19-1-21

Recall few things,

1. $(V, +, \cdot, \langle \cdot, \cdot \rangle)$ - real IPS (inner product space)

inner product has some properties that needed to be satisfied.

(a) $\langle x, x \rangle \geq 0$ & $\langle x, x \rangle = 0 \iff x = 0$

(b) $\langle x, y \rangle = \langle y, x \rangle$

(c) $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

(d) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$

for $\alpha \in \mathbb{R}$, $x, y, z \in V$

2. Norm / length

$$\|x\| = \sqrt{\langle x, x \rangle}$$

(a) $\|x\| \geq 0$, if $\|x\| = 0 \iff x = 0$

(b) $\|\alpha x\| = |\alpha| \cdot \|x\|$ for $\alpha \in \mathbb{R}$, $x \in V$.

$$\langle x, ay \rangle = \langle ay, x \rangle$$
$$= a \langle y, x \rangle$$
$$= a \langle x, y \rangle$$

$$\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$$
$$\langle y+z, x \rangle = \langle y, x \rangle + \langle z, x \rangle$$
$$= \langle x, y \rangle + \langle x, z \rangle$$

3. Orthogonal vectors :-

x is orthogonal to y ie $x \perp y$

if $\langle x, y \rangle = 0$

4. Cauchy-Schwartz inequality :

$u, v \in V$ (IPS)

$$|\langle u, v \rangle| \leq \|u\| \cdot \|v\|$$

Equality holds iff u & v are L.D.

5. $\|x+y\| \leq \|x\| + \|y\|$

$$\big| \|x\| - \|y\| \big| \leq \|x - y\|$$

6. Orthogonal set of vectors :.

$\{u_1, \dots u_n\}$ is orthogonal if

$$\langle u_i, u_j \rangle = 0 \quad \forall\ i \neq j$$

$$1 \leq i, j \leq n.$$

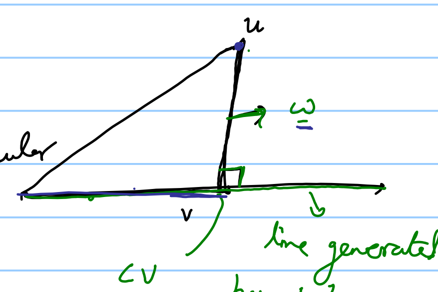
7) Non-zero Orthogonal set of vectors are L.I.

8) Unit vector $\frac{x}{\|x\|}$ is along x.

9) Decomposing a vector into 2 components —

let $v \neq 0$

u = a vector along v
+ vector perpendicular to v



$u = c \cdot v + (u - cv)$

component along v — this to be $\perp$er to v.

find c such that

$(u - cv) \perp v$

$\langle u - cv, v \rangle = 0$

$\langle u, v \rangle - c \langle v, v \rangle = 0$

$$\Rightarrow c = \frac{\langle u, v \rangle}{\|v\|^2}$$

Component of u along v is

$$cv = \frac{\langle u, v \rangle}{\|v\|^2} \cdot v$$

$u - cv \perp cv$ —①

$\Leftrightarrow u - cv \perp v$ —②

$c \in \mathbb{R}$

Orthogonal Projection of u along v

$$= \frac{\langle u, v \rangle}{\|v\|^2} \cdot v$$

ie $u - cv \perp v$ –

$$\langle u - cv, v \rangle = \langle u, v \rangle - c\langle v, v \rangle$$

$$= \langle u, v \rangle - \frac{\langle u, v \rangle}{\cancel{\|v\|^2}} \cdot \cancel{\|v\|^2}$$

$$= 0$$

$$u = \underbrace{\left(u - \frac{\langle u, v \rangle}{\|v\|^2} \cdot v\right)}_{\text{component } \perp\text{er to } v} + \underbrace{\left(\frac{\langle u, v \rangle}{\|v\|^2} \cdot v\right)}_{\text{component along } v}$$

v_1, v_2 st $v_1 \perp v_2$, $v_1 \neq 0$; $v_2 \neq 0$

u = component along v_1 + component along v_2 + component perpendicular to both v_1 & v_2.

Component of u along v_1

$$w_1 = \frac{\langle u, v_1 \rangle}{\|v_1\|^2} \cdot v_1$$

Component of u along v_2

$$w_2 = \frac{\langle u, v_2 \rangle}{\|v_2\|^2} \cdot v_2$$

Component of u which is orthogonal to v_1 & v_2

$$(u - w_1 - w_2)$$

$(u - w_1 - w_2) \perp v_1$ & $(u - w_1 - w_2) \perp v_2$

$$u = \frac{\langle u, v_1 \rangle}{\|v_1\|^2} v_1 + \frac{\langle u, v_2 \rangle}{\|v_2\|^2} v_2$$
$$+ \left(u - \frac{\langle u, v_1 \rangle}{\|v_1\|^2} \cdot v_1 - \frac{\langle u, v_2 \rangle}{\|v_2\|^2} \cdot v_2 \right)$$

<u>Gram-Schmidt process :-</u>

Given any basis of IPS V
$\Rightarrow$ convert it to a orthogonal basis

<u>Thm :-</u> Any finite dimensional IPS always has an orthogonal basis.

<u>Pf :-</u> $\{u_1, u_2, \ldots u_n\}$ is a basis of IPS 'V'.

Aim :- To construct a basis $\{w_1 \ldots w_n\}$ st it is orthogonal too.

<u>Step I :-</u> $w_1 = u_1$

<u>Step 2 :-</u> Find w_2 such that it is $\perp$er to $w_1 = u_1$

take u_2, find the component of u_2 along $u_1 = w_1$
it is
$$\frac{\langle u_2, w_1 \rangle}{\|w_1\|^2} w_1$$

$$w_2 = \left[u_2 - \frac{\langle u_2, w_1 \rangle}{\|w_1\|^2} w_1 \right]$$

$w_2 \perp w_1$

Check $\langle w_2, w_1 \rangle = 0$

$$\langle w_2, w_1\rangle = \left\langle u_2 - \left(\frac{\langle u_2, w_1\rangle}{\|w_1\|^2}\right) \cdot w_1 \, , \, w_1 \right\rangle$$

$$= \langle u_2, w_1\rangle - \frac{\langle u_2, w_1\rangle}{\cancel{\|w_1\|^2}} \cdot \cancel{\langle w_1, w_1\rangle}$$

$$= 0$$

$\{w_1, w_2\}$ – orthogonal set.

Step 3:- find w_3 such that $w_3 \perp w_1$, $w_3 \perp w_2$

take u_3 – find the component along w_1 & component along w_2

$w_3 = ($ u_3 – component along w_1 – component along w_2 $)$

Component of u_3 along w_1

$$\frac{\langle u_3, w_1\rangle}{\|w_1\|^2} \cdot w_1$$

Component of u_3 along w_2 is

$$\frac{\langle u_3, w_2\rangle}{\|w_2\|^2} \cdot w_2$$

$$w_3 = u_3 - \frac{\langle u_3, w_1\rangle}{\|w_1\|^2} w_1 - \frac{\langle u_3, w_2\rangle}{\|w_2\|^2} \cdot w_2 .$$

Check: $w_3 \perp w_1$ & $w_3 \perp w_2$

$$\langle w_3, w_1\rangle = 0 = \langle w_3, w_2\rangle .$$

$$\langle w_3, w_1\rangle = \langle u_3, w_1\rangle - \frac{\langle u_3, w_1\rangle}{\cancel{\|w_1\|^2}} \cdot \cancel{\langle w_1, w_1\rangle}$$

$$- \frac{\langle u_3, w_2\rangle}{\|w_2\|^2} \underbrace{\langle w_2, w_1\rangle}_{= 0}$$

$$= \langle u_3, w_1\rangle - \langle u_3, w_1\rangle = 0$$

$$w_4 = u_4 - \frac{\langle u_4, w_1\rangle}{\|w_1\|^2} \cdot w_1 - \frac{\langle u_4, w_2\rangle}{\|w_2\|^2} w_2$$

$$- \frac{\langle u_4, w_3\rangle}{\|w_3\|^2} \cdot w_3$$

then $w_4 \perp w_i \quad \forall\, 1 \le i \le 3$.

Proceeding like this

$$w_i = u_i - \sum_{j=1}^{i-1} \frac{\langle u_i, w_j\rangle}{\|w_j\|^2} \cdot w_j$$

$\forall\ 1 \le i \le n$.

$\{w_1, \ldots, w_n\}$ are orthogonal, non-zero & hence L-I.

$\Rightarrow$ they form a orthogonal basis

eg. $\mathbb{R}^2$, $e_1 = (1,0)$

$e_2 = (1,1)$

$\{e_1, e_2\}$ — L-I. $\to$ form a basis

$\langle e_1, e_2\rangle = 1 \neq 0$

$w_1 = e_1$

$$w_2 = e_2 - \frac{\langle e_2, w_1\rangle}{\|w_1\|^2} \cdot w_1$$

$$= (1,1) - \frac{1}{1} \cdot (1,0)$$

$$= (1,1) - (1,0) = (0,1)$$

$\left.\begin{array}{l} w_1 = (1,0) \\ w_2 = (0,1) \end{array}\right\}$ - orthogonal.

② $\mathbb{R}^2$, $v_1 = (1,1)$; $v_2 = (1,0)$

construct orthogonal basis

$w_1 = v_1$

$\|w_1\|^2 = 2. \quad \|(1,1)\| = \sqrt{\langle (1,1),(1,1)\rangle}$
$= \sqrt{1+1} = \sqrt{2}$

$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\|w_1\|^2} \cdot w_1$$

$$= (1,0) - \frac{1}{2}(1,1) = (1,0) - \left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

$$= \left(\tfrac{1}{2}, -\tfrac{1}{2}\right)$$

$w_1 = (1,1)$
$w_2 = \left(\tfrac{1}{2}, -\tfrac{1}{2}\right)$

$w_1 \perp w_2 \Rightarrow \langle w_1, w_2\rangle = 0$

Note:- Gram-Schmidt process doesn't give unique orthogonal basis.

H.W!- $\mathbb{R}^3$, $e_1 = (1,0,0)$
$e_2 = (1,1,0)$
$e_3 = (1,1,1)$

$\sqrt{\langle e_1, e_1\rangle}$

find orthogonal basis from $\{e_1, e_2, e_3\}$

& $\{e_2, e_3, e_1\}$

or $\{e_3, e_2, e_1\}$

(2) Find a basis of $\mathbb{R}^4$, from any 4×4 invertible matrix (take columns or rows to make basis vectors)

$e_1 = (1,0,0)$

$$\|e_1\|^2 = \langle e_1, e_1\rangle = \langle (1,0,0), (1,0,0)\rangle$$
$$= 1\cdot 1 + 0\cdot 0 + 0\cdot 0$$
$$= 1$$

$e_2 = (1,1,0)$

$$\|e_2\|^2 = 1^2 + 1^2 + 0^2$$
$$= 2$$

$$\langle (x_1, \dots x_n), (y_1 \dots y_n) \rangle$$
$$= (x_1 y_1 + x_2 y_2 + \cdots + x_n y_n)$$

if $x_i = y_i \;\forall\; 1 \le i \le n$

$$\|x\|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$$

$$x = (x_1, x_2, \dots, x_n)$$

Orthonormal set :-

orthogonal + norm $= 1$ or unit vector

$\{u_1 \dots u_n\}$ is said to orthonormal if

$$\langle u_i, u_j \rangle = 0 \quad \forall\; i \ne j$$

$$\langle u_i, u_i \rangle = 1 \quad \forall\; 1 \le i \le n.$$

orthogonal + norm $= 1$.

Remark:- $(1, 1)$, $\left(\frac{1}{2}, -\frac{1}{2}\right)$

$\{u_1 \dots u_n\}$ – orthogonal

$\left\{ \frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}, \dots \frac{u_n}{\|u_n\|} \right\}$ – orthonormal

② standard basis

$\{e_1 = (1, 0 \dots 0)$

$e_2 = (0, 1, 0 \dots 0)$

$\vdots$

$e_n = (0, \dots 1)$

– orthonormal

Gram schmidt $\Rightarrow$ find orthonormal basis