpha B)=\alpha(AB)$, $\forall A\in M_{m,n}(K)$, $B\in M_{n,p}(K)$ and $\alpha\in K$.

Proof: (i) Since A is a $(m\times n)$ matrix over K and BC is a $(n\times r)$ matrix over K, we then have A(BC) is a $(m\times r)$ matrix over K. Similarly, as AB is a $(m\times p)$ matrix over K and C is a $(p\times r)$ matrix over K, then (AB)C is a $(m\times r)$ matrix over K. Hence the two matrices A(BC) and (AB)C have the same number of rows and same number of columns.

Let $A=(a_{ij})$, $B=(b_{ij})$, $C=(c_{ij})$, $AB=(d_{ij})$, $BC=(e_{ij})$, $A(BC)=(f_{ij})$ and $(AB)C=(h_{ij})$. We have

$$f_{ij} = \sum_{t=1}^{n} a_{it}e_{tj} = \sum_{t=1}^{n} a_{it}(\sum_{s=1}^{p} b_{ts}c_{sj}) = \sum_{t=1}^{n}\sum_{s=1}^{p} a_{it}b_{ts}c_{sj}$$

and

$$h_{ij} = \sum_{s=1}^{p} d_{is}c_{sj} = \sum_{s=1}^{p}(\sum_{t=1}^{n} a_{it}b_{ts})c_{sj} = \sum_{s=1}^{p}\sum_{t=1}^{n} a_{it}b_{ts}c_{sj}.$$

But $\sum_{t=1}^{n}\sum_{s=1}^{p} a_{it}b_{ts}c_{sj} = \sum_{s=1}^{p}\sum_{t=1}^{n} a_{it}b_{ts}c_{sj}$, hence $f_{ij} = h_{ij}$, and so $A(BC)=(AB)C$.

(ii) We have that A(B+C) and AB+AC are two $(m\times p)$ matrices over K. Let $A=(a_{ij})$, $B=(b_{ij})$, $C=(c_{ij})$, $AB=(d_{ij})$, $AC=(e_{ij})$, $B+C=(f_{ij})$, $AB+AC=(h_{ij})$ and $A(B+C)=(x_{ij})$. We have that

$$x_{ij} = \sum_{t=1}^{n} a_{it}f_{tj} = \sum_{t=1}^{n} a_{it}(b_{tj}+c_{tj}) = \sum_{t=1}^{n}(a_{it}b_{tj}+a_{it}c_{tj}) = \sum_{t=1}^{n} a_{it}b_{tj} + \sum_{t=1}^{n} a_{it}c_{tj} = d_{ij}+e_{ij} = h_{ij}$$

hence $A(B+C) = AB+AC$.

(iii) The proof is similar to that of (ii).

(iv) Let $A=(a_{ij})$, $I_n=(b_{ij})$ and $AI_n=(c_{ij})$. We have that A and AI_n are two $(m\times n)$ matrices over K and $c_{ij} = \sum_{t=1}^{n} a_{it}b_{tj} = \sum_{\substack{t=1\\t\neq j}}^{n} a_{it}b_{tj} + a_{ij}b_{ij} = 0 + a_{ij}\times 1 = a_{ij}$, hence $AI_n = A$.

Similarly one can easily show that $I_m A = A$.

(v) Let $A=(a_{ij})$, $0=(b_{ij})$ and $A0=(c_{ij})$. As $b_{ij}=0$, $\forall i,j$, then $c_{ij} = \sum_{t=1}^{n} a_{it}b_{tj} = 0$, $\forall i,j$, and so $A0=0$. Similarly one can easily prove that $0A=0$.

(vi) Let $A=(a_{ij})$, $B=(b_{ij})$, $\alpha A=(c_{ij})$, $AB=(d_{ij})$, $\alpha(AB)=(e_{ij})$ and $(\alpha A)B=(f_{ij})$. We have that $\alpha(AB)$ and $(\alpha A)B$ are two $(m\times p)$ matrices over K and

$$e_{ij} = \alpha d_{ij} = \alpha \sum_{t=1}^{n} a_{it}b_{tj} = \sum_{t=1}^{n} \alpha(a_{it}b_{tj}) \text{ and } f_{ij} = \sum_{t=1}^{n} c_{it}b_{tj} = \sum_{t=1}^{n} (\alpha a_{it})b_{tj} = \sum_{t=1}^{n} \alpha(a_{it}b_{tj})$$

hence $e_{ij} = f_{ij}$, and so

$$\alpha(AB) = (\alpha A)B.$$

Let $A(\alpha B)=(h_{ij})$ and $\alpha B=(x_{ij})$. We have that $\alpha(AB)$ and $A(\alpha B)$ are two $(m\times p)$ matrices over K and

$$h_{ij} = \sum_{t=1}^{n} a_{it}x_{tj} = \sum_{t=1}^{n} a_{it}(\alpha b_{tj}) = \sum_{t=1}^{n} \alpha(a_{it}b_{tj}) = e_{ij}$$

hence $\alpha(AB) = A(\alpha B)$, and so $(\alpha A)B = A(\alpha B) = \alpha(AB)$. ■

Corollary 1.5.1.1. $M_n(K)$ is a unitary ring under the defined addition and multiplication of matrices.

Proof: By 1.3.1 and 1.5.1(i), (ii), (iii) and (iv). ■

1.5.2. Let $A=(a_{ij})$ be a $(m\times n)$ matrix and $B=(b_{ij})$ be $(n\times p)$ matrix over K. If F_i denotes the ith row of A and C_j denotes the jth column of B, then

(i) $F_i \times B$ is the ith row of AB.

(ii) $A\times C_j$ is the jth column of AB.

Proof: Put $AB=(c_{ij})$.

(i) $F_i \times B = \begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{pmatrix} = \begin{pmatrix} \sum_{t=1}^{n} a_{it}b_{t1} & \sum_{t=1}^{n} a_{it}b_{t2} & \cdots & \sum_{t=1}^{n} a_{it}b_{tp} \end{pmatrix}$

$= \begin{pmatrix} c_{i1} & c_{i2} & \cdots & c_{ip} \end{pmatrix}$, hence $F_i \times B$ is the ith row of AB.

(ii) $A\times C_j = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix} = \begin{pmatrix} \sum_{t=1}^{n} a_{1t}b_{tj} \\ \sum_{t=1}^{n} a_{2t}b_{tj} \\ \vdots \\ \sum_{t=1}^{n} a_{mt}b_{tj} \end{pmatrix} = \begin{pmatrix} c_{1j} \\ c_{2j} \\ \vdots \\ c_{mj} \end{pmatrix}$, hence $A\times C_j$ is the jth column of AB. ■

Thus

$$\text{ith row of } AB = (\text{ith row of } A)\times B$$
and
$$\text{jth column of } AB = A\times(\text{jth column of } B).$$

Example: Let $A=\begin{pmatrix}1&1&2\\0&1&3\\1&3&5\end{pmatrix}$ and $B=\begin{pmatrix}2&1&1&0\\3&0&0&1\\1&1&2&2\end{pmatrix}$, then

$$\text{the } 3^{rd} \text{ row of } AB \text{ is } \begin{pmatrix}1&3&5\end{pmatrix}\begin{pmatrix}2&1&1&0\\3&0&0&1\\1&1&2&2\end{pmatrix}=\begin{pmatrix}16&6&11&13\end{pmatrix}$$
and
$$\text{the } 4^{th} \text{ column of } AB \text{ is } \begin{pmatrix}1&1&2\\0&1&3\\1&3&5\end{pmatrix}\begin{pmatrix}0\\1\\2\end{pmatrix}=\begin{pmatrix}5\\7\\13\end{pmatrix}.$$

Corollary 1.5.2.1. Let $A=(a_{ij})$ be a $(m\times n)$ matrix. Then we have

(i) If F_i denote the ith row of A and M_i denotes that of I_m, then $M_i\times A=F_i$.

(ii) If C_j denotes the jth column of A and N_j denotes that of I_n, then $A\times N_j=C_j$.

Proof: (i) We have that $M_i\times A$ is the ith row of $I_m A$, by 1.5.2(i), and $I_m A=A$, hence $M_i\times A$ is the ith row of A, and so $M_i\times A=F_i$.

(ii) As $A\times N_j$ is the jth column of AI_n, by 1.5.2(ii) and $AI_n=A$, then $A\times N_j$ is the jth column of A, and so $A\times N_j=C_j$. ■

§ 1.6. TRANSPOSE OF A MATRIX.

Definition 7. Let $A=(a_{ij})$ be a $(m\times n)$ matrix over K. We define the **transpose** of the matrix A to be the $(n\times m)$ matrix over K, ${}^tA=(b_{ij})$, such that $b_{st}=a_{ts}$, $\forall 1\le t\le m$ and $\forall 1\le s\le n$. ■

The transpose of A is also denoted A^T.

Examples: If $A=\begin{pmatrix}1&i&0\end{pmatrix}$, then ${}^tA=\begin{pmatrix}1\\i\\0\end{pmatrix}$ and if $B=\begin{pmatrix}1&2\\i&3\end{pmatrix}$, then ${}^tB=\begin{pmatrix}1&i\\2&3\end{pmatrix}$.

1.6.1. The following hold

(i) ${}^t(A+B)={}^tA+{}^tB$, $\forall A,B\in M_{m,n}(K)$.

(ii) ${}^t(\alpha A)=\alpha\,{}^tA$ and ${}^t(A\alpha)=({}^tA)\alpha$, $\forall\alpha\in K$ and $\forall A\in M_{m,n}(K)$.

(iii) ${}^t(AB)={}^tB\,{}^tA$, $\forall A\in M_{m,n}(K)$ and $B\in M_{n,p}(K)$.

(iv) ${}^t({}^tA)=A$, $\forall A\in M_{m,n}(K)$.

Proof: (i) Let $A=(a_{ij})$, $B=(b_{ij})$, $A+B=(c_{ij})$, ${}^tA=(d_{ij})$, ${}^tB=(e_{ij})$, ${}^tA+{}^tB=(f_{ij})$ and ${}^t(A+B)=(h_{ij})$. We have that ${}^t(A+B)$ and ${}^tA+{}^tB$ are $(n\times m)$ matrices over K and

$$h_{ij}=c_{ij}=a_{ij}+b_{ij}=d_{ij}+e_{ij}=f_{ij}$$

hence ${}^tA+{}^tB={}^t(A+B)$.

(ii) Let $A=(a_{ij})$, $\alpha A=(b_{ij})$, ${}^tA=(c_{ij})$, $\alpha{}^tA=(d_{ij})$ and ${}^t(\alpha A)=(e_{ij})$. We have that ${}^t(\alpha A)$ and $\alpha{}^tA$ are two $(n\times m)$ matrices over K and

$$e_{ij} = b_{ji} = \alpha a_{ji} = \alpha c_{ij} = d_{ij}$$

so that ${}^t(\alpha A)=\alpha{}^tA$. Similarly one can easily prove that ${}^t(\alpha A) = ({}^tA)\alpha$.

(iii) Let $A=(a_{ij})$, $B=(b_{ij})$, $AB=(c_{ij})$, ${}^tA=(d_{ij})$, ${}^tB=(e_{ij})$, ${}^tB\,{}^tA=(f_{ij})$ and ${}^t(AB)=(h_{ij})$. We have ${}^t(AB)$ and ${}^tB\,{}^tA$ are $(p\times m)$ matrices over K and

$$h_{ij} = c_{ji} = \sum_{t=1}^{n} a_{jt}b_{ti} = \sum_{t=1}^{n} d_{tj}e_{it} = \sum_{t=1}^{n} e_{it}d_{tj} = f_{ij}$$

hence ${}^t(AB) = {}^tB\,{}^tA$.

(iv) Let $A=(a_{ij})$, ${}^tA=(b_{ij})$ and ${}^t({}^tA)=(c_{ij})$. We have that ${}^t({}^tA)$ and A are $(m\times n)$ matrices over K and $c_{ij} = b_{ji} = a_{ij}$, and so ${}^t({}^tA) = A$. ■

Definition 8. Let A be a square matrix of order n over K. We say that A is **symmetric** (resp. **anti-symmetric**) if ${}^tA=A$ (resp. ${}^tA=-A$). ■

Thus if $A=(a_{ij})$ is a square matrix of order n over K, then

(i) A is symmetric if and only if $a_{ij}= a_{ji}$, for all $1\le i\le n$ and all $1\le j\le n$,

(ii) A is anti-symmetric if and only if $a_{ij}= -a_{ji}$, for all $1\le i\le n$ and all $1\le j\le n$.

§ 1.7. TRACE OF A MATRIX.

Definition 9. Let $A = (a_{ij})\in M_n(K)$. We define **the trace** of A, written Tr(A) to be the sum of its diagonal entries, i.e $\mathrm{Tr}(A) = \sum_{t=1}^{n} a_{tt} = a_{11}+a_{22}+\cdots+a_{nn}$. ■

1.7.1. The following hold, for all $A,B\in M_n(K)$ and all $\alpha\in K$

(i) Tr(A+B) = Tr(A)+Tr(B).

(ii) $\mathrm{Tr}(\alpha A) = \alpha\mathrm{Tr}(A)$.

(iii) Tr(AB) = Tr(BA).

(iv) $\mathrm{Tr}({}^tA) = \mathrm{Tr}(A)$.

Proof: (i) Let $A=(a_{ij})$, $B=(b_{ij})$ and $A+B=(c_{ij})$, then

$$\mathrm{Tr}(A+B)= \sum_{t=1}^{n} c_{tt} = \sum_{t=1}^{n}(a_{tt}+b_{tt}) = \sum_{t=1}^{n} a_{tt} + \sum_{t=1}^{n} b_{tt} = \mathrm{Tr}(A)+\mathrm{Tr}(B).$$

(ii) Let $A=(a_{ij})$ and $\alpha A=(b_{ij})$, then $\mathrm{Tr}(\alpha A)= \sum_{t=1}^{n} b_{tt} = \sum_{t=1}^{n}\alpha a_{tt} = \alpha(\sum_{t=1}^{n} a_{tt})=\alpha\mathrm{Tr}(A)$.

(iii) Let $A=(a_{ij})$, $B=(b_{ij})$, $AB=(c_{ij})$ and $BA=(d_{ij})$, then

$$\mathrm{Tr}(AB)= \sum_{t=1}^{n} c_{tt} = \sum_{t=1}^{n}\sum_{s=1}^{n} a_{ts}b_{st} \text{ and } \mathrm{Tr}(BA)= \sum_{s=1}^{n} d_{ss} = \sum_{s=1}^{n}\sum_{t=1}^{n} b_{st}a_{ts}.$$

As $a_{ts}b_{st} = b_{st}a_{ts}$, then $\sum_{s=1}^{n}\sum_{t=1}^{n} b_{st}a_{ts} = \sum_{t=1}^{n}\sum_{s=1}^{n} a_{ts}b_{st}$, and so Tr(AB) = Tr(BA).

(iv) Let $A=(a_{ij})$ and ${}^tA=(b_{ij})$, then $Tr({}^tA)=\sum_{t=1}^{n} b_{tt} = \sum_{t=1}^{n} a_{tt} = Tr(A)$. ■

§ 1.8. INVERTIBLE MATRICES.

Definition 10. A matrix $A\in M_n(K)$ is said to be **invertible** (or **non-singular**) if there exists $B\in M_n(K)$, such that $AB=BA=I_n$. ■

1.8.1. If $A\in M_n(K)$ is an invertible matrix, then the matrix B in $M_n(K)$, such that $AB=BA=I_n$ is unique. B is called the inverse of A and denoted A^{-1}.

Proof: Let $C\in M_n(K)$, such that $AC=CA=I_n$. We have

$$B=BI_n=B(AC)=(BA)C=I_nC=C. ■$$

1.8.2. If $A,B\in M_n(K)$ and $AB=I_n$, then $BA=I_n$.

Proof: To be done in corollary 8.1.10.1 of chapter VIII. ■

Corollary 1.8.2.1. If $A\in M_n(K)$, then A is invertible if and only if there exists $B\in M_n(K)$, such that $AB = I_n$.

Proof: Follows easily from 1.8.2. ■

1.8.3. Let $A,B\in M_n(K)$. If A and B are invertible, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Proof: Easy, since the multiplication of matrices is associative. ■

Corollary 1.8.3.1. If $A_1,\ldots,A_r$ are a (n×n) invertible matrices, then $A_1\times\cdots\times A_r$ is invertible and $(A_1\times\cdots\times A_r)^{-1} = A_r^{-1}\times\cdots\times A_1^{-1}$.

Proof: Argue by induction on r and apply 1.8.3. ■

The set of invertible matrices of $M_n(K)$ is a group under the multiplication of matrices, called **the general linear group** and denoted $GL_n(K)$.

1.8.4. If $A\in M_n(K)$ and A is invertible, then ${}^tA$ is invertible and $({}^tA)^{-1} = {}^t(A^{-1})$. In particular ${}^tA\in GL_n(K)$, for all $A\in GL_n(K)$.

Proof: As ${}^tA\,{}^t(A^{-1}) = {}^t(A^{-1}A) = {}^t(I_n) = I_n$, then ${}^tA$ is invertible and $({}^tA)^{-1} = {}^t(A^{-1})$. ■

1.8.5. If a (n×n) matrix A is invertible, then every row (resp. column) of A is non-zero.

Proof: Assume that the ith row R_i of A is zero, then $R_i \times A^{-1}=0$. As $R_i \times A^{-1}$ is the ith row of AA^{-1}, by 1.5.2(i), and $AA^{-1}=I_n$, then the ith row of I_n is zero, which is impossible. Therefore $R_i \neq 0$, and so every row of A is non-zero.
Suppose that the ith column C_i of A is zero, then $A^{-1}\times C_i=0$. As $A^{-1}\times C_i$ is the ith column of $A^{-1}A$ and $A^{-1}A=I_n$, then the ith column of I_n is zero, which is impossible. Therefore $C_i \neq 0$, and so every column of A is non-zero. ■

§ 1.9. MULTIPLICATION BY BLOCKS.

Let $A=(a_{ij})$ be a $(m\times n)$ matrix over K.

Definition 11. Let $j_1, \dots, j_r$ be natural numbers, such that $1\leq j_1< \dots<j_r<n$. If $X_1, \dots, X_r$ are matrices, such that

$$X_1=\begin{pmatrix} a_{11} & \cdots & a_{1j_1} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj_1}\end{pmatrix}, X_2=\begin{pmatrix} a_{1j_1+1} & \cdots & a_{1j_2} \\ \vdots & & \vdots \\ a_{mj_1+1} & \cdots & a_{mj_2}\end{pmatrix}, \dots, X_r=\begin{pmatrix} a_{1j_r+1} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{mj_r+1} & \cdots & a_{mn}\end{pmatrix}$$

then each matrix X_i is called **a vertical sub-matrix** of A and the row matrix $X=\begin{pmatrix}X_1 & \cdots & X_r\end{pmatrix}$ is called **a vertical partition** of A. ■

Similarly if $Y_1, \dots, Y_s$ are matrices, such that

$$Y_1=\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{i_11} & \cdots & a_{i_1n}\end{pmatrix}, Y_2=\begin{pmatrix} a_{i_1+11} & \cdots & a_{i_1+1n} \\ \vdots & & \vdots \\ a_{i_21} & \cdots & a_{i_2n}\end{pmatrix}, \dots, Y_s=\begin{pmatrix} a_{i_s+11} & \cdots & a_{i_s+1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn}\end{pmatrix}$$

with $1\leq i_1< \dots<i_s<m$, then each Y_i is called **a horizontal sub-matrix** of A and the column matrix $Y=\begin{pmatrix}Y_1 \\ \vdots \\ Y_s\end{pmatrix}$ is called **a horizontal partition** of A.

Examples: If $A=\begin{pmatrix} 1 & 2 & 0 & 3 \\ -4 & -5 & 8 & 7 \\ 9 & -3 & 6 & -2\end{pmatrix}$, then $\left(\begin{pmatrix} 1 & 2 \\ -4 & -5 \\ 9 & -3\end{pmatrix} \begin{pmatrix} 0 & 3 \\ 8 & 7 \\ 6 & -2\end{pmatrix}\right)$ is a vertical partition of A and $\begin{pmatrix}\begin{pmatrix} 1 & 2 & 0 & 3 \\ -4 & -5 & 8 & 7\end{pmatrix} \\ \begin{pmatrix} 9 & -3 & 6 & -2\end{pmatrix}\end{pmatrix}$ is a horizontal partition of A.

Definition 12. Let $\begin{pmatrix}X_1 & \cdots & X_r\end{pmatrix}$ be a vertical partition of A and for each $1\leq j\leq r$ let $\begin{pmatrix}A_{1j} \\ \vdots \\ A_{sj}\end{pmatrix}$ be a horizontal partition of X_j. If in the matrix

$$B = \begin{pmatrix} A_{11} & \cdots & A_{1r} \\ \vdots & & \vdots \\ A_{s1} & \cdots & A_{sr} \end{pmatrix}$$

the matrices in each row (resp. column) have the same number of rows (resp. columns), then B is called **a partition of A by blocks**. ■

<u>Remark:</u> If B=$\begin{pmatrix} A_{11} & \cdots & A_{1r} \\ \vdots & & \vdots \\ A_{s1} & \cdots & A_{sr} \end{pmatrix}$ is a partition of A by blocks, we write A=B and we say that A is **subdivided** (or **partitioned**) **into blocks** and each A_{ij} is called a **block** of A.

<u>Example:</u> If A = $\begin{pmatrix} 1 & 2 & 0 & 3 \\ -4 & -5 & 8 & 7 \\ 9 & -3 & 6 & -2 \end{pmatrix}$, then $\begin{pmatrix} \begin{pmatrix} 1 & 2 \\ -4 & -5 \end{pmatrix} & \begin{pmatrix} 0 & 3 \\ 8 & 7 \end{pmatrix} \\ \begin{pmatrix} 9 & -3 \end{pmatrix} & \begin{pmatrix} 6 & -2 \end{pmatrix} \end{pmatrix}$ is a partition of A by blocks.

<u>1.9.1.</u> Let A be (m×n) and B be (n×p) matrices over K. If $\begin{pmatrix} A_{11} & \cdots & A_{1r} \\ \vdots & & \vdots \\ A_{s1} & \cdots & A_{sr} \end{pmatrix}$ is a partition of A and $\begin{pmatrix} B_{11} & \cdots & B_{1t} \\ \vdots & & \vdots \\ B_{r1} & \cdots & B_{rt} \end{pmatrix}$ is a partition of B, such that $A_{i\alpha} \times B_{\alpha j}$ is defined for all $1 \le i \le s$, $1 \le j \le t$ and $1 \le \alpha \le r$, then

$$AB = \begin{pmatrix} C_{11} & \cdots & C_{1t} \\ \vdots & & \vdots \\ C_{s1} & \cdots & C_{st} \end{pmatrix}$$

where $C_{ij} = \sum_{\alpha=1}^{r} A_{i\alpha} B_{\alpha j}$, $\forall 1 \le i \le s$ and $1 \le j \le t$.

<u>Proof:</u> To be admitted without proof. ■

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CHAPTER I

EXERCISES

1- Given the real matrices

$$A=\begin{pmatrix}1 & 3\\ 2 & -1\end{pmatrix},\ B=\begin{pmatrix}2 & 0 & -4\\ 3 & -2 & 0\end{pmatrix} \text{ and } C=\begin{pmatrix}1 & 2\\ 0 & 1\\ 1 & 0\end{pmatrix}.$$

Calculate, if possible, AB, BA, A+B, BC, CB, AC, CA, t_B, t_C and $t_{(BC)}$.

2- Find the real numbers x, y, z and t in the following cases:

(a) $\begin{pmatrix}3x+y & x-3y\\ 4z-2t & z+t\end{pmatrix}\begin{pmatrix}1 & -2\\ 3 & -5\end{pmatrix}=\begin{pmatrix}1 & 4\\ 2 & 8\end{pmatrix}$

(b) $2\begin{pmatrix}x & y\\ z & t\end{pmatrix}-5\begin{pmatrix}x-2 & y+3\\ z-2 & t+1\end{pmatrix}=\begin{pmatrix}2 & 1\\ 0 & 5\end{pmatrix}$.

3- Calculate

$$(1+i\ \ 2-i\ \ 3)\begin{pmatrix}0 & -2 & 3\\ 2 & 1 & i\\ -1 & 2 & 4\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ -i & 1 & 0\\ i & 0 & 2\end{pmatrix}.$$

4- Compute the nth power of the following matrices

$$A=\begin{pmatrix}1 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 1\end{pmatrix},\ B=\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix},\ C=\begin{pmatrix}0 & 1 & 1 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{pmatrix},\ D=\mathrm{diag}\begin{pmatrix}a_1 & \cdots & a_s\end{pmatrix} \text{ and}$$

$$E=\begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}.$$

5- Show that if A and B are two matrices, such that A is $(n\times p)$ and B is $(p\times n)$ and if $AB=I_n$, then $(BA)^2B=B$.

6- (a) Find two non-zero real matrices A and B, such that AB=0.

(b) Find three real matrices A, B and C, such that $A\neq B$, $C\neq 0$ and AC=BC.

7- Show that if $A\in M_n(K)$, then $A+t_A$ is symmetric and $A-t_A$ is anti-symmetric. Deduce that every real (resp. complex) matrix is the sum of two real (resp. complex) matrices with one of them is symmetric and the other is anti-symmetric.

8- Let $A\in M_{m,n}(\mathbb{R})$.

(a) Show that $At_A=0 \Rightarrow A=0$.

(b) Show that if $B\in M_{p,m}(\mathbb{R})$, then

$$BAt_A=0 \Rightarrow BA=0.$$

9- Given the real matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

Compare $(A+B)^2$ and $A^2+2AB+B^2$.

10- Let $A=\begin{pmatrix} 2 & 5 \\ 0 & 2 \end{pmatrix}$ and let $B=A-2I_2$. Calculate B^n and deduce A^n, where $n\in\mathbb{N}$.
(**Hint:** Use the binomial formula).

11- Let $A=\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and let $B= A-I_3$. Calculate B^n and deduce A^n, where $n\in\mathbb{N}$.

12- Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(K)$.

1- Show that $A^2-(a+d)A+(ad-bc)I_2 = 0$.

2- Show that if $ad-bc\neq 0$, then A is invertible and $A^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

13- Let

$$A=\begin{pmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{pmatrix}.$$

1- Show that $A^2 = I_3$,

2- Deduce that A is invertible and compute A^{-1}.

3- Determiner $({}^tA)^{-1}$.

14- Given the matrix

$$A=\begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix}.$$

1- Calculate $({}^tA)A$.

2- Find the conditions under which A is invertible and calculate A^{-1}.

15- Find the conditions satisfied by $a_1, \ldots, a_n$ under which $\mathrm{diag}\begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix}$ is invertible and find its inverse.

16- Let $A\in M_m(K)$, $B\in M_n(K)$ and C be $(m\times n)$ matrix over K. Show that if A and B are invertible, then so is $M=\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. Find M^{-1}.

Application: Show that the matrix

$$M = \begin{pmatrix} 2 & 0 & 1 & -1 \\ 0 & -3 & 2 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

is invertible and find its inverse.

17- Let $A \in M_n(K)$. We say that A is **nilpotent** if there exists a natural number p, such that $A^p = 0$. Show that if A is nilpotent, then $I_n - A$ is invertible and give its inverse.

18- Check if each of the following complex matrices is invertible or not and if yes, find its inverse :

$$A = \begin{pmatrix} 1 & -i & 0 \\ 1+i & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 4 & 2 & 1 \end{pmatrix}.$$

19- Show that if $A, B \in M_n(K)$ and AB is invertible, then A and B are invertible.

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CHAPTER II

ECHELON FORM OF A MATRIX.

Throughout this chapter, the letter K denotes a field.

§ 2.1. DEFINITION AND PROPERTIES.

Definition 1. Let $A=(a_{ij})$ be $(m\times n)$ matrix over K. We define the **leading entry** of every non-zero row of A to be the first non-zero entry from the left ∎

Thus if $R = \begin{pmatrix} a_{t1} & a_{t2} & \cdots & a_{tn}\end{pmatrix}$ is a non-zero row of A, then a_{ti} is the leading entry of R if $a_{ti}\neq 0$ and $a_{tj}=0$, $\forall 1\leq j\leq i-1$.

Similarly we define the leading entry of a non-zero column of A to be the first non-zero entry from above.

The leading entry of a row (resp. column) is also called **the pivot** of this row (resp. column).

Example: Let $A=\begin{pmatrix} 3 & 0 & 1 & 2\\ 0 & 1 & 0 & 3\\ 0 & 0 & 2 & 4\\ 0 & 0 & 0 & 0\end{pmatrix}$, then the leading entry of the first row of A is 3 and that of the 3rd row is 2, while the leading entry of the 2nd column of A is 1 and that of the 4th column is 2.

Definition 2. A matrix $A=(a_{ij})$ is called in **row echelon form** if A=0 or if $A\neq 0$ and r is the number of non-zero rows of A, then
(i) the first r rows of A are non-zero and all the others are zero,
(ii) if $a_{1j_1}, \ldots, a_{rj_r}$ are the leading entries of the first r rows of A, then $j_1 < j_2 < \cdots < j_r$. ∎

It follows that a non-zero matrix A is in row echelon form if
i) the non-zero rows of A are the first rows,
ii) in the column of every pivot the entries below the pivot are zero,
iii) the pivot of every row is strictly to the right of that of the previous row.

If furthermore the pivot of every non-zero row is 1 and is the only non-zero entry of its column, then we say that A is in **reduced row echelon form**.

Examples: 1) The matrix $A=\begin{pmatrix} 0 & 3 & 0 & 1\\ 0 & 0 & 0 & 6\\ 0 & 0 & 0 & 0\end{pmatrix}$ is in row echelon form, because :

$A\neq 0$ and the number of its non-zero rows is 2 and they are the first two and if j_t denotes the column of the leading entry of the tth row of A, where $1\leq t\leq 2$, then $j_1=2$ and $j_2=4$, and so as $j_1<j_2$, then A is in row echelon form.

2) The matrices $A=\begin{pmatrix}1&3&0&2\\2&1&0&3\\0&0&0&0\\0&0&0&0\end{pmatrix}$ and $B=\begin{pmatrix}0&0&2&2&0\\0&0&0&0&0\\0&0&0&3&1\\0&0&0&0&0\end{pmatrix}$ are not in row echelon form:

For A : We have that $A\neq 0$ and the number of non-zero rows of A is 2 and they are the first two. For each $1\leq t\leq 2$, let j_t denote the column of the leading entry of the tth row of A, then $j_1=1$ and $j_2=1$, and so as $j_1=j_2$, then A is not in row echelon form.
For B : As the number of the non-zero rows of B is 2 and they are not the first two, then B is not in row echelon form.

3) The matrix $A=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}$ is in reduced row echelon form, while the matrix

$B=\begin{pmatrix}0&0&0&1&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\end{pmatrix}$ is not in reduced row echelon form.

Remark: In a similar way we define a **column echelon form**.

2.1.1. A matrix A is in row echelon form if and only if its transpose is in column echelon form.

Proof: Follows easily from the fact that the transpose of every row (resp. column) of a matrix is a column (resp. row) of its transpose. ■

It follows from 2.1.1 that every property which is true for row echelon forms is true for column echelon forms.

Definition 3. We call **elementary row operation** on A, each of the following operations:
(i) interchanging two rows of A;
(ii) replacing a row R_t of A with $R_t+\alpha R_s$, with $s\neq t$ and $\alpha\in K$;
(iii) multiplying a row of A by α, where $\alpha\in K-\{0\}$. ■

Notation: (i) The operation where we interchange the rows R_t and R_s of A is denoted $R_t\leftrightarrow R_s$ and the matrix obtained from this operation is denoted $A(R_t\leftrightarrow R_s)$.
(ii) The operation where we replace the row R_t of A by $\alpha_1R_1+\cdots+\alpha_nR_n$, with $\alpha_1,\ldots,\alpha_n\in K$ and $\alpha_t\neq 0$, is denoted $R_t\to\alpha_1R_1+\cdots+\alpha_nR_n$ and the obtained matrix is denoted $A(R_t\to\alpha_1R_1+\cdots+\alpha_nR_n)$.
(iii) If $A_1,\ldots,A_k$ is a series of matrices, such that A_1 is obtained from A by a row operation and for each $2\leq i\leq k$, A_i is obtained from A_{i-1} by carrying on A_{i-1} a row operation, then each A_i is called **a matrix obtained from A by a finite sequence of row operations** and $A_1,\ldots,A_k$ is called a series of matrices obtained from A by a finite sequence of row operations. In this case the series is denoted

$$A\to A_1\to A_2\to\ldots.\to A_{k-1}\to A_k$$

where on the right of the rows of these matrices, we write the operations carried on these rows, for example if

$$A=\begin{pmatrix}0&1&3\\1&1&2\\3&2&2\end{pmatrix},\ A_1=A(R_1\leftrightarrow R_2)\text{ and } A_2=A_1(R_3\to 2R_3+3R_1)$$

then the series A_1, A_2 is written

$$\begin{pmatrix}0&1&3\\1&1&2\\3&2&2\end{pmatrix}\xrightarrow{R_1\leftrightarrow R_2}\begin{pmatrix}1&1&2\\0&1&3\\3&2&2\end{pmatrix}\xrightarrow[R_3\to 2R_3+3R_1]{}\begin{pmatrix}1&1&2\\0&1&3\\9&7&10\end{pmatrix}.$$

2.1.2. The matrix $B=A(R_t\to\alpha_1R_1+\cdots+\alpha_nR_n)$, with $\alpha_1,\ldots,\alpha_n\in K$ and $\alpha_t\neq 0$ is obtained from A by a finite sequence of elementary row operations.

Proof: For each $1\le i\le n$, let X_i denote the ith row of A**.** We argue by induction on the number s of the natural numbers j of $\{1,\ldots,n\}$, such that $\alpha_j\neq 0$. For s=1, we have $B=A(R_t\to\alpha_tR_t)$, because $\alpha_t\neq 0$, hence the property holds for s=1. Assume that the property holds up to s-1 and let's prove it for s. We have $s\ge 2$, because the case s=1 is already done, hence $\exists k\in\{1,\ldots,n\}$, such that $k\neq t$ and $\alpha_k\neq 0$. Let

$$B_1=A(R_t\to\beta_1R_1+\cdots+\beta_nR_n),\text{ with } \beta_i=\alpha_i \text{ if } i\neq k \text{ and } \beta_k=0.$$

Then the rows of B_1 are those of A except the tth which is $\beta_1X_1+\cdots+\beta_nX_n$, with $\beta_k=0$. As the number of natural numbers j of $\{1,\ldots,n\}$, such that $\beta_j\neq 0$ is s-1, then

B_1 is obtained from A by a finite sequence of elementary row operations

by induction hypothesis. Let

$$B_2=B_1(R_t\to R_t+\alpha_kR_k),$$

We have that B_2 is obtained from B_1 by one elementary row operations, and so as B_1 is obtained from A by a finite sequence of elementary row operations, then

B_2 is obtained from A by a finite sequence of elementary row operations.

As the tth row of B_2 is $\alpha_1X_1+\cdots+\alpha_nX_n$ and every other row of B_2 is equal to that of A, we get that $B_2=A(R_t\to\alpha_1R_1+\cdots+\alpha_nR_n)=B$. It follows that B is obtained from A by a finite sequence of elementary row operations, hence the property holds for s, and so it is true, for all $s\ge 1$. ■

Definition 4. A matrix B is said to be **a row echelon form** of A if B is a matrix in row echelon form obtained from A by a finite sequence of row operations. ■

2.1.3. Every $(m\times n)$ matrix A over K can be changed to a matrix in row echelon form by a finite sequence of elementary operations.

Proof: We argue by induction on m. It is true for m=1, because every row matrix is in row echelon form. Assume that the property holds up to m-1 and let's prove it for m. If A has a row equals to zero, then interchanging it with the last row we obtain a matrix of the form $\begin{pmatrix}B\\0\end{pmatrix}$, where B is a $((m-1)\times n)$ matrix, and so induction yields that B can be changed to a row echelon form B', say. Carrying the same elementary row operations on the rows of B in the matrix $\begin{pmatrix}B\\0\end{pmatrix}$, we obtain the matrix $\begin{pmatrix}B'\\0\end{pmatrix}$, which is a row echelon form of A. Assume that every row of A is non-zero. Let

$$C_t = \begin{pmatrix} a_{1t} \\ a_{2t} \\ \vdots \\ a_{mt} \end{pmatrix}$$

be the first non-zero column of A and let a_{it} be the first non-zero entry of C_t. Interchanging the first and the ith rows of A and replacing each row R_j with $j \geq i+1$ by $a_{it} R_j - a_{jt} R_1$, we obtain a matrix of the form

$$\begin{pmatrix} 0 & a_{it} & B \\ 0 & 0 & D \end{pmatrix}, \text{ if } t \neq 1 \text{ or } \begin{pmatrix} a_{i1} & B \\ 0 & D \end{pmatrix}, \text{ if } t=1$$

where B is a row matrix and D is a matrix whose number of rows is m-1, and so by induction, D can be changed to a matrix D' in row echelon form. Carrying the same operations on the rows of $\begin{pmatrix} 0 & a_{it} & B \\ 0 & 0 & D \end{pmatrix}$ (resp. $\begin{pmatrix} a_{i1} & B \\ 0 & D \end{pmatrix}$) containing the rows of D, we obtain the matrix $\begin{pmatrix} 0 & a_{it} & B \\ 0 & 0 & D' \end{pmatrix}$ (resp. $\begin{pmatrix} a_{i1} & B \\ 0 & D' \end{pmatrix}$) which is a row echelon form of A. Therefore the property holds for m, and so it is true for the matrix A. ■

To find a row echelon form of $A=(a_{ij})$, we proceed as follows: We search the first non-zero column of A, the tth say, and in this column we search the first non-zero entry, a_{it}, say. We interchange the ith row R_i containing this entry with the first row and we carry on each row R_j, where $j \geq i+1$, the operation $a_{it} R_j - a_{jt} R_1$, then we obtain a matrix A' of the form

$$A' = \begin{pmatrix} 0 & \cdots & 0 & a_{it} & a_{it+1} & \cdots & a_{in} \\ 0 & \cdots & 0 & 0 & b_{11} & \cdots & b_{1s} \\ \vdots & & \vdots & \vdots & b_{21} & \cdots & b_{2s} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & & 0 & 0 & b_{m-11} & \cdots & b_{m-1s} \end{pmatrix}$$

then we repeat the same procedure (all by staying in A') on the sub-matrix

$$B_1 = \begin{pmatrix} 0 & \cdots & 0 & b_{11} & \cdots & b_{1s} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & b_{m-11} & \cdots & b_{m-1s} \end{pmatrix}$$

obtained from A' by deleting the first row, and so on we obtain a row echelon form of A.

Example: Let's find a row echelon form of the real matrix $A = \begin{pmatrix} 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix}$. We have

$$\begin{pmatrix} 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 0 & 2 & 1 & -1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 3 & -2 \\ 0 & 1 & 1 & 2 \end{pmatrix} \xrightarrow{R_4 \to 2R_4 - R_1} \begin{pmatrix} 0 & 2 & 1 & -1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 4 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\begin{pmatrix} 0 & 2 & 1 & -1 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 4 \end{pmatrix} \xrightarrow{R_4 \to 3R_4 - R_2} \begin{pmatrix} 0 & 2 & 1 & -1 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 14 \end{pmatrix} \xrightarrow{R_4 \to 4R_4 - 14R_3} \begin{pmatrix} 0 & 2 & 1 & -1 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which is a row echelon form of A.

§ 2.2. INVERTIBLE MATRIX AND ECHELON FORM.

Let $A=(a_{ij})$ be $(n\times m)$ matrix over K.

2.2.1. The following hold

(i) $A(R_t \leftrightarrow R_s) = I_n(R_t \leftrightarrow R_s)\times A$.

(ii) $A(R_t \to R_t+\alpha R_s) = I_n(R_t \to R_t+\alpha R_s)\times A$, where $\alpha\in K$.

(iii) $A(R_t \to \alpha R_t) = I_n(R_t \to \alpha R_t)\times A$, with $\alpha\in K-\{0\}$.

Proof: (i) Let M_i be the ith row of I_n, F_i that of A, N_i that of $I_n(R_t \leftrightarrow R_s)$, D_i that of $A(R_t \leftrightarrow R_s)$ and E_i that of $I_n(R_t \leftrightarrow R_s)\times A$. We have

$$N_i = \begin{cases} M_i & \text{if } i\neq s \text{ and } i\neq t \\ M_s & \text{if } i=t \\ M_t & \text{if } i=s \end{cases} \quad \text{and} \quad D_i = \begin{cases} F_i & \text{if } i\neq s \text{ and } i\neq t \\ F_s & \text{if } i=t \\ F_t & \text{if } i=s. \end{cases}$$

By 1.5.2, we have

$$E_i = N_i\times A = \begin{cases} M_i\times A=F_i & \text{if } i\neq s \text{ and } i\neq t \\ M_s\times A=F_s & \text{if } i=t \\ M_t\times A=F_t & \text{if } i=s \end{cases}$$

hence $E_i=D_i$, for all $1\leq i\leq n$, and so $A(R_t \leftrightarrow R_s) = I_n(R_t \leftrightarrow R_s)\times A$.

(ii) Let M_i be the ith row of I_n, F_i that of A, N_i that of $I_n(R_t \to R_t+R_s)$, D_i that of $A(R_t \to R_t+R_s)$ and E_i that of $I_n(R_t \to R_t+R_s)\times A$. We have

$$N_i = \begin{cases} M_i & \text{if } i\neq t \\ M_t+\alpha M_s & \text{if } i=t \end{cases} \quad \text{and} \quad D_i = \begin{cases} F_i & \text{if } i\neq t \\ F_t+\alpha F_s & \text{if } i=t. \end{cases}$$

As

$$E_i = N_i\times A = \begin{cases} M_i\times A=F_i & \text{if } i\neq t \\ (M_t+\alpha M_s)A=M_t\times A+\alpha M_s\times A=F_t+\alpha F_s & \text{if } i=t \end{cases}$$

then $E_i=D_i$, for all $1\leq i\leq n$, and so $A(R_t \to R_t+\alpha R_s) = I_n(R_t \to R_t+\alpha R_s)\times A$..

(iii) Let M_i be the ith row of I_n, F_i that of A, N_i that of $I_n(R_t \to \alpha R_t)$, D_i that of $A(R_t \to \alpha R_t)$ and E_i that of $I_n(R_t \to \alpha R_t)\times A$. We have

$$N_i = \begin{cases} M_i & \text{if } i\neq t \\ \alpha M_t & \text{if } i=t \end{cases} \quad \text{and} \quad D_i = \begin{cases} F_i & \text{if } i\neq t \\ \alpha F_t & \text{if } i=t. \end{cases}$$

As

$$E_i = N_i\times A = \begin{cases} M_i\times A=F_i & \text{if } i\neq t \\ (\alpha M_t)A=\alpha(M_t\times A)=\alpha F_t & \text{if } i=t \end{cases}$$

then $E_i=D_i$, for all $1\leq i\leq n$, and so $A(R_t \to \alpha R_t) = I_n(R_t \to \alpha R_t)\times A$. ■

Definition 6. We call **elementary matrix** each of the following matrices

$I_n(R_t \leftrightarrow R_s)$, $I_n(R_t \to R_t+\alpha R_s)$ with $s\neq t$ and $\alpha\in K$ and $I_n(R_t \to \alpha R_t)$ with $\alpha\in K-\{0\}$. ■

The elementary matrix $I_n(R_t \leftrightarrow R_s)$ (resp. $I_n(R_t \to R_t+\alpha R_s)$, $I_n(R_t \to \alpha R_t)$) is called **the elementary matrix corresponding to** the operation $R_t \leftrightarrow R_s$ (resp. $R_t \to R_t+\alpha R_s$, $R_t \to \alpha R_t$).

Corollary 2.2.1.1. If B is obtained from A by one elementary operation and if E is the elementary matrix corresponding to that operation, then B=E×A.

Proof: We have that

$$B=A(R_t \leftrightarrow R_s) \text{ or}$$
$$B=A(R_t \to R_t+\alpha R_s), \text{ with } s\neq t, \text{ or}$$
$$B=A(R_t \to \alpha R_t) \text{ with } \alpha\in K\text{-}\{0\}.$$

If $B=A(R_t \leftrightarrow R_s)$, then $E=I_n(R_t \leftrightarrow R_s)$, hence E×A=B, by 2.2.1(i).
If $B=A(R_t \to R_t+\alpha R_s)$, then $E=I_n(R_t \to R_t+\alpha R_s)$, hence E×A=B, by 2.2.1(ii).
If $B=A(R_t \to \alpha R_t)$, then $E=I_n(R_t \to \alpha R_t)$, hence E×A=B, by 2.2.1(iii). ■

Corollary 2.2.1.2. Let $A_1,\ldots,A_k$ be a series of matrices obtained from A by a finite sequence of elementary operations. For each $1\le i\le k$, let E_i denote the elementary matrix corresponding to the elementary operation to obtain A_i, then

$$A_k = E_k \times E_{k-1} \times \cdots \times E_1 \times A.$$

Proof: We argue by induction on k. For k=1, we have that A_1 is obtained from A by one elementary operation and E_1 is the elementary matrix corresponding to that operation, hence

$$A_1 = E_1 \times A$$

by 2.2.1.1, and so the property holds for k=1. Suppose that it holds up to k-1 and let's prove it for k, then

$$A_{k-1} = E_{k-1} \times \cdots \times E_1 \times A$$

by induction hypothesis. Since A_k is obtained from A_{k-1} by one elementary operation and E_k is the elementary matrix corresponding to this operation, we get $A_k = E_k \times A_{k-1}$ by 2.2.1.1, hence $A_k = E_k \times E_{k-1} \times \ldots \times E_1 \times A$, and so the property holds for k, whence it is true, $\forall k\ge 1$. ■

2.2.2. Every elementary matrix is invertible. Moreover
(i) the inverse of $I_n(R_t \leftrightarrow R_s)$ is $I_n(R_t \leftrightarrow R_s)$;
(ii) if $s\neq t$ and $\alpha\in K$, then the inverse of $I_n(R_t \to R_t+\alpha R_s)$ is $I_n(R_t \to R_t-\alpha R_s)$;
(iii) if $\alpha\in K\text{-}\{0\}$, then the inverse of $I_n(R_t \to \alpha R_t)$ is $I_n(R_t \to \alpha^{-1} R_t)$.

Proof: (i) Put $A = I_n(R_t \leftrightarrow R_s)$. Then $A(R_t \leftrightarrow R_s) = I_n$. By 2.2.1, we have

$$A\times A = I_n(R_t \leftrightarrow R_s)\times A = A(R_t \leftrightarrow R_s) = I_n$$

hence $I_n(R_t \leftrightarrow R_s)$ is invertible and its inverse is $I_n(R_t \leftrightarrow R_s)$.
(ii) Put $A=I_n(R_t \to R_t-\alpha R_s)$. Then $A(R_t \to R_t+\alpha R_s)=I_n$. Since

$$I_n(R_t \to R_t+\alpha R_s)\times A = A(R_t \to R_t+\alpha R_s) = I_n$$

$I_n(R_t \to R_t+\alpha R_s)$ is invertible and its inverse is $I_n(R_t \to R_t-\alpha R_s)$.

(iii) Put A= $I_n(R_t \to \alpha^{-1}R_t)$. Then $A(R_t \to \alpha R_t) = I_n$. But

$$I_n(R_t \to \alpha R_t)\times A = A(R_t \to \alpha R_t)$$

hence $I_n(R_t \to \alpha R_t)$ is invertible and its inverse is $I_n(R_t \to \alpha^{-1}R_t)$. ■

Corollary 2.2.2.1. Let $A\in M_n(K)$. If A is invertible and B is the matrix obtained from A by a finite sequence of elementary row operations, then B is invertible.

Proof: Let $E_1, E_2, \ldots, E_r$ be the elementary matrices corresponding to the elementary row operations after which the matrix B is obtained. Then

$$B= (E_r \times E_{r-1}\times\ldots\times E_1)\times A.$$

As each E_i is invertible, by 2.2.2 and A is invertible, then B is invertible, by 1.8.3.1. ■

Corollary 2.2.2.2. Let $A\in M_n(K)$. If A is invertible and B is a row echelon form of A, then B is invertible.

Proof: Since B is obtained from A by a finite sequence of elementary row operations, we then have that B is invertible, by 2.2.2.1. ■

We shall need the following lemma

Lemma 1. If $A=\begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n+1} \\ 0 & b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & & \vdots \\ 0 & b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$ is invertible, then so is the matrix

$B=\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$.

Proof: As A is invertible, then every column of A is non-zero, by 1.8.5, hence $a_{11}\neq 0$. We have

$$^tA=\begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{12} & b_{11} & b_{21} & \cdots & b_{n1} \\ \vdots & b_{12} & b_{22} & \cdots & b_{n2} \\ \vdots & \vdots & & & \vdots \\ a_{1n+1} & b_{1n} & b_{2n} & \cdots & b_{nn} \end{pmatrix}$$

so that replacing the first row R_1 of ^tA by $(a_{11})^{-1}R_1$, and then replacing each row R_j, with $j\geq 2$, of the new matrix, by $R_j - a_{1j}R_1$, we obtain the matrix

$$N=\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & b_{11} & b_{21} & \cdots & b_{n1} \\ \vdots & b_{12} & b_{22} & \cdots & b_{n2} \\ \vdots & \vdots & & & \vdots \\ 0 & b_{1n} & b_{2n} & \cdots & b_{nn} \end{pmatrix}.$$

As t_A is invertible, then N is invertible, by 2.2.2.1. Put $N^{-1}=(c_{ij})$. We have that

$$\text{first row of } N = (1\ 0 \cdots 0) = \text{first row of } I_{n+1} \text{ and}$$

$$\text{first column of } N = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \text{first column of } I_{n+1}$$

hence

first row of NN^{-1} = (first row of N)$\times N^{-1}$ = (first row of I_{n+1})$\times N^{-1}$

= first row of $I_{n+1}N^{-1}$ = first row of N^{-1}

and

first column of $N^{-1}N = N^{-1}\times$(first column of N) $= N^{-1}\times$(first column of I_{n+1})

= first column of $N^{-1}I_{n+1}$ = first column of N^{-1}

and so as $NN^{-1} = N^{-1}N=I_{n+1}$, then N^{-1} is of the form

$$N^{-1}=\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & c_{22} & c_{23} & \cdots & c_{2n+1} \\ \vdots & c_{32} & c_{33} & \cdots & c_{3n+1} \\ \vdots & \vdots & & & \vdots \\ 0 & c_{n+12} & c_{n+13} & \cdots & c_{n+1n+1} \end{pmatrix}.$$

Put

$$D=\begin{pmatrix} c_{22} & c_{23} & \cdots & c_{2n+1} \\ c_{32} & c_{33} & \cdots & c_{3n+1} \\ \vdots & \vdots & & \vdots \\ c_{n+12} & c_{n+13} & \cdots & c_{n+1n+1} \end{pmatrix}.$$

Then $N^{-1}=\begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix}$, and so as $N=\begin{pmatrix} 1 & 0 \\ 0 & t_B \end{pmatrix}$, then

$$\begin{pmatrix} 1 & 0 \\ 0 & t_B D \end{pmatrix} = N\,N^{-1} = I_{n+1} = \begin{pmatrix} 1 & 0 \\ 0 & I_n \end{pmatrix}.$$

Thus $t_B D=I_n$, and so t_B is invertible, whence B is invertible. ■

<u>2.2.3.</u> If A is a (n×n) invertible matrix over K, then A can be changed to I_n by a finite sequence of elementary row operations.

<u>Proof:</u> We argue by induction on n. For n=1, we have $A=(a_{11})$ and $a_{11}\neq 0$, and so replacing the unique row R of A by $(a_{11})^{-1}R$, we get the matrix (1) which is I_1. Therefore the property holds for n=1. Assume that it holds for every invertible matrix over K of order ≤n-1 and let's prove it for A. As A is invertible, then the first column of A is non-zero, by 1.8.5. Let a_{t1} be the first non-zero entry of this column. We interchange first and tth rows of A, then in the obtained matrix we replace each row R_j, with j≥t+1, by $a_{t1}R_j - a_{j1}R_1$, and so we obtain a matrix A' of the form

$$A'=\begin{pmatrix} a_{t1} & a_{t2} & \cdots & \cdots & a_{tn} \\ 0 & b_{11} & b_{12} & \cdots & b_{1n-1} \\ 0 & b_{21} & b_{22} & \cdots & b_{2n-1} \\ \vdots & \vdots & & & \vdots \\ 0 & b_{n-11} & b_{n-12} & \cdots & b_{n-1n-1} \end{pmatrix}.$$

Let

$$B=\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n-1} \\ b_{21} & b_{22} & \cdots & b_{2n-2} \\ \vdots & \vdots & & \vdots \\ b_{n-11} & b_{n-12} & \cdots & b_{n-1n-1} \end{pmatrix}.$$

Since A' is invertible, by 2.2.2.1, so is B, by lemma 1, and so as the order of B is n-1, then induction yields that B can be changed to I_{n-1} by a finite sequence of elementary operations. Now carrying the same operations on the rows of A' containing those of B, we get the matrix

$$D=\begin{pmatrix} a_{t1} & a_{t2} & \cdots & \cdots & a_{tn} \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & & & \vdots \\ \vdots & \vdots & & & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

We replace the first row R_1 of D by R_1-($a_{t2}R_2+\cdots+a_{tn}R_n$) and multiplying the obtained row by a_{t1}^{-1}, we get the matrix I_n. Therefore A is changed to I_n by a finite sequence of elementary operations. It follows that the property holds for n, and so it is true, $\forall n\geq 1$. ■

Example: Change the matrix $A=\begin{pmatrix} -1 & 3 & 5 \\ 0 & 5 & 1 \\ 1 & 1 & -4 \end{pmatrix}$ to I_3.

$$\begin{pmatrix} -1 & 3 & 5 \\ 0 & 5 & 1 \\ 1 & 1 & -4 \end{pmatrix} \xrightarrow{R_3\to R_3+R_1} \begin{pmatrix} -1 & 3 & 5 \\ 0 & 5 & 1 \\ 0 & 4 & 1 \end{pmatrix} \xrightarrow{R_3\to 5R_3-4R_2} \begin{pmatrix} -1 & 3 & 5 \\ 0 & 5 & 1 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_2\to\frac{1}{5}R_2]{R_1\to -R_1}$$

$$\begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 1/5 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_2\to R_2-\frac{1}{5}R_3]{R_1\to R_1-5R_3} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1\to R_1-3R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

2.2.4. Let A be a (n×n) matrix over K. Then A is invertible if and only if A can be changed to I_n by a finite sequence of elementary row operations.

Proof: **N.C:** By 2.2.3.
S.C: As I_n is obtained from A by a finite sequence of elementary row operations, then let $E_1, E_2, \ldots, E_r$ be the elementary matrices corresponding to the elementary operations after which the matrix I_n is obtained. We have $(E_r\times E_{r-1}\times\ldots\times E_1)\times A=I_n$**,** hence A is invertible. ■ As I_n is obtained from A by a finite sequence of elementary row operations, then let

Corollary 2.2.4.1. If A is a (n×n) invertible matrix over K, then A^{-1} is obtained from I_n by carrying the same row operations after which A is changed to I_n.

Proof: Let $E_1,E_2, \ldots, E_r$ be the elementary matrices corresponding to the elementary operations after which A is changed to I_n. Then $(E_r\times E_{r-1}\times\ldots\times E_1)\times A= I_n$, and so

$$A^{-1} = E_r \times E_{r-1} \times \ldots \times E_1 .$$

As $A^{-1} = (E_r \times E_{r-1} \times \ldots \times E_1) \times I_n$, then A^{-1} is obtained from I_n by carrying the same elementary operations after which A is changed to I_n. ■

Remark: As A^{-1} is obtained from I_n by the same row operations after which A is changed to I_n, then we can write A and I_n in the same matrix by separating them by a column of points with A on the left side of the column of points and I_n on the right of that column and starting from this matrix we carry the necessary row operations to obtain I_n on the left side of the column of points. Thus the matrix obtained on the right side of the column of points is A^{-1}.

Examples: Check if the real matrix A is invertible or not and if yes, find its inverse in the following cases:

(i) $A=\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 1 & 3 & 5 \end{pmatrix}$ (ii) $A=\begin{pmatrix} 2 & 3 & 1 \\ 1 & 1 & 2 \\ 3 & 4 & 3 \end{pmatrix}$.

(i) We have

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 1 & 3 & 5 & 0 & 0 & 1 \end{array}\right)_{R_3 \to R_3 - R_1} \to \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 2 & 3 & -1 & 0 & 1 \end{array}\right)_{R_3 \to R_3 - 2R_2} \to$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & -3 & -1 & -2 & 1 \end{array}\right)_{R_3 \to \frac{-1}{3} R_3} \to \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1/3 & 2/3 & -1/3 \end{array}\right) \begin{matrix} R_1 \to R_1 - 2R_3 \\ R_2 \to R_2 - 3R_3 \to \end{matrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1/3 & -4/3 & 2/3 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 1/3 & 2/3 & -1/3 \end{array}\right)^{R_1 \to R_1 - R_2} \to \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 4/3 & -1/3 & -1/3 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 1/3 & 2/3 & -1/3 \end{array}\right)$$

hence A is invertible and $A^{-1} = \begin{pmatrix} 4/3 & -1/3 & -1/3 \\ -1 & -1 & 1 \\ 1/3 & 2/3 & -1/3 \end{pmatrix}$.

(ii) We have

$$\left(\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 3 & 4 & 3 & 0 & 0 & 1 \end{array}\right) \begin{matrix} R_2 \to 2R_2 - R_1 \to \\ R_3 \to 2R_3 - 3R_1 \end{matrix} \left(\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 0 & -1 & 3 & -1 & 2 & 0 \\ 0 & -1 & 3 & -3 & 0 & 3 \end{array}\right)_{R_3 \to R_3 - R_2} \to$$

$\left(\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 0 & -1 & 3 & -1 & 2 & 0 \\ 0 & 0 & 0 & -2 & -2 & 3 \end{array}\right)$, we stop here. We have obtained the matrix $A' = \begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$, which is not invertible, because it has a row equals to zero. But A' is obtained from A by a finite sequence of elementary row operations, hence A is not invertible, by 2.2.2.1.

2.2.5. If $B=(b_{ij})$ is a row echelon form of A, then A is invertible if and only if the number of non-zero rows of B is n.

Proof: **N.C:** As A is invertible, then B is invertible, by 2.2.2.2, and so every row of B is non-zero, whence the number of non-zero rows of B is n.

S.C: As B is (n×n) and the number of non-zero rows of B is n, then every row of B is non-zero. For each $1 \le t \le n$, let b_{tj_t} be the leading entry of the tth row of B. As B is in row echelon form, then

$$1 \le j_1 < j_2 < \cdots < j_n \le n.$$

Arguing by induction on n, we can easily prove that $j_t = t$, for all $1 \le t \le n$. Therefore b_{tt} is the leading entry of the tth row of B, and so B is upper triangular and every diagonal entry of B is non-zero. Let's show that B can be changed to I_n by a finite sequence of elementary row operations. We argue by induction on n. For n=1, we have $B=(b_{11})$ and $b_{11} \neq 0$, and so replacing the unique row R of B by $(b_{11})^{-1} R$, we get the matrix (1) which is I_1. Therefore the property holds for n=1. Assume that it holds up to n-1 and let's prove it for B. We have that $b_{nn} \neq 0$, so replacing the last row R_n of B by $(b_{nn})^{-1} R_n$, then the last row becomes (0 0 …0 1), and then replacing each row R_j of the obtained matrix by $R_j - b_{jn} R_n$, we get the matrix

$$B'=\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n-1} & 0 \\ 0 & b_{22} & \cdots & b_{2n-1} & 0 \\ \vdots & 0 & & & \vdots \\ \vdots & \vdots & & b_{n-1n-1} & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

We have that the matrix

$$D=\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n-1} \\ 0 & b_{22} & \cdots & b_{2n-2} \\ \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & b_{n-1n-1} \end{pmatrix}$$

is upper triangular of order n-1 and every diagonal entry of D is non-zero, hence D can be changed to I_{n-1} by a finite sequence of elementary row operations. Applying the same operations on the rows of $B'=\begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix}$ containing those of D, then B' is changed to $\begin{pmatrix} I_{n-1} & 0 \\ 0 & 1 \end{pmatrix}$. But $\begin{pmatrix} I_{n-1} & 0 \\ 0 & 1 \end{pmatrix} = I_n$, hence B' is changed to I_n, and so A is changed to I_n, whence A is invertible, by 2.2.4. ■

M1103

CHAPTER II

EXERCISES

1- Show that each of the following matrices is in row echelon form:

$$A=\begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ B=\begin{pmatrix} 1 & 1 & 2 & 3 & 1 \\ 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } C=\begin{pmatrix} -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

2- Show that each of the following matrices is not in row echelon form:

$$A=\begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix},\ B=\begin{pmatrix} 0 & 0 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } C=\begin{pmatrix} -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Find a row echelon form and a reduced row echelon form of each of them.

3- Find following the values of the real parameter m, a row echelon form of the real matrix A in the following cases:

(i) $A=\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & m+1 & 1 & 1 & 0 \\ 2 & 2 & 1 & 1 & 0 \\ m & m+2 & 0 & m & 0 \end{pmatrix}$ (ii) $A=\begin{pmatrix} m & 2 & 0 & 0 \\ m & 4 & 1 & 1 \\ 2m & 4 & 1 & 1 \\ m & m+2 & 1 & m \end{pmatrix}$.

4- Given the matrices

$$A=\begin{pmatrix} 1 & -i & 0 \\ 2 & 1-2i & 1 \\ 2 & i & 3i \end{pmatrix} \text{ and } B=\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 4 & 2 & 1 \end{pmatrix}.$$

Using the method of row echelon form, check if each of them is invertible or not and if yes find its inverse.

5- Using the method of row echelon form, find the values of m, for which the real matrix A is invertible and those for which A is not invertible and if A is invertible, then find A^{-1}, in the following cases:

(i) $A = \begin{pmatrix} m & 2 \\ -1 & 3 \end{pmatrix}$, (ii) $A=\begin{pmatrix} m & 0 & 1 \\ 1 & m & 1 \\ m & 1 & 1 \end{pmatrix}$.

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CHAPTER III

DETERMINANTS

Throughout this chapter the letter K is a field.

§ 3.1. DEFINITION AND PROPERTIES OF DETERMINANTS.

Let $A=(a_{ij})\in M_n(K)$. We denote by S_n the set of all the bijections (or **permutations**) of $I=\{1,...,n\}$ onto I. We have

$$\text{card}(S_n) = n!.$$

A permutation $\sigma\in S_n$ is usually denoted

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}.$$

Thus if we let e to be the identity mapping of I, then

$$S_1 = \left\{ e=\begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

$$S_2 = \left\{ e=\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \right\}$$

and

$$S_3 = \left\{ e=\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right\}.$$

For each $\sigma\in S_n$, let

$$I_\sigma = \{(i,j)\in I\times I \; ; \; i<j \text{ and } \sigma(i)>\sigma(j)\}$$

and

$$\varepsilon_\sigma = (-1)^{\text{card}(I_\sigma)}.$$

ε_σ is called the **signature** of σ and is usually denoted $\text{sgn}(\sigma)$. The permutation σ is said to be **even** (resp. **odd**) if $\varepsilon_\sigma = +1$ (resp. $\varepsilon_\sigma = -1$).

Examples: If $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ and $\delta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, then

$$I_\sigma = \{(1,2)\}$$

and

$$I_\delta = \{(1,3),(2,3)\}$$

so that $\varepsilon_\sigma = -1$ and $\varepsilon_\delta = +1$, and so σ is odd and δ is even.

If for each $1\le t\le n$, we let α_t to be the number of elements j of I, such that $t<j$ and $\sigma(t)>\sigma(j)$, then

$$\varepsilon_\sigma = (-1)^{\alpha_1+\alpha_2+\cdots+\alpha_{n-1}}.$$

Notice that α_t is the number of natural numbers in the second row of σ that are on the right of $\sigma(t)$ and less strictly than $\sigma(t)$. Thus if

$$\sigma=\begin{pmatrix}1&2&3&4&5&6\\3&1&2&6&4&5\end{pmatrix}$$

then we get that

$\alpha_1 = 2$, because $\sigma(1)=3$ and 1 and 2 are the only natural numbers in the second row of σ that are on the right of 3 and are less strictly than 3;

$\alpha_2 = 0$, because we have $\sigma(2)=1$ and all the natural numbers in the second row of σ that are on the right of 1 are greater strictly than 1;

$\alpha_3 = 0$, $\alpha_4 = 2$ and $\alpha_5 = 0$, hence

$$\varepsilon_\sigma = (-1)^{2+0+0+2+0} = (-1)^4 = +1.$$

Definition 1. We define the **determinant** of A, written det(A), by

$$\det(A) = \sum_{\sigma\in S_n} \varepsilon_\sigma a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)}\,. \blacksquare$$

The determinant of A is also denoted $|A|$ or $\begin{vmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\\vdots&\vdots&&\vdots\\a_{n1}&a_{n2}&\cdots&a_{nn}\end{vmatrix}$.

Examples: 1) $|a_{11}| = \sum_{\sigma\in S_1} \varepsilon_\sigma a_{1\sigma(1)} = \varepsilon_e a_{1e(1)} = a_{11}$, for $\varepsilon_e = +1$, because $I_e = \varnothing$. Thus

$$|-2| = -2 \text{ and } |+2| = +2.$$

2) Let $\lambda = \begin{pmatrix}1&2\\2&1\end{pmatrix}$. Then

$$\begin{vmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{vmatrix} = \sum_{\sigma\in S_2}\varepsilon_\sigma a_{1\sigma(1)}a_{2\sigma(2)} = \varepsilon_e\, a_{1e(1)}\, a_{2e(2)} + \varepsilon_\lambda\, a_{1\lambda(1)}a_{2\lambda(2)} = a_{11}a_{22} - a_{12}a_{21}$$

since $\varepsilon_e = +1$ and $\varepsilon_\lambda = -1$. Thus

$$\begin{vmatrix}a&b\\c&d\end{vmatrix} = ad-bc,\ \forall a,b,c,d\in K.$$

If $n\geq 2$, then for each $1\leq i\leq n$ and $1\leq j\leq n$, let A_{ij} be the matrix obtained from A by deleting the ith row and the jth column. Thus

$$A_{ij} = \begin{pmatrix}a_{11}&\cdots&a_{1j-1}&a_{1j+1}&\cdots&a_{1n}\\\vdots&&\vdots&\vdots&&\vdots\\a_{i-11}&\cdots&a_{i-1j-1}&a_{i-1j+1}&\cdots&a_{i-1n}\\a_{i+11}&\cdots&a_{i+1j-1}&a_{i+1j+1}&\cdots&a_{i+1n}\\\vdots&&\vdots&\vdots&&\vdots\\a_{n1}&\cdots&a_{nj-1}&a_{nj+1}&\cdots&a_{nn}\end{pmatrix}$$

which is a square matrix of order n-1 over K.

3.1.1. The following hold

(i) $|AB| = |A|\,|B|$, $\forall A,B\in M_n(K)$.

(ii) $|A| = (-1)^{i+1}a_{i1}|A_{i1}|+\cdots+(-1)^{i+n}a_{in}|A_{in}|$, $\forall 1\leq i\leq n$.

(iii) $|A| = (-1)^{1+j}a_{1j}|A_{1j}|+\cdots+(-1)^{n+j}a_{nj}|A_{nj}|$, $\forall 1\leq j\leq n$.

Proof: To be admitted without proof.■

Remark: If we write
$$|A| = (-1)^{i+1} a_{i1} |A_{i1}| + \cdots + (-1)^{i+n} a_{in} |A_{in}|$$
then we say that $|A|$ is expanded through the i-th row and if we write
$$|A| = (-1)^{1+j} a_{1j} |A_{1j}| + \cdots + (-1)^{n+j} a_{nj} |A_{nj}|$$
then we say that $|A|$ is expanded through the j-th column.

Example: Let D = $\begin{vmatrix} 1 & 2 & -3 \\ 2 & 5 & 6 \\ 2 & 3 & 2 \end{vmatrix}$. Expanding D through the 2nd row we get

$$D = (-1)^{2+1} 2 \begin{vmatrix} 2 & -3 \\ 3 & 2 \end{vmatrix} + (-1)^{2+2} 5 \begin{vmatrix} 1 & -3 \\ 2 & 2 \end{vmatrix} + (-1)^{2+3} 6 \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -2(4+9)+5(2+6)-6(3-4)$$
$$= -26+40+6 = 20.$$

If we expand $|A|$ through the i-th row (resp. j-th column), then we point out to this row (resp. column) by an arrow, here for D we write
$$D = \begin{vmatrix} 1 & 2 & -3 \\ 2 & 5 & 6 \\ 2 & 3 & 2 \end{vmatrix} \begin{matrix} \\ \leftarrow \\ \\ \end{matrix}.$$

If we expand D through the 3rd column we write $D = \begin{array}{c} \begin{matrix} & & \downarrow \end{matrix} \\ \begin{vmatrix} 1 & 2 & -3 \\ 2 & 5 & 6 \\ 2 & 3 & 2 \end{vmatrix} \end{array}$.

3.1.2. $|{}^tA| = |A|$.

Proof: We argue by induction on n. It holds for n=1, since $A = {}^tA = (a_{11})$. Assume that the property is true up to n-1 and let's show it for n. Let $B = {}^tA = (b_{ij})$. Then
$$B_{ij} = {}^tA_{ji} \text{ and } b_{ij} = a_{ji}, \ \forall i,j \in \{1,2,...,n\}.$$
We have
$$|{}^tA| = (-1)^{1+j} b_{1j} |B_{1j}| + \cdots + (-1)^{n+j} b_{nj} |B_{nj}| = (-1)^{j+1} a_{j1} |{}^tA_{j1}| + \cdots + (-1)^{j+n} a_{jn} |{}^tA_{jn}|.$$
But each A_{ji} is a square matrix of order n-1 over K, hence
$$|{}^tA_{ji}| = |A_{ji}|, \ \forall i \in \{1,2,...,n\}$$
by induction hypothesis, and so
$$|{}^tA| = (-1)^{j+1} a_{j1} |A_{j1}| + \cdots + (-1)^{j+n} a_{jn} |A_{jn}| = |A|$$
by 3.1.1. Therefore the property holds for n, and so it is true, for every $n \in \mathbb{N}^*$. ■

Remark: It follows from 3.1.2 that every property concerning the determinants which is true for the rows of A is also true for the columns and vice-versa.

3.1.3. If the entries of a row (resp. column) of A are zero, then $|A| = 0$.

Proof: Assume that the entries of the i-th row of A are zero. Then

$$a_{ij} = 0,\ \forall j \in \{1,2,...,n\}.$$

But $|A| = (-1)^{i+1} a_{i1} |A_{i1}| + \cdots + (-1)^{i+n} a_{in} |A_{in}|$, by 2.1.1(ii), hence $|A| = 0$. ■

3.1.4. If n≥2 and B is the matrix obtained from A by interchanging two rows (resp. columns) of A, then $|B| = -|A|$.

Proof: Assume that B is obtained from A by interchanging the rows $(a_{s1}\ a_{s2}\ ...\ a_{sn})$ and $(a_{t1}\ a_{t2}\ ...\ a_{tn})$, with s<t. We argue by induction on n. For n = 2, we have

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ and } B = \begin{pmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{pmatrix}$$

so that

$$|A| = a_{11}a_{22} - a_{21}a_{12} \text{ and } |B| = a_{21}a_{12} - a_{11}a_{22}$$

hence $|B| = -|A|$, and so the property is true for n = 2. Suppose that it is true for n-1 and let's show it for n. Since the property holds for n=2, we then get that n≥3, hence $\exists i \in \{1,2,...,n\}$, such that i≠s and i≠t. We have

$$|A| = (-1)^{i+1} a_{i1} |A_{i1}| + \cdots + (-1)^{i+n} a_{in} |A_{in}|$$

and

$$|B| = (-1)^{i+1} a_{i1} |B_{i1}| + \cdots + (-1)^{i+n} a_{in} |B_{in}|.$$

But B_{ij} is obtained from A_{ij} by interchanging the rows

$$(a_{s1}\ ...\ a_{sj-1}\ a_{sj+1}\ ...\ a_{sn}) \text{ and } (a_{t1}\ ...\ a_{tj-1}\ a_{tj+1}\ ...\ a_{tn})$$

and each A_{ij} is a square matrix of order n-1 over K, hence

$$|B_{ij}| = -|A_{ij}|,\ \forall j \in \{1,2,...,n\}$$

by induction hypothesis, and so

$$|B| = -|A|.$$

Thus the property is true for n, and so it is true, ∀n≥2. ■

Example: $\begin{vmatrix} 1 & 2 & -3 \\ 2 & 5 & 6 \\ -2 & 3 & 2 \end{vmatrix} = -\begin{vmatrix} -2 & 3 & 2 \\ 2 & 5 & 6 \\ 1 & 2 & -3 \end{vmatrix} = \begin{vmatrix} 2 & 3 & -2 \\ 6 & 5 & 2 \\ -3 & 2 & 1 \end{vmatrix}.$

3.1.5. If n≥2 and A has two equal rows (resp. columns), then $|A| = 0$.

Proof: Assume that the rows $(a_{s1}, a_{s2}, ..., a_{sn})$ and $(a_{t1}, a_{t2}, ..., a_{tn})$, with s<t, are equal. We argue by induction on n. For n = 2, we have

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{11} & a_{12} \end{pmatrix}$$

hence

$$|A| = a_{11}a_{12} - a_{11}a_{12} = 0$$

and so the property is true for n = 2. Suppose that it is true for n-1 and let's show it for n. Since the property holds for n=2, we then have n≥3, whence $\exists i\in\{1,...,n\}$, such that $i\neq s$ and $i\neq t$. We have

$$|A| = (-1)^{i+1} a_{i1} |A_{i1}| +\cdots+ (-1)^{i+n} a_{in} |A_{in}|$$

and so as the rows

$$(a_{s1} \ ...\ a_{sj-1}\ a_{sj+1}\ ...\ a_{sn}) \text{ and } (a_{t1}\ ...\ a_{tj-1}\ a_{tj+1}\ ...\ a_{tn})$$

of A_{ij} are equal and each A_{ij} is a square matrix of order n-1 over K, then induction hypothesis yields that $|A_{ij}| = 0$, $\forall j\in\{1,2,...,n\}$, and so $|A| = 0$. Therefore the property is true for n, and so it is true, $\forall n\geq 2$. ■

3.1.6. If

$$A=\begin{pmatrix} a_{11} & \cdots & a_{1j-1} & \alpha x_1 & a_{1j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj-1} & \alpha x_n & a_{nj+1} & \cdots & a_{nn} \end{pmatrix}$$

where $\alpha, x_1,...,x_n \in K$, then

$$|A| = \alpha \begin{vmatrix} a_{11} & \cdots & a_{1j-1} & x_1 & a_{1j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj-1} & x_n & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}.$$

Proof: Let $B=\begin{pmatrix} a_{11} & \cdots & a_{1j-1} & x_1 & a_{1j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj-1} & x_n & a_{nj+1} & \cdots & a_{nn} \end{pmatrix}$, then $B_{ij} = A_{ij}$, for all $i\in\{1,2,...,n\}$.

we have

$$|A| = (-1)^{1+j}(\alpha x_1)|A_{1j}| +\cdots+ (-1)^{n+j}(\alpha x_n)|A_{nj}|$$
$$= \alpha[(-1)^{1+j}\ x_1\ |A_{1j}| +\cdots+ (-1)^{n+j}\ x_n\ |A_{nj}|\,] = \alpha|B|.\ ■$$

Example Let $D = \begin{vmatrix} 1 & 2 & -3 \\ 2 & 5 & 6 \\ 2 & 3 & 2 \end{vmatrix}$. If $\alpha\in K$, then

$$\alpha D=\begin{vmatrix} \alpha & 2\alpha & -3\alpha \\ 2 & 5 & 6 \\ 2 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2\alpha & -3 \\ 2 & 5\alpha & 6 \\ 2 & 3\alpha & 2 \end{vmatrix} \text{ and } \alpha^2 D=\begin{vmatrix} \alpha & 2\alpha & -3\alpha \\ 2 & 5 & 6 \\ 2\alpha & 3\alpha & 2\alpha \end{vmatrix} = \begin{vmatrix} 1 & 2\alpha^2 & -3 \\ 2 & 5\alpha^2 & 6 \\ 2 & 3\alpha^2 & 2 \end{vmatrix}.$$

3.1.7. If

$$A=\begin{pmatrix} a_{11} & \cdots & a_{1j-1} & x_1+y_1 & a_{1j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj-1} & x_n+y_n & a_{nj+1} & \cdots & a_{nn} \end{pmatrix}$$

with $x_1,...,x_n, y_1,..., y_n \in K$, then

$$|A| = \begin{vmatrix} a_{11} & \cdots & a_{1j-1} & x_1 & a_{1j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj-1} & x_n & a_{nj+1} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & \cdots & a_{1j-1} & y_1 & a_{1j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj-1} & y_n & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}.$$

Proof: Let

$$B=\begin{pmatrix} a_{11} & \cdots & a_{1j-1} & x_1 & a_{1j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj-1} & x_n & a_{nj+1} & \cdots & a_{nn} \end{pmatrix} \text{ and } C=\begin{pmatrix} a_{11} & \cdots & a_{1j-1} & y_1 & a_{1j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj-1} & y_n & a_{nj+1} & \cdots & a_{nn} \end{pmatrix}.$$

Then

$$B_{ij} = C_{ij} = A_{ij}, \text{ for all } i\in\{1,2,...,n\}.$$

But

$$|B| = (-1)^{1+j} x_1 |B_{1j}| + \cdots + (-1)^{n+j} x_n |B_{nj}|,$$

$$|C| = (-1)^{1+j} y_1 |C_{1j}| + \cdots + (-1)^{n+j} y_n |C_{nj}|$$

and

$$|A| = (-1)^{1+j} (x_1+y_1) |A_{1j}| + \cdots + (-1)^{n+j} (x_n+y_n) |A_{nj}|$$

hence $|A| = |B| + |C|$. ■

3.1.8. If

$$A=\begin{pmatrix} a_{11} & \cdots & a_{1j-1} & \sum_{t=1}^{m} \alpha_t x_{t1} & a_{1j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj-1} & \sum_{t=1}^{m} \alpha_t x_{tn} & a_{nj+1} & \cdots & a_{nn} \end{pmatrix}$$

then $|A| = \sum_{t=1}^{m} \alpha_t |B_t|$, where

$$B_t = \begin{pmatrix} a_{11} & \cdots & a_{1j-1} & x_{t1} & a_{1j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj-1} & x_{tn} & a_{nj+1} & \cdots & a_{nn} \end{pmatrix}.$$

Proof: Argue by induction on m and apply 3.1.6 and 3.1.7. ■

Corollary 3.1.8.1. Denote by C_i the ith column of A. Let $1\le j\le n$, and let $\alpha_1,...,\alpha_n \in K$, such that $\alpha_j=1$. If B is the matrix obtained from A by replacing the jth column by $\alpha_1 C_1 + ... + \alpha_n C_n$, that is

$$B=\begin{pmatrix} a_{11} & \cdots & a_{1j-1} & \sum_{t=1}^{n} \alpha_t a_{1t} & a_{1j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj-1} & \sum_{t=1}^{n} \alpha_t a_{nt} & a_{nj+1} & \cdots & a_{nn} \end{pmatrix}$$

then $|B| = |A|$.

Proof: By 3.1.8, we have $|B| = \sum_{t=1}^{n} \alpha_t |B_t|$, where

$$B_t = \begin{pmatrix} a_{11} & \cdots & a_{1j-1} & a_{1t} & a_{1j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nt} & a_{nj+1} & \cdots & a_{nn} \end{pmatrix}$$

hence if $t \neq j$, then B_t has two equal columns and if t=j, then B_t =A, and so $|B_t|=0$, if $t\neq j$ and $|B_t| = |A|$ if t=j, whence $|B| = |A|$. ■

Thus if $A=\begin{pmatrix} 1 & 2 & 1 & 4 \\ 2 & -1 & 3 & 1 \\ 0 & -3 & 2 & 5 \\ 1 & 0 & 0 & 2 \end{pmatrix}$, then if we add to the first column all the others, then $|A|$ does not change, that is

$$|A| = \begin{vmatrix} 1 & 2 & 1 & 4 \\ 2 & -1 & 3 & 1 \\ 0 & -3 & 2 & 5 \\ 1 & 0 & 0 & 2 \end{vmatrix} \overset{c_1 \to c_1 + c_2 + c_3 + c_4}{=} \begin{vmatrix} 8 & 2 & 1 & 4 \\ 5 & -1 & 3 & 1 \\ 4 & -3 & 2 & 5 \\ 3 & 0 & 0 & 2 \end{vmatrix}$$

Also if we replace the second row R_2 by R_2 -2R_1+R_3, we get

$$|A| = \begin{vmatrix} 1 & 2 & 1 & 4 \\ 2 & -1 & 3 & 1 \\ 0 & -3 & 2 & 5 \\ 1 & 0 & 0 & 2 \end{vmatrix} \overset{R_2 \to R_2 - 2R_1 + R_4}{=} \begin{vmatrix} 1 & 2 & 1 & 4 \\ 1 & -5 & 1 & -5 \\ 0 & -3 & 2 & 5 \\ 1 & 0 & 0 & 2 \end{vmatrix}.$$

3.1.9. If A is lower (resp. upper) triangular, then

$$|A| = a_{11} a_{22} \ldots a_{nn}$$

i.e the determinant of a triangular matrix is equal to the product of its diagonal entries.

Proof: Suppose that A is lower triangular. We proceed by induction on n. It is true for n=1, since A=(a_{11}). Suppose that it holds for n-1 and let's show it for n. By expanding $|A|$ through the first row we get

$$|A| = (-1)^{1+1} a_{11} |A_{11}| + (-1)^{1+2} a_{12} |A_{12}| + \cdots + (-1)^{1+n} a_{1n} |A_{1n}|.$$

But

$$a_{1t} = 0, \forall t \neq 1$$

hence

$$|A| = a_{11} |A_{11}|.$$

Since

$$A_{11} = \begin{pmatrix} a_{22} & 0 & \cdots & \cdots & 0 \\ a_{32} & a_{33} & 0 & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & 0 \\ a_{n2} & a_{n3} & \cdots & \cdots & a_{nn} \end{pmatrix}$$

is a lower triangular matrix of order n-1 over K, it then follows from the hypothesis of induction that

$$|A_{11}| = a_{22} a_{33} \ldots a_{nn}$$

whence

$$|A| = a_{11}a_{22}...a_{nn}.$$

Thus the property is true for n, and so it is true, $\forall n \geq 1$.

Suppose that A is upper triangular. As ${}^tA$ is lower triangular, then

$$|{}^tA| = a_{11}a_{22}...a_{nn}.$$

But $|A| = |{}^tA|$, by 3.1.2, hence $|A| = a_{11}a_{22}...a_{nn}$. ■

Corollary 3.1.9.1. If A is a diagonal matrix, then

$$|A| = a_{11}a_{22}...a_{nn}$$

i.e the determinant of a diagonal matrix is equal to the product of its diagonal entries.

Proof: As A is triangular, then $|A| = a_{11}a_{22}...a_{nn}$, by 3.1.9. ■

Definition 2. We call **(i,j)-th cofactor** of A the scalar $(-1)^{i+j}|A_{ij}|$. ■

Definition 3. We define the **classical adjoint** of A to be the ($n\times n$) matrix over K

$$\mathrm{adj}(A) = \begin{pmatrix} M_{11} & M_{21} & \cdots & M_{n1} \\ M_{12} & M_{22} & \cdots & M_{n2} \\ \vdots & \vdots & & \vdots \\ M_{1n} & M_{2n} & \cdots & M_{nn} \end{pmatrix}$$

where for all $i,j \in \{1,...,n\}$, M_{ij} is the (i,j)-th cofactor of A. ■

Remark: The transpose of adj(A) is called the **cofactor matrix** of A and denoted A^{cof}.

Example: If $A=\begin{pmatrix} 1 & 0 & 3 \\ 2 & 4 & 0 \\ 1 & 0 & 5 \end{pmatrix}$, then

$$\mathrm{adj}(A)=\begin{pmatrix} M_{11} & M_{21} & M_{31} \\ M_{12} & M_{22} & M_{32} \\ M_{13} & M_{23} & M_{33} \end{pmatrix},$$

with

$$M_{11} = (-1)^{1+1}\begin{vmatrix} 4 & 0 \\ 0 & 5 \end{vmatrix} = 20,\ M_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 0 \\ 1 & 5 \end{vmatrix} = -10,\ M_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 4 \\ 1 & 0 \end{vmatrix} = -4,$$

$$M_{21} = (-1)^{2+1}\begin{vmatrix} 0 & 3 \\ 0 & 5 \end{vmatrix} = 0,\ M_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} = 5-3=2,\ M_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = 0,$$

$$M_{31} = (-1)^{3+1}\begin{vmatrix} 0 & 3 \\ 4 & 0 \end{vmatrix} = -12,\ M_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = -(-6)=6 \text{ and } M_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 0 \\ 2 & 4 \end{vmatrix} = 4,$$

hence $\mathrm{adj}(A)=\begin{pmatrix} 20 & 0 & -12 \\ -10 & 2 & 6 \\ -4 & 0 & 4 \end{pmatrix}$.

3.1.10. $A[\mathrm{adj}(A)] = [\mathrm{adj}(A)]A = |A|I_n$.

Proof: For every $i,j\in\{1,...,n\}$, let M_{ij} be the (i,j)-th cofactor of A. Let $A[\text{adj}(A)] = (c_{ij})$. Then

$$c_{ij} = a_{i1}M_{j1}+\cdots+a_{in}M_{jn}.$$

Let B be the matrix obtained from A by replacing the j-th row of A by $(a_{i1} \ldots a_{in})$. Expanding B through the j-th row we get that

$$|B| = a_{i1}M_{j1}+\cdots+a_{in}M_{jn}$$

hence

$$c_{ij} = |B|,\ \forall i,j\in\{1,...,n\}.$$

If $i\neq j$, then the i-th and j-th rows of B are equal, hence $|B|=0$, and so

$$c_{ij} = 0, \text{ if } i\neq j.$$

If $i=j$, then $B=A$, hence $|B|=|A|$, and so

$$c_{ij} = |A|, \text{ if } i=j.$$

It follows that

$$A[\text{adj}(A)] = \begin{pmatrix} |A| & 0 & \cdots & \cdots & 0 \\ 0 & |A| & 0 & \cdots & 0 \\ \vdots & 0 & & & \vdots \\ \vdots & \vdots & & & 0 \\ 0 & 0 & \cdots & 0 & |A| \end{pmatrix} = |A|I_n.$$

Similarly we show that $[\text{adj}(A)]A=|A|I_n$. ■

3.1.11. A is invertible if and only if $|A|\neq 0$. Moreover if $n\geq 2$ and A is invertible, then

$$A^{-1}=\frac{1}{|A|}\text{adj}(A).$$

Proof: **N.C:** We have $AA^{-1} = A^{-1}A = I_n$, hence

$$|AA^{-1}| = |A^{-1}A| = |I_n| = 1$$

because I_n is a diagonal matrix whose diagonal entries are all equal to 1, and so $|A|\,|A^{-1}| = |A^{-1}|\,|A| = 1$, by 3.1.1(i), whence $|A|$ is invertible in K.

S.C: Suppose that n=1. Then $A=(a_{11})$, and so $|A|=a_{11}$. It follows that a_{11}^{-1} exists in K. Let $B=(a_{11}^{-1})$. Then $B\in M_1(K)$ and $AB=(1)=I_1$, and so A is invertible. Assume that $n\neq 1$. Then $n\geq 2$, and so adj(A) exists. Let

$$B = \frac{1}{|A|}\text{adj}(A).$$

We have $\text{adj}(A)\in M_n(K)$, and so $B\in M_n(K)$. But

$$AB = A[\frac{1}{|A|}\text{adj}(A)] = \frac{1}{|A|}(A[\text{adj}(A)]) = (\frac{1}{|A|})|A|I_n = I_A$$

hence A is invertible and $A^{-1} = B = \frac{1}{|A|}\text{adj}(A)$. ■

§ 3.2. RANK OF A MATRIX.

Let $A=(a_{ij})$ be a $(m\times n)$ matrix over K.

Definition 4. Let $r\in\mathbb{N}^*$, such that $r\le m$ and $r\le n$. We call **minor** of A of order r the determinant of every matrix $B\in M_r(K)$ of the form

$$B=\begin{pmatrix} a_{i_1j_1} & a_{i_1j_2} & \cdots & a_{i_1j_r} \\ a_{i_2j_1} & a_{i_2j_2} & \cdots & a_{i_2j_r} \\ \vdots & \vdots & & \vdots \\ a_{i_rj_1} & a_{i_rj_2} & \cdots & a_{i_rj_r} \end{pmatrix}$$

where $1\le i_1<\ldots<i_r\le m$ and $1\le j_1<\ldots<j_r\le n$. ■

Examples: Let

$$A=\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 9 \\ 8 & -1 & 0 & -2 \end{pmatrix}.$$

Then

$$\begin{vmatrix} 1 & 2 \\ 5 & 6 \end{vmatrix},\ \begin{vmatrix} 2 & 4 \\ 6 & 9 \end{vmatrix},\ \begin{vmatrix} 7 & 9 \\ 0 & -2 \end{vmatrix} \text{ and } \begin{vmatrix} 1 & 3 \\ 8 & 0 \end{vmatrix} \text{ are minors of A of order 2}$$

and

$$\begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 8 & -1 & 0 \end{vmatrix},\ \begin{vmatrix} 1 & 3 & 4 \\ 5 & 7 & 9 \\ 8 & 0 & -2 \end{vmatrix} \text{ and } \begin{vmatrix} 2 & 3 & 4 \\ 6 & 7 & 9 \\ -1 & 0 & -2 \end{vmatrix} \text{ are minors of A of order 3.}$$

Definition 5. If $A\neq 0$, we define the **rank** of A, denoted rank(A) to be the greatest order of non-zero minors of A and if $A=0$, we define rank(A)=0. ■

Remark: We have that

(i) rank(A)≤the number of rows of A,

(ii) rank(A)≤the number of columns of A,

(iii) if Δ is a non-zero minor of A, then

$$\text{rank}(A)\ge \text{ order of } \Delta.$$

3.2.1. If $A\in M_n(K)$, then the following are equivalent:

(i) A is invertible,

(ii) $|A|\neq 0$,

(iii) rank(A)=n.

Proof: (i)⇒(ii): By 3.1.11.

(ii)⇒(iii): As $|A|\neq 0$, then $|A|$ is a non-zero minor of A of order n, hence rank(A)≥n. But rank(A)≤n, and so rank(A)=n.

(iii)⇒(i): We have that rank(A)=n and n≥1, hence $A\neq 0$, and so A has a non-zero minor of order n. But $|A|$ is the only minor of A of order n, hence $|A|\neq 0$, and so A is invertible, by 3.1.11. ■

3.2.2. $\text{rank}({}^tA) = \text{rank}(A)$.

Proof: If A=0, then $^tA = 0$, hence rank(^tA) = rank(A) = 0. Suppose that A≠0 and set

$$r = \text{rank}(^tA) \text{ and } s = \text{rank}(A).$$

Let $|D|$ be a non-zero minor of A of order r and $|D'|$ be a non-zero minor of ^tA of order s. Since $|^tD|$ is a non-zero minor of ^tA of order r, because $|^tD| = |D|$, we then have

$$r \leq s.$$

Similarly, as $|^tD'|$ is a non-zero minor of A of order s, then s≤r. It then follows that r = s, and so rank(^tA) = rank(A). ■

3.2.3. The rank of A is equal to the number of non-zero rows of a row echelon form of A.

Proof: To be given in Appendix I at the end of the course. ■

CHAPTER III

EXERCISES

1- Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 5 & 6 & 4 & 1 \end{pmatrix}$ and $\delta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 6 & 2 & 4 & 5 \end{pmatrix}$.

Calculate σ^2, σ^3, δ^2, δ^3, $\sigma o \delta$, σ^{-1} and δ^{-1} and find their signatures.

2- Find the positive integers i and j so that the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & i & 7 & 4 & 9 & j & 5 & 6 \end{pmatrix}$$

is even (resp. odd).

3- Show without expansion that

a) $\begin{vmatrix} 5 & 6 & 20 \\ 15 & 5 & 40 \\ 40 & 2 & 60 \end{vmatrix}$ is a multiple of 100; b) $\begin{vmatrix} 34 & 35 & 62 \\ 3 & 3 & 6 \\ 4 & 5 & 2 \end{vmatrix} = 0$;

c) $\begin{vmatrix} a & ab & a^2 \\ b & bc & b^2 \\ c & ca & c^2 \end{vmatrix} + \begin{vmatrix} b & ab & b^2 \\ c & bc & c^2 \\ a & ca & a^2 \end{vmatrix} = 0$, where a, b and c are real numbers.

4- Calculate the following determinants:

a) $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2i \\ i+1 & 0 & 0 \end{vmatrix}$ b) $\begin{vmatrix} 3 & -1 & -2 & 3 \\ 1 & 1 & 6 & 7 \\ 5 & 1 & 2 & 3 \\ 3 & 1 & -4 & 2 \end{vmatrix}$.

5- Show without expansion that if a, b and c are real numbers, then

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3.$$

6- Calculate the following determinants:

a) $D_n = \begin{vmatrix} x & a & \cdots & \cdots & a \\ a & x & a & \cdots & a \\ \vdots & a & & & \vdots \\ \vdots & \vdots & & & a \\ a & a & \cdots & a & x \end{vmatrix}$.

b) $D_n = \begin{vmatrix} a+b & 1 & 0 & \cdots & \cdots & 0 \\ ab & a+b & 1 & 0 & \cdots & 0 \\ 0 & ab & & & & \vdots \\ \vdots & 0 & & & & 0 \\ \vdots & \vdots & & & & 1 \\ 0 & 0 & \cdots & 0 & ab & a+b \end{vmatrix}$.

(**Hint:** Show that $D_n = (a+b)D_{n-1} - abD_{n-2}$).

c) $D_n=\begin{vmatrix} 2 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & & & & \vdots \\ \vdots & 0 & & & & 0 \\ \vdots & \vdots & & & & 1 \\ 0 & 0 & \cdots & 0 & 1 & 2 \end{vmatrix}$

(**<u>Hint:</u>** use b)).

7- Compute the following determinants

a) $\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix}$; b) $\begin{vmatrix} b & c & a+1 & d \\ b & c+1 & a & d \\ b+1 & c & a & d \\ b & c & a & d+1 \end{vmatrix}$;

c) $\begin{vmatrix} n^2 & (n+1)^2 & (n+2)^2 \\ (n+1)^2 & (n+2)^2 & (n+3)^2 \\ (n+2)^2 & (n+3)^2 & (n+4)^2 \end{vmatrix}$; d) $\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}$.

8- Find the values of x, for which the following matrices are invertible:

$A=\begin{pmatrix} 1 & -1 & 2 \\ x+1 & x & 1 \\ 6 & -4 & 4-x \end{pmatrix}$ and $B=\begin{pmatrix} 1-x & -1 & 2 & -2 \\ 0 & -x & 1 & -1 \\ 1 & -1 & 1-x & 0 \\ 1 & -1 & 1 & -x \end{pmatrix}$.

9- Show that if $A\in M_n(K)$, then $|\alpha A|=\alpha^n|A|$, for all $\alpha\in K$, and deduce that if A is an anti-symmetric real matrix of odd order, then $|A|=0$.

10- Let $A\in M_n(\mathbb{R})$. Show that

(i) If $A=A^{-1}$, then $|A|=\pm1$.

(ii) If ${}^tA=A^{-1}$, then $|A|=\pm1$.

11- Show that the matrix

$$A=\begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 2 & 1 & 3 \end{pmatrix}$$

is invertible and find its inverse by using the classical adjoint of A.

12- Calculate following the values of the real parameter m, the rank of the matrix A in the following cases:

(i) $A=\begin{pmatrix} 2 & m & 1 \\ m & 1 & 2 \\ 1 & 2 & m \end{pmatrix}$, (ii) $A=\begin{pmatrix} m & 1 & 0 & 1 & 0 \\ 1 & 1 & m & 1 & 1 \\ 1 & 1 & m & 2m & 1 \end{pmatrix}$, (iii) $A=\begin{pmatrix} m & 1 & 1 & 1 \\ 1 & m & 1 & 1 \\ 1 & 1 & m & 1 \\ 1 & 1 & 1 & m \end{pmatrix}$.

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CHAPTER IV

SYSTEM OF LINEAR EQUATIONS

Throughout this chapter the letter K is a field.

§ 4.1. LINEAR EQUATIONS.

Definition 1. We call **linear equation** with n unknowns $x_1,\dots,x_n$ over K, every equation of the form

$$a_1x_1+\dots+a_nx_n=b$$

where $a_1,\dots,a_n,b\in K$. ■

For the rest of this section let (I) be the following system of linear equations:

$$E_1 : a_{11}x_1+\dots+a_{1n}x_n=b_1$$
$$E_2 : a_{21}x_1+\dots+a_{2n}x_n=b_2$$
$$\vdots$$
$$E_m : a_{m1}x_1+\dots+a_{mn}x_n=b_m .$$

Let $A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ and $B=\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$. The system (I) can be written in the form

$$A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}=B$$

and if for each $1\le i\le n$, we denote by C_i the i-th column of A, then the system (I) can be written in the form $x_1C_1+\dots+x_nC_n=B$.

The matrix A is called the **matrix of the system**.

Definition 2. We call **solution** of system (I) in K, every column matrix $\Omega=\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ over K, such that $A\Omega=B$. ■

Remark: When we ask you to solve a system, we in fact ask you to see if the system has solutions or not and if it has solutions, then you find all of them.

Thus a system of linear equations is said to be **consistent** if it has at least one solution, else it will be called **inconsistent** (or **impossible**).

Suppose that $A\neq 0$ and let $r=\text{rank}(A)$.

Definition 3. We call **principal determinant** of the system (I), every non-zero minor of A of order r. ■

Since A≠0, r is the greatest order of non-zero minors of A, hence there exists at least one non-zero minor of A of order r, i.e there exists at least one principal determinant of system (I). If

$$\Delta=\begin{vmatrix} a_{i_1j_1} & a_{i_1j_2} & \cdots & a_{i_1j_r} \\ a_{i_2j_1} & a_{i_2j_2} & \cdots & a_{i_2j_r} \\ \vdots & \vdots & & \vdots \\ a_{i_rj_1} & a_{i_rj_2} & \cdots & a_{i_rj_r} \end{vmatrix}$$

with $1\le i_1 < i_2 < \cdots < i_r \le m$ and $1 \le j_1 < j_2 < \cdots < j_r \le n$, is a principal determinant, then the equations $E_{i_1}, E_{i_2}, \ldots, E_{i_r}$ are called **the principal equations** and the variables x_{j_1}, $x_{j_2}, \ldots, x_{j_r}$ are called **the principal variables** (or **the principal unknowns**) and the other variables are called **the secondary variables** (or **the secondary unknowns**).

Since we can arrange the equations in a way that the first r equations are as follows: E_{i_1} the fist, E_{i_2} the second, and so on E_{i_r} the r-th, and each equation is written in a way that the first r variables are as follows: x_{j_1} the first, x_{j_2} the second, and so on x_{j_r} the r-th, then we may suppose that the entries of Δ are chosen from the first r rows and the first r columns of A. Thus we may assume that

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rr} \end{vmatrix}.$$

Example: Given the real system

$E_1 : 2x_1+x_2+3x_3-x_4=2$

$E_2 : 2x_1+x_2+5x_3-2x_4=3$

$E_3 : 4x_1+2x_2+8x_3-3x_4=5$

$E_4 : 6x_1+3x_2+11x_3-4x_4=7.$

The matrix of the system is

$$A=\begin{pmatrix} 2 & 1 & 3 & -1 \\ 2 & 1 & 5 & -2 \\ 4 & 2 & 8 & -3 \\ 6 & 3 & 11 & -4 \end{pmatrix}.$$

We have

$$\begin{pmatrix} 2 & 1 & 3 & -1 \\ 2 & 1 & 5 & -2 \\ 4 & 2 & 8 & -3 \\ 6 & 3 & 11 & -4 \end{pmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - 3R_1 \end{matrix} \to \begin{pmatrix} 2 & 1 & 3 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 2 & -1 \end{pmatrix} \begin{matrix} R_3 \to R_3 - R_2 \\ R_4 \to R_4 - R_2 \end{matrix} \to \begin{pmatrix} 2 & 1 & 3 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

hence

$$\text{rank}(A)=2.$$

Let

$$\Delta=\begin{vmatrix} 1 & -1 \\ 3 & -4 \end{vmatrix}.$$

We have Δ=-1, so that Δ is a non-zero minor of A of order 2, hence

Δ is a principal determinant of the system.

The principle unknowns are x_2 and x_4 and the principle equations are E_1 and E_4. The system can be written as follows

E_1 : $x_2 - x_4 + 2x_1 + 3x_3 = 2$
E_4 : $3x_2 - 4x_4 + 6x_1 + 11x_3 = 7.$
E_2 : $x_2 - 2x_4 + 2x_1 + 5x_3 = 3$
E_3 : $2x_2 - 3x_4 + 4x_1 + 8x_3 = 5.$

Here the matrix of this system is

$$A' = \begin{pmatrix} 1 & -1 & 2 & 3 \\ 3 & -4 & 6 & 11 \\ 1 & -2 & 2 & 5 \\ 2 & -3 & 4 & 8 \end{pmatrix}$$

and the entries of Δ are taken from the first two rows and the first two columns of A'.

Let

$$D = \begin{pmatrix} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{pmatrix}$$

and for each $1 \le t \le m$, let

$$D_t = \begin{pmatrix} a_{11} & \cdots & a_{1r} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{r1} & \cdots & a_{rr} & b_r \\ a_{t1} & \cdots & a_{tr} & b_t \end{pmatrix}.$$

Definition 4. We call **characteristic determinant** of system (I), every determinant $\Delta_j = \left|D_j\right|$, where $r+1 \le j \le m$. ■

Thus the number of characteristic determinants of the system is (m-r).

4.1.1. The following are equivalent:
(i) The system (I) is consistent.
(ii) rank(D) = r.
(iii) Every characteristic determinant of system (I) is zero.
(iv) $\left|D_t\right| = 0, \forall 1 \le t \le m$.

Proof: The proof is given in Appendix I. ■

Corollary 4.1.1.1. If rank(A)=m, then the system (I) is consistent.

Proof: Since the t-th and last rows of $\left|D_t\right|$ are equal, we then have $\left|D_t\right| = 0, \forall 1 \le t \le m$, whence the system (I) is consistent, by 4.1.1. ■

4.1.2. If the system (I) is consistent, then the solutions are all the column matrices $\Omega = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ over K, such that $\alpha_{r+1}, \ldots, \alpha_n$ are arbitrary in K and $\alpha_i = \dfrac{\Delta_{\alpha_i}}{\Delta}$, $\forall 1 \le i \le r$, where Δ_{α_i} is the determinant obtained from Δ by replacing the i-th column by $\begin{pmatrix} d_1 \\ \vdots \\ d_r \end{pmatrix}$, with

$$d_s = b_s - (a_{s\,r+1}\alpha_{r+1} + \ldots + a_{sn}\alpha_n)$$

for all $1 \le s \le r$.

Proof: The proof is given in Appendix I. ■

Définition 5. The system (I) is said to be a **Cramer's system** if m = n and $|A| \neq 0$. ■

4.1.3. If the system (I) is a Cramer's system, then it has a unique solution $\Omega = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$, such that $\alpha_i = \dfrac{\Delta_{\alpha_i}}{\Delta}$, $\forall 1 \le i \le n$, where Δ_{α_i} is the determinant obtained from Δ by replacing the i-th column by $\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$.

Proof: Since $|A| \neq 0$, rank(A) = n = number of rows of A, hence the system has solutions, by 4.1.1.1. Let $\Omega = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ be such a solution. Then $\alpha_i = \dfrac{\Delta_{\alpha_i}}{\Delta}$, $\forall 1 \le i \le n$, where Δ_{α_i} is the determinant obtained from Δ by replacing the i-th column by $B = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$, by 4.1.2. Now let Ω' be a solution of the system. Then $A\Omega = B$ and $A\Omega' = B$. But A is invertible, because $|A| \neq 0$, hence $\Omega = A^{-1}B$ and $\Omega' = A^{-1}B$, and so $\Omega = \Omega'$. ■

4.1.4. Let $A \in M_n(K)$. If the system

$$A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 0$$

has a non-zero solution, then $|A| = 0$.

Proof: If $|A| \neq 0$, then the system becomes a Cramer's system, and so it has a unique solution. But $\Omega = 0$ is a solution of the system, and so the system has no non-zero solutions, which is impossible. Hence $|A| = 0$. ■

§ 4.2. ECHELON FORM AND SYSTEM OF LINEAR EQUATIONS.

Consider the system of linear equations

$$A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = B$$

where $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ and $B = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$.

4.2.1. If A' is a row echelon form of A and B' is the matrix obtained from B by carrying the same elementary operations as those carried on A, then the systems $A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = B$ and

$A'\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = B'$ are equivalent.

Proof: Let $M_1,\ldots,M_k$ be the elementary matrices corresponding to the elementary operations carried on A to obtain A', then

$$M_k \times \cdots \times M_1 \times A = A'$$

and as B' is obtained from B by carrying on B the same elementary operations as those carried on A, we get that $M_k \times \cdots \times M_1 \times B = B'$, but $M_1,\ldots,M_k$ are invertible, hence

$$A'\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = B' \Leftrightarrow M_k \times \cdots \times M_1 \times A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = M_k \times \cdots \times M_1 \times B \Leftrightarrow A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = B. \blacksquare$$

As we carry the same elementary operations on A and B we combine them in one matrix which we shall denote by $\overline{A}$ or $\widehat{A}$, and we separate them by a column of points as follows:

$$\overline{A} = \left(\begin{array}{ccc:c} a_{11} & \cdots & a_{1n} & b_1 \\ a_{21} & \cdots & a_{2n} & b_2 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right).$$

The matrix $\overline{A}$ is called **the augmented matrix of the system**. It is also written $(A \vdots B)$. When we carry on $(A \vdots B)$ the same elementary operations that we need to carry on A to obtain A', we get the matrix $(A' \vdots B')$.

Suppose that $A \neq 0$ and let

$$\text{rank}(A) = s.$$

We have that the first s rows of A' are non-zero and all the others are zero. Let

$$A' = (\alpha_{ij}) \text{ and } B' = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_s \\ \beta_{s+1} \\ \vdots \\ \beta_m \end{pmatrix},$$

then

$$(A' \vdots B') = \left(\begin{array}{ccc:c} \alpha_{11} & \cdots & \alpha_{1n} & \beta_1 \\ \vdots & & \vdots & \vdots \\ \alpha_{s1} & \cdots & \alpha_{sn} & \beta_s \\ 0 & \cdots & 0 & \beta_{s+1} \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & \beta_m \end{array}\right)$$

and so the system becomes

$$\begin{aligned} D_1 \quad &: \alpha_{11}x_1 + \alpha_{12}x_2 + \cdots + \alpha_{1n}x_n = \beta_1 \\ D_2 \quad &: \alpha_{21}x_1 + \alpha_{22}x_2 + \cdots + \alpha_{2n}x_n = \beta_2 \end{aligned}$$

$$\begin{array}{llcl}
\vdots & & \vdots & \vdots \\
D_s & : & \alpha_{s1}x_1+\alpha_{s2}x_2+\cdots+\alpha_{sn}x_n & = \beta_s \\
D_{s+1} & : & 0 & = \beta_{s+1} \\
\vdots & & \vdots & \vdots \\
D_m & : & 0 & = \beta_m .
\end{array}$$

4.2.2. The system has solutions if and only if $\beta_i=0$, for all $s+1\le i\le m$.

Proof: **N.C:** Let $s+1\le i\le m$. As the i-th equation is $0=\beta_i$, then $\beta_i=0$, for all $s+1\le i\le m$.
S.C: As $\beta_i=0$, for all $s+1\le i\le m$, then the system becomes

$$\begin{array}{ll}
D_1 & : \alpha_{11}x_1+\alpha_{12}x_2+\cdots+\alpha_{1n}x_n = \beta_1 \\
D_2 & : \alpha_{21}x_1+\alpha_{22}x_2+\cdots+\alpha_{2n}x_n = \beta_2 \\
\vdots & \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
D_s & : \alpha_{s1}x_1+\alpha_{s2}x_2+\cdots+\alpha_{sn}x_n = \beta_s .
\end{array}$$

The matrix of the system is

$$A'' = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha_{s1} & \alpha_{s2} & \cdots & \alpha_{sn} \end{pmatrix}$$

which is a matrix in row echelon form, with s non-zero rows, hence

$$\text{rank}(A'') = s = \text{number of rows of } A''$$

and so the system has solutions, by 4.1.1.1. ■

Remark: If during the search of a row echelon form of A in $(A \,\vdots\, B)$, a row of the form $(0 \cdots 0 \,\vdots\, a)$, with $a\neq 0$, appears, then we stop the search, because we get $0=a$, which is impossible, and we deduce that the system is inconsistent in this case.

For each $1\le t\le s$, let α_{tj_t} be the leading entry of the tth row of A' and let

$$\Delta=\begin{pmatrix} \alpha_{1j_1} & \alpha_{1j_2} & \cdots & \cdots & \alpha_{1j_s} \\ 0 & \alpha_{2j_2} & \cdots & \cdots & \alpha_{2j_s} \\ \vdots & 0 & & & \\ \vdots & \vdots & & & \\ 0 & 0 & \cdots & 0 & \alpha_{sj_s} \end{pmatrix} \text{ and } d_i = \beta_i - \sum_{t\in\{1,\ldots,n\}-\{j_1,\ldots,j_r\}} \alpha_{it}x_t \text{ , with } 1\le i\le s.$$

We have that Δ is a matrix in row echelon form of order s, hence rank(Δ)=s, and so Δ is invertible, by 2.2.5.

4.2.3. If the system is consistent, then the solutions are all the column matrices $\Omega = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ over K, such that x_t is arbitrary in K, for all $t\in\{1,\ldots,n\}-\{j_1,j_2,\ldots,j_s\}$ and $x_{j_1}, \ldots, x_{j_s}$ are given by $\begin{pmatrix} x_{j_1} \\ \vdots \\ x_{j_s} \end{pmatrix} = \Delta^{-1}\begin{pmatrix} d_1 \\ \vdots \\ d_s \end{pmatrix}$.

Proof: As Δ is invertible and the system is equivalent to the system $\Delta\begin{pmatrix} x_{j_1} \\ \vdots \\ x_{j_s} \end{pmatrix}=\begin{pmatrix} d_1 \\ \vdots \\ d_s \end{pmatrix}$, then the solutions are all the column matrices $\Omega=\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ over K, such that x_t is arbitrary in K, for all $t\in\{1,\ldots,n\}-\{j_1,\ldots,j_s\}$ and $x_{j_1},\ldots,x_{j_s}$ are given by $\begin{pmatrix} x_{j_1} \\ \vdots \\ x_{j_s} \end{pmatrix}=\Delta^{-1}\begin{pmatrix} d_1 \\ \vdots \\ d_s \end{pmatrix}$. ■

It follows from 4.2.3 that if the system is consistent, then $|\Delta|$ is a principal determinant of the system, and so $x_{j_1},\ldots,x_{j_s}$ are the principal variables and each x_t, with $t\in\{1,\ldots,n\}-\{j_1,\ldots,j_s\}$ is a secondary variable. Thus to find the solutions, we need to calculate $x_{j_1},\ldots,x_{j_s}$ in terms of $d_1,\ldots,d_s$. This can be done by either finding the inverse of Δ, or by using the following method, known as **the method of back substitution**:
As the system becomes

$$\begin{array}{lrcl}
D_1: & \alpha_{1j_1}x_{j_1}+\alpha_{1j_2}x_{j_2}+\cdots+\alpha_{1j_s}x_{j_s} & = & d_1 \\
D_2: & \alpha_{2j_1}x_{j_1}+\cdots+\alpha_{2j_s}x_{j_s} & = & d_2 \\
\vdots & \vdots & & \vdots \\
D_{s-1}: & \alpha_{s-1j_{s-1}}x_{j_{s-1}}+\alpha_{s-1j_s}x_{j_s} & = & d_{s-1} \\
D_s: & \alpha_{sj_s}x_{j_s} & = & d_s
\end{array}$$

then from D_s, we get x_{j_s}, and substituting x_{j_s} in D_{s-1} we get $x_{j_{s-1}}$, and so on, we continue back substitution, till we get $x_{j_1},\ldots,x_{j_s}$.

4.2.4. If the system is consistent, then
(i) if rank(A)<n, then the system has more than one solution (when K=$\mathbb{R}$ or $\mathbb{C}$, we say that the system has infinitely many solutions),
(ii) if rank(A)=n, then the system has a unique solution.

Proof: (i) Since rank(A)<n, there is at least one secondary variable, and as every secondary variable is arbitrary, by 4.2.3, we then have that the system has more than one solution.
(ii) As s=n, then there is no secondary variable, and so $d_1,\ldots,d_n$ are constant, whence the unique solution is $\Omega=\Delta^{-1}\begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}$. ■

4.2.5. If m=n, then the system has unique solution if and only if rank(A)=m.

Proof: **N.C:** If rank(A)$\neq$ m, then as m=n, we get rank(A)<n, and so the system has more than one solution, by 4.2.4(i), impossible. Therefore rank(A)=m.
S.C: As rank(A)=m, then the system is consistent, by 4.1.1.1, and so it has a unique solution, by 4.2.4(ii). ■

Examples: 1) Solve the real system

$$x+y-z=1$$

$$x-2y+z=2$$
$$-x+2y+2z=4.$$

The augmented matrix of the system is $\overline{A}=\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 1 & -2 & 1 & 2 \\ -1 & 2 & 2 & 4 \end{array}\right)$. We have

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 1 & -2 & 1 & 2 \\ -1 & 2 & 2 & 4 \end{array}\right) \begin{array}{l} R_2 \to R_2 - R_1 \to \\ R_3 \to R_3 + R_1 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -3 & 2 & 1 \\ 0 & 3 & 1 & 5 \end{array}\right) R_3 \to R_3 + R_2 \to$$

$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -3 & 2 & 1 \\ 0 & 0 & 3 & 6 \end{array}\right)$, hence the system becomes

$$x+y-z=1$$
$$-3y+2z=1$$
$$3z=6$$

and so z=2, y=(1-2z)/-3 = 1 and x=1-y+z = 2, whence the system is consistent and it has a unique solution $\Omega=\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$.

2) Solve the real system

$$x+y+2z=2$$
$$3x+y+3z=4$$
$$4x+2y+5z=5.$$

The augmented matrix of the system is $\overline{A}=\left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 3 & 1 & 3 & 4 \\ 4 & 2 & 5 & 5 \end{array}\right)$. We have

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 3 & 1 & 3 & 4 \\ 4 & 2 & 5 & 5 \end{array}\right) \begin{array}{l} R_2 \to R_2 - 3R_1 \to \\ R_3 \to R_3 - 4R_1 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & -2 & -3 & -2 \\ 0 & -2 & -3 & -3 \end{array}\right) R_3 \to R_3 - R_2 \to$$

$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & -2 & -3 & -2 \\ 0 & 0 & 0 & -1 \end{array}\right)$, hence 0 = -1, impossible, and so the system is inconsistent.

3) Solve the real system

$$x+3y+z=2$$
$$2x+y+4z=4$$
$$3x+4y+5z=6.$$

The augmented matrix of the system is $\overline{A}=\left(\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 2 & 1 & 4 & 4 \\ 3 & 4 & 5 & 6 \end{array}\right)$. We have

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 2 & 1 & 4 & 4 \\ 3 & 4 & 5 & 6 \end{array}\right) \begin{array}{l} R_2 \to R_2 - 2R_1 \to \\ R_3 \to R_3 - 3R_1 \end{array} \left(\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 0 & -5 & 2 & 0 \\ 0 & -5 & 2 & 0 \end{array}\right) R_3 \to R_3 - R_2 \to \left(\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 0 & -5 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

hence the system is consistent. The principal variables are x and y and z is the only secondary variable. The system becomes

$$x+3y+z=2$$
$$-5y+2z=0.$$

By using the method of back substitution, we get from the last equation that y=(2/5)z, then substituting y in the equation above the last, we get

$$x=2-z-3y=2-z-(6/5)z=2-(11/5)z$$

and so the solutions are the matrices $\Omega = \begin{pmatrix} 2-(11/5)z \\ (2/5)z \\ z \end{pmatrix}$, with z arbitrary in $\mathbb{R}$.

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CHAPTER IV

EXERCISES

1- Given the real system:

$$(S):\begin{cases}2y-z=1\\-y+z=1\\x+z=4\end{cases}.$$

a) Write (S) in the form AX=B.
b) Find the inverse of A by the method of row echelon form.
c) Deduce the solutions of (S).

2- Show that the following system is a Cramer's system and find its solution:

$$(S):\begin{cases}x+y-z=0\\x-2y+3z=7\\2x+3y+z=2\end{cases}.$$

3- Given the real system:

$$(S):\begin{cases}x+y+(2m-1)z=1\\mx+y+z=1\\x+my-5z=3(m+1)\end{cases}.$$

Find the values of m, for which the system (S) is a Cramer's system, then solve it.

4- Discuss following the values of the real parameter m, the real system

$$(S):\begin{cases}mx+2y+6z=0\\2x+y+3z=0\\x+y-5z=0\end{cases}.$$

5- Given the real system

$$(S):\begin{cases}-x+2y-3z=a\\2x+6y-11z=b\\x-2y+7z=c\end{cases}.$$

Find the relations satisfied by a, b and c, so that the system is consistent and then give the solutions.

6- Solve and discuss each of the following real systems:

$$(S_1):\begin{cases}3x+4y+2z=m\\x+y+mz=2\\2x+3y-z=1\end{cases};\qquad (S_2):\begin{cases}x-2y+3z=2\\x+3y-2z=5\\2x-y+mz=1\end{cases};$$

$$(S_3):\begin{cases}mx+y+mz=2-m\\x+y+z=1\\mx+2y+2z=m\end{cases};\qquad (S_4):\begin{cases}(2m+1)x-my+(m+1)z=m-1\\(m-2)x+(m-1)y+(m-2)z=m\\(2m-1)x+(m-1)y+(2m-1)z=m\end{cases}.$$

7- Let a be a complex number, such that $a^3-a+1=0$ and let $b=a^2+a+2$. Find a non-zero real polynomial f(x), such that f(b)=0.

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CHAPTER V

VECTOR SPACES

Throughout this chapter K is a field.

§ 5.1. INTRODUCTION.

Let E be a non-empty set.

Definition 1. We call **action** (or **scalar multiplication**) of K on E, every mapping of $K\times E$ to E. ■

NOTATION: If f is an action of K on E, then the image by f of every element (a,x) of $K\times E$ is denoted ax.

Definition 2. We say that E is a **vector space over K** (or a **K-vector space**) if E is an additive abelian group and there is an action of K on E satisfying:
(i) $(a+b)x = ax+bx$,
(ii) $a(x+y) = ax+ay$,
(iii) $a(bx) = (ab)x$,
(iv) $1_K x = x$,
for all $a,b\in K$ and all $x,y\in E$. ■

Examples: 1) The set E of free vectors in the space, is a $\mathbb{R}$-vector space under the addition of vectors and the following action of $\mathbb{R}$: if $\vec{V}$ is a free vector and $\alpha\in\mathbb{R}$, then $\alpha\vec{V}$ is the vector having the same direction of $\vec{V}$ and same sense (resp. opposite sense) of $\vec{V}$ if α is poisitive (resp. negative) and whose module is $|\alpha\vec{V}| = |\alpha|\,|\vec{V}|$.

2) $M_{m,n}(K)$, endowed with the addition and the scalar multiplication of matrices is a K-vector space.

3) The set

$$K^n = \{(a_1,...,a_n) ; a_1,...,a_n \in K\}$$

is a K-vector space under the following addition and action of K

$$(a_1,...,a_n)+(b_1,...,b_n) = (a_1+b_1,...,a_n+b_n)$$

and

$$\alpha(a_1,...,a_n) = (\alpha a_1,...,\alpha a_n)$$

such that $0_E = (0,0,...,0)$ and if $x = (a_1,...,a_n)$, then $-x = (-a_1,...,-a_n)$.

4) If K' is a subfield of K, then K, endowed with its addition and the following action of K'

$$\begin{aligned} f : K'\times K &\longrightarrow K \\ (\alpha, x) &\longrightarrow \alpha x \end{aligned}$$

where αx is the product in K of α by x, is a K'-vector space. Thus; $\mathbb{C}$ is a vector space over $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$. Also $\mathbb{R}$ is a vector space over $\mathbb{Q}$ and over $\mathbb{R}$.

5) K[x] is a K-vector space under the addition and scalar multiplication of polynomials.

6) Let S be a non-empty set and V be a K-vector space and set $E=V^S$, where V^S is the set of all the mappings of S to V. Define on E the following addition and action: if $f,g\in E$, then

$$f+g : S\longrightarrow V \text{ is defined by } (f+g)(x) = f(x)+g(x),\ \forall x\in S$$

and if $\alpha\in K$ and $f\in V$, then

$$\alpha f : S\longrightarrow V \text{ is defined by } (\alpha f)(x) = \alpha f(x),\ \forall x\in S.$$

Then E is a K-vector space, for the above addition and action, such that 0_E is the zero mapping of S to V, i.e

$$0_E : S\longrightarrow V \text{ is defined by } 0_E(x) = 0_V,\ \forall x\in S.$$

In particular as $\mathbb{R}^{\mathbb{N}}$ is the set of all real sequences, then it is an $\mathbb{R}$-vector space.

For the rest of this chapter, E is a K-vector space. The zero of E will be denoted 0_E and if $x,y\in E$, we shall write x-y for x+(-y) and if $a\in K$, we write –ax for -(ax).

5.1.1. The following hold for all $x,y,z\in E$ and all $a,b\in K$:
(i) $-(-x) = x$.
(ii) $-(x+y) = -x-y$.
(iii) $x+y = z \Leftrightarrow x = z-y$.
(iv) $x+y = 0_E \Leftrightarrow x = -y$
(v) $0_K x = 0_E$.
(vi) $a0_E = 0_E$.
(vii) $(-1_K)x = -x$.
(viii) $a(-x) = (-a)x = -ax$.
(ix) $(a-b)x = ax-bx$.
(x) $a(x-y) = ax-ay$.
(xi) $ax = 0_E \Rightarrow a = 0_K$ or $x = 0_E$.

Proof: (i) through (iv) follow from the fact that (E,+) is a group.
(v) We have $0_K x =(0_K+0_K)x =0_K x+0_K x$, hence $0_K x =0_K x-0_K x =0_E$.
(vi) Since $a0_E =a(0_E+0_E)=a0_E+a0_E$, we get $a0_E = a0_E-a0_E = 0_E$.
(vii) As $(-1_K)x+x = (-1_K)x+1_K x = (-1_K+1_K)x = 0_K x = 0_E$, then $(-1_K)x = -x$.
(viii) We have $a(-x)+ax = a(-x+x) = a0_E = 0_E$, by (vi), hence $a(-x) = -ax$.
Similarly as $(-a)x+ax = (-a+a)x = 0_K x = 0_E$, then $(-a)x = -ax$. Hence

$$a(-x) = (-a)x = -ax.$$

(ix) $(a-b)x = (a+(-b))x = ax+(-b)x = ax+(-bx) = ax-bx$.
(x) $a(x-y) = a(x+(-y)) = ax+a(-y) = ax+(-ay) = ax-ay$.
(xi) Suppose that $a\neq 0_K$ and $x\neq 0_E$. Then a^{-1} exists in K. Since $ax = 0_E$, we then have

$$a^{-1}(ax) = a^{-1}0_E = 0_E.$$

But $a^{-1}(ax) = (a^{-1}a)x = 1_K x=x$, hence $x=0_E$, impossible. Therefore $a=0_K$ or $x=0_E$. ■

5.1.2. If $E_1,...,E_n$ are a K-vector spaces, then the set $V=E_1\times\cdots\times E_n$ is a K-vector space under the follwing addition and scalar multiplication

$$(a_1,...,a_n)+(b_1,...,b_n) = (a_1+b_1,...,a_n+b_n)$$

and

$$\alpha(a_1,...,a_n) = (\alpha a_1,...,\alpha a_n)$$

such that $0_V = (0_{E_1},...,0_{E_n})$ and if $x = (a_1,...,a_n)$, then $-x = (-a_1,...,-a_n)$.

This space is called **the direct product** (or **the product space**) of $E_1,...,E_n$.

Proof: Easy enough. ■

§ 5.2. SUB-SPACES OF A VECTOR SPACE.

Definition 3. A subset F of E is said to be **closed** under the action of K if $\alpha x\in F$, for all $x\in F$ and all $\alpha\in K$. ■

Definition 4. A subset F of E is said to be a **subspace** of E over K (or **K-subspace** of E) if F is a subgroup of (E,+), closed under the action of K. ■

5.2.1. Let F be subset of E, then the following are equivalent;
(i) F is a subspace of E,
(ii) $F\neq\varnothing$ and $\alpha x+\beta y\in F$, for all $x,y\in F$ and all $\alpha,\beta\in K$,
(iii) $F\neq\varnothing$ and $\alpha x+y\in F$, for all $x,y\in F$ and all $\alpha\in K$,
(iv) $F\neq\varnothing$ and $x+y\in F$ and $\alpha x\in F$, for all $x,y\in F$ and all $\alpha\in K$.

Proof: (i)⇒(ii): Follows easily from definition 4.
(ii)⇒(iii): Let $x,y\in F$ and $\alpha\in K$. We have $\alpha x+y=\alpha x+1_K y$, hence $\alpha x+y\in F$.
(iii)⇒(iv): Let $x,y\in F$ and $\alpha\in K$. We have $x+y=1_K x+y$, hence

$$x+y\in F.$$

As $0_E = (-1_K)x+x$, then $0_E\in F$. But $\alpha x=\alpha x+0_E$, hence $\alpha x\in F$.
(iv)⇒(i): As $-x =(-1_K)x$, then $-x\in F$, for all $x\in F$, and so F is a subgroup of (E,+). But F is closed under the action of K, hence F is a subspace of E over K. ■

Examples: 1) $\{0_E\}$ and E are a subspaces of E over K, called the **trivial** subspaces of E.
2) The set of symmetric (resp. anti-symmetric) matrices of $M_n(K)$ is a subspace of $M_n(K)$.
3) The set of matrices of $M_n(K)$ of trace zero is a subspace of $M_n(K)$.
4) The set of upper (resp. lower, diagonal) of $M_n(K)$ is a subspace of $M_n(K)$.
5) The set of convergent real sequences is a subspace over $\mathbb{R}$ of $\mathbb{R}^{\mathbb{N}}$.
6) For every non-zero natural number n, the set

$$K_n[x]=\{f(x)=a_0+a_1x+\cdots+a_nx^n;\ a_0, a_1, \ldots, a_n\in K\}$$

is a subspace of K[x].

5.2.2. If F is a subspace of E, then
(i) $0_E\in F$
(ii) $-x\in F$, for all $x\in F$.
(iii) If $x_1,...,x_n\in F$, then $(\alpha_1x_1+\cdots+\alpha_nx_n)\in F$, $\forall\alpha_1,...,\alpha_n\in K$.
(iv) The correspondence f from $K\times F$ to F that associates every element (a,x) of $K\times F$ with

the element ax of F is an action of K on F. Moreover F is a K-vector space under this action and the addition induced by that of E on F, such that $0_F = 0_E$, that is F is itself a K-vector space for the same laws defined on E, such that $0_F = 0_E$.

Proof: As F is a subgroup of (E,+), then (i) and (ii) hold.

(iii) We argue by induction on n. It is true for n = 1, for if $x_1 \in F$, then $\alpha_1 x_1 \in F$, $\forall \alpha_1 \in K$. Suppose that the property holds for (n-1) and let's show it for n. Let $x_1,...,x_n \in F$ and $\alpha_1,...,\alpha_n \in K$. We have that

$$\alpha_1 x_1 + \cdots + \alpha_{n-1} x_{n-1} \in F$$

by induction hypothesis. Since $\alpha_n \in K$ and $x_n \in F$ and F is a subspace of E, we then get that $\alpha_n x_n \in F$, and so

$$\alpha_1 x_1 + \cdots + \alpha_{n-1} x_{n-1} + \alpha_n x_n \in F.$$

Therefore the property is true for n, and so it is true, for all n≥1.

(iv) Since F is closed under the action of K on E, one can easily show that f is a mapping, and so it is an action of K on F. Also one can easily show that F is a group under the addition induced by that of E and as the action of K on F is the same as that of K on E, then conditions (i) through (iv) of definition 2 are satisfied, and so F is a K-vector space. Now as F is a subgroup of (E,+), then $0_F = 0_E$. ■

5.2.3. If $(F_i)_{i\in I}$ is a non-empty family of subspaces of E, then $\bigcap_{i\in I} F_i$ is a subspace of E.

Proof: Since F_i is a subspace of E, $0_E \in F_i$, for all $i\in I$, by 5.2.2(i), and so $0_E \in \bigcap_{i\in I} F_i$, whence $\bigcap_{i\in I} F_i \neq \varnothing$. Let $x,y \in \bigcap_{i\in I} F_i$ and $\alpha \in K$. Then $x,y \in F_i$, for all $i\in I$, and as F_i is a subspace of E, we get $\alpha x+y \in F_i$, for all $i\in I$, hence $\alpha x+y \in \bigcap_{i\in I} F_i$, and so $\bigcap_{i\in I} F_i$ is a subspace of E. ■

5.2.4. If S is a subset of E, then there exists one and only subspace W of E satisfying:

(i) W contains S,

(ii) W is contained in every subspace of E containing S.

This subspace is called **the smallest subspace** of E containing S.

Proof: **Existence:** Let $(F_i)_{i\in I}$ be the family of the subspaces of E containing S. As E is a member of this family, then the family is non-empty, and so $\bigcap_{i\in I} F_i$ is a subspace of E satisfying (i) and (ii), by 5.2.3.

Uniqueness: Let W' be a subspace of E satisfying (i) and (ii). As W is a subspace of E containing S, then $W' \subseteq W$. Also as W' is a subspace of E containing S, then $W \subseteq W'$, and so W=W'. ■

Remark: It follows from the proof of 5.2.4 that the smallest subspace of E containing S is the intersection of all the subspaces of E containing S.

5.2.5. If F and V are two subspaces of E, then

(i) $F \cap V$ is a subspace of E.

(ii) If $V \subseteq F$, then V is a subspace of the K-vector space F.

(iii) If W is a K-subspace of F over K, then W is a K-subspace of E over K.

Proof: (i) By 5.2.3.
(ii) Since V is a subspace of E, we then have that

$$V\neq\emptyset \text{ and } \alpha x+y\in V,\ \forall x,y\in V \text{ and } \forall\alpha\in K.$$

But $V\subseteq F$, hence V is a subspace of F over K.
(iii) We have that $W\subseteq E$, hence $W\neq\emptyset$ and $\alpha x+y\in W$, for all $x,y\in W$ and all $\alpha\in K$, since W is a subspace of F over K, hence W is a subspace of E over K. ■

§ 5.3. SUM OF SUBSPACES.

If A and B are two non-empty subsets of E, we let

$$A+B = \{x+y\ ;\ x\in A \text{ and } y\in B\}.$$

Thus;

$$z\in A+B \Leftrightarrow \exists x\in A \text{ and } \exists y\in B, \text{ such that } z = x+y.$$

5.3.1. If A and B are two non-empty subsets of E, then
(i) $A+B = B+A$.
(ii) If W is a K-subspace of E containing A and B, then $A+B\subseteq W$.

Proof: (i) We have

$$\begin{aligned} z\in A+B &\Leftrightarrow \exists x\in A \text{ and } \exists y\in B, \text{ such that } z = x+y \\ &\Leftrightarrow \exists y\in B \text{ and } \exists x\in A, \text{ such that } z = y+x \\ &\Leftrightarrow z\in B+A \end{aligned}$$

hence $A+B = B+A$.
(ii) Let $z\in A+B$. Then $\exists x\in A$ and $\exists y\in B$, such that $z = x+y$. Since $A\subseteq W$ and $B\subseteq W$, we get that $x,y\in W$. But W is a subspace of E over K, hence $x+y\in W$, and so $z\in W$, whence $A+B\subseteq W$. ■

5.3.2. If U and V are two subspaces of E, then U+V is a subspace of E, called the **sum** of U and V. Moreover U+V is the smallest subspace of E containing $U\cup V$.

Proof: Since U and V are two subspaces of E, we then have that $0_E\in U$ and $0_E\in V$, by 5.2.2(i), hence $0_E+0_E\in U+V$, and so

$$U+V\neq\emptyset.$$

Let $u,v\in U+V$ and $\alpha,\beta\in K$. Then

$$\exists x\in U \text{ and } \exists x'\in V, \text{ such that } u = x+x'$$
$$\text{and}$$
$$\exists y\in U \text{ and } \exists y'\in V, \text{ such that } v = y+y'.$$

We have

$$\alpha u+v = \alpha(x+x')+(y+y') = (\alpha x+y)+(\alpha x'+y')$$

and so as $\alpha x+y\in U$ and $\alpha x'+y'\in V$, then $\alpha u+v\in U+V$, whence U+V is a subspace of E.
<u>$U\subset U+V$</u>: Lett $x\in U$. As $x=x+0_E$ and $0_E\in V$, then $x\in U+V$, and so $U\subseteq U+V$.
<u>$V\subset U+V$</u>: Let $x\in V$. As $x=0_E+x$ and $0_E\in U$, then $x\in U+V$, and so $V\subseteq U+V$.
Therefore

$$U\cup V\subseteq U+V.$$

Let W be a subspace of E containing $U\cup V$, then $U+V\subseteq W$, by 5.2.6(ii), and so U+V is the smallest subspace of E containing $U\cup V$. ■

Definition 5. Let U and V be two subspaces of E over K. We say that E is the **direct sum**

of U and V and we write $E = U \oplus V$ if
(i) $E = U+V$,
(ii) $U \cap V = \{0_E\}$. ■

Example: Let $E = \mathbb{R}^2$,

$$U = \{(a,0) ; a \in \mathbb{R}\} \text{ and } V = \{(0,b) ; b \in \mathbb{R}\}.$$

Then U and V are two subspaces of E over $\mathbb{R}$ and $E = U \oplus V$.

Definition 6. Let U be a subspace of E over K. We call **supplement** of U every K-subspace V of E, such that $E = U \oplus V$. ■

Later in chapter VI, we shall show that every subspace of E has a supplement.

Remark: Let U and V be two subspaces of E over K. As $U+V \subseteq E$, then to show that $E=U+V$, it is enough to prove that every element z of E can be written in the form $z = x+x'$, for some $x \in U$ and $x' \in V$.
Also as $\{0_E\} \subseteq U \cap V$, then to show that $U \cap V = \{0_E\}$, it suffices to prove that $z = 0_E$, for all $z \in U \cap V$.

5.3.3. If U and V are two subspaces of E over K, then $E = U \oplus V$ if and only if every element z of E is uniquely written in the form $z = x+x'$, where $x \in U$ and $x' \in V$.

Proof: **N.C:** Let $z \in E$. Since $E = U \oplus V$, we get that $E = U+V$, hence $z \in U+V$, and so

$$\exists x \in U \text{ and } \exists x' \in V, \text{ such that } z = x+x'.$$

Suppose that $z = y+y'$, with $y \in U$ and $y' \in V$. We have $x+x' = y+y'$, and so

$$x-y = y'-x'.$$

But $x,y \in U$, $x',y' \in V$ and U and V are two subspaces of E over K, hence $x-y \in U$ and $y'-x' \in V$, and as $x-y = y'-x'$, then

$$x-y \in U \cap V \text{ and } y'-x' \in U \cap V.$$

We have $U \cap V = \{0_E\}$, hence $x-y=0_E$ and $y'-x'=0_E$, and so $x=y$ and $x'=y'$.
S.C: Let $z \in E$. Then $\exists x \in U$ and $\exists x' \in V$, such that $z = x+x'$, hence

$$E = U+V.$$

Let $x \in U \cap V$. Then $x \in U$ and $x \in V$. We have that

$$x = x+0_E, \text{ with } x \in U \text{ and } 0_E \in V$$

and

$$x = 0_E+x, \text{ with } 0_E \in U \text{ and } x \in V$$

hence $x = 0_E$ and $0_E = x$, and so $x = 0_E$, whence

$$U \cap V = \{0_E\}.$$

Therefore $E = U+V$ and $U \cap V = \{0_E\}$, and so $E = U \oplus V$. ■

§ 5.4. SYSTEM OF GENERATORS.

Definition 5. Let S be a non-empty subset of E. An element x of E is said to be a **linear combination** of elements of S over K if $\exists x_1,...,x_n \in S$ and $\exists \alpha_1,...,\alpha_n \in K$, such that $x = \alpha_1 x_1 + \cdots + \alpha_n x_n$. ■

Let L(S) be the set of all linear combinations of elements of S over K. Then

$$x \in L(S) \Leftrightarrow \exists x_1,...,x_n \in S \text{ and } \exists \alpha_1,...,\alpha_n \in K, \text{ such that } x = \alpha_1 x_1 + \cdots + \alpha_n x_n.$$

5.4.1. If S is a non-empty subset of E, then
(i) $S \subseteq L(S)$.
(ii) L(S) is a subspace of E over K.
(iii) If W is a subspace of E over K containing S, then $L(S) \subseteq W$. In particular L(S) is the smallest subspace of E containing S.
(iv) L(S) is the intersection of the family of all the subspaces of E containing S.
(v) If S' is a non-empty subset of E, such that $S \subseteq S'$, then $L(S) \subseteq L(S')$.

Proof: (i) Let $x \in S$. As $x = 1_K x$ and $x \in S$ and $1_K \in K$, then $x \in L(S)$, and so $S \subseteq L(S)$.
(ii) We have $S \neq \varnothing$ and $S \subseteq L(S)$, hence

$$L(S) \neq \varnothing.$$

Let $u,v \in L(S)$ and $\alpha \in K$. Then

$$\exists x_1,\dots,x_n \in S \text{ and } \exists a_1,\dots,a_n \in K, \text{ such that } u = a_1 x_1 + \cdots + a_n x_n$$
$$\text{and}$$
$$\exists y_1,\dots,y_t \in S \text{ and } \exists b_1,\dots,b_t \in K, \text{ such that } v = b_1 y_1 + \cdots + b_t y_t.$$

This implies that

$$\alpha u + v = (\alpha a_1)x_1 + \cdots + (\alpha a_n)x_n + b_1 y_1 + \cdots + b_t y_t.$$

But $x_1,\dots,x_n,y_1,\dots,y_t \in S$ and $\alpha a_1,\dots,\alpha a_n, b_1,\dots, b_t \in K$, hence $\alpha u + v \in L(S)$, and so L(S) is a subspace of E.
(iii) Let $x \in L(S)$. Then $\exists x_1,\dots,x_n \in S$ and $\exists a_1,\dots,a_n \in K$, such that

$$x = a_1 x_1 + \cdots + a_n x_n.$$

We have $S \subseteq W$, hence $x_1,\dots,x_n \in W$, and so as W is a subspace of E over K, then $x \in W$, by 5.2.2(ii), whence $L(S) \subseteq W$.
(iv) See the remark after the proof of 5.2.4.
(v) Let $x \in L(S)$. Then $\exists x_1,\dots,x_n \in S$ and $\exists a_1,\dots,a_n \in K$, such that

$$x = a_1 x_1 + \cdots + a_n x_n.$$

As $S \subseteq S'$, then $x_1,\dots,x_n \in S'$, and so $x \in L(S')$, whence $L(S) \subseteq L(S')$. ■

Corollary 5.4.1.1. The following hold
(i) If W is a subspace of E over K, then L(W) = W.
(ii) If S is a non-empty subset of E, then L(L(S)) = L(S).

Proof: (i) We have that $W \subseteq L(W)$, by 3.3.1(i) and W is a subspace of E over K and $W \subseteq W$, hence $L(W) \subseteq W$, by 3.3.1(iii), and so L(W) = W.
(ii) As L(S) is a subspace of E over K, by 3.3.1(ii), then L(L(S)) = L(S), by (i). ■

For every $(x_1,\dots,x_n) \in E^n$, let

$$\mathrm{Vect}(x_1,\dots,x_n) = \{x = a_1 x_1 + \cdots + a_n x_n \ ; \ a_1,\dots,a_n \in K\}.$$

5.4.2. If $(x_1,\dots,x_n) \in E^n$, then $\mathrm{Vect}(x_1,\dots,x_n)$ is a subspace of E over K and

$$L(\{x_i \ ; \ 1 \leq i \leq n\}) = \mathrm{Vect}(x_1,\dots,x_n).$$

Proof: Let $S = \{x_i \ ; \ 1 \leq i \leq n\}$ and

$$F = \mathrm{Vect}(x_1,\dots,x_n).$$

First we show that F is a subspace of E. As $0_K x_1 + \cdots + 0_K x_n \in F$, we then have that

$$F\neq\emptyset.$$

Let $u,v\in F$ and $\alpha\in K$. Then $\exists a_1,...,a_n, b_1,...,b_n \in K$, such that

$$u = a_1x_1+\cdots+a_nx_n \text{ and } v = b_1x_1+\cdots+b_nx_n.$$

We have $\alpha u+v = (\alpha a_1+b_1)x_1+\cdots+(\alpha a_n+b_n)x_n$, hence $\alpha u+v\in F$, and so

F is a subspace of E over K.

Since every element of F is a linear combination of elements of S over K, we then have

$$F\subseteq L(S).$$

For each $1\leq i\leq n$, we have

$$x_i = 0_Kx_1+\cdots+0_Kx_{i-1}+1_Kx_i+0_Kx_{i+1}+\cdots+0_Kx_n$$

so that $x_i\in F$, for all $1\leq i\leq n$, hence $S\subseteq F$, and so $L(S)\subseteq F$, by 5.4.1(iii), whence $F=L(S)$. ■

Definition 6. Let W be a subspace of E. A non-empty subset S of E is said to be a **system of generators** of W (or that S **generates** W, or that W is **generated** by S) over K if W=L(S). ■

Definition 7. Let W be a subspace of E and let $x_1,...,x_n\in W$. We say that $x_1,...,x_n$ form a **system of generators** of W (or that $x_1,...,x_n$ **generate** W) over K if W= Vect($x_1,...,x_n$). ■

Examples: 1) E is a system of generators of E over K, because L(E)=E.

2) The set S = {(1,0,0), (0,1,0), (0,0,1)} is a system of generators of K^3 over K

5.4.3. Let W be a subspace of E. If $x_1,...,x_n\in W$, then $x_1,...,x_n$ form a system of generators of W over K if and only if every element x of W is written in the form $x=a_1x_1+\cdots+a_nx_n$, for some $a_1,...,a_n\in K$.

Proof: Follows easily from the fact that

$$\mathrm{Vect}(x_1,...,x_n) = \{x = a_1x_1+\cdots+a_nx_n \ ; a_1,...,a_n\in K\}. \blacksquare$$

5.4.4. Let W be a subspace of E over K. If S is a system of generators of W over K, then every subset S' of W containing S is also a system of generators of W over K.

Proof: We have $S\subseteq S'\subseteq W$, hence $L(S)\subseteq L(S')\subseteq W$, by 5.4.1(iii). But W=L(S), hence $W\subseteq L(S')$, and so W=L(S'), whence S' is a system of generators of W over K. ■

Corollary 5.4.4.1. Let W be a subspace of E. If $x_1,...,x_n$ form a system of generators of W over K, then so do the elements $x_1,...,x_n, y_1,...,y_t$, for all $y_1,...,y_t\in W$.

Proof: Let $x_{n+1} = y_1,..., x_{n+t} = y_t$ and let

$$S = \{x_i \ ; 1\leq i\leq n\} \text{ and } S' = \{x_i \ ; 1\leq i\leq n+t\}.$$

As W=L(S) and $S\subseteq S'\subseteq W$, then S' is a system of generators of W over K, by 5.4.4, and so

W=L(S').

But L(S')=Vect($x_1,...,x_n, y_1,...,y_t$), by 5.4.2, hence W=Vect($x_1,...,x_n, y_1,...,y_t$), and so $x_1,...,x_n, y_1,...,y_t$ form a system of generators of W over K. ■

Corollary 5.4.4.2. If $x_1,...,x_n$ are elements of E, then

$$\mathrm{Vect}(x_1,\ldots,x_n, y_1,\ldots,y_t) = \mathrm{Vect}(x_1,\ldots,x_n)$$
for all $y_1,\ldots,y_t \in \mathrm{Vect}(x_1,\ldots,x_n)$.

Proof: Let $W=\mathrm{Vect}(x_1,\ldots,x_n)$. As $y_1,\ldots,y_t \in W$, then the elements $x_1,\ldots,x_n, y_1,\ldots,y_t$ form a system of generators of W, by 5.4.4.1, and so
$$\mathrm{Vect}(x_1,\ldots,x_n, y_1,\ldots,y_t) = \mathrm{Vect}(x_1,\ldots,x_n). \ ■$$

5.4.5. If S and S' are two non-empty subsets of E, then
$$L(S\cup S') = L(S)+L(S').$$
In particular if U and V are two subspaces of E, such that S is a system of generators of U and S' is a system of generators of V, then $S\cup S'$ is a system of generators of U+V.

Proof: As $S\subseteq S\cup S'$ and $S'\subseteq S\cup S'$, then $L(S)\subseteq L(S\cup S')$ and $L(S')\subseteq L(S\cup S')$, and so
$$L(S)+L(S')\subseteq L(S\cup S').$$
But $U=L(S)$ and $V=L(S')$, hence
$$U+V\subseteq L(S\cup S').$$
We have $S\subseteq L(S)$ and $L(S)\subseteq L(S)+L(S')$, hence
$$S\subseteq L(S)+L(S').$$
Also, as $S'\subseteq L(S')$ and $L(S')\subseteq L(S)+L(S')$, then $S'\subseteq L(S)+L(S')$, and so
$$S\cup S'\subseteq L(S)+L(S').$$
It follows from 5.4.1(iii) that $L(S\cup S')\subseteq L(S)+L(S')$, hence
$$L(S\cup S') = L(S)+L(S').$$
As $U= L(S)$ and $V=L(S')$, then $U+V=L(S)+L(S')=L(S\cup S')$, and so
$S\cup S'$ is a system of generators of U+V over K. ■

Corollary 5.4.5.1. If $x_1,\ldots,x_n, y_1,\ldots,y_t \in E$, then
$$\mathrm{Vect}(x_1,\ldots,x_n, y_1,\ldots,y_t) = \mathrm{Vect}(x_1,\ldots,x_n)+\mathrm{Vect}(y_1,\ldots,y_t).$$

Proof: Let $S = \{x_i \ ; 1\le i\le n\}$, $S' = \{y_i \ ; 1\le i\le t\}$ and $S'' = \{x_i \ ; 1\le i\le n+t\}$. We have $S''=S\cup S'$, hence $L(S\cup S')=L(S)+L(S')$, by 5.4.5, and so
$$\mathrm{Vect}(x_1,\ldots,x_n, y_1,\ldots,y_t) = \mathrm{Vect}(x_1,\ldots,x_n)+\mathrm{Vect}(y_1,\ldots,y_t)$$
by 5.4.2. ■

5.4.6. Let W be a subspace of E over K. If $S\cup S'$ is a system of generators of W over K and $S'\subseteq L(S)$, then S is a system of generators of W over K.

Proof: We have $S'\subseteq L(S)$ and $S\subseteq L(S)$, hence $S\cup S'\subseteq L(S)$, and so
$$L(S\cup S')\subseteq L(S)$$
by 5.4.1(iii). As $S\subseteq S\cup S'$, then $L(S)\subseteq L(S\cup S')$, by 5.4.1(iv), and so $L(S\cup S')=L(S)$. Therefore $W=L(S)$, and so S is a system of generators of W. ■

Corollary 5.4.6.1. If S is a system of generators of a subspace W of E, such that $0_E \in S$ and $S-\{0_E\}\neq\varnothing$, then $S-\{0_E\}$ is a system of generators of W over K.

Proof: Let $S'=S-\{0_E\}$, then $S=S'\cup\{0_E\}$, and so $S'\cup\{0_E\}$ is a system of generators of W over K. But $\{0_E\}\subseteq L(S')$, hence S' is a system of generators of W over K, by 5.4.6. ■

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CHAPTER V

EXERCISES

1- Let S be a non-empty set and V be a K-vector space. Let $E=V^S$ be the set of all mappings of S to V. We define on E an addition and a scalar multiplication by:
if $f,g\in E$, then

$$f+g : S\longrightarrow V \text{ is defined by } (f+g)(x) = f(x)+g(x),\ \forall x\in S$$

and if $\alpha\in K$ and $f\in E$, then

$$\alpha f : S\longrightarrow V \text{ is defined by } (\alpha f)(x) = \alpha f(x),\ \forall x\in S.$$

Show that E is a K-vector space under these addition and scalar multiplication, such that 0_E is the zero mapping of S to V, i.e

$$0_E : S\longrightarrow V \text{ is defined by } 0_E(x) = 0_V,\ \forall x\in S.$$

2- Prove that if K' is a subfield of K, then every K-vector space E is a K'-vector space under the same addition and scalar multiplication of E.

3- Let E = $\mathbb{R}^2$. Show that E is not a vector space over $\mathbb{R}$ under the following addition and scalar multiplication :
(i) $(a,b)+(c,d) = (a , b+d)$ and $\alpha(a,b) = (\alpha a , \alpha b)$.
(ii) $(a,b)+(c,d) = (a+c , b+d)$ and $\alpha(a,b) = (\alpha a , 0)$.
(iii) $(a,b)+(c,d) = (a+c , b+d)$ and $\alpha(a,b) = (\alpha^2 a , \alpha b)$.
(iv) $(a,b)+(c,d) = (a+c+1 , b+d)$ and $\alpha(a,b) = (\alpha a , \alpha b)$.

4- Show that W is a subspace of $M_n(K)$ over K in the following cases:
(i) W= the set of matrices of $M_n(K)$ of trace zero.
(ii) W = the set of symmetric matrices of $M_n(K)$.
(iii) W = the set of antisymmetric matrices of $M_n(K)$.
(iv) W = the set of upper triangular matrices of $M_n(K)$.
(v) W = the set of lower triangular matrices of $M_n(K)$.
(vi) W = the set of diagonale matrices of $M_n(K)$.

5- Say if W is or is not a subspace of $\mathbb{R}^3$ over $\mathbb{R}$ in the following cases:
(i) $W = \{(a,b,c)\in\mathbb{R}^3 ; a-3b+3c = 0\}$.
(ii) $W = \{(a,b,c)\in\mathbb{R}^3 ; a^2-c^2=0 \}$.
(iii) $W = \{(a,b,c)\in\mathbb{R}^3 ; a+b+1= 0 \text{ and } a+3c=0\}$.
(iv) $W = \{(a,b,c)\in\mathbb{R}^3 ; a+2b-5c = 0 \text{ and } c\geq 0\}$.
(v) $W = \{(a,b,c)\in\mathbb{R}^3 ; a^2+b^2+c^2\leq 1\}$.

6- Let $I=[0,1]$ and $E=\mathbb{R}^I$. Is W a subspace of E over $\mathbb{R}$ in the following cases?
(i) $W = \{f\in E ; f(x) = f(1-x),\ \forall x\in I \}$.
(ii) $W = \{f\in E ; f \text{ is differentiable and } f(1)= f'(0)+1\}$.

(iii) $W = \{f \in E \ ; \ f$ is differentiable at 0 and $f'(0) \leq f(0)\}$.
(iv) $W = \{f \in E \ ; \ f(1)=0\}$.
(v) $W = \{f \in E \ ; \ f(x) \neq 0, \ \forall x \in I\}$.

7- Show that if F and F' are two subspaces of a K-vector space E, then $F \cup F'$ is a subspace of E over K if and only if one of them is contained into the other.

8- Let $K_n[x]$ denotes the set of polynomials f(x) of the form

$$f(x)=a_0+a_1x+\cdots+a_nx^n$$

where $a_0, a_1, \ldots, a_n \in K$. Show that $K_n[x]$ is a subspace of K[x] over K.

9- Let S(n) (resp. A(n)) be the set of symmetric (resp. anti-symmetric) matrices of $M_n(\mathbb{R})$. Show that $M_n(\mathbb{R}) = S(n) \oplus A(n)$.

10- Let E and F be two K-vector spaces and let

$$U = \{(x, 0_F) \ ; \ x \in E\} \text{ and } V = \{(0_E, x) \ ; \ x \in F\}.$$

Show that U and V are K-subspaces of $E \times F$ and that $E \times F = U \oplus V$.

11- Let $E = \mathbb{R}^{\mathbb{R}}$ and let

$$F = \{f \in E \ ; \ f \text{ is constant on } \mathbb{R}\} \text{ and } G = \{f \in E \ ; \ f(0)=0\}.$$

a) Show that F and G are two subspaces of E over $\mathbb{R}$.
b) Show that if $u \in E$, then the mapping g defined on $\mathbb{R}$ by $g(x)=u(x)-u(0)$, is an element of G, and deduce that $E = F \oplus G$.

12- Given the real matrices

$$A=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ B=\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } C=\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Can the real matrix $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ be an element of Vect(A, B, C)?

13- Find the values of the real number a, so that the vector $u=(1,-2,a)$ of $\mathbb{R}^3$ is a linear combination over $\mathbb{R}$ of the elements $x=(1,1,1)$ and $y=(1,2,3)$. Give in this case two reals α and β, such that $u=\alpha x+\beta y$.

14- Let

$$U = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) \ ; \ b-c+3d=0\right\}$$

and

$$V = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) \ ; \ a+c = 0 \text{ and } b=3d\right\}.$$

1- Show that U and V are two subspaces of $M_2(\mathbb{R})$ over $\mathbb{R}$.
2- Find a system of generators of U, V, U+V and $U \cap V$.
3- Let W=U+V. Is $W = U \oplus V$? Justify your answer.

15- Which of the following sets are system of generators of $\mathbb{R}^3$?
(i) {(1,1,1), (1,2,3)}; (ii) {(1,1,0), (1,0,1), (0,1,1)}.

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CHAPTER VI

BASIS OF VECTOR SPACE

Throughout this chapter K is a field and E be a K-vector space.

§ 6.1. LINEAR DEPENDENCE.

Definition 1. Let $x_1,...,x_n \in E$.

(i) We say that $x_1,...,x_n$ are **linearly dependent** over K if there exist $\alpha_1,...,\alpha_n$ in K, not all zero and $\alpha_1 x_1+\cdots+\alpha_n x_n = 0_E$.

(ii) We say that $x_1,...,x_n$ are **linearly independent** over K if they are not linearly dependent over K. ■

Thus; if $x_1,...,x_n \in E$, then $x_1,...,x_n$ are linearly independent over K if and only if the implication

$$\alpha_1 x_1+\cdots+\alpha_n x_n = 0_E \Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$$

is true, $\forall \alpha_1,...,\alpha_n \in K$.

Examples: 1) The elements $x_1 = (1,0,0)$, $x_2 = (0,1,0)$ and $x_3 = (0,0,1)$ of $E=K^3$ are linearly independent over K.

2) The elements $x_1 = 1$ and $x_2 = i$ of $\mathbb{C}$ are linearly independent over $\mathbb{R}$.

3) The elements $x_1 =1$ and $x_2 =i$ of $\mathbb{C}$ are linearly dependent over $\mathbb{C}$, for $\alpha_1 = 1$ and $\alpha_2 = i$ are elements of $\mathbb{C}$ that are not all zero and $\alpha_1 x_1+\alpha_2 x_2 = 0$.

Definition 2. Let X be subset of E. We say that X is **free** over K if the elements of every finite subset of X are linearly independent over K. ■

Example: $\{1, x, \ldots, x^n, \ldots.\}$ is a free subset of K[x].

Remark: If $(x_1,...,x_n)$ is a family of elements of E, then we say that this family is free over K if $x_1,...,x_n$ are pairwise distinct and the set $\{x_1,...,x_n\}$ is free over K.

6.1.1. Every element $x\in E-\{0_E\}$ is linearly independent over K.

Proof: $\forall \alpha \in K$, we have

$$\alpha x = 0_E \Rightarrow \alpha = 0_K \text{ or } x = 0_E \Rightarrow \alpha = 0_K \text{ (as } x\neq 0_E)$$

hence x is linearly independent over K. ■

6.1.2. If $n\geq 2$ and $x_1,...,x_n \in E$ are linearly independent over K, then they are pairwise distinct.

Proof: Suppose that $x_1,...,x_n$ are not pairwise distinct. $\exists i,j\in\{1,2,...,n\}$, such that $i\neq j$ and $x_i = x_j$. Let $\alpha_i = 1_K$, $\alpha_j = -1_K$ and $\alpha_t = 0_K$, $\forall t\in\{1,...,n\}-\{i,j\}$. As $\alpha_1 x_1+\cdots+\alpha_n x_n = 0_E$, then $x_1,...,x_n$ are linearly dependent over K, a contradiction, and so $x_1,...,x_n$ are pairwise distinct. ■

6.1.3. If $x_1,...,x_n \in E$ are linearly dependent over K and if $y_1,...,y_m \in E$, then the elements $x_1,...,x_n, y_1,...,y_m$ are linearly dependent over K.

Proof: We have that $x_1,...,x_n$ are linearly dependent over K, hence there exist $\alpha_1,...,\alpha_n$ in K, such that $\alpha_1,...,\alpha_n$ are not all zero and

$$\alpha_1 x_1 + \cdots + \alpha_n x_n = 0_E .$$

Let $\alpha_{n+1} = \cdots = \alpha_{n+m} = 0_K$. Then $\alpha_1,...,\alpha_n, \alpha_{n+1},...,\alpha_{n+m}$ are not all zero and

$$\alpha_1 x_1 + \cdots + \alpha_n x_n + \alpha_{n+1} y_1 + ... + \alpha_{n+m} y_m = \alpha_1 x_1 + \cdots + \alpha_n x_n = 0_E ,$$

and so $x_1,...,x_n, y_1,...,y_m$ are linearly dependent over K. ■

6.1.4. If $n \geq 2$ and $x_1,...,x_n$ are pairwise distinct elements of E, then $x_1,...,x_n$ are linearly dependent over K if and only if one of them is a linear combination over K of the others.

Proof: **N.C:** Since $x_1,...,x_n$ are linearly dependent over K, $\exists \alpha_1,...,\alpha_n \in K$, such that $\alpha_1,...,\alpha_n$ are not all zero and $\alpha_1 x_1 + \cdots + \alpha_n x_n = 0_E$. As $\alpha_t \neq 0_K$, for some t, then

$$x_t = -\alpha_t^{-1}(\alpha_1 x_1 + \cdots + \alpha_{t-1} x_{t-1} + \alpha_{t+1} x_{t+1} + \cdots + \alpha_n x_n)$$

and so x_t is a linear combination over K of the others.
S.C: We have that for some t, x_t is a linear combination over K of the others, hence $\exists \beta_1,...,\beta_{t-1}, \beta_{t+1},...,\beta_n \in K$, such that

$$x_t = \beta_1 x_1 + \cdots + \beta_{t-1} x_{t-1} + \beta_{t+1} x_{t+1} + \cdots + \beta_n x_n .$$

Let $\beta_t = -1_K$. Then $\beta_1,...,\beta_n$ are not all zero and $\beta_1 x_1 + \cdots + \beta_n x_n = 0_E$, and so $x_1,...,x_n$ are linearly dependent over K. ■

Corollary 6.1.4.1. Let $n \geq 2$ and $x_1,...,x_n$ be pairwise distinct elements of E. If $x_1,...,x_n$ are linearly dependent over K, then so are the elements $y_1,...,y_n$ with $\{y_1,...,y_n\} = \{x_1,...,x_n\}$.

Proof: As $x_1,...,x_n$ are linearly dependent, then there exists t, such that x_t is a linear combination over K of the elements of $\{x_1,...,x_n\} - \{x_t\}$. But $\{y_1,...,y_n\} = \{x_1,...,x_n\}$, hence there exists s, such that $x_t = y_s$. We get that

$$\{x_1,...,x_n\} - \{x_t\} = \{y_1,...,y_n\} - \{y_s\},$$

hence y_s is a linear combination over K of the elements of $\{y_1,...,y_n\} - \{y_s\}$, and so $y_1,...,y_n$ are linearly dependent over K, by 6.1.4. ■

6.1.5. If $x_1,...,x_n \in E$ are linearly independent over K, then so are any m elements $y_1,...,y_m$, with $\{y_1,...,y_m\} \subseteq \{x_1,...,x_n\}$.

Proof: Let

$$\{y_{m+1}, y_{m+2},...,y_n\} = \{x_1,...,x_n\} - \{y_1,...,y_m\}.$$

If $y_1,...,y_m$ are not linearly independent over K, then they are linearly dependent over K, and so $y_1,...,y_m, y_{m+1},...,y_n$ are also linearly dependent over K, by 6.1.3. But

$$\{x_1,\ldots,x_n\} = \{y_1,\ldots,y_m, y_{m+1},\ldots,y_n\}$$

hence $x_1,\ldots,x_n$ are linearly dependent over K, by 6.1.4.1, which is impossible. Therefore $y_1,\ldots,y_m$ are linearly independent over K. ■

Corollary 6.1.5.1. If $B=\{x_1,\ldots,x_n\}$ is a finite subset of E, then B is free over K if and only if $x_1,\ldots, x_n$ are linearly independent over K.

Proof: This is an immediate consequence of 6.1.5. ■

6.1.6. If $B=\{x_1,\ldots,x_n\}$ is a free subset of E and if $x\in E\text{-}L(B)$, then $B\cup\{x\}$ is free over K.

Proof: Let $a_1,\ldots,a_n,\alpha\in K$, such that

$$a_1x_1+\cdots+a_nx_n+\alpha x = 0_E.$$

If $\alpha\neq 0$, we get $x = -\alpha^{-1}(a_1x_1+\cdots+a_nx_n)$, and so $x\in L(B)$, a contradiction. hence

$$\alpha=0.$$

This implies that $a_1x_1+\cdots+a_nx_n = 0_E$, and so as $x_1,\ldots,x_n$ are linearly independent over K, then $a_1=\cdots=a_n=0$, whence $a_1=\cdots=a_n=\alpha=0$, and consequently $x_1,\ldots,x_n$, x are linearly independent over K. Therefore $B\cup\{x\}$ is free over K, by 6.1.5.1. ■

6.1.7. If A is a $(n\times n)$ matrix over K and $|A|\neq 0$, then the rows (resp. columns) of A are linearly independent over K.

Proof: For each $1\leq i\leq n$, let L_i denote the ith row of A. Suppose that the rows of A are linearly dependent, then one of them, for example L_t, is a linear combination of the others, hence $\exists\alpha_1,\ldots,\alpha_n\in K$, such that $\alpha_t=1$ and $\alpha_1L_1+\ldots+\alpha_nL_n=0$. Replacing the row L_t by $\alpha_1L_1+\ldots+\alpha_nL_n$, then $|A|$ is equal to the obtained determinant, by 3.1.8.1. But the tth row of the obtained determinant is zero, hence this determinant is zero, and so $|A|=0$, which is impossible. Therefore the rows of A are linearly independent over K.
Applying the same proof, with the word "row" replaced by the word "column", then we get that the columns of A are linearly independent over K. ■

Corollary 6.1.7.1. Let A be $(m\times n)$ matrix over K. Denote by C_i (resp. L_i) the ith column (resp. row) of A. If

$$\Delta=\begin{vmatrix} a_{i_1j_1} & a_{i_1j_2} & \cdots & a_{i_1j_r} \\ a_{i_2j_1} & a_{i_2j_2} & \cdots & a_{i_2j_r} \\ \vdots & \vdots & & \vdots \\ a_{i_rj_1} & a_{i_rj_2} & \cdots & a_{i_rj_r} \end{vmatrix}$$

is a non-zero minor of A of order r, then
(i) the columns $C_{j_1},\ldots,C_{j_r}$ of A are linearly independent over K and
(ii) the rows $L_{i_1},\ldots,L_{i_r}$ of A are linearly independent over K.

Proof: Suppose that $C_{j_1},\ldots,C_{j_r}$ are not linearly independent, then they are linearly dependent, and so $\exists\alpha_1,\ldots,\alpha_r\in K$, not all zero, such that

$$\alpha_1 C_{j_1} + ... + \alpha_r C_{j_r} = 0.$$

For each $1 \le i \le r$, let T_i denote the ith column of Δ, then

$$\alpha_1 T_1 + ... + \alpha_r T_r = 0$$

and so the columns of Δ are linearly dependent over K, but $\Delta \ne 0$, hence the columns of Δ are linearly independent over K, by 6.1.7, a contradiction. Therefore

$$C_{j_1}, ..., C_{j_r} \text{ are linearly independent over K.}$$

Similarly, we show that $L_{i_1}, ..., L_{i_r}$ are linearly independent over K. ■

<u>§ 6.2. BASIS OF A VECTOR SPACE.</u>

<u>Definition 3.</u> A subset B of E is said to be a **basis** of E over K if B is a system of generators of E and B is free over K. ■

<u>Examples :</u> 1) {1} is a basis of K over K, for 1 is linearly independent over K and {1} is a system of generators of K over K, since $\forall x \in K$, we have x = x1.

2) {1,i} is a basis of $\mathbb{C}$ over $\mathbb{R}$.

3) Let $E = K^n$. For each $1 \le i \le n$, let

$$e_i = (0,\dots,0,1,0,\dots,0), \text{ with 1 is the ith entry.}$$

Then $\{e_1, e_2, ..., e_n\}$ is a basis of E over K, called the **canonical basis** of K^n over K. Thus

$$\{(1,0),(0,1)\} \text{ is the canonical basis of } K^2 \text{ over K and}$$

$$\{(1,0,0),(0,1,0),(0,0,1)\} \text{ is the canonical basis of } K^3 \text{ over K.}$$

4) Let $E = M_{m,n}(K)$. For every natural numbers s and t, such that $1 \le s \le m$ and $1 \le t \le n$, let

$$E_{st} = \begin{array}{cl} \begin{array}{ccccccc} & & & \text{t-th column} & & & \end{array} & \\ \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} & \begin{array}{l} \\ \\ \\ \text{s-th row} \\ \\ \\ \\ \end{array} \end{array}.$$

We have

$$B = \{E_{ij} \ ; \ 1 \le i \le m \text{ and } 1 \le j \le n\}$$

is a basis of E over K, called the **canonical basis** of E over K. Thus

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \text{ is the canonical basis of } M_2(K) \text{ and}$$

$$\left\{ \begin{pmatrix} 1\,0\,0 \\ 0\,0\,0 \end{pmatrix}, \begin{pmatrix} 0\,1\,0 \\ 0\,0\,0 \end{pmatrix}, \begin{pmatrix} 0\,0\,1 \\ 0\,0\,0 \end{pmatrix}, \begin{pmatrix} 0\,0\,0 \\ 1\,0\,0 \end{pmatrix}, \begin{pmatrix} 0\,0\,0 \\ 0\,1\,0 \end{pmatrix}, \begin{pmatrix} 0\,0\,0 \\ 0\,0\,1 \end{pmatrix} \right\} \text{is the canonical basis of } M_{2,3}(K).$$

5) $\{1, x, \dots, x^n, \dots.\}$ is a basis of K[x].

6) $\{1, x, \dots, x^n\}$ is a basis of $K_n[x]$, called the **canonical basis** of $K_n[x]$.

<u>6.2.1.</u> If $x_1, ..., x_n \in E$, then $\{x_1, ..., x_n\}$ is a basis of E over K if and only if

(i) $x_1, ..., x_n$ are linearly independent over K and

(ii) $\{x_1,...,x_n\}$ is a system of generators of E over K.

Proof: By 6.1.5.1. ■

Remark: We say that the family $(x_1,...,x_n)$ of E is a basis of E over K if the set $\{x_1,...,x_n\}$ is a basis of E over K.

6.2.2. If $\{x_1,...,x_n\}$ is a basis of E over K, then every element u of E is uniquely written in the form $u = a_1x_1+\cdots+a_nx_n$, where $a_1,...,a_n \in K$.

Proof: Let $u\in E$. Since $\{x_1,...,x_n\}$ is a system of generators of E over K, $\exists a_1,...,a_n \in K$, such that

$$u = a_1x_1+\cdots+a_nx_n .$$

Suppose that $u = b_1x_1+\cdots+b_nx_n$, where $b_1,...,b_n \in K$. Then

$$a_1x_1+\cdots+a_nx_n = b_1x_1+\cdots+b_nx_n$$

and so $(a_1-b_1)x_1+\cdots+(a_n-b_n)x_n=0_E$. But $x_1,...,x_n$ are linearly independent over K, hence $a_1-b_1=\cdots= a_n-b_n=0_K$, and so $a_1= b_1,..., a_n = b_n$. ■

If $\{x_1,...,x_n\}$ is a basis of E over K and if u is an element of E, then the elements $a_1,...,a_n$ of K, such that $u = a_1x_1+\cdots+a_nx_n$ are called the **components** (or the **coordinates**) of u relative to the basis $\{x_1,..,x_n\}$ of E.

Definition 4. We say that E is **finite dimensional** over K if E has a finite system of generators over K. ■

Examples: K^n and $M_{m,n}(K)$ are finite dimensional over K and $\mathbb{C}$ is finite dimensional over $\mathbb{R}$.

In the rest of this section, E is a finite dimensional vector space.

6.2.3. Let W be a subspace of E and let S be a system of generators of W over K. Suppose that there exists a non-zero natural number m, such that

$$\text{card}(X)\leq m, \text{ for every free subset X of S.}$$

If $A\subseteq S$ and A is free over K, then there exists a finite basis B of W, such that $A\subseteq B\subseteq S$.

Proof: The proof is given in Appendix I. ■

Corollary 6.2.3.1. If $E\neq\{0_E\}$, then E has a finite basis.

Proof: Let S be a finite system of generators of E over K. Let card(S)=m, then

$$\text{card}(X)\leq m, \text{ for every free subset X of S.}$$

We have $S\not\subset\{0_E\}$, for otherwise $E=L(S)\subseteq\{0_E\}$, and so as $\{0_E\}\subseteq E$, we get $E=\{0_E\}$, impossible. Therefore $\exists x\in S-\{0_E\}$. As $x\neq 0_E$, then $\{x\}$ is free over K, by 6.1.1 and 6.1.5.1. But $\{x\}\subseteq S$ and S is a system of generators of E over K, hence there exists a basis B of E, such that $\{x\}\subseteq B\subseteq S$, by 6.2.3. As $B\subseteq S$ and S is finite, then B is a finite basis of E. ■

§ 6.3. ROW SPACE AND BASIS OF A VECTOR SPACE.

Definition 5. Let A be (m×n) matrix over K. We call **row** (resp. **column**) **space** of A, the subspace of $M_{1,n}(K)$ (resp. $M_{m,1}(K)$) generated by the rows (resp. columns) of A. ■

6.3.1. Let $A=(a_{ij})$ be (m×n) matrix over K. Denote by C_i (resp. L_i) the ith column (resp. row) of A. Let W (resp W') be the column (resp. row) space of A. If rank(A)=r and

$$\Delta=\begin{vmatrix} a_{i_1j_1} & a_{i_1j_2} & \cdots & a_{i_1j_r} \\ a_{i_2j_1} & a_{i_2j_2} & \cdots & a_{i_2j_r} \\ \vdots & \vdots & & \vdots \\ a_{i_rj_1} & a_{i_rj_2} & \cdots & a_{i_rj_r} \end{vmatrix}$$

is a non-zero minor of A of order r, then
(i) $\{C_{j_1},...,C_{j_r}\}$ is a basis of W over K and
(ii) $\{L_{i_1},...,L_{i_r}\}$ is a basis of W' over K.

Proof: The proof is given in Appendix I. ■

6.3.2. If $\{x_1,...,x_n\}$ is a basis of E over K and if m is a natural number with m>n, then any m elements $y_1, \ldots, y_m$ of E are linearly dependent over K.

Proof: The proof is given in Appendix I. ■

Corollary 6.3.2.1. If $\{x_1,...,x_n\}$ is a basis of E over K, then every free subset X of E is finite and card(X)≤n.

Proof: If X contains more than n elements, then it contains at least n+1 elements $y_1,...,y_{n+1}$. Since X is free, $y_1,...,y_{n+1}$ are linearly independent, but $y_1,...,y_{n+1}$ are linearly dependent, by 6.3.2, hence we have a contradiction, and so X does not contain more than n elements, whence X is finite and card(X)≤n. ■

Corollary 6.3.2.2. If $\{x_1,...,x_n\}$ is a basis of E over K, and if B is a basis of E, then B is finite and card(B)=n. In particular any two bases of E have the same cardinal.

Proof: As B is a free subset of E, then B is finite and card(B)≤n, by 6.3.2.1. Let m=card(B). If n≠m, then n>m, and so as B is a basis of E, then any n elements of E are linearly dependent, by 6.3.2, whence $x_1,...,x_n$ are linearly dependent over K, impossible. Therefore n=m, and so card(B)=n. ■

Corollary 6.3.2.3. If $E\neq\{0_E\}$, then the following hold
(i) Every system of generators S of E over K contains a basis of E.
(ii) Every free subset $A=\{x_1,...,x_r\}$ of E is contained in some basis of E.

Proof: (i) We have $S\not\subset\{0_E\}$, for otherwise $E=L(S)\subseteq\{0_E\}$, and so as $\{0_E\}\subseteq E$, we get $E=\{0_E\}$, impossible. Therefore $\exists x\in S-\{0_E\}$. As $x\neq 0_E$, then x is linearly independent over K, by 6.1.1, and so there exists a basis B of E, such that $\{x\}\subseteq B\subseteq S$, by 6.2.3.
(ii) We have $E\neq\{0_E\}$, hence E has a finite basis B, by 6.2.3.1. Put card(B)=n, then

$$\text{card}(X)\leq n, \text{ for every free subset } X \text{ of } E$$

by 6.3.2.1, and so there exists a basis B' of E over K, such that $A\subseteq B'$, by 6.2.3. ■

Definition 6. We define **the dimension** of E, denoted $\dim_K(E)$ to be the cardinal of a basis of E over K. ■

Examples: $\dim_{\mathbb{R}}(\mathbb{C})=2$, $\dim_K(M_{m,n}(K))=mn$, $\dim_K(K^n)=n$ and $\dim_K(K_n[x])=n+1$.

If $E=\{0_E\}$, we define $\dim_K(E) = 0$.

6.3.3. $\dim_K(E) = 0 \Leftrightarrow E = \{0_E\}$.

Proof: $\Rightarrow$): If $E\neq\{0_E\}$, then $\exists x\in E-\{0_E\}$, and so as x is linearly independent over K, then there exists a basis B of E over K, such that $x\in B$, by 6.3.2.3(ii). Since $\text{card}(B)=\dim_K(E)$, $\text{card}(B)=0$, and so $B=\emptyset$, impossible, for $x\in B$. Therefore $E=\{0_E\}$.
$\Leftarrow$): By definition, we have $\dim_K(\{0_E\})=0$, hence $\dim_K(E)=0$. ■

Remark: If $E=\{0_E\}$, we say that $B=\emptyset$ is a basis of E over K and we write

$$L(\emptyset)=\{0_E\}.$$

Definition 7. A subspace F of E is said to be a **vector straight line** (resp. **vector plane**, **vector hyperplane**) if the dimension of F is 1 (resp. 2, n-1), where $n=\dim_K(E)$. ■

6.3.4. If $\dim_K(E) = n$, where $n\in\mathbb{N}^*$, then
(i) if $x_1,...,x_n\in E$ are linearly independent over K, then $\{x_1,...,x_n\}$ is a basis of E over K,
(ii) if the elements $x_1,...,x_n$ form a system of generators of E over K, then they are pairwise distinct and $\{x_1,...,x_n\}$ is a basis of E over K.

Proof: (i) Since $x_1,...,x_n$ are linearly independent over K, there exists a basis B of E over K, such that $\{x_1,...,x_n\}\subseteq B$, by 6.3.2.3(ii). As

$$\text{card}(\{x_1,...,x_n\}) = n = \dim_K(E) = \text{card}(B)$$

then $\{x_1,...,x_n\}=B$, and so $\{x_1,...,x_n\}$ is a basis of E over K.
(ii) As $\dim_K(E)\neq 0$, then $E\neq\{0_E\}$, by 6.3.3. Since the set $S=\{x_i\,;\,1\leq i\leq n\}$ is a system of generators of E over K, there exists a basis B of E over K, such that $B\subseteq S$, by 6.3.2.3(i). We have $\text{card}(B)\leq\text{card}(S)$ and $\text{card}(B)=\dim_K(E)=n$, hence $n\leq\text{card}(S)$. But $\text{card}(S)\leq n$, and so $\text{card}(S) = n = \text{card}(B)$. As S is finite and $B\subseteq S$, then

$$S = B.$$

On the other hand, we have $S=\{x_i\,;\,1\leq i\leq n\}$ and $\text{card}(S) = n$, hence $x_1,...,x_n$ are pairwise distinct, and so $B=S=\{x_1,...,x_n\}$. Thus $\{x_1,...,x_n\}$ is a basis of E over K. ■

6.3.5. If W is a subspace of E over K, then
(i) W is finite dimensional over K and $\dim_K(W)\leq\dim_K(E)$.
(ii) If W contains a basis of E over K, then W=E.
(iii) If $\dim_K(W)=\dim_K(E)$, then W=E.

Proof: (i) If $W=\{0_E\}$, then W is finite dimensional over K and $\dim_K(W)=0\leq\dim_K(E)$. Assume that $W\neq\{0_E\}$. Then $\exists x\in W-\{0_E\}$. As $\{x\}$ is free over K and W is a system of generators of W over K, then there exists a finite basis B of W, such that $\{x\}\subset B$, by 6.2.3, hence W is finite dimensional over K and $\dim_K(W)=\mathrm{card}(B)\leq\dim_K(E)$, by 6.3.2.1.

(ii) Let B be basis of E, such that $B\subseteq W$. Since W is a subspace of E over K, we then have $L(B)\subseteq W$. But $E=L(B)$, hence $E\subseteq W$, and so $W=E$.

(iii) If $\dim_K(W)=0$, then $\dim_K(E)=0$, hence $W=E=\{0_E\}$, by 6.3.3. Suppose that $W\neq\{0_E\}$. Then W has a finite basis B over K, by 6.2.3.1. Set $B=\{x_1,...,x_s\}$, then

$$\dim_K(E)=\dim_K(W)=\mathrm{card}(B)=s.$$

As $x_1,...,x_s$ are linearly independent over K, then $\{x_1,...,x_s\}$ is a basis of E over K, by 6.3.4(i). Now W contains the basis $\{x_1,...,x_s\}$ of E, hence W=E, by (ii). ■

§ 6.4. DIMENSION OF A VECTOR SPACE AND RANK.

6.4.1. Let A be a $(m\times n)$ matrix over K. If W (resp. W') denotes the row (resp. column) space of A, then $\dim_K(W) = \dim_K(W') = \mathrm{rank}(A)$.

Proof: By 6.3.1. ■

6.4.2. Let A be a $(n\times m)$ matrix over K. If $E_1, \ldots, E_r \in M_n(K)$ are elementary matrices, then the matrices A and $(E_1\times\cdots\times E_r)\times A$ have the same row space. In particular if B is a row echelon form of A, then A and B have the same row space.

Proof: The proof is given in Appendix I. ■

Corollary 6.4.2.1. If B is a row echelon form of a $(m\times n)$ matrix A over K, then
(i) the non-zero rows of B form a basis of the row space of A,
(ii) rank(A) = rank(B) = the number of non-zero rows of B.

Proof: Let W be the row space of A, then

W = the row space of B

by 6.4.2, and so $\mathrm{rank}(B) = \dim_K(W)$. But $\mathrm{rank}(A) = \dim_K(W)$, by 6.4.1, hence

$$\mathrm{rank}(A) = \mathrm{rank}(B) = \dim_K(W). \qquad (*)$$

Let s be the number of non-zero rows of B and put $B=(\alpha_{ij})$. As B is a matrix in row echelon form, then the first s rows of B are non-zero and all the others are zero. For each $1\leq t\leq s$, let L_t denote the tth row of B and let α_{tj_t} be the leading entry of L_t. Put

$$\Delta=\begin{vmatrix} \alpha_{1j_1} & \alpha_{1j_2} & \cdots\cdots & \alpha_{1j_s} \\ 0 & \alpha_{2j_2} & \cdots\cdots & \alpha_{2j_s} \\ \vdots & 0 & & \\ \vdots & \vdots & & \\ 0 & 0 & \cdots\ 0 & \alpha_{sj_s} \end{vmatrix}.$$

As Δ is upper triangular, then $\Delta=\alpha_{1j_1}\times\alpha_{2j_2}\times\ldots\times\alpha_{sj_s}$, and so $\Delta\neq0$, hence

the rows $L_1, \ldots, L_s$ of B are linearly independent over K

by 6.1.7.1. We have $L_t=0$, for all $s+1\leq t\leq m$, hence

$$W=\mathrm{Vect}(L_1, \ldots, L_s, L_{s+1}, \ldots, L_m) = \mathrm{Vect}(L_1, \ldots, L_s)$$

by 5.4.6.1, and so $L_1, \ldots, L_s$ form a system of generators of W, whence $\{L_1, \ldots, L_s\}$ is a basis of W over K, and consequently

the non-zero rows of B form a basis of the row space of A.

Therefore (i) holds.

As $\dim_K(W)=s$ and s is the number of non-zero rows of B, then (*) yields

rank(A) = rank(B) = the number of non-zero rows of B

and so (ii) holds. ■

Definition 8. Let $y_1, \ldots, y_m \in E$. We define the **rank** of $y_1, \ldots, y_m$, denoted $\mathrm{rank}(y_1,\ldots, y_m)$ to be the dimension over K of $\mathrm{Vect}(y_1, \ldots, y_m)$. ■

6.4.3. Let $\{x_1,...,x_n\}$ be basis of E over K and let $y_1, \ldots, y_m$ be elements of E. Put

$$\begin{aligned} y_1 &= a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n \\ y_2 &= a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n \\ &\vdots \\ y_m &= a_{m1}x_1+a_{m2}x_2+\cdots+a_{mn}x_n \end{aligned}$$

where $a_{ij} \in K$, $\forall i,j$ and set

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

Let r=rank(A) and suppose $r\neq 0$, then we have

(i) If $B=(\alpha_{ij})$ is a row echelon form of A and if for each $1\leq i\leq r$,

$$z_i = \alpha_{i1}x_1+\alpha_{i2}x_2+\cdots+\alpha_{in}x_n$$

then $\{z_1,...,z_r\}$ is a basis of $\mathrm{Vect}(y_1, \ldots, y_m)$.

(ii) $\mathrm{rank}(y_1,\ldots, y_m)=\mathrm{rank}(A)$.

(iii) $y_1,\ldots, y_m$ form a system of generators of E over K if and only if rank(A)=n.

(iv) If m=n, then $y_1,\ldots, y_n$ form a basis of E if and only if rank(A)=n.

Proof: (i) For each $\leq i\leq r$, let L_i be the ith row of B, then

$$L_i = (\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in}).$$

As $\{L_1, ..., L_r\}$ is a basis of the row space of A, by 6.4.2.1(i), then one can easily show that $\{z_1,...,z_r\}$ is a basis of $\mathrm{Vect}(y_1, \ldots, y_m)$.

(ii) By (i)

$$\mathrm{rank}(y_1,\ldots, y_m) = \dim_K(\mathrm{Vect}(y_1,\ldots, y_m)) = \mathrm{card}(\{z_1,...,z_r\}) = r = \mathrm{rank}(A).$$

(iii) **N.C:** We have $E=\mathrm{Vect}(y_1,\ldots, y_m)$, so $\mathrm{rank}(A)=\dim_K(E) = n$, by (ii).

S.C: We have

$$\dim_K(\mathrm{Vect}(y_1,\ldots, y_m)) = \mathrm{rank}(y_1,\ldots, y_m) = \mathrm{rank}(A) = n = \dim_K(E)$$

by (ii), hence $E=\mathrm{Vect}(y_1,\ldots, y_m)$, by 6.3.5(iii), and so $y_1,\ldots, y_m$ form a system of generators of E over K.

(iv) **N.C:** We have $E=\mathrm{Vect}(y_1,\ldots, y_n)$, so $\mathrm{rank}(A)=\dim_K(E) = n$, by (ii).

S.C: We have

$$\dim_K(\mathrm{Vect}(y_1,\ldots,y_n)) = \mathrm{rank}(y_1,\ldots,y_n) = \mathrm{rank}(A) = n = \dim_K(E)$$

by (ii), hence E=Vect($y_1,\ldots,y_n$), by 6.3.5(iii), and so $y_1,\ldots,y_n$ form a system of generators of E over K, whence $y_1,\ldots,y_n$ form a basis of E over K, by 6.3.4(ii). ■

Application: Let $\{x_1, x_2, x_3, x_4\}$ be basis of a real vector space E. Let

$$y_1 = x_1+x_2+x_4,\ y_2 = 2x_1+3x_2+x_3+2x_4 \text{ and } y_3 = x_1+2x_2+x_3+x_4$$

Find a basis of W=Vect(y_1, y_2, y_3) and complete it to a basis of E.

Answer: Consider the matrix

$$A=\begin{pmatrix}1&1&0&1\\2&3&1&2\\1&2&1&1\end{pmatrix}.$$

We have

$$\begin{pmatrix}1&1&0&1\\2&3&1&2\\1&2&1&1\end{pmatrix}\ \begin{matrix}R_2\to R_2-2R_1\\R_3\to R_3-R_1\end{matrix}\to\begin{pmatrix}1&1&0&1\\0&1&1&0\\0&1&1&0\end{pmatrix}\ R_3\to R_3-R_2\ \to\begin{pmatrix}1&1&0&1\\0&1&1&0\\0&0&0&0\end{pmatrix},$$

which is a row echelon form of A, hence

$$\{x_1+x_2+x_4,\ x_2+x_3\} \text{ is a basis of W over } \mathbb{R}$$

by 6.4.3(i). Consider the matrix

$$B=\begin{pmatrix}1&1&0&1\\0&1&1&0\\0&0&1&0\\0&0&0&1\end{pmatrix}.$$

We have B is a matrix in row echelon form, hence rank(B)=4=$\dim_{\mathbb{R}}$(E), and so

$$\{x_1+x_2+x_4,\ x_2+x_3,\ x_3,\ x_4\}$$

is a basis of E over $\mathbb{R}$, completing the obtained basis of W, by 6.4.3(iv).

Corollary 6.4.3.1. If $\{x_1,\ldots,x_n\}$, $y_1,\ldots,y_m$ and A are as in 6.4.3, then

(i) $y_1,\ldots,y_m$ are linearly independent over K if and only if rank(A)=m.

(ii) $y_1,\ldots,y_m$ are linearly dependent over K if and only if rank(A)≠m.

Proof: (i) **N.C:** As $y_1,\ldots,y_m$ are linearly independent over K, then $\{y_1,\ldots,y_m\}$ is a basis of W=Vect($y_1,\ldots,y_m$) over K, and so $\dim_K$(W)=m, whence rank(A)=m, by 6.4.3(ii).

S.C: As $y_1,\ldots,y_m$ form a system of generators of W=Vect($y_1,\ldots,y_m$) over K and $\dim_K$(W)=rank($y_1,\ldots,y_m$)=m, then $\{y_1,\ldots,y_m\}$ is a basis of W over K, by 6.3.4(ii), and so $y_1,\ldots,y_m$ are linearly independent over K.

(ii) We have that $y_1,\ldots,y_m$ are linearly dependent over K if and only if they are not linearly independent over K, hence they are linearly dependent over K if and only if rank(A)≠m, by (i). ■

§ 6.5. BASIS AND DIRECT SUM.

6.5.1. Let U and V be subspaces of E. If $\{x_1,\ldots,x_r\}$ is a basis of U and $\{y_1,\ldots,y_s\}$ is a basis of V and U∩V=$\{0_E\}$, then $\{x_1,\ldots,x_r, y_1,\ldots,y_s\}$ is a basis of U+V and

$$\dim_K(U+V)=\dim_K(U)+\dim_K(V).$$

Proof: By 5.4.5, we have that

$$\{x_1,\ldots,x_r\}\cup\{y_1,\ldots,y_s\} \text{ is a system of generators of } U+V.$$

Let $a_1,\ldots,a_r,b_1,\ldots,b_s \in K$, such that

$$a_1x_1+\cdots+a_rx_r+b_1y_1+\cdots+b_sy_s=0_E.$$

We have

$$a_1x_1+\cdots+a_rx_r = -(b_1y_1+\cdots+b_sy_s)$$

so that as $a_1x_1+\cdots+a_rx_r \in U$ and $b_1y_1+\cdots+b_sy_s \in V$, then

$$a_1x_1+\cdots+a_rx_r \in U\cap V \text{ and } b_1y_1+\cdots+b_sy_s \in U\cap V$$

and so $a_1x_1+\cdots+a_rx_r=0_E$ and $b_1y_1+\cdots+b_sy_s=0_E$. But $x_1,\ldots,x_r$ (resp. $y_1,\ldots,y_s$) are linearly independent over K, hence

$$a_1=\cdots=a_r=0 \text{ and } b_1=b_2=\cdots=b_s=0.$$

Therefore $x_1,\ldots,x_r,y_1,\ldots,y_s$ are linearly independent over K, and so they are pairwise distinct, by 6.1.2, whence

$$\{x_1,\ldots,x_r, y_1,\ldots,y_s\} \text{ is a basis of } U+V \text{ over } K.$$

We have $\dim_K(U+V) = r+s = \dim_K(U)+\dim_K(V)$. ■

Corollary 6.5.1.1. Every subspace U of E has a supplement.

Proof: If $U=\{0_E\}$, then E is a supplement of U, since $E=\{0_E\}\oplus E$. Suppose that $U\neq\{0_E\}$. As U is finite dimensional over K, by 6.3.5(i), then U has a basis $\{x_1,\ldots,x_r\}$, by 6.2.3.1. We have that $x_1,\ldots,x_r$ are linearly independent over K, hence there exists a basis B of E containing $\{x_1,\ldots,x_r\}$, by 6.3.4(i). Put

$$B-\{x_1,\ldots,x_r\}=\{x_{r+1},\ldots,x_n\}$$

and let $V=\mathrm{Vect}(x_{r+1},\ldots,x_n)$. As $x_{r+1},\ldots,x_n$ are linearly independent over K, then

$$\{x_{r+1},\ldots,x_n\} \text{ is a basis of } V.$$

Let $x\in U\cap V$, then $x\in U$ and $x\in V$, and so $\exists a_1, \ldots, a_r, a_{r+1},\ldots, a_n \in K$, such that

$$x=a_1x_1+\cdots+a_rx_r \text{ and } x=a_{r+1}x_{r+1}+\cdots+a_nx_n.$$

This implies that $a_1x_1+\cdots+a_rx_r+(-a_{r+1})x_{r+1}+\cdots+(-a_n)x_n=0_E$, hence

$$a_1=\cdots= a_r = -a_{r+1}=\cdots= -a_n=0$$

and so $x=0_E$. Therefore

$$U\cap V=\{0_E\}.$$

On the other hand

$$U+V=\mathrm{Vect}(x_1,\ldots,x_r)+\mathrm{Vect}(x_{r+1},\ldots,x_n)=\mathrm{Vect}(x_1,\ldots,x_r,x_{r+1},\ldots,x_n)=E$$

hence $E=U\oplus V$, and so V is a supplement of U. ■

6.5.2. If U and V are two subspaces of E, then

$$\dim_K(U+V) = \dim_K(U)+\dim_K(V)-\dim_K(U\cap V).$$

Proof: Let W be a supplement of $U\cap V$ in U. We shall show that

$$U+V=W\oplus V.$$

We have $W\subseteq U$, hence $W= W\cap U$, and so

$$W\cap V = (W\cap U)\cap V = W\cap(U\cap V) = \{0_E\}.$$

Also as $W\subseteq U$, then

$$W+V \subseteq U+V.$$

Let $x \in U+V$, then $x=u+v$, for some $u \in U$ and $v \in V$. As $U=W+(U \cap V)$, then

$$u=y+z, \text{ for some } y \in W \text{ and } z \in U \cap V.$$

As $x=y+z+v$ and $u \in W$ and $z+v \in V$, then $x \in W+V$, and so $U+V \subseteq W+V$, whence $W+V=U+V$, and consequently $U+V=W \oplus V$. It follows that

$$\dim_K(U+V) = \dim_K(W+V) = \dim_K(W)+\dim_K(V).$$

But $\dim_K(U) = \dim_K(W)+\dim_K(U \cap V)$, hence $\dim_K(W)= \dim_K(U)-\dim_K(U \cap V)$, and so $\dim_K(U+V) = \dim_K(U)+\dim_K(V)-\dim_K(U \cap V)$. ■

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CHAPTER VI

EXERCISES

1- Say if the following elements of the real space E are linearly independent or linearly dependent over $\mathbb{R}$ in the following cases:

(i) u = (2,1,4), v = (1,-3,2), w=(3,-2,6) and z=(3,3,3) and E = $\mathbb{R}^3$.

(ii) u = (2,1,4) and v = (3,3,3) and E = $\mathbb{R}^3$.

(iii) u = $\begin{pmatrix} 2 & -2 & 0 \\ 3 & 5 & 1 \end{pmatrix}$, v = $\begin{pmatrix} 4 & 2 & 1 \\ -1 & 1 & 1 \end{pmatrix}$ and w = $\begin{pmatrix} 8 & -2 & 1 \\ 5 & 11 & 3 \end{pmatrix}$ and E=$M_{2,3}(\mathbb{R})$.

2- Precise if the family (x_1, x_2, x_3, x_4) is free over $\mathbb{R}$ or not, where

x_1 = (1, -1, 3), x_2 = (2, 1, 4), x_3 = (1, 2, 1) and x_4 = (3, 1, 2).

3- In the following cases find a basis of the subspace W=Vect(x_1, x_2, x_3) and complete it to a basis of E:

(i) x_1 = (2,1,4), x_2 = (1,-3,2) and x_3 = (3,-2,6) and E=$\mathbb{R}^3$.

(ii) $x_1 = \begin{pmatrix} 5 & 1 \\ 1 & -1 \end{pmatrix}$, $x_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $x_3 = \begin{pmatrix} 1 & 1 \\ 1 & 6 \end{pmatrix}$ and E=$M_2(\mathbb{R})$.

4- Let u=(0,1,1), v=(1,0,1) and w=(1,1,0) be three elements of $\mathbb{R}^3$.

1- Show that {u, v, w} is a basis of $\mathbb{R}^3$ over $\mathbb{R}$.
2- Find the components of the element x = (1,1,1) relative to this basis.

5- (i) Find a subset of $\mathbb{R}^3$ which is free over $\mathbb{R}$ but is not a system of generators of $\mathbb{R}^3$.

(ii) Find a system of generators of $\mathbb{R}^3$ which is not free over $\mathbb{R}$.

6- Find the values of the real number m, under which the subset

S={(1, 0, 0), (0, m, -1), (0, 1, m)}

of $\mathbb{R}^3$ is free over $\mathbb{R}$.

7- Find following the values of the real number α the rank of S=(u,v,w) , where

u=(α,1,1), v=(1,α,1) and w=(1,1,α).

8- Let A = $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, B = $\begin{pmatrix} 3 & -1 \\ 2 & 2 \end{pmatrix}$ and C = $\begin{pmatrix} 1 & -5 \\ -4 & 0 \end{pmatrix}$.

(a) Show that A, B and C are linearly independent over $\mathbb{R}$.
(b) Find a basis of the subspace of $M_2(\mathbb{R})$ generated by A, B and C.

9- Let U = {(a,b,c)$\in\mathbb{R}^3$; 3a+2b+c=0} and V={(a,b,c)$\in\mathbb{R}^3$; a+5b-7c=0}.

1- Show that U and V are subspaces of $\mathbb{R}^3$ and find bases and dimensions of U and V.

2- Determine U$\cap$V and deduce the dimension of U+V. Deduce that U+V=$\mathbb{R}^3$.

10- Let U = {(a, 3b, a+2b) ; a,b$\in\mathbb{R}$} and V={(c, c+2d, c-d) ; c,d$\in\mathbb{R}$}.

1- Show that U and V are subspaces of $\mathbb{R}^3$ and give a basis of U and a basis of V.

2- Find the dimension of U+V and deduce the dimension and a basis of U$\cap$V.

11- Let U =Vect(u_1,u_2,u_3) and V=Vect(v_1,v_2), where

u_1 = (1,-1,0,2), u_2 = (2,1,3,1), u_3 = (4,5,9,-1), v_1 = (3,2,2,-2) and v_2 = (3,-4,4,2).

Compute a bases of U, V and U+V and deduce the dimension of U$\cap$V.

12- Let U = {f(x)=$ax^3+bx^2+cx+d\in\mathbb{R}_3[x]$; a=c and d=-2b } and

V = {f(x)=$ax^3+bx^2+cx+d\in\mathbb{R}_3[x]$; c=d}.

1- Show that U and V are two subspaces of $\mathbb{R}_3[x]$ over $\mathbb{R}$.

2- Find a bases of U, V, U+V and U$\cap$V and find their dimensions. Is $\mathbb{R}_3[x]$=U+V?

13- Let $U=\left\{\begin{pmatrix} a & a \\ c & 0 \end{pmatrix} ; a,c\in\mathbb{R}\right\}$ and $V=\left\{\begin{pmatrix} a & -a \\ a & d \end{pmatrix} ; a,d\in\mathbb{R}\right\}$.

1- Show that U and V are two subspaces of $M_2(\mathbb{R})$ over $\mathbb{R}$.

2- Compute the dimensions of U, V, U+V and U$\cap$V. Is U+V a direct sum?

14- Find a basis of $\mathbb{C}^2$ over $\mathbb{R}$ and calculate $\dim_{\mathbb{R}}(\mathbb{C}^2)$.

15- Let E=$\mathbb{R}_4[x]$, $f_1=x+x^2$, $f_2=1+x+x^2+x^4$ and $f_3=2-x+2x^2-3x^3+2x^4$.

1- Find a basis B of F=Vect(f_1, f_2, f_3) over $\mathbb{R}$.

2- Complete B to a basis of E and deduce a supplement of F in E.

16- Let E be a finite dimensional K-vector space of dimension n$\geq$1 and let F be a subspace of E of dimension n-1. Show that if x$\in$E-F, then E=F$\oplus$Vect(x).

17- Show that W={(x,y,z)$\in\mathbb{C}^3$; x-iy+2iz=0} is a subspace of $\mathbb{C}^3$ over $\mathbb{C}$ and find a basis of it over $\mathbb{C}$ and a basis over $\mathbb{R}$.

18- Given the real matrix

$$A=\begin{pmatrix} 1 & 1 & 2 & 1 & 0 \\ -1 & 0 & 0 & 2 & 1 \\ 1 & 2 & 4 & 4 & 1 \\ 2 & 3 & 1 & 0 & 1 \end{pmatrix}.$$

Find a non-zero minor of A of order r = rank(A) and deduce a basis of the row space of A and a basis of the column space of A.

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CHAPTER VII

LINEAR MAPPINGS

Throughout this chapter the letter K denotes a field and E and F are two vector spaces over K.

§ 7.1. DEFINITION AND PROPERTIES.

Definition 1. We define a **K-linear mapping** (or simply **linear mapping**) of E to F to be every mapping $f : E \longrightarrow F$ satisfying:
(i) $f(x+y) = f(x)+f(y)$, $\forall x,y \in E$ and
(ii) $f(\alpha x) = \alpha f(x)$, $\forall \alpha \in K$ and $\forall x \in E$. ■

A K-linear mapping is also called **an homomorphism of vector spaces** (or **vector space homomorphism**). The set of all K-linear mappings of E to F is denoted $Hom_K(E,F)$ (or $L_K(E,F)$).

If E=F, then a K-linear mapping is called a **linear operator** of E. It is also called **an endomorphism** of E. In this case $Hom_K(E,E)$ is denoted $End_K(E)$. It is also denoted $GL_K(E)$.

7.1.1. If $f : E \longrightarrow F$ is a mapping, then the following are equivalent:
(i) f is K-linear,
(ii) $f(\alpha x+\beta y) = \alpha f(x)+\beta f(y)$, $\forall x,y \in E$ and $\forall \alpha,\beta \in K$,
(iii) $f(\alpha x+y) = \alpha f(x)+f(y)$, $\forall x,y \in E$ and $\forall \alpha \in K$,

Proof: (i)⇒(ii): Let $x,y \in E$ and $\alpha,\beta \in K$, then

$$f(\alpha x+\beta y) = f(\alpha x)+f(\beta y) = \alpha f(x)+\beta f(y).$$

(ii)⇒(iii): Let $x,y \in E$ and $\alpha \in K$, then

$$f(\alpha x+y) = f(\alpha x+1_K y) = \alpha f(x)+1_K f(y) = \alpha f(x)+f(y).$$

(iii)⇒(i): Let $x,y \in E$ and $\alpha \in K$, then

$$f(x+y) = f(1_K x+y) = 1_K f(x)+f(y) = f(x)+f(y)$$

and

$$f(\alpha x) = f(\alpha x+0_K 0_E) = \alpha f(x)+0_K f(0_E) = \alpha f(x)+0_F = \alpha f(x)$$

hence f is K-linear. ■

7.1.2. If $f : E \longrightarrow F$ is K-linear, then
(i) $f(0_E) = 0_F$,
(ii) $f(-x) = -f(x)$, $\forall x \in E$,
(iii) $f(a_1 x_1+\cdots+a_n x_n) = a_1 f(x_1)+\cdots+a_n f(x_n)$, where $a_1,\ldots,a_n \in K$, $x_1,\ldots,x_n \in E$ and $n \geq 1$.

Proof: (i) We have $f(0_E)=f(0_K 0_E) = 0_K f(0_E) = 0_F$.
(ii) As $f(x)+f(-x) = f(x+(-x)) = f(0_E) = 0_F$, hence $f(-x) = -f(x)$.
(iii) We argue by induction on n. The relation is true for n = 1, since f is K-linear. Suppose that it is true for (n-1) and let's show it for n. We have

$$f(a_1x_1+\cdots+a_nx_n) = f(a_1x_1+\cdots+a_{n-1}x_{n-1}+a_nx_n) = f(a_1x_1+\cdots+a_{n-1}x_{n-1})+f(a_nx_n)$$
$$= [a_1f(x_1)+\cdots+a_{n-1}f(x_{n-1})]+a_nf(x_n) = a_1f(x_1)+\cdots+a_nf(x_n)$$
hence the relation is true for n, and so it is true, for all $n\geq 1$. ■

<u>Definition 2.</u> We define an **isomorphism of vector spaces** (or **a vector space isomorphism**) of E onto F to be every K-linear bijection of E onto F. ■

If E=F, then a vector space isomorphism is called **an automorphism** of E and the set of all automorphisms of E is denoted $\mathrm{Aut}_K(E)$.

<u>7.1.3.</u> If $f : E \longrightarrow F$ is a vector space isomorphism of E onto F, then f^{-1} is a vector space isomorphism of F onto E.

<u>Proof:</u> Since f is bijective, so is f^{-1}. Let $x,y\in F$ and $\alpha\in K$. We have
$$f(f^{-1}(\alpha x+y)) = \alpha x+y = \alpha f(f^{-1}(x))+f(f^{-1}(y)) = f(\alpha f^{-1}(x)+f^{-1}(y))$$
so that as f is injective, then
$$f^{-1}(\alpha x+y) = \alpha f^{-1}(x)+f^{-1}(y)$$
and so f^{-1} is K-linear, by 7.1.1, whence f^{-1} is a vector space isomorphism of F onto E. ■

<u>Definition 3.</u> We say that E is **isomorphic** to F and we write $E\cong F$ (read: "E isomorphic to F) if there exists an isomorphism of vector spaces of E onto F. ■

<u>7.1.4.</u> Let G be a K-vector space. If $f : E \longrightarrow F$ and $g : F \longrightarrow G$ are K-linear, then so is $g\circ f$.

<u>Proof:</u> Let $x,y\in E$ and $\alpha\in K$. Then
$$g\circ f(\alpha x+y)=g(f(\alpha x+y))=g(\alpha f(x)+f(y))= \alpha g(f(x))+g(f(y))= \alpha g\circ f(x)+g\circ f(y)$$
and so $g\circ f$ is K-linear, by 7.1.1. ■

<u>Corollary 7.1.4.1.</u> If G is a K-vector space and $f : E \longrightarrow F$ and $g : F \longrightarrow G$ are two vector space isomorphisms, then so is $g\circ f$.

<u>Proof:</u> Since f and g are bijective and K-linear, so is $g\circ f$, by 7.1.4, hence $g\circ f$ is a vector space isomorphism. ■

<u>7.1.4.</u> The following hold
(i) $E\cong E$.
(ii) If $E\cong F$, then $F\cong E$.
(iii) If G is a K-vector space and if $E\cong F$ and $F\cong G$, then $E\cong G$.

<u>Proof:</u> (i) We can easily show that the identity mapping id_E of E is a vector space isomorphism of E onto E, hence $E\cong E$.
(ii) Since $E\cong F$, there exists a vector space isomorphism f of E onto F. We have that f^{-1} is a vector space isomorphism of F onto E, by 7.1.3, hence $F\cong E$.
(iii) We have $E\cong F$ and $F\cong G$, so that there exist two vector space isomorphisms f of E onto F and g of F onto G. As $g\circ f$ is an isomorphism of E onto G, by 7.1.4.1, then $E\cong G$. ■

7.1.6. If E is finite dimensional and $\dim_K(E) = n$, with $n \in \mathbb{N}^*$, then $E \cong K^n$.

Proof: Since $\dim_K(E) \neq 0$, $E \neq \{0_E\}$, by 6.3.3, hence E has a basis over K, by 6.2.3.1. Let $\{x_1,\ldots,x_n\}$ be a basis of E over K and let $f : K^n \longrightarrow E$ be the mapping defined by

$$f(a_1,\ldots,a_n) = a_1 x_1 + \cdots + a_n x_n .$$

f K-linear: Let $(a_1,\ldots,a_n)$, $(b_1,\ldots,b_n) \in K^n$ and $\alpha \in K$. Then

$$\begin{aligned} f[\alpha(a_1,\ldots,a_n)+(b_1,\ldots,b_n)] &= f(\alpha a_1+b_1,\ldots,\alpha a_n+b_n) = (\alpha a_1+b_1)x_1+\cdots+(\alpha a_n+b_n)x_n \\ &= (\alpha a_1 x_1+b_1 x_1)+\cdots+(\alpha a_n x_n+b_n x_n) \\ &= (\alpha a_1 x_1+\cdots+\alpha a_n x_n)+(b_1 x_1+\cdots+b_n x_n) \\ &= \alpha(a_1 x_1+\cdots+a_n x_n)+(b_1 x_1+\cdots+b_n x_n) \\ &= \alpha f(a_1,\ldots,a_n)+f(b_1,\ldots,b_n) \end{aligned}$$

and so f is K-linear, by 7.1.1.

f injective: Let $(a_1,\ldots,a_n)$, $(b_1,\ldots,b_n) \in K^n$. Then

$$\begin{aligned} f(a_1,\ldots,a_n) = f(b_1,\ldots,b_n) &\Rightarrow a_1 x_1+\cdots+a_n x_n = b_1 x_1+\cdots+b_n x_n \\ &\Rightarrow a_1 = b_1,\ldots,a_n = b_n \text{ (by 4.2.1)} \\ &\Rightarrow (a_1,\ldots,a_n) = (b_1,\ldots,b_n) \end{aligned}$$

and so f is injective.

f surjective: Let $u \in E$. Since $\{x_1,\ldots,x_n\}$ is a basis of E over K, there exist $a_1,\ldots,a_n$ in K, such that $u = a_1 x_1+\cdots+a_n x_n$, by 4.2.1. We have that $(a_1,\ldots,a_n) \in K^n$ and

$$f(a_1,\ldots,a_n) = a_1 x_1+\cdots+a_n x_n = u,$$

hence f is surjective. Therefore f is K-linear and bijective, and so it is an isomorphism of vector spaces of K^n onto E. This implies that $K^n \cong E$, whence $E \cong K^n$, by 7.1.5(ii). ■

Corollary 7.1.6.1. If E and F are finite dimensional and $\dim_K(E) = \dim_K(F)$, then $E \cong F$.

Proof: Put $n = \dim_K(E)$. If n=0, then $E=\{0_E\}$ and $F=\{0_F\}$, by 6.3.3, and so as the mapping f of E to F defined by $f(0_E)=0_F$ can be easily shown to be an isomorphism of vector spaces, then $E \cong F$. Suppose that $n \neq 0$, then $n \in \mathbb{N}^*$, and so $E \cong K^n$ and $F \cong K^n$, by 7.1.6. As $K^n \cong F$, by 7.1.5(ii), we then obtain that $E \cong K^n$ and $K^n \cong F$, and so $E \cong F$, by 7.1.5(iii). ■

§ 7.2. KERNEL AND IMAGE.

Definition 4. Let $f : E \longrightarrow F$ be a K-linear mapping.

(i) We define the **kernel** of f, denoted Ker(f), to be the set

$$\mathrm{Ker}(f) = \{x \in E \; ; \; f(x) = 0_F\}.$$

(ii) We define the **image** of f, written Im(f), to be the set

$$\mathrm{Im}(f) = f(E) = \{f(x) \; ; \; x \in E\}. \; ■$$

Thus

$$x \in \mathrm{Ker}(f) \Leftrightarrow x \in E \text{ and } f(x) = 0_F$$

and

$$y \in \mathrm{Im}(f) \Leftrightarrow \exists x \in E, \text{ such that } y = f(x).$$

7.2.1. If $f : E \longrightarrow F$ is a K-linear mapping, then
(i) Ker(f) is a subspace of E over K,
(ii) If W is a subspace of E over K, then f(W) is a subspace of F over K. In particular Im(f) is a subspace of F over K.

Proof: (i) Since $f(0_E) = 0_F$, by 7.1.2(i), we get $0_E \in Ker(f)$, and so
$$Ker(f) \neq \emptyset.$$
Let $x,y \in Ker(f)$ and $\alpha \in K$. We have
$$f(\alpha x+y) = \alpha f(x)+f(y) = \alpha 0_F + 0_F = 0_F$$
hence $\alpha x+y \in Ker(f)$, and so Ker(f) is a subspace of E over K.
(ii) We have that $0_E \in W$, hence $f(0_E) \in f(W)$ and so
$$f(W) \neq \emptyset.$$
Let $x,y \in f(W)$ and $\alpha \in K$. Then $\exists u,v \in W$, such that
$$x = f(u) \text{ and } y = f(v).$$
We have
$$\alpha x+y = \alpha f(u)+f(v) = f(\alpha u+v)$$
so that as $\alpha u+v \in W$, then $\alpha x+y \in f(W)$, and so f(W) is a subspace of F over K.
As E is a subspace of E over K and Im(f)=f(E), then Im(f) is a subspace of F over K. ■

7.2.2. If $f : E \longrightarrow F$ is a K-linear mapping, then
(i) f is injective if and only if $Ker(f)=\{0_E\}$,
(ii) f is surjective if and only if Im(f)=F.

Proof: (i) **N.C:** We have $\{0_E\} \subseteq Ker(f)$. Let $x \in Ker(f)$, then $f(x)=0_F=f(0_E)$, but f is injective, hence $x=0_E$, and so $x \in \{0_E\}$. It follows that $Ker(f) \subseteq \{0_E\}$, whence $Ker(f)=\{0_E\}$.
S.C: We have
$$f(x)=f(y) \Rightarrow f(x)-f(y)=0_F \Rightarrow f(x)+f(-y)=0_F \Rightarrow f(x-y)=0_F \Rightarrow (x-y) \in Ker(f) \Rightarrow x-y=0_E \Rightarrow x=y$$
hence f is injective.
(ii) **N.C:** We have $Im(f) \subseteq F$. Let $y \in F$. Then $\exists x \in E$, such that $y=f(x)$, hence $y \in Im(f)$, and so $F \subseteq Im(f)$. Therefore Im(f)=F.
S.C: Let $y \in F$. Then $y \in Im(f)$, and so $\exists x \in E$, such that $y = f(x)$. Hence f is surjective. ■

7.2.3. If $f : E \longrightarrow F$ is a K-linear mapping and $\{x_1,...,x_n\}$ be basis of E over K, then
$$Im(f) = Vect(f(x_1),...,f(x_n)).$$

Proof: Let $y \in Im(f)$. Then
$$\exists x \in E, \text{ such that } y = f(x).$$
As $\{x_1,...,x_n\}$ is a basis of E over K, then $\exists a_1,...,a_n \in K$, such that $x = a_1x_1+\cdots+a_nx_n$, by 6.2.2. We have $y = f(a_1x_1+\cdots+a_nx_n) = a_1f(x_1)+\cdots+a_nf(x_n)$, by 7.1.2(iii), hence $y \in Vect(f(x_1),...,f(x_n))$, and so
$$Im(f) \subseteq Vect(f(x_1),...,f(x_n)).$$
As $f(x_1),...,f(x_n) \in Im(f)$ and Im(f) is a subspace of F, by 7.2.1(ii), then
$$Vect(f(x_1),...,f(x_n)) \subseteq Im(f)$$
and so $Im(f)=Vect(f(x_1),...,f(x_n))$. ■

7.2.4. If $f : E \longrightarrow F$ is an injective K-linear mapping and if E is finite dimensional over K, then

$$\dim_K (E) = \dim_K (\mathrm{Im}(f)).$$

Proof: If $E = \{0_E\}$, then $\mathrm{Im}(f) = \{0_F\}$, and so

$$\dim_K (\mathrm{Im}(f)) = \dim_K (E) = 0.$$

Assume that $E \neq \{0_E\}$. Then E has a basis $\{x_1,...,x_n\}$ over K. We shall show that $\{f(x_1),...,f(x_n)\}$ is a basis of Im(f). As $\mathrm{Im}(f)=\mathrm{Vect}(f(x_1),...,f(x_n))$, by 7.2.3, it is enough to show that $f(x_1),...,f(x_n)$ are linearly independent over K. Let $a_1,...,a_n \in K$. Then

$$\begin{aligned} a_1 f(x_1)+\cdots+a_n f(x_n) = 0_F &\Rightarrow f(a_1 x_1+\cdots+a_n x_n) = 0_F \\ &\Rightarrow f(a_1 x_1+\cdots+a_n x_n) = f(0_E) \\ &\Rightarrow a_1 x_1+\cdots+a_n x_n = 0_E \\ &\Rightarrow a_1 = ... = a_n = 0_K \end{aligned}$$

and so $f(x_1),...,f(x_n)$ are linearly independent over K, whence $\{f(x_1),...,f(x_n)\}$ is a basis of Im(f) over K. Therefore $\dim_K (\mathrm{Im}(f)) = n$, and so $\dim_K (E) = \dim_K (\mathrm{Im}(f))$. ■

7.2.5. (Theorem of dimensions): If E is finite dimensional over K and $f : E \longrightarrow F$ is K-linear, then

$$\dim_K (E) = \dim_K (\mathrm{Ker}(f))+\dim_K (\mathrm{Im}(f)).$$

Proof: The proof is given in Appendix I. ■

Definition 5. Let $f : E \longrightarrow F$ be a K-linear mapping. We define the **rank** of f, denoted rank(f) to be the dimension over K of Im(f). ■

It follows from 7.2.5 that if E is finite dimensional over K, then

$$\dim_K (E) = \dim_K (\mathrm{Ker}(f))+\mathrm{rank}(f).$$

§ 7.3. OPERATIONS ON LINEAR MAPPINGS.

7.3.1. If $f,g \in \mathrm{Hom}_K(E,F)$ and $\alpha \in K$, then the mappings $f+g$ and αf of E to F defined by

$$(f+g)(x) = f(x)+g(x) \text{ and } (\alpha f)(x) = \alpha f(x)$$

are K-linear.

Proof: Let $x,y \in E$ and $\beta \in K$. Then

$$\begin{aligned} (f+g)(\beta x+y) &= f(\beta x+y)+g(\beta x+y) = \beta f(x)+f(y)+ \beta g(x)+g(y) = \beta f(x)+\beta g(x)+f(y)+g(y) \\ &= \beta(f+g)(x)+(f+g)(y) \end{aligned}$$

and

$$\begin{aligned} (\alpha f)(\beta x+y) &= \alpha f(\beta x+y) = \alpha[\beta f(x)+f(y)] = (\alpha\beta)f(x)+\alpha f(y) = (\beta\alpha)f(x)+(\alpha f)(y) \\ &= \beta(\alpha f(x))+(\alpha f)(y) = \beta[(\alpha f)(x)]+(\alpha f)(y) \end{aligned}$$

hence $f+g$ and αf are K-linear. ■

7.3.2. The set $\mathrm{Hom}_K(E,F)$ is a K-vector space under the addition and the scalar multiplication defined in 7.3.1, whose zero is the mapping $O : E \longrightarrow F$, defined by

$$O(x) = 0_F, \forall x \in E.$$

Proof: The proof is easy enough. ■

7.3.3. If E and F are finite dimensional over K and $\dim_K(E)=\dim_K(F)$ and $f : E \longrightarrow F$ is a linear mapping, then the following are equivalent:
(i) f is bijective, (ii) f is injective, (iii) f is surjective.

Proof: (i)⇒(ii): As f is bijective, then f is injective.
(ii)⇒(iii): We have that E is finite dimensional over K and f is injective, hence $\dim_K(E)=\dim_K(\mathrm{Im}(f))$, by 7.2.4, and so $\dim_K(\mathrm{Im}(f))=\dim_K(F)$. But Im(f) is a subspace of F over K, hence Im(f)=F, by 6.3.5(iii), and so f is surjective, by 7.2.2(ii).
(iii)⇒(i): Since f is surjective, F = Im(f), by 7.2.2(ii), and so

$$\dim_K(F) = \dim_K(\mathrm{Im}(f)).$$

But $\dim_K(E) = \dim_K(\mathrm{Ker}(f))+\dim_K(\mathrm{Im}(f))$, by 7.2.5, hence $\dim_K(\mathrm{Ker}(f))=0$, and so $\mathrm{Ker}(f)=\{0_E\}$, by 6.3.3, whence f is injective, by 7.2.2(i). Therefore f is bijective. ■

Corollary 7.3.3.1. If V is a finite dimensional K-vector space and $f : V \longrightarrow V$ is a linear operator of V, then the following are equivalent:
(i) f is bijective, (ii) f is injective, (iii) f is surjective.

Proof: Take E=F=V and apply 7.3.3. ■

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CHAPTER VII

EXERCISES

1- Study the linearity of the mappings in the following cases:

(i) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, defined by $f(x,y) = (2x+5y , 3x-2y)$,

(ii) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, defined by $f(x,y) = (0 , |y|)$,

(iii) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, defined by $f(x,y) = (x+y , x-y+1)$,

(iv) $f : \mathbb{R} \longrightarrow \mathbb{R}$, defined by $f(x) = x^3$,

(v) $\varphi : E \longrightarrow \mathbb{R}^2$, defined by $\varphi(f) = \int_0^1 f(x)dx$, where E is the real space of continuous mappings of $[0,1]$ to $\mathbb{R}$.

2- Show that the mapping $f : \mathbb{C} \longrightarrow \mathbb{C}$, defined by $f(z) = \bar{z}$ is $\mathbb{R}$-linear and that it is not $\mathbb{C}$-linear.

3- Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be an $\mathbb{R}$-linear mapping, such that

$$f(1,1,1) = 1,\ f(0,1,1) = 2 \text{ and } f(1,1,0) = 3.$$

Calculate $f(a,b,c)$, where $(a,b,c) \in \mathbb{R}^3$.

4- Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in M_2(\mathbb{R})$ and $f : M_2(\mathbb{R}) \longrightarrow M_2(\mathbb{R})$ be defined by $f(X) = XA-AX$.

(a) Show that f is $\mathbb{R}$-linear.
(b) Find Ker(f) and Im(f).

5- Let E and F be two K-vector spaces and $f : E \longrightarrow F$ a K-linear mapping.

1- Show that if $x_1,...,x_n \in E$ and $f(x_1),...,f(x_n)$ are linearly independent over K, then so are $x_1,...,x_n$.

2- Let H be a subspace of E of finite dimension over K.

(i) Show that the mapping $g : H \longrightarrow F$, defined by $g(x)=f(x)$, $\forall x \in H$ is K-linear and $Im(g) = f(H)$.

(ii) Deduce that f(H) is a subspace of F and $\dim_K(f(H)) \leq \dim_K(H)$.

(iii) Show that if f is injective, then $\dim_K(f(H)) = \dim_K(H)$.

6- Let

$$F = \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} ; a,b,c \in \mathbb{R} \right\}.$$

1- Show that F is a subspace of $\in M_2(\mathbb{R})$. Compute a basis of F and find its dimension.

2- Show that the mapping $f : F \longrightarrow F$ defined by $f\begin{pmatrix} a & b \\ c & a \end{pmatrix} = \begin{pmatrix} a & b+c \\ b+c & a \end{pmatrix}$ is an endomorphism of F.

3- Compute a basis of Ker(f) and a basis of Im(f).

7- Answer by true or false and justify the following:

(a) A $\mathbb{C}$-linear mapping $f : \mathbb{C}^4 \longrightarrow \mathbb{C}^2$ cannot be injective.

(b) An $\mathbb{R}$-linear mapping $f : \mathbb{R}^3 \longrightarrow \mathbb{C}^2$ can be surjective.

(c) If $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$ is $\mathbb{R}$-linear, then Ker(f) cannot be isomorphic to $\mathbb{R}$.

8- Let u=(1,0,1), v=(0,-1,1) and w=(2,-1,1).

1- show that (u, v, w) is a basis of $\mathbb{R}^3$ over $\mathbb{R}$.

2- Determine the linear mapping $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, satisfying

$$f(u) = e_1,\ f(v) = e_2 \text{ and } f(w) = e_3$$

where $\{e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{R}^3$.

3- Is f injective?

4- Compute rank(f) and find a basis of Ker(f).

9- Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be the mapping defined by

$$f(x,y,z,t) = (2x-3y+2z-t\ ,\ 3x-y+4z\ ,\ 2x+y-t\ ,\ 3x+2y+4z-3t).$$

and let

$$u_1 = (2,3,2,3),\ u_2 = (-3,-1,1,2),\ u_3 = (2,4,0,4) \text{ and } u_4 = (-1,0,-1,-3).$$

(i) Show that $f(x,y,z,t) = xu_1 + yu_2 + zu_3 + tu_4$.

(ii) Prove that f is $\mathbb{R}$-linear and find a basis of Im(f) over $\mathbb{R}$.

(iii) Calculate $\dim_{\mathbb{R}}(\mathrm{Ker}(f))$.

10- Let $\varphi : \mathbb{R}_3[x] \longrightarrow \mathbb{R}_4[x]$ be the mapping defined by

$$\varphi(f(x)) = f'(x) - xf(x).$$

1- Show that φ is $\mathbb{R}$-linear.

2- Find a basis of Ker(φ) and a basis of Im(φ).

3- Calculate the rank of φ.

4- Is φ injective ? surjective ?

11- Let E be a K-vector space. We call **projector** of E, every linear operator of E, satisfying that $p^2 = p$, where $p^2 = p \circ p$. Let p be linear operator of E.

1- Show that $x \in \mathrm{Im}(p) \Leftrightarrow p(x) = x$.

2- Prove that $(x-p(x)) \in \mathrm{Ker}(p),\ \forall x \in E$.

3- Deduce that $E = \mathrm{Im}(p) \oplus \mathrm{Ker}(p)$.

4- Show that p is injective if and only if $p = id_E$.

12- Let E and F be two real vector spaces and let $f : E \longrightarrow F$ be a mapping, satisfying

$$f(x+y)=f(x)+f(y) \text{ and } f(\alpha x)=\alpha f(x), \text{ for all } x,y \in E \text{ and all } \alpha \geq 0.$$

Show that f is $\mathbb{R}$-linear.

13- Show that if f is a linear operator of a finite dimensional K-vector space V, then $V = \mathrm{Im}(f) \oplus \mathrm{Ker}(f)$ if and only if $\mathrm{Im}(f) \cap \mathrm{Ker}(f) = \{0\}$.

<u>**Applications:**</u> Is $V = \mathrm{Im}(f) \oplus \mathrm{Ker}(f)$ in each of the following cases ?

(i) $V = \mathbb{R}^2$ and f is defined by $f(x,y) = (y, y)$.

(ii) $V = \mathbb{R}^3$ and f is defined by $f(x,y,z) = (2x-y, z, 0)$.

14- Let f be an endomorphism of $\mathbb{R}^n$.

1- Show that

(i) $f \circ f = 0 \Leftrightarrow \mathrm{Im}(f) \subseteq \mathrm{Ker}(f)$;

(ii) $f \circ f = 0$ and $n = 2 \times \mathrm{rank}(f) \Leftrightarrow \mathrm{Im}(f)=\mathrm{Ker}(f)$.

2- If n is odd, can we find an endomorphism g of $\mathbb{R}^n$, such that Im(g)=Ker(g) ?

15- Let f be an endomorphism of a K-vector space V.

1- Show that

(i) $\mathrm{Im}(f^2) \subseteq \mathrm{Im}(f)$ and $\mathrm{Ker}(f) \subseteq \mathrm{Ker}(f^2)$.

(ii) $V = \mathrm{Im}(f)+\mathrm{Ker}(f) \Leftrightarrow \mathrm{Im}(f^2) = \mathrm{Im}(f)$.

(iii) $\mathrm{Im}(f) \cap \mathrm{Ker}(f)=\{0\} \Leftrightarrow \mathrm{Ker}(f^2) = \mathrm{Ker}(f)$.

2- Assume that V is finite dimensional over K and $\mathrm{rank}(f) = \mathrm{rank}(f^2)$.

(i) Show that $\mathrm{Im}(f^2) = \mathrm{Im}(f)$.

(ii) Deduce that $V = \mathrm{Im}(f) \oplus \mathrm{Ker}(f)$.

CHAPTER VIII

MATRIX REPRESENTATION

Throughout this chapter the letter K denotes a field.

§ 8.1. DEFINITION AND PROPERTIES.

Let E and F be two K-vector spaces with bases $\{x_1,....,x_n\}$ and $\{y_1,....,y_m\}$ respectively and let $f : E \longrightarrow F$ be a K-linear mapping.

Definition 1. A $(m\times n)$ matrix $A=(a_{ij})$ over K is said to be **associated with** f **relative to the ordered bases** $\{x_1,....,x_n\}$ and $\{y_1,....,y_m\}$ if

$$f(x_t) = a_{1t}y_1 + a_{2t}y_2 + \cdots + a_{mt}y_m$$

for all $1\leq t\leq n$. ■

8.1.1. There exists one and only one matrix associated with f relative to the ordered bases $\{x_1,....,x_n\}$ and $\{y_1,....,y_m\}$.

Proof: **Existence:** Let $1\leq t\leq n$. Since $\{y_1,....,y_m\}$ is a basis of F over K, there exist $a_{1t}, a_{2t},..., a_{mt}$ in K, such that

$$f(x_t) = a_{1t}y_1 + a_{2t}y_2 + \cdots + a_{mt}y_m .$$

Let A be the $(m\times n)$ matrix whose t-th column is

$$\begin{pmatrix} a_{1t} \\ a_{2t} \\ \vdots \\ a_{mt} \end{pmatrix}.$$

As

$$f(x_t) = a_{1t}y_1 + a_{2t}y_2 + \cdots + a_{mt}y_m , \forall 1\leq t\leq n$$

then A is associated with f relative to the ordered bases $\{x_1,....,x_n\}$ and $\{y_1,....,y_m\}$.

Uniqueness: Let $A=(a_{ij})$ and $B=(b_{ij})$ be two matrices associated with f relative to the ordered bases $\{x_1,....,x_n\}$ and $\{y_1,....,y_m\}$. Then A and B are $(m\times n)$ matrices over K. Let $j\in\{1,2,...,n\}$. We have

$$f(x_j) = a_{1j}y_1 + a_{2j}y_2 + \cdots + a_{mj}y_m \text{ and } f(x_j) = b_{1j}y_1 + b_{2j}y_2 + \cdots + b_{mj}y_m$$

and $\{y_1,....,y_m\}$ is a basis of F over K, hence

$$a_{1j} = b_{1j}, a_{2j} = b_{2j},..., a_{mj} = b_{mj}$$

by 6.2.2, and so $a_{sj} = b_{sj}$, $\forall s\in\{1,2,...,m\}$ and $\forall j\in\{1,2,...,n\}$, whence A=B. ■

The unique matrix associated with f relative to the ordered bases $\{x_1,....,x_n\}$ and $\{y_1,....,y_m\}$ is called **the matrix of** f **relative to** $\{x_1,....,x_n\}$ and $\{y_1,....,y_m\}$ and is denoted $M(f ; (x_i), (y_j))$. Also if the basis of E (resp. F) is denoted B (resp. B'), then the matrix of f relative to B and B' is denoted M(f; B, B').

If E=F and $\{x_1,....,x_n\}=\{y_1,....,y_m\}$, we refer to $M(f\,;(x_i),(y_j))$ as **the matrix of f relative to** $\{x_1,...,x_n\}$ and we denote it by $M(f\,;(x_i))$. If the basis of E is denoted B, then the matrix of f relative to B is denoted $M(f\,;B)$. Thus

$$M(f\,;(x_i)) = M(f\,;(x_i),(x_i)) \text{ and } M(f\,;B) = M(f\,;B,B).$$

Examples: 1) Let $f:\mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the linear operator defined by

$$f(a,b,c) = (a+b+c, a+c).$$

then the matrix of f relative to the canonical bases $\{e_1, e_2, e_3\}$ and $\{f_1, f_2\}$ is

$$M(f\,;(e_i),(f_j)) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

because $f(e_1) = f(1,0,0) = (1,1) = f_1+f_2$, $f(e_2) = f(0,1,0) = (1,0) = f_1$ and $f(e_3) = f(0,0,1) = (1,1) = f_1+f_2$.

2) Let $\alpha\in K$ and $f_\alpha : V \longrightarrow V$ be the mapping defined by

$$f_\alpha(x) = \alpha x, \ \forall x\in V.$$

Then f_α is a linear operator of V and if $\{x_1,....,x_n\}$ is a basis of V over K, then

$$M(f_\alpha\,;(x_i)) = \begin{pmatrix} \alpha & 0 & \cdots & \cdots & 0 \\ 0 & \alpha & 0 & \cdots & 0 \\ \vdots & 0 & & & \vdots \\ \vdots & \vdots & & & 0 \\ 0 & 0 & \cdots & 0 & \alpha \end{pmatrix}.$$

For the rest of this section let $B=\{x_1,....,x_n\}$ and $B'=\{y_1,....,y_m\}$.

8.1.2. If $A=M(f\,;B,B')$, then $\mathrm{rank}(f) = \mathrm{rank}(A)$.

Proof: Put $A=(a_{ij})$, then

$$f(x_1) = a_{11}y_1 + a_{21}y_2 +\cdots+ a_{m1}y_m$$
$$f(x_2) = a_{12}y_1 + a_{22}y_2 +\cdots+ a_{m2}y_m$$
$$\vdots$$
$$f(x_n) = a_{1n}y_1 + a_{2n}y_2 +\cdots+ a_{mn}y_m.$$

We have

$$\mathrm{rank}(f(x_1),...,f(x_n)) = \mathrm{rank}({}^tA) = \mathrm{rank}(A)$$

by 6.4.3(ii) and 3.2.2, and so as $\mathrm{Im}(f)=\mathrm{Vect}(f(x_1),...,f(x_n))$, by 7.2.3, then

$$\mathrm{rank}(f) = \mathrm{rank}(A). \ ■$$

8.1.3. If V is a K-vector space of finite dimension n over K, with $n\geq 1$ and if $C=\{u_1,....,u_n\}$ is a basis of V over K, then

$$M(id_V\,;C) = M(id_V\,;C,C) = I_n.$$

Proof: As $id_V(u_i) = u_i$, $\forall 1\leq i\leq n$, then $M(id_V\,;C) = M(id_V\,;C,C) = I_n$. ■

If V is a K-vector space of finite dimension $n\geq 1$ over K, and if $C=\{u_1,....,u_n\}$ is a basis of V over K, then every element x of V is uniquely written in the form

$$x = a_1 u_1 + \cdots + a_n u_n$$

where $a_1, \ldots, a_n \in K$, hence for each $x \in E$, there exists a unique column matrix over K

$$A = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

such that $x = a_1 u_1 + \cdots + a_n u_n$.

This matrix is called the matrix of x relative to C and denoted M(x ; C) or M(x ; (u_i)). Thus

$$M(x\,;\,C) = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \Leftrightarrow x = a_1 u_1 + \cdots + a_n u_n .$$

8.1.4. If V is a K-vector space and $C=\{u_1, \ldots, u_n\}$ is a basis of V over K, then the following hold, for all $x,y \in V$ and $\alpha \in K$:

(i) $M(x\,;\,C) = M(y\,;\,C) \Leftrightarrow x = y$,

(ii) $M(x+y\,;\,C) = M(x\,;\,C)+M(y\,;\,C)$,

(iii) $M(\alpha x\,;\,C) = \alpha M(x\,;\,C)$.

Proof: Let

$$M(x\,;\,C) = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \text{ and } M(y\,;\,C) = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}.$$

Then

$$x = a_1 u_1 + \cdots + a_n u_n \text{ and } y = b_1 u_1 + \cdots + b_n u_n .$$

(i) $M(x\,;\,C) = M(y\,;\,C) \Leftrightarrow \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \Leftrightarrow a_1 = b_1, \ldots, a_n = b_n \Leftrightarrow x = y.$

(ii) We have $x+y = (a_1+b_1)u_1 + \cdots + (a_n+b_n)u_n$, hence

$$M(x+y\,;\,C) = \begin{pmatrix} a_1+b_1 \\ \vdots \\ a_n+b_n \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = M(x\,;\,C)+M(y\,;\,C).$$

(iii) Since $\alpha x = (\alpha a_1)u_1 + \cdots + (\alpha a_n)u_n$, we then have

$$M(\alpha x\,;\,C) = \begin{pmatrix} \alpha a_1 \\ \vdots \\ \alpha a_n \end{pmatrix} = \alpha \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \alpha M(x\,;\,C). \blacksquare$$

8.1.5. Let V be a K-vector space and $C=\{u_1, \ldots, u_n\}$ be basis of V over K. If A and B are $(m \times n)$ matrices over K, then

$$A \times M(x\,;\,C) = B \times M(x\,;\,C), \text{ for all } x \in V \Leftrightarrow A = B.$$

Proof: $\Rightarrow$): Set $A=(a_{ij})$ and $B=(b_{ij})$. Let $t \in \{1,\ldots,n\}$ and $x = u_t$. We have

$$M(x\,;\,C) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \\ \\ \\ \leftarrow \text{t-th entry} \\ \\ \\ \\ \end{matrix} = \text{tth column of } I_n$$

hence

$$A\times M(x\ ;\ C) = \text{tth column of } A$$
$$\text{and}$$
$$B\times M(x\ ;\ C) = \text{tth column of } B$$

by 1.5.2.1(ii), and so tth column of A= tth column of B, for all $t\in\{1,...,n\}$, whence A = B, by 1.1.1(ii).

<u>⇐):</u> Since A = B, $A\times M(x\ ;\ C) = B\times M(x\ ;\ C)$, for all $x\in V$. ■

<u>8.1.6.</u> $M(f\ ;\ B, B')\times M(u\ ;\ B) = M(f(u)\ ;\ B')$, $\forall u\in E$.

<u>Proof:</u> Let $u\in E$. Let $M(f\ ;\ B, B') = (a_{ij})$ and $M(u\ ;\ B) = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$. Then

$$u = b_1 x_1 + \cdots + b_n x_n$$
$$\text{and}$$
$$f(x_t) = a_{1t} y_1 + a_{2t} y_2 + \cdots + a_{mt} y_m,\ \forall 1\le t\le n$$

hence

$$\begin{aligned} f(u) &= f(b_1 x_1 + \cdots + b_n x_n) = b_1 f(x_1) + \cdots + b_n f(x_n) \\ &= b_1(a_{11} y_1 + \cdots + a_{m1} y_m) + \cdots + b_n(a_{1n} y_1 + \cdots + a_{mn} y_m) \\ &= (b_1 a_{11} + b_2 a_{12} + \cdots + b_n a_{1n}) y_1 + \cdots + (b_1 a_{m1} + b_2 a_{m2} + \cdots + b_n a_{mn}) y_m \end{aligned}$$

and so

$$M(f(u)\ ;\ B') = \begin{pmatrix} b_1 a_{11} + \cdots + b_n a_{1n} \\ b_1 a_{21} + \cdots + b_n a_{2n} \\ \vdots \\ b_1 a_{m1} + \cdots + b_n a_{mn} \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = M(f\ ;\ B, B')\times M(u\ ;\ B).\ ■$$

<u>8.1.7.</u> The following hold, for all $f,g\in Hom_K(E,F)$ and $\alpha\in K$

(i) $M(f\ ;\ B, B') = M(g\ ;\ B, B') \Leftrightarrow f = g$,

(ii) $M(f+g\ ;\ B, B') = M(f\ ;\ B, B') + M(g\ ;\ B, B')$,

(iii) $M(\alpha f\ ;\ B, B') = \alpha M(f\ ;\ B, B')$.

<u>Proof:</u> (i) We have:

$$\begin{aligned} M(f\ ;\ B, B') = M(g\ ;\ B, B') &\Leftrightarrow M(f\ ;\ B, B')\times M(u\ ;\ B) = M(g\ ;\ B, B')\times M(u\ ;\ B),\ \forall u\in E \\ &\Leftrightarrow M(f(u)\ ;\ B') = M(g(u)\ ;\ B'),\ \forall u\in E \qquad \text{(by 8.1.6)} \\ &\Leftrightarrow f(u) = g(u),\ \forall u\in E \qquad \text{(by 8.1.4(i))} \\ &\Leftrightarrow f = g. \end{aligned}$$

(ii) $\forall u\in E$, we have

$$\begin{aligned} M(f+g\ ;\ B, B')\times M(u\ ;\ B) &= M((f+g)(u)\ ;\ B') = M(f(u)+g(u)\ ;\ B') \\ &= M(f(u)\ ;\ B') + M(g(u)\ ;\ B') \\ &= M(f\ ;\ B, B')\times M(u\ ;\ B) + M(g\ ;\ B, B')\times M(u\ ;\ B) \\ &= [M(f\ ;\ B, B') + M(g\ ;\ B, B'),]\times M(u\ ;\ B) \end{aligned}$$

hence $M(f+g\ ;\ B, B') = M(f\ ;\ B, B') + M(g\ ;\ B, B')$, by 8.1.5.

(iii) For every $u\in E$, we have

$$\begin{aligned} M(\alpha f\ ;\ B, B')\times M(u\ ;\ B) &= M((\alpha f)(u)\ ;\ B') = M(\alpha f(u)\ ;\ B') \\ &= \alpha M(f(u)\ ;\ B') = \alpha[M(f\ ;\ B, B')\times M(u\ ;\ B)] \\ &= [\alpha M(f\ ;\ B, B')]\times M(u\ ;\ B) \end{aligned}$$

hence $M(\alpha f\ ;\ B, B') = \alpha M(f\ ;\ B, B')$, by 8.1.5. ■

<u>8.1.8.</u> Let G be a K-vector space with finite dimension. If B” is a basis of G over K and

$g : F \longrightarrow G$ is a K-linear mapping, then

$$M(g \circ f ; B, B'') = M(g ; B', B'') \times M(f ; B, B').$$

Proof: $\forall u \in E$, we have

$$\begin{aligned} M(g \circ f ; B, B'') \times M(u ; B) &= M(g \circ f(u) ; B'') = M(g(f(u)) ; B'') \\ &= M(g ; B', B'') \times M(f(u) ; B') \\ &= M(g ; B', B'') \times [M(f ; B, B') \times M(u ; B)] \\ &= [M(g ; B', B'') \times M(f ; B, B')] \times M(u ; B) \end{aligned}$$

hence $M(g \circ f ; B, B'') = M(g ; B', B'') \times M(f ; B, B')$, by 8.1.5. ■

8.1.9. Whenever $A \in M_{m,n}(K)$, there exists a K-linear mapping $g : E \longrightarrow F$, such that

$$M(g ; B, B') = A.$$

Proof: Let g be the correspondence from E to F which associates every element u of E with the elements v of F satisfying

$$M(v ; B') = A \times M(u ; B).$$

g mapping: Let $u \in E$. We have that $A \times M(u ; B)$ is a $(m \times 1)$ matrix, hence $\exists b_1, \ldots, b_m \in K$, such that

$$A \times M(u ; B) = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}.$$

Let $v = b_1 y_1 + \ldots + b_m y_m$. Then

$$v \in F \text{ and } M(v ; B') = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} = A \times M(u ; B)$$

and so v is an image of u by g. Let v_1 and v_2 be two images of u by g. Then

$$(v_1 ; B') = A \times M(u ; B) \text{ and } (v_2 ; B') = A \times M(u ; B)$$

hence $(v_1 ; B') = (v_2 ; B')$, and so $v_1 = v_2$. We deduce that g is a mapping and

$$M(g(u) ; B') = A \times M(u ; B), \ \forall u \in E.$$

g K-linear: Let $u, v \in E$ and $\alpha \in K$. Then

$$\begin{aligned} M(g(\alpha u+v) ; B') &= A \times M(\alpha u+v ; B) = A \times [M(\alpha u ; B)+M(v ; B)] \\ &= A \times [\alpha M(u ; B)] + A \times M(v ; B) = \alpha A \times M(u ; B) + M(g(v) ; B') \\ &= \alpha M(g(u) ; B') + M(g(v) ; B') = M(\alpha g(u) ; B') + M(g(v) ; B') \\ &= M(\alpha g(u)+g(v) ; B') \end{aligned}$$

and so

$$g(\alpha u+v) = \alpha g(u)+g(v)$$

by 8.1.4(i), whence g is K-linear. Therefore

$$M(g(u) ; B') = M(g ; B, B') \times M(u ; B), \ \forall u \in E$$

by 8.1.6. But $M(g(u) ; B') = A \times M(u ; B)$, $\forall u \in E$, hence

$$M(g ; B, B') \times M(u ; B) = A \times M(u ; B), \ \forall u \in E$$

and so $M(g ; B, B') = A$, by 8.1.5. ■

Corollary 8.1.9.1. The mapping $\varphi : Hom_K(E,F) \longrightarrow M_{m,n}(K)$ defined by

$$\varphi(f) = M(f ; B, B')$$

is a vector space isomorphism of $Hom_K(E,F)$ onto $M_{m,n}(K)$. In particular $Hom_K(E,F)$ is finite dimensional over K and its dimension is $m \times n$.

Proof: By 8.1.7 and 8.1.9, we have that φ is injective, surjective and K-linear, hence φ is an isomorphism of K-vector spaces. ■

8.1.10. If m=n, then a linear mapping $g : E \longrightarrow F$ is invertible if and only if M(g ; B, B') is an invertible matrix. Moreover if g is invertible, then

$$M(g^{-1} ; B', B) = [M(g ; B, B')]^{-1}.$$

Proof: **N.C:** We have that g^{-1} is an isomorphism, by 7.1.3, hence $M(g^{-1} ; B', B)$ exists and $M(g^{-1} ; B', B) \in M_n(K)$. But

$$M(g ; B, B') \times M(g^{-1} ; B', B) = M(g \circ g^{-1} ; B', B') = M(id_F ; B', B') = I_n$$

by 8.1.3 and 8.1.8, hence M(g ; B, B') is an invertible matrix and

$$M(g^{-1} ; B', B) = [M(g ; B, B')]^{-1}.$$

S.C: We have M(g ; B, B') is invertible and $M(g ; B, B') \in M_n(K)$, so that $\exists A \in M_n(K)$, such that

$$A \times M(g ; B, B') = M(g ; B, B') \times A = I_n .$$

By 8.1.9, there exists h in $Hom_K(F,E)$, such that

$$A = M(h ; B', B).$$

Using 8.1.8, we get

$$M(h \circ g ; B, B) = M(h ; B', B) \times M(g ; B, B') = A \times M(g ; B, B') = I_n = M(id_E ; B, B)$$

hence $h \circ g = id_E$, by 8.1.7(i), and so g is injective. As $\dim_K(E)=\dim_K(F)=n$, then g is bijective, by 7.3.3, and so g is invertible. ■

Corollary 8.1.10.1 (Proof of 1.8.2). If $A,B \in M_n(K)$ and $AB=I_n$, then $BA=I_n$.

Proof: Let $V=K^n$ and let C be the canonical basis of V, then $\exists f,g \in Hom_K(V,V)$, such that

$$A=M(f ; C) \text{ and } B=M(g ; C)$$

by 8.1.9. We have

$$I_n = AB = M(f ; C) \times M(g ; C) = M(f \circ g ; C)$$

by 8.1.8, hence $M(f \circ g ; C) = M(id_V ; C)$, and so

$$f \circ g = id_V$$

by 8.1.7(i). It follows that f is surjective, and so f is bijective, by 7.3.3.1. As $f \circ g = id_V$ and f is bijective, then $g = f^{-1}$, and so

$$BA = M(g ; C) \times M(f ; C) = M(g \circ f ; C) = M(id_V ; C) = I_n . \ ■$$

§ 8.2. TRANSITION MATRIX.

Let V be a K-vector space of finite dimension n, where n≥1 and let $B=\{x_1,...,x_n\}$ and $B'=\{y_1,...,y_n\}$ be two bases of V over K.

Definition 2. We define **the transition matrix** from B to B' to be the (n×n) matrix

$$P = M(id_V ; B', B). \ ■$$

Thus if $P=(a_{ij})$ is $(n\times n)$ over K, then P is the transition matrix from B to B' if and only if $y_t = a_{1t}x_1+a_{2t}x_2+\cdots+a_{nt}x_n$, $\forall 1\le t\le n$.

8.2.1. If P is the transition matrix from B to B', then P is invertible and P^{-1} is the transition matrix from B' to B.

Proof: Let Q be the transition matrix from B' to B. Then $Q = M(id_V ; B, B')$. We have

$$PQ = M(id_V ; B', B)\times M(id_V ; B, B')) = M(id_V ; B, B) = I_n$$

by 8.1.3 and 8.1.8, hence P is invertible and $P^{-1}= Q$, which is the transition matrix from B' to B. ■

Remark: Let $P=(a_{ij})\in M_n(K)$ and suppose that P is invertible. Let $u_1,...,u_n \in V$, such that

$$u_t = a_{1t}x_1+a_{2t}x_2+\cdots+a_{nt}x_n,\ \forall 1\le t\le n.$$

We have that ${}^tP$ is the matrix of the system

$$\begin{gathered} a_{11}x_1+a_{21}x_2+\cdots+a_{n1}x_n = u_1 \\ a_{12}x_1+a_{22}x_2+\cdots+a_{n2}x_n = u_2 \\ \vdots \\ a_{1n}x_1+a_{2n}x_2+\cdots+a_{nn}x_n = u_n \end{gathered}$$

and $\left|{}^tP\right| \neq 0$, hence the system becomes a Cramer's system, and so it has a solution, whence $x_1,...,x_n$ can be calculated in terms of $u_1,...,u_n$. Therefore, whenever $1\le t\le n$, there exist $b_{1t}, b_{2t},\dots, b_{nt}$ in K, such that

$$x_t = b_{1t}u_1+b_{2t}u_2+\cdots+b_{nt}u_n,\ \forall 1\le t\le n.$$

We have $\text{rank}({}^tP)=n$, by 3.2.1, hence $\{u_1,...,u_n\}$ is a basis of V over K, by 6.4.3(iv). But P^{-1} is the transition matrix from $\{u_1,...,u_n\}$ to $\{x_1,...,x_n\}$, by 8.2.2, hence

$$P^{-1}=\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}.$$

Applications: Find the inverse of each of the matrices

$$A=\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } B=\begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 3 & 4 \end{pmatrix}.$$

Answer: inverse of A: Consider the system

$$x_1 = y_1,\ x_1+x_2 = y_2 \text{ and } x_2+x_3 = y_3.$$

We have

$$x_1 = y_1,\ x_2 = y_2 - x_1 = y_2 - y_1 = -y_1+y_2 \text{ and } x_3 = y_3-x_2 = y_3-(-y_1+y_2) = y_1-y_2+y_3$$

hence

$$A^{-1}=\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

inverse of B: Consider the system

$$x_1 = y_1, \qquad 2x_1+x_2+3x_3 = y_2, \qquad -x_1+x_2+4x_3 = y_3.$$

We have

$$x_1 = y_1 \text{ and } y_3 - y_2 = -3x_1+x_3 = -3y_1+x_3$$

hence

$$x_1 = y_1,\ x_3 = 3y_1-y_2+y_3 \text{ and } x_2 = y_2-2x_1-3x_3 = -11y_1+4y_2-3y_3$$

and so

$$B^{-1} = \begin{pmatrix} 1 & -11 & 3 \\ 0 & 4 & -1 \\ 0 & -3 & 1 \end{pmatrix}.$$

8.2.2. Let E and F be finite dimensional K-vector spaces. Suppose that B and C are two bases of E and B' and C' are two bases of F. Let $f : E \longrightarrow F$ be a K-linear mapping. If P is the transition matrix from B to C and Q is that from B' to C', then

(i) $M(f ; B, B')\times P = M(f ; C, B')$;

(ii) $Q^{-1}\times M(f ; B, B') = M(f ; B, C')$;

(iii) $Q^{-1}\times M(f ; B, B')\times P = M(f ; C, C')$.

Proof: (i) $M(f ; B, B')\times P = M(f ; B, B')\times M(id_E ; C, B) = M(f\circ id_E ; C, B') = M(f ; C, B')$,

(ii) $Q^{-1}\times M(f ; B, B') = M(id_F ; B', C')\times M(f ; B, B') = M(id_F \circ f ; B, C') = M(f ; B, C')$.

(iii) $Q^{-1}\times M(f ; B, B')\times P = Q^{-1}\times M(f ; C, B') = M(f ; C, C')$, by (i) and (ii). ■

Corollary 8.2.2.1. Let V be a K-vector space and let B and C be bases of V over K. If $f : V \longrightarrow V$ is a linear operator of V and P is the transition matrix from B to C, then

(i) $M(f ; B)\times P = M(f ; C, B)$;

(ii) $P^{-1}\times M(f ; B) = M(f ; B, C)$;

(iii) $M(f ; C) = P^{-1}M(f ; B)P$.

Proof: Take B'=B and C'=C, then Q=P, and so using 8.2.2, we get

(i) $M(f ; B)\times P = M(f ; B, B)\times P = M(f ; C, B)$;

(ii) $P^{-1}\times M(f ; B) = P^{-1}\times M(f ; B, B) = M(f ; B, C)$;

(iii) $P^{-1}M(f ; B)P = P^{-1}M(f ; B, B)P = M(f ; C)$. ■

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CHAPTER VIII

EXERCISES

1- Compute the matrix of the linear mapping φ relative to the canonical basis in the following cases:

(i) $\varphi : M_2(K) \longrightarrow M_2(K)$ defined by $\varphi(A) = {}^tA$.

(ii) $\varphi : \mathbb{R}_2[x] \longrightarrow M_2(\mathbb{R})$ defined by $\varphi(f(x)) = \begin{pmatrix} f'(0) & f(1) \\ f''(2) & 0 \end{pmatrix}$.

(iii) $\varphi : \mathbb{R}_2[x] \longrightarrow \mathbb{R}^4$ defined by $\varphi(f(x)) = (f(1),\ f'(1),\ \int_0^1 f(x)dx,\ \int_{-1}^0 f(x)dx)$.

2- Let f be the endomorphism of $\mathbb{R}^3$ whose matrix relative to the canonical basis is

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

1- Show that f is an automorphism of $\mathbb{R}^3$.

2- Let $u \in \mathbb{R}^3$ and let x, y and z be the components of u and x', y' and z' be those of f(u) relative to the canonical basis of $\mathbb{R}^3$.

(i) Calculate x', y' and z' in terms of x, y and z.

(ii) Calculate x, y and z in terms of x', y' and z' and deduce A^{-1}.

3- Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the linear mapping, such that

$$f(e_1) = 2e_1 - e_2 + e_3,\ f(e_2) = -e_1 + e_2 - 2e_3 \text{ and } f(e_3) = e_3.$$

1- Compute $A = M(f\,;(e_i))$ and show that f is invertible.

2- Find $f^{-1}(e_1)$, $f^{-1}(e_2)$ and $f^{-1}(e_3)$ in terms of e_1, e_2 and e_3 and deduce A^{-1}**.**

4- Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the mapping defined by

$$f(x,y,z) = (2x-3z\ ,\ x-y+z).$$

1- Show that f is $\mathbb{R}$-linear.

2- Compute M(f ; B, B'), where B and B' are the canonical bases of $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively.

3- Let $x_1 = (1,2,1)$, $x_2 = (0,-1,2)$ and $x_3 = (2,1,3)$. Show that $C=(x_1,x_2,x_3)$ is a basis of $\mathbb{R}^3$ over $\mathbb{R}$.

4- Let $u_1 = (1,-1)$ and $u_2 = (-1,2)$. Show that $C'=(u_1,u_2)$ is a basis of $\mathbb{R}^2$ over $\mathbb{R}$.

5- Find the transition matrix Q from B' to C' and compute Q^{-1}.

6- Find the transition matrix P from B to C and deduce M(f ; C, B').

7- Compute M(f ; C, C').

5- Let f be the endomorphism of $\mathbb{R}^3$, such that

$$f(e_1) = (1,1,0),\ f(e_2) = (2,0,1) \text{ and } f(e_3) = (1,0,-1)$$

where $B=(e_1, e_2, e_3)$ is the canonical basis of $\mathbb{R}^3$.

1- Find the matrix A of f relative to the basis B.
2- Compute f(2,2,-1), by using A.
3- Show that C = $(f(e_1), f(e_2), f(e_3))$ is a basis of $\mathbb{R}^3$ and deduce that f is an automorphism of $\mathbb{R}^3$.
4- Calculate M(f ; B, C) and M(f ; C, B).

6- Let B=(x_1, x_2, x_3) be a basis of a finite dimensional $\mathbb{R}$-vector space V. Let f be the endomorphism of V, such that

$$f(x_1) = x_1+x_2+2x_3,\ f(x_2) = x_1-x_3 \text{ and } f(x_3) = 2x_1+x_2+x_3.$$

Let $u_1 = x_1+x_2-2x_3$, $u_2 = 2x_1+x_2$ and $u_3 = x_2+3x_3$.
1- Show that C = (u_1, u_2, u_3) is a basis of V over $\mathbb{R}$.
2- Find the transition matrices P from B to C and Q from C to B.
3- Compute M(f ; B), M(f ; C), M(f ; B, C) and M(f ; C, B).

7- Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the mapping defined by

$$f(x,y,z) = (2x-3z\ ,\ x-y+z\ ,\ 3z).$$

1- Show that f is $\mathbb{R}$-linear and find a system of generators of Im(f) over $\mathbb{R}$.
2- Find the matrix of f relative to the canonical basis of $\mathbb{R}^3$ over $\mathbb{R}$.
3- Show that if $u\in\mathbb{R}^3$, and f(u)=u, then u=0.
4- Does there exist a basis B=$\{x_1,x_2,x_3\}$ of $\mathbb{R}^3$ over $\mathbb{R}$, such that

$$M(f\ ;B)=\begin{pmatrix}1&2&3\\0&1&1\\0&0&2\end{pmatrix}?$$

Justify your answer.
5- Let x_1 = (1,2,1), x_2 = (0,-1,2) and x_3 = (2,1,3).
(i) Show that $e_1=-x_1-x_2+x_3$, $e_2=\frac{1}{5}(4x_1+x_2-2x_3)$ and $e_3=\frac{1}{5}(2x_1+3x_2-x_3)$.
(ii) Deduce that $\{x_1,x_2,x_3\}$ is a basis of $\mathbb{R}^3$ over $\mathbb{R}$ and compute the inverse of the matrix

$$A=\begin{pmatrix}1&0&2\\2&-1&1\\1&2&3\end{pmatrix}.$$

(iii) Compute M(f ; (x_i)).
(iv) Calculate $f(x_1)$, $f(x_2)$ and $f(x_3)$ in terms of x_1, x_2 and x_3.

8- Let $\varphi : \mathbb{R}_2[x] \longrightarrow \mathbb{R}_2[x]$ be the mapping defined by

$$\varphi(f(x))=2xf(x)-(x^2-1)f'(x).$$

1- Show that φ is $\mathbb{R}$-linear.
2- Find the matrix A of φ relative to the canonical basis of $\mathbb{R}_2[x]$.
3- Let $f_1=1$, $f_2=1+x$ and $f_3=1+x^2$. Show that (f_1, f_2, f_3) is a basis of $\mathbb{R}_2[x]$ over $\mathbb{R}$.
4- Compute the transition matrix P from the canonical basis of $\mathbb{R}_2[x]$ to $\{f_1, f_2, f_3\}$.
5- Compute P^{-1} and deduce M(φ ; (f_i)).

9- Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a non-zero linear mapping, such that $f^2 = 0$.

1- Show that if B is a basis of $\mathbb{R}^3$ and A=M(f ; B), then $A^2 = 0$.
2- Show that $Im(f) \subseteq Ker(f)$.
3- Deduce that $\dim_{\mathbb{R}}(Im(f)) = 1$ and $\dim_{\mathbb{R}}(Ker(f)) = 2$.
4- Let $\{u_1, u_2\}$ be a basis of Ker(f) over $\mathbb{R}$ and let $\{u_1, u_2, u_3\}$ be a basis of $\mathbb{R}^3$ containing the basis $\{u_1, u_2\}$ of Ker(f).
(i) Show that $f(u_3)$ is a non-zero vector of Ker(f).
(ii) Deduce that $\exists a,b \in \mathbb{R}$, such that $a \neq 0$ or $b \neq 0$ and $f(u_3) = au_1 + bu_2$.
(iii) Assume that $a \neq 0$. Let $v_1 = u_3$, $v_2 = f(u_3)$ and $v_3 = u_2$. Show that (v_1, v_2, v_3) is a basis of $\mathbb{R}^3$ over $\mathbb{R}$ and compute M(f ; (v_i)).

10- Let V be a finite dimensional K-vector space.

1- Show that If $P \in M_n(K)$ and B and C are two bases of V over K, then P is the transition matrix from B to C if and only if M(u ; B) = P×M(u ; C), $\forall u \in V$.
2- Deduce that if B, B' and B'' are bases of V over K and if P is the transition matrix from B to B' and Q that from B' to B'', then PQ is the transition matrix from B to B''.

11- Let $P = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & 2 \end{pmatrix}$ be the transition matrix from the basis B = (x_1, x_2, x_3) to the basis C = (y_1, y_2, y_3) of $\mathbb{R}^3$.

(a) Find y_1, y_2 and y_3 if $x_1 = (1,-1,1)$, $x_2 = (2,0,1)$ and $x_3 = (0,1,-2)$.
(b) Find x_1, x_2 and x_3 if $y_1 = (2,-1,1)$, $y_2 = (1,1,0)$ and $y_3 = (1,-3,0)$.

12- Let B = (e_1, e_2, e_3) be the canonical basis of $\mathbb{R}^3$. Let

$$x_1 = (2,1,0),\ x_2 = (1,1,0) \text{ and } x_3 = (1,1,1)$$
and
$$y_1 = (1,-1,2),\ y_2 = (1,0,2) \text{ and } y_3 = (0,1,1).$$

1- Show that the families C = (x_1, x_2, x_3) and D = (y_1, y_2, y_3) are two bases of $\mathbb{R}^3$.
2- Find the transition matrices P from B to C, Q from B to D, H from C to B and L from D to B.
3- Deduce the transition matrix T from C to D.

13- Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ and $g : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$ be the linear mappings defined by

$$f(x, y, z, t) = (2x-y+z-t, -x+y-z, -y+z-t)$$
and
$$g(x, y, z) = (x+z, -y+2z, 2y+3z, 2x+3y+2z).$$

1- Compute M(f ; B, C) and M(g ; C, B), where B and C are the canonical bases of $\mathbb{R}^4$ and $\mathbb{R}^3$ respectively.
2- Compute the matrix of $h = f \circ g$ relative to the basis C of $\mathbb{R}^3$.
3- Deduce that h is bijective.

14- Let E, F and G be finite dimensional K-vector spaces.

1- Show that if $f : E \longrightarrow F$ and $g : F \longrightarrow G$ are K-linear mappings, then

$$\operatorname{rank}(g \circ f) \le \operatorname{rank}(f) \text{ and } \operatorname{rank}(g \circ f) \le \operatorname{rank}(g).$$

2- Deduce that if A is a $(m\times n)$ matrix and B is a $(n\times p)$ matrix over K, then

$$\operatorname{rank}(AB) \le \operatorname{rank}(A) \text{ and } \operatorname{rank}(AB) \le \operatorname{rank}(B).$$

3- Deduce that if A is a $(m\times n)$ matrix over K and if $P \in M_m(K)$ and $Q \in M_n(K)$ are invertible matrices, then

$$\operatorname{rank}(PA) = \operatorname{rank}(AQ) = \operatorname{rank}(PAQ) = \operatorname{rank}(A).$$

APPENDIX I

PROOF OF SOME THEOREMS

4.1.1. The following are equivalent:
(i) The system is consistent.
(ii) rank(D) = r.
(iii) Every characteristic determinant of system (I) is zero.
(iv) $|D_t| = 0, \forall 1 \le t \le m$.

Proof: Let W be the space column of A. Since Δ is a non-zero minor of A of order r = rank(A), we then have that

$$\{C_1,...,C_r\} \text{ is a basis of W over K} \quad (*)$$

by 5.2.2. Let's now prove that (i) through (iv) are equivalent.

<u>(i)⇒(ii):</u> Let $\Omega = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ be a solution of system (I). Then

$$\alpha_1 C_1 + \cdots + \alpha_n C_n = B$$

so that $B \in W$. Let W' be the space column of D. Then $W \subseteq W'$. But $B \in W$ and $C_1,...,C_n \in W$, hence $W' \subseteq W$, and so W = W', whence rank(D) = $\dim_K(W)$ = r.

<u>(ii)⇒(iii):</u> Let $r+1 \le j \le m$. We have $\Delta_j = |D_j|$ is a minor of D of order r+1 and rank(D) = r, hence $\Delta_j = 0, \forall r+1 \le j \le m$.

<u>(iii)⇒(iv):</u> If $1 \le t \le r$, then $|D_t|$ has two equal rows, hence $|D_t| = 0, \forall 1 \le t \le r$. But $|D_t| = 0$, For all $r+1 \le t \le m$, hence $|D_t| = 0, \forall 1 \le t \le m$.

<u>(iv)⇒(i):</u> Let $1 \le t \le m$. Expanding $|D_t|$ through the last row, we get

$$|D_t| = a_{t1}M_1 + \cdots + a_{tr}M_r + b_t M_{r+1}$$

where for each $1 \le i \le r+1$, M_i is the (r+1,i)-th cofactor of D_t, hence

$$a_{t1}M_1 + \cdots + a_{tr}M_r + b_t M_{r+1} = 0, \forall 1 \le t \le m.$$

Since $M_{r+1} = \Delta$, $M_{r+1} \ne 0$. For each $1 \le i \le r$, let

$$\alpha_i = -M_i (M_{r+1})^{-1}.$$

Then

$$b_t = \alpha_1 a_{t1} + \cdots + \alpha_r a_{tr}, \forall 1 \le t \le m$$

and so

$$B = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} = \begin{pmatrix} \alpha_1 a_{11} + \cdots + \alpha_r a_{1r} \\ \vdots \\ \alpha_1 a_{m1} + \cdots + \alpha_r a_{mr} \end{pmatrix} = \alpha_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} + \cdots + \alpha_r \begin{pmatrix} a_{1r} \\ \vdots \\ a_{mr} \end{pmatrix} = \alpha_1 C_1 + \cdots + \alpha_r C_r.$$

Letting $\alpha_{r+1} = \cdots = \alpha_n = 0$, then $\Omega = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ is a solution of system (I). ■

4.1.2. If the system is consistent, then the solutions are all the column matrices $\Omega = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ over K, such that $\alpha_{r+1},\dots,\alpha_n$ are arbitrary in K and $\alpha_i = \frac{\Delta_{\alpha_i}}{\Delta}$, $\forall 1\le i\le r$, where Δ_{α_i} is the determinant obtained from Δ by replacing the i-th column by $\begin{pmatrix} d_1 \\ \vdots \\ d_r \end{pmatrix}$ with

$$d_s = b_s - (a_{s\,r+1}\alpha_{r+1} + \dots + a_{sn}\alpha_n)$$

for all $1\le s\le r$.

Proof: Let

$$T = \begin{pmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & & \vdots \\ a_{r1} & \cdots & a_{rr} \end{pmatrix}.$$

Let $\Omega = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ be a solution of system (I). Then

$$T\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix} = \begin{pmatrix} d_1 \\ \vdots \\ d_r \end{pmatrix}.$$

where $d_s = b_s - (a_{s\,r+1}\alpha_{r+1} + \dots + a_{sn}\alpha_n)$, $\forall 1\le s\le r$. Since $|T| = \Delta \neq 0$, T is invertible and $T^{-1} = \frac{1}{|T|}\operatorname{adj}(T)$, hence

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix} = T^{-1}\begin{pmatrix} d_1 \\ \vdots \\ d_r \end{pmatrix} = \frac{1}{\Delta}\operatorname{adj}(T)\begin{pmatrix} d_1 \\ \vdots \\ d_r \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} N_{11} & \cdots & N_{1r} \\ \vdots & & \vdots \\ N_{r1} & \cdots & N_{rr} \end{pmatrix}\begin{pmatrix} d_1 \\ \vdots \\ d_r \end{pmatrix} \qquad (*)$$

where for all $i,j\in\{1,\dots,r\}$, N_{ij} is the (i,j)-th cofactor of T, and so

$$\alpha_i = \frac{1}{\Delta}(N_{1i}d_1 + N_{2i}d_2 + \dots + N_{ri}d_r),\ \forall 1\le i\le r.$$

For each $1\le i\le r$ let Δ_{α_i} be the determinant obtained from Δ by replacing the i-th column by $\begin{pmatrix} d_1 \\ \vdots \\ d_r \end{pmatrix}$. Expanding Δ_{α_i} through the i-th column, we get that

$$\Delta_{\alpha_i} = N_{1i}d_1 + N_{2i}d_2 + \dots + N_{ri}d_r$$

hence

$$\alpha_i = \frac{\Delta_{\alpha_i}}{\Delta},\ \forall 1\le i\le r.$$

Now let $\alpha_1,\dots,\alpha_n \in K$, such that

$$\alpha_i = \frac{\Delta_{\alpha_i}}{\Delta},\ \forall 1\le i\le r$$

and let's show that $\Omega = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ is a solution of the system. Since

$$\Delta_{\alpha_i} = N_{1i}d_1 + N_{2i}d_2 + ... + N_{ri}d_r, \ \forall 1 \le i \le r$$

we then have that

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix} = \frac{1}{\Delta} \operatorname{adj}(T) \begin{pmatrix} d_1 \\ \vdots \\ d_r \end{pmatrix} = T^{-1} \begin{pmatrix} d_1 \\ \vdots \\ d_r \end{pmatrix}$$

hence

$$\begin{pmatrix} d_1 \\ \vdots \\ d_r \end{pmatrix} = T \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix}.$$

This yields that

$$a_{s1}\alpha_1 + \cdots + a_{sr}\alpha_r = d_s, \ \forall 1 \le s \le r$$

hence

$$a_{s1}\alpha_1 + \cdots + a_{sr}\alpha_r + a_{s\,r+1}\alpha_{r+1} + ... + a_{sn}\alpha_n = b_s, \ \forall 1 \le s \le r$$

and so the first r equations of the system are satisfied by $\alpha_1, ..., \alpha_n$. Let's show that the other equations of the system are satisfied by $\alpha_1, ..., \alpha_n$. For each $1 \le i \le m$ let R_i denote the i-th row of

$$D = \begin{pmatrix} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{pmatrix}.$$

Since the system has solutions, rank(D)= r, by 4.1.1. But Δ is a non-zero minor of D of order r and the entries of Δ are chosen from the first r rows of D, hence $\{R_1, ..., R_r\}$ is a basis of the space row of D. Let $r+1 \le t \le m$. Then $\exists \beta_1, ..., \beta_r \in K$, such that

$$R_t = \beta_1 R_1 + \cdots + \beta_r R_r$$

hence

$$(a_{t1}, \ldots, a_{tn}, b_t) = \beta_1(a_{11}, ..., a_{1n}, b_1) + \cdots + \beta_r(a_{r1}, \ldots, a_{rn}, b_r)$$

and so

$$a_{ti} = \beta_1 a_{1i} + \cdots + \beta_r a_{ri}, \ \forall 1 \le i \le n$$

and

$$b_t = \beta_1 b_1 + \cdots + \beta_r b_r.$$

But

$$\begin{gathered} a_{11}\alpha_1 + \cdots + a_{1n}\alpha_n = b_1 \\ \vdots \\ \vdots \\ a_{r1}\alpha_1 + \cdots + a_{rn}\alpha_n = b_r \end{gathered}$$

hence

$$\beta_1(a_{11}\alpha_1 + \cdots + a_{1n}\alpha_n) + \cdots + \beta_r(a_{r1}\alpha_1 + \cdots + a_{rn}\alpha_n) = \beta_1 b_1 + \cdots + \beta_r b_r$$

and so

$$(\beta_1 a_{11} + \cdots + \beta_r a_{r1})\alpha_1 + \cdots + (\beta_1 a_{1n} + \cdots + \beta_r a_{rn})\alpha_n = b_t.$$

However $a_{ti} = \beta_1 a_{1i} + \cdots + \beta_r a_{ri}$, $\forall 1 \le i \le n$. Therefore

$$a_{t1}\alpha_1+\cdots+a_{tn}\alpha_n = b_t,\ \forall r+1\le t\le m.$$

It follows that the equations $E_{r+1},\dots,E_m$ of the system are satisfied by $\alpha_1,\dots,\alpha_n$, and so $\Omega = \begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{pmatrix}$ is a solution of the system. ■

6.2.3. Let W be a subspace of E and let S be a system of generators of W over K. Suppose that there exists a non-zero natural number m, such that

$$\text{card}(X)\le m, \text{ for every free subset X of S.}$$

If $A\subseteq S$ and A is free over K, then there exists a finite basis B of W, such that $A\subseteq B\subseteq S$.

Proof: Let H be the set of all free subsets X of S, satisfying $A\subseteq X\subseteq S$ and let

$$T=\{\text{card}(X)\ ;\ X\in H\}.$$

As $A\in H$, then $\text{card}(A)\in T$, and so

$$T\neq\varnothing.$$

We have $\text{card}(X)\le m$, for all $X\in H$, hence T is a finite subset of $\mathbb{N}$, and so it has a greatest element n, say. Let $B\in H$, such that card(B) = n. We shall show that B is a basis of W over K. To do this Put $B=\{x_1,\dots,x_n\}$ and let $x\in S$. Suppose that $x\notin L(B)$, then $B\cup\{x\}$ is free, by 6.1.6. But $A\subseteq B\cup\{x\}\subseteq S$, hence $B\cup\{x\}\in H$, and so

$$\text{card}(B\cup\{x\})\le n.$$

We have that $x\notin L(B)$, hence $x\notin B$, and so $\text{card}(B\cup\{x\})= \text{card}(B)+\text{card}(\{x\}) = n+1$, whence $n+1\le n$, a contradiction. Therefore $x\in L(B)$, for all $x\in S$, and so $S\subseteq L(B)$. This implies that $L(S)\subseteq L(B)$. But L(S)=W, hence $W\subseteq L(B)$, and as $L(B)\subseteq W$, then W=L(B), and so B is a system of generators of W, and finally B is a finite basis of W over K, as required. ■

6.3.1. Let $A=(a_{ij})$ be $(m\times n)$ matrix over K. Denote by C_i (resp. L_i) the ith column (resp. row) of A. Let W (resp W') be the column (resp. row) space of A. If rank(A)=r and

$$\Delta=\begin{vmatrix} a_{i_1j_1} & a_{i_1j_2} & \cdots & a_{i_1j_r}\\ a_{i_2j_1} & a_{i_2j_2} & \cdots & a_{i_2j_r}\\ \vdots & \vdots & & \vdots\\ a_{i_rj_1} & a_{i_rj_2} & \cdots & a_{i_rj_r}\end{vmatrix}$$

is a non-zero minor of A of order r, then

(i) $\{C_{j_1},\dots,C_{j_r}\}$ is a basis of W over K and

(ii) $\{L_{i_1},\dots,L_{i_r}\}$ is a basis of W' over K.

Proof: As $\Delta\neq 0$, then

$$C_{j_1},\dots,C_{j_r} \text{ are linearly independent over K}$$

and

$$L_{i_1},\dots,L_{i_r} \text{ are linearly independent over K}$$

by 6.1.7.1. For each $1\le q\le m$ and $1\le s\le n$, let

$$D_{qs} = \begin{pmatrix} a_{i_1j_1} & \cdots & a_{i_1j_r} & a_{i_1s}\\ \vdots & & \vdots & \vdots\\ a_{i_rj_1} & \cdots & a_{i_rj_r} & a_{i_rs}\\ a_{qj_1} & \cdots & a_{qj_r} & a_{qs}\end{pmatrix}.$$

If $q\in\{i_1,\dots,i_r\}$ (resp. $s\in\{j_1,\dots,j_r\}$), then D_{qs} has two equal rows (resp. columns), hence

$$\left|D_{qs}\right| = 0 \text{ if } q\in\{i_1,...,i_r\} \text{ or } s\in\{j_1,...,j_r\}. \quad (1)$$

Suppose that $q\notin\{i_1,...,i_r\}$ and $s\notin\{j_1,...,j_r\}$. Let T be the minor of A whose entries are chosen from the columns $C_{j_1},...,C_{j_r}$ and C_s and from the rows $L_{i_1},...,L_{i_r}$ and L_q. In T consider the column C of T containing the entry a_{qs}. First we shall move C to the right until it becomes the last one. If it is not in that place we interchange it with the column on its right and we repeat this as necessarily until we get it in that place. Let T' be the obtained determinant, then

$$T'=(-1)^{\alpha}T,$$

where α is the number of the columns of T that are on the right of C. In T', consider the row L containing the entry a_{qs}. We shall move L to the bottom of T'. If it is not at the bottom, t we interchange it with row under it and we repeat this as necessarily until L becomes at the bottom. When L is at the bottom, then the obtained determinant is $\left|D_{qs}\right|$. Let β be the number of the rows of T' that are below L, then

$$\left|D_{qs}\right|=(-1)^{\beta}T'.$$

As T is a minor of A of order r+1 and rank(A)=r, then T=0, and so T'=0, and consequently $\left|D_{qs}\right|=0$. Therefore

$$\left|D_{qs}\right| = 0 \text{ if } q\notin\{i_1,...,i_r\} \text{ and } s\notin\{j_1,...,j_r\}. \quad (2)$$

Combining (1) and (2), we get

$$\left|D_{qs}\right| = 0,\ \forall 1\le q\le m \text{ and } \forall 1\le s\le n.$$

Fixing s and expanding $\left|D_{qs}\right|$ through the last row, we obtain

$$\left|D_{qs}\right| = a_{qj_1}M_1+\cdots+a_{qj_r}M_r+a_{qs}M_{r+1}$$

where for each $1\le i\le r+1$, M_i denotes the (r+1,i)-th cofactor of D_{qs}, hence

$$a_{qj_1}M_1+\cdots+a_{qj_r}M_r+a_{qs}M_{r+1} = 0,\ \forall 1\le q\le m \text{ and } \forall 1\le s\le n.$$

Since $M_{r+1}=\left|B\right|$, $M_{r+1}\neq 0$, and so letting for each $1\le i\le r$

$$\alpha_i = -M_i(M_{r+1})^{-1}$$

then $\alpha_1,...,\alpha_r$ depend upon s and are independent of q, and so

$$a_{qs} = \alpha_1 a_{qj_1}+\cdots+\alpha_r a_{qj_r},\ \forall 1\le q\le m.$$

This implies that

$$C_s = \begin{pmatrix} a_{1s} \\ \vdots \\ a_{ms} \end{pmatrix} = \begin{pmatrix} \alpha_1 a_{1j_1}+\cdots+\alpha_r a_{1j_r} \\ \vdots \\ \alpha_1 a_{mj_1}+\cdots+\alpha_r a_{mj_r} \end{pmatrix} = \alpha_1\begin{pmatrix} a_{1j_1} \\ \vdots \\ a_{mj_1} \end{pmatrix}+\cdots+\alpha_r\begin{pmatrix} a_{1j_r} \\ \vdots \\ a_{mj_r} \end{pmatrix}$$

$$= \alpha_1 C_{j_1}+\cdots+\alpha_r C_{j_r}$$

hence $C_s \in \mathrm{Vect}(C_{j_1},...,C_{j_r})$, for all $s\in\{1,\ldots,n\}-\{j_1,...,j_r\}$, and so

$$\mathrm{Vect}(\{C_1,...,C_n)=\mathrm{Vect}(C_{j_1},...,C_{j_r})$$

by 5.4.4.2, whence $\{C_{j_1},...,C_{j_r}\}$ is a system of generators of W over K, and thence

$\{C_{j_1},...,C_{j_r}\}$ is a basis of W over K.

Similarly, by expanding $|D_{qs}|$ through the last column, we get that every row is a linear combination over K of $L_{i_1},\dots,L_{i_r}$, whence $\{L_{i_1},\dots,L_{i_r}\}$ is a system of generators of W', and so it is a basis of W' over K. ■

6.3.2. If $\{x_1,\dots,x_n\}$ is a basis of E over K and if m is a natural number with m>n, then any m elements $y_1, \dots, y_m$ of E are linearly dependent over K.

Proof: Suppose that $y_1, \dots, y_m$ are linearly independent and Let

$$\begin{aligned} y_1 &= a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n \\ y_2 &= a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n \\ &\vdots \\ y_m &= a_{m1}x_1+a_{m2}x_2+\cdots+a_{mn}x_n \end{aligned}$$

where $a_{ij}\in K$, $\forall i,j$ and set

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

Put r = rank(A) and let

$$\Delta=\begin{vmatrix} a_{i_1j_1} & a_{i_1j_2} & \cdots & a_{i_1j_r} \\ a_{i_2j_1} & a_{i_2j_2} & \cdots & a_{i_2j_r} \\ \vdots & \vdots & & \vdots \\ a_{i_rj_1} & a_{i_rj_2} & \cdots & a_{i_rj_r} \end{vmatrix}$$

be a non-zero minor of A of order r. Let W be the row space of A and for each $1\le i\le m$, let L_i denote the ith row of A, then

$$\{L_{i_1},\dots,L_{i_r}\} \text{ is a basis of W over K}$$

by 6.3.1. As m>n and r≤n, then r<m, and so $\exists q\in\{1, \dots, m\}-\{\{i_1, \dots, i_r\}$. We have that L_q is a linear combination over K of $L_{i_1}, \dots, L_{i_r}$ hence $L_q, L_{i_1}, \dots, L_{i_r}$ are linearly dependent over K, and so one can easily verify that

$$y_q, y_{i_1}, \dots, y_{i_r} \text{ are linearly dependent over K.}$$

But $y_q, y_{i_1}, \dots, y_{i_r}$ are pairwise distinct elements of $\{y_1, \dots, y_m\}$, hence they are linearly independent over K, by 6.1.5, impossible. Thus $y_1,\dots, y_m$ are linearly dependent over K. ■

6.4.2. Let A be a (n×m) matrix over K. If $E_1, \dots, E_r \in M_n(K)$ are elementary matrices, then the matrices A and $(E_1\times\cdots\times E_r)\times A$ have the same row space. In particular if B is a row echelon form of A, then A and B have the same row space.

Proof: We argue by induction on r. Consider the case r=1 and put $C= E_1\times A$.
For each $1\le i\le m$, let L_i (resp. D_i) be the ith row of A (resp. C). Let W be the row space of A and W' be that of C. We have

$$C=A(R_t\leftrightarrow R_s) \text{ or } C=A(R_t\to R_t+\alpha R_s) \text{ with } s\ne t$$
$$\text{or } C=A(R_t\to \alpha R_t) \text{ with } \alpha\in K-\{0\}.$$

If $C=A(R_t\leftrightarrow R_s)$, then A and C have the same rows, and so W'=W.

Assume that $C=A(R_t \to R_t+\alpha R_s)$ with $s\neq t$. We have

$$D_i=L_i,\ \forall i\neq t \text{ and } D_t=L_t+\alpha L_s$$

hence $D_i \in W$, $\forall 1\leq i\leq m$, and so

$$W'\subseteq W.$$

We have $L_i=D_i$, $\forall i\neq t$, hence $L_i \in W'$, $\forall i\neq t$. As $s\neq t$, then $L_s \in W'$, and so $L_t \in W'$, whence $L_i \in W'$, $\forall 1\leq i\leq m$, and consequently $W\subseteq W'$. Thus W'=W' in this case.

Assume that $C=A(R_t \to \alpha R_t)$ with $\alpha\in K-\{0\}$. We have

$$D_i=L_i,\ \forall i\neq t \text{ and } D_t=\alpha L_t$$

hence $L_i \in W$, $\forall 1\leq i\leq m$, and so

$$W'\subseteq W.$$

We have $L_i=D_i$, $\forall i\neq t$, hence $L_i \in W'$, $\forall i\neq t$. As $\alpha\neq 0$, then $L_t=\alpha^{-1}D_t$, hence $L_t \in W'$, and so $L_i \in W'$, $\forall 1\leq i\leq m$, whence $W\subseteq W'$. Therefore W'=W, and so the property is true for r=1. Assume that it holds up to r-1 and let's prove it for r. Let $C=E_r\times A$, then

C and A have the same row space.

But C and $(E_1\times\cdots\times E_{r-1})\times C$ have the same row space, by induction hypothesis, hence

A and $(E_1\times\cdots\times E_{r-1})\times C$ have the same row space.

As $(E_1\times\cdots\times E_{r-1})\times C=(E_1\times\cdots\times E_r)\times A$, then the property holds for r, and so it holds, for all $r\geq 1$.

As B is obtained from A by a finite sequence of elementary row operations, then $B=(E_1\times\cdots\times E_r)\times A$, for some elementary matrices $E_1, \ldots, E_r \in M_n(K)$, and so A and B have the same row space. ■

7.2.5. (Theorem of dimensions): If E is finite dimensional over K and $f: E \longrightarrow F$ is K-linear, then

$$\dim_K(E) = \dim_K(\mathrm{Ker}(f))+\dim_K(\mathrm{Im}(f)).$$

Proof: Since Ker(f) is a subspace of E, by 7.2.1(i), we then have that

$$\dim_K(\mathrm{Ker}(f))\leq\dim_K(E)$$

by 6.3.5(i). Let

$$n = \dim_K(E) \text{ and } s = \dim_K(\mathrm{Ker}(f)).$$

If s=n, then Ker(f)=E, by 6.3.5(iii), whence $\mathrm{Im}(f)=\{0_F\}$, and so the relation is true in this case. Suppose that $s\neq n$. Then $s<n$. If s=0, then $\mathrm{Ker}(f)=\{0_E\}$, whence f is injective, by 7.2.2(i), and so $\dim_K(E)=\dim_K(\mathrm{Im}(f))$, by 7.2.4, and as $\dim_K(\mathrm{Ker}(f)) = 0$, then the relation is true.

Assume that $s\neq 0$. Then $\mathrm{Ker}(f)\neq\{0_E\}$, and so Ker(f) has a basis over K. Let $\{x_1,\ldots,x_s\}$ be a basis of Ker(f) over K. Since $x_1,\ldots,x_s$ are elements of E linearly independent over K, we then have that there exists a basis $B=\{x_1,\ldots,x_s,x_{s+1},\ldots,x_n\}$ of E over K containing $\{x_1,\ldots,x_s\}$, by 6.3.2.3(ii). We shall show that $\{f(x_{s+1}),\ldots,f(x_n)\}$ is a basis of Im(f) over K. Let $a_{s+1},\ldots,a_n \in K$, such that

$$a_{s+1}f(x_{s+1})+\cdots+a_nf(x_n) = 0_F.$$

Then

$$f(a_{s+1}x_{s+1}+\cdots+a_nx_n) = 0_F$$

by 7.1.2(iii), and so $a_{s+1}x_{s+1}+\cdots+a_n x_n \in \mathrm{Ker}(f)$. But $\{x_1,...,x_s\}$ is a basis of Ker(f) over K, hence $\exists b_1,...,b_s \in K$, such that

$$a_{s+1}x_{s+1}+\cdots+a_n x_n = b_1 x_1+\cdots+b_s x_s .$$

This yields that

$$b_1 x_1+\cdots+b_s x_s+(-a_{s+1})x_{s+1}+\cdots+(-a_n)x_n = 0_F$$

hence

$$b_1=\cdots=b_s = -a_{s+1} =\cdots= -a_n = 0_K$$

and so

$$a_{s+1} =\cdots= a_n = 0_K .$$

It follows that

$$f(x_{s+1}),...,f(x_n) \text{ are linearly independent over K.}$$

Let $y\in \mathrm{Im}(f)$. Then $\exists x\in E$, such that

$$y = f(x).$$

But $\{x_1,...,x_s,x_{s+1},...,x_n\}$ is a basis of E over K, hence there exist $\alpha_1,...,\alpha_s,\alpha_{s+1},...,\alpha_n$ in K, such that

$$x = \alpha_1 x_1+\cdots+\alpha_s x_s+\alpha_{s+1}x_{s+1}+\cdots+\alpha_n x_n .$$

This implies that

$$y = \alpha_1 f(x_1)+\cdots+\alpha_s f(x_s)+\alpha_{s+1}f(x_{s+1})+\cdots+\alpha_n f(x_n)$$

by 7.1.2(iii). As $x_1,...,x_s \in \mathrm{Ker}(f)$, then

$$f(x_1) =\cdots= f(x_s) = 0_F$$

and so

$$y = \alpha_{s+1}f(x_{s+1})+\cdots+\alpha_n f(x_n).$$

Therefore $f(x_{s+1}),...,f(x_n)$ form a system of generators of Im(f) over K, by 5.4.3, and so $\{f(x_{s+1}),...,f(x_n)\}$ is a basis of Im(f) over K. It follows that

$$\dim_K(\mathrm{Im}(f)) = \mathrm{card}(\{f(x_{s+1}),...,f(x_n)\}) = n-s.$$

However $\dim_K(E) = n$ and $\dim_K(\mathrm{Ker}(f)) = s$. Therefore

$$\dim_K(E) = \dim_K(\mathrm{Ker}(f))+\dim_K(\mathrm{Im}(f)). \blacksquare$$

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