Lebanese University

Faculty of Sciences

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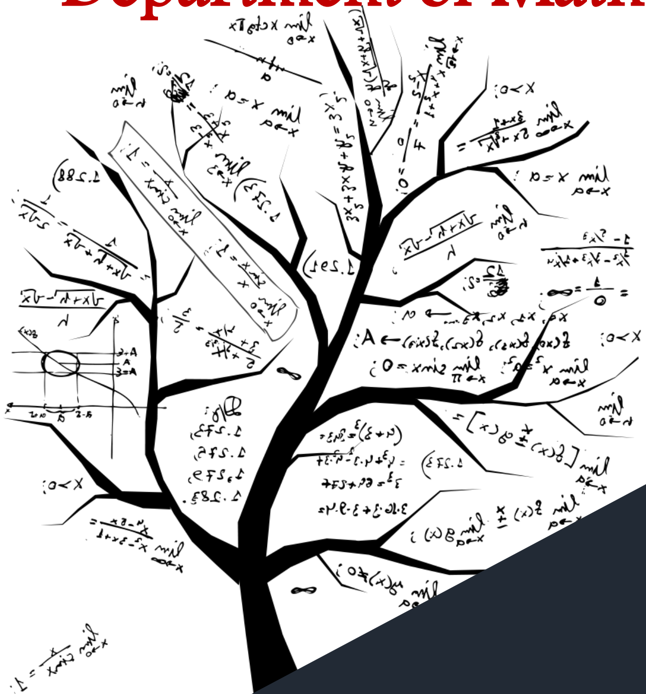
M1106
Analysis IV

Department of Mathematics



Spring Semester

2018 ~ 2019

The work of the authors is dedicated to
their families ...

Preface

All **MIS** students taking the course Math **M1101** in the fall, also follow the course Math **M1106** in the spring, which mainly aims to consolidate the achievements of high school to serve the exact sciences, as a result of teaching mathematical. **In contrast**, here we adopt a mathematicians style: highlighting of foundations like the axioms and definitions, abstract concepts, demonstrations. This course aimed to **MIS** students of the Faculty of Sciences at the Lebanese University. It is quite detailed and contains supplements that sometimes go beyond the planned program. Like any course of mathematics, it must be read with a pen and a blank sheet of paper by hand to check setp by step that all assertions are correct. Each chapter of the course contains uncorrected exercises **"without moderation"**. Indeed, in addition to the exercises in this book, which will be corrected in plenary **tutorials**, it is advisable to practice solve by itself these exercises without having a solution: it is through this essential personal work that we can go further in the understanding and assimilation of mathematical concepts introduced. **It is the only known method to date to progress in mathematics**.

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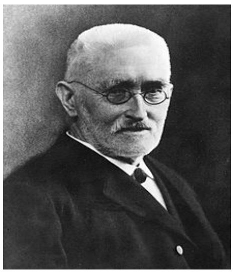
Chapter 1

Introduction to the Topology of $\mathbb{R}$

Motivation



Julius Wilhelm Richard Dedekind (1831-1916) is a German mathematician. Pioneer axiomatization of arithmetic, he proposed an axiomatic definition of the set of integers and a rigorous construction of real numbers from real rational numbers.

The **topology of the real line** is a mathematical structure that gives, for all real numbers, precise definitions to the concepts of limit and continuity. The subsets of $\mathbb{R}$ which play an essential role in the topology are on one hand the set of rational numbers, on which it is constructed, and the other intervals, in which the topology is constructed. We will simply give some definitions of $\mathbb{R}$:

- $\mathbb{R}$ is the set of all Dedekind cuts, where a Dedekind cut is a part D of $\mathbb{Q}$ having the following properties:

  1. $D \neq \varnothing$ et $D \neq \mathbb{Q}$;
  2. If $r \in D$ and $r' \in \mathbb{Q}$ with $r' < r$ then $r' \in D$;
  3. D does not have a greatest element.

- An axiomatic definition:
  $\mathbb{R}$ is the totally ordered field in which any nonempty part bounded from above admits an upper bound.

1.1 Definitions and proprerties in $\mathbb{R}$

Definition 1.1.1. *Let A be a subset of $\mathbb{R}$.*

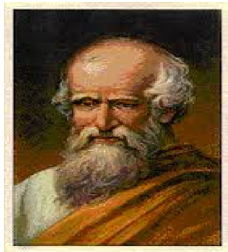
1. *A point x in $\mathbb{R}$ is said to be an* **interior** *point of A if there exists $\delta > 0$ such that*
$$]x-\delta, x+\delta[\subseteq A.$$

2. *A point x in $\mathbb{R}$ is said to be an* **adherent** *point of A if for all $\delta > 0$, we have*
$$]x-\delta,\, x+\delta[\cap A \neq \phi.$$

3. *A point x in $\mathbb{R}$ is said to be an* **accumulation** *point of A if*
$$\forall r > 0,\ \Big(]x-r,\, x+r[\setminus\{x\}\Big) \cap A \neq \phi.$$

4. *A point x in $\mathbb{R}$ is said to be a* **boundary** *point of A if it is an adherent point but not an interior point.*

We would like to draw readers' attention to the following definition: Each interval in the form $]x-r,\, x+r[$ is called **open ball** of center x and radius r.



Archimedes of Syracuse, born in Syracuse to 287 BC and died in this same city in 212 BC, is a great Greek scientist Sicily (Magna Grecia) of antiquity, physicist, mathematician and engineer.

Property 1.1.1. (Archimedean property)
$\mathbb{R}$ *is* ***Archimedean****, that is:*
$$\forall x > 0,\ \forall y \in \mathbb{R},\ \exists n \in \mathbb{N} \,/\, y < nx.$$

Proof. Let's argue by contradiction. Suppose that there are $x > 0$ and $y \in \mathbb{R}$ such that $y \geq nx$ for all $n \in \mathbb{N}$. Thus, the subset $A = \{nx;\ n \in \mathbb{N}\}$ of $\mathbb{R}$ is nonempty and bounded from above. Let M its upper bound. Then, we have $M - x < nx$ and $M < (n+1)x$. It is a contradiction because $(n+1)x \in A$. $\square$

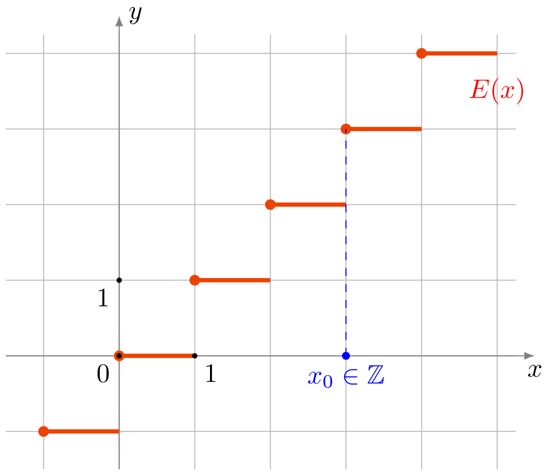
Example 1.1.1.

- $\forall \varepsilon > 0,\ \exists n \in \mathbb{N}^* \,/\, \dfrac{1}{n} < \varepsilon.$
- $\forall \varepsilon > 0,\ \exists n \in \mathbb{N} \,/\, \dfrac{1}{(2n+1)^3} < \varepsilon.$
- $\forall \varepsilon > 0,\ \exists n \in \mathbb{N}^* \,/\, \dfrac{7}{\sqrt[3]{n}+4n} < \varepsilon.$

The Archimedean property has the following consequence:

Proposition 1.1.1. *Let x be a real number. There exists a unique relative integer n such that*
$$n \leq x < n+1.$$
The integer n is called, by definition, the **integer part** *of x. We shall denote it by $E(x)$.*

Proof. Let $x \in \mathbb{R}$ be fixed and $A := \{q \in \mathbb{Z};\ q \leq x\}$.
By the Archimedean property, there exists $n_0 \in \mathbb{N}$ such that $|x| \leq n$. Thus, $A \neq \varnothing$ because $-n_0 \in A$. Therefore, it has a greatest element n. Hence, $n \leq x$ since $n \in A$. On the other hand, $x < n+1$, because if $x \geq n+1$ then $n+1 \in A$ and $n+1 \leq n$. This is a contradiction. Hence existence.
Now, we will prove the uniqueness of n. Indeed, if there exists $m \in \mathbb{Z}$ such that $m \leq x < m+1$ then $n < m+1$ and $m < n+1$. Thus $n \leq m$ and $m \leq n$ and so $m = n$. $\square$



Graph of $y = E(x)$

We can remark that:

$$E(x) = \max\{n \in \mathbb{Z};\ n \leq x\}.$$

Example 1.1.2.

1. *The point* $\dfrac{1}{2}$ *is an interior point of* $[0,\, 1]$*, but the points* 0 *and* 1 *do not.*
2. *The point* 1 *is an adherent and a boundary point of* $]0,\, 1[$.
3. *The point* 0 *is accumulation point of the set* $A = \left\{\dfrac{1}{n};\ n \in \mathbb{N}^*\right\}$. *Indeed, for every* $r > 0$*, there exists* $n_0 \in \mathbb{N}^*$ *such that* $\dfrac{1}{n_0} < r$. *Then,*

$$\big(A \cap]-r,\, r[\big) \setminus \{0\} \supseteq \big(A \cap]-\frac{1}{n_0},\, \frac{1}{n_0}[\big) \setminus \{0\} \ni \frac{1}{2n_0}.$$

Consequently, $\big(A \cap]-r,\, r[\big) \setminus \{0\} \neq \varnothing$ *and* 0 *is an accumulation of* A.

One can remark easily that accumulation point of a part is an adherent point of the same part.

Definition 1.1.2. *Every limit of a subsequence of a sequence* (u_n) *is called* ***cluster point*** *of* (u_n).

Example 1.1.3. *For the sequence* $\Big((-1)^n\Big)$ *the values* ± 1 *are cluster points, but they are not accumulation points of the set* $\{-1,\, +1\}$.

Remark 1.1.1. *The cluster points of a sequence* (u_n) *are adherent points of the set* $\{u_n;\; n \in \mathbb{N}\}$.

Exercise 1.1.1. (Mini-PW)
Show that an adherent point of the set $\{u_n;\; n \in \mathbb{N}\}$ *is not necessarily a cluster point of the sequence* (u_n).

Definition 1.1.3.

1. *A subset* $\mathcal{V}$ *of* $\mathbb{R}$ *is a* ***neighborhood*** *of a point* x *if there exists* $\delta > 0$ *such that*
$$]x-\delta,\, x+\delta[\subseteq \mathcal{V}.$$
2. *A subset* A *of* $\mathbb{R}$ *is an* ***open*** *if for all* $x \in A$*, there exists* $\delta_x > 0$ *such that*
$$]x-\delta_x,\, x+\delta_x[\subseteq A.$$
3. *A subset* A *of* $\mathbb{R}$ *is* ***closed*** *if its complement is an open.*

Example 1.1.4. *Each interval* $]a,b[\subseteq \mathbb{R}$ *is an open because if* $x \in]a,b[$ *then*
$$]x-\delta,\, x+\delta[\, \subset \,]a,\, b[$$
if we take $\delta = \min\{b-x,\, x-a\}$.

Proposition 1.1.2. *A subset* A *of* $\mathbb{R}$ *is closed if and only if the limit of any convergent sequence of* A *belongs to* A.

Proof. Let (x_n) be a sequence of a closed subset A converges to a real x. Then each neighborhood of x contains the points x_n of A. Thus, $x \in A$, because if $x \in A^c$ complement of A in $\mathbb{R}$ which is open, then A^c is a neighborhood of x, does not contain points of A. Which is impossible.
Conversely, suppose that the limit of each convergent sequence (x_n) of A belongs to A. Let $x \in A^c$. If for every $\delta > 0$, we have
$$]x-\delta,\, x+\delta[\, \nsubseteq A^c$$

then there exists $y_\delta \in A \cap]x-\delta, x+\delta[$. Hence, for all $n \in \mathbb{N}$ there exists $x_n \in A$ such that
$$x_n \in]x - \frac{1}{n}, x + \frac{1}{n}[.$$
This implies that $x = \lim_{n \longrightarrow \infty} x_n \in A$. Which is a contradiction. Consequently, A^c is open and A is closed. □

Example 1.1.5.

1. *The set of integers* $\mathbb{N}$ *(resp.* $\mathbb{Z}$*) is a closed of* $\mathbb{R}$ *since the convergent sequences* (x_n) *in* $\mathbb{N}$ *(resp.* $\mathbb{Z}$*) are stationary, then they converge in* $\mathbb{N}$ *(resp.* $\mathbb{Z}$*).*

2. *Each segment* $[a, b] \subseteq \mathbb{R}$ *is a closed, because if we consider a convergent sequence* (x_n) *of* $[a, b]$*, we have for all integer* n*,*

$$a \leq x_n \leq b \Rightarrow a \leq \lim_{n \longrightarrow \infty} x_n \leq b \Rightarrow \lim_{n \longrightarrow \infty} x_n \in [a, b].$$

1. Show that the unbounded intervals $]-\infty, a]$ and $[a, +\infty[$ are closed in $\mathbb{R}$.

2. Show that the intervals $]-\infty, a[$ and $]a, +\infty[$ are open in $\mathbb{R}$.

3. Show that the set of all rationnel numbers $\mathbb{Q}$ is not closed, nor open.

Proposition 1.1.3. (Characterisation of the accumulation points using the sequences)
Let A *be a subset of* $\mathbb{R}$*. A point* x *of* $\mathbb{R}$ *is an accumulation point of* A*, if and only if, there exists a sequence of elements of* A *different from* x *which converges to* x*.*

Proof. Let x be an accumulation point of A. Then, for all $n \in \mathbb{N}$, there exists $x_n \in A \setminus \{x\}$ such that
$$x_n \in]x - \frac{1}{n}, x + \frac{1}{n}[.$$
This allows us to conclude that $\lim_{n \longrightarrow \infty} x_n = x$.
Conversely, let (x_n) be a sequence of elements of A different from x which converges to x. Then, for all $r > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, we have $x_n \in]x - r, x + r[\setminus \{x\}$. This allows us to conclude that x is an accumulation point. □

Proposition 1.1.4. *A subset* A *of* $\mathbb{R}$ *is closed, if and only if, it contains all its accumulation points.*

Proof. Let A be a closed subset of $\mathbb{R}$ and x be an accumulation point. By the proposition 1.1.3, there exists a sequence (x_n) of A such that $x_n \neq x$ for all n and $\lim\limits_{n\longrightarrow\infty} x_n = x$. Hence, we conclude from the proposition 1.1.2 that $x \in A$.
Conversely, wen suppose that each accumulation point of A belongs to A. Let $x \in A^c$. If, for every $\delta > 0$, we have

$$]x-\delta,\, x+\delta[\, \not\subseteq A^c$$

then

$$\Big(]x-\delta,\, x+\delta[\backslash\{x\}\Big) \cap A \neq \phi$$

and x is an accumulation point of A. Thus, $x \in A$. This is a contradiction. Hence A^c is open and A is closed. □

Definition 1.1.4.

1. *An open* ***cover*** *of a subset A of $\mathbb{R}$ is a family of open $(\mathcal{O}_i)_{i\in I}$ such that $A \subseteq \bigcup\limits_{i\in I} \mathcal{O}_{i\in I}$, where I is a subset of $\mathbb{N}$.*
2. *A subset A of $\mathbb{R}$ is said to be* ***compact****, if and only if, from all cover of A by open subsets, we can extract a finite subcovert i.e. if $(\mathcal{O}_i)_{i\in I}$ is a family of open such that $A \subseteq \bigcup\limits_{i\in I} \mathcal{O}_{i\in I}$, then there exists $i_0,\, i_1,\, ...,\, i_n \in I$ for some $n \in \mathbb{N}$ such that $A \subseteq \bigcup\limits_{k=1}^{n} \mathcal{O}_{i_k}$.*

Definition 1.1.5. (Sequential compactness)
A subset A of $\mathbb{R}$ is said to be ***compact*** *if every sequence of A has a convergent subsequence in A.*

Example 1.1.6.

1. $\Big(]-n,\, n[\Big)_{n\in\mathbb{N}}$ *is a cover of $\mathbb{R}$.*
2. $\mathbb{R}$ *is not compact.*
3. *The interval $]0,\, 1[$ is not compact, since the sequence $\left(\frac{1}{n}\right)_{n\geq 1}$ converges to 0 and all its subsequences converge to $0 \notin]0,\, 1[$.*
4. *The subset $\mathbb{N}$ is not compact, since the sequence $(n)_n$ convrges to infinity and so all its subsequences converge to infinity.*

Proposition 1.1.5. ***(Admitted)***
Let A be a subset of $\mathbb{R}$. Then, A is compact if and only if A is bounded and closed.

1.1.1 Density of $\mathbb{Q}$ in $\mathbb{R}$

Definition 1.1.6. *Let $A \subseteq X \subseteq \mathbb{R}$. We say that A is* **dense** *in X, if each nonempty open interval I of X intersects A, so that it contains at least an element of A.*

Proposition 1.1.6. *Let A be a subset in $\mathbb{R}$. Then, A is dense in $\mathbb{R}$, if and only if, each point in $\mathbb{R}$ is the limit of a sequence of elements of A. In other word, for all $x \in \mathbb{R}$, there exists a sequence (x_n) in A such that $\lim\limits_{n \longrightarrow \infty} x_n = x$.*

Proof. Let A be a subset in $\mathbb{R}$ and $x \in \mathbb{R}$. Suppose that A is dense in $\mathbb{R}$. Then, for every $n \in \mathbb{N}$, there exists a real $x_n \in A$ such that

$$x - \frac{1}{2^n} < x_n < x + \frac{1}{2^n} \quad i.e. \quad \left| x_n - x \right| < \frac{1}{2^n}.$$

Hence, we find a sequence (x_n) of A converges to x.
Conversely, suppose that each element of $\mathbb{R}$ is the limit of a sequence of elements of A. Let us prove that A is dense in $\mathbb{R}$. Let x and y be two reals such that $x < y$. Therefore, there exists a sequence (x_n) of A such that $\lim\limits_{n \longrightarrow \infty} x_n = \dfrac{x+y}{2}$. This implies that there exists $n_0 \in \mathbb{N}$ such that

$$\left| x_{n_0} - \frac{x+y}{2} \right| < \frac{y-x}{2}.$$

Thus $x < x_{n_0} < y$ and A is dense in $\mathbb{R}$. $\square$

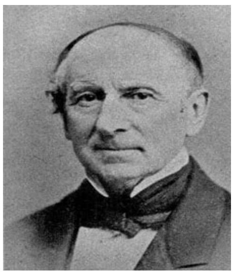
Theorem 1.1.1.

1. *$\mathbb{Q}$ is dense in $\mathbb{R}$.*
2. *$\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.*

Proof. Firstly, we want to prove the assertion 1. Let x and y be two reals such that $x < y$. We must prove that there exists $r \in \mathbb{Q}$ such that $x < r < y$. By the Archimedean property, there exists $n \in \mathbb{N}^*$ such that $\dfrac{1}{n} < y - x$. On the other hand, we have $E(nx) \leq nx < E(nx) + 1$. This implies that

$$\frac{E(nx)}{n} \leq x < \frac{E(nx)}{n} + \frac{1}{n} < x + (y - x) = y.$$

Therefore, we find $r = \dfrac{E(nx)}{n} + \dfrac{1}{n} \in \mathbb{Q}$ such that $x < r < y$.



Augustin Louis Cauchy (1789-1857) was a great French mathematician of the early 19th-century. Professor at the Polytechnic School and the Sorbonne.

Now, we want to prove the assertion 2. Let x and y be two reals such that $x < y$. As the previous argument, We must prove that there exists $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ such that $x < \alpha < y$. The density of $\mathbb{Q}$ in $\mathbb{R}$ ensures the existence of $r \in \mathbb{Q}$ such that $\dfrac{x}{\sqrt{2}} < r < \dfrac{y}{\sqrt{2}}$. Thus, $x < \sqrt{2}r < y$. Hence, we find $\alpha = \sqrt{2}r \in \mathbb{R} \setminus \mathbb{Q}$ such that $x < \alpha < y$. □

1.2 Classical Inequalities

Some inequalities have become famous for their great help. They are also the main ingredients for solving many mathematical, physical, and chemical problems. In this section, we are interested to present two famous inequalities: **Cauchy-Schwarz** inequality and **Minkowski** inequality.

Theorem 1.2.1. (Cauchy-Schwarz inequality)
For all n-tuples $(x_1, x_2, ..., x_n)$, $(y_1, y_2, ..., y_n)$ of $\mathbb{R}^n$, we have

$$\left|\sum_{k=1}^{n} x_k y_k\right| \leq \left(\sum_{k=1}^{n} x_k^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} y_k^2\right)^{\frac{1}{2}}.$$

Proof. Let t un réel et $(x_1, x_2, ..., x_n)$, $(y_1, y_2, ..., y_n)$ n-tuples of $\mathbb{R}^n$. The trinomial

$$\sum_{k=1}^{n}(x_k + ty_k)^2 = \Big(\sum_{k=1}^{n} y_k^2\Big)t^2 + 2\Big(\sum_{k=1}^{n} x_k y_k\Big)t + \sum_{k=1}^{n} x_k^2 \geq 0.$$

This implies that its reduced discriminant

$$\triangle' = \left(\sum_{k=1}^{n} x_k y_k\right)^2 - \left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n} y_k^2\right) \leq 0,$$

then

$$\left|\sum_{k=1}^{n} x_k y_k\right| \leq \left(\sum_{k=1}^{n} x_k^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} y_k^2\right)^{\frac{1}{2}}.$$

□

Theorem 1.2.2. (Minkowski inequality)
For all n-tuples $(x_1, x_2, ..., x_n)$, $(y_1, y_2, ..., y_n)$ of $\mathbb{R}^n$, we have

$$\left(\sum_{k=1}^{n}(x_k + y_k)^2\right)^{\frac{1}{2}} \leq \left(\sum_{k=1}^{n} x_k^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{n} y_k^2\right)^{\frac{1}{2}}.$$

Bernard Bolzano (G man Bohem 1781-1848) p vides the first st definition of concept of limits and the first pur analytical proof of Bolzano-Weierstrass theorem.

Karl Weierstrass (a mand 1815-1897) one of the founders modern analysis.

Proof. We have

$$\begin{aligned}
\sum_{k=1}^{n}(x_k+y_k)^2 &= \sum_{k=1}^{n} x_k^2 + 2\Big(\sum_{k=1}^{n} x_k y_k\Big) + \sum_{k=1}^{n} y_k^2 \\
&\leq \sum_{k=1}^{n} x_k^2 + 2\left|\sum_{k=1}^{n} x_k y_k\right| + \sum_{k=1}^{n} y_k^2 \\
&\leq \sum_{k=1}^{n} x_k^2 + 2\left(\sum_{k=1}^{n} x_k^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n} y_k^2\right)^{\frac{1}{2}} + \sum_{k=1}^{n} y_k^2 \quad \text{(CS)} \\
&= \left(\left(\sum_{k=1}^{n} x_k^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{n} y_k^2\right)^{\frac{1}{2}}\right)^2.
\end{aligned}$$

Hence, the proof is completed. □

1.3 Bolzano-Weierstrass Theorem

Theorem 1.3.1. (Bolzano-Weierstrass theorem)
Every infinite and bounded subset of $\mathbb{R}$ has an accumulation point.

Proof. To complete the proof of this theorem, we shall need the following lemma which admitted without proof.

Lemma 1.3.1. *Let $\Big([a_n,\, b_n]\Big)_{n\geq 1}$ be a sequence of decreasing closed intervals .i.e.*

$$[a_1,\, b_1] \supseteq [a_2,\, b_2] \supseteq \dots \supseteq [a_n,\, b_n] \supseteq [a_{n+1},\, b_{n+1}] \supseteq \dots.$$

and $a_n \neq b_n,\ \forall n \geq 1$. Then,

$$\bigcap_{n=1}^{+\infty} [a_n,\, b_n] \neq \varnothing.$$

Moreover, if $\lim\limits_{n\longrightarrow\infty}(b_n - a_n) = 0$ then the intersection is reduced to a single point.

Now, we return to the proof of the theorem.
Let A be an infinite and bounded subset of $\mathbb{R}$. Then, there exists $M > 0$ such that $A \subseteq [-M,\, M]$.
Indeed, we have $[-M, M] = [-M, 0] \cup [0, M]$, So at least one interval of this union is infinite. Let A_1 be this interval. The length of A_1 is $l_1 = M$. Similarly, we divide A_1 into two closed intervals of the same length. At least one of them is infinite. Let A_2 be this subset. The length of A_2 is $l_2 = \dfrac{M}{2}$.

Hence, we construct and step by step a subset $A_n \subseteq A$ such that A_n is infinite, $A_{n+1} \subseteq A_n$ and of length $l_n = \dfrac{M}{2^{n-1}}$. But, A_n is in the form $[a_n, b_n]$, then $\lim_{n\longrightarrow\infty}(b_n - a_n) = \lim_{n\longrightarrow\infty} l_n = 0$.
On the other hand, for $\varepsilon > 0$, by the Archimedean property, there exists $n_0 \in \mathbb{N}^*$ such that $\dfrac{M}{2^{n_0-1}} < \varepsilon$. Since $a \in A_{n_0}$ and the length of A_{n_0} is $\dfrac{M}{2^{n_0-1}}$, then

$$]a-\varepsilon,\, a+\varepsilon[\, \supseteq\,]a - \frac{M}{2^{n_0-1}},\, a + \frac{M}{2^{n_0-1}}[\, \supseteq\, A_{n_0}.$$

Hence, $]a-\varepsilon,\, a+\varepsilon[$ contains infinite points of A. Therefore, a is an accumulation point of A. □

Corollary 1.3.1. *From any bounded sequence, we can extract a convergent subsequence.*

Edmund Georg Hermann Landau (1877-1938) was a German mathematician, author of 253 publications of mathematics. Landau studied mathematics at the University of Berlin and received his doctorate in 1899 and his habilitation in 1901.

Proof. Let (x_n) be a bounded sequence and $A = \{x_n\ n \in \mathbb{N}\}$. We will have two possible cases.

Case 1: A is finite.
Then, there exists a subsequence $(x_{n_k})_k$ such that $x_{n_k} = x_{n_0}$ for some $n_0 \in \mathbb{N}$. Hence, $(x_{n_k})_k$ converges to x_{n_0}.

Case 2: A is infinite.
Then A is infinite and bounded. Thus, by the Bolzano - Weierstrass theorem A has an accumulation point a. Therefore, the proposition 1.1.3 ensures the existence of a sequence $(x_{n_k})_k$ of A which converges to a.

□

1.4 Landau Notations

Our main goal in this paragraph is the comparison of two real functions f and g in the neighborhood of a point x_0, where f and g are defined on a neighborhood of x_0 except perhaps at x_0. We will study also the comparison of two real sequences.

Definition 1.4.1. *We say that $f = o(g)$ if for $g(x) \neq 0$ in a neighborhood of x_0 we have $\lim_{x\longrightarrow x_0} \dfrac{f(x)}{g(x)} = 0$.*
Then, we can write that $f = o(g)$ in the neighborhood of x_0 if

$$f(x) = \varepsilon(x)g(x), \text{ where } \lim_{x\longrightarrow x_0} \varepsilon(x) \longrightarrow 0.$$

Example 1.4.1.

1. $x^3 = o(x^2)$ *in the neighborhood of* 0.
2. $\ln x = o(x)$ *in the neighborhood of* $+\infty$.
3. $\sin(x-1) = o(1)$ *in the neighborhood of* 1.

Definition 1.4.2. *We say that* $f = \mathcal{O}(g)$ *if for* $g(x) \neq 0$ *in the neighborhood of* x_0 *we have* $\dfrac{f}{g}$ *is bounded i.e. if there exists a neighborhood of* $\mathcal{V}_{x_0}$ *of* x_0 *such that* $g(x) \neq 0$ *for all* $x \in \mathcal{V}_{x_0}$, *there exists a constant* $M > 0$ *such that* $\left|\dfrac{f(x)}{g(x)}\right| \leq M$ *for all* $x \in \mathcal{V}_{x_0}$.

Example 1.4.2.

1. $\sin(x^5) = \mathcal{O}(1)$ *in the neighborhood of* $+\infty$.
2. $E(x) = \mathcal{O}(x)$ *in the neighborhood of* $x_0 \in \mathbb{R}$.

Proposition 1.4.1.

1. *If* $f = o(g)$ *then* $f = O(g)$.
2. *If* $f = o(g)$ *and* $g = o(h)$ *then* $f = o(h)$.
3. *If* $f = o(h)$ *and* $g = o(h)$ *then* $f + g = o(h)$. *This does not mean that* $o(h) + o(h) = o(h)$.
4. *If* $f = o(g)$ *and* $u = o(v)$ *then* $fu = o(gv)$.

Proof. Left to the readers. □

Definition 1.4.3. *Two real functions* f *and* g *are* ***equivalent*** *in the neighborhood of* $x_0 \in \mathbb{R}$ *if* $f - g = o(x_0)$ *in the neighborhood of* x_0 *i.e.*

$$f(x) = (1 + \varepsilon(x))\, g(x), \text{ where } \lim_{x \longrightarrow x_0} \varepsilon(x) \longrightarrow 0.$$

We write $f \underset{x_0}{\sim} g$.

In practice, if g does not vanish in the neighborhood of x_0,

$$f \underset{x_0}{\sim} g \quad \text{if and only if} \quad \lim_{x \longrightarrow x_0} \frac{f(x)}{g(x)} = 1.$$

We can remark that the relation $\underset{x_0}{\sim}$ between two functions is reflexive.

Definition 1.4.4. *Let $(u_n)_{\mathbb{N}}$ and $(v_n)_{\mathbb{N}}$ be two real sequences.*

1. *We say that the sequence $(u_n)_{\mathbb{N}}$ is* ***dominated*** *by the sequence $(v_n)_{\mathbb{N}}$ if there exists $A > 0$ such that $(\forall n \in \mathbb{N})(|u_n| \leq A|v_n|)$. We write $(u_n)_{\mathbb{N}} \prec\prec (v_n)_{\mathbb{N}}$.*

2. *We say that the sequence $(u_n)_{\mathbb{N}}$ is* ***negligible*** *compared to the sequence $(v_n)_{\mathbb{N}}$ if there exists a sequence $(\epsilon_n)_{\mathbb{N}}$ of limit zero, such that $u_n = \epsilon_n v_n$ from a certain rank. We write $(u_n)_{\mathbb{N}} << (v_n)_{\mathbb{N}}$.*

Exercise 1.4.1. (Mini-PW)
Prove that a $(u_n)_{\mathbb{N}}$ negligible compared to a sequence $(v_n)_{\mathbb{N}}$ is also dominated by $(v_n)_{\mathbb{N}}$.

Definition 1.4.5. (Landau Notations)
Soit $(v_n)_{\mathbb{N}}$ une suite réelle.

1. *We denote by $\mathcal{O}(v_n)$ the set of sequences $(u_n)_{\mathbb{N}}$ dominated by the sequence $(v_n)_{\mathbb{N}}$ i.e.*
$$\mathcal{O}(v_n) = \Big\{(u_n)_{\mathbb{N}};\ (u_n)_{\mathbb{N}} \prec\prec (v_n)_{\mathbb{N}}\Big\}.$$

2. *We denote by $o(v_n)$ the set of sequences $(u_n)_{\mathbb{N}}$ negligible compared to the sequence $(v_n)_{\mathbb{N}}$ i.e.*
$$o(v_n) = \Big\{(u_n)_{\mathbb{N}};\ (u_n)_{\mathbb{N}} << (v_n)_{\mathbb{N}}\Big\}.$$

Example 1.4.3. *If α and β are two reals such that $\alpha < \beta$ then $n^\alpha \in o(n^\beta)$. Indeed, $\lim\limits_{n\longrightarrow\infty} \dfrac{n^\alpha}{n^\beta} = \lim\limits_{n\longrightarrow\infty} n^{\alpha-\beta} = 0$ because $\alpha - \beta < 0$.*

Exercise 1.4.2. (Mini-PW)

1. *Show that $n^\alpha \in o(e^n),\ \forall \alpha \in \mathbb{R}$.*

2. *Determine $\mathcal{O}(1, 0, 0, ...)$ and $o(1, 1, 1, ...)$.*

1.5 Exercises

Exercise 1.5.1.

1. *Show that the set of interior points of the following subsets is empty:*
$$\mathbb{N}, \quad \mathbb{Z}, \quad \mathbb{Q}, \quad \Big\{\frac{1}{n};\, n \in \mathbb{N}^*\Big\}, \quad \Big\{x \in \mathbb{Q};\, x^2 < 2\Big\}.$$

2. *Determine the sets of adherents points and boundary points of the subset of part 1.*

Exercise 1.5.2.

1. *Determine the accumulation points of the set*

$$A = \left\{ \frac{1}{n} + \frac{1}{m};\ (n,\, m) \in \mathbb{N}^* \times \mathbb{N}^* \right\}.$$

2. *Is A closed, open and compact?*

Exercise 1.5.3. *Let*

$$A = \left\{ \frac{(-1)^n}{1+\frac{1}{n}};\ n \in \mathbb{N}^* \right\}.$$

1. **a)** *Determine the interior, adherent, and accumulation points of A.*

   b) *Is A closed, open and compact?*

2. *Same question for $B = \{2^{-p} + 7^{-q};\ p,\, q \in \mathbb{N}^*\}$.*

Exercise 1.5.4. *Let $x_1,\, x_2,\, ...,\, x_n$ be reals. Show that the set*

$$A = \{x_1,\, x_2,\, ...,\, x_n\}$$

has no accumulation points.

Exercise 1.5.5. *Show that*

1. *$\varnothing$ and $\mathbb{R}$ are closed sets.*

2. *All intersection of closed sets is a closed set.*

3. *All finite union of closed sets is a closed set.*

Exercise 1.5.6.

1. *$\varnothing$ and $\mathbb{R}$ are open sets.*

2. *All union of open sets is an open set.*

3. *All finite intersection of open sets is an open set.*

Exercise 1.5.7.

1. *Show that $A = \mathbb{Q} \cap [0,\, 1]$ is not compact.*

2. *Show that the set* $A = \left\{\frac{1}{n};\ n \in \mathbb{N}^*\right\}$ *is not compact, but* $B = A \cup \{0\}$ *is compact.*

Exercise 1.5.8.

1. *Show that* $A = \{\cos(n);\ n \in \mathbb{N}\}$ *is dense in* $[-1,\ 1]$.
   (Hint: It is admitted that the set $\mathbb{Z} + 2\pi\mathbb{Z}$ *is dense in* $\mathbb{R}$*)*

2. *Show that if* $\frac{\theta}{\pi} \notin \mathbb{Q}$ *then the set* $B = \{\cos(n\theta);\ \in \mathbb{Z}\}$ *is dense in* $]-1,\ 1[$.
   (Hint: It is admitted that the set $\mathbb{Z}\theta + 2\pi\mathbb{Z}$ *is dense in* $\mathbb{R}$ *if and only if* $\frac{\theta}{\pi} \notin \mathbb{Q}$*)*

3. *Show that* $C = \{\cos(\ln n);\ \in \mathbb{N}^*\}$ *is dense in* $[-1,\ 1]$.

Exercise 1.5.9. *Show that* $A = \{m - \ln(n);\ m \in \mathbb{Z},\ n \in \mathbb{N}^*\}$ *is dense in* $\mathbb{R}$.

Exercise 1.5.10. *Let* $A = \left\{(\sqrt{2} - 1)^n;\ n \in \mathbb{N}\right\}$.

1. *Show that* $0 < (\sqrt{2} - 1)^n < \frac{1}{n}$*, for all* $n \geq 1$.

2. *Deduce that* 0 *is an adherent point of* A.

Exercise 1.5.11. *Let* $x \in \mathbb{R}$.

1. *Show that the sequence* $\left(\frac{E(2^n x)}{2^n}\right)$ *converges to* x.

2. *Deduce that the set* $\left\{\frac{m}{2^n};\ (m,\ n) \in \mathbb{Z} \times \mathbb{N}\right\}$ *is dense in* $\mathbb{R}$.

Exercise 1.5.12. *Let* (x_n) *be a sequence of real numbers such that*

$$\lim_{n \longrightarrow \infty} x_n = +\infty \quad et \quad \lim_{n \longrightarrow \infty} (x_{n+1} - x_n) = 0.$$

1. *Let* $\varepsilon > 0$ *and* $p \in \mathbb{N}$ *such that for all* $n \geq p$, $|x_{n+1} - x_n| < \varepsilon$. *Show that, for every* $x \geq x_p$*, there exists* $p_0 \geq p$ *such that* $|x_{p_0} - x| < \varepsilon$.

2. *Deduce that the set* $\{x_n - E(x_n);\ n \in \mathbb{N}\}$ *is dense in the interval* $[0,\ 1]$.

Exercise 1.5.13. *Show that, for every positive reals* a, b *we have :*

$$\sqrt{ab} \leq \frac{1}{2}(a + b).$$

In which case is the equality realized?

Exercise 1.5.14. *Show that if*

$$a_1 \geq a_2 \geq ... \geq a_n > 0 \quad et \quad b_1 \geq b_2 \geq ... \geq b_n > 0,$$

then

$$\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right) \leq n \sum_{k=1}^{n} a_k b_k.$$

Exercise 1.5.15.

1. *Let* $(x_1, x_2, ..., x_n) \in \mathbb{R}^n$. *Show that*

$$\left(\sum_{k=1}^{n} x_k\right)^2 \leq n \sum_{k=1}^{n} x_k^2 \leq n\sqrt{n}\sqrt{\sum_{k=1}^{n} x_k^4}.$$

2. *Let* $(x_1, x_2, ..., x_n) \in (\mathbb{R}^*)^n$ *such that*

$$|x_1| + |x_2| + ... + |x_n| = 1.$$

   Show that $n^2 \leq \displaystyle\sum_{k=1}^{n} \frac{1}{|x_k|}$.

Exercise 1.5.16.

1. *Show that every infinite sequence in an interval* $[a, b]$ *has an accumulation point.*

2. *Show that the set* $\{\sin(n);\, n \in \mathbb{N}\}$ *has an accumulation point.*

Exercise 1.5.17. *Prove that:*

1. $\Big[x^m = o(x^n) \quad \text{as} \quad x \longrightarrow +\infty\Big]$ *if and only if* $m < n$.

2. $\Big[x^m = o(x^n) \quad \text{as} \quad x \longrightarrow 0\Big]$ *if and only if* $m > n$.

Exercise 1.5.18. *Let* $f(x) = \dfrac{3x^5 + x^3 - x^2 + 1}{x^3}$ *and* $g(x) = x^2$. *Prove that* $f = \mathcal{O}(g)$ *as* $x \longrightarrow +\infty$, *but* $f \neq o(g)$ *as* $x \longrightarrow 0$.

Exercise 1.5.19. *Prove that:*

1. $\cos h - 1 + \dfrac{h^2}{2} - \dfrac{h^4}{24} = \mathcal{O}(h^6) \quad \text{as} \quad h \longrightarrow 0.$

2. $\sin\left(\dfrac{1}{h}\right) - \dfrac{1}{h} + \dfrac{1}{6h^3} = o\left(\dfrac{1}{h^4}\right) \quad \text{as} \quad h \longrightarrow +\infty.$

Exercise 1.5.20. *Let* $[a, b] \subset \mathbb{R}$ *and* $f \in \mathcal{C}^3([a, b])$. *Prove that*

$$\left| \frac{f(x+h) - f(x-h)}{2h} - f'(x) \right| = \mathcal{O}(h^2) \qquad h \longrightarrow 0.$$

Exercise 1.5.21. *Prove that:*

1. $n^2 \in \mathcal{O}(10^{-5} n^3)$.

2. $2018n^8 - 33n^5 + 1 + \sin(n) \in \mathcal{O}(n^8)$.

3. $2^{n+1000} \in \mathcal{O}(2^n)$.

Exercise 1.5.22. *Let* $f, g, S, T : \mathbb{N} \longrightarrow \mathbb{N}$, *where* $S(n) \in \mathcal{O}(f(n))$ *and* $T(n) \in \mathcal{O}(g(n))$.

1. *Show that if* $f(n) \in \mathcal{O}(g(n))$, *then* $f(n) + g(n) \in \mathcal{O}(g(n))$.

2. *Show that if* $\mathcal{O}(f(n) + g(n)) = \mathcal{O}(\max(f(n), g(n)))$.

3. *Show that if* $f(n) \in \mathcal{O}(g(n))$, *then* $S(n) + T(n) \in \mathcal{O}(g(n))$.

4. *Show that* $S(n)T(n) \in \mathcal{O}(f(n)g(n))$.

Chapter 2

Complement to the Numerical Sequences

Motivation

The concept of lower limit and upper limit first appear in the book (Analyse Algébrique) written by Cauchy in 1821. But until 1882, Paul du Bois-Reymond gave explanations on them, it becomes well-known.

2.1 Limit Superior and Limit Inferior

Definition 2.1.1. *Let $(x_n)_{n\in\mathbb{N}}$ be a real sequence.*

1. *The **upper limit** of $(x_n)_{n\in\mathbb{N}}$ is defined by the expression*

$$\inf_{n\in\mathbb{N}} \sup_{p\geq n} x_p,$$

denoted by, $\lim\limits_{n\longrightarrow\infty} \sup x_n$ or $\overline{\lim} x_n$.

2. *The **lower limit** of $(x_n)_{n\in\mathbb{N}}$ is defined by the expression*

$$\sup_{n\in\mathbb{N}} \inf_{p\geq n} x_p,$$

denoted by, $\lim\limits_{n\longrightarrow\infty} \inf x_n$ or $\underline{\lim} x_n$.

Example 2.1.1.

- *Consider the sequence (x_n) defined by $x_n = (-1)^n$, $\forall n \in \mathbb{N}$.*

$$y_0 = \sup_{p\geq 0} x_p = \sup\{1,\, -1,\, 1,\, -1,\, ...\} = 1,$$

$$y_1 = \sup_{p \geq 1} x_p = \sup\{-1,\, 1,\, -1,\, 1,\, ...\} = 1,$$

$$y_2 = \sup_{p \geq 2} x_p = \sup\{1,\, -1,\, 1,\, -1,\, ...\} = 1, ...$$

Then

$$\lim_{n \longrightarrow \infty} \sup x_n = \inf_{n \in \mathbb{N}} \{y_0,\, y_1,\, y_2,\, ...\} = 1.$$

Repeating the same procedure yields $\lim\limits_{n \longrightarrow \infty} \inf x_n = -1.$

- *Consider the sequence* (x_n) *defined by* $x_n = n$, $\forall n \in \mathbb{N}$.

$$y_0 = \sup_{p \geq 0} x_p = \sup\{0,\, 1,\, 2,\, 3,\, 4,\, ...\} = +\infty,$$

$$y_1 = \sup_{p \geq 1} x_p = \sup\{1,\, 2,\, 3,\, 4,\, ...\} = +\infty,$$

$$y_2 = \sup_{p \geq 2} x_p = \sup\{2,\, 3,\, 4,\, ...\} = +\infty,\, ...$$

Then

$$\lim_{n \longrightarrow \infty} \sup x_n = \inf_{n \in \mathbb{N}} \{y_0,\, y_1,\, y_2,\, ...\} = +\infty.$$

Repeating the same procedure yields $\lim\limits_{n \longrightarrow \infty} \inf x_n = 0.$

Proposition 2.1.1. $\lim\limits_{n \longrightarrow \infty} \sup x_n$ *(respectively* $\lim\limits_{n \longrightarrow \infty} \inf x_n$*) is the greatest (respectively smallest) cluster point of the sequence* (x_n).

Proposition 2.1.2. *Let* $(x_n)_{n \in \mathbb{N}}$ *be a real sequence and* $p \in \mathbb{N}^*$ *such that for each* $0 \leq q \leq p-1$, $\lim\limits_{k \longrightarrow \infty} x_{pk+q} = L_q \in \overline{\mathbb{R}}$. *Then, the set of all cluster points in*

$$A = \{L_q;\ 0 \leq q \leq p-1\}.$$

Example 2.1.2. *Find* $\lim\limits_{n \longrightarrow \infty} \sup x_n$ *and* $\lim\limits_{n \longrightarrow \infty} \inf x_n$, *where* (x_n) *is defined by*

$$x_n = (-1)^n \frac{n}{n+1}.$$

For $p = 2$, *we have* $q \in \{0,\, 1\}$ *and so*

$$\lim_{k \longrightarrow \infty} x_{2k} = 1 \quad \text{and} \quad \lim_{k \longrightarrow \infty} x_{2k+1} = -1.$$

Hence, $A = \{-1,\, 1\}$. *Therefore,*

$$\lim_{n \longrightarrow \infty} \sup x_n = 1 \quad \text{and} \quad \lim_{n \longrightarrow \infty} \inf x_n = -1.$$

Proposition 2.1.3. *Let $(x_n)_{n\in\mathbb{N}}$ be a real sequence. Then, (x_n) converges to $L \in \mathbb{R}$ if and only if $\lim_{n\longrightarrow\infty} \inf x_n = \lim_{n\longrightarrow\infty} \sup x_n = L$.*

Proof. Suppose that $\lim_{n\longrightarrow\infty} \inf x_n = \lim_{n\longrightarrow\infty} \sup x_n = L$. Indeed, $\forall n \in \mathbb{N}$, we have

$$\inf_{p\geq n} x_p \leq x_n \leq \sup_{p\geq n} x_p.$$

Therefore, by the Sandwich rule, $\lim_{n\longrightarrow\infty} x_n = L$.
Conversely, if $\lim_{n\longrightarrow\infty} x_n = L$, then

$$\forall\varepsilon > 0,\ \exists n_0 \in \mathbb{N}/\forall n \in \mathbb{N}, n \geq n_0 \Rightarrow |x_n - L| < \varepsilon$$

and so

$$\forall n \in \mathbb{N}, n \geq n_0 \Rightarrow |\sup_{p\geq n} x_p - L| < \varepsilon.$$

Thus,

$$|\inf_{n\geq n_0} \sup_{p\geq n} x_p - L| < \varepsilon,\ \forall\varepsilon > 0 \quad \text{ans so} \quad \inf_{n\geq n_0} \sup_{p\geq n} x_p = L.$$

Therefore, $\lim_{n\longrightarrow\infty} \sup x_n \leq L$. Similarly, we prove that $\lim_{n\longrightarrow\infty} \inf x_n \geq L$. But, $\lim_{n\longrightarrow\infty} \inf x_n \leq \lim_{n\longrightarrow\infty} \sup x_n$. Consequently,

$$\lim_{n\longrightarrow\infty} \inf x_n = \lim_{n\longrightarrow\infty} \sup x_n = L.$$

We can prove the converse by other easier idea. In fact, all the subsequences of (x_n) converge to L i.e. all cluster point are equal to L. But, $\lim_{n\longrightarrow\infty} \inf x_n$ and $\lim_{n\longrightarrow\infty} \sup x_n$ are the smallest and greatest cluster points respectively. Hence,

$$\lim_{n\longrightarrow\infty} \inf x_n = \lim_{n\longrightarrow\infty} \sup x_n = L.$$

□

2.2 Adjacent Sequences

Definition 2.2.1. *Two reel sequences (u_n) and (v_n) are said to be adjacent, if:*

1. $u_n \leq u_{n+1} \leq v_{n+1} \leq v_n,\ \forall n \in \mathbb{N}$.
2. $\lim_{n\longrightarrow\infty} (v_n - u_n) = 0$.

Proposition 2.2.1. *Two adjacent sequences (u_n) and (v_n) converge to the same limit L such that $u_n \leq L \leq v_n,\ \forall n \in \mathbb{N}$.*

Proof. The sequence (u_n) is increasing and bounded from above by v_0. Then converges to L. Similarly, the sequence (v_n) is decreasing and bouded from blow by u_0. So, it converges to L'. On the other hand, since

$$u_0 \le u_n \le L \le L' \le v_n \le v_0,\ \forall n \in \mathbb{N},$$

then

$$0 \le L' - u_n \le L - L' \le v_n - u_n,\ \forall n \in \mathbb{N}.$$

Thus, by passing to the limit $u_0 \le L - L' \le 0$ and so $L = L'$. □

Example 2.2.1. *Consider the sequence (x_n) defined by*

$$x_n = 1 - \frac{1}{2} + \frac{1}{3} - \ ...\ + \frac{(-1)^n}{n}.$$

Then, the sequences (u_n) and (v_n) defined by $u_n = x_{2n}$ and $v_n = x_{2n+1}$ are adjacent.
We attract the attention of the reader that the sequence (x_n) is called alternate sequence.

2.3 Cauchy Sequences

Definition 2.3.1. *A reel sequence (u_n) is said to be Cauchy sequence if it satifies*

$$\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N}/\forall p,\ q \in \mathbb{N}, p \ge q \ge n_0 \Rightarrow |u_p - u_q| < \varepsilon.$$

Proposition 2.3.1. *A sequence (u_n) is Cauchy if and only if it converges.*

Proof. Let (u_n) converges to L. Then,

$$\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N}/\forall n \in \mathbb{N}, n \ge n_0 \Rightarrow |u_n - L| < \frac{\varepsilon}{2}.$$

Thus, $\forall p,\ q \in \mathbb{N}$, we have

$$\begin{aligned} p \ge q \ge n_0 \quad \Rightarrow & \quad |u_p - u_q| \\ \le & \quad |u_p - L| + |u_q - L| \\ \le & \quad \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence, (u_n) is Cauchy. □

Now, let us prove the converse. Suppose that (u_n) is a Cauchy sequence. Then,

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}/\forall p,\, q \in \mathbb{N}, p \geq q \geq n_0 \Rightarrow |u_p - u_q| < \varepsilon.$$

In particular, we have

$$\forall n \in \mathbb{N}, n \geq n_0 \Rightarrow |u_n - u_{n_0}| < 1.$$

Thus,

$$\forall n \in \mathbb{N}, n \geq n_0 \Rightarrow |u_n| \leq |u_{n_0}| + 1$$

and so (u_n) is bounded. Therefore, by Bolzano- Weierestrass theorem, (u_n) has a convergent subsequence (u_{n_k}). Let L be the limit of (u_{n_k}). Then,

$$\forall k \in \mathbb{N}, k \geq k_0 \Rightarrow |u_{n_k} - L| < \frac{\varepsilon}{2}.$$

Consequently,

$$\begin{aligned} n \geq \max\{n_0,\, n_{k_0}\} &\Rightarrow |u_n - L| \\ &\leq |u_n - u_{n_k}| + |u_{n_k} - L| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus, (u_n) is convergent.

2.4 Recurrent Sequences

An essential category of sequences are the recurrent sequences defined by a function. This chapter is the outcome of our study on the sequences, but need also the study of functions.(see"Limits and continuous functions").

2.4.1 Recurrent Sequence defined by a Function

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. A recurrent sequence is defined by its first term and a relation allows to calculate the terms de proche en proche :

$$u_0 \in \mathbb{R} \quad \text{et} \quad u_{n+1} = f(u_n) \quad \text{pour } n \geq 0$$

A recuurent sequence is the defined by two givens: a initial term u_0 and a recursive relation $u_{n+1} = f(u_n)$. Then, the sequence can be written as:

$$u_0, \quad u_1 = f(u_0), \quad u_2 = f(u_1) = f(f(u_0)), \quad u_3 = f(u_2) = f(f(f(u_0))), \ldots$$

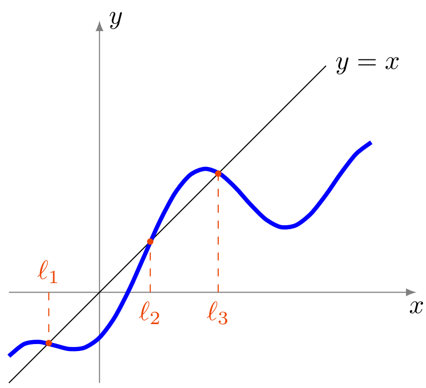
The behavior can quickly becomes complex. Soit $f(x) = 1 + \sqrt{x}$. Fixons $u_0 = 2$ et définissons pour $n \geq 0$: $u_{n+1} = f(u_n)$. C'est-dire $u_{n+1} = 1 + \sqrt{u_n}$. Alors les premiers termes de la suite sont :

$$2, \quad 1+\sqrt{2}, \quad 1+\sqrt{1+\sqrt{2}}, \quad 1+\sqrt{1+\sqrt{1+\sqrt{2}}}, \quad 1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{2}}}}, \ldots$$

Here is an essential result concerning the limit if it exists.

Proposition 2.4.1. *If f is a continuous function and the recurrent sequence (u_n) converges to ℓ, then ℓ is a solution of the equation : $f(\ell) = \ell$*

If we can prove that the limit exists then this proposition allows to calculate candidates to be this limit.



A value ℓ, verifying $f(\ell) = \ell$ is a fixed point of f. The proof is very simple and It deserves to be redone every time.

Proof. When $n \to +\infty$, $u_n \to \ell$ and then also $u_{n+1} \to \ell$. Since $u_n \to \ell$ and that f is continuous then the sequence $(f(u_n)) \to f(\ell)$. The relation $u_{n+1} = f(u_n)$ becomes at the limit (as $n \to +\infty$): $\ell = f(\ell)$. □

We shall study in details two fundamental particular cases: when the function is continuous, then when the function is decreasing.

2.4.2 Case of Increasing Function

Let us start by remark that for an increasing function, the behavior of the sequence (u_n) defined by induction is quite simple

If $u_1 \geq u_0$ then (u_n) is increasing.

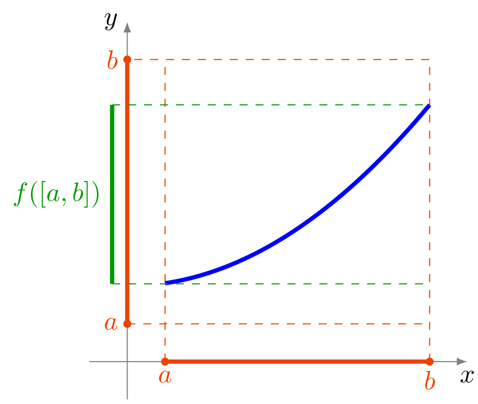
If $u_1 \leq u_0$ then (u_n) is decreasing.

The proof is a simple induction: for example if $u_1 \geq u_0$, then since f is increasing we have $u_2 = f(u_1) \geq f(u_0) = u_1$. Starting from $u_2 \geq u_1$ we deduce $u_3 \geq u_2$,...

Here is the principal result:

Proposition 2.4.2. *If $f : [a, b] \to [a, b]$ a continuous function and increasing, then for every $u_0 \in [a, b]$, the recurrent sequence (u_n) is monotone and converges to $\ell \in [a, b]$ verifying $f(\ell) = \ell$.*

There is an important hypothesis which is a bit hidden: f go from the interval $[a, b]$ to itself. In the practice, for apply this pour proposition, you have to start by choosing $[a, b]$ and verifying that $f([a, b]) \subset [a, b]$.

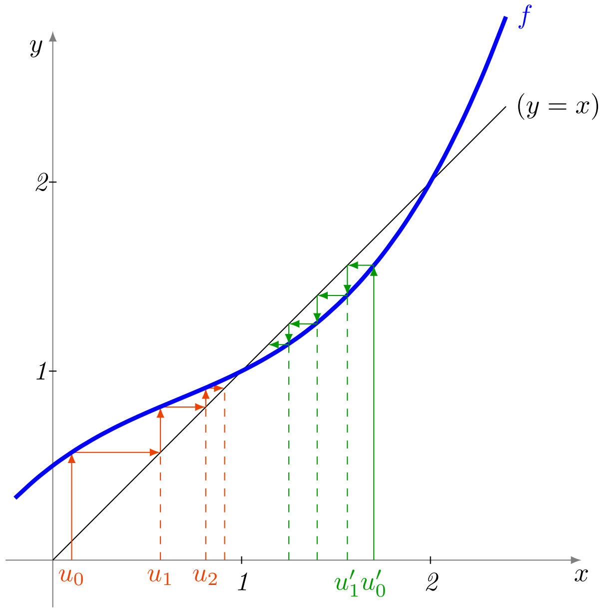


Proof. The proof is a consequence of previous results. For example if $u_1 \geq u_0$ then the sequence (u_n) is increasing, it is bouded from above by b, then it converges to a real ℓ. By the proposition 2.4.1, then $f(\ell) = \ell$. If $u_1 \leq u_0$, then (u_n) is decreasing and bounded from below by a, and the conclusion is the same. □

Example 2.4.1. *Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{1}{4}(x^2 - 1)(x - 2) + x$ and $u_0 \in [0, 2]$. Let us study the sequence (u_n) dedined by induction by: $u_{n+1} = f(u_n)$ (for all $n \geq 0$).*

1. *Study of f*
   (a) *f is contnuous over $\mathbb{R}$.*
   (b) *f is differentiable over $\mathbb{R}$ et $f'(x) > 0$.*
   (c) *On the interval $[0, 2]$, f is strictly increasing.*
   (d) *And since $f(0) = \frac{1}{2}$ and $f(2) = 2$ then $f([0, 2]) \subset [0, 2]$.*

2. *Graph of f*



See how trace a sequence: we trace the graph of f and the first bisector $(y = x)$. We start from a value u_0 (in red) on the axis of abscissas, the value $u_1 = f(u_0)$ reads on the axis of ordinates, but we report the value of u_1 on the axis of abscissas by symmetry with respect to the first bisector. We restart : $u_2 = f(u_1)$ reads on the axis of ordinates and we report it on the axis of abscissas, etc. We obtain then a kind of staircase and graphically we conjecture that the sequence is increasing and tends to 1. If we start from another initial value u'_0 (in green), this is the same principle, but this time we get a staircase going down.

3. *Calculation of the fixed points.*

Looking for values x which verify $(f(x) = x)$, in other words $(f(x) - x = 0)$, but

$$f(x) - x = \frac{1}{4}(x^2 - 1)(x - 2). \tag{2.1}$$

Thus, the fixed points are $\{-1, 1, 2\}$. The limit of (u_n) is then to find among these 3 values.

4. *First case : $u_0 = 1$ or $u_0 = 2$.*

Then $u_1 = f(u_0) = u_0$ and by induction the sequence (u_n) is constant (and converges then to u_0).

5. *Second case :* $0 \leq u_0 < 1$.

   - *Comme* $f([0,1]) \subset [0,1]$, *la fonction* f *se restreint sur l'intervalle* $[0,1]$ *en une fonction* $f : [0,1] \to [0,1]$.
   - *In addition, on* $[0,1]$, $f(x) - x \geq 0$. *This is deduced from the study of* f *or directly from the expression (2.1).*
   - *For* $u_0 \in [0,1[$, $u_1 = f(u_0) \geq u_0$ *by the previous point. Since* f *is increasing, by induction, as we have seen it, the sequence* (u_n) *is increasing.*
   - *The sequence* (u_n) *is increasing and bounded from above by* 1, *then it converges. We denote by* ℓ *its limit.*
   - *From a hand,* ℓ *must be a fixed point of* f : $f(\ell) = \ell$. *Then,* $\ell \in \{-1, 1, 2\}$.
   - *On the other hand, the sequence* (u_n) *is increasing with* $u_0 \geq 0$ *and bounded from above by* 1, *then* $\ell \in [0,1]$.
   - *Conclusion: if* $0 \leq u_0 < 1$ *then* (u_n) *converges to* $\ell = 1$.

6. *Third case :* $1 < u_0 < 2$.

   The function f *is restricted to* $f : [1,2] \to [1,2]$. *On the interval* $[1,2]$, f *is increasing but this time* $f(x) \leq x$. *Then* $u_1 \leq u_0$ *and the sequence* (u_n) *is decreasing. The sequence* (u_n) *is bounded from below by* 1, *it converges. If we denote by* ℓ *its limit then from a hand* $f(\ell) = \ell$, *then* $\ell \in \{-1, 1, 2\}$ *and on the other hand* $\ell \in [1, 2[$. *Conclusion:* (u_n) *converges to* $\ell = 1$.

The graph of f plays a very important role, It must be traced even if it is not explicitly requested. It allows to get a very precise idea of the behavior of the sequence: Is it increasing? Is it positive ? Does it seem to converge? To what limit? These indications are essential to know what to show during the study of the sequence.

2.4.3 Case of Decreasing Function

Proposition 2.4.3. *Let* $f : [a,b] \to [a,b]$ *be a continuous function and decreasing. Let* $u_0 \in [a,b]$ *and the recurrent sequence* (u_n) *be defined by* $u_{n+1} = f(u_n)$. *Then:*

- *The subsequence* (u_{2n}) *converges to a limit* ℓ *verifying* $f \circ f(\ell) = \ell$*.*
- *The subsequence* (u_{2n+1}) *converges to a limit* ℓ' *verifying* $f \circ f(\ell') = \ell'$*.*

IIt is possible (or not!) that $\ell = \ell'$.

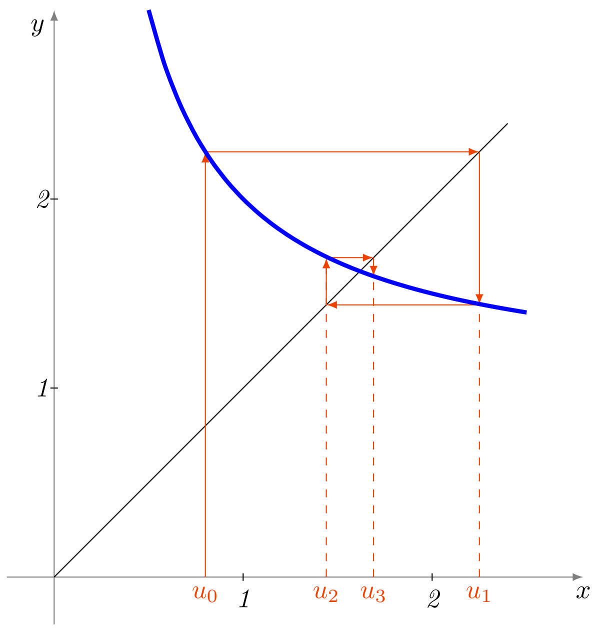
Proof. The proof is deduced from the case increasing. The function f is decreasing, the function $f \circ f$ is increasing. Then, applying the proposition 2.4.2 to the function $f \circ f$ and to the subsequence (u_{2n}) defined by induction by $u_2 = f \circ f(u_0)$, $u_4 = f \circ f(u_2)$,...

Similarly, by starting from u_1 and $u_3 = f \circ f(u_1)$,... □

Example 2.4.2.

$$f(x) = 1 + \frac{1}{x}, \qquad u_0 > 0, \qquad u_{n+1} = f(u_n) = 1 + \frac{1}{u_n}$$

1. *Study of* f*. The function* $f :]0, +\infty[\to]0, +\infty[$ *is a continuous function and strictly decreasing.*
2. *Graph of* f*.*



The principle for tracing is the same as before: we plot u_0*, we trace* $u_1 = f(u_0)$ *on the axis of ordinates and we report by symmetry on the axis of abscissas,... We obtain then a kind of snail, and graphically we conjecture that the sequence converges to the fixed point of* f*. In addition, we denote that the sequence of terms of even ranks seems to aa increasing sequence, thus the sequence of terms of odd ranks is decreasing.*

3. *fixed points of $f \circ f$.*

$$f \circ f(x) = f\Big(f(x)\Big) = f\Big(1 + \frac{1}{x}\Big) = 1 + \frac{1}{1+\frac{1}{x}} = 1 + \frac{x}{x+1} = \frac{2x+1}{x+1}$$

Then

$$f \circ f(x) = x \iff \frac{2x+1}{x+1} = x \iff x^2 - x - 1 = 0 \iff x \in \left\{\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right\}$$

Since the limit must be positive, the unique fixed point to considered is $\ell = \frac{1+\sqrt{5}}{2}$.
Attention ! There is a unique fixed point, but we can not conclude at this stage because f is defined on $]0, +\infty[$ which is not a compact interval.

4. *First case $0 < u_0 \leq \ell = \frac{1+\sqrt{5}}{2}$.*
then, $u_1 = f(u_0) \geq f(\ell) = \ell$. By a study of $f \circ f(x) - x$, we obtain that : $u_2 = f \circ f(u_0) \geq u_0$; $u_1 \geq f \circ f(u_1) = u_3$.
Since $u_2 \geq u_0$ and $f \circ f$ is increasing, the sequence (u_{2n}) is increasing. Similarly, $u_3 \leq u_1$, then the sequence (u_{2n+1}) is decreasing. Moreover, since $u_0 \leq u_1$, by applying f an even *number of times, we obtain that $u_{2n} \leq u_{2n+1}$.*
The situation is then the following :

$$u_0 \leq u_2 \leq \cdots \leq u_{2n} \leq \cdots \leq u_{2n+1} \leq \cdots \leq u_3 \leq u_1$$

The sequence (u_{2n}) is increasing and bounded from above by u_1, then it converges Its limit can not be the unique fixed point of $f \circ f$: $\ell = \frac{1+\sqrt{5}}{2}$.
The sequence (u_{2n+1}) is decreasing and bounded from below by u_0, so it converges also to $\ell = \frac{1+\sqrt{5}}{2}$.
Therefore, we conclude that the sequence (u_n) converges to $\ell = \frac{1+\sqrt{5}}{2}$.

5. *Second case $u_0 \geq \ell = \frac{1+\sqrt{5}}{2}$.*
By the same way, we prove that (u_{2n}) is decreasing and converges to $\frac{1+\sqrt{5}}{2}$, and that (u_{2n+1}) is increasing and converges also to $\frac{1+\sqrt{5}}{2}$.

1. Let $f(x) = \frac{1}{9}x^3 + 1$, $u_0 = 0$ and for $n \geq 0$: $u_{n+1} = f(u_n)$. Study in details the sequence (u_n) : (a) Prove that $u_n \geq 0$; (b) Study and trace the graphe of g; (c) Plot the first terms of (u_n) ; (d) Prove that (u_n) is increasing ; (e) Study the function $g(x) = f(x) - x$; (f) Prove that f has two fixed points on $\mathbb{R}_+$, $0 < \ell < \ell'$; (g) Prove that $f([0, \ell]) \subset [0, \ell]$; (h) Deduce that (u_n) converges to ℓ.

2. Let $f(x) = 1 + \sqrt{x}$, $u_0 = 2$ and for $n \geq 0$: $u_{n+1} = f(u_n)$. Study in details the sequence (u_n).

3. Let $(u_n)_{n\in}$ be the sequence defined by : $u_0 \in [0, 1]$ et $u_{n+1} = u_n - u_n^2$. Study in details the sequence (u_n).

4. Study the sequence defined by $u_0 = 4$ et $u_{n+1} = \frac{4}{u_n+2}$.

Exercise 2.4.1. (Mini-PW) .

2.5 Exercises

Exercise 2.5.1. *Calculate* $\lim_{n\longrightarrow\infty} \sup$ *and* $\lim_{n\longrightarrow\infty} \inf$ *of the sequence* (x_n) *in the following cases:*

1. $x_n = (-1)^n n$, $n \in \mathbb{N}$.

2. $x_n = \cos \frac{n\pi}{3} + \dfrac{(-1)^n}{n}$, $n \geq 1$.

3. $x_n = \sin \frac{n\pi}{2} + \dfrac{(-1)^n}{n}$, $n \geq 1$.

4. $x_n = \sin \frac{n\pi}{2} \cos \frac{n\pi}{3}$, $n \in \mathbb{N}$.

Exercise 2.5.2. *Let* (x_n) *and* (y_n) *be bounded sequences in* $\mathbb{R}$.

1. Prove that

   (a) $\lim_{n\longrightarrow\infty} \sup(\lambda x_n) = \lambda \lim_{n\longrightarrow\infty} \sup x_n, \ \forall\lambda > 0.$

   (b) $\lim_{n\longrightarrow\infty} \inf(\lambda x_n) = \lambda \lim_{n\longrightarrow\infty} \inf x_n, \ \forall\lambda > 0.$

   (c) $\lim_{n\longrightarrow\infty} \sup(x_n + y_n) \leq \lim_{n\longrightarrow\infty} \sup x_n + \lim_{n\longrightarrow\infty} \sup y_n.$

   (d) $\lim_{n\longrightarrow\infty} \inf(x_n + y_n) \geq \lim_{n\longrightarrow\infty} \inf x_n + \lim_{n\longrightarrow\infty} \inf y_n.$

2. If x_n *is convergente, prove that*

   (a) $\lim_{n\longrightarrow\infty} \sup(x_n + y_n) = \lim_{n\longrightarrow\infty} x_n + \lim_{n\longrightarrow\infty} \sup y_n.$

   (b) $\lim_{n\longrightarrow\infty} \inf(x_n + y_n) = \lim_{n\longrightarrow\infty} x_n + \lim_{n\longrightarrow\infty} \inf y_n.$

3. Prove that

   (a) $\lim_{n\longrightarrow\infty} \sup(-x_n) = - \lim_{n\longrightarrow\infty} \inf x_n.$

(b) $\lim_{n\longrightarrow\infty} \inf(-x_n) = -\lim_{n\longrightarrow\infty} \inf x_n$.

Exercise 2.5.3. *Let (x_n) and (y_n) be sequences of positives terms and bounded from above .*

1. *Prove that*
2. $\lim_{n\longrightarrow\infty} \sup(x_n y_n) \leq \lim_{n\longrightarrow\infty} \sup x_n \lim_{n\longrightarrow\infty} \sup y_n$.
3. *If x_n is convergente, prove that*

$$\lim_{n\longrightarrow\infty} \sup(x_n y_n) \leq \lim_{n\longrightarrow\infty} x_n \lim_{n\longrightarrow\infty} \sup y_n.$$

Exercise 2.5.4. *Consider the two sequences $(u_n)_{n\geq 1}$ and $(v_n)_{n\geq 1}$ defined, for all $n \geq 1$ by*

$$u_n = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \dots + \frac{1}{n^3}$$

and

$$v_n = u_n + \frac{1}{n^2}.$$

Show that the two sequences are adjacent.

Exercise 2.5.5. *Let a and b be two real numbers such that $0 < a < b$. Let (u_n) and (v_n) be two sequences defind by $u_0 = \sqrt{ab}$, $v_0 = \frac{a+b}{2}$, and the recurrent relations*

$$u_{n+1} = \sqrt{u_n v_n} \quad \text{and} \quad v_{n+1} = \frac{u_n + v_n}{2} \ \forall n \in \mathbb{N}.$$

Show that the two sequences are adjacent.

Exercise 2.5.6. *Let $a > 0$ and let (u_n) and (v_n) be two sequences defind by $u_0 > v_0 > 0$, and the recurrent relations*

$$u_{n+1} = \frac{(a+1)u_n v_n}{a u_n + v_n} \quad \text{and} \quad v_{n+1} = \frac{u_n + a v_n}{a+1}, \ \forall n \in \mathbb{N}.$$

Show that the two sequences are adjacent.

Exercise 2.5.7. *Let $\theta \in]0, \frac{\pi}{2}[$ and*

$$u_n = 2^n \sin\left(\frac{\theta}{2^n}\right), \ v_n = 2^n \tan\left(\frac{\theta}{2^n}\right).$$

Prove that the sequences (u_n) et (v_n) are adjacent. What is their commun limit?

Exercise 2.5.8. ***(Irrationality of** e)*
Let (u_n) and (v_n) be two sequences defind by

$$u_n = \sum_{k=1}^{n} \frac{1}{k!} \quad \text{and} \quad v_n = u_n + \frac{1}{n.n!}.$$

1. *Show that (u_n) and (v_n) are strictly monotone and adjacent. We admit that their common limit is e.*
2. *Our goal in this part is to prove that $e \notin \mathbb{Q}$. For that, suppose that $e = \frac{p}{q}$, where $p \in mathbbN$ and $q \in mathbbN^*$.*
   Show that $u_q < e < v_q$. Conclude.

Exercise 2.5.9. *Show that the sequences (x_n) and (y_n) defined by*

$$x_n = \sum_{k=1}^{n} \frac{1}{k} \quad \text{and} \quad y_n = \sum_{k=1}^{n} \frac{1}{k^2}.$$

1. *Prove that (x_n) is not a Cauchy sequence.*
2. *Prove that (y_n) is a Cauchy sequence. Conclude.*

Exercise 2.5.10. *Let (u_n) be the sequence defind by $u_0 = 1$ and the recurrent relation*

$$u_{n+1} = \sqrt{u_n^2 + \frac{1}{2^n}},\ \forall n \geq 1.$$

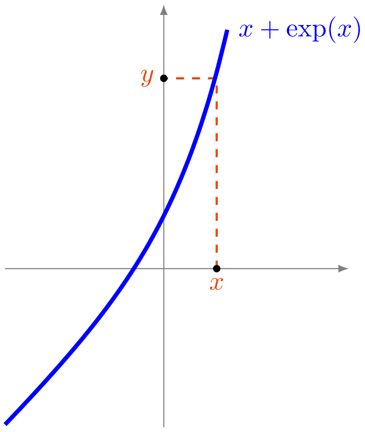
Show that it is Cauchy.

Chapter 3

The Continuous Functions

Motivation

We know solve many equations (eg $ax + b = 0$, $ax^2 + bx + c =$ 0, ...) but these equations are very particulars. For most equations we will not know to solve, in fact it is not obvious to say whether there is a solution, or how many there are. Consider for example the extremely easy equation: $x + \exp x = 0$. There is no known formula (with sums, products, ... of common functions) to find the solution x. In this chapter we will see that according to a study of the function $f(x) = x + \exp x$ it is possible to obtain much information about the solution of the equation $x + \exp x = 0$ and even the more general equation $x + \exp x = y$ (or $y \in$ is fixed).



We will be able to prove that for every $y \in$ the equation

$$x + \exp x = y$$

admits a unique solution x; and we will know how x varies in terms of y. The key-point is the study of the function f and especially its continuity. Although *it is not possible to find the exact expression of the solution x in terms of y, we will develop the theoretical tools to find an approximate solution.*

3.1 Continuity on an interval

3.1.1 Continuous functions on a segment

Theorem 3.1.1. *Let $f : [a, b] \longrightarrow$ be a continuous function on $[a, b]$; then the range $f[a, b] = \{f(x); x \in [a, b]\}$ is bounded in , i.e there exists $c, d \in$ such that $\forall\, x \in [a, b]$ we have: $c \leq f(x) \leq d$.*

Proof. We will prove that $B = f[a, b]$ is bounded from above, if it's not there exists a sub-sequence $(x_n) \subset A = [a, b]$ such that $f(x_n) \to +\infty$, the set $X = \{x_n; n \in \mathbb{N}\}$ is infinity and bounded in A, so, by using the **Bolzano** theorem we deduce that this set has a sub-sequence (x_{n_k}) which converges to a limit $\alpha \in A$; therefore $f(x_{n_k}) \to f(\alpha)$ and $f(x_{n_k}) \to +\infty$. Since the limit of a sequence is unique one gets a contradiction. $\square$

Exercise 3.1.1. *We use the same way, proving that B is bounded from below in .*

The last step showed us that $f[a, b]$ is bounded in , therefore it admits a lower bound which will be denoted by c and an upper bound which will be denoted by d. The following theorem shows that our f reaches these boundaries.

Theorem 3.1.2. (Extrema theorem for continuous functions)
Let $f : [a, b] \longrightarrow$ be a continuous function over $[a, b]$, let $c = \inf f[a, b]$ and $d = \sup f[a, b]$, then there exists at least $\xi, \eta \in [a, b]$ such that $f(\xi) = c, f(\eta) = d$.

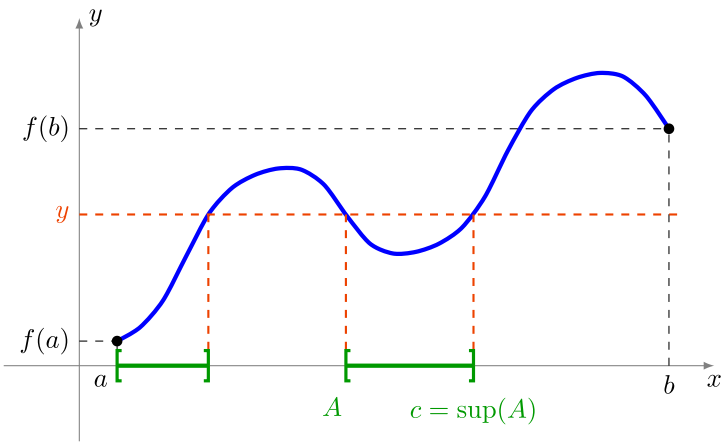
Proof. We will prove that f attains its bounds i.e there exists $\xi, \eta \in [a, b]$ such that $f(\xi) = c, f(\eta) = d$. If $f[a, b]$ is a finite set then there is no problem. We suppose that this set is infinity. Or $d = \sup f[a, b]$, by the characteristic property of the upper bound, we verify that there exist a sequence (y_n) such that $y_n \to d$ and $y_n \in f[a, b]$ (d is an accumulation point of $f[a, b]$). Let $(x_n) \subset [a, b]$ an infinity sequence such that $f(x_n) = y_n$, then by using the **Bolzano** theorem, one can find a convergent sub-sequence $x_{n_k} \to \eta$, therefore $f(x_{n_k}) \to f(\eta)$ and $f(x_{n_k}) \to d$. Since the limit of a sequence is unique one gets $f(\eta) = d$. Similiarly, one can prove that there exists at least ξ such that $f(\xi) = c$. $\square$

3.1.2 The intermediate value theorem

Theorem 3.1.3. (The intermediate value theorem)
Let $f : [a,b] \to$ be a continuous function on a segment. For each real y between $f(a)$ and $f(b)$, there exists $c \in [a,b]$ such that $f(c) = y$.

Proof. We prove the theorem in the case $f(a) < f(b)$. We consider a real number y such that $f(a) \leq y \leq f(b)$ and we want to show that it has an antecedent by f.

1. We introduce the following set
$$A = \Big\{x \in [a,b] \;|/\; f(x) \leq y\Big\}.$$
First of all the set A is no empty (since $a \in A$) and it is bounded above (since $A \subset [a,b]$). It admits an upper bound, which is denoted $c = \sup A$. Show that $f(c) = y$.



2. We want to show here that $f(c) \leq y$. Since $c = \sup A$, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ include in A such that (u_n) converges to c. We have for all $n \in \mathbb{N}$, since $u_n \in A$, that $f(u_n) \leq y$. Since f is continuous at c, the sequence $(f(u_n))$ converges to $f(c)$. One can deduce, by using the limit of a sequence, that $f(c) \leq y$.

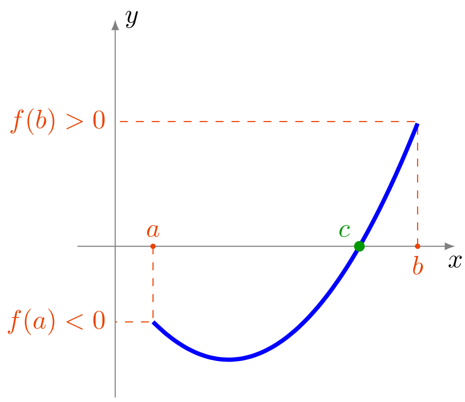
3. We want to show here that $f(c) \geq y$. We remark first that if $c = b$, then there is no problem, since $f(b) \geq y$. Else, for all $x \in]c,b]$, since $x \notin A$, we have $f(x) > y$. But, by using the fact that f is continuous at c, f admits limit from right at c, which is equal to $f(c)$ and one obtains that $f(c) \geq y$.

$\square$

3.1.3 Applications of the intermediate value theorem

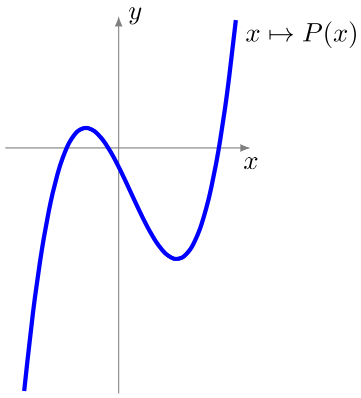
Here are the most widely used version of the intermediate value theorem.

Corollary 3.1.1. *Let $f : [a, b] \to$ be a continuous function on a segment. If $f(a) \cdot f(b) < 0$, then there exists $c \in]a, b[$ such that $f(c) = 0$.*



Proof. This is a direct application of the intermediate value theorem with $y = 0$. The hypothesis $f(a) \cdot f(b) < 0$ means that $f(a)$ and $f(b)$ have opposite signs . □

Example 3.1.1. Each odd degree polynomial has at least one real root .



Indeed, such a polynomial is written as $P(x) = a_n x^n + \cdots + a_1 x + a_0$ where n is an integer odd uumber. We can assume that the coefficient a_n is strictly positive. Then, we have $\lim_{-\infty} P = -\infty$ and $\lim_{+\infty} P = +\infty$. In particular , there are two real numbers a and b such that $f(a) < 0$ and $f(b) > 0$ and it is concluded by the previous corollary .

4

Remark 3.1.1. *Let $f : [a, b] \to$ be a continuous and strictly monotonic function on a segment. If $f(a) \cdot f(b) < 0$, then there exists a unique $c \in]a, b[$ such that $f(c) = 0$.*

Corollary 3.1.2. *Let $f : I \to$ be a continuous function on an interval I. Then $f(I)$ is an interval.*

Proof. Let $y_1, y_2 \in f(I)$, $y_1 \leq y_2$. Show that if $y \in [y_1, y_2]$, then $y \in f(I)$. By hypothesis , there are $x_1, x_2 \in I$ such that $y_1 = f(x_1)$, $y_2 = f(x_2)$ and therefore y is between $f(x_1)$ and $f(x_2)$. According to the intermediate value theorem, as f is continuous, there exists $x \in I$ such that $y = f(x)$, and finally, $y \in f(I)$. □

Exercise 3.1.2. (Mini-PW)

1. *Let $P(x) = x^5 - 3x - 2$ and $f(x) = x2^x - 1$ two defined functions on . Show that the equation $P(x) = 0$ has at least a root in $[1, 2]$; the equation $f(x) = 0$ has at least a root in $[0, 1]$; the equation $P(x) = f(x)$ has at least a root in $]0, 2[$.*

2. *Prove that there exists $x > 0$ such that $2^x + 3^x = 7^x$.*

3. *Draw the graph of a continuous function $f :\to$ such that $f() = [0, 1]$. and $f() =]0, 1[$; $f() = [0, 1[$; $f() =]-\infty, 1]$, $f() =]-\infty, 1[$.*

4. *Let f and g be two continuous functions on $[0, 1]$ such that for all $x \in [0, 1]$, we have $f(x) < g(x)$. Show that there exists $m > 0$ such that for all $x \in [0, 1]$, we have $f(x) + m < g(x)$. This result is it true if we replace $[0, 1]$ by ?*

3.2 Monotonic and bijection functions

3.2.1 Monotonic and bijections functions

Here is an important result which provides bijective functions.

Theorem 3.2.1. (Bijection theorem)
Let $f : I \to$ be a function defined on an interval I to . If f is a continuous and strictly monotonic function on I, then

1. *f establishes a bijection of the interval I to the image interval $J = f(I)$,*

2. *The inverse function $f^{-1} : J \to I$ is continuous and strictly monotonic on J and has the same sense of variation of f.*

To prove this theorem, we shall need the following lemma .

Lemma 3.2.1. *Let $f : I \to$ be a function defined on an interval I to . If f is strictly monotonic on I, then f is one-to-one on I.*

Proof. For all $x, x' \in I$ for which $f(x) = f(x')$. Let us show that $x = x'$. If we had $x < x'$, then it would necessarily $f(x) < f(x')$ or $f(x) > f(x')$, depending on whether f is strictly increasing or strictly decreasing. As this is impossible, we deduce that $x \geq x'$. By exchanging the roles of x and x' is also shows that $x \leq x'$. It follows that $x = x'$ and therefore f is one-to-one. □

Now we return to the proof of the theorem.

Proof. Indeed, by using the previous lemma, f is one-to-one on I. Therefore f establishes a bijection of the interval I to the image interval $J = f(I)$. Since f is continuous, by using the intermediate value theorem, the set J is an interval. From another side and for the case of simplicity, we suppose that f is increasing function.

1. Show that f^{-1} is strictly increasing on J. Let $y, y' \in J$ such that $y < y'$. We denote by $x = f^{-1}(y) \in I$ and $x' = f^{-1}(y') \in I$. Then $y = f(x)$, $y' = f(x')$ and then
$$\begin{aligned} y < y' &\implies f(x) < f(x') \\ &\implies x < x' \qquad \text{(since } f \text{ strictly increasing)} \\ &\implies f^{-1}(y) < f^{-1}(y'), \end{aligned}$$
i.e f^{-1} is strictly increasing on J.

2. Show that f^{-1} is continuous on J. We consider here the case where I is of the form $]a, b[$, the other cases proved in the same why. Let $y_0 \in J$. We denote by $x_0 = f^{-1}(y_0) \in I$. Let $\epsilon > 0$. We can always suppose that $]\,x_0 - \epsilon, x_0 + \epsilon\,[\subset I$. We seek $\delta > 0$ such that for all $y \in J$ one has
$$y_0 - \delta < y < y_0 + \delta \implies f^{-1}(y_0) - \epsilon < f^{-1}(y) < f^{-1}(y_0) + \epsilon$$
i.e for all $x \in I$ one has
$$y_0 - \delta < f(x) < y_0 + \delta \implies f^{-1}(y_0) - \epsilon < x < f^{-1}(y_0) + \epsilon.$$
Or, since f is strictly increasing one has for all $x \in I$,
$$\begin{aligned} f(x_0 - \epsilon) < f(x) < f(x_0 + \epsilon) &\implies x_0 - \epsilon < x < x_0 + \epsilon \\ &\implies f^{-1}(y_0) - \epsilon < x < f^{-1}(y_0) + \epsilon. \end{aligned}$$

Since $f(x_0 - \epsilon) < y_0 < f(x_0 + \epsilon)$, one can choose the real $\delta > 0$ such that

$$f(x_0 - \epsilon) < y_0 - \delta \quad \text{and} \quad f(x_0 + \epsilon) > y_0 + \delta$$

and one then gets, for all $x \in I$,

$$\begin{aligned} y_0 - \delta < f(x) < y_0 + \delta &\implies f(x_0 - \epsilon) < f(x) < f(x_0 + \epsilon) \\ &\implies f^{-1}(y_0) - \epsilon < x < f^{-1}(y_0) + \epsilon. \end{aligned}$$

The function f^{-1} is then continuous on J.

$\square$

Exercise 3.2.1. (Mini-PW)

1. *Show that each of the hypothesis "continuity" and "strictly monotonic" is necessary in the announcement of the theorem.*
2. *Let $f :\to$ defined by $f(x) = x^3 + x$. Show that f is bijective, draw the graph of f and f^{-1}.*
3. *Let $n \geq 1$. Show that $f(x) = 1 + x + x^2 + \cdots + x^n$ establishes a bijection from $[0, 1]$ to an interval to be determined.*
4. *Is there a continuous function:$f : [0, 1[\to]0, 1[$ that is bijective?* *$f : [0, 1[\to]0, 1[$ that is injective? $f :]0, 1[\to [0, 1]$ that is surjective?*
5. *For $y \in$ we consider the equation $x + \exp x = y$. Prove that there exists a unique solution $x \in$.*

3.3 Uniform Continuity on an interval

3.3.1 Uniform continuity

Definition 3.3.1. *Let I be an interval of , the function f is said uniformly continuous on I when*

$$\forall \varepsilon > 0, \quad \exists \delta > 0, \quad \forall x', x'' \in I, \quad \Big(|x' - x''| < \delta \Longrightarrow |f(x') - f(x'')| < \varepsilon \Big).$$

This definition implies the continuity of f in each point $x \in I$; one can give the following theorem:

Theorem 3.3.1. *Every uniformly continuous function on I is continuous at each point in I.*

Can we ask about the truth of the converse of this theorem ?

Example 3.3.1. *$f(x) = \frac{1}{x}$ is uniformly continuous on $[0.5, 1]$. But it is not uniformly continuous on $]0, 1]$ even that it is continuous on $]0, 1]$.*

3.3.2 Heine-Borel theorem

In this section, we will see the sufficient conditions on the interval I so continuity ensures uniform continuity.

Eduard Heine (1821-1881) was a famous German mathematician for its results on special functions and real analysis.

Theorem 3.3.2. (Heine-Borel theorem)
If f is continuous on a closed bounded interval $[a,b]$ then f is uniformly continuous on this interval.

Proof. Let $\varepsilon > 0$. Seek a $\delta > 0$. Since f is continuous at each point $x \in [a,b]$, then for this ε there exists $\delta_x > 0$ such that

$$|y-x| < \delta_x \Longrightarrow |f(y)-f(x)| < \frac{\varepsilon}{2}.$$

Since the family $\left(I_x(\frac{\delta_x}{2})\right)_x$ of open intervals of centre x and of radius $\frac{\delta_x}{2}$ is an open recovery family of the interval $[a,b]$, then by the **Borel** theorem there exists a finite number of this family covering $[a,b]$, denoted it by :

$$I_{x_1}(\frac{\delta_{x_1}}{2}), \quad I_{x_2}(\frac{\delta_{x_2}}{2}), \quad ...., \quad I_{x_n}(\frac{\delta_{x_n}}{2}).$$

Émile Borel (1871-1956) was a French mathematician, specialist in the theory of functions and probability.

Let $\delta = \min(\frac{\delta_{x_1}}{2}, \frac{\delta_{x_2}}{2}, ..., \frac{\delta_{x_n}}{2})$. Now start from two points $x', x'' \in [a,b]$ such that $|x'-x''| < \delta$. Or x'' is exist in a certain $I_{x_k}(\frac{\delta_{x_k}}{2})$, then $|x''-x_k| < \frac{\delta_{x_k}}{2}$, this implies

$$|x'-x_k| \leq |x'-x''| + |x''-x_k| \leq \delta + \frac{\delta_{x_k}}{2} < \delta_{x_k}.$$

Then x', x'' are in the interval of centre x_k and of radius δ_{x_k}. Then

$$|f(x')-f(x'')| \leq |f(x')-f(x_k)| + |f(x_k)-f(x'')| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, the proof of the theorem is complete. □

Exercise 3.3.1. (Mini-PW)

1. *Prove that the function $f(x) = \frac{x^2+\sin x}{x+1}$ is uniformly continuous on $[0,1]$.*

2. *Prove that the function $f(x) = \frac{1-\cos^2(5x)}{x^2}$ is uniformly continuous on $]0,1]$.*

3. *Prove that the function $f(x) = \frac{1}{x}$ is uniformly continuous on $[1,+\infty[$.*

3.4 Exercises

Exercise 3.4.1. *Let a be a strictly positive number.*

1. *Without using the derivative, prove that the polynomial function*

$$p(x) = x^3 - 5x$$

has one only minimal value in the interval $I = [0, \sqrt{5}]$; determine the variations of p in this interval.

2. *Same question, for the function*

$$p_a(x) = x^3 + (a-2)x^2 + (1-2a)x$$

on $I_a = [0, \sqrt{a}]$.

Exercise 3.4.2. *Let f be a continuous function on $[0,1]$ to $[0,1]$. prove that there exists a point $\xi \in [0,1]$ such that $f(\xi) = \xi$.*

Exercise 3.4.3. *Prove that there exists $x > 0$ such that $3^x + 5^x = 10^x$.*

Exercise 3.4.4. *Without using the derivative, prove that the equation*

$$x + \ln x = 3,$$

admits one only root in the interval $[1, e]$.

Exercise 3.4.5.

1. *Prove that the function $f(x) = x \sin(\frac{1}{x})$ is uniformly continuous on $]0, +\infty[$.*

2. *same question for the function*

$$g(x) = \frac{x}{1 + 10^{-2016} + \sin x},$$

over $I_\alpha = [-\alpha, \alpha]$, where $\alpha \in$.

Exercise 3.4.6.

1. *Let $f :\to$ defined by $f(x) = e^{-x^{2n+1}}$, where $n \in \mathbb{N}$. Prove that f is bijective.*

2. *Let $n \geq 1$. Prove that $f(x) = 1 + x + x^2 + \cdots + x^n$ establishes a bijection from the interval $[0,1]$ to an interval to be determined.*

3. *Let* $y \in$, *we consider the equation* $x + |\cos x| = y$. *Prove that there exists a unique solution* $x \in$.

Exercise 3.4.7. *Let* f *be a monotonic function on* $[a, b]$ *and has the property of the intermediate value theorem, prove that* f *is continuous.*

Exercise 3.4.8. *Let* I *be an interval on and* f *is continuous function on* I, *such that* $\forall x \in I$, $f(x)^2 = 1$. *Prove that* f *is constant on* I.

Exercise 3.4.9. *Let* $f :^{+} \longrightarrow$ *is continuous and it admits a finite limit at* $+\infty$. *Prove that* f *is bounded. Does it attain its bounds?*

Exercise 3.4.10.

1. *Let* g *is a continuous function from* $[a, b]$ *to itself. Prove that* g *has a fixed point.*

2. *Let* g *is an increasing function from* $[a, b]$ *to itself. Prove that* g *has a fixed point.*

Exercise 3.4.11. *Let* $f :^{+} \longrightarrow$ *is continuous and it admits a finite limit at* $+\infty$. *Prove that* f *is uniformly continuous on* $^{+}$.

Exercise 3.4.12. *Find all the continuous linear mappings of* $(, +)$.

Exercise 3.4.13. *Let* f *be a periodic and continuous function over . Prove that* f *is bounded and uniformly continuous on .*

Chapter 4

The Differentiable Functions

Motivation

In this chapter, for any function, we will find the polynomial of degree n that best approach the function. The results are only valid for x around a set value, this will often around 0. This polynomial will be calculated from successive derivatives at the considered point. Here is the formula, called formula of ***Taylor-Young****:*

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \cdots + f^{(n)}(0)\frac{x^n}{n!} + x^n\epsilon(x).$$

The polynomial part $f(0) + f'(0)x + \cdots + f^{(n)}(0)\frac{x^n}{n!}$ is a polynomial of degree n approaching $f(x)$ near to $x = 0$. The part $x^n\epsilon(x)$ is the «remainder» for which $\epsilon(x)$ is a function tending to 0 as x tends to 0 and it is negligible compared with the polynomial part.

Michel Rolle (1652-1719) was a French mathematician. He invented to designate the nth root of a real x, the standard notation: $\sqrt[n]{x}$.

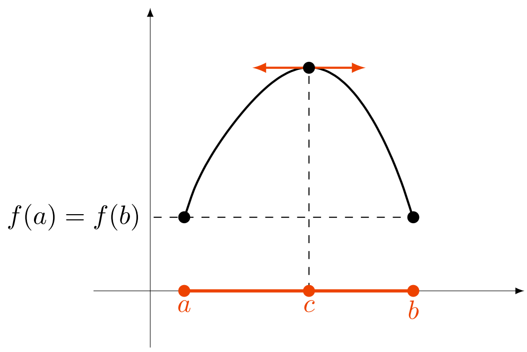
4.1 Generalized mean value theorem

4.1.1 Rolle's theorem

Theorem 4.1.1. (Rolle's theorem)
Let $f : [a, b] \to$ such that

- *f is continuous on $[a, b]$,*
- *f is differentiable on $]a, b[$,*
- *$f(a) = f(b)$.*

Then, there exists $c \in]a, b[$ such that $f'(c) = 0$.



<u>Geometric interpretation:</u> there exists at least one point of the graph of f at which the tangent is horizontal.

Proof. First of all, if f is constant on $[a, b]$ then every $c \in]a, b[$ verifies the result. If not, there exists $x_0 \in [a, b]$ such that $f(x_0) \neq f(a)$. Suppose for example $f(x_0) > f(a)$. Then f is continuous over the closed bounded interval $[a, b]$, so it admits a maximum value at a point $c \in [a, b]$. But $f(c) \geq f(x_0) > f(a)$ then $c \neq a$. Similarly as $f(a) = f(b)$ then $c \neq b$. Hence, $c \in]a, b[$. At c, f is a differentiable function and admits a local maximum value then $f'(c) = 0$. $\square$

Example 4.1.1. *Let*

$$P(X) = (X - \alpha_1)(X - \alpha_2) \cdots (X - \alpha_n)$$

is a polynomial has n different real roots: $\alpha_1 < \alpha_2 < \cdots < \alpha_n$.

1. Show that P' has $n - 1$ distincts roots.

   *We consider P as a polynomial function $x \mapsto P(x)$. P is continuous and differentiable function over . As $P(\alpha_1) = 0 = P(\alpha_2)$, then by **Rolle's** theorem there exists $c_1 \in]\alpha_1, \alpha_2[$ such that $P'(c_1) = 0$. More general, for $1 \leq k \leq n - 1$, as $P(\alpha_k) = 0 = P(\alpha_{k+1})$ then **Rolle's** theorem ensures the existence of $c_k \in]\alpha_k, \alpha_{k+1}[$ such that $P'(c_k) = 0$. We found $n - 1$ roots of P' : $c_1 < c_2 < \cdots < c_{n-1}$. As P' is a polynomial of degree $n-1$, then all its roots are real and distincts.*

2. Show that $P + P'$ has $n - 1$ distincts roots.

   *The trick, consist of considering an auxiliaire function $f(x) = P(x) \exp x$. f is a continuous and differentiable function over . f vanishes as P at $\alpha_1, \ldots, \alpha_n$. The derivative of f is $f'(x) = \big(P(x) + P'(x)\big) \exp x$. Then, by **Rolle's** theorem, for each $1 \leq k \leq n - 1$ and since $f(\alpha_k) = 0 = f(\alpha_{k+1})$, there exists $\gamma_k \in]\alpha_k, \alpha_{k+1}[$ such that $f'(\gamma_k) = 0$. But,* *the fact that the exponentiel function not vanishes any more one gets* $(P+P')(\gamma_k)=0$. *We have found* $n-1$ *distincts roots of* $P+P'$: $\gamma_1<\gamma_2<\cdots<\gamma_{n-1}$.

3. Deduce that all the roots of $P+P'$ are real.

   $P+P'$ *is a polynomial with real coefficients admitting* $n-1$ *real roots. Then,*

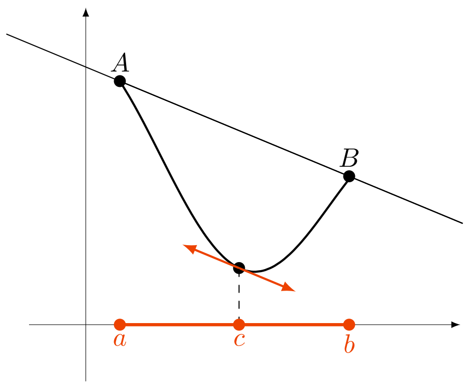
$$(P+P')(X)=(X-\gamma_1)\cdots(X-\gamma_{n-1})Q(X),$$

   where $Q(x)=X-\gamma_n$ *is a polynomial of degree* 1. *As* $P+P'$ *is with real coefficients and the* γ_i *are also real, then* $\gamma_n\in$. *So, we obtain the* n*-th real root* γ_n *(not necessarily distinct from* γ_i*).*

4.1.2 Mean value theorem

Theorem 4.1.2. (Mean value theorem)
Let $f:[a,b]\to$ *be a continuous function on* $[a,b]$ *and differentiable on* $]a,b[$. *Then, there exists* $c\in]a,b[$ *such that*

$$f(b)-f(a)=f'(c)\ (b-a).$$



Geometric interpretation: there exists at least a point of the graph of f *where the tangent is parallel to the straight line* (AB) *where* $A=(a,f(a))$ *and* $B=(b,f(b))$.

Proof. Set $\ell=\frac{f(b)-f(a)}{b-a}$ and $g(x)=f(x)-\ell\cdot(x-a)$. Then, $g(a)=f(a)$ and $g(b)=f(b)-\frac{f(b)-f(a)}{b-a}\cdot(b-a)=f(a)$. By the **Rolle's** theorem, there exists $c\in]a,b[$ such that $g'(c)=0$. Or $g'(x)=f'(x)-\ell$. Therefore, $f'(c)=\frac{f(b)-f(a)}{b-a}$. □

4.1.3 Mean value theorem inequality

Corollary 4.1.1. (Mean value theorem inequality)
Let $f : I \to$ be a differentiable function on an open interval I. If there exists a constant M such that for all $x \in I$, $\left|f'(x)\right| \leq M$ then

$$\forall x, y \in I \qquad \left|f(x) - f(y)\right| \leq M|x - y|.$$

Proof. We fix $x, y \in I$, then there exists $c \in]x, y[$ or $]y, x[$ such that

$$f(x) - f(y) = f'(c)(x - y),$$

and since $|f'(c)| \leq M$, then

$$\left|f(x) - f(y)\right| \leq M|x - y|.$$

□

Example 4.1.2. *Let $f(x) = \sin(x)$. Since $f'(x) = \cos x$ then $|f'(x)| \leq 1$ for all $x \in$. For all x, $y \in$, the mean value theorem can be written as:*

$$|\sin x - \sin y| \leq |x - y|.$$

In particular, if one fix $y = 0$ then we obtain $|\sin x| \leq |x|$ which is particularly interesting for x close to 0.

4.1.4 Generalized mean value theorem of Cauchy

Theorem 4.1.3. (Generalized mean value theorem of Cauchy GMVT)
Let f and g be two continuous functions on $[a, b]$ and differentiable on $]a, b[$ and g' does not vanish on $]a, b[$. Then, there exists at least a point $c \in]a, b[$ such that:

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof. We define the function

$$\psi(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}(g(x) - g(a)),$$

The continuity and differentiability of f and g imply that ψ is continuous on $[a, b]$ and differentiable on $]a, b[$. In addition $\psi(a) = \psi(b) = 0$, then by the **Rolle's** theorem there exists a point $c \in]a, b[$ such that $\psi'(x) = 0$ so, we get the following pseudo-equality of **TAFG**:

$$f'(c) - g'(c)\frac{f(b) - f(a)}{g(b) - g(a)} = 0.$$

Since in addition g' not vanishes on $]a,b[$, then there exists at least a point $c \in]a,b[$ such that:

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

□

Guillaume François Antoine de L'Hôspital (1661-1704) was a French mathematician. He is known for the rule that bears his name. He is also the author of the first book in french on the differential calculus: Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes. Published in 1696.

4.1.5 L'Hôspital's rule

Theorem 4.1.4. (L'Hôspital's rule HR)

Let $f, g : I \to$ be two differentiable functions and let $x_0 \in I$. We suppose that

- $f(x_0) = g(x_0) = 0$,
- $\forall x \in I \setminus \{x_0\} \quad g'(x) \neq 0$.

If $\displaystyle\lim_{x\to x_0} \frac{f'(x)}{g'(x)} = \ell \in$ *then* $\displaystyle\lim_{x\to x_0} \frac{f(x)}{g(x)} = \ell.$

Proof. We fix $a \in I \setminus \{x_0\}$ with for example $a < x_0$. Let $h : I \to$ defined by $h(x) = g(a)f(x) - f(a)g(x)$. Then

- h is continuous on $[a, x_0] \subset I$,
- h is differentiable on $]a, x_0[$,
- $h(x_0) = h(a) = 0$.

Then by **Rolle's** theorem there exists $c_a \in]a, x_0[$ such that $h'(c_a) = 0$. Or $h'(x) = g(a)f'(x) - f(a)g'(x)$ then $g(a)f'(c_a) - f(a)g'(c_a) = 0$. As g' not vanishes on $I \setminus \{x_0\}$ this implies $\frac{f(a)}{g(a)} = \frac{f'(c_a)}{g'(c_a)}$. As $a < c_a < x_0$ when we tend a to x_0 we obtain $c_a \to x_0$. this implies

$$\lim_{a\to x_0} \frac{f(a)}{g(a)} = \lim_{a\to x_0} \frac{f'(c_a)}{g'(c_a)} = \lim_{c_a\to x_0} \frac{f'(c_a)}{g'(c_a)} = \ell.$$

□

Remark 4.1.1. *In this remark we will clarify many interested points:*

1. ***L'Hôspital's** rule remains true even that x_0 is a limit point such that*

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = 0.$$

2. ***L'Hôspital's** rule remains true even that the limit point $x_0 = \infty$.*

3. ***L'Hôspital's*** *rule remains true for the form* $\frac{\infty}{\infty}$.

4. *In practice, this rule can be applied several times to derivatives as we suitable with the indeterminate forms.*

Example 4.1.3. *Calculate the limit at* 1 *of* $\frac{\ln(x^2+x-1)}{\ln(x)}$. *We verify that:*

- $f(x) = \ln(x^2 + x - 1)$, $f(1) = 0$, $f'(x) = \frac{2x+1}{x^2+x-1}$,
- $g(x) = \ln(x)$, $g(1) = 0$, $g'(x) = \frac{1}{x}$,
- *We consider* $I =]0, 1]$, $x_0 = 1$, *then* g' *not vanishes over* $I \setminus \{x_0\}$.

$$\frac{f'(x)}{g'(x)} = \frac{2x+1}{x^2+x-1}x = \frac{2x^2+x}{x^2+x-1} \xrightarrow[x\to 1]{} 3.$$

Then

$$\frac{f(x)}{g(x)} \xrightarrow[x\to 1]{} 3.$$

Exercise 4.1.1. (Mini-PW)

1. *Let* $f(x) = \sqrt{x}$. *Apply the MVT theorem over* $[100, 101]$. *Deduce that*
$$10 + \frac{1}{22} \leq \sqrt{101} \leq 10 + \frac{1}{20}.$$

2. *Let* $f(x) = e^x$. *What does the MVT inequality give over* $[0, x]$? *Deduce that for all* $x \geq 0$, $e^x - 1 \leq xe^x$.

3. *Apply* ***L'Hôspital's*** *rule to calculate the following limits (when* $x \to 0$*):*
$$\frac{x}{(1+x)^n - 1} \; ; \; \frac{\ln(x+1)}{\sqrt{x}} \; ; \; \frac{1-\cos x}{\tan x} \; ; \; \frac{x - \sin x}{x^3}.$$

4.2 Taylor' formulas

Brook Taylor (1685-1731) was an English man of science. Best known mathticien as he was also interested in music, painting and religion.

We will see three ***Taylor's*** *formulas. They will all have the same polynomial part but give more or less information on the rest . We begin by* ***Taylor's*** *formula with integral remainder which gives an exact expression of the rest. Then* ***Taylor's*** *formula with remainder* $f^{(n+1)}(c)$ *which provides an estimation of rest and we end with the* ***Taylor -Young's*** *formula handy if you do not need information on the rest.*

Let $I \subset$ *be an open interval. For* $n \in \mathbb{N}^*$, *we say that* $f : I \to$ *is a function of class* $\mathcal{C}^n$ *if* f *is* n *times differentiable on* I *and* $f^{(n)}$ *is continuous.* f *is of class* $\mathcal{C}^0$ *if* f *is continuous over* I. f *is of class* $\mathcal{C}^\infty$ *if* f *is of class* $\mathcal{C}^n$ *for all* $n \in \mathbb{N}$.

4.2.1 Taylor's formula with integral remainder

Theorem 4.2.1. (Taylor's formula with integral remainder)
Let $f : I \to$ be a function of class $\mathcal{C}^{n+1}$ $(n \in \mathbb{N})$ and let $a, x \in I$. Then

$$f(x) = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n dt.$$

We denote by $T_n(x)$ the polynomial part of **Taylor's** *formula (It is depend of n and also of f and a) :*

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Proof. We prove the **Taylor's** formula by induction on $k \leq n$:

$$f(b) = f(a)+f'(a)(b-a)+\frac{f''(a)}{2!}(b-a)^2+\cdots+\frac{f^{(k)}(a)}{k!}(b-a)^k+\int_a^b f^{(k+1)}(t)\frac{(b-t)^k}{k!}dt.$$

(to avoid confusion between what varies and what is fixed in this proof we replace x by b.)
Initialisation. for $n = 0$, a primitive of $f'(t)$ is $f(t)$ then

$$\int_a^b f'(t)\, dt = f(b) - f(a),$$

then

$$f(b) = f(a) + \int_a^b f'(t)\, dt.$$

(Remember that by convention $(b-t)^0 = 1$ et $0! = 1$)
Heredity. We suppose that the formula is true to the rank $k-1$. Therefore,

$$f(b) = f(a)+f'(a)(b-a)+\cdots+\frac{f^{(k-1)}(a)}{(k-1)!}(b-a)^{k-1}+\int_a^b f^{(k)}(t)\frac{(b-t)^{k-1}}{(k-1)!}dt.$$

We do an integration by parts in the integral $\int_a^b f^{(k)}(t)\frac{(b-t)^{k-1}}{(k-1)!}\, dt$. Setting

$$u(t) = f^{(k)}(t) \quad \text{and} \quad v'(t) = \frac{(b-t)^{k-1}}{(k-1)!},$$

we get

$$u'(t) = f^{(k+1)}(t) \quad \text{and} \quad v(t) = -\frac{(b-t)^k}{k!};$$

then

$$\begin{aligned}\int_a^b f^{(k)}(t)\frac{(b-t)^{k-1}}{(k-1)!}\,dt &= \left[-f^{(k)}(t)\frac{(b-t)^k}{k!}\right]_a^b + \int_a^b f^{(k+1)}(t)\frac{(b-t)^k}{k!}\,dt \\ &= f^{(k)}(a)\frac{(b-a)^k}{k!} + \int_a^b f^{(k+1)}(t)\frac{(b-t)^k}{k!}\,dt.\end{aligned}$$

So when replacing this expression in the formula to rank $k-1$ we obtain the formula to rank k.
Conclusion. By the induction principle the **Taylor's** formula is true for all integers n for which f is of class $\mathcal{C}^{n+1}$.

□

Remark 4.2.1. *Writing $x = a + h$ (and so, $h = x - a$) the previous* ***Taylor's*** *formula becomes (for all a and $a + h$ in I) :*

$$f(a+h) = f(a)+f'(a)h+\frac{f''(a)}{2!}h^2+\cdots+\frac{f^{(n)}(a)}{n!}h^n+\int_0^h \frac{f^{(n+1)}(a+t)}{n!}(h-t)^n dt.$$

Example 4.2.1. *The function $f(x) = \exp x$ is of class $\mathcal{C}^{n+1}$ over $I =$ for each n. Fix $a \in$. As $f'(x) = \exp x$, $f''(x) = \exp x, \ldots$ then for each $x \in$:*

$$\exp x = \exp a + \exp a \cdot (x-a) + \cdots + \frac{\exp a}{n!}(x-a)^n + \int_a^x \frac{\exp t}{n!}(x-t)^n dt.$$

Of course, if we place $a = 0$, then we find the beginning of our approximation of the exponential function at $x = 0$: $\exp x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

4.2.2 Taylor's formula with remainder $f^{(n+1)}(c)$

Theorem 4.2.2. (Taylor's formula with remainder $f^{(n+1)}(c)$)
Let $f : I \to$ be a function of class $\mathcal{C}^{n+1}$ ($n \in \mathbb{N}$) and let $a, x \in I$. there exists a real number c between a and x such that:

$$f(x) = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

Example 4.2.2. *Let $a, x \in$. For each integer $n \geq 0$ there exists c between a and x such that*

$$\exp x = \exp a + \exp a \cdot (x-a) + \cdots + \frac{\exp a}{n!}(x-a)^n + \frac{\exp c}{(n+1)!}(x-a)^{n+1}.$$

In most cases, will not be known that c . But, this theorem is used to give information about the rest . This is expressed by the following corollary :

Corollary 4.2.1. *If in addition the function $|f^{(n+1)}|$ is bounded from above over I by M, then for each $a, x \in I$, one has:*

$$\Big|f(x) - T_n(x)\Big| \leq M\frac{|x-a|^{n+1}}{(n+1)!}.$$

Example 4.2.3. (Approximation of the sin function) *Approximation of* $\sin(0,01)$. *Let $f(x) = \sin x$. Then $f'(x) = \cos x$, $f''(x) = -\sin x$, $f^{(3)}(x) = -\cos x$, $f^{(4)}(x) = \sin x$. So, one obtains $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f^{(3)}(0) = -1$. The above **Taylor's** formula of order 3 at $a = 0$ becomes:*

$$f(x) = 0 + 1 \cdot x + 0 \cdot \frac{x^2}{2!} - 1\frac{x^3}{3!} + f^{(4)}(c)\frac{x^4}{4!},$$

i.e

$$f(x) = x - \frac{x^3}{6} + f^{(4)}(c)\frac{x^4}{24},$$

for some c between 0 and x.
Take here $x = 0,01$. Since the rest is small we then get

$$\sin(0,01) \approx 0,01 - \frac{(0,01)^3}{6} = 0,00999983333\ldots$$

We can also find the precision of this approximation: as $f^{(4)}(x) = \sin x$ then $|f^{(4)}(c)| \leq 1$. So,

$$\Big|f(x) - \Big(x - \frac{x^3}{6}\Big)\Big| \leq \frac{x^4}{4!}.$$

For $x = 0,01$ this gives :

$$\Big|\sin(0,01) - \Big(0,01 - \frac{(0,01)^3}{6}\Big)\Big| \leq \frac{(0,01)^4}{24}.$$

As $\frac{(0,01)^4}{24} \approx 4,16 \cdot 10^{-10}$, then our approximation give at least eight exacts digits after the comma.

Remark 4.2.2. (Mini-Remarks)

- *In this theorem hypothesis f of class $\mathcal{C}^{n+1}$ may be weakened by f is «n+1 times differentiable on I».*
- *«The real c is between a and x» means «$c \in]a, x[$ or $c \in]x, a[$».*

- *For $n = 0$ this is exactly the mean value theorem: there exists $c \in]a, b[$ such that*
$$f(b) = f(a) + f'(c)(b - a).$$
- *If I is a closed bounded interval and f of class $\mathcal{C}^{n+1}$, then $f^{(n+1)}$ is continuous function on I so there exists M such that $|f^{(n+1)}(x)| \leq M$ for each $x \in I$. Hence, it's always possible to apply the corollary.*

To prove the theorem we need a preliminary result.

Lemma 4.2.1. (MVT for integration)
Let $a < b$ and $u, v : [a, b] \to$ be two continuous functions with $v \geq 0$. Then, there exists $c \in [a, b]$ such that
$$\int_a^b u(t)v(t)\,dt = u(c)\int_a^b v(t)\,dt.$$

Proof. We denote by $m = \inf\limits_{t\in[a,b]} u(t)$ and $M = \sup\limits_{t\in[a,b]} u(t)$. One gets
$$m\int_a^b v(t)\,dt \leq \int_a^b u(t)v(t)\,dt \leq M\int_a^b v(t)\,dt \quad \text{(since } v \geq 0\text{)}.$$
So,
$$m \leq \frac{\displaystyle\int_a^b u(t)v(t)\,dt}{\displaystyle\int_a^b v(t)\,dt} \leq M.$$
Since u is continuous on $[a, b]$ then all its values are between m and M (IVT). Then there exists $c \in [a, b]$ with
$$u(c) = \frac{\displaystyle\int_a^b u(t)v(t)\,dt}{\displaystyle\int_a^b v(t)\,dt}.$$
□

Now we return to the proof of the theorem 4.2.2

Proof. For the proof we show the **Taylor's** formula for $f(b)$ supposing $a < b$. We prove only that $c \in [a, b]$ instead of $c \in]a, b[$.
We set $u(t) = f^{(n+1)}(t)$ and $v(t) = \frac{(b-t)^n}{n!}$. The **Taylor's** formula with integral remainder can be written as
$$f(b) = T_n(a) + \int_a^b u(t)v(t)\,dt.$$

by the lemma, there exists $c \in [a, b]$ such that

$$\int_a^b u(t)v(t)\,dt = u(c)\int_a^b v(t)\,dt.$$

Thus, the rest is

$$\begin{cases} \displaystyle\int_a^b u(t)v(t)\,dt = f^{(n+1)}(c)\int_a^b \frac{(b-t)^n}{n!}\,dt = f^{(n+1)}(c)\left[-\frac{(b-t)^{n+1}}{(n+1)!}\right]_a^b \\ = f^{(n+1)}(c)\frac{(b-a)^{n+1}}{(n+1)!} \end{cases}$$

Which gives the desired formula. □

William Henry Young (1863-1942) was an English mathematician from the University of Cambridge and has worked at the University of Liverpool and that of Lausanne. His studies focused on the measure theory, Lebesgue integrals, Fourier series, and differential calculus. He brought brilliant contributions to complex analysis.

4.2.3 Taylor-Young's formula

Theorem 4.2.3. (Taylor-Young's formula)
Let $f : I \to$ be a function of class $\mathcal{C}^n$ and let $a \in I$. Then, for each $x \in I$, we have:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + (x-a)^n\epsilon(x),$$

where ϵ is a function defined over I such that $\epsilon(x) \xrightarrow[x\to a]{} 0$.

Proof. f is a function of class $\mathcal{C}^n$. The application of **Taylor's** formula with remainder $f^{(n)}(c)$ to the rank $n-1$ allows us to conclude that for all x, there exists $c = c(x)$ between a and x such that

$$\begin{cases} f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} \\ \qquad + \frac{f^{(n)}(c)}{n!}(x-a)^n. \end{cases}$$

We rewrite:

$$\begin{cases} f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\ \qquad + \frac{f^{(n)}(c) - f^{(n)}(a)}{n!}(x-a)^n \end{cases}$$

We set $\epsilon(x) = \frac{f^{(n)}(c) - f^{(n)}(a)}{n!}$. Since $f^{(n)}$ is continuous and that $c(x) \to a$, then $\lim\limits_{x\to a} \epsilon(x) = 0$. □

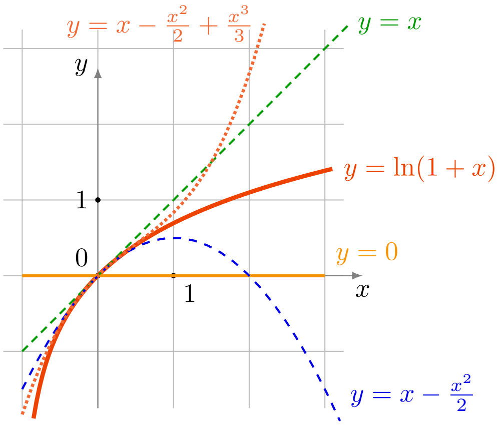
Example 4.2.4. *Let $f :]-1, +\infty[\to$, $x \mapsto \ln(1+x)$; f is infinitely differentiable. We will calculate the **Taylor's** formulas at 0 for the first orders. First of all $f(0) = 0$. $f'(x) = \frac{1}{1+x}$, then $f'(0) = 1$. As $f''(x) = -\frac{1}{(1+x)^2}$, then $f''(0) = -1$. Also, $f^{(3)}(x) = +2\frac{1}{(1+x)^3}$, then $f^{(3)}(0) = +2$. Hence, by* *induction principle, we show that* $f^{(n)}(x) = (-1)^{n-1}(n-1)!\frac{1}{(1+x)^n}$ *and then* $f^{(n)}(0) = (-1)^{n-1}(n-1)!$. *Thus, for* $n > 0$, *we have*

$$\frac{f^{(n)}(0)}{n!}x^n = (-1)^{n-1}\frac{(n-1)!}{n!}x^n = (-1)^{n-1}\frac{x^n}{n}.$$

We get the first order polynomials of **Taylor** *:*

$$T_0(x) = 0 \qquad T_1(x) = x \qquad T_2(x) = x - \frac{x^2}{2} \qquad T_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}$$

Taylor's formulas tell us that the remains are becoming smaller as n *increases. On the graphs the polynomials* T_0, T_1, T_2, T_3 *approach increasingly the graph of* f. *Please note this is true that around* 0.



For any n *we calculate the* **Taylor's** *polynomial at* 0 *which is*

$$T_n(x) = \sum_{k=1}^{n}(-1)^{k-1}\frac{x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n}.$$

4.2.4 Summary

There are three **Taylor** *formulas that are written all in the form*

$$f(x) = T_n(x) + R_n(x),$$

where $T_n(x)$ *is always the same* **Taylor's** *polynomial:*

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

It is the expression of the remainder $R_n(x)$ which changes. Pay attention that the rest has no reason to be a polynomial.

$R_n(x) = \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n dt$ **Taylor's formula with integral remainder**

$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ **Taylor 's with remainder** $f^{(n+1)}(c)$, *c between a and x*

$R_n(x) = (x-a)^n \epsilon(x)$ **Taylor-Young's with** $\epsilon(x) \xrightarrow[x \to a]{} 0$

Depending on the situation one of the formulations is more suitable than others. Often we do not need a lot of information on the rest, and so is the ***Taylor-Young's*** *formula that will be most useful. Note that the three formulas do not require exactly the same assumptions:* ***Taylor's*** *with integral remainder to the order n requires a function of class $\mathcal{C}^{n+1}$,* ***Taylor's*** *with remainder function $n+1$ times differentiable, and* ***Taylor-Young's*** *a function of class $\mathcal{C}^n$. A more restrictive assumption logically gives a stronger conclusion. That said, for the functions of class $\mathcal{C}^\infty$ that are handled often, the three assumptions are always verified.*

Colin Maclaurin (1698-1746) was a Scottish mathematician. He was professor of mathematics at Marischal College in Aberdeen from 1725 to 1717 and at the university of Édimbourg from 1725 to 1745. He did remarkable work in geometry, more specifically in the study of plane curves.

Notation. *The term $(x-a)^n\epsilon(x)$ where $\epsilon(x) \xrightarrow[x \to 0]{} 0$ is often abbreviated by small o of $(x-a)^n$ and is denoted by $o((x-a)^n)$. We must get used to this notation that simplifies writing, but always keep in mind what it means (see Chapter 1).*

Particular case: ***Taylor -Young's*** *formula near to 0.*
We often brings to the particular case where $a = 0$, the ***Taylor-Young's*** *formula or often called* ***Maclaurin's*** *formula is then written*

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \cdots + f^{(n)}(0)\frac{x^n}{n!} + x^n\epsilon(x),$$

where $\lim_{x \to 0} \epsilon(x) = 0$.
Thus, with the notation «small o» that gives :

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \cdots + f^{(n)}(0)\frac{x^n}{n!} + o(x^n),$$

It's the finite expansion ***FE*** *of f to the order n near to 0.*

4.2.5 Behavior of a curve

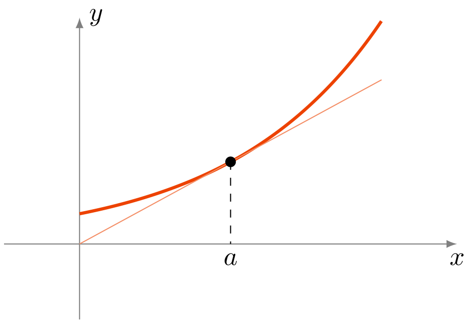
Proposition 4.2.1. *Let $f : I \to$ be a function admitting a FE at a :*

$$f(x) = c_0 + c_1(x-a) + c_k(x-a)^k + (x-a)^k \epsilon(x),$$

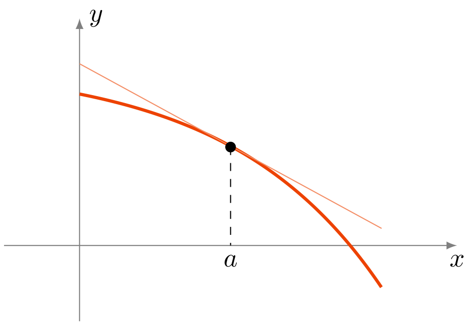
where k is the smaller integer ≥ 2 such that c_k is non zero. Then, the equation of the tangent of the curve of f at a is: $y = c_0 + c_1(x-a)$ and the relative position of the curve with respect to the tangent for x close to a is given by the sign of $f(x) - y$, i.e the sign of $c_k(x-a)^k$.

There are three possible cases.

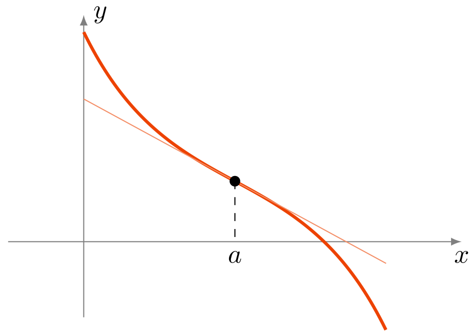
- *If the sign is positive then the curve is above the tangent.*



- *If the sign is negative then the curve is below the tangent.*



- *If the sign changes (when going from $x < a$ ou $x > a$), then the curve crosses the tangent at the point of abscissa a. This is an* **inflection** *point.*



As the FE of f at a to order 2 can also be written

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + (x-a)^2\epsilon(x).$$

Then the equation of the tangent is also $y = f(a)+f'(a)(x-a)$. If in addition $f''(a) \neq 0$ then $f(x) - y$ keeps a constant sign around a. Accordingly, if a is an inflection point , so $f''(a) = 0$. (The converse is not true.)

Example 4.2.5. *Let $f(x) = x^4 - 2x^3 + 1$.*

1. *Determine the tangent at $\frac{1}{2}$ of the graph of f and specify the position of graph with respect to the tangent. We have $f'(x) = 4x^3 - 6x^2$, $f''(x) = 12x^2 - 12x$, then $f''(\frac{1}{2}) = -3 \neq 0$ and $k = 2$.*

   We deduce the FE of f at $\frac{1}{2}$ by using the ***Taylor-Young's*** *formula:*

$$\begin{aligned} f(x) &= f(\frac{1}{2}) + f'(\frac{1}{2})(x-\frac{1}{2}) + \frac{f''(\frac{1}{2})}{2!}(x-\frac{1}{2})^2 + (x-\frac{1}{2})^2\epsilon(x) \\ &= \frac{13}{16} - (x-\frac{1}{2}) - \frac{3}{2}(x-\frac{1}{2})^2 + (x-\frac{1}{2})^2\epsilon(x). \end{aligned}$$

   Then, the tangent at $\frac{1}{2}$ is $y = \frac{13}{16} - (x - \frac{1}{2})$ and the graph of f is bellow the tangent since

$$f(x) - y = \Big(-\frac{3}{2} + \epsilon(x)\Big)(x-\frac{1}{2})^2$$

   is negative around $x = \frac{1}{2}$.

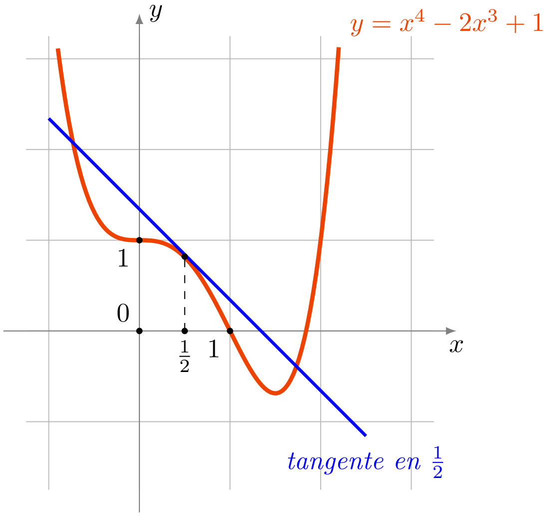
2. *Determine the inflection points.*

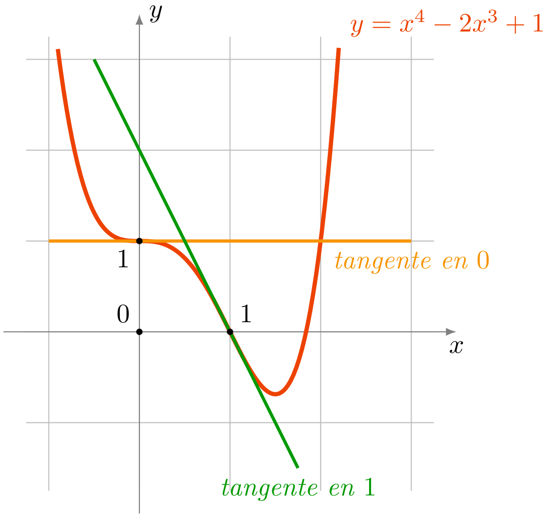
   The inflection points are to be found among the solutions of $f''(x) = 0$. Therefore, $x = 0$ and $x = 1$.

- *The FE at* 0 *is* $f(x) = 1 - 2x^3 + x^4$ *(this is just to write monomials by increasing degrees !). The equation of the tangent at* 0 *is* $y = 1$ *(a horizontal tangent). As* $-2x^3$ *changes its sign around* 0 *then* 0 *is an inflection point of* f*.*
- *The FE at* 1 *: We calculate* $f(1)$*,* $f'(1)$*,* $\ldots$ *to find the FE at* 1

$$f(x) = -2(x-1) + 2(x-1)^3 + (x-1)^4.$$

The equation of the tangent at 1 *is* $y = -2(x-1)$*. As* $2(x-1)^3$ *changes its sign around* 1*, then* 1 *is an inflection point of* f*.*





4.3 Exercises

Exercise 4.3.1. *Prove the following points : (using GMVT)*

1. ***L'Hôspital's*** *rule remains true even that x_0 is a limit point such that*
$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = 0.$$
2. ***L'Hôspital's*** *rule remains true even that the limit point $x_0 = \infty$.*
3. ***L'Hôspital's*** *rule remains true for the form $\frac{\infty}{\infty}$.*

Exercise 4.3.2. *Calculate the following limits: (using HR)*

1. $\displaystyle\lim_{x\to 0} \frac{x^2 - \sin^2 x}{x^2 \sin^2 x}$ *2.* $\displaystyle\lim_{x\to 1} \frac{1}{\tan(x-1)} - \frac{3}{x^3 - 1}$ *3.* $\displaystyle\lim_{x\to \frac{\pi}{2}^-} \frac{\tan x}{\tan 3x}$

4. $\displaystyle\lim_{x\to 0^+} x e^{\frac{1}{x}}$ *5.* $\displaystyle\lim_{x\to +\infty} \frac{e^x}{x^n}, \quad n \in \mathbb{N}$ *6.* $\displaystyle\lim_{x\to +\infty} \frac{\ln^n x}{x}, \quad n \in \mathbb{N}$

7. $\displaystyle\lim_{x\to +\infty} \frac{x^\alpha}{\beta^x}, \quad (\alpha, \beta) \in^2$ *8.* $\displaystyle\lim_{x\to \infty} x - \sqrt{x^2 + \sqrt{x}}$ *9.* $\displaystyle\lim_{x\to +\infty} \left(1 - \frac{3}{x}\right)^x$

10. $\displaystyle\lim_{x\to 0^-} \left(1 - \frac{3}{x}\right)^x$ *11.* $\displaystyle\lim_{x\to \frac{\pi}{2}} (\cos x)^{\frac{\pi}{2} - x}$.

Exercise 4.3.3. *Prove that the following limits cannot be calculated by using L'Hôspital's rule, then calculate them by another method.*

1. $\displaystyle\lim_{x\to 0} \frac{x^2 \sin(\frac{1}{x})}{\sin x}$.
2. $\displaystyle\lim_{x\to +\infty} \frac{x - \sin x}{x + \sin x}$.

Exercise 4.3.4. *Let φ be a mapping $\longrightarrow$ defined by $\varphi(t) = \frac{t^3}{1+t^6}$. Calculate $\varphi^{(n)}(0)$ for each $n \in \mathbb{N}$. Deduce the value of $\varphi^{(2016)}(0)$.*

Exercise 4.3.5. *Let a be a real number and $f :]a, +\infty[\longrightarrow$ a mapping of class C^2. We suppose that f and f'' are bounded; we set $M_0 = \sup_{x>a} |f(x)|$ and $M_2 = \sup_{x>a} |f''(x)|$.*

1. *By applying a Taylor's formula linking $f(x)$ and $f(x+h)$, show that, for each $x > a$ and $h > 0$, we have*
$$|f'(x)| \leq \frac{h}{2} M_2 + \frac{2}{h} M_0.$$

2. *Deduce that f' is bounded over $]a, +\infty[$.*

3. *Establishe the following result; let $g :]0, +\infty[\longrightarrow$ a mapping of class C^2 with its second derivative is bounded such that $\lim_{x \to +\infty} g(x) = 0$. Then $\lim_{x \to +\infty} g'(x) = 0$.*

Exercise 4.3.6. *By using a Taylor's formula, find $a, b \in$ such that*

$$d(x) = \cos x - \frac{1 + ax^2}{1 + bx^2},$$

is an $o(x^n)$ at 0 with n maximum.

Exercise 4.3.7. *By applying a Taylor's formula, calculate*

$$l = \lim_{x \to +\infty} \left(\frac{\ln(x+1)}{\ln x} \right)^x,$$

then give an equivalent of $\left(\frac{\ln(x+1)}{\ln x} \right)^x - l$ as $x \to +\infty$.

Exercise 4.3.8. *Give through a Maclaurin's expansion an approximate value of $\sin 20^\circ$ with an error up to 5° decimal after the comma.*

Exercise 4.3.9. *Calculate the Taylor-Young's expansion of the following functions. and give a simple equivalent of each them:*

1. $2e^x - \sqrt{1 + 4x} - \sqrt{1 + 6x^2}$, *at 0.*

2. $(\cos x)^{\sin x} - (\cos x)^{\tan x}$, *at 0.*

3. $\arctan x + \arctan \frac{3}{x} - \frac{2\pi}{3}$, *at $\sqrt{3}$.*

4. *Argch*$(\frac{1}{\cos x})$, *at 0.*

5. $\sqrt{1 + x^2} - 2\sqrt[3]{x^3 + x} + \sqrt[4]{x^4 + x^2}$, *at $+\infty$.*

Gottfried Wilhelm Leibniz (1646-1716) was a philosopher, scientist, mathematician, logician, diplomat, lawyer, librarian and German philologist who wrote in Latin, German and French.

Exercise 4.3.10. *Let u and v two differentiable functions to the order n.*

1. *Prove the following Leibniz's formula:*

$$(uv)^{(n)} = \sum_{k=0}^{n} C_n^k u^{(k)} v^{(n-k)}.$$

2. *Calculate $\frac{d^n y}{dx^n}$ where $y = (1 - x^2) \cosh x$.*

3. *Calculate the derivatives to the order 3 for* $f(t) = \sin t \ln(2+t)$. *Then find its values at 0. Verify the obtained results by using the Maclaurin's expansion.*

Exercise 4.3.11. *Let* I *be an interval of and* $f : I \longrightarrow$ *of class* C^{2n} $(n \geq 1)$ *over* I. *Let* $x_0 \in I$ *such that* $f'(x_0) = f''(x_0) = ... = f^{(2n-1)}(x_0) = 0$. *Show that:*

1. *If* $f^{(2n)}(x_0) > 0$ *then* x_0 *is a local minimum of* f.
2. *If* $f^{(2n)}(x_0) < 0$ *then* x_0 *is a local maximum of* f.

Exercise 4.3.12 (*)**.** *Let* f *be a function of class* C^3 *over verifying :*

$$\forall (x,y) \in^2,\ f(x+y)f(x-y) \leq (f(x))^2.$$

Show that $\forall x \in,\ f(x)f''(x) \leq (f'(x))^2$.
Hint: Apply the Taylor's formula with integral remainder between x *and* $x+y$ *then between* x *et* $x-y$.

Exercise 4.3.13. Function $\mathcal{C}^\infty$ with compact support
Let $f :^+ \rightarrow$ *of class* $\mathcal{C}^\infty$ *such that* $f(0) = 1$, *and :* $\forall\, x \geq \frac{1}{2}$, $f(x) = 0$.

1. *Show that* $\sup\limits_{+}\left|f^{(n)}\right| \geq 2^n n!$.
2. *Prove that for all* $n \geq 1$, $\sup\limits_{+}\left|f^{(n)}\right| > 2^n n!$.

Chapter 5

Numerical Resolution of an Equation

Motivation

It is known that for most of the equations we will not know to solve them, in fact it is not easy to say whether there is a solution, or how many there are. Then if this solution is available (as in the case of equation $x + \exp x = 0$), it's more likely that there is no known formula (with sums , products, ... of common functions) to find the solution x. In this chapter we will apply the concepts of sequences and functions, to find the zeros of a given function. Specifically, we will see three methods to find approximate solutions of an equation of the type $(f(x) = 0)$.

5.1 The bisection method

5.1.1 The bisection method

The bisection method based on the next version of the intermediate value theorem:

Theorem 5.1.1. *Let $f : [a, b] \to$ be a continuous function on a segment. If $f(a) \cdot f(b) \leq 0$, then there exists $\ell \in [a, b]$ such that $f(\ell) = 0$.*

The condition $f(a) \cdot f(b) \leq 0$ means that $f(a)$ and $f(b)$ are of opposite sign (or one of the two is zero). The assumption of continuity is essential ! This theorem states that there is at least one solution of equation $(f(x) = 0)$ in the interval $[a, b]$. To make it effective, and find an approximate solution of $(f(x) = 0)$, the challenge now is to apply it on a small enough interval. We

will see that it provides a solution ℓ of the equation $(f(x) = 0)$ as a limit of a sequence.
Here is how to build a more nested intervals, the length tends to 0, and each containing a solution of the equation $(f(x) = 0)$. We begin with a continuous function $f : [a, b] \to$ with $a < b$, and
$f(a) \cdot f(b) \leq 0$. This is the first stage of construction: looking at the sign of the value of the function f applied to the midpoint $\frac{a+b}{2}$.

- *If $f(a) \cdot f(\frac{a+b}{2}) \leq 0$, so there exists $c \in [a, \frac{a+b}{2}]$ such that $f(c) = 0$.*
- *If $f(a) \cdot f(\frac{a+b}{2}) > 0$, then $f(\frac{a+b}{2}) \cdot f(b) \leq 0$. Therefore, there exists $c \in [\frac{a+b}{2}, b]$ such that $f(c) = 0$.*

We got a half length of the interval in which the equation $(f(x) = 0)$ has a solution. Then iterates the process to divide again the interval in half.
Here the complete procedure:

- **At rank 0 :**

  We set $a_0 = a$, $b_0 = b$. there exists a solution x_0 of the equation $(f(x) = 0)$ in the interval $[a_0, b_0]$.

- **At rank 1 :**
  - *If $f(a_0) \cdot f(\frac{a_0+b_0}{2}) \leq 0$, then we set $a_1 = a_0$ and $b_1 = \frac{a_0+b_0}{2}$,*
  - *else, we set $a_1 = \frac{a_0+b_0}{2}$ and $b_1 = b$.*
  - *In the two cases, there exists a solution x_1 of the equation $(f(x) = 0)$ in the interval $[a_1, b_1]$.*

- *...*

- **At rank n :** *suppose we construct an interval $[a_n, b_n]$, with length $\frac{b-a}{2^n}$, and has a solution x_n of the equation $(f(x) = 0)$. Then :*
  - *If $f(a_n) \cdot f(\frac{a_n+b_n}{2}) \leq 0$, then we set $a_{n+1} = a_n$ and $b_{n+1} = \frac{a_n+b_n}{2}$,*
  - *else we set $a_{n+1} = \frac{a_n+b_n}{2}$ and $b_{n+1} = b_n$.*
  - *In the two cases, there exists a solution x_{n+1} of the equation $(f(x) = 0)$ in the interval $[a_{n+1}, b_{n+1}]$.*

At each step we have

$$a_n \leq x_n \leq b_n.$$

We stop the procedure when $b_n - a_n = \frac{b-a}{2^n}$ is less then the demanded precision.

As (a_n) is by construction an increasing sequence, and (b_n) is a decreasing sequence, and $(b_n - a_n) \to 0$ as $n \to +\infty$, then (a_n) and (b_n) are adjacent sequences and so they have the same limit. By the "sandwich" theorem, it is also the limit denoted it ℓ of the sequence (x_n). The continuity of f proves that $f(\ell) = \lim\limits_{n\to+\infty} f(x_n) = \lim\limits_{n\to+\infty} 0 = 0$. Then the sequences (a_n) and (b_n) goes to ℓ, which is a solution of the equation $(f(x) = 0)$.

5.1.2 Numerical results for $\sqrt{10}$

We will calculate an approximation of $\sqrt{10}$. Let f be a function defined by $f(x) = x^2 - 10$, it is a continuous function on which vanishes at $\pm\sqrt{10}$. In addition $\sqrt{10}$ is the unique positive solution of the equation $(f(x) = 0)$. We can restrict the function f on the interval $[3, 4]$: indeed $3^2 = 9 \leq 10$ so $3 \leq \sqrt{10}$ and $4^2 = 16 \geq 10$ then $4 \geq \sqrt{10}$. In other hand $f(3) \leq 0$ and $f(4) \geq 0$, then the equation $(f(x) = 0)$ admits a solution in the interval $[3, 4]$ by the IVT, and by the uniqueness of the solution it's $\sqrt{10}$, so $\sqrt{10} \in [3, 4]$. Note that we do not choose to f the function $x \mapsto x - \sqrt{10}$ because it knows not the value of $\sqrt{10}$. That's what we try to calculate ! Here are the first few steps:

1. *We set $a_0 = 3$ and $b_0 = 4$, we have $f(a_0) \leq 0$ and $f(b_0) \geq 0$. We calculate $\frac{a_0+b_0}{2} = 3,5$ and $f(\frac{a_0+b_0}{2})$: $f(3,5) = 3,5^2 - 10 = 2,25 \geq 0$. Then $\sqrt{10}$ is in the interval $[3; 3,5]$ ans we set $a_1 = a_0 = 3$ and $b_1 = \frac{a_0+b_0}{2} = 3,5$.*

2. *So we know that $f(a_1) \leq 0$ and $f(b_1) \geq 0$. We calculate $f(\frac{a_1+b_1}{2}) = f(3,25) = 0,5625 \geq 0$, we set $a_2 = 3$ et $b_2 = 3,25$.*

3. *We calculate $f(\frac{a_2+b_2}{2}) = f(3,125) = -0,23\ldots \leq 0$. As $f(b_2) \geq 0$ then this time f vanishes in the second half of the interval $[\frac{a_2+b_2}{2}, b_2]$ and we set $a_3 = \frac{a_2+b_2}{2} = 3,125$ and $b_3 = b_2 = 3,25$.*

At this stage, it has been proved : $3,125 \leq \sqrt{10} \leq 3,25$.

Here is the sequence of steps:

$$\begin{array}{ll}
a_0 = 3 & b_0 = 4 \\
a_1 = 3 & b_1 = 3,5 \\
a_2 = 3 & b_2 = 3,25 \\
a_3 = 3,125 & b_3 = 3,25 \\
a_4 = 3,125 & b_4 = 3,1875 \\
a_5 = 3,15625 & b_5 = 3,1875 \\
a_6 = 3,15625 & b_6 = 3,171875 \\
a_7 = 3,15625 & b_7 = 3,164062\ldots \\
a_8 = 3,16015\ldots & b_8 = 3,164062\ldots
\end{array}$$

So eight-step and the boundaries are obtained:

$$3,160 \leq \sqrt{10} \leq 3,165$$

In particular, it has obtained the first two digits:
$\sqrt{10} = 3,16\ldots$

5.1.3 Calculation error

The bisection method has the huge advantage of providing boundaries to a solution ℓ of the equation $(f(x) = 0)$. So it's easy to have a bounded above of the error. Indeed, at each step, the size interval containing ℓ is divided by 2. Initially, we know that $\ell \in [a, b]$ (of length $b - a$) ; and $\ell \in [a_1, b_1]$ (of length $\frac{b-a}{2}$) ; and $\ell \in [a_2, b_2]$ (of length $\frac{b-a}{4}$) ; ... ; $[a_n, b_n]$ being length $\frac{b-a}{2^n}$.
If, for example, it is desired to approximate ℓ up to 10^{-N}, as we know that $a_n \leq \ell \leq b_n$, we obtain $|\ell - a_n| \leq |b_n - a_n| = \frac{b-a}{2^n}$. So for get $|\ell - a_n| \leq 10^{-N}$, just choose n such that $\frac{b-a}{2^n} \leq 10^{-N}$.
We will use the logarithm:

$$\begin{aligned}
\frac{b-a}{2^n} \leq 10^{-N} &\iff (b-a)10^N \leq 2^n \\
&\iff \log(b-a) + \log(10^N) \leq \log(2^n) \\
&\iff \log(b-a) + N \leq n \log 2 \\
&\iff n \geq \frac{N + \log(b-a)}{\log 2}
\end{aligned}$$

Knowing $\log 2 = 0,301\ldots$, if for example $b-a \leq 1$, here the sufficient number of iterations to get a precision up to 10^{-N} (this corresponds, roughly, to N exact decimal).

10^{-10} *(~ 10 decimals)*	34 *iterations*
10^{-100} *(~ 100 decimals)*	333 *iterations*
10^{-1000} *(~ 1000 decimals)*	3322 *iterations*

It takes three to four more iterations to get a new decimal place..

Remark 5.1.1. *Strictly speaking it should not confuse precision and number of exact decimal, for example $0,999$ is an approximation of $1,000$ up to 10^{-3}, but no decimal point is accurate. In practice, it is the accuracy that is most important, but it is most striking about the number of exact decimal.*

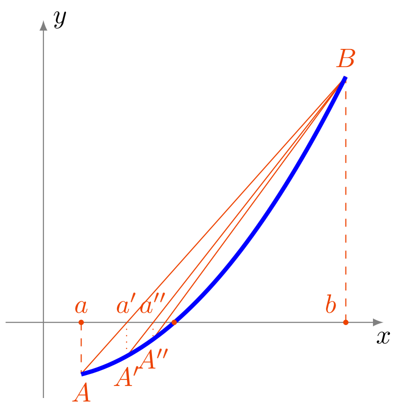
Exercise 5.1.1. (Mini-PW)

1. *By hand, calculate a boundary around $0,1$ to $\sqrt{3}$. Idem for $\sqrt[3]{2}$.*
2. *Calculate an approximation of the solutions to the equation $x^3+1=3x$.*
3. *Is it more effective to divide the interval into four instead of two? (At each iteration, the classical bisection method means evaluating f in a new value $\frac{a+b}{2}$ for improved accuracy by a factor of 2.)*
4. *Write an algorithm to compute several solutions $(f(x)=0)$.*

5.2 The secant method

5.2.1 The secant principle

The idea of the secant method is very simple: for a continuous function f on an interval $[a,b]$, and verifying $f(a)\leq 0$, $f(b)>0$, we draw the segment $[AB]$ where $A=(a,f(a))$ and $B=(b,f(b))$. If the segment is left above the graph of f then the function vanishes on the interval $[a',b]$ where $(a',0)$ is the intersection point of the straight line (AB) with the x-axis. The line (AB) is called secant. Now, it is repeated starting from the interval $[a',b]$ to obtain a value a''.

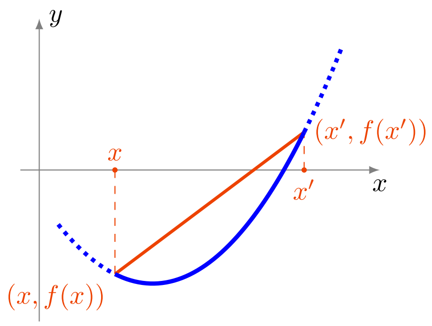


Proposition 5.2.1. *Let $f : [a,b] \to$ be a continuous function, strictly increasing and convex such that $f(a) \leq 0$, $f(b) > 0$. Then the sequence defined by:*

$$a_0 = a \qquad \text{and} \qquad a_{n+1} = a_n - \frac{b - a_n}{f(b) - f(a_n)} f(a_n)$$

is increasing and converges to the solution ℓ of $(f(x) = 0)$.

The hypothesis that f is convex means exactly that for each x, x' in $[a,b]$ the secant (or cord) between $(x, f(x))$ and $(x', f(x'))$ is above the graph of f.



Proof.

1. First justify the construction of the recursive sequence. The equation of the line passing through the two points $(a, f(a))$ et $(b, f(b))$ is

$$y = (x - a)\frac{f(b) - f(a)}{b - a} + f(a)$$

This straight line cuts the x-axis at $(a', 0)$ which then verifies

$$0 = (a' - a)\frac{f(b) - f(a)}{b - a} + f(a),$$

therefore

$$a' = a - \frac{b - a}{f(b) - f(a)} f(a).$$

2. <u>Increase of (a_n):</u>

Show by induction that $f(a_n) \leq 0$. It's true at rank 0 then $f(a_0) = f(a) \leq 0$ by given. Suppose that the hypothesis is true to the rank n. If $a_{n+1} < a_n$ then since f is strictly increasing, we have $f(a_{n+1}) < f(a_n)$, and in particular $f(a_{n+1}) \leq 0$. If not $a_{n+1} \geq a_n$. As f is convex, the secant between $(a_n, f(a_n))$ et $(b, f(b))$ is above the graph of f. In particular the point $(a_{n+1}, 0)$ (which is by definition of a_{n+1} on this secant) is above the point $(a_{n+1}, f(a_{n+1}))$, and so $f(a_{n+1}) \leq 0$ also in this case, concludes the induction. As $f(a_n) \leq 0$ and f is increasing, then by the formula

$$a_{n+1} = a_n - \frac{b - a_n}{f(b) - f(a_n)} f(a_n),$$

we obtain that $a_{n+1} \geq a_n$.

3. Convergency of (a_n):

   The sequence of (a_n) is increasing and bounded above by b, so it is convergent. Denoted by ℓ its limit. By Continuity $f(a_n) \to f(\ell)$. As for each n, $f(a_n) \leq 0$, we deduce that $f(\ell) \leq 0$. In particular, as we suppose that $f(b) > 0$, we have $\ell < b$. As $a_n \to \ell$, $a_{n+1} \to \ell$, $f(a_n) \to f(\ell)$, the equality

   $$a_{n+1} = a_n - \frac{b - a_n}{f(b) - f(a_n)} f(a_n)$$

   becomes when $n \to +\infty$:

   $$\ell = \ell - \frac{b - \ell}{f(b) - f(\ell)} f(\ell),$$

   which implies $f(\ell) = 0$.
   Conclusion: (a_n) converges to the solution of $(f(x) = 0)$.

5.2.2 Numerical results for $\sqrt{10}$

For $a = 3$, $b = 4$, $f(x) = x^2 - 10$ here the numerical results, and a bounded above for the corresponding error $\epsilon_n = \sqrt{10} - a_n$ (see below).

$a_0 = 3$	$\epsilon_0 \leq 0,1666\ldots$
$a_1 = 3,14285714285\ldots$	$\epsilon_1 \leq 0,02040\ldots$
$a_2 = 3,16000000000\ldots$	$\epsilon_2 \leq 0,00239\ldots$
$a_3 = 3,16201117318\ldots$	$\epsilon_3 \leq 0,00028\ldots$
$a_4 = 3,16224648985\ldots$	$\epsilon_4 \leq 3,28\ldots \cdot 10^{-5}$
$a_5 = 3,16227401437\ldots$	$\epsilon_5 \leq 3,84\ldots \cdot 10^{-6}$
$a_6 = 3,16227723374\ldots$	$\epsilon_6 \leq 4,49\ldots \cdot 10^{-7}$
$a_7 = 3,16227761029\ldots$	$\epsilon_7 \leq 5,25\ldots \cdot 10^{-8}$
$a_8 = 3,16227765433\ldots$	$\epsilon_8 \leq 6,14\ldots \cdot 10^{-9}$

5.2.3 Calculation error

The secant method provides a the inequalities $a_n \leq l \leq b$. But, as b is fixed it provides no usable information for $|l - a_n|$. Here, a general way of estimating the error, using the mean value theorem..

Proposition 5.2.2. *Let $f : I \to$ be a differentiable function and ℓ such that $f(\ell) = 0$. If there exists $m > 0$ such that for each $x \in I$, $|f'(x)| \geq m$ then*

$$|x - \ell| \leq \frac{|f(x)|}{m} \qquad \textit{for each}\, x \in I.$$

Proof. By the MVT between x and ℓ:

$$|f(x) - f(\ell)| \geq m|x - \ell|.$$

As $f(\ell) = 0$, we obtain the desired upper bound. □

Example 5.2.1. (Error for $\sqrt{10}$)
Let $f(x) = x^2 - 10$ and the interval $I = [3, 4]$. Then $f'(x) = 2x$ so $|f'(x)| \geq 6$ over I. We then set $m = 6$, $\ell = \sqrt{10}$, $x = a_n$. We obtain the estimation of the error:

$$\epsilon_n = |\ell - a_n| \leq \frac{|f(a_n)|}{m} = \frac{|a_n^2 - 10|}{6}$$

For example, we found $a_2 = 3,16... \leq 3,17$ then

$$\sqrt{10} - a_2 \leq \frac{|3,17^2 - 10|}{6} = 0,489.$$

For a_8 We found $a_8 = 3,1622776543347473\ldots$ then

$$\sqrt{10} - a_8 \leq \frac{|a_8^2 - 10|}{6} = 6,14\ldots \cdot 10^{-9}.$$

We made seven exact decimal places.

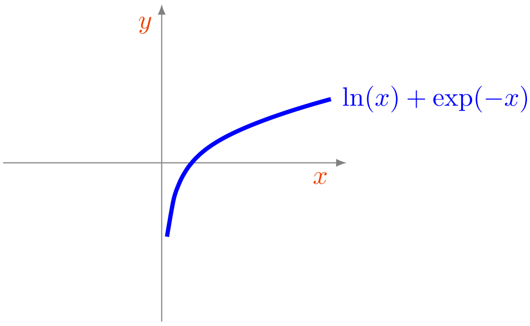
In practice, this is the sufficient number of iterations for a precision up to 10^{-n} for this example. Roughly speaking , an iteration over gives an additional decimal place.

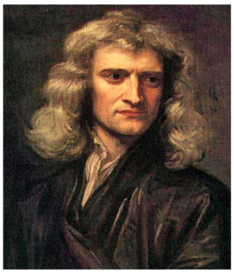
10^{-10} ($\sim$ 10 decimals)	10 iterations
10^{-100} ($\sim$ 100 decimals)	107 iterations
10^{-1000} ($\sim$ 1000 decimals)	1073 iterations

□

Exercise 5.2.1. (Mini-PW)

1. *By hand, calculate a boundary around* $0,1$ *to* $\sqrt{3}$. *Idem for* $\sqrt[3]{2}$.
2. *Calculate an approximation of the solutions to the equation* $x^3+1=3x$.
3. *Calculate an approximation of the solution to the equation* $(\cos x=0)$ *on* $[0,\pi]$. *Idem with* $(\cos x=2\sin x)$.
4. *Study the equation* $(\exp(-x)=-\ln(x))$. *Give an approximation of the solution and an upper bound of the corresponding error.*





IIsaac Newton (16431727) was a philosopher, mathematician, physicist, alchemist, astronomer and English theologian and British. Emblematic of science, he is best known for having founded classical mechanics, for its theory of universal gravitation and the creation, in competition with Gottfried Wilhelm Leibniz's calculus.

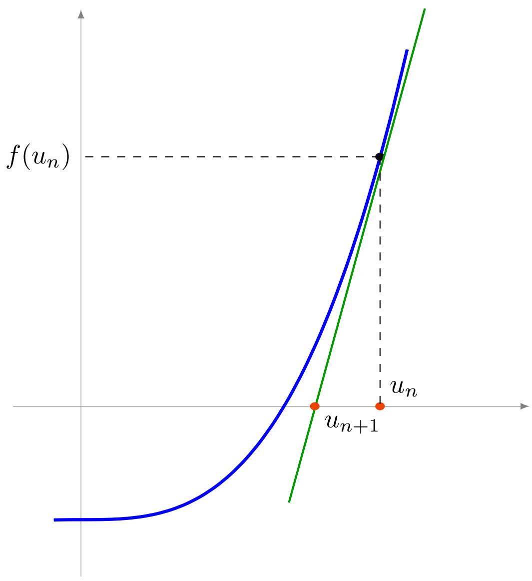
5.3 Newton Method

Newton's method consists in replacing the secant of the previous method by the tangent. It is extremely effective. Let us start from a differentiable function $f:[a,b]\rightarrow$ *and from a point* $u_0\in[a,b]$. *Let* $(u_1,0)$ *be the intersection of the tangent to the graph of* f *at* $(u_0,f(u_0))$ *with the axis of abscissas. If* $u_1\in[a,b]$ *then the operation is repeated with the tangent at the point of abscissa* u_1. *This process leads to the definition of a recurrent sequence:* $u_0\in[a,b]$ *and* $u_{n+1}=u_n-\frac{f(u_n)}{f'(u_n)}$.

Proof. Indeed, the tangent at the point of abscissa u_n has the equation:

$$y=f'(u_n)(x-u_n)+f(u_n).$$

Then, the point $(x,0)$ belonging to the tangent (and to the axis of abscissas) verifies $0=f'(u_n)(x-u_n)+f(u_n)$. Hence, $x=u_n-\frac{f(u_n)}{f'(u_n)}$. $\square$



5.3.1 Results for $\sqrt{10}$

Héron d'Alexandria was an engineer, a mechanic and a Greek mathematician of the first century after Christ.

To calculate $\sqrt{a}$, we set $f(x) = x^2 - a$, with $f'(x) = 2x$. The sequence derived from **Newton's method** *is determined by $u_0 > 0$ and the recurrence relation $u_{n+1} = u_n - \frac{u_n^2 - a}{2u_n}$. Sequence which, for this example, is called* **Héron** *sequence and which is often rewritten*

$$u_0 > 0 \quad \text{and} \quad u_{n+1} = \frac{1}{2}\left(u_n + \frac{a}{u_n}\right).$$

Proposition 5.3.1. *This sequence (u_n) converges to $\sqrt{a}$.*

For the calculation of $\sqrt{10}$, we set for example $u_0 = 4$, and we can even start the calculations by hand:

$$\begin{aligned}
u_0 &= 4 \\
u_1 &= \tfrac{1}{2}\left(u_0 + \tfrac{10}{u_0}\right) = \tfrac{1}{2}\left(4 + \tfrac{10}{4}\right) = \tfrac{13}{4} = 3,25 \\
u_2 &= \tfrac{1}{2}\left(u_1 + \tfrac{10}{u_1}\right) = \tfrac{1}{2}\left(\tfrac{13}{4} + \tfrac{10}{\frac{13}{4}}\right) = \tfrac{329}{104} = 3,1634\ldots \\
u_3 &= \tfrac{1}{2}\left(u_2 + \tfrac{10}{u_2}\right) = \tfrac{216\,401}{68\,432} = 3,16227788\ldots \\
u_4 &= 3,162277660168387\ldots
\end{aligned}$$

For u_4, we obtain $\sqrt{10} = 3,1622776601683\ldots$ with already thirteen decimal places!
Here is the proof of convergence of the sequence (u_n) to $\sqrt{a}$.

Proof.

$$u_0 > 0 \quad \text{and} \quad u_{n+1} = \frac{1}{2}\left(u_n + \frac{a}{u_n}\right).$$

1. Prove that $u_n \geq \sqrt{a}$ for $n \geq 1$.

   Firstly,

   $$u_{n+1}^2 - a = \frac{1}{4}\left(\frac{u_n^2 + a}{u_n}\right)^2 - a = \frac{1}{4u_n^2}(u_n^4 - 2au_n^2 + a^2) = \frac{1}{4}\frac{(u_n^2 - a)^2}{u_n^2}.$$

   Then, $u_{n+1}^2 - a \geq 0$. As it is clear that for all $n \geq 0$, $u_n \geq 0$, we deduce that for all $n \geq 0$, $u_{n+1} \geq \sqrt{a}$. (Note that u_0 is arbitrary.)

2. Show that $(u_n)_{n\geq 1}$ is a decreasing sequence which converges.

   Since $\frac{u_{n+1}}{u_n} = \frac{1}{2}\left(1 + \frac{a}{u_n^2}\right)$ and for $n \geq 1$ we have just seen that $u_n^2 \geq a$ (then $\frac{a}{u_n^2} \leq 1$), so $\frac{u_{n+1}}{u_n} \leq 1$, for all $n \leq 1$.

   Consequence: The sequence $(u_n)_{n\geq 1}$ is decreasing and bounded from below by 0, then it converges.

3. (u_n) converges to $\sqrt{a}$.

   Note ℓ the limit of (u_n). Hence, $u_n \to \ell$ and $u_{n+1} \to \ell$. When $n \to +\infty$ in the relation $u_{n+1} = \frac{1}{2}\left(u_n + \frac{a}{u_n}\right)$, we obtain $\ell = \frac{1}{2}\left(\ell + \frac{a}{\ell}\right)$. This leads to the relation $\ell^2 = a$ and by positivity of the sequence, $\ell = \sqrt{a}$.

□

5.3.2 The Error Calculation for $\sqrt{10}$

Proposition 5.3.2.

1. *Let k such that: $u_1 - \sqrt{a} \leq k$. Then, for all $n \geq 1$:*

   $$u_n - \sqrt{a} \leq 2\sqrt{a}\left(\frac{k}{2\sqrt{a}}\right)^{2^{n-1}}$$

2. *For $a = 10$, $u_0 = 4$, we have :*

   $$u_n - \sqrt{10} \leq 8\left(\frac{1}{24}\right)^{2^{n-1}}$$

*We admire the power of **Newton**'s method: Eleven iterations already give one thousand exact decimal places. This convergence speed is justified by calculating the error: the precision is multiplied by 2 at each step, thus at each iteration, the number of double decimal places!*

10^{-10} *(~ 10 decimal places)*	4 *iterations*
10^{-100} *(~ 100 decimal places)*	8 *iterations*
10^{-1000} *(~ 1000 decimal places)*	11 *iterations*

Proof.

1. In the proof of the proposition 5.3.1, we saw the equality:

$$u_{n+1}^2 - a = \frac{(u_n^2 - a)^2}{4u_n^2},$$

then

$$(u_{n+1} - \sqrt{a})(u_{n+1} + \sqrt{a}) = \frac{(u_n - \sqrt{a})^2(u_n + \sqrt{a})^2}{4u_n^2}.$$

Hence, since $u_n \geq \sqrt{a}$ for $n \geq 1$:

$$\begin{aligned} u_{n+1} - \sqrt{a} &= (u_n - \sqrt{a})^2 \times \frac{1}{u_{n+1} + \sqrt{a}} \times \frac{1}{4}\left(1 + \frac{\sqrt{a}}{u_n}\right)^2 \\ &\leq (u_n - \sqrt{a})^2 \times \frac{1}{2\sqrt{a}} \times \frac{1}{4} \cdot (1+1)^2 \\ &= \frac{1}{2\sqrt{a}}(u_n - \sqrt{a})^2. \end{aligned}$$

If k verifies $u_1 - \sqrt{a} \leq k$, we will deduce by induction, for all $n \geq 1$, the formula

$$u_n - \sqrt{a} \leq 2\sqrt{a}\left(\frac{k}{2\sqrt{a}}\right)^{2^{n-1}}$$

That is true for $n = 1$. Suppose the true formula to rank n, then:

$$u_{n+1} - \sqrt{a} \leq \frac{1}{2\sqrt{a}}(u_n - \sqrt{a})^2 = \frac{1}{2\sqrt{a}}(2\sqrt{a})^2\left(\left(\frac{k}{2\sqrt{a}}\right)^{2^{n-1}}\right)^2 = 2\sqrt{a}\left(\frac{k}{2\sqrt{a}}\right)^{2^n}$$

The formula is then true to the next rank.

2. For $a = 10$ with $u_0 = 4$, we have $u_1 = 3,25$. Since $3 \leq \sqrt{10} \leq 4$ then $u_1 - \sqrt{10} \leq u_1 - 3 \leq \frac{1}{4}$. So, we fix $k = \frac{1}{4}$. Always by $3 \leq \sqrt{10} \leq 4$, the obtained formula above becomes

$$u_n - \sqrt{10} \leq 2 \cdot 4 \left(\frac{\frac{1}{4}}{2 \cdot 3} \right)^{2^{n-1}} = 8 \left(\frac{1}{24} \right)^{2^{n-1}} .$$

□

Exercise 5.3.1. (Mini-PW)

1. *Let $a > 0$. How to calculate $\frac{1}{a}$ by a method of* ***Newton****?*

2. *Calculate n so that $u_n - \sqrt{10} \leq \frac{1}{3}$, with $u_0 = 4$, $u_{n+1} = \frac{1}{2}\left(u_n + \frac{10}{u_n}\right)$.*

5.4 Exercises

Exercise 5.4.1.

1. *Prove that the equation $x \ln x = 1$ admits in $[1, e]$ a unique solution $\bar{x}$.*

2. *Use the Newton method to approach $\bar{x}$ $(x_0 = 1.5)$. Do two iterations.*

3. *Use the bisection method to approach $\bar{x}$ with error up to 10^{-2}.*

Exercise 5.4.2.

1. *Prove that the equation $xe^x = 1$ admits in $[0, 1]$ a unique solution $\bar{x}$.*

2. *Use the bisection method to find $\bar{x}$. Do three iterations.*

3. *Use the secant method to find $\bar{x}$. Do two iterations.*

Exercise 5.4.3. *we consider the following function: $f(x) = 3x - \cos x - 1$.*

1. *Prove that $f(x) = 0$ admits in $[0, \frac{\pi}{3}]$ a unique solution α.*

2. *By using the bisection method, calculate the number of used iterations to obtain*
$$|x_n - \alpha| \leq 10^{-5}.$$

3. *Use the bisection method to find α. Do two iterations.*

4. *Use the Newton method to find α $(x_0 = \frac{\pi}{3})$. Do two iterations.*

Exercise 5.4.4 (*).
We consider the function $f(x) = e^{-x} - x^2$.

1. *Show that there exists a unique* $\alpha \in [0, 1]$ *such that* $f(\alpha) = 0$.
2. *Find the necessary minimal number of iterations to approximate* α *with an accuracy of* 2^{-20} *by using the bisection method.*
3. *Let the function* $\varphi(x) = x - \frac{1}{4}(x^2 - e^{-x})$.

   a) *Verify that* α *is a fixed point of* φ.

   b) *Prove that* $0 \leq \varphi(x) \leq 1$ *for each* $x \in [0, 1]$.

   c) *We define the sequence* $(x_k)_k$ *by*

   $$x_0 \in [0, 1] \quad et \quad x_{k+1} = \varphi(x_k) \ \forall\ k \geq 0.$$

   Show that there exists $0 < C < 1$ *such that*

   $$|x_{k+1} - \alpha| \leq C|x_k - \alpha|, \quad k \geq 0.$$

   Deduce that the limit of the sequence $(x_k)_k$ *is* α. *Give the necessary minimal number of iterations to estimate* α *with accuracy of* 2^{-20}.

Exercise 5.4.5. *By using the bisection method, give in terms of* p *the necessary number of iterations to approach* $\sqrt{p}$ *up to order* $0,1$ *where* p *is a prime integer number.*

Exercise 5.4.6. *By using the bisection method, give in terms of* p *and* ℓ *the number of necessary iterations to approach* $\sqrt[3]{p}$ *to within* $2^{-\ell}$, *where* p *is a prime integer and* ℓ *is a nonzero positive integer.*

Exercise 5.4.7. *Calculate the error to approach* $\sqrt{17}$ *by using Newton's method.*

Exercise 5.4.8. *Approach* $(1, 10)^{1/12}$ *by comparing the three used methods.*

Exercise 5.4.9. *Using the three proposed methods in this chapter, calculate an approximation of solutions of equation* $x^2 - 9 = 2x$. *Then compare the used methods.*

Chapter 6

Differential Equations

Motivation

In mathematics, a differential equation is a relationship between one or several unknown functions and their derivatives. The differential equations are used to construct mathematical models of physical and biological phenomena. Therefore, the differential equations are a vast field of study, both in pure mathematics and in applied mathematics.

Rudolf Otto Sigismund Lipschitz (1832-1903) was a German mathematician. Lipschitz left his name with bounded derivative applications (Lipschitz continuity). In reality, his work covers diverse areas as number theory, analysis, differential geometry.

6.1 Definition and Existence

Definition 6.1.1.

- *An ordinary differential equation (ODE) is a relation between the real variable x, an unknown function $x \longrightarrow y(x)$, and its derivatives y', y'', ..., $y^{(n)}$ at a point x defined as*

$$F(x,\, y,\, y',\, y'',\, ...,\, y^{(n)}) = 0.$$

- *A normal differential equation of order n is an equation in the form*

$$y^{(n)} = f(x,\, y,\, y',\, y'',\, ...,\, y^{(n-1)}).$$

The function y verifies one of the equations above is said to be ***solution*** *or* ***integral*** *of the differential equation. We can denote by $y' = \dfrac{d\,y}{d\,x}$.*

Example 6.1.1. *$y = \ln x$ is a solution of the differential equation*

$$y' = \frac{1}{x(\ln x)^2}\, y^2$$

defined on $]1,\, +\infty[$.

Theorem 6.1.1. (Cauchy-Lipschitz Theorem)
Let I be an interval of $\mathbb{R}$ and f be a real valued function of two real variables:

$$f: \begin{array}{ccc} I\times\mathbb{R} & \longmapsto & \mathbb{R} \\ (x,\,y) & \longmapsto & f(x,\,y(x)). \end{array}$$

Consider the following Cauchy problem:

$$\begin{cases} y'(x) = f(x,y) \\ y(x_0) = y_0,\ x_0 \in I. \end{cases} \tag{6.1}$$

If the function f is continuous and k-Lipschitz at y, i.e. if f verifies the Lipschitz condition:

Charles Émile Picard (1856-1941) was a French mathematician, specialist in mathematical analysis. He left his name to an iterative method of resolution integral equations.

$$\exists\, k>0\,/\,\forall y\in\mathbb{R},\,\forall x_1,\,x_2\in I^2,\quad \text{we have}\quad \Big|f(x_1,\,y)-f(x_2,\,y)\Big| \leq k\Big|x_1-x_2\Big|,$$

then, there exists one and only one solution $y(x)$ of the differential equation defined for all $x\in I$, verifies the given initial condition.

Proof. For the proof, we will need the following lemma which admitted without proof:

Lemma 6.1.1. (Picard's fixed-point lemma)
Let $T:\mathbb{R}\longmapsto\mathbb{R}$ be a contraction mapping i.e.

$$\forall 0<k<1\,/\,\forall x_1,\,x_2\in\mathbb{R},\quad \text{we have}\quad \Big|T(x_1)-T(x_2)\Big| \leq k\Big|x_1-x_2\Big|,$$

then, there exists q unique fixed point $x_0\in\mathbb{R}$ such that $T(x_0)=x_0$.

Let $f: I\times\mathbb{R}\longmapsto\mathbb{R}$ be a continuous function and Lipschitz with respect to y. Consider the Cauchy problem (6.1). Then,

$$\exists\, k>0\,/\,\forall y\in\mathbb{R},\,\forall x_1,\,x_2\in I^2,\quad \text{we have}\quad \Big|f(x_1,\,y)-f(x_2,\,y)\Big| \leq k\Big|x_1-x_2\Big|.$$

By Taylor, the system (6.1) is equivalent to integral form

$$y(x) = y(x_0) + \int_x^{x_0} y'(s)d\,s \quad \forall x\in I,$$

then

$$y(x) = y(x_0) + \int_x^{x_0} f\Big(s, y(s)\Big)d\,s.$$

Define now the mapping $T_y:\mathbb{R}\longrightarrow\mathbb{R}$ by

$$T_y(x) = y(x_0) + \int_x^{x_0} f\Big(s, y(s)\Big)d\,s.$$

Then $y(x) = T_y(x)$, $\forall x \in I$. Hence, $\forall\, y,\ z\ \in \mathbb{R}$, $\forall\, x \leq x_0$, we will have

$$\begin{aligned}
e^{-k(x-x_0)}|T_y(x) - T_z(x)| &\leq e^{-k(x-x_0)} \int_x^{x_0} |f(s, y(s)) - f(s, z(s))| d\, s \\
&\leq e^{-k(x-x_0)} \int_x^{x_0} k|y(s) - z(s)| d\, s \\
&\leq e^{-k(x-x_0)} \int_x^{x_0} k e^{k(s-x_0)} \Big(e^{-k(s-x_0)}|y(s) - z(s)|\Big) d\, s \\
&\leq e^{-k(x-x_0)} \int_x^{x_0} k e^{k(s-x_0)} \sup_{t\in I} |y(t) - z(t)| d\, s \\
&\leq e^{-k(x-x_0)} \sup_{t\in I} |y(t) - z(t)| \Big(1 - e^{-k(x-x_0)}\Big) d\, s \\
&\leq \Big(1 - e^{-k(x-x_0)}\Big) \sup_{t\in I} |y(t) - z(t)|.
\end{aligned}$$

We obtain exactly the same result for $x \leq x_0$ by replacing $e^{-k(x-x_0)}$ by $e^{-k(x_0-x)}$. Thus, $\forall\, y,\ z\ \in \mathbb{R}$, $\forall\, x \in\ I$, we have

$$\begin{aligned}
e^{-k|x-x_0|}|T_y(x) - T_z(x)| &\leq \Big(1 - e^{-k|x-x_0|}\Big) \sup_{t\in I} |y(t) - z(t)| \\
&\leq \left(1 - \exp\left(-k \min_{x\in I} |x - x_0|\right)\right) \sup_{t\in I} |y(t) - z(t)|.
\end{aligned}$$

Consequently, $\forall\, y,\ z\ \in \mathbb{R}$, $\forall\, x \in\ I$, we have

$$|T_y(x) - T_z(x)| \leq \left(1 - \exp\left(-k \min_{x\in I} |x - x_0|\right)\right) |y(x) - z(x)|.$$

Therefore, T is a contraction because $0 < 1 - \exp\left(-k \min_{x\in I} |x - x_0|\right) < 1$. Hence, by the Picard's fixed-point lemma, T has a unique fixed point and the proof is completed. $\square$

Exercise 6.1.1. (Mini-PW)
Let $y' = \sin(xy)$. Prove that there exists one and only one solution $x \longrightarrow y(x)$ of this differential equation defined for all $x \in \mathbb{R}$, verified $y(x_0) = y_0$, where $x_0,\ y_0 \in \mathbb{R}$.

6.2 Differential Equation of the First Order

6.2.1 Differential Equations with Separable Variables

Definition 6.2.1. *A differential equation with* ***separable variables*** *is a differential equation in the form*

$$a\,(y(x))\, y'(x) = b(x). \tag{6.2}$$

The solutions are obtained by calculating primitive with respect to x of both sides of the equation.

Example 6.2.1. *Solve the equation*

$$y' = (\cos x)y^3, \quad y \neq 0.$$

We have

$$\frac{dy}{y^3} = \frac{\cos x}{dx} \Rightarrow \int \frac{dy}{y^3} = \int \cos x dx \Rightarrow \frac{-1}{2y^2} = \sin x + cte \Rightarrow y = \pm\sqrt{\frac{-1}{2\sin x + cte}}.$$

A special case of differential equations with separable variables is the differential equation of the form

$$y' = f\left(\frac{y}{x}\right), \ x \neq 0. \tag{6.3}$$

To solve the equation (6.3), we use the change of variable

$$z(x) = \frac{y(x)}{x}.$$

6.2.2 Linear Differential Equations of the First Order

Definition 6.2.2. *Let a, b, and c be three functions defined on the interval I of $\mathbb{R}$ and y be the unkown function, defined and differentiable on the interval I. Moreover, suppose that the function a does not vanish on the interval I. A linear differential equation of the first order LDE is an equation in the form*

$$a(x)y'(x) + b(x)y(x) = c(x). \tag{6.4}$$

Definition 6.2.3. *A* ***particular solution*** *of the differential equation*

$$a(x)y'(x) + b(x)y(x) = c(x)$$

is a function y which satisfies the equation.

Example 6.2.2. $y(x) = \sin x$ *is particular solution of the equation*

$$(\cos x)y' + (\sin x)y = 1.$$

Proposition 6.2.1. *The set of solutions of equation (6.4) is obtained by adding all the solutions of the equation without second member to any particular solution of (6.4).*

Resolution of the equation without second member:
We have

$$\begin{aligned} a(x)y'(x)+b(x)y(x)=0 \quad &\Rightarrow \quad y'(x)+\frac{b(x)}{a(x)}y=0 \Rightarrow \frac{y'}{y}=-\frac{b(x)}{a(x)} \\ &\Rightarrow \quad \int \frac{y'}{y}d\,x=\int -\frac{b(x)}{a(x)}d\,x \Rightarrow \ln|y|=-\int \frac{b(x)}{a(x)}d\,x+C \\ &\Rightarrow \quad y=K\exp\left(-\int \frac{b(x)}{a(x)}d\,x\right), \quad \text{where} \quad K{=}{\pm}e^{C}. \end{aligned} \tag{6.6}$$

Thus, the found solution is called the **general solution** *of the equation without second member.*

Particular solution by variation of the constant:
Firstly, set $F(x)=-\int \frac{b(x)}{a(x)}d\,x$. *Let us find the particular solution in the form* $y=K(x)e^{F(x)}$, *with* K *is a function to be determined. Then, by substitution of* y *in the equation (6.3), we obtain*

$$K'(x)=\frac{c(x)}{a(x)}e^{-F(x)} \Leftrightarrow K(x)=\int \frac{c(x)}{a(x)}e^{-F(x)}d\,x.$$

The constant can be forgotten, because it corresponds to a solution of the equation without second member. Therefore, a particular solution is

$$y=\left(\int \frac{c(x)}{a(x)}e^{-F(x)}d\,x\right)e^{F(x)}.$$

This method is called the method of **variation of the constant**.
Finally, the proposition 6.2.1 allows us to conclude that the general solution of the complete equation is given by

$$y=\left(K+\int \frac{c(x)}{a(x)}e^{-F(x)}d\,x\right)e^{F(x)}, \quad \text{where} \quad K\in\mathbb{R}.$$

Example 6.2.3. $y=e^x$ *is the unique solution to the equation*

$$\begin{cases} y'+2y=3e^x, \\ y(0)=1. \end{cases}$$

The equation without second member

$$a(x)y'(x) + b(x)y(x) = 0. \tag{6.7}$$

It is also called the homogeneous equation associated to (6.3). It can be written in the form

$$y'(x) + \frac{b(x)}{a(x)}y = 0. \tag{6.8}$$

Jacques or Jacob Bernoulli (1654-1705) was a Swiss physicist and mathematician. He earned through his work and discoveries to be made a partner of the Academy of Science of Paris (1699) and one in Berlin (1702).

Jacopo Francesco Riccati (1676-1754) was an Italian physicist and mathematician. His works were published after his death by his son from 1764 under the title Opere del conte Jacopo Riccati.

6.2.3 Differential Equations Bernoulli type and Riccati type

Bernoulli's equation:
The Bernoulli equation is in the form:

$$y' + a(x)y + b(x)y^n = 0 \qquad n \in \mathbb{Z} \quad n \neq 0, n \neq 1$$

It reduces to a linear equation by the function change $z(x) = \frac{1}{y(x)^{n-1}}$. Indeed, it is assumed that a solution y does not vanish. Dividing the equation

$$y' + a(x)y + b(x)y^n = 0$$

by y^n yields

$$\frac{y'}{y^n} + a(x)\frac{1}{y^{n-1}} + b(x) = 0.$$

Set $z(x) = \frac{1}{y^{n-1}}$. So, $z'(x) = (1-n)\frac{y'}{y^n}$. The Bernoulli equation becomes a differential linear equation

$$\tfrac{1}{1-n}z' + a(x)z + b(x) = 0.$$

Example 6.2.4. *To find the solutions of the equation $xy' + y - xy^3 = 0$, we look for the solutions y that are do not vanish. We can then divide by y^3 to obtain:*

$$x\frac{y'}{y^3} + \frac{1}{y^2} - x = 0.$$

Set $z(x) = \frac{1}{y^2(x)}$, then $z'(x) = -2\frac{y'(x)}{y(x)^3}$. The differential equation is then expressed as

$$\frac{-1}{2}xz' + z - x = 0,$$

so that:

$$xz' - 2z = -2x.$$

The solutions of this equation on are

$$z(x) = \begin{cases} \lambda_+ x^2 + 2x \text{ if } x \geq 0 \\ \lambda_- x^2 + 2x \text{ if } x < 0 \end{cases} \quad , \quad \lambda_+, \lambda_- \in$$

As we set $z(x) = \frac{1}{y^2(x)}$, we restrict to an interval I in which $z(x) > 0$: necessarily $0 \notin I$, then we consider $z(x) = \lambda x^2 + 2x$, which is strictly positive on I_λ, where

$$I_\lambda = \begin{cases}]0; +\infty[& \textit{if } \lambda = 0 \\ \left]0; -\frac{2}{\lambda}\right[& \textit{if } \lambda < 0 \\ \left]-\infty; -\frac{2}{\lambda}\right[\text{ or }]0; +\infty[& \textit{if } \lambda > 0. \end{cases}$$

We have $(y(x))^2 = \frac{1}{z(x)}$, for all $x \in I_\lambda$. So, $y(x) = \epsilon(x)\frac{1}{\sqrt{z(x)}}$, where $\epsilon(x) = \pm 1$. y is continuous on the interval I_λ and does not vanish hypothetically: By the intermediate value theorem, y can not take both strictly positive values and strictly negative values, then $\epsilon(x)$ is either constant equal to 1 or is constant equal to à -1. Hence, the desired solutions are :

$$y(x) = \frac{1}{\sqrt{\lambda x^2 + 2x}} \text{ or } y(x) = \frac{-1}{\sqrt{\lambda x^2 + 2x}} \quad \text{on } I_\lambda \qquad (\lambda \in)$$

Note that the zero solution is also solution.

Riccati's Equation:
The **Riccati** equation is in the form:

$$y' + a(x)y + b(x)y^2 = c(x).$$

Let y_0 be a particular solution of the **Riccati**'s equation, then the function defined by $u(x) = y(x) - y_0(x)$ satisfies the **Bernoulli** equation (with $n = 2$). indeed, set $u(x) = y(x) - y_0(x)$, then $y = u + y_0$. The equation becomes :

$$u' + y_0' + a(x)(u + y_0) + b(x)(u^2 + 2uy_0 + y_0^2) = c(x)$$

Since y_0 is a particular solution, then

$$y_0' + a(x)y_0 + b(x)y_0^2 = c(x).$$

Therefore, the equation can be simplified to :

$$u' + \Big(a(x) + 2y_0(x)b(x)\Big)u + b(x)u^2 = 0$$

which is an equation of **Bernoulli** type.

Example 6.2.5. *Let us solve $x^2(y' + y^2) = xy - 1$, where $y_0(x) = \frac{1}{x}$ is a solution. We proceed:*

- *After division by x^2, it is a **Riccati** equation on $I =]-\infty; 0[$ ou $I =]0; +\infty[$.*
- $y_0 = \frac{1}{x}$ *is a particular solution.*
- *We have $u(x) = y(x) - y_0(x)$ and so $y = u + \frac{1}{x}$. The equation*

$$x^2(y' + y^2) = xy - 1$$

becomes

$$x^2\left(u' - \frac{1}{x^2} + u^2 + 2\frac{u}{x} + \frac{1}{x^2}\right) = x\left(u + \frac{1}{x}\right) - 1,$$

which simplified to

$$x^2\left(u' + u^2 + 2\frac{u}{x}\right) = xu,$$

*which corresponds to the **Bernoulli** equation:*

$$u' + \frac{1}{x}u + u^2 = 0.$$

- *if u does not vanish, by dividing by u^2, this equation becomes*

$$\frac{u'}{u^2} + \frac{1}{x}\frac{1}{u} + 1 = 0.$$

Set $z(x) = \frac{1}{u}$, the equation becomes

$$-z' + \frac{1}{x}z + 1 = 0.$$

Its solutions on I are

$$z(x) = \lambda x + x \ln|x|,$$

$\lambda \in$. Hence,

$$u(x) = \frac{1}{z(x)} = \frac{1}{\lambda x + x \ln|x|},$$

but there is also the zero solution $u(x) = 0$.

- ***Conclusion**. Since $y = u + \frac{1}{x}$, we obtain then solutions of the original equation on $]-\infty; 0[$ and $]0; +\infty[$:*

$$y(x) = \frac{1}{x} \qquad or \qquad y(x) = \frac{1}{x} + \frac{1}{\lambda x + x \ln|x|} \qquad (\lambda \in).$$

6.3 Differential Equation of Second Order with Constant Coefficients

Definition 6.3.1. *Let $a \neq 0$, b, and c be three real constants, f be a differentiable function over I, and y be the unknown function, defined and twice differentiable over I. A linear differential equation of second order with constant coefficients is an equation in the form*

$$ay''(x) + by'(x) + cy(x) = f(x). \tag{6.9}$$

The equation

$$ay''(x) + by'(x) + cy(x) = 0 \tag{6.10}$$

is called the homogeneous equation associated to the equation (6.9).

Theorem 6.3.1. *Each solution of (6.9) can be written in the form $y = y_h + y_p$, where y_h is the general solution of the homogeneous equation (6.10) and y_p is a particular solution of the equation "with second member" (6.9).*

Proof. Let y_1 and y_2 be two solutions of the equation (6.9). Then, $z = y_1 - y_2$ is a solution of (6.10). Hence, if y_1 represents the general solution of the equation with second member and y_2 is a particular solution, as $y_1 = z + y_2$, we prove that the general solution of the equation (6.9) is the sum of the general solution of the equation without second member and a particular solution of the equation with second member. □

6.3.1 General Solution of the Equation without Second Member

To find the general solution of the differential equation without second member $ay'' + by' + cy = 0$, we see if there exist solutions $y(x)$ having the same form as those of homogeneous equations associated with the LDE of first order with constant coefficient, so that in the form: $y(x) = e^{rx}$, where r is a real coefficient. For such a function, we have : $y'(x) = re^{rx}$ and $y''(x) = r^2e^{rx}$. By replacing in (6.10), we obtain:

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0$$

and so r must be solution of the equation

$$ar^2 + br + c = 0. \tag{6.11}$$

The equation (6.11) is called the **characteristic equation** associated to (6.10). The general solution $y_h(x)$ of the equation without second member (6.10) depends then on the value of the roots of this characteristic equation according to the following theorem that is accepted without proof:

Theorem 6.3.2. *Let $\Delta = b^2 - 4ac$, the discriminant. The real solutions of (6.10) depend on the sign of the discriminant:*

* $\Delta > 0$: *The general solution of the equation (6.10) is in the form*

$$y(x) = Ae^{r_1 x} + Be^{r_2 x},$$

where r_1 et r_2 are the two real roots of the characteristic equation (6.11), A and B are two arbitrary constants.

* $\Delta = 0$: *The general solution of the equation (6.10) is in the form*

$$y(x) = (Ax + B)e^{rx},$$

where r is the double root of the characteristic equation (6.11), A and B are two arbitrary constants.

* $\Delta < 0$: *The real general solution of the equation (6.10) is in the form*

$$y(x) = \Big(Acos(\beta x) + B\sin(\beta x)\Big)e^{\alpha x},$$

where $\alpha \pm i\beta$ are the complex roots of the characteristic equation.

Gabriel Cramer (1704-1752) was a Swiss mathematician, professor of mathematics and philosophy at the Academy of Geneva.

6.3.2 Particular Solution of the Equation with Second Member

We will seek for a particular solution of the equation with second member (6.9) using the method of variation of the constant. This method consists in assuming that $A = A(x)$ and $B = B(x)$. We set also

$$y(x) = A(x)y_1(x) + B(x)y_2(x),$$

where $y_1(x)$ and $y_2(x)$ are the particular solutions of the equation without second member (6.10) and satisfies $\Delta = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \neq 0.$
Moreover, We impose the condition

$$A'(x)y_1(x) + B'(x)y_2(x) = 0. \tag{6.12}$$

By substituting y, y' and y'' in (6.9) and by using the condition (6.12), we get

$$A'(x)y_1'(x) + B'(x)y_2'(x) = f(x).$$

Therefore, we obtain the following system

$$\begin{cases} A'(x)y_1(x) + B'(x)y_2(x) = 0, \\ A'(x)y_1'(x) + B'(x)y_2'(x) = f(x). \end{cases} \tag{6.13}$$

By the Cramer's theorem, we have

$$A'(x) = \frac{-y_2(x)f(x)}{\Delta} \quad \text{and} \quad B'(x) = \frac{-y_1(x)f(x)}{\Delta}.$$

So, we will seek to find a primitive of $A(x)$ and a primitive of $B(x)$.

Example 6.3.1. *Solve the following differential equation*

$$y'' + y = \frac{1}{\sin x}.$$

The general solution of the homogeneous equation is

$$y(x) = A(x)\cos x + B(x)\sin x.$$

On the other hand, $y_1(x) = \cos x$ and $y_2(x) = \sin x$ are two particular solutions of the equation with second member and $\Delta = 1 \neq 0$. Then, by the method of variation of the constant, we have

$$A'(x) = -1 \quad \text{and} \quad B'(x) = \frac{\cos x}{\sin x}.$$

Hence,

$$A(x) = x \quad \text{and} \quad B(x) = \ln|\sin x|.$$

Therefore, a particular solution of the equation with second member is

$$y_p = -x\cos x + \sin x \ln|\sin x|.$$

Finally, the general solution of the equation with second member is

$$y = y(x) = A\cos x + B\sin x + y_p.$$

Proposition 6.3.1. *Let I be an interval in $\mathbb{R}$. Then, for every $(x_0, y_0, y_1) \in I \times \mathbb{R}^2$, there exists a unique solution of (6.9) defined on I such that $y(x_0) = y_0$ and $y'(x_0) = y_1$.*

Proposition 6.3.2. Superposition principle
If y_1 is a solution of $ay''(x) + by'(x) + cy(x) = f_1(x)$ and y_2 is a solution of $ay''(x) + by'(x) + cy(x) = f_2(x)$, then $y_1 + y_2$ is a solution of $ay''(x) + by'(x) + cy(x) = f_1(x) + f_2(x)$.

Remark 6.3.1. *There are methods to find more quickly a particular solution of the equation with the second member when the latter has a certain expression.*

1. *Second member in the form* $f(x) = P(x)$*, where* P *is a polynomial: we seek a particular solution in the form of a polynomial. Reflect about the degree of this polynomial according as* c *is zero or not.*

2. *Second member in the form* $f(x) = \lambda\cos(\alpha x)+\mu\sin(\alpha x)$*. We must then distinguish two cases:*

    - *The function* $\cos(\alpha x)$ *(and so* $\sin(\alpha x)$*) is not a solution of the homogeneous equation. We can then seek a particular solution in the form:*
    $$z = A\cos(\alpha x) + B\sin(\alpha x).$$
    - *The function* $\cos(\alpha x)$ *(and so* $\sin(\alpha x)$*) is a solution of the homogeneous equation. We must then seek a particular solution in the form:*
    $$z = x(A\cos(\alpha x) + B\sin(\alpha x)).$$

3. *Second member in the form* $f(x) = g(x)e^{\alpha x}$*, where* g *is a polynomial or a function of* $\sin$ *and* $\cos$*, and* α *is a real. The general method is to make the change of unknown function* $y = z(x)e^{\alpha x}$ *and one is brought back to previous situations.*

6.4 Exercises

Exercise 6.4.1. *Solve:*

1. $y'' - 3y' + 2y = 0$
2. $y'' + 2y' + 2y = 0$
3. $y'' - 2y' + y = 0$
4. $y'' - 2y' + y = \ln x$
5. $y'' + y = 2\cos^2 x$
6. $y'' + y' - 2y = \tan x$

Exercise 6.4.2. *For the following differential equations, find the solutions defined on :*

1. $x^2y' - y = 0$ (E_1)

2. $xy' + y - 1 = 0$ (E_2)

Exercise 6.4.3. *We consider the differential equation*

$$y' - e^x e^y = a$$

Determine its solutions, carefully indicate their intervals of definition, for

1. $a = 0$
2. $a = -1$ *(make the change of the unknown function $z(x) = x + y(x)$)*

In each case, construct the integral curve passing through the origin.

Exercise 6.4.4 (Variation of the constant)**.** *Solve the following differential equations by finding a particular solution by the method of variation of the constant:*

1. $y' - (2x - \frac{1}{x})y = 1$ *sur* $]0; +\infty[$
2. $y' - y = x^k \exp(x)$ *sur , avec* $k \in \mathbb{N}$
3. $x(1 + \ln^2(x))y' + 2\ln(x)y = 1$ *sur* $]0; +\infty[$

Exercise 6.4.5. *Dtermine all the functions $f : [0; 1] \to$, differentiable, such that*

$$\forall x \in [0; 1],\ f'(x) + f(x) = f(0) + f(1)$$

Exercise 6.4.6. *Solve the following differential equations:*

1. $y' + 2y = x^2$ (E_1)
2. $y' + y = 2\sin x$ (E_2)
3. $y' - y = (x + 1)e^x$ (E_3)
4. $y' + y = x - e^x + \cos x$ (E_4)
5. $y' = \frac{x-y}{x+y}$
6. $xyy' = x^2 + y^2$

Exercise 6.4.7 (*)**.** *Find all the mappings $f :\to$, continuous over such that* $\forall (x, y) \in^2,\ f(x)f(y) = \displaystyle\int_{x-y}^{x+y} f(t)\, dt$.

Exercise 6.4.8. *Find all the functions f differentiable over verifiying $\forall x \in$, $f'(x) + f(-x) = e^x$.*

Exercise 6.4.9 (*)**.** *Solve on the proposed interval I:*

1. $xy' - 2y = 0$ *($I =$)*
2. $xy' - y = 0$ *($I =$)*
3. $xy' + y = 0$ *($I =$)*
4. $xy' - 2y = x^3$ *($I =]0, +\infty[$)*
5. $x^2 y' + 2xy = 1$ *($I =$)*
6. $2x(1-x)y' + (1-x)y = 1$ *($I =]-\infty, 0[,\ ]0,1[,\ ]1,+\infty[,\ ]-\infty, 1[,\ ]0, +\infty[,$)*
7. $|x|y' + (x-1)y = x^3$ *($I =$).*

Exercise 6.4.10. *Solve over the proposed differential equation:*

1. $\begin{cases} y' + y = 1, \\ y(0) = 2. \end{cases}$
2. $2y' - y = \cos x$
3. $y' - 2y = xe^{2x}$
4. $y'' - 4y' + 4y = e^{2x} + x$
5. $y'' + 4y = \cos(2x)$
6. $\begin{cases} y'' + 2y' + 2y = \cos x \cosh x, \\ y(0) = 0,\ y'(0) = 1. \end{cases}$

Exercise 6.4.11. *Solve the following differential equation:*

1. $xy' + y = xy^3$.
2. $2xy' + y = \frac{2x^2}{y^3}$.
3. $\sqrt{x}y' - y + (x + 2\sqrt{x})\sqrt{y} = 0$.
4. $xy' + y = (xy)^{3/2}$.
5. $x^3 y' = y(3x^2 + y^2)$.
6. $x^2(y' + y^2) = xy - 1$.

Chapter 7

Exams Sessions

Partial Exam - 2016

Exercise 1.

I- *Consider the subset A of $\mathbb{R}$ defined by*

$$A = \left\{5 + \frac{1}{2^n};\, n \in \mathbb{N}\right\}.$$

(a) *Prove that the set of all interior points of A is an empty set.*

(b) *Determine the set of all adherent points of A.*

(c) *Determine the set of all boundary points of A.*

(d) *Prove that A is not compact and $A \cup \{5\}$ is compact.*

(e) *Prove that 5 is an accumulation point of A. Deduce that A is not closed.*

II- *Consider now the subset B of $\mathbb{R}$ defined by*

$$B = \left\{\frac{5}{n} + \frac{1}{2^m};\, n \in \mathbb{N}^*,\, m \in \mathbb{N}\right\}.$$

Prove that $0,\, 1,\, \frac{1}{2},\, \frac{5}{2},\, \frac{1}{16}$ *are accumulation points of B.*

Exercise 2.

Consider the subset A of $\mathbb{R}$ defined by $A = \{1 + \sin n;\, n \in \mathbb{Z}\}$.

Using the fact that $\mathbb{Z} + 2\pi\mathbb{Z}$ *is dense in* $\mathbb{R}$, *show that A is dense in* $[0,\, 2]$.

SOLUTION

Exercise 1.

I- $A = \left\{5 + \frac{1}{2^n};\, n \in \mathbb{N}\right\}$.

We denote by:

$\overset{\circ}{A}$*: the set of all interior points of* A.

$\overline{A}$*: the set of all adherent points of* A.

∂A*: the set of all boundary points of* A.

(a) *Suppose that* $\overset{\circ}{A} \neq \varnothing$. *Let* $x \in \overset{\circ}{A}$. *Then, there exists* $\delta > 0$ *such that*

$$]x - \delta,\, x + \delta[\subset A.$$

By the density of $\mathbb{R} \setminus \mathbb{Q}$ *in* $\mathbb{R}$, *there exists* $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ *such that* $x - \delta < \alpha < x + \delta$. *Thus,* $\alpha \in A \subset \mathbb{Q}$, *which is impossible.*

Therefore, $\overset{\circ}{A} = \varnothing$.

(b) *Fistly,* $A \subset \overline{A}$.

Prove that 5 *is an adherent point of* A, *because* $5 = \lim\limits_{n \longmapsto +\infty} \left(5 + \frac{1}{2^n}\right)$ *and* $5 + \frac{1}{2^n} \in A$, *for all* $n \in \mathbb{N}$.

Now, prove that if $x \notin A$ *is an adherent point of* A, *then* $x = 5$. *In fact,* $x = \lim\limits_{n \longmapsto +\infty} x_n$, *where* $x_n \in A$ *for all* $n \in \mathbb{N}$. *But, the limit of all subsequences of* A *is equal to* 5. *Hence,* $x = 5$. *Consequently,* $\overline{A} = A \cup \{5\}$.

(c) *We have* $\partial A = \overline{A} \setminus \overset{\circ}{A} = \overline{A}$.

(d) A *is not compact, because* $5 = \lim\limits_{n \longmapsto +\infty} \left(5 + \frac{1}{2^n}\right)$ *and* $5 \notin A$.

$A \cup \{5\}$ *is compact, because the limit of all subsequences of* $A \cup \{5\}$ *is equal to* $5 \in A \cup \{5\}$.

(e) 5 *is an accumulation point of* A, *because* $5 = \lim\limits_{n \longmapsto +\infty} \left(5 + \frac{1}{2^n}\right)$ *and* $5 \neq 5 + \frac{1}{2^n}$, *for all* $n \in \mathbb{N}$.

A *is not closed, because* 5 *is an accumulation point of* A, *but* $5 \notin A$.

II- . $B = \left\{\frac{5}{n} + \frac{1}{2^m};\, n \in \mathbb{N}^*,\, m \in \mathbb{N}\right\}$.

- $0 = \lim\limits_{n \longmapsto +\infty} \left(\frac{5}{n} + \frac{1}{2^n}\right)$, $\frac{5}{n} + \frac{1}{2^n} \in B$, *and* $0 \neq \frac{5}{n} + \frac{1}{2^n}$, $\quad n \in \mathbb{N}^*$.

- $1 = \lim\limits_{n \longmapsto +\infty} \left(1 + \frac{1}{2^n}\right)$, $1 + \frac{1}{2^n} \in B$, *and* $1 \neq 1 + \frac{1}{2^n}$, $\quad n \in \mathbb{N}$.
- $\frac{1}{2} = \lim\limits_{n \longmapsto +\infty} \left(\frac{5}{10} + \frac{1}{2^n}\right)$, $\frac{5}{10} + \frac{1}{2^n} \in B$, *and* $\frac{1}{2} = \frac{5}{10} \neq \frac{5}{10} + \frac{1}{2^n}$, $\quad n \in \mathbb{N}$.
- $\frac{5}{2} = \lim\limits_{n \longmapsto +\infty} \left(\frac{5}{2} + \frac{1}{2^n}\right)$, $\frac{5}{2} + \frac{1}{2^n} \in B$, *and* $\frac{5}{2} \neq \frac{5}{10} + \frac{1}{2^n}$, $\quad n \in \mathbb{N}$.
- $\frac{1}{16} = \lim\limits_{n \longmapsto +\infty} \left(\frac{5}{n} + \frac{1}{2^4}\right)$, $\frac{5}{n} + \frac{1}{2^4} \in B$, *and* $\frac{5}{n} \neq \frac{5}{n} + \frac{1}{2^4}$, $\quad n \in \mathbb{N}^*$.

Therefore, 0, 1, $\frac{1}{2}$, $\frac{5}{2}$, $\frac{1}{16}$ *are accumulation points of* B.

Exercise 2. *We have* $A = \{1 + \sin n;\, n \in \mathbb{Z}\} = \{1 + \sin(n + 2k\pi);\, n,\, k \in \mathbb{Z}\}$. *Let* $x \in [0,\, 2]$. *Then* $x - 1 \in [-1,\, 1]$. *So, there exists* $\theta \in \left[-\frac{\pi}{2},\, \frac{\pi}{2}\right]$ *such that* $x - 1 = \sin\theta$ *(The function* $\sin$ *is bijective from* $\left[-\frac{\pi}{2},\, \frac{\pi}{2}\right]$ *into* $[-1,\, 1]$*).*
On the other hand, we have $\mathbb{Z} + 2\pi\mathbb{Z}$ *is dense in* $\mathbb{R}$. *Then, there exists a sequence* (x_n) *of* $\mathbb{Z} + 2\pi\mathbb{Z}$ *such that* $\lim\limits_{n \longmapsto +\infty} x_n = \theta$. *Then,* $\lim\limits_{n \longmapsto +\infty} \sin x_n = \sin\theta = x - 1$ *and so* $\lim\limits_{n \longmapsto +\infty} (1 + \sin x_n) = x$, *where* $1 + \sin x_n \in A$, $\quad \forall n \in \mathbb{N}$.
Consequently, A *is dense in* $[0,\, 2]$.

Final Exam 1- 2016

Exercise 1. *(12 points)*

1. ***Bolzano-Weierstrass.*** *(course)*
   Prove that every bounded sequence (x_n) has a convergent subsequence (x_{n_k}).

2. ***Application.***
   Consider the recurrent sequence (x_n) defined by $x_0 \in [1, e]$ and $x_{n+1} = e^{\frac{1}{x_n}}$, $n \in \mathbb{N}$.

   (a) Show that $x_n \in [1, e]$, for all n.

   (b) Deduce that the equation $1 - x \ln x = 0$ has a solution in the interval $[1, e]$.

Exercise 2. *(10 points)*
Consider the following differential equation:

$$(E) \qquad y' - \frac{y}{x} - y^2 = -9x^2.$$

1. *Verify that $y_0 = 3x$ is a particular solution of (E).*

2. *Prove that the change of function $y = y_0 - \frac{1}{z}$ transforms the equation (E) into the differential equation*

$$(E_1) \qquad z' + (6x + \frac{1}{x})z = 1.$$

3. *Solve (E_1) and give the general solution of (E).*

Exercise 3. *(14 points)*
Let f be a real function defined by $f(x) = 1 + \frac{x^2}{4}$, where $x > 0$.

1. *Prove that f has a unique fixed point to be determined.*

2. *Let $E =]0, 2[$. Show that $f(E) \subset E$.*

3. *Consider the sequence (u_n) defined by $u_0 > 0$ and $u_{n+1} = f(u_n)$, $n \in \mathbb{N}$.*

   (a) *If the sequence (u_n) is convergent, calculate its limit.*

   (b) *In this part, suppose that $u_0 \in E$.*

      i. *Show that the sequence (u_n) is bounded.*

      ii. *Study the monotonicity of the sequence (u_n). Deduce that it is convergent.*

4. *Study the monotonicity and the convergence of (u_n) in the following cases:*

   (a) $u_0 = 2$.

   (b) $u_0 > 2$.

Exercise 4.*(14 points)*
Consider the function $f : [0, \frac{\pi}{2}[\to \mathbb{R}$ defined by:

$$f(x) = \frac{x}{e^x \cos x}.$$

1. *By using the Taylor-Young formula at $x = 0$ (F.E.), find $f(0)$, $f'(0)$, $f''(0)$ et $f'''(0)$.*

2. (a) *Prove that $\lim_{x \to 0} \frac{\arctan x}{x} = 1$.*

   (b) *Let $\alpha \in \mathbb{R}$ such that $\alpha \neq 1$. Discuss, following the values of α, the following limit:*
   $$\lim_{x \to 0} \frac{\arctan x}{x^\alpha}.$$

3. (a) *Let g be the function defined by*
   $$g(x) = \frac{x \sin(\frac{1}{x})}{\arctan x} f(x).$$
   Can we apply the L'Hôspital rule to calculate $\lim_{x \to 0} g(x)$? Justify your answer.

   (b) *Find this limit.*

Final Exam 2- 2016

Exercise 1.
Let $(x_1, x_2, ..., x_n) \in \mathbb{R}^n$, where $n \in \mathbb{N}^*$ and $x_i > 0$, $\forall i \in \{1, ..., n\}$.

1. Use the Caushy-Shwarz inequality to show that:
   (a) $\left(\sum_{k=1}^{n} x_k\right)^2 \leq n \sum_{k=1}^{n} x_k^2.$
   (b) $\sum_{k=1}^{n} x_k\sqrt{x_k} \leq \sqrt[4]{n}\left(\sum_{k=1}^{n} x_k^2\right)^{\frac{3}{4}}.$
2. Deduce that
$$\lim_{n\to+\infty} \frac{\sum_{k=1}^{n} x_k\sqrt{x_k}}{n\left(\sum_{k=1}^{n} x_k^2\right)^{\frac{3}{4}}} = 0.$$

Exercise 2.

Consider the set
$$A = \left\{\sqrt{2} + \frac{1}{n};\ n \in \mathbb{N}^*\right\}.$$

1. Prove that the interior of A, denoted by $\overset{\circ}{A}$, is an empty set.
2. Prove that $\sqrt{2}$ is an accumulation point of A and deduce that A is not closed.

Exercise 3.
Consider the following differential equation:
$$y' - \frac{y}{x} - y^3 = 0 \qquad (E).$$

1. Use the change of variable $z = \dfrac{1}{y^2}$ to prove that (E) becomes:

$$\frac{1}{2}z' + \frac{1}{x}z = -1 \qquad (E_1).$$

2. Solve (E_1) and give the general solution of (E).

Exercise 4. Consider the function f defined over $\mathbb{R}$ by

$$f(x) = \frac{x^3 + 3ax}{3x^2 + a},$$

where $a > 0$. Let (u_n) be a sequence defined by:

$$\begin{cases} u_0 > 0, \\ u_{n+1} = f(u_n), \ \forall n \in \mathbb{N}. \end{cases}$$

a) Study the variations of f.

b) Find in terms of a the solution of the equation $f(x) = x$.

c) Study the sign of $f(x) = x$.

d) Study the monotonicity and the convergence of (u_n) in the following cases:

   (a) $u_0 \in]0, \sqrt{a}[$.
   (b) $u_0 = \sqrt{a}$.
   (c) $u_0 \in]\sqrt{a}, +\infty[$.

Partial Exam - 2017

Exercise 1.

Consider the following subsets of $\mathbb{R}$ defined by:

$A = \left\{ \dfrac{2}{n} + \dfrac{3}{m};\ m,\ n \in \mathbb{N}^* \right\}$, $B = \left\{ \dfrac{1}{\sqrt{n}} + \dfrac{1}{\sqrt{m}};\ m,\ n\,\text{are prime numbers} \right\}$, and $C =]1,\ 2]$.

1. Prove that the sets of all interior points of A and B are empty sets.
2. Prove that 2 is not an interior point of C and give $\mathring{C}$.
3. Determine the set of all adherent points of $A_1 = \left\{ \dfrac{2}{n} + \dfrac{3}{4};\ n \in \mathbb{N}^* \right\}$.
4. Determine the set of all boundary points of A_1.
5. Prove that the set of all boundary points of C is $\{1,\ 2\}$.
6. Prove that A_1 is not closed. Is A_1 compact?
7. Prove that $\dfrac{1}{\sqrt{2}}$ is an accumulation point of B. Deduce that B is not closed.

Exercise 2.
Consider the subset A of $\mathbb{R}$ defined by $A = \{\sin n;\ n \in \mathbb{Z}\}$.
Using the fact that $\mathbb{Z} + 2\pi\mathbb{Z}$ is dense in $\mathbb{R}$, show that A is dense in $[-1,\ 1]$.

Exercise 3.
Calculate the $\limsup$ and $\liminf$ of the sequence (x_n) defined by

$$x_n = \cos\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right) + \frac{(-1)^{n+1}}{n}.$$

SOLUTION

Exercise 1.

1. $\underline{\mathring{A} = \varnothing :}$
   Suppose that $\mathring{A} \neq \varnothing$ and let $x \in \mathring{A}$. Then, there exists $\delta > 0$ such that
   $$]x-\delta, x+\delta[\subset A \subset \mathbb{Q}.$$
   By the density of $\mathbb{R} \setminus \mathbb{Q}$ in $\mathbb{R}$, there exists $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ such that $x - \delta < \alpha < x + \delta$. So, $\alpha \in \mathbb{Q}$, which is a contradiction. Therefore, $\mathring{A} = \varnothing$.

   $\underline{\mathring{B} = \varnothing :}$
   Suppose that $\mathring{B} \neq \varnothing$ and let $x \in \mathring{B}$. Then, there exists $\delta > 0$ such that
   $$]x-\delta, x+\delta[\subset B \subset \mathbb{R} \setminus \mathbb{Q},$$
   because m and n are prime.
   By the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists $\alpha \in \mathbb{Q}$ such that $x-\delta < \alpha < x+\delta$. So, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, which is a contradiction. Therefore, $\mathring{B} = \varnothing$.

2. The point 2 is not an interior point of C because $]2-\varepsilon, 2+\varepsilon[\not\subseteq]1,2]$, for all $\varepsilon > 0$ since $2+\varepsilon > 2$.
   The interior of C is $\mathring{C} =]1,\, 2[$.

3. The set of all adherent points of A_1 is $\overline{A_1} = A_1 \cup \left\{\dfrac{3}{4}\right\}$. Indeed, the limit of any convergent sequence (non-trivial) is equal to $\dfrac{3}{4}$

4. The set of all boundary points of A_1 is $\partial A_1 = \overline{A_1} \setminus \mathring{A}_1 = \overline{A_1}$.

5. The set of all boundary points of C is $\partial C = \overline{C} \setminus \mathring{C} = [1,\, 2] \setminus]1,\, 2[= \{1,\, 2\}$.

6. A_1 is not closed because $\lim\limits_{n \longrightarrow +\infty} \underbrace{\frac{2}{n} + \frac{3}{4}}_{\in A_1} = \dfrac{3}{4} \notin A_1$.
   A_1 is not compact since it is not closed.

7. We have $\dfrac{1}{\sqrt{2}} = \lim\limits_{n \longrightarrow +\infty} \underbrace{\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{2}}}_{\in B}$ and $\dfrac{1}{\sqrt{n}} + \dfrac{1}{\sqrt{2}} \neq \dfrac{1}{\sqrt{2}}$, for all $n \in \mathbb{N}^*$.
   Thus, $\dfrac{1}{\sqrt{2}}$ is an accumulation point of B.
   B is not closed since $\dfrac{1}{\sqrt{2}}$ is an accumulation point of B, but it does not belong to B.

Final Exam 1- 2017

Exercise 1.

1. Write and prove the Cauchy-Schwarz inequality.
2. Recall that $f(x) = \cos(\alpha x)$ satisfies the identity $\big(f(x)\big)^2 = \frac{1}{2}\big(1+f(2x)\big)$.
   Show that if $p_k \in \mathbb{R}^+$ for $1 \leq k \leq n$ and $p_1 + p_2 + ... + p_n = 1$ then
   $g(x) = \sum\limits_{k=1}^{n} p_k \cos(\alpha_k x)$ satisfies $\big(g(x)\big)^2 \leq \frac{1}{2}\big(1 + g(2x)\big)$.
   `Hint:` $p_k \cos(\alpha_k x) = \sqrt{p_k}\sqrt{p_k}\cos(\alpha_k x)$.

Exercise 2.
Consider the subset A of $\mathbb{R}$ defined by $A = \{\sin n;\ n \in \mathbb{Z}\}$.
Use the fact that $\mathbb{Z} + 2\pi\mathbb{Z}$ is dense in $\mathbb{R}$ to show that A is dense in $[-1,\ 1]$.

Exercise 3.
Calculate the lim sup and lim inf of the sequence (x_n) defined by

$$x_n = \cos\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right) + \frac{(-1)^{n+1}}{n}, \quad n \in \mathbb{N}^*.$$

Exercise 4.

1. Use the mean value theorem to show that

$$|\arctan x - \arctan y| \leq |x - y|, \quad \forall x,\ y \in \mathbb{R}.$$

2. Show that the function f defined by $f(x) = \dfrac{\arctan x}{x}$ is uniformly continuous on $[\frac{1}{2}\,, 1]$ and on $[1,\ +\infty[$.

3. Deduce that f is uniformly continuous on $[\frac{1}{2}, +\infty[$.

Exercise 5.
Consider the function f defined on $\mathbb{R}$ by $f(x) = \dfrac{\arctan x}{\sin x}$.

1. Use the L'Hôspital rule to find $\lim\limits_{x\longrightarrow 0} f(x)$.

2. Explain why we cannot apply the L'Hôspital rule to calculate $\lim\limits_{x\longrightarrow +\infty} f(x)$.

3. Give the finite expansion to order 3 in a neighborhood of 0 of the function g defined by $g(x) = \dfrac{\arctan x}{1+\sin x}$. Deduce the value of $g^{(3)}(0)$.

4. Find an equivalent function of the function g in the neighborhood of 0. Deduce $\lim\limits_{x\longrightarrow 0} \dfrac{g(x)}{x}$.

Exercise 6.
Let f be a uniformly continuous and increasing function on $\mathbb{R}$. Let (x_n) and (y_n) be two adjacent sequences such that (x_n) is increasing and (y_n) is decreasing.

1. Show that the sequence (U_n) defined by $U_n = f(x_n)$ is increasing and the sequence (V_n) defined by $V_n = f(y_n)$ is decreasing.

2. Show that $\lim\limits_{n\rightarrow +\infty} (f(x_n) - f(y_n)) = 0$. Conclude.

SOLUTION

Exercise 1.

1. See the course.

2. $f(x) = \cos(\alpha x)$ satisfies the identity $\Big(f(x)\Big)^2 = \dfrac{1}{2}\Big(1 + f(2x)\Big)$: $\qquad (*)$.

By Cauchy-Shwarz inequality, we have:

$$\begin{aligned}
\Big(g(x)\Big)^2 &= \left(\sum_{k=1}^{n} p_k \cos(\alpha_k x)\right)^2 \\
&\leq \left(\sum_{k=1}^{n} \sqrt{p_k}^2\right)\left(\sum_{k=1}^{n} \Big(\sqrt{p_k}\cos(\alpha_k x)\Big)^2\right) \\
&= \left(\sum_{k=1}^{n} p_k\right)\left(\sum_{k=1}^{n} p_k \cos^2(\alpha_k x)\right) \\
&= (1)\left(\sum_{k=1}^{n} p_k \frac{1}{2}\Big(1+\cos(2\alpha_k x)\Big)\right) \quad \text{by } (*) \\
&= \frac{1}{2}\sum_{k=1}^{n} p_k + \frac{1}{2}\sum_{k=1}^{n} p_k \cos(2\alpha_k x)) \\
&= \frac{1}{2} + \frac{1}{2} g(2x) \\
&= \frac{1}{2}\Big(1+g(2x)\Big).
\end{aligned}$$

Exercise 2.
We have $A = \{\sin n;\, n \in \mathbb{Z}\} = \{\sin(n+2k\pi);\, n,\, k \in \mathbb{Z}\}$.
Let $x \in [-1,\, 1]$, then there exists $\theta \in \left[-\frac{\pi}{2},\, \frac{\pi}{2}\right]$ such that $x = \sin\theta$ (The function sin is bijective from $\left[-\frac{\pi}{2},\, \frac{\pi}{2}\right]$ into $[-1,\, 1]$).
On the other hand, we have $\mathbb{Z} + 2\pi\mathbb{Z}$ is dense in $\mathbb{R}$. Then, there exists a sequence (x_p) of $\mathbb{Z} + 2\pi\mathbb{Z}$ such that $\lim\limits_{p \longmapsto +\infty} x_p = \theta$. Then, $\lim\limits_{p \longmapsto +\infty} \sin x_p = \sin\theta = x$, where
$\sin x_p \in A, \quad \forall p \in \mathbb{N}$. Consequently, A is dense in $[-1,\, 1]$.

Exercise 3.
Take $p = 4$. Then, $q \in \{0,\, 1,\, 2,\, 3\}$.
• $x_{4k} = \cos\left(\frac{4k\pi}{2}\right) + \sin\left(\frac{4k\pi}{2}\right) + \frac{(-1)^{4k+1}}{4k} = 1 + 0 - \frac{1}{4k} \underset{k \to +\infty}{\longrightarrow} 1.$
• $x_{4k+1} = \cos\left(\frac{(4k+1)\pi}{2}\right) + \sin\left(\frac{(4k+1)\pi}{2}\right) + \frac{(-1)^{4k+2}}{4k+1} = 0 + 1 + \frac{1}{4k+1} \underset{k \to +\infty}{\longrightarrow}$
1.

- $\begin{cases} x_{4k+2} = \cos\left(\dfrac{(4k+2)\pi}{2}\right) + \sin\left(\dfrac{(4k+2)\pi}{2}\right) + \dfrac{(-1)^{4k+3}}{4k+2} \\ \qquad\quad = -1 + 0 - \dfrac{1}{4k+2} \underset{k\to+\infty}{\longrightarrow} -1. \end{cases}$
- $\begin{cases} x_{4k+3} = \cos\left(\dfrac{(4k+3)\pi}{2}\right) + \sin\left(\dfrac{(4k+3)\pi}{2}\right) + \dfrac{(-1)^{4k+4}}{4k+3} \\ \qquad\quad = -1 + 0 + \dfrac{1}{4k+3} \underset{k\to+\infty}{\longrightarrow} 1. \end{cases}$

Thus, the set of all cluster points is $A = \{-1,\, +1\}$. Consequently, $\limsup x_n = 1$ and $\liminf x_n = -1$.

Exercise 4.

1. Let x and y be two elements in $\mathbb{R}$ such that $x < y$. the function arctan is continuous over $[x,\, y]$ and differentiable over $]x,\, y[$. Then, by MVT, there exists $c \in]x,\, y[$ such that

$$\frac{\arctan y - \arctan x}{y - x} = \arctan'(c) = \frac{1}{1+c^2}.$$

Hence,

$$|\arctan x - \arctan y| = \frac{1}{1+c^2}|x-y| \leq |x-y|.$$

2. • The function f is continuous on $[\frac{1}{2}\,, 1]$. Then by Heine - Borel's Theorem f is uniformly continuous on $[\frac{1}{2}\,, 1]$.
• Let $x,\, y \in [1,\, +\infty[$ and let $\varepsilon > 0$ be given. Let us try to find $\delta > 0$ such that

$$|x-y| < \delta| \Rightarrow |f(x) - f(y)| < \varepsilon.$$

Indeed, we have:

$$\begin{aligned} |f(x) - f(y)| &= \left|\frac{\arctan x}{x} - \frac{\arctan x}{y} + \frac{\arctan x}{y} - \frac{\arctan y}{y}\right| \\ &\leq |\arctan x|\left|\frac{x-y}{xy}\right| + \frac{1}{|y|}|\arctan x - \arctan y| \\ &\leq \frac{\pi}{2}|x-y| + |x-y| \qquad \text{(by part 1.)} \\ &= (\frac{\pi}{2}+1)|x-y| < (\frac{\pi}{2}+1)\delta. \end{aligned}$$

If $(\frac{\pi}{2}+1)\delta < \varepsilon$ the $\delta < \dfrac{\varepsilon}{\frac{\pi}{2}+1}$. So, let us choose $\delta > 0$ such that $\delta < \dfrac{\varepsilon}{\frac{\pi}{2}+1}$.

Consequently, f is uniformly continuous on $[1, +\infty[$.

3. Let $x \in [\frac{1}{2}, 1]$ and $y \in [1, +\infty[$ and let $\varepsilon > 0$ be given.
   • f is uniformly continuous on $[\frac{1}{2}, 1]$, then:

$$\exists \delta' > 0 / |x-1| < \delta' \Rightarrow |f(x) - f(1)| < \frac{\varepsilon}{2}.$$

   • f is uniformly continuous on $[1, +\infty[$, then:

$$|y-1| < \delta \Rightarrow |f(y) - f(1)| < \frac{\varepsilon}{2}.$$

Let $\delta'' = \min \delta, \delta'$. then if $|x-y| < \delta''$, we have $|x-1| < \delta'$ and $|y-1| < \delta$. Therefore, we have:

$$\begin{aligned}
|x-y| < \delta'' \quad &\Rightarrow \quad \begin{cases} 0 \leq 1-x \leq y-x \\ x-y \leq 1-y \leq 0 \end{cases} \\
&\Rightarrow \quad \begin{cases} |x-1| \leq |x-y| < \delta' \\ |y-1| \leq |x-y| < \delta \end{cases} \\
&\Rightarrow \quad |f(x)-f(y)| \leq |f(x)-f(1)| + |f(y)-f(1)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Exercise 5. Consider the function f defined on $\mathbb{R}$ by $f(x) = \dfrac{\arctan x}{\sin x}$.

1. Let$\varphi(x) = \arctan x$ and $\psi(x) = \sin x$. We have:
   • $\varphi(0) = \psi(0) = 0$.
   • $\forall x \in I \setminus \{0\},\ \psi'(x) = \cos(x) \neq 0$. ($I =]0, \frac{\pi}{4}[$ for example).
   Then, by L'Hôspital's rule, we have:

$$\lim_{x \longrightarrow 0} f(x) = \lim_{x \longrightarrow 0} \frac{\varphi'(x)}{\psi'(x)} = \lim_{x \longrightarrow 0} \frac{1}{(1+x^2)\cos x} = 1.$$

2. We cannot apply the L'Hôspital rule to calculate $\lim\limits_{x \longrightarrow +\infty} f(x)$ because $\lim\limits_{x \longrightarrow +\infty} \psi(x)$ does not exist.

3. We have
$$\begin{aligned} g(x) &= \frac{\arctan x}{1+\sin x} \\ &= (x-\frac{x^3}{3})\frac{1}{1+x-\frac{x^3}{6}} + o(x^3) \\ &= (x-\frac{x^3}{3})\Big(1-(x-\frac{x^3}{6})+(x-\frac{x^3}{6})^2-\Big(x-\frac{x^3}{6})^3\Big)+o(x^3) \\ &= (x-\frac{x^3}{3})\Big(1-x+\frac{x^3}{6}+x^2-x^3\Big)+o(x^3) \\ &= (x-\frac{x^3}{3})\Big(1-x+x^2-\frac{5x^3}{6}\Big)+o(x^3) \\ &= x-x^2+\frac{2}{3}x^3+o(x^3). \end{aligned}$$
Then $\dfrac{g^{(3)}(0)}{3!} = \dfrac{2}{3}$ and so $g^{(3)}(0) = 4$.

4. We have $g(x) \underset{0}{\sim} x$. Then $\frac{g(x)}{x} \underset{0}{\sim} \dfrac{x}{x} \underset{0}{\sim} 1$. Therefore, $\displaystyle\lim_{x\longrightarrow 0}\frac{g(x)}{x} = 1$.

Exercise 6.

1. • For all n, $x_{n+1} \geq x_n$ and f is increasing, then For all n, $f(x_{n+1}) \geq f(x_n)$ i.e. $U_{n+1} \geq U_n$. So (U_n) is increasing.
   • For all n, $y_{n+1} \leq y_n$ and f is increasing, then For all n, $f(y_{n+1}) \leq f(y_n)$. So (V_n) is decreasing.

2. F is uniformly continuous, then:
$$\forall \varepsilon > 0,\ \exists \delta > 0 / \forall x,\ y \in \mathbb{R},\ |x-y| < \delta \Rightarrow |f(x)-f(y)| < \varepsilon. \qquad (*)$$
On the other hand, since (x_n) and (y_n) are adjacent, then $\displaystyle\lim_{n\longrightarrow+\infty}(x_n - y_n) = 0$. Thus:
$$\exists n_0 \in \mathbb{N} / \forall n,\ n \geq n_0 \Rightarrow |x_n - y_n| < \delta.$$
Therefore, by (*):
$$\exists n_0 \in \mathbb{N} / \forall n,\ n \geq n_0 \Rightarrow |x_n - y_n| < \delta \Rightarrow |f(x_n) - f(y_n)| < \varepsilon.$$
Consequently, $\displaystyle\lim_{n\longrightarrow+\infty}\Big(f(x_n) - f(y_n)\Big) = \lim_{n\longrightarrow+\infty}(U_n - V_n) = 0$.

`Conclusion`: The sequences (U_n) and (V_n) are adjacent.

Final Exam 2- 2017

Exercise 1.
Let $f : [a, b] \to \mathbb{R}$ be a continuous function over $[a, b]$ and $d = \sup f[a, b]$. Use the Bolzano - Weierstrass theorem to show that there exists at least $\eta \in [a, b]$ such that $f(\eta) = d$.

Exercise 2: Show, using Cauchy-Schwartz inequality, that

$$\sum_{k=1}^{n} k\sqrt{k} \leq \frac{n(n+1)}{2\sqrt{3}}\sqrt{2n+1}.$$

Hint: $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

Exercise 3.
Consider the subset A of $\mathbb{R}$ defined by $A = \{2\cos n;\ n \in \mathbb{N}\}$.

1. Prove that for all $x \in [-2, 2]$, there exists $\theta \in [0, \pi]$ such that $\cos\theta = \dfrac{x}{2}$.
2. Use the fact that $\mathbb{Z} + 2\pi\mathbb{Z}$ is dense in $\mathbb{R}$ to show that A is dense in $[-2,\ 2]$.

Exercise 4.
Calculate the $\limsup$ and $\liminf$ of the sequence (x_n) defined by

$$x_n = \sin\left(\frac{n\pi}{3}\right) + \frac{(-1)^n}{n}, \quad n \in \mathbb{N}^*.$$

Exercise 5.
Consider the function f defined on $\mathbb{R}$ by $f(x) = xe^{\frac{1}{x}}$.

1. Use the L'Hôspital rule to find $\displaystyle\lim_{x \longrightarrow 0} f(x)$.

2. Explain why we cannot apply the L'Hôspital rule to calculate $\lim\limits_{x \longrightarrow +\infty} f(x)$.

Exercise 6 Let $\alpha \in]0,1[$. Show that $f(x) = x^{\alpha}$ is uniformly continuous on $\mathbb{R}^{+}$.
Hint: If $0 \leq x \leq y$ then $0 \leq y^{\alpha} - x^{\alpha} \leq (y-x)^{\alpha}$.

Exercise 7 Let $\boldsymbol{U_n = \sum\limits_{k=0}^{n} \frac{1}{k!}}$, $\boldsymbol{V_n = U_n + \frac{1}{n.n!}}$.

1. Show that $\boldsymbol{U_n}$ and $\boldsymbol{V_n}$ are adjacent.

2. By using Taylor's formula, prove that there exists a positive constant $\boldsymbol{M}$ such that

$$\boldsymbol{\left| f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k \right| \leq M \frac{|x|^{n+1}}{(n+1)!}.}$$

3. Deduce that $\boldsymbol{U_n \to e}$.