Ex1

1) $AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 3 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 4 & 4 & 3 \end{pmatrix}$

$AC = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 3 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 4 & 4 & 3 \end{pmatrix}$

on remarque que $AB = AC$

2) Non, car si A était inversible
alors
$AB = AC \Rightarrow B = C$ ce qui est faux

Ex2 $\begin{cases} x + y + 2z = 3 \\ x + 2y + z = 1 \\ 2x + y + z = 0 \end{cases}$

1) $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$ $|A| = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & -1 & -3 \end{vmatrix}$

$= -3 - 1 = -4 \neq 0$

alors A est inv.
et le sys. est de Cramer.

2) $x = \dfrac{\begin{vmatrix} 3 & 1 & 2 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix}}{-4} = \dfrac{3(2-1) - 1(1-2)}{-4} = \dfrac{3+1}{-4} = -1$

$y = \dfrac{\begin{vmatrix} 1 & 3 & 2 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix}}{-4} = \dfrac{2(3-2) + 1(1-3)}{-4} = \dfrac{2-2}{-4} = 0$

$z = \dfrac{\begin{vmatrix} 1 & 1 & 3 \\ 1 & 2 & 1 \\ 2 & 1 & 0 \end{vmatrix}}{-4} = \dfrac{2(1-6) - 1(1-3)}{-4} = \dfrac{-10+2}{-4} = 2$

Ex3

1) Base(F) = ??

$$\begin{pmatrix} -1 & 0 & 1 & 1 \\ 2 & -1 & 0 & 1 \\ 0 & -1 & 2 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & -1 & 2 & 3 \\ 0 & -1 & 2 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & -1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$rg(v_1, v_2, v_3) = 2$$

$$Base(F) = \{v_1, v_2\}. \quad (v_1 \neq \lambda v_2, \ \forall \lambda \in \mathbb{R})$$

2) Base(G) = ??

$$\begin{pmatrix} -1 & 1 & -1 & 1 \\ -1 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ -2 & 3 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 1 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 2 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 1 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$rg(u_1, u_2, u_3, u_4) = 3$$

$$Base(G) = \{u_1, u_2, u_3\}$$

3) $F + G = Vect(v_1, v_2, u_1, u_2, u_3)$

$$\left(\begin{array}{cccc|l} -1 & 0 & 1 & 1 & v_1 \\ 2 & -1 & 0 & 1 & v_2 \\ -1 & 1 & -1 & 1 & u_1 \\ -1 & 2 & 1 & 0 & u_2 \\ 0 & 0 & 2 & 1 & u_3 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|l} -1 & 0 & 1 & 1 & v_1 \\ 0 & -1 & 2 & 3 & v_2 + 2v_1 \\ 0 & 1 & -2 & 0 & u_1 - v_1 \\ 0 & 2 & 0 & -1 & u_2 - v_1 \\ 0 & 0 & 2 & 1 & u_3 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|l} -1 & 0 & 1 & 1 & v_1 \\ 0 & -1 & 2 & 3 & v_2 + 2v_1 \\ 0 & 0 & 0 & 3 & v_1 + v_2 + u_1 \\ 0 & 0 & 4 & 5 & u_2 + 2v_2 + 3v_1 \\ 0 & 0 & 2 & 1 & u_3 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|l} -1 & 0 & 1 & 1 & v_1 \\ 0 & -1 & 2 & 3 & v_2 + 2u_1 \\ 0 & 0 & 4 & 5 & u_2 + 2v_2 + 3v_1 \\ 0 & 0 & 2 & 1 & u_3 \\ 0 & 0 & 0 & 3 & v_1 + v_2 + u_1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|l} -1 & 0 & 1 & 1 & v_1 \\ 0 & -1 & 2 & 3 & v_2 + 2v_1 \\ 0 & 0 & 4 & 5 & u_2 + 2v_2 + 3v_1 \\ 0 & 0 & 0 & 1 & u_2 + 2v_2 + 3v_1 - 2u_3 \\ 0 & 0 & 0 & 3 & v_1 + v_2 + u_1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} -1 & 0 & 1 & 1 & \\ 0 & -1 & 2 & 3 & \\ 0 & 0 & 4 & 5 & \\ 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & \underbrace{v_1 + v_2 + u_1 - 3u_2 - 6v_2 - 9v_1 + 6u_3}_{-8v_1 - 5v_2 + u_1 - 3u_2 + 6u_3} \end{array}\right)$$

$rg(v_1, v_2, \cancel{v_3}, u_1, u_2, u_3) = 4 = \dim \mathbb{R}^4$ et $F + G \subset \mathbb{R}^4$

$$\boxed{F + G = \mathbb{R}}$$

4) $\dim F \cap G = \dim F + \dim G - \dim F+G$
$= 2 + 3 - 4 = 1$

De plus, $-8v_1 - 5v_2 + u_1 - 3u_2 + 6u_3 = 0$

$$\rightarrow \underbrace{8v_1 + 5v_2}_{\in F} = \underbrace{u_1 - 3u_2 + 6u_3}_{\in G} \in F \cap G.$$

$8v_1 + 5v_2 = 8(-1, 0, 1, 1) + 5(2, -1, 0, 1)$
$= (2, -5, 8, 13)$

$u_1 - 3u_2 + 6u_3 = \cancel{(\ldots)}$
$= (-1, 1, -1, 1) - 3(-1, 2, 1, 0) + 6(0, 0, 2, 1)$
$= (2, -5, 8, 7)$

<u>Ex 4</u>

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$(x, y, z) \longmapsto (x - y - z, y, x - z)$$

1) $$A = \mathrm{Mat}_B f = \begin{array}{cc} & \begin{array}{ccc} f(e_1) & f(e_2) & f(e_3) \end{array} \\ & \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{array}{c} e_1 \\ e_2 \\ e_3 \end{array} \end{array}$$

2) $\ker f = ??$

$X = (x, y, z) \in \ker f \Rightarrow f(x,y,z) = 0$

$$\Rightarrow \begin{cases} x - y - z = 0 \\ y = 0 \\ x - z = 0 \end{cases} \Rightarrow \begin{cases} y = 0 \\ x = z \end{cases}$$

$X = (x, 0, x) = x(1,0,1)$

$\ker f = \mathrm{Vect}\{(1,0,1)\}$

$\mathrm{Base}(\ker f) = \{(1,0,1)\}$ $\quad$ f n'est pas inj' car $\ker f \neq \{(0,0,0)\}$

3) $\dim \mathbb{R}^3 = \dim \mathrm{Im} f + \dim \ker f$

$\Rightarrow \dim \mathrm{Im} f = 2.$

on a $\mathrm{Im} f = \mathrm{Vect}(f(e_1), f(e_2), f(e_3))$

$\dim \mathrm{Im} f = 2$

$f(e_1) \neq \lambda f(e_2)$

$\Rightarrow \mathrm{Base}(\mathrm{Im} f) = \{(1,0,1), (-1,1,0)\}$

$f \neq$ surj car $\mathrm{Im} f \neq \mathbb{R}^3$

4) $\mathbb{R}^3 = \mathrm{Im} f + \ker f$??

~~$\dim \mathbb{R}^3 = \dim \mathrm{Im} f + \dim \ker f.$~~

on a $(1,0,1) \in \mathrm{Im} f \cap \ker f$

~~et~~ ~~$\mathrm{vect}\{(1,0,1), (-1,1,0)\}$~~

donc $\mathrm{Im} f + \ker f = \mathrm{Im} f.$

donc $\mathbb{R}^3 \neq \mathrm{Im} f + \ker f.$

5) $\begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix} = 1(0+1) + 1(1+1)$

$= 1 + 2 = 3 \neq 0$

donc $\{e_1', e_2', e_3'\}$ libre $\Longrightarrow B'$ base de $\mathbb{R}^3$.

Card $\{e_1', e_2', e_3'\} = 3 = \dim \mathbb{R}^3$

$$P_{BB'} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

$$|P_{BB'}| = 3$$

$$P^{-1}_{BB'} = \frac{1}{3}\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 2 \\ 2 & -1 & 1 \end{pmatrix}$$

On a $\quad A' = \text{Mat}_{B'} f = P^{-1}_{BB'} \, A \, P_{BB'}$

$$= \frac{1}{3}\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 2 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & -3 \\ 3 & -3 & -3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & -1 & 1 \\ 1 & -2 & 2 \end{pmatrix}$$

7) $\quad v = (2, -1, 1)$

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = P^{-1}_{BB} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 2 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

$$v = (2, -1, 1) = 0e'_1 + 1e'_2 + 2e'_3$$

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