4-1-2021

Note Title 28-12-2020

2 quiz + 1 mid sem

3 quiz → best 2

1 mid sem

Thm:- Let S be a L.I. subset of a V.S. 'V' (FD). If $\exists$ $v \notin L(S)$.
Then $\{x \cup S\}$ is L.I in V.

Pf:- Given: Let $S = \{x_1 \dots x_n\}$ – L.I in V.
$x \notin L\{x_1 \dots x_n\}$

To show:- $x \cup \{x_1 \dots x_n\} = \{x, x_1, \dots x_n\}$ is L.I. in V.

Let $a, a_1 \dots a_n \in \mathbb{R}$.

$$ax + a_1 x_1 + \dots + a_n x_n = 0 \quad \text{—— (1)}$$

if $a \neq 0$ $\quad x = -\sum_{i=1}^{n} \left(\frac{a_i}{a}\right) x_i$

$$x \in L\{x_1 \dots x_n\}$$

$a \Rightarrow \in$

$\Rightarrow a = 0$

(1) becomes $a_1 x_1 + \dots + a_n x_n = 0$

$\Rightarrow a_i = 0 \quad \forall 1 \leq i \leq n$

$$a = 0 = a_1 = a_2 \dots = a_n$$

$\Rightarrow \{x, x_1, \dots x_n\}$ are L.I.

Implications $\{v_1 \neq 0\} \in V$
↳ extend it to a basis of V if V is (FD)

Thm:- Let V be a FDVS.

(a) Any finite L.I. set of vectors in V can be extended to a basis of V by adding more vectors, if necessary.

(b) Any finite set of vectors that span V can be reduced to a basis of V by discarding vectors, if necessary.

Pf:- (a) S – L.I. set in V

If $L(S) = V$ – stop here, B is a basis

if $L(S) \neq V \rightarrow \exists\, x \in V$ st $x \notin L(S)$

$S_1 = S \cup \{x\}$

If $L(S_1) = V \rightarrow S_1$ is a basis

if not find $x \in V \setminus L(S_1)$

& keep on doing this

we will stop as V is FD.

(b) B st $B \subseteq V$

$L(B) = V$

If B is L.I. set $\Rightarrow$ B is a basis.

If B is not L.I. $\Rightarrow$ L.D.

$v_1 \in B$ st v_1 is l.c. of other vectors in B.

$\underline{L(B \setminus \{v_1\}) = V}$?

Note Title 28-12-2020

$\underline{\alpha v_1} = \alpha$ ($\underline{\text{l.c. of } v_1}$)

$B_1 = B \setminus \{v_1\} \to$ L.I.

$L(B_1) = V$. - B_1 is a basis

If B_1 is not L.I.

continue the process till dim of V is reached.

$\{(1,2), (3,4), \cancel{(5,6)}, \cancel{(7,8)}, \cancel{(9,10)}\} = B$

B_2, B_1

Cor 1:- If W is a proper subspace of a $\underline{\text{FDVS}}$ 'V'. Then $\underline{W}$ is also $\underline{\text{FD}}$. & $\underline{\dim W < \dim V}$.

Pf:-

$\dim W = \dim V$

or

$\boxed{\dim W \cancel{\geq \dim V}}$

$W \subsetneq V$

$\exists\ x \in V$ st $x \notin W$.

$\dim W = \underline{\dim V} = \underline{n}$

$W = L\{v_1 \dots v_n\}$. let $\underline{\{v_1 \dots v_n\}}$ - basis of W

~~then~~ $\{v_1 \dots v_n\}$ - form a basis of V also.

$\Rightarrow W = L\{v_1 \dots v_n\} = V$

$\Rightarrow W = V$ a $\Rightarrow\Leftarrow$

Cor 2:- If V is a FDVS. with $\dim V = n$. Then the following are true:-

(a) Any subset of V which contains more than n vectors is L.D.

(b) No subset of V which contains fewer than $\underline{n}$ vectors span V.

$L(B) \neq V$

Cor 3:- Let A be an $n \times n$ matrix over $\mathbb{R}$. Suppose that row vectors of A are L.I. vectors in $\mathbb{R}^n$. Then A is invertible.

Pf:-

$$A = \begin{bmatrix} \boxed{1 \quad 2} \\ \boxed{3 \quad 4} \end{bmatrix} \qquad \begin{aligned} R_1 &= (1, 2) \in \mathbb{R}^2 \\ R_2 &= [3, 4] \in \mathbb{R}^2 \end{aligned}$$

$\underline{R_1, R_2}$ are L.I. in $\mathbb{R}^2$

REF of A will be I_n $\leftarrow$ $\underline{R_1, R_2}$ will form a basis of $\mathbb{R}^2$.

$\Rightarrow$ A is invertible.

Pf:- Let $A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{bmatrix}$

$R_1 = [a_{11}, a_{12}, \dots, a_{1n}]$; $R_i = [a_{i1}, \dots, a_{in}] \in \mathbb{R}^n$

$R_1, \underbrace{R_2 \cdots R_n}_{n \text{ vectors}} \in \mathbb{R}^n$ and are L-I.

Note Title 28-12-2020

then $R_1, \cdots R_n$ form a basis of $\mathbb{R}^n$.

Standard basis of $\mathbb{R}^n$ = $e_1 = (1,0,\cdots 0)$; $e_2 = (0,1,0 \cdots 0)$

$e_k = (0, \cdots 0, \underset{k^{th} \text{ coordinate}}{1}, 0 \cdots 0)$, $e_n = (0, \cdots 0, 1)$

$e_1 \cdots e_n \in \mathbb{R}^n$

$e_1 = (1, 0, \cdots 0) = b_{11} R_1 + \cdots + b_{1n} R_n \quad ; \quad b_{1i} \in \mathbb{R}, \; 1 \le i \le n$

$= \begin{bmatrix} b_{11} & \cdots & b_{1n} \end{bmatrix} \begin{bmatrix} R_1 \\ \vdots \\ R_n \end{bmatrix} \longrightarrow$

$e_i = (0, \cdots 0, \underset{i^{th} \text{ position}}{1}, 0 \cdots 0) = b_{i1} R_1 + \cdots + b_{in} R_n$

$= \begin{bmatrix} b_{i1} & \cdots & b_{in} \end{bmatrix} \begin{bmatrix} R_1 \\ \vdots \\ R_n \end{bmatrix}$

$$I_n = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} = \underbrace{\begin{bmatrix} b_{11} & \cdots & b_{1n} \\ b_{21} & \cdots & b_{2n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix}_{n \times n}}_{B} \underbrace{\begin{bmatrix} R_1 \\ \vdots \\ R_n \end{bmatrix}}_{= A}$$

$b_{ij} \in \mathbb{R}$, $1 \le i, j \le n$.

$e_1 = b_{11} R_1 + \cdots + b_{1n} R_n$

$e_2 = b_{21} R_1 + \cdots + b_{2n} R_n$

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ b_{21} & \cdots & b_{2n} \end{bmatrix} \begin{bmatrix} R_1 \\ \vdots \\ R_n \end{bmatrix}$$

$BA = I$

$B = A^{-1}$.

$\Rightarrow A$ is invertible.

$$\left.\begin{aligned} e_1 &= (1, 0, 0 \cdots 0) \\ e_2 &= (0, 1, \cdots 0) \\ &\vdots \\ e_n &= (0, 0, \cdots 1) \end{aligned}\right\} \sim n \text{ vectors in } \mathbb{R}^n.$$

$(0, 0, \cdots 0) \notin$

$a_1 e_1 + a_2 e_2 + \cdots + a_n e_n = 0$

$v_1 \cdots v_n$

$(a_1, a_2, \cdots, a_n) = 0 \Rightarrow a_i = 0$

$0 = 0 \cdot v_1 + \cdots + 0 \cdot v_n$

Row space of $A_{m \times n} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix} = \begin{bmatrix} C_1 & \cdots & C_n \end{bmatrix}$

$R_1 = [a_{11}, \cdots a_{1n}] \in \mathbb{R}^n$

m rows in $\mathbb{R}^n$.

$C_1 = \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}$

n columns in $\mathbb{R}^m$.

Row space $= L\{R_1, \cdots R_m\} \subseteq \mathbb{R}^n$

dim (Row space) = No. of L-I rows in A.

Column space of A = $L\{C_1, \cdots C_n\} \subseteq \mathbb{R}^m$

dim (Column space) = No. of L-I columns in A

Note Title 28-12-2020

Thm:- If W_1 & W_2 are FD. subspaces of a V.S. V. Then $W_1+W_2 = \{w_1+w_2 \mid w_1 \in W_1, w_2 \in W_2\}$ is a FD. subspace of V. &

$$\dim(W_1+W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2).$$

Pf:- W_1, W_2

$W_1 \cap W_2$ ↓ subspace

$W_1 \cup W_2$ → not a subspace if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

A, B
$A \cap B$
$A \cup B$
V, W
$\subseteq V \cap W$
$V \cup W$

$W_1+W_2 = \{w_1+w_2 \mid w_1 \in W_1, w_2 \in W_2\}$

If W_1 is FD, W_2 is FD ⇒ W_1+W_2 is FD.

V − FD → W_1+W_2 − FD

$W_1+W_2 \subseteq V$

V − VS, W_1, W_2 − FD, → W_1+W_2 is FD.

W_1, W_2 − FD

$W_1 \cap W_2$ is also FD as it is contained in W_1 & W_2

$W_1 \cap W_2 \subseteq W_1$.

$W_1 \cap W_2 \subseteq W_2$

$\{v_1 \dots v_n\}$ − basis of $W_1 \cap W_2 \subseteq W_1$

$\{v_1 \dots v_n\}$ − L.I in W_1 also

$\{v_1 \dots v_n, w_1 \dots w_k\}$ − basis of W_1.

$\dim W_1 = n+k$.

$\{v_1 \dots v_n\}$ − LI in W_2 as $W_1 \cap W_2 \subseteq W_2$

$\{(v_1 \dots v_n), p_1 \dots p_m\}$ − basis of W_2, → $\dim W_2 = n+m$.

W_1+W_2 → $\{(v_1 \dots v_n), w_1 \dots w_k, p_1 \dots p_m\}$ − form a basis of W_1+W_2

$L\{v_1 \dots v_n, w_1 \dots w_k, p_1 \dots p_m\} = W_1+W_2$ (H.W)

$\{v_1 \dots v_n, w_1 \dots w_k, p_1 \dots p_m\}$ are L.I in W_1+W_2

$\dim W_1+W_2 = n+k+m$

$= (k+n) + (m+n) - n$

$= \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$.

$W_1 = L\{\underset{v_1}{(1,1,0,-1)}, \underset{v_2}{(0,1,3,1)}\}$

$W_2 = \{\overset{v_3}{(0,-1,-2,1)}, \overset{v_4}{(1,2,2,-2)}\}$

$x = a v_1 + b v_2 = c v_3 + d v_4$

a

find basis of W_1+W_2

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \to \text{which vectors are LI?}$$

To find intersection. Let $x \in W_1 \cap W_2 \Rightarrow \exists\, a,b,c,d \in \mathbb{R}$ st $W_1 \ni x = a v_1 + b v_2$ & $x = c v_3 + d v_4 \in W_2 \Rightarrow a v_1 + b v_2 = c v_3 + d v_4$

$\Rightarrow a v_1 + b v_2 - c v_3 - d v_4 = 0 \Rightarrow$ solve for a, b, c, d ?