25-1-2021

Recall

Eigen values & eigenvector of a matrix $A_{n\times n}$.

λ (eigen values), $(x \neq 0)$ (eigenvector)

$\lambda \in \mathbb{R}$ is said to be an eigen value of a matrix $A_{n\times n}$ if there exist a non-zero vector $X \in \mathbb{R}^n$ st

$$AX = \lambda X$$

$X (\neq 0)$ is called the eigenvector corresponding to the eigenvalue $\lambda (\in \mathbb{R})$

How to find λ & x

find $\lambda, x \neq 0$ such that $Ax = \lambda x$

$\Leftrightarrow (A-\lambda I)x = 0$

$\Leftrightarrow (A-\lambda I)x = 0$ has a nontrivial solution $x \neq 0$

$\Leftrightarrow (A-\lambda I)$ is not invertible (or singular)

$\Leftrightarrow \det(A-\lambda I) = 0$

↓

Solve it to find a polynomial of degree n in λ

↓

find roots to find λ.

↓

use $(A-\lambda I)x = 0$ to find $x \neq 0$.

Eigen space corresponding to an eigen value λ is Null space of $(A-\lambda I)$

eg:) $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = A$; $\lambda_1 = -1$, $\lambda_2 = 3$ → 2 distinct eigenvalues

eigen corresponding to eigenvalue $\lambda_1 = -1$ is

$L\{(1,-1)\}$

$= \{(x,-x) \mid x \in \mathbb{R}\}$

eigenvector $= (1,-1) = X_1$

Eigen space corresponding to eigenvalue $\lambda_2 = 3$

$L\{(1,1)\} = \{(x,x) \mid x \in \mathbb{R}\}$

eigenvector $= (1,1) = X_2$

X_1, X_2 - L.I. - form a basis of $\mathbb{R}^2$.

eg: $A = \begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ - eigenvalues

$$|A-\lambda I| = \begin{vmatrix} 3-\lambda & -2 & 0 \\ -2 & 3-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0.$$

$$(5-\lambda)\left[(3-\lambda)^2-2^2\right]=0$$

$\Rightarrow \lambda = 5, 5, 1 \quad ; \lambda_1=\lambda_2=5 \; ; \lambda_3 = 1$

$\downarrow$

Repeated eigenvalues

Eigen space corresponding to $\lambda_1=\lambda_2=5$ is

$(A-5I)x=0$

$$\begin{pmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

x_3 - free variable

$x_1 = -x_2$

sol of this system of eq $= \{(x_1, -x_1, x_3) \mid x_1, x_3 \in \mathbb{R}\}$

$= \{x_1 \underbrace{(1,-1,0)}_{v_1} + x_3 \underbrace{(0,0,1)}_{v_2} \mid x_1, x_3 \in \mathbb{R}\}$

$= L\{\underbrace{(1,-1,0)}_{v_1}, \underbrace{(0,0,1)}_{v_2}\}$

$\{v_1, v_2\}$ - eigen vectors L-I

$v_1, v_2, \alpha v_1 + \beta v_2, \alpha, \beta \in \mathbb{R}$ will work as a eigenvector

Eigen space corresponding to the eigenvalue $\lambda_3 = 1$.

$(A-\lambda_3 I)x = (A-I)x = 0$

$$\Rightarrow \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow x_1 = x_2, \; x_3 = 0$

sol- of this eq is $\{(x_1, x_1, 0) \mid x_1 \in \mathbb{R}\}$

eigenspace $= L\{\underbrace{(1,1,0)}_{v_3}\} = \{(x_1, x_1, 0) \mid x_1 \in \mathbb{R}\}$

v_3 - eigen vector corresponding to $\lambda_3 = 1$.

$$\left.\begin{array}{l} v_1 = (1,-1,0) \\ v_2 = (0,0,1) \\ v_3 = (1,1,0) \end{array}\right\} \text{- L-I - form a basis of } \mathbb{R}^3.$$

eg: $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ - eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = 3$

eigenspace corresponding to $\lambda_1 = 3 = \lambda_2 = \lambda_3$

$(A-\lambda_1 I)x = 0$

$$\Rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ x_1 = free variable

$x_2 = x_3 = 0$

sol. of this eq is $= \{(x_1, 0, 0) \mid x_1 \in \mathbb{R}\}$

$= L\{(1,0,0)\}$

eigen space $= L\{\underset{v_1}{(1,0,0)}\}$

v_1 - eigenvector

$\lambda_1 = 3$
$\lambda_2 = 3$
$\lambda_3 = 3$
$v_1 = (1,0,0)$ — Repeated eigen values with insufficient L.I eigen vectors.

Thm!- For an upper / lower triangular matrix, eigen values are the diagonal entries.

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & \ddots & \vdots \\ 0 & & a_{nn} \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ & a_{22}-\lambda & & \vdots \\ 0 & & \ddots & \vdots \\ & & & a_{nn}-\lambda \end{vmatrix}$$

$$= \prod_{i=1}^{n} (a_{ii} - \lambda) = 0$$

$\Rightarrow \lambda = a_{ii} \quad \forall 1 \le i \le n$

eigen values are diagonal entry.

eg $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$

$|A - \lambda I| = 0 \Rightarrow \lambda_1 = 0$

$\lambda_2 = 3.$

eigenspace corresponding to $\lambda_1 = 0$
$L\{(1,-1)\} = L\{v_1\}$

eigen space corresponding to $\lambda_2 = 3$
$L\{(1,2)\} = L\{v_2\}$

$\{v_1, v_2\}$ - L.I.

$\Rightarrow$ form a basis of $\mathbb{R}^2$.

$\lambda_1 = 0 \iff |A - \lambda_1 I| = 0$

$\iff |A - 0I| = 0$

$\iff |A| = 0$

if $\lambda = 0$ is an eigenvalue of A $\iff$ A is not invertible.

Thm!- A $n \times n$ matrix is invertible iff 0 is not an eigenvalue of it.

Pf!- let $A_{n\times n}$ matrix

A is invertible $\Leftrightarrow$ $\det A \neq 0$

$\Leftrightarrow$ $\det(A - 0 \cdot I) \neq 0$

$\Leftrightarrow$ 0 is not an eigenvalue of A.

Thm:- Let $A_{n \times n}$ be a matrix with distinct eigenvalues $\lambda_1, \lambda_2, \cdots \lambda_n$. Then the corresponding eigenvectors $x_1, x_2, \cdots x_n$ are L.I.

Pf:- $(\lambda_1, x_1), (\lambda_2 x_2) \cdots (\lambda_n, x_n) \to$ eigen pairs

$\lambda_i \neq \lambda_j \quad \forall\, 1 \leq i, j \leq n$
$(i \neq j)$

P.T. $\underbrace{\{x_1, x_2, \cdots x_n\}}_{= S}$ - L.I. in $\mathbb{R}^n$.

Let $x_1 \cdots x_n$ be L.D.

Choose $\{x_1 \cdots x_k\}$ s.t. it is the largest L.I. subset of S. i.e. if we add x_{k+1}, then $\{x_1, x_2, \cdots x_k, x_{k+1}\}$ is L.D.

$\Rightarrow \underline{x_{k+1}} = \sum_{i=1}^{k} c_i x_i$ for $c_i \in \mathbb{R}$, $1 \leq i \leq k$. — ①

$A x_{k+1} = A\left(\sum_{i=1}^{k} c_i x_i\right)$

$= \sum_{i=1}^{k} c_i A(x_i)$

$(A x_i = \lambda_i x_i)$ $\forall\, 1 \leq i \leq n$

$\lambda_{k+1} x_{k+1} = \sum_{i=1}^{k} c_i \lambda_i x_i$ — ②

from ①, we also have

$\lambda_{k+1} x_{k+1} = \sum_{i=1}^{k} c_i \lambda_{k+1} x_i$ — ③

from ② & ③ we have

$\sum_{i=1}^{k} c_i (\lambda_{k+1} - \lambda_i) x_i = 0$

$x_1 \cdots x_k$ - L.I.

$\Rightarrow c_i(\lambda_{k+1} - \lambda_i) = 0 \quad \forall\, 1 \leq i \leq k.$

but $\lambda_{k+1} \neq \lambda_i \quad \forall\, 1 \leq i \leq k.$
$i \neq k+1$

$\Rightarrow c_i = 0 \quad \forall\, 1 \leq i \leq k$

from ①, we have

$x_{k+1} = 0$ a contradiction

$\Rightarrow \{x_1 \cdots x_n\}$ is L.I.

$A_{n\times n} \to \lambda_1 \dots \lambda_n$ - distinct

$x_1 \dots x_n$

$\underbrace{x_1 \dots x_n}$ L.I. $\Rightarrow$ form a basis of $\mathbb{R}^n$.

$$A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$ ~~$Ax_i = \lambda_i x_i$~~

eigen vector $\leftarrow \{x_1 \dots x_n\}$ $\qquad \{x_1 \dots x_n\}$

$Ax_1 = \lambda_1 x_1$

$Ax_2 = \lambda_2 x_2$

$\vdots$

$Ax_n = \lambda_n x_n$

matrix of A wrt the basis (ordered) $\{x_1 \dots x_n\}$

$$[Ax_1 \quad Ax_2 \dots Ax_n]$$

$$= \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

diagonal matrix with all eigenvalues on the diagonal.

Cayley-Hamilton theorem:- $A_{n\times n}$

$|A - \lambda I|$ = polynomial in λ say

$p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0.$

characteristic polynomial of matrix A

$p(A) = A^n + a_{n-1}A^{n-1} + \dots + a_1 A + a_0 I = 0$

Cayley-Hamilton theorem:- Every matrix satisfies its characteristic equation.

- this can be used to calculate powers & inverse of a matrix.

$A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \rightarrow p(\lambda)$

$p(A) = 0$

Similarity:-

diagonal matrix → eigenvalues = diagonal
upper Δular matrix matrix:

$A \longrightarrow \Delta$ular matrix / diagonal matrix D

e.g.

$A \xrightarrow{RREF} D$

doesn't preserve eigenvalues

$A \xrightarrow{\text{preserve eigenvalues.}} D$

Similar matrices:-

Let A & B be 2 $n\times n$ matrices.

We say A is similar to B if there is an invertible matrix $P_{n\times n}$ such that

$$P^{-1}AP = B$$

Notation: $A \sim B$ = A is similar to B

Result:- let A & B be 2 similar matrices or A is similar to B

$\Rightarrow \exists\, P$ st $P^{-1}AP = B$

$$A = PBP^{-1}$$
$$= (P^{-1})^{-1} B (P^{-1})$$
$$= Q^{-1} BQ$$

$B \sim A$.

(i) $A \sim B \Leftrightarrow B \sim A$.

(ii) P is not a unique matrix for a given pair of similar matrices.

eg $A = B = I$. $\Rightarrow I \sim I$

any P (invertible)

$P^{-1}IP = I$

eg- $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$

$A \sim B$.

$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ as $P^{-1}AP = B$

or $AP = PB$

Result:- A, B, C – $n\times n$ matrices

(a) $A \sim A$ – reflexive $(I^{-1}AI = A)$

(b) $A \sim B \Rightarrow B \sim A$ – symmetric

(c) $A \sim B$ & $B \sim C$

$\Rightarrow A \sim C$ – transitive

($A \sim B \Rightarrow \exists\, P$ st $P^{-1}AP = B$

$B \sim C \Rightarrow \exists\, Q$ st $Q^{-1}BQ = C$

$Q^{-1}(P^{-1}AP)Q = Q^{-1}BQ$
$= C$

$\Rightarrow A \sim C$)

Similarity is an equivalence relation

Thm:- Let A, B be $n\times n$ matrices such that $A \sim B$. Then the following holds

(i) $\det A = \det B$

$\det(AB) = \det A \cdot \det B$
$= \det(BA)$.
$\det(AA^{-1}) = 1$

$\Rightarrow \exists\, P$ st

$P^{-1}AP = B$

$\det B = \det(P^{-1}AP) = \det(APP^{-1}) = \det A$.

(ii) A is invertible iff B is invertible

$\because$ if $\det A \neq 0$ & $\det B = \det A \neq 0$

(iii) A & B have same charcteristic polynomial.

Characteristic polynomial of B is

$$\det(B - \lambda I)$$

$$= \det(P^{-1}AP - \lambda I)$$

$$= \det(P^{-1}AP - \lambda \cdot P^{-1}P)$$

$$= \det(P^{-1}AP - P^{-1}(\lambda I)P)$$

$$= \det[P^{-1}(A - \lambda I)P]$$

$$= \det([A - \lambda I]PP^{-1})$$

$\det(AB) = \det(BA)$

$$\det(B - \lambda I) = \det(A - \lambda I)$$

$\Rightarrow$ same characteristic polynomial

(iv) They have same same eigen values.

(v) $A^m \sim B^m$ for $m \in \mathbb{N}$.

$$B^m = (P^{-1}AP)^m$$

$$= P^{-1}A^mP$$

$\Rightarrow B^m \sim A^m$.